# THE NEPALI MATHEMATICAL SCIENCES REPORT

INSTITUTE OF SCIENCE
DEAN'S OFFICE
TRIBHUVAN UNIVERSITY
KIRTIPUR
NEPAL

Vol. 6 No. 1 & 2 1981

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1. Erdelyi, A., Tables of Integral Transforms, Vol. II, McGraw Hill, (1954
2. Stokes, G.G., On the effect of the internal friction on the motion of pendulums, Trans. Camb. Phil. Soc., 9 (1851), 8-106.

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# Contents

Nep. Math. Sc. Rep.
Vol. 6, Nos. 1 & 2 (1981), 1-8

# A Characterization of the Family of Power Distributions Through Structural Setup

T.S.K. Moothathu

Summary

The family of power distributions is defined. The random variables such as Lognormal, Pareto, Gamma and Beta belong to this family. When the conditional distribution of X given XY has a certain structural setup, X and Y are shown to belong to the family of power distributions. The corollaries of this main result characterizes the Lognormal, Pareto, Gamma and Beta distributions.

Key Words: Family of Power distributions; Lognormal, Pareto (including discrete Pareto and inverse Pareto), Gamma and Beta (of first and second kind) distributions; conditional density.

## 1. Introduction

In general the marginal distributions of X and Y cannot be determined by knowing the conditional distribution of X given $Z = U(X,Y)$, where $U(x,y)$ is a Borel measurable function of $(x,y)$. Patil and Seshadri (1964) considered $U(X,Y) = X + Y$ (i.e. $Z = X+Y$) and showed that the marginal distributions of the independent random variables X and Y (either both discrete or both continuous) are determined when the conditional distribution of X given $Z = X + Y$ has a certain structural setup. We consider the case $U(X,Y) = XY$ (i.e. $Z = XY$). In what follows it is shown that the two non-negative independent continuous random variables X and Y belong to the family of power distributions, to be defined now, when the conditional density of X given XY, $c(x,z)$ has a certain structural setup.

Definition: If a random variable X has the probability function

$$p(x) = \begin{cases} b\, u(x)\, x^{\theta}, & \text{for } x \in A,\ b > 0,\ \theta \in \Omega \\ 0, & \text{elsewhere} \end{cases} \tag{1.1}$$

where A is a subset of the set of positive real numbers, and b is the normalizing factor, that is,

$$b^{-1} = \int_A u(x)\, x^{\theta}$$

where $\int_A$ denotes the integral or summation depending upon X is continuous or discrete, and $\Omega$ is some parameter space, which is a non-empty subset of the set of real numbers, then $\{p(x),\ \theta \in \Omega\}$ is said to be a family of power distributions.

A random variable X (or its distribution) is said to belong to the family of power distributions when the probability function of X belongs to some family of power distributions.

Since, when X belongs to a family of power distributions log X belongs to a 'linear exponential family of distributions' (Mathai and Pederzoli (1977), the family of power distributions may also be named as 'log-linear exponential family of distributions.'

Among the continuous random variables the Lognormal, Pareto, inverse Pareto (if X is Pareto, 1/X is called inverse Pareto), Gamma and Beta of first and second kind belong to the family of power distributions. For, the density functions $f_1$, $f_2$, $f_3$, $f_4$, $f_5$ and $f_6$ of these random variables respectively are given by,

$$f_1(x) = (x^2 \sigma^2 2\pi)^{-\frac{1}{2}} \exp -(2\sigma^2)^{-1} (\log x - \mu)^2 ,$$

$$-\infty < \mu < \infty;\ x, \sigma > 0$$

$$= b \exp\left[-(2\sigma^2)^{-1} (\log x)^2\right] x^{\theta},\ x > 0 \qquad (1.2)$$

where $\sigma > 0$ is fixed but arbitrary, $\theta = \mu\sigma^{-2} - 1$,

$$b = \left\{2\pi\sigma^2 \exp\left[(\theta\sigma + \sigma)^2\right]\right\}^{-\frac{1}{2}},\quad -\infty < \theta < \infty.$$

$$f_2(x) = a\, k^a x^{-(a+1)},\ x \geq k > 0,\quad a > 0$$

$$= b\ x^{\theta},\ x \geq k > 0,\quad \theta < -1 \qquad (1.3)$$

where k is fixed, but arbitrary, $\theta = -(a+1)$ and

$$b = -(\theta + 1)\, k^{-(\theta + 1)}$$

$$f_3(x) = a\, k^a x^{a-1} \quad 0 < x \leq k^{-1};\ a, k > 0$$

$$= b\, x^{\theta},\quad 0 < x \leq k^{-1},\quad \theta > -1 \qquad (1.4)$$

where k is fixed, but arbitrary, $\theta = a-1$ and $b = (1+\theta)\, k^{(1+\theta)}$

$$f_4(x) = b\, e^{-mx} x^{\theta},\ x > 0,\quad \theta > -1 \qquad (1.5)$$

where $m > 0$ is fixed but arbitrary and $b = m^{(\theta+1)} (\Gamma(\theta+1))^{-1}$

$$f_5(x) = [B(p,q)]^{-1} x^{p-1} (1-x)^{q-1},\ 0 < x \leq 1;\ p, q > 0$$

$$= b(1-x)^{q-1} x^{\theta},\ 0 < x \leq 1,\quad \theta > -1 \qquad (1.6)$$

where q is fixed but arbitrary, $\theta = p-1$ and $b = [B(\theta+1,\ q)]^{-1}$

$$f_6(x) = [B(p,q)]^{-1} (1+x)^{-(p+q)} x^{p-1}, \quad x, p, q > 0$$

$$= b(1+x)^{-(\theta+q+1)} x^{\theta}, \quad x > 0, \quad \theta > -1 \qquad (1.7)$$

where q is fixed, but arbitrary and $\theta = p-1$; $b = [B(\theta+1,q)]^{-1}$

Reimann zeta (which is also known as discrete Pareto) variate is an example of the discrete random variable that belongs to the family of power distributions. For its probability function $p(x)$ is given by

$$p(x) = \left[\sum_{r=1}^{\infty} r^{-(a+1)}\right]^{-1} x^{-(a+1)}, \quad x = 1, 2, \ldots \quad a > 0$$

$$= b\, x^{\theta}, \ \theta < -1, \quad x = 1, 2, \ldots \qquad (1.8)$$

where $\theta = -(a+1)$ and $b = \left[\sum_{r=1}^{\infty} r^{\theta}\right]^{-1}$

Since there are not many familiar discrete random variables which belong to the family of power distributions, we restrict our attention to continuous random variables, and establish the following characterization theorem.

## 2. A Characterization Theorem

Let the following assumptions be made about the random variables X and Y.

(i) X and Y are non-negative independent continuous random variables.

(ii) The density functions $f(x)$ and $g(y)$ of X and Y do not vanish at unity.

(iii) The conditional density function of X given XY, denoted by $c(x,z)$ is such that

$$\frac{c(xy,xy)\ \ c(1,y)}{c(x,\ xy)\ \ c(y,y)} \text{ is of the form } \frac{u(xy)}{u(x)\ u(y)}$$

where $u(.)$ is an arbitrary non-negative function.

The following theorem can now be established to characterize several families of power distributions.

Theorem: Under the assumptions (i) - (iii)

$$f(x) = f(1)\ u(x)\ x^{\theta}$$

where $\theta$ is an arbitrary constant and $f(1)$ a suitable normalizer which makes $f(x)$ a density function,

$$g(y) = g(1)\, k(y)\, y^{\theta} \quad \text{where}$$

$$k(y) = u(y)\, c(1,y)/y\, c(y,y)$$

and $g(1)$ is the corresponding normalizer for $g(y)$. Further if

(iv) $B = \{x: u(x) > 0\} = \{y: k(y) > 0\}$ and

$$c(y,y) = y^{-1}\, c(1,y), \quad y \in B$$

then X and Y are identifically distributed.

Proof: We have

$$c(x,xy) = \frac{f(x)\, g(y)}{x\;\; r(xy)}$$

where $r(z) = \int x^{-1} f(x)\;\; g(z/x)dx$.

Hence

$$f(x)\;\; g(y) = x\; r(xy)\;\; c(x,xy) \tag{2.1}$$

Put $y = 1$ in (2.1) and change x to xy to obtain

$$f(xy)\;\; g(1) = xy\;\; r(xy)\;\; c(xy,xy) \tag{2.2}$$

Thus

$$\frac{f(xy)\;\; g(1)}{f(x)\;\;\; g(y)} = \frac{y\; c(xy,\; xy)}{c(x,\; xy)} \tag{2.3}$$

Setting $x = 1$ in (2.3) yields

$$\frac{f(y)\;\; g(1)}{f(1)\;\; g(y)} = \frac{y\; c(y,y)}{c(1,y)} \tag{2.4}$$

From (2.3) and (2.4) it is clear that

$$\frac{f(xy)\;\; f(1)}{f(x)\;\;\; f(y)} = \frac{c(xy,xy)\;\; c(1,y)}{c(x,xy)\;\; c(y,y)}$$

and by assumption (iii),

$$\frac{f(xy)\;\; f(1)}{f(x)\;\;\; f(y)} = \frac{u(xy)}{u(x)\;\; u(y)} \tag{2.5}$$

The transformation $\phi(x) = \frac{f(x)}{f(1)\, u(x)}$ reduces (2.5) to a Cauchy functional equation $\phi(xy) = \phi(x)\, \phi(y)$ whose solution is $x^{\theta}$ for some constant $\theta$. Thus

$$f(x) = f(1)\, u(x)\, x^{\theta},$$

$f(1)$ being a suitable normalizer that makes $f(x)$ a density function. And from (2.4)

$$f(y) = g(1)\, k(y)\, y^{\theta}$$

where $k(y) = u(y)\, c(1,y)/y\, c(y,y)$ and $g(1)$ is the corresponding normalizer of $g(y)$.

Further under the assumption (iv), we have

$k(y) = u(y), \quad y \in B$ and the requirement

$1 = \int_B f(x)\, dx = \int_B g(y)\, dy$ yields

$$f(x) = g(x), \quad x \in B.$$

Thus X and Y are identically distributed.

Notice that once the function $c(x,z)$ is specified the domain of the functions $u(x)$ and $k(y)$ - the sets $\{x : u(x) > 0\}$ and $\{y : k(y) > 0\}$ - can be determined since the transformation from $(x,z)$ to $(x,y)$ where $z = xy$, is one-to-one.

The Beta distribution, with density function (1.6) though belongs to the family of power distributions, does not come under the perview of the above theorem, since its density function vanishes at unity. In what follows the random variables X and Y are assumed to satisfy assumptions (i) and (ii).

Corollary 1: Let

$$c(x,z) = \text{const. } x^{-1} \exp\left[-(2\sigma^2)^{-1} (\log x - \tfrac{1}{2} \log z)^2\right], \quad x, z > 0.$$

Then

$$u(x) = \exp\left[-(2\sigma_1^2)^{-1} (\log x)^2\right], \quad x > 0$$

$$k(y) = \exp\left[-(2\sigma_1^2)^{-1} (\log y)^2\right], \quad y > 0$$

where $\sigma_1^2 = 2\sigma^2$. Thus X and Y are identically distributed and the common density function $f(.)$ is given by

$$f(x) = f(1) \exp\left[-(2\sigma_1^2)^{-1} (\log x)^2\right] x^{\theta}, \quad x > 0, \ -\infty \ \theta < \infty$$

<u>Corollary 2:</u> Let

$$c(x,z) = \left[x \log (z/k)\right]^{-1}, \quad k \leq x \leq z < \infty, \ 0 < k \leq 1, \ z \neq 1$$

Then

$$u(x) = 1, \quad x \geqslant k$$

$$k(y) = 1, \quad y \geqslant 1$$

Thus $f(x) = f(1)\, x^{\theta}, \ x \geqslant k$

$$g(y) = g(1)\, y^{\theta}, \quad y \geqslant 1 \qquad \theta < -1$$

Thus X and Y are distributed (identically when $k = 1$) as Pareto.

<u>Corollary 3:</u> Let

$$c(x,z) = \left[x \log (k/z)\right]^{-1}, \quad 0 < z \leq x \leq k, \ z \neq 1, \ k \geqslant 1$$

Then

$$u(x) = 1, \quad 0 < x \leqslant k$$

$$k(y) = 1, \quad 0 < y \leqslant 1$$

Thus

$$f(x) = f(1)\, x^{\theta}, \quad 0 < x \leqslant k$$

$$g(y) = g(1)\, y^{\theta}, \quad 0 < y \leqslant 1 \qquad k \geqslant 1, \ \theta > -1$$

Thus X and Y are distributed (identically when $k = 1$) as inverse Pareto.

<u>Corollary 4:</u> Let

$$c(x,z) = \left\{x\, s_1(z) \exp\left[m(x+z/x)\right]\right\}^{-1}, \quad x, z, m > 0$$

where $s_1(z) = \int_0^{\infty} x^{-1} \exp\left[-m(x+z/x)\right] dx$

Then

$$u(x) = \exp\left[-m(x+1)\right], \quad x > 0$$

$$k(y) = \exp\left[-m(y+1)\right], \quad y > 0$$

Thus X and Y are identically distributed as Gamma and the common density function f(.) is given by, after some adjustments,

$$f(x) = f(1)\, e^{-mx}\, x^{\theta}, \quad x > 0, \quad \theta > -1$$

Corollary 5: Let

$$c(x,z) = \left[x\, s_2(z)\right]^{-1} z^{m-1} \left[(1+x)(1+z/x)\right]^{-(m+n)},$$

$$x,\ z,\ m,\ n > 0.$$

where

$$s_2(z) = \int_0^{\infty} x^{-1}\, z^{m-1} \left[(1+x)(1+z/x)\right]^{-(m+n)} dx$$

Then

$$u(x) = \left(\frac{1+x}{2}\right)^{-(m+n)}, \quad x > 0$$

$$k(y) = \left(\frac{1+y}{2}\right)^{-(m+n)}, \quad y > 0$$

Thus X and Y are identically distributed as Beta (of second kind) and the common density function f(.) is given by,

$$f(x) = f(1)\, (1+x)^{-(m+n)}\, x^{\theta}, \quad x > 0\ ;\ \theta < m+n-1$$

Corollary 6: Let

$$c(x,z) = m \exp\left[m(1-x)\right], \quad x \geq z > 0; \quad m > 0,$$

Then

$$u(x) = \exp\left[m(1-x)\right], \quad x > 0$$

$$k(y) = y^{-1}, \quad 0 < y \leq 1$$

Thus

$$\left.\begin{aligned} f(x) &= f(1)\, \exp(-mx)\, x^{\theta}, \quad x > 0 \\ g(y) &= g(1)\, y^{\theta-1}, \quad 0 < y \leq 1 \end{aligned}\right. \qquad \theta > 0$$

Thus X is distributed as Gamma and Y is distributed as inverse Pareto.

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Nep. Math. Sc. Rep.
Vol. 6, No. 1 & 2 (1981), 9-14

# Childhood Mortality Estimation for Hilly Region of Nepal

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The present paper deals with estimating the childhood mortality on the basis of the information on total number of children ever born and children surviving for different age groups of mothers. The estimation is based on the data from Nepal Fertility Survey which was conducted in the international level in the year 1976 from April to June.

## Introduction

The main objective of the World Fertility Survey was to study human fertility and the factors which affect it in as many countries of the world as are willing to participate. In addition to the division of Nepal into four development regions, the country is divided into three geographic areas, the mountains, the hills and the Terai. The mountains range in altitude from 16000 ft. to 29028 ft. including 35% of the total land area and 10% of the total population of the country.

The hill region range in altitude from above 1000 ft. to about 16000 ft. including Kathmandu Valley and Pokhara. This area includes 44% of the total area and includes more than 52% of the total population. The Terai area range from about 200 ft. to 1000 ft. above sea level and includes about 1/5th of the total land area and little over 37% of the total population.

Our study is based on the data for Hill region only.

It is a well known fact that the level of infant mortality and childhood mortality depends on the proportion of children surviving to the mothers of different age groups. Thus the census reports in which data on surviving children are included often contain comments that variations within the enumerated population in proportions surviving can be considered as an index of differential mortality.

Willian Brass has greatly increased the usefulness of data of this sort by developing a method of translating proportions surviving and proportions dead among the children ever born to women in different age groups into conventional measure of mortality. His technique makes it possible under certain circumstances to estimate the proportion of children who survive to age 1,2,3,5,10,15,20, ... etc. from the proportion as surviving among children ever born to women of age groups 15-19, 20-24, 25-29, ... 60-64.

## Application

The data from World Fertility Survey, Nepal for hill region was used to compute the proportion of children dying to mothers of the above age-groups as described in Table 1.

Table 1
Childhood Mortality Estimation

| Age group | Proportion ever married | No. ever married | Total No. of women | Children ever born (CEB) | Children surviving | Proportion dying | Mean CEB $P_i$ |
|---|---|---|---|---|---|---|---|
| 15-19 | .6269 | 361 | 575.85 | 105 | 91 | .1333 | .1823 |
| 20-24 | .9397 | 583 | 620.41 | 835 | 671 | .1964 | 1.3459 |
| 25-29 | .9823 | 582 | 592.49 | 1628 | 1275 | .2168 | 2.7477 |
| 30-34 | .988 | 429 | 434.21 | 1745 | 1353 | .2246 | 4.0188 |
| 35-39 | | | | 1845 | 1356 | .2650 | |
| | | | | 2202 | 1573 | .2856 | |
| | | | | 1600 | 1158 | .2763 | |

Source:- Data from Brall 7125178 Jot 0715.

It has been seen that the proportion dead for a given age group of mother ($D_i$) depends on the length of time since the children were born and may be equated to the probability of dying between birth and some age r denoted by q(r), r will depend on the differences between the ages of mother at the time of the survey and those at which the children were born that is on how early the child bearing starts. It will also be affected by the way fertility changes with age of woman and the pattern of mortality in childhood. Since the effect of these are subsidiary only the estimation is based on some knowledge of the location of the fertility distribution. This was done by the numerical calculation of the values of r for a fixed standard pattern of mortality and a model fertility distribution of a simple form which varied only with the mean age of child bearing. It can be assumed that the proportion of children dying before their first birthday is close to the proportion dead among those ever born to women in the age group 15-19, the proportion dying before their second birth day close to the proportion dead among the children ever born to women 20-24, the proportion dying before their third birth day close to the proportion dead among the children ever born to women 25-29, the proportion dying before their fifth birth close to the proportion dead among the children ever born to women 30-34, before the tenth birth day to the proportion dead among the children ever born to women 35-39 and so on.

To comput q(r) that is the proportion of children born alive who die by age r, the proportion of children dead $D_i$ were multiplied by their corresponding multiplying f-ctors where $D_i$ is given by

$$D_i = \frac{CEB-CS}{CEB}$$

The result is presented in Table II. In order to select the right series of the multiplying factors by age group for deriving q(r) from the observed $D_i$ an estimate of the location of child bearing is needed.

In this case the ratio $P_2/P_3$ is considered a suitable choice of a location parameter, where $P_2$ and $P_3$ are the mean children ever born in the age groups 20-24 and 25-29 respectively, since $P_1$ the mean children ever born in the age group 15-19 is supposed to be sensitive to age reporting errors at the start of child bearing and also sample fluctuation due to the relatively small number of births to women in that age group. $P_2/P_3$ is particularly satisfactory for the estimates of q(2), q(3) and q(5). Hence the multiplying factors are selected corresponding to $P_2/P_3$ which was .490 in this case.

Table II
Estimation of Childhood Mortality Nepal WFS 1976

| Age group 15-19 | Multiplying factor K | Proportion dying $D_i$ | Estimated | | $Y_s(r)$* | Graduated | |
|---|---|---|---|---|---|---|---|
| | | | q(r) $KD_i$ | Y(r) logit (1-q(r)) | | Y(r) | q(r) |
| 15-19 | 0.977 | .1333 | .1302 | .9496 | .8670 | .8861 | .1453 |
| 20-24 | 1.101 | .1964 | .1984 | .6982 | .7152 | .7183 | .1921 |
| 25-29 | 0.994 | .2168 | .2155 | .6420 | .6552 | .6519 | .2135 |
| 30-34 | 1.002 | .2246 | .2250 | .6184 | .6015 | .5925 | .2342 |
| 35-39 | 1.011 | .2650 | .2679 | .5027 | .5498 | .5354 | .2553 |
| 40-44 | 0.988 | .2856 | .2822 | .4668 | .5131 | .4948 | .2710 |
| 45-59 | 0.986 | .2763 | .2724 | .4912 | .4551 | .4306 | .2971 |

*Logit of the general standard life table.

It has been found that the probability of dying from birth to age one is estimated as .1333 that is the infant mortality is 133 per thousand which seems to be quite smaller than what it should be. Usually in developing countries it has been supposed that the estimates of q(r) derived in this way may be affected by some errors. It seems that children who die very young may be omitted from the records, also the absence of the children of mother who died in child birth may be missed from reporting and thus will lower the rates since these births have a smaller chance of surviving. Such graduation also helps in reducing sampling errors when the data are from small surveys.

In the graduation method first of all the q(r) values are transferred to logit values given by

$$Y(x) = 1/2 \log_e \frac{q(x)}{1-q(x)}$$

In the logit model system

$$Y(x) = \alpha + \beta Y_s(x)$$

which is assumed to be linear. Here $Y_s(x)$ is the logit taken from the standard life table and $\alpha$ and $\beta$ are constants. The average of the three

estimated logits in age group 20-24, 25-29, 30-34 and the corresponding average of the same three logits from the standard values were computed. Also the average of the logits of the later three age groups for the estimated and standard values were computed. These values were respectively found to be .6542, .6573, .4869 and .5060. The parameters $\alpha$ and $\beta$ were computed by solving the two equations

$$.6542 = \alpha + .6573\,\beta$$

$$.4869 = \alpha + .5060\,\beta$$

which were obtained as

$$\alpha = -.0726,\ \beta = 1.1058$$

The graduated logit is obtained by the relation

$$\text{logit}\quad 1-q(r) \quad = \quad + \quad \log \quad 1-q_s(r)$$

and the graduated $q(r)$ is given by

$$q(r) = \frac{1}{1 + e^{2\ \text{logit}\quad 1-q(r)}}$$

The graduated estimates are also presented in table number II. The graduated value of $q(1)$ i.e. .1453 seem to be better than the estimated $q(1)$ which is .1333. In the Nepal Fertility Survey Report the infant deaths in the years 2028, 2029 and 2030 and dividing by the total number of live births during that period was 152 per thousand which is considered to be small than our actual infant mortality. The infant mortality from Demographic sample survey 1976 is 52.79 per thousand for urban Nepal and 136.10 per thousand for rural Nepal which seem to be still smaller and more unreliable than the former one.

The estimate of $q(1)$ obtained in this study may not be reliable because of the difficulty of locating fertility at the beginning of reproduction precisely and the sampling fluctuations from small numbers. The estimate may be more representative if $q(2)$ may be considered as $q(1)$ which are .1984 as estimated value and .1921 as the graduated value so that the infant mortality will come as 198 per thousand and 192 per thousand in the two cases.

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Nep. Math. Sc. Rep.
Vol. 6, No. 1 & 2 (1981), 15-20

# On a Distributional Poison Transform

Bhanu Chandra Bajracharya

## 1. Abstract

A complex inversion formula is obtained for the Poison Transform and is extended to generalized functions interpreting convergence in the weak distributional sense.

## Introduction

Let $f(y)$ belong to L in $(-R < y < R)$ for every positive R and be such that the integral

$$F(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1 + (x-y)^2} f(y)\, dy \tag{1.1}$$

converges for $|\mathrm{Im}.x| < 1$.

The inversion operator for (1.1) as defined by Pollard, H. [2] is as follows:

$$T_t(F(x)) = \left(\frac{\sin tD}{D}\right) \hat{F}(x) + (\cos tD)F(x) \tag{1.2}$$

where

$\wedge$. is defined by

$$\hat{F}(x) = -\frac{1}{\pi} \int_{0}^{\infty} u^{-2} [F(x+u) + F(x-u) - 2\, F(x)]\, du. \tag{1.3}$$

Pollard, H. [2] has shown that

$$\lim_{t \to 1-} T_t(F(x)) = f(x).$$

The purpose of the present paper is to find out a complex inversion formula for (1.1) and to extend it to generalized functions in the weak distributional sense.

## 2. Notation and Terminology

R denotes the real one-dimensional Euclidean space. A function is said to be smooth if its derivatives of all orders are continuous at all points of its domain. $\langle f, \emptyset \rangle$ denotes the number assigned to some element $\emptyset$ in a certain testing function space by a member of the dual space. D is the space of smooth functions on R having compact support. D' is the dual space of D.

## 3. Testing Function Space $H_{n,\alpha}$.

An infinitely differentiable complex valued function $\phi(y)$ defined over $I(-\infty, \infty)$ satisfying the relation

$$\gamma_n(\phi(y)) = \gamma_{n,\alpha}(\phi) = \sup_{-\infty < y < \infty} \left| (1+y^2)^{\alpha/2} y^n D^n \phi(y) \right| < \infty$$

belongs to $H_{n,\alpha}$ for any fixed n where n assumes the values 0,1,2,3, ..... and $\alpha \leq 1$. $\gamma_0$ is a norm and hence the collection of semi-norms is separating [4, p. 8]. The topology of $H_{n,\alpha}$ is generated by the collection of semi-norms $\{\gamma_n\}_{n=0}^{\infty}$. A sequence $\{\phi_\nu\}_{\nu=1}^{\infty}$ where each $\phi_\nu \in H_{n,\alpha}$ is said to converge in $H_{n,\alpha}$ to the limit $\phi$ in $H_{n,\alpha}$ if $\gamma_n(\phi_\nu - \phi) \longrightarrow 0$ as $\nu \longrightarrow \infty$ for each n = 0,1,2, ..... A sequence $\{\phi_\nu\}_{\nu=1}^{\infty}$ is said to be a Cauchy sequence if $\gamma_n(\phi_\nu - \phi_\mu) \longrightarrow 0$ as $\nu$ and $\mu$ both tend to infinity independently of each other.

Lemma 3.1

For any $\alpha \leq 1$ and for any complex x such that $|\mathrm{Im}.x| < 1$, ($\mathrm{Re}.x = \sigma$, $\mathrm{Im}.x = \tau$)

$$\phi(y) = \frac{1}{\pi} \frac{1}{1 + (x-y)^2} \text{ belongs to } H_{n,\alpha}.$$

Proof:

$$\gamma_{n,\alpha}(\phi) = \sup_{-\infty < y < \infty} \left| (1+y^2)^{\alpha/2} y^n D^n \phi(y) \right|$$

$$= \sup_{-\infty < y < \infty} \left| (-1)^n n! (1+y^2)^{\alpha/2} y^n \frac{\sin\left\{(n+1) \arctan \frac{1}{x-y}\right\}}{\left[1 + (x-y)^2\right]^{\frac{1}{2}(n+1)}} \right|$$

$$\leq \sup_{-\infty < y < \infty} \left(\frac{1+y^2}{y^2}\right)^{\alpha/2} |y|^{\alpha-1} \frac{1}{\left\{\frac{1-\tau^2}{y^2} + (\sigma/y - 1)^2\right\}^{\frac{1}{2}(n+1)}}$$

which is bounded for all values of y for $\alpha \leq 1$. This completes the proof of the lemma.

Lemma 3.2

For an arbitrary element $f(t) \in H'_{n,\alpha}$ and $\alpha \leq 1$, let

$$F(x) = \left\langle f(t), \ \frac{1}{\pi} \ \frac{1}{1+(x-y)^2} \right\rangle$$

for any x in the complex plane for which $|\text{Im. } x| < 1$, then for $k = 0,1,2, \ldots\ldots$

$$F^{(k)}(x) = \left\langle f(t), \ (\partial/\partial x)^n \left\{ \frac{1}{\pi} \ \frac{1}{1+(x-y)^2} \right\} \right\rangle$$

The proof of this lemma follows from Bajracharya, B.C. [1].

4. Complex Inversion Formula

If $F(x)$ and $\widehat{F}(x)$, defined by (1.1) and (1.3), converge, then

$$\lim_{t \to 1} \left[ 2^{-1} \left\{ F(x+it) + F(x-it) \right\} + (2iD)^{-1} \left\{ \widehat{F}(x+it) - \widehat{F}(x-it) \right\} \right] = f(x)$$

where x is a real variable.

Proof:

Since

$$F(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+(x-y)^2} \ f(y) \ dy,$$

so

$$2^{-1} \left[ F(x+it) + F(x-it) \right] = (2\pi)^{-1} \int_{-\infty}^{\infty} \left\{ \frac{1}{1+(x-y+it)^2} + \frac{1}{1+(x-it-y)^2} \right\} f(y) \ dy$$

$$(4.1) \qquad = \pi^{-1} \int_{-\infty}^{\infty} \frac{1-t^2 + (x-y)^2}{1+2(x-y)^2 - 2t^2 + (x-y)^4 + 2t^2(x-y)^2 + t^4} \ f(y) \ dy$$

Also since

$$1 + 2(x-y)^2 - 2t^2 + (x-y)^4 + 2t^2 (x-y)^2 + t^4$$

$$= \left\{ (1+t)^2 + (x-y)^2 \right\} \left\{ (1-t)^2 + (x-y)^2 \right\},$$

setting $1+t = a$, $1-t = b$ and $x-y = z$ in

$$\frac{1-t^2 + (x-y)^2}{\left\{ (1+t)^2 + (x-y)^2 \right\} \left\{ (1-t)^2 + (x-y)^2 \right\}},$$

We have

$$\frac{ab+z^2}{(a^2+z^2)\,(b^2+z^2)} = \frac{Az+B}{a^2+z^2} + \frac{Cz+D}{b^2+z^2}$$

where A,B,C,D are constants to be determined.

Therefore,

$$z^2 + ab = z^3\,(A+C) + z^2\,(B+D) + z(Ab^2 + Ca^2) + (Bb^2 + Da^2)$$

Equating the coefficients of like terms,

$$A+C=o \tag{4.2}$$

$$B+D=1 \tag{4.3}$$

$$Ab^2 + Ca^2 = 0 \tag{4.4}$$

$$Bb^2 + Da^2 = ab \tag{4.5}$$

Solving the above four equations we have,

$$A=C=0 \text{ and } B = \frac{1+t}{2}, \quad D = \frac{1-t}{2}$$

Thus,

$$2^{-1}\,[F(x+it) + F(x-it)]$$
$$= (2\pi)^{-1} \int_{-\infty}^{\infty} \left[\frac{1-t}{(1-t)^2 + (x-y)^2} + \frac{1+t}{(1+t)^2 + (x-y)^2}\right] f(y)\,dy \tag{4.6}$$

Next, using Pollard, H. [2]

$$F(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1-(x-y)^2}{[1+(x-y)^2]^2}\, f(y)\,dy$$

and a process similar to the above case we obtain

$$(2iD)^{-1}\,[\hat{F}(x+it) - \hat{F}(x-it)]$$
$$= (2\pi)^{-1} \int_{-\infty}^{\infty} \left[\frac{1-t}{(1-t)^2 + (x-y)^2} - \frac{1+t}{(1+t)^2 + (x-y)^2}\right] f(y)\,dy \tag{4.7}$$

Now in view of (4.6) and (4.7)

$$\lim_{t\to 1-} \left[2^{-1}\left\{F(x+it) + F(x-it)\right\} + (2iD)^{-1}\left\{\hat{F}(x+it) - \hat{F}(x-it)\right\}\right]$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1 - t}{(1-t)^2 + (x-y)^2} f(y)\, dy. \quad \text{as } t \to 1-$$

$\longrightarrow f(x)$ as $t \to 1-$ which has been shown by Pollard, H. [2]

5. Distributional Complex Inversion for the Poison Transform

Theorem 5.1

If $f(y) \in H'_{n,\alpha}$ and $F(x)$ be the Poisson transform of $f(y)$, then for an arbitrary element $\phi(x) \in D(I)$, we have

$$\left\langle \left[ 2^{-1} \left\{ F(x+it) + F(x-it) \right\} + (2iD)^{-1} \left\{ \hat{F}(x+it) - \hat{F}(x-it) \right\} \right], \phi(x) \right\rangle$$

$$\longrightarrow \langle f(y), \phi(y) \rangle \text{ as } t \to 1-$$

Proof: Since

$$F(x) = \left\langle f(y), \frac{1}{\pi} \frac{1}{1 + (x-y)^2} \right\rangle ,$$

the theorem will be proved if the steps in the following manipulations are justified.

$$\left\langle \left[ 2^{-1} \left\{ F(x+it) + F(x-it) \right\} + (2iD)^{-1} \left\{ \hat{F}(x+it) - \hat{F}(x-it) \right\} \right] \phi(x) \right\rangle$$

$$(5.1) \qquad = \left\langle \left\langle f(y), \frac{1}{\pi} \frac{1 - t}{(1-t)^2 + (x-y)^2} \right\rangle , \phi(x) \right\rangle$$

$$(5.2) \qquad = \left\langle f(y), \left\langle \frac{1}{\pi} \frac{1 - t}{(1 - t)^2 + (x-y)^2} \right\rangle, \phi(x) \right\rangle$$

$$\longrightarrow \langle f(y), \phi(y) \rangle \text{ as } t \to 1- .$$

Some simple calculations justify the relation (5.1). The equality of relation (5.1) to the relation (5.2) is justified by a process very similar to Zemanian [3, p. 237]

Now, to complete the proof, it suffices to show that

$$\left\langle \frac{1}{\pi} \frac{1 - t}{(1-t)^2 + (x-y)^2}, \phi(x) \right\rangle \longrightarrow \phi(y) \text{ as } t \to 1 -$$

Therefore,

$$(1+y^2)^{\alpha/2}\, y^n D_y^n \left[ \left\langle \frac{1}{\pi} \frac{1-t}{(1-t)^2+(x-y)^2}, \phi(x) \right\rangle - \phi(y) \right]$$

$$= (1+y^2)^{\alpha/2}\, y^n \left[ \left\langle (-D_x)^n \left\{ \frac{1}{\pi} \frac{1-t}{(1-t)^2+(x-y)^2} \right\}, \phi(x) \right\rangle - D_y^n \phi(y) \right]$$

Integration by parts leads to

$$(1+y^2)^{\alpha/2}\, y^n \left[ \left\langle \frac{1}{\pi} \frac{1-t}{(1-t)^2+(x-y)^2}, D_x^n \phi(x) \right\rangle - D_y^n \phi(y) \right]$$

$$= (1+y^2)^{\alpha/2}\, y^n \left[ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1-t}{(1-t)^2+(x-y)^2} (D_x^n \phi(x))\, dx - D_y^n \phi(y) \right]$$

$$= \frac{(1+y^2)^{\alpha/2}\, y^n}{\pi} \int_{-\infty}^{\infty} \frac{1-t}{(1-t)^2+(x-y)^2} (\phi^n(x) - \phi^n(y))\, dx \tag{5.3}$$

where $\phi^n(x) \triangleq D_x^n \phi(x)$ and

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1-t}{(1-t)^2+(x-y)^2}\, dx = 1$$

The right hand side of (.3) tends to zero as $t \to 1-$ which has already been shown in Bajracharya, B.C. [1].

This completes the proof of the theorem.

Acknowledgement

I wish to express my gratitude to Dr. R.P. Manandhar for his help in the preparation of this paper.

The author is also thankful to the Institute of Science, T.U. for the award of Ph.D. Research fellowship.

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Nep. Math. Sc. Rep.
Vol. 6, No. 1-2 (1981), 21-26

# A Characterization of Hypodivergence Measure

Parvinder Kaur

Summary

The hypodivergence measure between two discrete distribution is characterized in this paper with the help of a functional inequality.

## 1. Introduction

Let $P = (p_1, p_2, \ldots, p_n)$ $Q = (q_1, q_2, \ldots q_n)$, $p_i, q_i \geqslant 0$ with $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$ be two finite discrete probability distributions. Then the directed divergence defined by Kullback (1959) between two distributions P and Q is

$$I_n(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots q_n) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i} \tag{1.1}$$

where the convention followed is that whenever a $q_i = o$ the corresponding $p_i = o$ and that $o \log \frac{o}{o} = o$. This convention however, is not always very appealing.

Carlo Fereeri [1] has defined a hypodivergence measure between two probability distributions P and Q as

$$J_\lambda(P:Q) = \frac{1}{\lambda} \sum_{i=1}^{n} (1 + \lambda p_i) \log \frac{1 + \lambda p_i}{1 + \lambda q_i} \tag{1.2}$$

$$\lambda > o, \quad o \leq J_\lambda(P:Q) \leq \log(1 + \lambda)$$

The above hypodivergence measure (1.2) reduces to (1.1) as $\lambda \to \infty$ while for finite it permits to overcome the difficulty in defining (1.1) that (1.1) is defined only for $q_i > o$. Thus, the convention followed in defining (1.1) is totally removed in defining (1.2).

The above quantity (1.2) is characterized in the next section with the help of the functional inequality

$$\sum_{i=1}^{n} (1 + \lambda p_i) \left[ f(p_i) - f(q_i) \right] \geqslant o \tag{1.3}$$

## 2. Characterization

Theorem: If a function $H_\lambda(P:Q)$ satisfies the following postulates

$$A_1 : H_\lambda(p_1, p_2, \dots p_n; q_1, q_2, \dots q_n) = \sum_{i=1}^{n} (1+\lambda p_i) [f(p_i) - f(q_i)]$$

$$A_2 : H_\lambda(p_1, p_2, \dots, p_n; q_1, q_2, \dots q_n) \geqslant 0$$

and

$$A_3 : H_2(1, 0; 0, 1) = \log(1+\lambda)$$

then

$$H_\lambda(p_1, p_2, \dots, p_n; q_1, q_2 \dots, q_n) = J_\lambda(P:Q)$$

$$\frac{1}{\lambda} \sum_{i=1}^{n} (1+\lambda p_i) \log \frac{1+\lambda p_i}{1+\lambda q_i}, \lambda > 0$$

Before proving the theorem we shall first find the solution of the functional inequality (1.3) in the following lemma.

Lemma: Every solution of functional inequality (1.3) is differentable every where in (0,1) and the solutions are given by

$$f(x) = \frac{a}{\lambda} \log(1+\lambda x)$$

where a is any constant.

Proof of Lemma: Let us first show that f is monotone increasing in $[0,1]$

In the inequality (1.3) taking $p_3 = q_3$, $p_4 = q_4$, $\dots$ $p_n = q_n$, we have

$$(1+\lambda p_1) [f(p_1) - f(q_1)] + (1+\lambda p_2) [f(p_2) - f(q_2)] \geqslant 0 \qquad (2.1)$$

with $p_1 + p_2 = q_1 + q_2 < 1$

or

$$\frac{1+\lambda p_1}{1+\lambda p_2} [f(p_1) - f(q_1)] + [f(p_2) - f(q_2)] \geqslant 0 \qquad (2.2)$$

Interchanging $p_1$ and $q_1$ and $p_2$ and $q_2$ in (2.2), we get

$$\frac{1+\lambda q_1}{1+\lambda q_2}\left[f(q_1)-f(p_1)\right]+\left[f(q_2)-f(p_2)\right]\geqslant 0 \tag{2.3}$$

Adding (2.2) and (2.3) we have

$$\frac{1+\lambda p_1}{1+\lambda p_2}-\frac{1+\lambda q_1}{1+\lambda q_2}\left[f(p_1)-f(q_1)\right]\geqslant 0$$

Now let $q_1 > p_1$, $q_2 < p_2$

so that

$$\frac{1+\lambda p_1}{1+\lambda p_2} < \frac{1+\lambda q_1}{1+\lambda q_2}$$

implying $f(p_1) \leqslant f(q_1)$ for $p_1 < q_1$ (2.4)

which shows that f is monotone increasing in $[0, 1]$

Now in order to prove that f is differentiable every where in (0,1) let us prove that at all those points $p \in (0,1)$ at which f is diffrentiable

$$(1+\lambda p)\ f'(p) = a \geqslant 0 \tag{2.5}$$

holds

Since f is monotone increasing in (0,1), it is differentiable almost every where in (0,1) (cf. Natanson 1964, p. 211). The inequality (2.1) may be written as

$$(1+\lambda p_1)\left[f(p_1)-f(q_1)\right]\geqslant(1+\lambda p_2)\left[f(q_2)-f(p_2)\right] \tag{2.6}$$

Putting $q_1 = p_1 + \delta$, $q_2 = p_2 - \delta$, $\delta > 0$ in (2.6) and dividing both sides by $\delta$, we have

$$(1+\lambda p_1)\left[\frac{f(p_1)-f(p_1+\delta)}{\delta}\right]\geqslant(1+\lambda p_2)\left[\frac{f(p_2-\delta)-f(p_2)}{\delta}\right] \tag{2.7}$$

with $o < \delta < p_2 < 1-p_1$

Let $p_1$ and $p_2$ be the points where f is differentiable. Taking $\delta \longrightarrow o$ in (2.7) and using symmetry in $p_1$ and $p_2$, we get

$$(1+\lambda p)\ f'(p) = a \tag{2.8}$$

where a is some constant.

Since f is monotone increasing in (0,1), $f'(p) \geqslant 0$ giving $a \geqslant 0$.

Now let p be an arbitrary point and let $D^+$, $D^-$, $D_+$, $D_-$ be the four Dini-derivatives at p. Let $p_1$ be a point at which f is differentiable.

In (2.7) with $p_2 = p$, we obtain on taking Inf $\delta \longrightarrow 0$, from (2.7) and (2.8)

$$a \leq (1 + \lambda p)\, D_- f(p) \tag{2.9}$$

Again taking $p_1 = p$ in (2.7) and supposing that f is differentiable at $p_2$ and taking Sup $\delta \longrightarrow 0$, we get

$$(1 + \lambda p)\, D^+ f(p) \leq a \tag{2.10}$$

Taking $p_1 = q_1 + \delta$, $p_2 = q_2 - \delta$, $\delta > 0$ in (2.6), we have on division by

$$(1+ \lambda(q_1 + \delta))\left[\frac{f(q_1 + \delta) - f(q_1)}{\delta}\right] \geqslant (1 + \lambda(q_2 - \delta))\left[\frac{f(q_2) - f(q_2 - \delta)}{\delta}\right] \tag{2.11}$$

with $0 < \delta < q_2 < 1 - q_1$

Now assuming f to be differentiable at $q_1$ and taking $q_2 = p$,

then taking sup $\delta \longrightarrow 0$, (2.11) gives

$$(1 + \lambda P)\, D^- f(p) \leqslant a \tag{2.12}$$

Lastly, taking $q_1 = p$ and f to be differentiable at $q_2$ in (2.11) and taking Inf $\delta \longrightarrow 0$

$$(1 + \lambda p)\, D_+ f(p) \geqslant a \tag{2.13}$$

Hence for an arbitrary $p \in (0,1)$ we have from (2.9), (2.10), (2.12) and (2.13) that

$$D_- f = D^- f = D^+ f = D_+ f$$

Therefore, f is differentiable every where in (0,1) and we have

$$f'(p) = \frac{a}{a + \lambda p}$$

giving

$$f(p) = \frac{a}{\lambda} \log (1 + \lambda p) \tag{2.14}$$

This completes the proof of lemma.

Proof of theorem

Using postulates $A_1$ and $A_2$

$$H_\lambda(p_1, p_2, \dots p_n;\ q_1, q_2 \dots q_n) = \sum_{i=1}^{n} (1+\lambda p_i) \left[ f(p_i) - f(q_i) \right] \geq 0 \tag{2.15}$$

From (2.14) and (2.15)

$$H_\lambda(p_1, p_2, \dots p_n;\ q_1, q_2, \dots q_n) = \sum_{i=1}^{n} (1+\lambda p_i) \frac{a}{\lambda} \log \frac{1+\lambda p_i}{1+\lambda q_i} \tag{2.16}$$

Using postulate $A_3$, we get

$a = 1$

Hence

$$H_\lambda(p_1, p_2 \dots, p_n;\ q_1, q_2 \dots q_n) = \frac{1}{\lambda} \sum_{i=1}^{n} (1+\lambda p_i) \log \frac{1+\lambda p_i}{1+\lambda q_i} = J_\lambda(P:Q)$$

This proves the theorem.

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Nep. Math. Sc. Rep.
Vol. 6, No. 1 & 2 (1981), 27-32

# Hydromagnetic Pulsating Flow of Visco-Elastic Fluid Between Two Parallel Porous Surfaces

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Abstract

The exact solution is obtained for unsteady flow of an incompressible visco-elastic (Revlin-Ericksen model) conducting fluid between two non-conducting parallel porous surfaces with oscillating pressure gradient in time when there is uniform suction and injection on both the surfaces in the presence of uniform transverse magnetic field.

## 1. Introduction

The exact solution for unsteady flow of an incompressible viscous fluid between two parallel surfaces with oscillating pressure gradient in time has been obtained in [1]. The exact solution for unsteady flow of an incompressible viscous conducting fluid between two non-conducting parallel surfaces with oscillating pressure gradient in time has been obtained by the author [3] in the presence of uniform transverse magnetic field. This analysis has been extended by the author [4] to visco-elastic (Revlin-Ericksen model) fluid. The exact solution for unsteady flow of an incompressible viscous conducting fluid between two non-conducting parallel porous surfaces with oscillating pressure gradient in time has also been obtained by the author [5] when there is an equal uniform suction and injection on both the surfaces in the presence of uniform transverse magnetic field. In the present note, we extend the analysis further to visco-elastic (Revlin-Ericksen model) fluid.

## 2. Formulation of the Problem and Its Solution

Let u be the component of velocity in the direction of x-axis. A magnetic field of uniform strength $H_o$ is applied in the direction of y-axis taken perpendicular to both the non-conducting parallel porous surfaces which are placed at $y = \pm a$. The fluid being injected through the surface $y = -a$ and is being sucked through the surface $y = a$ with uniform velocity V. The induced magnetic field may be neglected assuming that the conductivity of the fluid to be very small. All the parameters are independent of x except the pressure since the surfaces are infinite. The pressure gradient is assumed to oscillate in time in the direction of x-axis. The only non-zero component of velocity will be u(y,t). The governing equation describing the flow of an incompressible visco-elastic conducting fluid in the presence of uniform transverse magnetic field is [2].

$$(1) \qquad \frac{\partial u}{\partial t} + V\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} + \beta\frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial y}\right) - \frac{\sigma}{\rho}B_o^2 u$$

where $\nu$ is the kinematic viscosity, $\beta$ the kinematic visco-elasticity, p the pressure, $\rho$ the density of the fluid, $\sigma$ the electrical conductivity and $B_o = \mu_e H_o$ (constant) the component of electromagnetic induction.

The boundary conditions are

(2) $$u(a,t) = u(-a,t) = 0, \quad t > 0.$$

As in [1] the pressure gradient and the non-zero component of velocity are assumed to be of the form

(3) $$\frac{\partial p}{\partial x} = \mathrm{Re}\,[P_x \exp(int)]$$

and

(4) $$u(y,t) = \mathrm{Re}\,[\omega(y) \exp(int)],$$

where $P_x$ is a constant, which represents the magnitude of pressure gradient oscillation.

Substituting (3) and (4) in (1) we get

(5) $$\beta\lambda \frac{d^3w}{dy^3} + (1 + \beta k^2)\frac{d^2w}{dy^2} - \lambda\frac{dw}{dy} - (m + k^2)w = \frac{P_x}{\rho\nu},$$

where $k^2 = \frac{in}{\nu}$, $m = \frac{\sigma B_o^2}{\rho\nu}$, $\lambda = \frac{V}{\nu}$.

The corresponding boundary conditions are

(6) $$w(a,t) = w(-a,t) = 0, \quad t > 0.$$

Since (5) is the differential equation of order three and we have only two boundary conditions, therefore we solve the equation regarding $\beta$ as small. This is consistent with the derivation of Revlin-Ericksen constitutive equation where only small values of $\beta$ is contempleted. Thus we set

(7) $$w = w_o + \beta w_1 + O(\beta^2).$$

Substituting (7) in (5) and equating the coefficients of different powers of $\beta$, we get

(8) $$\frac{d^2w_o}{dy^2} - \lambda\frac{dw_o}{dy} - (m + k^2)w_o = \frac{P_x}{\rho\nu}$$

and

(9) $$\frac{d^2w_1}{dy^2} - \lambda\frac{dw_1}{dy} - (m + k^2)w_1 = -\lambda\frac{d^3w_o}{dy^3} - k^2\frac{d^2w_o}{dy^2}.$$

The corresponding boundary conditions for $w_o$ and $w_1$ are

$$\left.\begin{array}{ll} (10) & w_o(a,t) = w_o(-a,t) = 0, \\ (11) & w_1(a,t) = w_1(-a,t) = 0 \end{array}\right\} \quad t > 0.$$

The solution of (8) subject to the condition (10) is

$$(12) \qquad w_o(y) = \frac{P_x}{\rho\nu(m+k^2)} \left[ \exp(\tfrac{\lambda y}{2}) \left\{ \frac{ch(\frac{\lambda a}{2})}{ch\ Ba}\ ch\ By\ \frac{sh(\frac{\lambda a}{2})}{sh\ Ba}\ sh\ By \right\} - 1 \right],$$

where $B = \sqrt{\frac{\lambda^2}{4} + m + k^2}$.

Substituting the value of $w_o(y)$ in (9) we get

$$(13) \qquad \frac{d^2 w_1}{dy^2} - \frac{dw_1}{dy} - (m+k^2)w_1 = \frac{P_x}{\rho\nu(m+k^2)} - \exp(\tfrac{\lambda y}{2})\ (M\ ch\ By + N\ sh\ By),$$

where $\quad m = -g\,\frac{ch\ (\frac{\lambda a}{2})}{ch\ Ba} + h\,\frac{sh\ (\frac{\lambda a}{2})}{sh\ Ba},\ N = -h\,\frac{sh\ (\frac{\lambda a}{2})}{ch\ Ba} + g\,\frac{sh\ (\frac{\lambda a}{2})}{sh\ Ba},$

$$g = \frac{k^2\lambda^2}{4} + k^2B^2 + \frac{\lambda^4}{8} + 2\lambda^2B^2,\ h = k^2\lambda + \frac{3\ \lambda^3 B}{4} + \lambda B^3.$$

The solution of (13) subject to the condition (11) is

$$w_1(y) = \frac{P_x}{2\ \rho\nu(m+k^2)B} - \exp(\tfrac{\lambda y}{2}) \left\{ (M\ ch\ By + N\ sh\ By)y \right.$$
$$\left. - a(M\frac{ch\ Ba}{sh\ Ba}\ sh\ By + N\ \frac{sh\ Ba}{ch\ Ba}\ ch\ By) \right\}.$$

Thus the solution for $u(y,t)$ is

$$(14) \qquad u(y,t) = Re \left\langle \left\{ w_o(y) + \beta w_1(y) \right\} \exp\ (int) \right\rangle$$
$$= Re \left\langle \frac{P_x}{\rho\nu\ (m+k^2)} \left[ \exp\ (\tfrac{\lambda y}{2}) \left\{ (\frac{ch(\frac{\lambda a}{2})}{ch\ Ba}\ ch\ By - \frac{sh(\frac{\lambda a}{2})}{sh\ Ba}\ sh\ By) - \right.\right.\right.$$
$$\left. + \frac{\beta}{2B}\ (M\ ch\ By + N\ sh\ By)y - (M\frac{ch\ Ba}{sh\ Ba}\ sh\ By + N\frac{sh\ Ba}{ch\ Ba}\ ch\ By)a \right\}$$
$$\left. \left. - 1 \right] \exp\ (int) \right\rangle.$$

It is evident from the above result that the velocity oscillates with the same frequency as the pressure gradient but that a phase lag, which depends on y exists. Thus the motion of the fluid which is adjacent to the boundaries will have a time wise phase shift relative to the motion near the centre line of the boundaries. The amplitude of the motion near the boundaries will differ from that near the centre line and in order to satisfy this condition, the amplitude will still approach zero as the boundaries are approached.

For $\lambda \neq 0$, $\beta = 0$ and $m \neq 0$ we get

$$u(y,t) = \mathrm{Re}\left\langle \frac{P_x}{\rho\nu(m+k^2)}\left[\exp\left(\frac{\lambda y}{2}\right)\left\{\frac{\mathrm{ch}(\frac{\lambda a}{2})}{\mathrm{ch}\,Ba}\,\mathrm{ch}\,By - \frac{\mathrm{sh}(\frac{\lambda a}{2})}{\mathrm{sh}\,Ba}\,\mathrm{sh}\,By\right\} - 1\right]\exp(int)\right\rangle,$$

which is the solution for hydromagnetic plusating flow of viscous incompressible fluid between parallel porous surfaces [5].

For $\lambda = 0$, $\beta = 0$ and $m \neq 0$ we get

$$u(y,t) = \mathrm{Re}\left\langle \frac{P_x}{\rho\nu(m+k^2)}\left[\frac{\mathrm{ch}\sqrt{(m+k^2)}\,y}{\mathrm{ch}\sqrt{(m+k^2)}\,a} - 1\right]\exp(int)\right\rangle,$$

which is the solution for hydromagnetic pulsating flow of viscous incompressible fluid between parallel surfaces [3].

For $\lambda \neq 0$, $\beta \neq 0$ and $m = 0$ we get

$$u(y,t) = \mathrm{Re}\left\langle \frac{P_x}{\rho\nu k^2}\left[\exp\left(\frac{\lambda y}{2}\right)\left\{\frac{\mathrm{ch}(\frac{\lambda a}{2})}{\mathrm{ch}\sqrt{(\frac{\lambda^2}{4}+k^2)}\,a}\,\mathrm{ch}\sqrt{(\frac{\lambda^2}{4}+k^2)}\,y - \frac{\mathrm{sh}(\frac{\lambda a}{2})}{\mathrm{sh}\sqrt{(\frac{\lambda^2}{4}+k^2)}\,a}\,\mathrm{sh}\sqrt{(\frac{\lambda^2}{4}+k^2)}\,y\right\} - 1\right]\exp(int)\right\rangle,$$

which is the solution for the pulsating flow of viscous incompressible fluid between parallel porous surfaces

For $\lambda = 0$, $\beta = 0$ and $m = 0$ we get

$$u(y,t) = \mathrm{Re}\left[\frac{P_x}{\rho\nu k^2}\left(\frac{\mathrm{ch}\,ky}{\mathrm{ch}\,ka} - 1\right)\exp(int)\right],$$

which is the solution for the pulsating flow of viscous incompressible fluid between parallel surfaces [1].

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Nep. Math. Sci. Rep.
Vol. 6, No. 1 & 2 (1981), 33-36

# On the Growth of a Function Analytic in the Unit Disc

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Let $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be analytic in the unit disc $D = \{z : |z| < 1\}$, $\lambda_0 = 0$ and $\{\lambda_n\}_{n=1}^{\infty}$ is a strictly increasing sequence of natural numbers such that no element of the sequence $\{a_n\}_{n=1}^{\infty}$ is zero. Set

$$M(r) = \max_{|z|=r} |f(z)|, \quad \mu(r) = \max_{n \geq 0} \left\{ |a_n| \, r^{\lambda_n} \right\}$$

and

$$\nu(r) = \max \left\{ \lambda_n : \mu(r) = |a_n| \, r^{\lambda_n} \right\}.$$

$M(r)$, $\mu(r)$ and $\nu(r)$ are called, respectively, the maximum modulus, the maximum term and the rank of the maximum term of $f(z)$ for $|z| = r$. It is known ([3]) that,

$$(1) \quad \log \mu(r) = \log \mu(r_0) + \int_{r_0}^{r} (\nu(t)/t)\, dt, \quad r_0 < r < 1,$$

if $\mu(r)$ and $\nu(r)$ are unbounded functions of $r$.

Let $f(z)$ be analytic in D with $M(r) \to \infty$ as $r \to 1$. Set

$$\begin{matrix} \rho(q) \\ \lambda(q) \end{matrix} = \lim_{r \to 1} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log_q M(r)}{-\log (1-r)}$$

where $\log_1 x = \log x$ and $\log_q x = \log(\log_{q-1} x)$ for $q \geq 2$. If $\rho(q) < \infty$ and $\rho(q-1) = \infty$, $q = 2,3,\ldots$, then $f(z)$ is said to be of index q. Further $\rho(q)$ and $\lambda(q)$ are called, respectively, q-order and lower q-order of $f(z)$ (see [2]).

A coefficient characterisation of $\rho(q)$ was recently obtained by Kapoor and Gopal ([2, Theorem 1]). Further, if $\rho(q) > 0$, then they have also obtained a coefficient characterisation of $\lambda(q)$ ([2, Theorem 2]) for those functions which satisfy the condition

$$(2) \qquad \Lambda(q) + \lambda(q) = \liminf_{r \to 1} \frac{\log_{q-1} \nu(r)}{-\log (1-r)} .$$

Here $\Lambda(q) = 1$ if $q = 2$, otherwise $\Lambda(q) = 0$. It is known [1] that there exist functions analytic in D having index 2 with $\rho(2) > o$ and for which (2) does not hold for $q = 2$.

In the present note we prove.

<u>Theorem:</u> <u>Let</u> $f(z)$ <u>be analytic in D, having index</u> $q$, $q \geq 3$, $q$-<u>order</u> $\rho(q)$ $(> o)$ <u>and lower</u> $q$-<u>order</u> $\lambda(q)$. <u>Then,</u>

$$\lambda(q) = \liminf_{r \to 1} \frac{\log_{q-1} \nu(r)}{-\log (1-r)} .$$

Our theorem shows that, for $q \geq 3$, the above condition (2) in Theorems 2 and 3 of [2] is redundant.

<u>Proof of the theorem.</u> It is known [2, Lemma 1] that

$$(3) \qquad \lambda(q) = \liminf_{r \to 1} \frac{\log_q \mu(r)}{-\log (1-r)}$$

Now, since $\rho(q) > o$, using (1) we have

$$\log \mu(r) < \log \mu(r_o) + \nu(r) \log(r/r_o)$$

$$< k\, \nu(r)$$

k is a constant. This easily gives that

$$(4) \qquad \liminf_{r \to 1} \frac{\log_q \mu(r)}{-\log (1-r)} \leq \liminf_{r \to 1} \frac{\log_{q-1} \nu(r)}{-\log (1-r)}$$

On the other hand, again by (1), we have

$$\nu(r)\ ((1-r)/2) \leq \nu(r) \log [(r+(1-r)/2)/r]$$

$$\leq \int_r^{r+(1-r)/2} (\nu(t)/t)\, dt$$

$$\leq \log \mu(r+(1-r)/2)$$

or

$$\log_{q-1} \nu(r) \leq \log_q \mu(r+(1-r)/2) + \log_{q-1}(2/(1-r))$$

for all r sufficiently near to 1. Now, $r+(1-r)/2 \rightarrow 1$ as $r \rightarrow 1$ and $\log(1/(1-r)) \sim -\log(1-(r+(1-r)/2))$ as $r \rightarrow 1$. Thus, dividing the above inequality by $-\log(1-r)$ and passing to limits we get

$$(5) \quad \liminf_{r \rightarrow 1} \frac{\log_{q-1} \nu(r)}{-\log(1-r)} \leq \liminf_{r \rightarrow 1} \frac{\log_q \mu(r)}{-\log(1-r)}$$

Proof of the theorem is complete in view of (3), (4) and (5).

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Nep. Math. Sc. Rep.
Vol. 6, No. 1 & 2 (1981), 37-44

# The Distribution of Population Growth Rate in Nepal

Dr. Mrigendra Lall Singh

## 1. Background

The population of Nepal which was 5.6 millions in 1911 and which had remained virtually constant during the period of 1911-1930, started to show the growth trend only after 1930. The annual growth rate which was only 1.3%* during 1930-1941 has increased to 2.1%** by 1971-1981. This has resulted in a population of 14.1 millions for Nepal in 1981.

Though the recent observed rate is explosive*** and is likely to double Nepal's population in next 30 years, yet, it is far behind the expected rate of 2.4% for developing countries during 1980-1985; recorded rate of 2.4% in India for 1981 and 2.5% in Nepal as indicated by mid-term census taken in 1976.

Due to this fact, many demographers in Nepal are doubting the authenticity of the rate. But their doubts have yet not been substanciated or rebuted.

A review of intercensus growth rates so far observed in Nepal shows that, since 1930, it has been increasing slowly and steadly. Also, it shows that the reported rate for 1981 is the second highest next only to those (2.2%) for 1941-1952/54. The growth rates reported for the periods prior to 1961 are not of intrinsic values, for they have not been adjusted to equal census periods.

The population growth rates of Nepal adjusted to equal census periods are shown in Table 1.

Table 1
(Adjusted Population growth rates based on equal census periods Nepal, 1911-1981)

In %

| Period | Growth rate | Prior | Growth rate |
|---|---|---|---|
| 1911-1921 | 0.35 | 1951-1961 | 2.03 |
| 1921-1931 | 0.35 | 1961-1971 | 2.07 |
| 1931-1941 | 0.35 | 1971-1981 | 2.10+ |

Source: Singh, M.L., "Population Dynamics of Nepal' Ph.D. thesis, Tribhuvan University, 1978, p. 22.
+Press report released by CBS.

* Estimated from Nepal population figures for 1930 and 1941.
** Figures obtained from press report released by CBS.
***Bogue has classified the population by different categories of growth rates. According to him, a population with growth rate of 2% and higher is explosive. See Bogue, D.J., 'Principles of Demography' John Willey and Sons, New York, 1969, p. 36.

The table clearly shows that the present growth rate of 2.1% is highest of all intercensus growth rates.

In this context, some few relevant questions arise. They are:

a. Whether the present reported rate of 2.1% is reasonable or not?

b. What is maximum growth rate that is possible for Nepal?

c. What are the confidence limits of the variation of population growth rate in Nepal?

Though theoretically it can be proved that growth rate can have value as high as 10%, but in practice, a population with growth rate as high as 4% is hardly known.[2] Keeping in view of above fact, attempts are made in this paper to answer above questions.

For this, population growth rate matrix i.e. r-matrix is first constructed and from the matrix so constructed, confidential limits for growth rate 'r' are estimated and accordingly answers are obtained.

## 2. r-matrix and probabilities of growth rates

In case of a closed population, the growth rate 'r' is estimated as the differences between the CBR 'b' and CDR 'd'. Due to invariancy of Age structure of Nepal's population, the population of Nepal can be considered a closed one. Therefore, ignoring the effects of migration, the growth rate is estimated as $r = b - d$.

Since 'b' and 'd' are non-negative numbers, the minimum values of 'b' and 'd' that can be expected are zero's. On the assumption that 'b' is never less than 'd' for Nepalese population, the minimum value of 'r' becomes zero. The population with above values of vital rates is the stationary population. Therefore, keeping in view of existing high IMR (140 per thousand) in Nepal, the minimum values for 'b' and 'd' are taken as 15* per thousand. Of course the maximum possible values for them are 50 per thousand.**

The r-matrix, where, $r = b - d$ is constructed for $50 \geq b \geq 15$ ; $50 \geq d \geq 15$ by dividing the range of variation of values of 'b' and 'd' into equal class intervals of width. The matrix so constructed is shown in Table 2.

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* Due to high proportion of young population (40% in the age group 0-14 years in Nepal), it may be argued that the CDR may go down as low as 4%. But at present, due to high FMR, such a drastic fall is not expected.

**CBR and CDR higher than 50 lead a population whose life expectancy is less than 20 years and such population ultimately terminates.

Table 2
(A hypothetical r-matrix)

| Death rate in thousand | Birth rate in thousand | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 50 | 45 | 40 | 35 | 30 | 25 | 20 | 15 |
| 50 | 0 | | | | | | | |
| 45 | 5 | 0 | | | | | | |
| 40 | 10 | 5 | 0 | | | | | |
| 35 | 15 | 10 | 5 | 0 | | | | |
| 30 | 20 | 15 | 10 | 5 | 0 | | | |
| 25 | 25 | 20 | 15 | 10 | 5 | 0 | | |
| 20 | 30 | 25 | 20 | 15 | 10 | 5 | 0 | |
| 15 | 35 | 30 | 25 | 20 | 15 | 10 | 5 | 0 |

The matrix shows the maximum possible value of 'r' in Nepal, when the population is closed, as 35 per thousand or 3.5%.

Assuming that b, d and r take the discrete values as specified in the table, it is found that there are eight states at which 'r' becomes zero i.e. the population attains the stationary character. Therefore, it can be deduced that the probability of 'r' becoming 0% i.e. P(r=0%) equals to 0.22. Also by using other sample points of the matrix, it can be shown that P(r=x) decreases as x increases. In fact, it is found that P(r=2.5%) equals to 0.083 and P(r=3%) equals to 0.027%. The above findings show that the probabilities of getting higher values of 'r' is very small.

3. Distribution function for 'r'

One of the statistical tool to express the character of a parameter is the distribution function. Therefore, the distribution for 'r' is derived in the following way:

Let u = upper limit of 'b' and 'd' i.e. 50 per thousand
v = lower limit of 'b' and 'd' i.e. 15 per thousand

$\frac{u-v}{h} = k$, where h = width of the class interval
k = number of class intervals.

Then P(r) values are estimated as:

$$f(hx) = \frac{(k-x)}{\sum k}, \quad \text{where } x=0,\ 1,\ 2\ \ldots..7$$

$$\therefore\ f(r) = \frac{(k-\frac{r}{h})}{\sum k} = \frac{2(hk-r)}{hk(k+1)}$$

Now hk= u-v = 50-15 = 35

$$\therefore \quad f(r) = \frac{2(35-r)}{35(k+1)} \qquad 0 \leq r \leq 35$$

$$\therefore \quad \int f(r)\, dr = \frac{2}{35(k+1)} \qquad (35-r)\, dr \qquad (1)$$

$$= \frac{35}{k+1}$$

f(r) becomes distribution function if $\int f(r)\, dr = 1$

i.e. $\frac{35}{k+1} = 1$ or $k=34$

$$\therefore \quad f(r) = \frac{2}{35^2}(35-r), \qquad 0 \leq r \leq 35 \qquad (2)$$

$$\frac{u-v}{h} = k$$

$$P(R=hx) = \frac{2(k-x)}{k(k+1)}, \qquad x = 0,1,2,\ldots\ldots, k$$

$$= \frac{2(hk-hx)}{hk\ (k+1)}$$

$$\therefore \; P(R=r) \frac{2(hk-r)}{hk\ (k+1)}, \qquad r = 0,h,2h,\ldots, kh$$

Let $k \rightarrow \infty$, keeping $kh = 35$,

then probability that R assumes values in an interval of small length h around r, is

$$f_R(r)\ h = \frac{2(35-r)}{35(35+h)}\ h$$

$$\simeq \frac{2(35-r)}{(35)^2}, \qquad 0 \leq r \leq 35$$

An alternative derivation of f(r) is given below, which does not limit the value of k to 34, and which makes for nearly continuous.

<u>Additions to be made to explain the facts shown in my paper</u>

<u>Estimation of E(r) and Var(r) in descrete case</u>

From the matrix shown above,

The P(r) is found as $P(r) = \frac{(k-x)}{\sum k}$ where, $x=o,1,2,\ldots\ldots 7$

$k=8$

$$\therefore \; E(r) = r\,P(r)$$

$$= 0.\frac{8}{\Sigma 8} + 5.\frac{7}{\Sigma 8} + 10.\frac{6}{\Sigma 8} + \ldots\ldots 35.\frac{1}{\Sigma 8}$$

= 11.666 per thousand or 1.16%.

$$E(r^2) = r^2 P(r)$$

$$= 5^2.\frac{7}{\Sigma 8} + 10^2.\frac{6}{\Sigma 8} + \ldots\ldots\ldots\ldots 35^2.\frac{1}{\Sigma 8}$$

$$= 233.33$$

$$\therefore \; Var(r) = E(r^2) - [E(r)]^2 = 97.223$$

$$\therefore \quad s_r = + \sqrt{V\ var(r)} = 9.86 \text{ per thousand}$$

<u>Distribution function of 'r'</u>

Suppose $\frac{V-U}{h} = k$ where V+= Maximum value of b or d
U = Minimum value of b or d
h = Width of the class interval
k = Number of class intervals

Then $f(hx) = \frac{(k-x)}{\sum k}$

so that $f(r) = \frac{k-\frac{r}{h}}{\sum k} = \frac{2(hk-r)}{hk(k+1)}$

Since, in reality, b, d and r are continuous variable, the probability function derived above is special case where b and d were supposed to take only descrete values such as 50,45 etc.

Now extending to the continuous case,

$$f(r) = \frac{2(35-r)}{35(k+1)}, \; 0 \leq r \leq 35, \; hk = V-U = 35$$

$$\therefore \; f(r) = \frac{2}{35(k+1)} \int_0^{35} (35-r)\,dr$$

$$= \frac{2}{35(k+1)} \left[ (35r)_{0}^{35} - (\tfrac{r^2}{2})_{0}^{35} \right]$$

$$= \frac{2.612.5}{35(k+1)} = \frac{35}{k+1}$$

Now f(r) becomes distribution function only if f(r) = 1

i.e. $\frac{35}{k+1} = 1$

$\therefore$ k+1 = 35 or k=34

(Note that: in descrete case as shown in the matrix k=8).

$\therefore$ The distribution function of 'r' is found as

$$f(r) = \frac{2}{35^2} (35-r)\, dr \qquad 0 \leq r \leq 35$$

$\therefore$ $E(r) = \int r\, dr = 11.66$ per thousand = 1.16

$Var(r) = E(r^2) - E(r)^2 = 97.32.$

The values of E(r) and Var(r) estimated were found to be exactly the same as in case of the discrete case.

The expression (2) is derived distribution function or 'r' for Nepal.

This function is used to find the E(r) and Var(r)

$$\text{Mean } (r) = E(r) = \int r\, dr \qquad 0 \leq r \lesssim 35 \qquad (3)$$

$$= \frac{2}{35^2} \cdot \int_{o}^{35} r(35-r)\, dr = 1.166\%$$

The result shows that mean value of the growth rate in Nepal is 1.17%.

Similarly by using the relation,

$$E(r^2) = \int r^2 f(r)\, dr \qquad (4)$$

and

$$Var(r) = E(r^2) - [E(r)]^2 \text{ and } s_r = +\sqrt{Var(r)} \qquad (5)$$

the standard error of estimation of mean is estimated as 9.86 per thousand i.e. 0.98%.

4. Confidence limits for 'r'

For simplicity the confidence limits for the variation of 'r' are defined as:

$$E(r) \pm s_r^* \qquad (6)$$

.˙. The limits for the variation of 'r' are obtained as 2.15% and 0.18%.

A glance at the values of the growth rate 'r' estimated[3] and observed so far in Nepal shows that except for two or three values, most of the values lie within the limits found above.

5. Conclusion

Since the reported population growth rate of Nepal for 1981 is found to be uppermost limit of the confidence limits found in the present study, it is concluded that there is no ground to suspect the validity of present value. As regards maximum possible value of 'r', it is found that 7 in 10 chances (68%), the observed value of population growth rate is likely to fluctuate between 0.18% and 2.15%. The conclusions are however not conclusive, for, once the magnitude of migration comes to vital role in Nepal's population phenomenon, the picture will be entirely different.

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Nep. Math. Sc. Rep.
Vol. 6, No. 1 & 2 (1981), 45-48

# A Type of Semi-ring

Prahlad Singh & N.S. Yadav

## Introduction

This paper investigates a generalization of a ring, to be called R-semi-ring and explores certain properties thereof.

## 1. Definitions

An R-semi-ring is a system R with two operations, multiplication and commutative addition (+) with the following properties:

(i) $(a+b)+c = a+(b+c)$, $a,b,c \in R$ ;

(ii) $a(b+c) = ab+ac$ , $a,b,c \in R$ ;

(iii) There exists an element 0 (zero) in R such that

$0+a = a$, $a \in R$

and (iv) $0a = 0$, $a \in R$.

If multiplication is commutative, R is commutative and if it is associative, R is an associative R-semi-ring. If there is an element 1 in R such that $1a = a1 = a$, $a \in R$, then R is an R-semi-ring with unity 1.

Hereinafter we shall use simply semi-ring for R-semi-ring.

## 2. Theorem

Every abelian semi-group G with identity is the additive semi-group of some semi-ring.

## Proof

We can presuppose that the semi-group operation of G is written additively. If $a,b \in G$, let us define $ab = 0$. That all the properties of a semi-ring are satisfied is evident. This zero-semi-ring is obviously commutative-associative.

It can be seen that in the semi-ring R defined here, $a^2 = 0$ and $ab = -ba$, $a,b \in R$. If R does not consist of zero alone, then R is a semi-ring without unity.

## 3. Definitions

Regular element: An element a in an associative semi-ring R is regular if there exists an element $x \in R$ such that $a\ x\ a = a$.

A non-empty subset S of R will be called regular if each element of S is regular.

4. Theorem

If R be (associative) regular, then for each non-zero element $a \in R$, there exists an idempotent b such that $aR = bR$. Conversely, if R be associative with unity such that for each non-zero element a of R there is an idempotent b such that $aR = bR$, then R is regular.

Proof: Let R be regular and a a non-zero element of R. Then there exists an element x of R such that $a\,x\,a = a$. Since $(ax)\,(ax) = (axa)x = ax$, it follows that ax is an idempotent.

We shall now show that $(ax)\,R = aR$.

Let $(ax)r \in (ax)R$. Hence $axr = a(xr) \in aR$. If $ar \in aR$, then $axar \in aR$.

But, $(ax)\,(ar) \in axR$.

Conversely, let a be any non-zero element of R and let b be an idempotent such that $aR = bR$.

Thus $b = ax$ for some $x \in R$.

Since $a \in bR$, $a = by$ for some $y \in R$.

Now $a = by = b^2y = b(by) = ba$ and hence $axa = (ax)a = ba = a$.

5. Definition

An element b of a semi-ring R is called nilpotent if and only if there exists a positive integer n such that $b^n = 0$. A subset $B(\subseteq R)$ is called nil if every element of R is nilpotent.

6. Theorem

An (associative) regular semi-ring with unity contains no non-zero nil subset B such that $BR \subseteq B$.

Proof

Let be be a non-zero element of B with $BR \subseteq B$. Then, by Theorem 4, there exists an idempotent c such that $cR = bR$. Hence $c = bx$ for some x in R, that is, $c \in B$, a contradiction.

7. Definition

If a and b are non-zero elements of a semi-ring R with $ab = 0$ or $ba = 0$, then a and b are called zero divisors of R.

8. Theorem

Let R be (associative) regular with unity with no zero-divisors. If the additive structure of R is that of a group, then the non-zero elements of R form a multiplicative group.

Proof

Let a be a non-zero element of R. Then there exists an element $x \in R$ such that $axa = a$, i.e. $a(xa-1) = 0$; but $a \neq 0$. Hence $xa = 1$.

Similarly $xyx = x$ for some $y \in R$ i.e. $yx = 1$. Now, $ax = (yx)(ax) = y(xa)x = yx = 1$; that is a has a multiplicative inverse x.

Remark

If R is commutative, then it is a field.

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Nep. Math. Sc. Rep.
Vol. 6, No. 1 & 2 (1981), 49-52

# On Certain Sets in Pasch Geometric Modules of Dimension Three

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The concept of a Pasch Geometry is given in [1] or [3] and that of geometric module in [2]. Geometric modules over geometric Sfields give rise to projective spaces. In this short paper, we prove two basic propositions on the cardinality of certain subsets which are technical in proving results on geometric modules of dimension three.

Suppose V is a geometric module over a geometric Sfield A, the dimension of V over A being 3 [2]. Since $A^* = A - \{0\}$ acts on V, we consider the geometry $V/_{A}^{*}$ [2]. For $v \in V$, $\bar{v}$ denotes the element of $V/A^*$. Since $v^{\#} = 1^{\#}v$, we have $\bar{v}^{\#} = \bar{v}$. Also, let $(\bar{v}, \bar{v}, \bar{w}) \in \triangle_{V/A^*}$. Then $\exists\, \alpha$, $\beta \in A^*$ such that $(v, \alpha v, \beta w) \in \triangle_V$. Suppose $\beta w \neq 0$. Then $\beta w \in Sp(v)$ $\Longrightarrow \beta w = \gamma v$ for some $\gamma \in A$, $\gamma \neq 0$. So $\bar{w} = \bar{v}$. Hence $V/A^*$ as a geometry is projective [1]. Since $\dim_A V$ is 3, it is easy to see that $\dim V/_{A^*}$ is also 3. So $V/_{A^*}$ corresponds naturally to a projective space of projective dimension two i.e. a projective plane.

In this paper V will be a geometric module over a geometric Sfield A with dimension V equal to 3.

1.1 <u>Definition</u>

(i) For $\alpha \in A^*$, let $C_\alpha = \{\beta : (\alpha, \alpha^{\#}, \beta) \in \triangle_A\}$.

(ii) For $v_1, v_2 \in V$, $v_1, v_2$ independent, and $\alpha \in A^*$, we set

$$L_\alpha(v_1, v_2) = \{u : u \in V,\ (u, v_1, \alpha v_2) \in \triangle_V\}.$$

Note that $L_\alpha(v_1, v_2) \neq \emptyset$. We fix the elements $v_1$, $v_2$, and write $L_\alpha$ for $L_\alpha(v_1, v_2)$ $\forall\, \alpha \in A^*$. For any set S, card(S) will denote the cardinality of the set S.

1.2 <u>Proposition</u>

(i) Card $(C\alpha)$ = Card$(C\beta)$ $\forall\, \alpha, \beta \in A^*$.

(ii) Card $(C_\alpha) \leq$ Card $(L_\alpha)$ $\forall\, \alpha \in A^*$.

Proof

(i) We note that $\gamma \in C_\alpha$ if and only if $(\alpha, \alpha^{\#}, \gamma) \in \Delta_A$ if and only if $(\beta, \beta^{\#}, \beta\alpha^{-1}\gamma) \in \Delta_A$ i.e. $\beta\alpha^{-1}\gamma \in C\beta$. Now the association $\gamma \longleftrightarrow \beta\alpha^{-1}\gamma$ is clearly a bijective mapping.

(ii) Choose $u_0 \in V$ such that $(u_0, v_1, \alpha v_2) \in \Delta_V$. Let $\gamma \in C_\alpha$, $\gamma \neq 0$. Then

$(\alpha, \alpha^{\#}, \gamma) \in \Delta_A \Longrightarrow (1, 1^{\#}, \alpha^{-1}\gamma) \in \Delta_A \Longrightarrow (v_1, v_1^{\#}, \alpha^{-1}\gamma v_1) \in \Delta$. So $(v_1, \alpha v_2, u_o)$, $(v_1, v_1^{\#}, \alpha^{-1}\gamma v_1) \in \Delta_V$
$\Longrightarrow \exists\, u \in V$ such that $(u, v_1, \alpha v_2)$, $(u, \alpha^{-1}\gamma v_1, u_0^{\#}) \in \Delta_V$.

Note that $u \in L_\alpha$. Thus give $\gamma \in C_\alpha$, $\gamma \neq 0$, $\exists\, u \in L$ such that $(u, \alpha^{-1}\gamma_1 v_1, u_0^{\#}) \in \Delta_V$.

Note that since $v_1, u_0$ are independent, $u_0 \neq u$. Let $\phi(\gamma) = \{u \in L_\alpha :$ $(u, \alpha^{-1}\gamma v_1, u_0^{\#}) \in \Delta_V$

So $\phi(\gamma) \neq$ null set. Let $\phi(0) = \{u_0\}$. Suppose $u \in \phi(\gamma_1) \cap \phi(\gamma_2)$. Then $(u, \alpha^{-1}\gamma_1 v_1, u_0^{\#})$, $(u, \alpha^{-1}\gamma_2 v_1, u_0^{\#}) \in \Delta$. Since $v_1, u_0$ are independent, we get $\alpha^{-1}\gamma_1 = \alpha^{-1}\gamma_2$ i.e. $\gamma_1 = \gamma_2$. Thus,

$$\phi : C_\alpha \longrightarrow \text{Set of subsets of } L_\alpha$$

is a map such that $\phi(\gamma_1)$ and $\phi(\gamma_2)$ are disjoint for $\gamma_1 \neq \gamma_2$. Clearly this defines, using axiom of choice, an injective mapping from $C_\alpha$ to $L_\alpha$.

1.3 Proposition

Card $(L_\alpha) =$. Card $(L_\beta)$ $\forall\, \alpha, \beta \in A^*$.

Proof

Let $v_3 \in V^*$, $v_3 \notin Sp(v_1, v_2)$. Note that such $v_3$ exists, since dimV is greater than 2. Let $w_0 \in V$ such that $(w_0, v_1, \beta\alpha v_3^{\#}) \in \Delta_V$. Note $w_0 \notin Sp(v_1, v_2)$, since otherwise $\beta\alpha v_3^{\#}$ and hence $v_3$ would be in $Sp(v_1, v_2)$ as $\alpha \neq 0$, $\beta \neq 0$. We define a map

$$\phi_{(w_0)} : L_\beta (v_1, v_2) = L_\beta \longrightarrow L_\alpha (v_2, v_3)$$

as follows: Let $u \in L_\beta$ so that $(u, v_1, \beta v_2) \in \Delta_V$. Then $(v_1, w_0, \beta\alpha v_3^{\#})$, $(v_1, u, \beta v_2) \in \Delta_V \Longrightarrow \exists\ s \in V$ such that $(s, u^{\#}, w_0)$, $(s, \beta v_2, \beta\alpha v_3) \in \Delta_V$ i.e. $\beta^{-1} s \in L_\alpha(v_2, v_3)$. We set $\phi_{(w_0)}(u) = \beta^{-1} s \in L_\alpha(v_2, v_3)$ where $(s, u^{\#}, w_0) \in \Delta_V$.

$\phi_{(w_0)}$ is well defined: Suppose also $\beta^{-1} t \in L_\alpha(v_2, v_3)$ such that $(t, u^{\#}, w_0) \in \Delta_V$. Then $s, t \in Sp(u, w_0) \cap Sp(v_2, v_3)$. Note $u \notin Sp(v_2, v_3)$, otherwise since $u, v_2$ are independent, $v_3$ would be in $Sp(u, v_2) = Sp(v_1, v_2)$. So $Sp(u, w_0) \cap Sp(v_2, v_3)$ has dimension at most one. So $\exists\ \delta \in A^*$ with $t = \delta s$. But then $(\delta s, u^{\#}, w_0)$, $(s, u^{\#}, w_0) \in \Delta_V$. But $u, w_0$ are independent, so $\delta = 1$ and $t = s$.

$\phi_{(w_0)}$ is one-one: Let $(\phi_{(w_0)}(u_1) = \phi_{(w_0)}(u_2) = \beta^{-1} s$ so that $\beta^{-1} s \in L(v_2, v_3)$ and $(s, u_1^{\#}, w_0)$, $(s, u_2^{\#}, w_0) \in \Delta$. The $\exists\ x \in V$ such that $(x, u_2, u_1^{\#})$, $(x, w_0, w_0^{\#}) \in \Delta$. So $s = \gamma w_0$, $\gamma \in A$. If $\gamma \neq 0$, then $\gamma w_0$ and so $w_0$ would be in $Sp(u_1, u_2) \subseteq Sp(v_1, v_2)$, a contradiction. So $\gamma = 0$ i.e. $x = 0$ and so $u_1 = u_2$.

Thus $\phi_{(w_0)}$ is an injection from $L_\beta$ into $L_\alpha(v_2, v_3)$. Note that since $v_1 \notin Sp(v_2, v_3)$, the symmetry implies there is an injection from $L_\alpha(v_2, v_3)$ to $L_\beta$. So by Bernstein-Schoder theorem, $Card(L_\beta) = Card(L_\alpha(v_2, v_3))$. Taking $\alpha = \beta$, $Card(L_\alpha) = Card(L_\alpha(v_2, v_3)) = Card(L_\beta)$. This completes the proof.

Remark

If A is sharp (i.e. naturally a skewfield), then $C_\alpha = \{0\}\ \forall \alpha \in A$ if and only if $C_1 = \{0\}$. Also, if $C_1 = \{0\}$, then A is sharp, so V is sharp and hence $Card(L_\alpha) = 1\ \forall \alpha \in A$. Conversely, if $Card(L_1) = 1$, then $C_1 = \{0\}$ so V is sharp.

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Nep. Math. Sc. Rep.
Vol. 6, No. 1 & 2 (1981), 53-56

# A Class of Generating Functions

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Abstract

A new class of generating functions of the form

$$(1-t)^{-a}\, H[(-4m^2 t)^m\, x(1-t)^{-2m}] = \sum_{n=0}^{\infty} g^{a}_{n,m}(x)\, t^n$$

is used to obtain known and new generating functions.

## 1. Introduction

Various extensions of the generating function [3]

$$(1.1)\qquad (1-t)^{-1}\, {}_2F_1\, [\zeta, \tfrac{1}{2};\ p;\ -4x(1-t)^{-2}]$$

$$= \sum_{n=0}^{\infty} {}_3F_2\, (-n, n+1, \zeta;\ 1, p;\ x)\, t^n$$

where

$$(1.2)\qquad {}_3F_2\, (-n, n+1, \zeta;\ 1, p;\ x) = R_n(x)$$

is the Rice polynomial [4].

Brafman's extension of (1.1) is of the form [1]

$$(1.3)\qquad (1-t)^{-1}\, {}_pF_q\, [(a_p);\ (b_q);\ -4xt\, (1-t)^{-2}]$$

$$= \sum_{n=0}^{\infty} {}_{p+2}F_{q+2}\, [-n, n+1,\ (a_p);\ \tfrac{1}{2}, 1,\ (b_q);\ x]\, t^n,$$

where $(a_p)$ stands for the sequence of parameters $a_1, a_2, \dots, a_p$.

Further extensions in the forms

$$(1.4)\qquad (1-t)^{-1}\, H[-4xt\, (1-t)^{-2}] = \sum_{n=0}^{\infty} g_n(x)\, t^n,$$

and

$$(1.5)\qquad (1-t)^{-1}\, H[(-4m^2 t)^m\, x(1-t)^{-2m}] = \sum_{n=0}^{\infty} g_{n,m}(x)\, t^n,$$

where

$$(1.6) \qquad H(x) = \sum_{n=0}^{\infty} a_n x^n,$$

$$(1.7) \qquad g_n(x) = \sum_{n=0}^{\infty} \frac{(-n)_k (1+n)_k}{(\frac{1}{2})_k \, k!} a_k x^k,$$

and

$$(1.8) \qquad g_{n,m}(x) = \sum_{n=0}^{[n/m]} \frac{(-n)_{mk} (1+n)_{mk}}{(1)_{2mk}} a_k [(2m)^{2m} x]^k ,$$

were furnished by Fasenmeyer [2] and Shrestha [5] respectively.

In the present paper, we give an elementary extension covering all the above extensions and indicate how we obtain known and new generating functions. Since our method of proof is classical, we just give a short outline of the proof.

## 2. Main result

If

$$(2.1) \qquad H(x) = \sum_{n=0}^{\infty} a_n x^n,$$

and

$$(2.2) \qquad g^{\alpha}_{n,m}(x) = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk} (\alpha)_n (\alpha+n)_{mk}}{(\alpha)_{2mk} \, n!} a_k [(2m)^{2m} x]^k,$$

then

$$(2.3) \qquad (1-t)^{-\alpha} H[(-4m^2 t)^m x(1-t)^{-2m}] = \sum_{n=0}^{\infty} g^{\alpha}_{n,m}(x) \, t^n,$$

where m is a non-zero positive integer.

Proof. Use (2.1) in the left-side of (2.3) and simplify with the help of

$$(2.4) \qquad \sum_{n=0}^{\infty} \sum_{k=0}^{[n/m]} A(n-mk,k) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} A(n,k) ,$$

$$(2.5) \qquad (a)_m (a+m)_n = (a)_{m+n} ,$$

$$(2.6) \qquad (a)_{mk} = m^{mk} \left(\frac{a}{m}\right)_k \left(\frac{a+1}{m}\right)_k \cdots \left(\frac{a+m-1}{m}\right)_k , \; m > 0 , \quad k \geqslant 0$$

and

$$(2.7) \qquad (-n)_k = (-1)^k \; n! \,/(n-k)! \;, \quad 0 \le k \le n \;,$$

to find

$$(1-t)^{-\alpha} H[(-4m^2 t)^m \, x(1-t)^{-2m}]$$

$$= \sum_{k=0}^{\infty} \frac{(-4m^2 t)^{mk} \, a_k \, x^k}{(1-t)^{\alpha+2mk}}$$

$$= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-4m^2)^{km} \, (\alpha+2mk)_n \, a_k}{n!} \, x^k \, t^{n+mk}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{[n/m]} \frac{(-n)_{mk} \, (\alpha)_n \, (\alpha+n)_{mk}}{(\alpha)_{2mk} \, n!} \, a_k \, [(2m)^{2m} \, x]^k \, t^n$$

$$= \sum_{n=0}^{\infty} g_{n,m}^{\alpha} (x) \; t^n .$$

In particular, when $\alpha = 1$, we arrive at the result (1.6) due to Shrestha [5]. Results due to Sister Celine follows from (1.6) by taking $m = 1$. The other results due to Rice and Brafmann may be obtained similarly.

## 3. Applications

As a first application of (2.3), we derive the following generating function [6]

$$(3.1) \qquad \sum_{n=0}^{\infty} \frac{(2\lambda)_n \Gamma(\lambda+\frac{1}{2})}{n!} R_n^*(x) \; t^n = (1-t)^{-2} \; {}_2F_1^* \left[ \begin{matrix} \lambda, \; a \\ c \end{matrix} ; -\frac{4xt}{(1-t)^2} \right]$$

where $\quad {}_2F_1^*(\lambda, a;\; c;\; -\frac{4xt}{(1-t)^2}) = \frac{1}{\Gamma(c)} \, {}_2F_1 \, (\lambda, a;\; c;\; -\frac{4xt}{(1-t)^2}) \;,$

and

$$(3.2) \qquad R_n^*(x) = {}_3F_2^*(-n, n+2\lambda\,,\; a; \lambda+\tfrac{1}{2},\; c;\; x)$$

denotes the hypergeometric polynomial appearing as coefficients appearing in the expansion of the confluent hypergeometric function in series of Bessel functions. Further properties of $R_n^*(x)$ can be found in Shrestha [6].

In order to obtain (3.1), we take $m = 1$, $\alpha = 2$ and

$$(3.3) \qquad H\left[-\frac{4xt}{(1-t)^2}\right] = {}_2F_1^*\left(\lambda, a; c; -\frac{4xt}{(1-t)^2}\right)$$

in (2.3).

As a second example, we have the following interesting generating function

$$(3.4) \qquad (1-t)^{-\alpha} {}_pF_q\left[(a_p); (b_q); (-4m^2 t)^m x(1-t)^{-2m}\right]$$

$$= \sum_{n=0}^{\infty} {}_{p+2m+1}F_{q+2m}\left[\begin{matrix}\Delta(m;-n), \ \Delta(m;\alpha+n), \alpha, \ (a_p) \\ \Delta(2m;\alpha), \ (b_q)\end{matrix}; x\right] t^n$$

where $\Delta(m;a)$ stands for the sequence of parameters

$$\frac{a}{m}, \frac{a+1}{m}, \ \ldots \ , \frac{a+m-1}{m}, \text{ m being a positive integer.}$$

This result is obtained from (2.3) by taking

$$(3.5) \qquad H(u) = {}_pF_q\left[(a_p); (b_q); u\right].$$

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Nep. Math. Sc. Rep.
Vol. 6, No. 1 & 2 (1981), 57-60

# An Iteration of the Mk,m-Transform

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Abstract

In this paper we deal with the iteration of the well-known $M_{k,m}$-Transform. As Stieltjes Transform is obtained by the first iteration of the Laplace Transform, by iterating its generalisation, we expect a generalised Stieltjes Transform.

## 1. Introduction

If the determining function of the Laplace Toansform

$$(1.1) \quad f(s) = \int_0^\infty e^{-st} \phi(x)\, dt \quad (s > 0)$$

is itself the Laplace Transform of another function, say $\alpha(x)$, that is, if

$$(1.2) \quad \phi(t) = \int_0^\infty e^{-xt} \alpha(x)\, dx ,$$

then

$$(1.3) \quad f(s) = \int_0^\infty \frac{\alpha(x)}{s+x}\, dx ,$$

which is known as the Stieltjes Transform [4]. In this note, we shall obtain the first iteration of the $M_{k,m}$-Transform

$$(1.4) \quad f(s) = \int_0^\infty e^{-\lambda st} (st)^{-m-\frac{1}{2}} M_{k,m}(st)\, \phi(t)\, dt,$$

where $\lambda$ is a real positive parameter greater than $\frac{1}{2}$, k & m may be real or complex and $2m \neq -1, -2, -3, \dots$ . This transform was given by the author [3]. This transform is still true for $\lambda = \frac{1}{2}$ when

$$|t^{+k-m-\frac{1}{2}} \phi(t)| \in L(o,\infty)$$

It reduces to (1.1) for $\lambda = 3/2$ and $k + m = \frac{1}{2}$.

## 2. Iteration

In this section we obtain the generalisation of the Stieltjes transform by iteration of the $M_{k,m}$-Transform.

Theorem. If

$$(2.1)\qquad f(s) = \int_0^\infty e^{-\lambda sx}\,(sx)^{-m-\frac{1}{2}}\,M_{k,m}(sx)\,\phi(x)\,dx$$

where

$$(2.2)\qquad \phi(s) = \int_0^\infty e^{-\lambda' xt}\,(xt)^{-m'-\frac{1}{2}}\,M_{k',m'}(xt)\,\alpha(t)\,dt,$$

then

$$(2.3)\qquad f(s) = \int_0^\infty [(\lambda+\tfrac{1}{2})s + (\lambda'+\tfrac{1}{2})t]^{-1}$$

$$F_2\left[\begin{matrix}1, m-k-\frac{1}{2}, m'-k'-\frac{1}{2}, 2m+1 \\ 2m+1,\ 2m'+1\end{matrix};\ \frac{s}{(\lambda+\frac{1}{2})s+(\lambda'+\frac{1}{2})t},\ \frac{t}{(\lambda+\frac{1}{2})s+(\lambda'+\frac{1}{2})t}\right]\alpha(t)dt$$

provided that

a) 2m and 2m' are not negative integers,

b) $\lambda,\ \lambda' > \frac{1}{2}$ , $s > 0$

c) $\alpha(t) = O(t^{\mu}),\ \mu > 0$, for $t \to 0$

$\qquad = O(e^{-\gamma t}),\ \gamma > 0$, for $t \to \infty$,

and $F_2$ standing for the hypergeometric function of two variables.

Proof. On substituting from (2.2) in (2.1), we get

$$f(s) = s^{-m-\frac{1}{2}} \int_0^\infty e^{-\lambda sx}\,x^{-m-\frac{1}{2}}\,M_{k,m}(sx) \left\{ \int_0^\infty e^{-\ \ 'xt}(xt)^{-m'-\frac{1}{2}} M_{k',m'}(xt)\right.$$

$$\left.\alpha(t)\,dt\right\} dx$$

$$= s^{-m-\frac{1}{2}} \int_0^\infty \frac{\alpha(t)\,dt}{t^{m'+\frac{1}{2}}} \int_0^\infty e^{-(\lambda s+\lambda' t)x}\,x^{-m-m'-1}\,M_{k,m}(sx)\,M_{k',m'}(t)\,dx,$$

provided that the change of order of integration is permissible.

Now, evaluating the inner integral with the help of the result [3]

$$(2.4)\qquad \int_0^\infty e^{-pt}\,t^{q-1}\,M_{k,m-\frac{1}{2}}(at)\,M_{k',m'-\frac{1}{2}}(bt)\,dt$$

$$= a^{m} b^{m'} \left[p+\tfrac{1}{2}(a+b)\right]^{-q-m-m'} \Gamma(q+m+m')\, F_2 \left[\begin{matrix} q+m+m', m-k, m'-k' \\ 2m \;,\; 2m' \end{matrix} ; \frac{a}{p+\frac{1}{2}(a+b)}, \frac{b}{p+\frac{1}{2}(a+b)}\right]$$

provided that $(q + m + m') > 0$ , $(p \pm \frac{1}{2}a \pm \frac{1}{2}b) > 0$,

we arrive at the required result.

As regards the validity of the change of order of integration, the t-integral and the x-integral are absolutely convergent under the conditions stated in the theorem. Moreover the resulting integral is also convergent under the same conditions. Hence the change of order of integration is permissible.

Corollary. If $\lambda = \lambda' = 3/2$, and $k = m+\frac{1}{2}$, $k' = m' + \frac{1}{2}$, the $F_2$ in (2.3) reduces to

$$F_2 = \frac{2(s + t)}{s + 2t}\, {}_1F_0 \left[1; - ; \frac{t}{s + 2t}\right] = 2.$$

Hence (2.3) reduces to the well-known Stieltjes transform obtained by the iteration of (1.1).

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Nep. Math. Sc. Rep.
Vol. 6, No. 1 & 2 (1981), 61-64

# Integrals Involving Generalised Struve's Function

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Abstract

The object of the present paper is to give a new generalisation of Struve's function and evaluate some integrals involving the generalised Struve's function.

## 1. Introduction

Struve's function $H_\nu(z)$ of order $\nu$ is defined by [1]

$$(1.1) \quad H_\nu(z) = \frac{2(\frac{1}{2}z)^\nu}{\Gamma(\frac{1}{2})\,\Gamma(\nu+\frac{1}{2})} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \sin zt \, dt$$

$$= \frac{2\,(\frac{1}{2}z)^\nu}{\Gamma(\frac{1}{2})\,\Gamma(\nu+\frac{1}{2})} \int_0^{\frac{1}{2}\pi} \sin^{2\nu}\theta \, \sin(z\cos\theta)\, d\theta, \quad \mathrm{Re}(\nu+\tfrac{1}{2}) > 0$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\,(\frac{1}{2}z)^{\nu+2r+1}}{\Gamma(r+3/2)\,\Gamma(\nu+r+3/2)}, \text{ for all values of } \nu.$$

In the year 1962, Bhowmick defined the generalised Struve's function $H_\nu^\lambda(z)$ by the formula [2]

$$(1.2) \quad H_\nu^\lambda(z) = \sum_{r=0}^{\infty} \frac{(-1)^r\,(\frac{1}{2}z)^{\nu+2r+1}}{\Gamma(r+3/2)\,\Gamma(\nu+\lambda r+3/2)}, \quad (\lambda > 0).$$

We now consider a new generalisation of Struve's function $H_\nu^{\lambda,k}(z)$ defined by

$$(1.3) \quad H_\nu^{\lambda,k}(z) = \sum_{r=0}^{\infty} \frac{(-1)^r\,(\frac{1}{2}z)^{\nu+2r+1}}{\Gamma(kr+3/2)\,\Gamma(\nu+\lambda r+3/2)}, \; k > 0, \; \lambda > 0.$$

## 2. Finite Integral Representations

We now express the generalised Struve's function as integrals involving another generalised Struve's function and also as integrals involving generalised hypergeometric function.

$$(2.1) \quad H_{\nu+\rho+1}^{\lambda,k}(z) = \frac{z^{\rho+1}}{2^\rho\,\Gamma(\rho+1)} \int_0^{\frac{1}{2}\pi} \sin^{(\nu+1)(2-\lambda)}\theta \, \cos^{2\rho+1}\theta \; H_\nu^{\lambda,k}(z)\,(z\sin^\lambda\theta)\, d\theta.$$

Proof. We have

$$\frac{z^{\rho+1}}{2^{\rho}\Gamma(\rho+1)}\int_0^{\frac{1}{2}\pi}\sin^{(\nu+1)(2-\lambda)}\theta\,\cos^{2\rho+1}\theta\; H_{\nu}^{\lambda,k}(z\sin^{\lambda}\theta)\,d\theta$$

$$=\frac{z^{\rho+1}}{2^{\rho}\Gamma(\rho+1)}\int_0^{\frac{1}{2}\pi}\sin^{(\nu+1)(2-\lambda)}\theta\,\cos^{2\rho+1}\theta\sum_{r=0}^{\infty}\frac{(-1)^r(\frac{1}{2}z\sin^{\lambda}\theta)^{\nu+2r+1}}{\Gamma(kr+3/2)\,\Gamma(\nu+\lambda r+3/2)}\,d\theta$$

$$=\sum_{r=0}^{\infty}\frac{(-1)^r\,2\,(\frac{1}{2}z)^{\nu+\rho+1+2r+1}}{\Gamma(\rho+1)\,\Gamma(kr+3/2)\,\Gamma(\nu+\lambda r+3/2)}\int_0^{\frac{1}{2}\pi}\sin^{2(\nu+\lambda r+1)}\theta\,\cos^{2\rho+1}\theta\,d\theta$$

$$=\sum_{r=0}^{\infty}\frac{(-1)^r\,2\,(\frac{1}{2}z)^{\nu+\rho+1+2r+1}}{\Gamma(\rho+1)\,\Gamma(kr+3/2)\,\Gamma(\nu+\lambda r+3/2)}\;\frac{\Gamma(\nu+\lambda r+3/2)\,\Gamma(\rho+1)}{2\,\Gamma(\nu+\lambda r+\rho+5/2)}$$

$= H_{\nu+\rho+1}^{\lambda,k}(z)$ , which is the required result.

The term by term integration in the above simplification is justified because of the fact that the series is uniformly convergent in $0 \leq \theta \leq \frac{1}{2}\pi$ , and this infinite series is multiplied by an absolutely integrable function, and then integrated term by term

In particular, if $k = 1$, we get the known result due to Bhowmick [2]. Two other interesting results derivable from (2.1) are

$$(2.2)\quad H_{\nu}^{\lambda,k}(z)=\frac{z^{\nu+\frac{1}{2}}}{2^{\nu-\frac{1}{2}}\,\Gamma(\nu+\frac{1}{2})}\int_0^{\frac{1}{2}\pi}\sin^{1-\frac{1}{2}\lambda}\theta\,\cos^{2\nu}\theta\; H_{-\frac{1}{2}}^{\lambda,k}(z\sin^{\lambda}\theta)\,d\theta$$

where $\mathrm{Re}(\nu+3/2) > 0$,

and

$$(2.3)\quad H_{\nu+\rho+1}^{\lambda,k}(z)=\frac{z^{\rho+\frac{1}{2}}}{2^{\rho}\,\Gamma(\rho+\frac{1}{2})}\int_0^{1}x^{(\nu+1)(2-\lambda)}\,(1-x^2)^{\rho}\,H_{\nu}^{\lambda,k}(zx^{\lambda})\,dx,$$

where $\mathrm{Re}(\nu+3/2) > 0$, $\mathrm{Re}(\rho+1) > 0$.

The result follows from (2.1) by taking $\nu = -\frac{1}{2}$ and replacing $\rho + \frac{1}{2}$ by $\nu$; and the result (2.3) follows similarly by putting $\sin\theta = x$ in (2.1). Known results can be obtained by specialising the parameters in these results.

Representation II. For $\mathrm{Re}(\nu+3/2) > 0$, $k$ and $\lambda$ are positive integers,

$$(2.4)\quad H_{\nu}^{\lambda,k}(z) = \frac{\Gamma(3/2k)\ (\tfrac{1}{2}z)^{\nu+1}}{\Gamma(3/2)\,\Gamma(3/2k-1)\,\Gamma(\nu+3/2)} \int_0^1 (1-t)^{3/2k-2}\,.$$

$$_0F_{\lambda+k-1}\left(-;\ \frac{5}{2k},\ldots,\frac{3+2k-2}{2k},\ \frac{\nu}{\lambda}+\frac{3}{2\lambda},\ldots,\ \frac{\nu}{\lambda}+\frac{3+2\lambda-2}{2\lambda},\ \frac{-z^2t}{4\lambda^{\lambda}k^k}\right)dt$$

Proof. Using $\Gamma(a+k) = (a)_k\ \Gamma(a)$ and

$$(a)_{mk} = m^{mk}\ (\tfrac{a}{m})_k\ (\tfrac{a+1}{m})_k\ \ldots\ (\tfrac{a+m-1}{m})_k\ ,\ m > 0,\ \ k \geqslant 0,$$

in (1.3), we arrive at

$$(2.5)\quad H_{\nu}^{\lambda,k}(z) = \frac{(\tfrac{1}{2}z)^{\nu+1}}{\Gamma(3/2)\,\Gamma(\nu+3/2)}\ {}_1F_{k+\lambda}\left(1;\ 3/2k,\ \ldots,\ (3+2k-2)/2k\ ,\right.$$

$$\left.\frac{\nu+3/2}{\lambda}\ ,\ \frac{\nu+3/2+\lambda-1}{\lambda}\ ;\ \frac{-z^2}{4\lambda^{\lambda}k^k}\right).$$

It is easy to show that [4]

$$(2.6)\quad \frac{(1)_r}{(3/2k)_r} = \frac{\Gamma(3/2k)}{\Gamma(3/2k-1)}\cdot\frac{\Gamma(r+1)\,\Gamma(3/2k-1)}{\Gamma(3/2k+r)}$$

$$= \frac{\Gamma(3/2k)}{\Gamma(3/2k-1)}\ B\ (r+1,\ 3/2k-1)$$

$$= \frac{\Gamma(3/2k)}{\Gamma(3/2k-1)} \int_0^1 t^r\ (1-t)^{3/2k-2}\ dt.$$

The required result follows on using (2.6) in (2.5) and integrating term by term.

Two particular cases of special interest are as follows:

$$(2.7)\quad H_{\nu}^{\lambda}(z) = \frac{(\tfrac{1}{2}z)^{\nu+1}}{\Gamma(\tfrac{1}{2})\,\Gamma(\nu+3/2)} \int_0^1 (1-t)^{-\frac{1}{2}}\ {}_0F_{\lambda}\left(-;\frac{2\lambda+3}{2\lambda},\ldots,\frac{2\nu+2\lambda+1}{2\lambda};\ \frac{-z^2t}{4\lambda^{\lambda}}\right)dt$$

and

$$(2.8)\quad H_{\nu}(z) = \frac{(\tfrac{1}{2}z)^{\nu+1}}{\Gamma(\tfrac{1}{2})\,\Gamma(\nu+3/2)} \int_0^1 (1-t)^{-\frac{1}{2}}\ {}_0F_1\left(-;\ \frac{2\nu+3}{2};\ -z^2t/4\right)\ dt,$$

These results seem to be new. The first result is obtained from (2.4) by taking k=1, and the second follows from (2.7) by taking $\lambda = 1$.

Acknowledgement

I am obliged to Dr. Girdhari Behary, Professor of Mathematics, Patna Science College, for his guidance in the preparation of this article.

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GLOSSARY OF MATHEMATICAL TERMS
(Proposed)
(N)

Naive सरल, अकृत्रिम
Natural यथार्थ, स्वाभाविक, प्राकृतिक
Naught शून्य
Nautical नाविक, जहाजी, समुद्री
n-dimension एन-डाइमेन्सन्
Nebula नेबुला, नीहारिका
Negative रीनात्मक, निषेधार्थक
Negater निषेधक
Negation निषेध
Neighbourhood सामिप्य, समीपता
Neptune नेप्च्यून, वरूण
Nest नेष्ट्
Neutral तटस्थ, उदासीन
Nil शून्य
Nilpotent शून्यभावी
Nine नौ
Ninefold नौगुना
Ninety नब्बे
Nineteen उन्नाइस
Nineteenth उन्नाइसौं
Ninetieth नब्बेऔं
Ninth नवौं
Nonagon नौकोण, नौभुज
Non-algebraic अबीजिय
Norm नर्म, मानक
Normal अभिलंब, लम्बरूप, यथाक्रम
Nonmalization अभिलम्बिकरण
Notation संकेट
Nought शून्य, गणना, संख्या पद्धति
Null शून्य
Number संख्या, गणना
Numerable गणना योग्य
Numeral अंक, संख्या चिन्ह
Numerator अंश
Numerical संख्या-सूचक, संख्यात्मक
Numerically संख्यानुसार

(O)

Object वस्तु
Oblate ध्रुवतीर च्याप्तो परेको, लुच्चुप्पा, चेप्टो
Oblique टेढो, वक्र, तेर्सो, फुकेको
Oblong आयताकार
Observation निरूपण, निरीक्षण
Obtuse अधिक
Occurrence घट्ना
Octagon अष्टकोण, अष्टभुज
Octagonal अष्टकोणिय, अष्टभुजीय
Octahedral अष्ट फलकीय

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यो शब्दाली त्रि. वि. कीर्तिपुर वहुमुखी क्याम्पस, गणित तथा नेपाली शिक्षण समितिले संयुक्त रूपमा तयार गरिएको हो ।

Octahedron अष्टफलक
Octant अष्टांशक
Octuple आठगुना
Odd विषम, विजोर
Omega ओमेगा, (ω)
One एक
Open खुला, विवृत्त
Operand संकार्य
Operate संक्रिया
Operation संक्रिया
Operational संक्रियात्मक
Operative संकारी
Operator संकारक
Opposite विपरीत
Opposition वियुति
Optimal इष्टतम
Optimization इष्टतमीकरण
Optimizing इष्टतमीकरण
Optimum इष्टतम
Orbit कक्षा
Orbital अर्विटल
Order क्रम, कोटि, घात, मूलांक संमूहांक
Ordered क्रमित
Ordinal क्रमसूचक
Ordinarily साधारणतया
Ordinary साधारण, सामान्य

Ordinate कोटि
Orientability अभिविन्यसनीयता
Orientable अभिविन्यसनीय
Origin मूलविन्दु, उद्गम
Original मूल, प्रारम्भिक
Orthocentre लंबकेन्द्र
Orthocentric लंबकेन्द्रीय
Orthogonal लंबकोणिय, लांविक
Orthogonality लंबकोणियता, लांविकता
Orthogonalization लांविकीकरण
Orthogonally लांविकत
Orthonormal अथोंर्नमल, लांविक
Oscillate दोलन गर्नु
Oscillating दोलायमान
Oscillation दोलन
Oscillator दोलित्र
Oscillatory दोलन, दोलनी
Oscillogram अस्सिलोग्राम
Osculating आश्लेषी
Osculation आश्लेषन, आश्लेष्णी
Osculatory स्पर्शी
Osmotic परासरणी
Outer वाह्य
Oval अंडाकार
Ovaloid अंडवक्राभ
Overlap अतिव्याप्ति, अतिव्यापन

Overstability अतिस्थायित्व

Overstairs अतिप्रक्रम

(P)

Pair जोडा, युग्म

Paired युग्मित, युगलित

Pairing युग्मन

Parwise युगलत:

Parabola परवलय, अनुवृत्त

Parabolic परवलियक, अनुवृत्तीय

Paraboloid पैराबोल्वायड्

Paraboloidal पैराबोल्वाइडी

Paracompact अनुसंहत

Paradox विरोघाभास

Parallax पैरेलैक्स

Parallel समानान्तर

Parallelogram समानान्तर चतुर्भुज

Parallelopiped समानान्तर षट्फलक

Parameter प्राचल

Parametric प्राचालिक

Parametrix प्राचालक

Parenthesis सानोकोष्ठ ( )

Part अंश, खण्ड, भाग

Partial आंशिक

Partially अंशत:

Particle कण

Particular विशेष, विशिष्ट

Partition विभाजन

Path पथ

Pattern प्रतिरूप

Pedal पदिक, पाद

Pendulum दोलक

Penetrate वेधन

Pentagon पंचभुज

Per प्रति

Per cent प्रतिशत

Percentage प्रतिशतता

Percentile शततमक

Perfect पूर्ण, परिपूर्ण

Perihelion उपसौर

Perimeter परिमाप

Period आवर्तकाल, अवधि

Periodic आवर्ती

Periodical आवर्ती

Permanent स्थायी

Permanence स्थायीत्व

Permiable पारगम्य

Permute क्रमचय गर्नु

Permutation क्रमचय

Perpendicular लंब

Perpendicularity लम्बता

Perpendicularly लंबत, अनुलम्ब

Perpetual सतत, शाश्वत

Perturbation क्षोभ
Perturbed क्षुब्ध
Perturbing क्षोभकारी
Phase कला, कोणांक
Phi फाइ ($\phi$)
Physical भौतिक
Physics भौतिकशास्त्र
Pi पाइ ($\pi$)
Piece टुक्रा, खण्ड
Pitch पिच
Pivot धुराग्न, धुरी
Plane समतल
Planet ग्रह
Planetary ग्रहीय
Plus धन
Point विन्दु
Pointwise विन्दुश:
Polar ध्रुवीय
Pole ध्रुव
Polynomial वहुपद, वहुपदीय
Population जनसंख्या
Position स्थान, स्थिति
Positive धनात्मक, धन
Positivity धनता
Postulate अभिगृहीत
Power घात, सामर्थ्य, क्षमता

Predecessor पूर्ववर्ती, पूर्वग
Predicted प्रागुक्त
Prediction प्रागुक्ति
Prefix पूर्वलग्न
Press चाप, चाप लगाउनु, दवाउनु
Pressure चाप, दाव
Price मुल्य
Primary प्राथमिक
Prime अभाज्य, दृढ
Primitive आदि, आद्य
Principal मूल, मुख्य
Principle सिद्धान्त
Probability प्रायिकता, सम्भाव्यता
Probable संभाव्य
Problem समस्या
Process प्रक्रम
Produce उत्पादन गर्नु
Product उत्पादय, गुणनफल
Progression श्रेढी
Project प्रक्षेपन गर्नु
Projected प्रक्षिप्त, प्रक्षेपित
Projectile प्रक्षेपि
Projection प्रक्षेपण, प्रक्षेप
Proof प्रमाण, उपपति
Proper उचित
Proportion समानुपात

Proportional समानुपातिक

Proportionality समानुपातिकता

Proportionate समानुपातिक

Proposition साध्य

Prove प्रमाणित गर्नु

Psi साइ ($\psi$)

Pull अभिकर्ष, अभिकर्षण

Pulley घिरनी

Pulsating स्पंदमान

Pure शुद्ध

Push कर्ष, धकेलाई

Pyramid पिरैमिड