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2. Stokes, G.G., On the effect of the internal friction on the motion of pendulums, Trans. Camb. Phil. Soc., 9 (1851), 8-106.

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Hydromagnetic Pulsating Flow of Visco-Elastic Fluid Between two Parallel Surfaces

Y.R. Sthapit

Abstract

The exact solution for unsteady flow of an incompressible visco-elastic (Revlin-Ericksen model) conducting fluid between two parallel surfaces with oscillating pressure gradient in time is obtained in the presence of uniform transverse magnetic field.

1. Introduction

The exact solution for unsteady flow of an incompressible viscous fluid between two parallel surfaces with oscillating pressure gradient in time has been obtained in [1]. This analysis has been extended by the author [3] for the case of unsteady flow of an incompressible viscous conducting fluid between two non-conducting parallel surfaces in the presence of uniform transverse magnetic field. In the present note, we extend the analysis further to visco-elastic (Revlin-Erickson model) fluid.

2. Equations and Solution

Let u be the component of velocity in the direction of x-axis. A magnetic field of uniform strength H_o is applied in the direction of y-axis taken perpendicular to both the surfaces which are placed at $y=\pm a$. Induced magnetic field may be neglected by assuming the conductivity of the fluid to be very small. All the parameters are independent of x except the pressure since the plates are infinite. The pressure gradient is assumed to oscillate in time in the direction of x-axis. The only non-zero component of velocity will be $u(y,t)$. The governing equation describing the flow of an incompressible conducting visco-elastic fluid in the presence of uniform transverse magnetic field [2] is

$$(1) \qquad \frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} + \beta\frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial t}\right) - \frac{\sigma B_o^2 u}{\rho},$$

where ν is the kinematic viscosity, β, the kinematic visco-elasticity, p, the pressure, ρ, the density of the fluid, σ, the electrical conductivity and $B_o = \mu_e H_o$ (constant), the component of electro-magnetic induction.

The boundary conditions are

$$(2) \qquad u(a,t) = u(-a,t) = 0, \quad t > 0.$$

As in [1] the pressure gradient and the non-zero component of velocity are assumed to be of the form

(3) $$\frac{\partial p}{\partial x} = \mathrm{Re}\left[P_x \exp(int)\right]$$

and

(4) $$u(y,t) = \mathrm{Re}\left[w(y)\exp(int)\right],$$

where P_x is a constant, which represents the magnitude of pressure gradient oscillation. Substituting (3) and (4) in (1) we get the following non-homogeneous differential equation

(5) $$\frac{d^2w}{dy^2} - S^2 w(y) = \frac{P_x}{(1+\beta k^2)},$$

where

$$S^2 = \frac{m+k^2}{1+\beta k^2}, \quad k^2 = \frac{in}{\nu} \text{ and } m = \frac{\sigma B_o^2}{\rho\nu}.$$

The corresponding boundary conditions are

(6) $$w(a,t) = w(-a,t) = 0, \quad t > 0.$$

The solution of (5) subject to the boundary conditions (6) is

$$w(y) = \frac{P_x}{\rho\nu(m+k^2)}\left[\frac{\mathrm{ch}\ Sy}{\mathrm{ch}\ Sa} - 1\right],$$

where chθ stands for coshθ.

Thus the solution for $u(y,t)$ is

$$u(y,t) = \mathrm{Re}\left(\frac{P_x}{\rho\nu(m+k^2)}\left[\frac{\mathrm{ch}\ Sy}{\mathrm{ch}\ Sa} - 1\right]\exp(int)\right).$$

For $\beta = 0$, $m \neq 0$, we get

$$u(y,t) = \mathrm{Re}\left(\frac{P_x}{\rho\nu(m+k^2)}\left[\frac{\mathrm{ch}\ (m+k^2)^{\frac{1}{2}}y}{\mathrm{ch}\ (m+k^2)^{\frac{1}{2}}a} - 1\right]\exp(int)\right)$$

which is the solution obtained in [3] for hydromagnetic pulsating flow between two non-conducting parallel surfaces.

For $m = 0$, $\beta \neq 0$, we get

$$u(y,t) = \mathrm{Re}\left(\frac{P_x}{\rho\nu k^2}\left[\frac{\mathrm{ch}\ k(1+\ k^2)^{-\frac{1}{2}}y}{\mathrm{ch}\ k(1+\ k^2)^{-\frac{1}{2}}a} - 1\right]\exp(int)\right),$$

which is the solution for the flow of non-conducting visco-elastic fluid between two parallel surfaces with oscillating pressure gradient in time.

For $m = 0$, $\beta = 0$, we get

$$u(y,t) = \mathrm{Re}\left(\frac{P_x}{\rho\nu k^2}\left[\frac{\mathrm{ch}\ ky}{\mathrm{ch}\ ka} - 1\right]\exp(int)\right)$$

$$= \mathrm{Re}\left(\frac{iP_x}{\rho n}\left[1 - \frac{\mathrm{ch}\ (\frac{in}{\nu})^{\frac{1}{2}}y}{\mathrm{ch}\ (\frac{in}{\nu})^{\frac{1}{2}}a}\right]\exp(int)\right),$$

which is the solution for the flow of viscous incompressible non-conducting fluid between two parallel surfaces with oscillating pressure gradient in time (1, pp. 237).

Acknowledgement

The author is thankful to the referee for his fruitful comment.

References

[1] Currie, I.G., Fundamental mechanics of fluids, McGraw-Hill, Inc., 1974.

[2] Revlin, R.S. and Erickson, J.L., Rat. Mech. Analysis, 4 (1955), 323.

[3] Sthapit, Y.R., Nep. Math. Sci. Rep., 5, No. 1 (1980), 5.

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Flow of Walters Liquid B' Through Contracting and Expanding Cylinder

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Abstract

A similar solution with respect to time and space is obtained for nonlinear flow problem of Elastico-Viscous Walters liquid B' in a circular cylinder whose radius varies with time by employing perturbation scheme in α which represents the rate of contraction or expansion of the cylinder. It is seen that for any α, positive or negative, the numerical value of shearing stress and pressure difference between the wall and centre line of cylinder increase on account of elasticity.

Introduction

The flow problems of viscous Newtonian fluid in pipes have been widely studied. Recently, the study of flow problems of Walters liquid B' has gathered great momentum because of their close resemblance with real fluids like blood and high polymer solution. The present investigation dealing with the study of elastico-viscous Walters liquid B' in contracting and expanding circular tubes may find application in biophysics, since blood veins and arteries being elastic in nature have time varying diameter.

Uchid and Aoki [1] studied the flow of viscous fluid in a cylinder whose radius varies with time. They supposed that the pipe is closed at one end by an elastic membrane which prevents axial motion but allows radial motion. Our aim, here, is to study above problem for Elastico-Viscious Walters liquid B' with short memories.

The constitutive equation [2] for Walters liquid B' with short memories is

$$p^{ik} = -p\, g_{ik} + 2\eta_o e^{ik} - 2K_o \frac{\delta e^{ik}}{\delta t}$$

where $e^i_k = \frac{1}{2}\left\{v^l_{,k} + v^k_{,l}\right\}$ and η_o is the limiting viscosity, g_{ik} the metric tensor with respect to fixed co-ordinate system, k_o the elastic parameter, v^i the velocity vector, $\frac{\delta}{\delta t}$ denotes the convicted derivative of a tensor quantity for any contravarient tensor b^{ik} and is given by

$$\frac{\delta b^{ik}}{\delta t} = \frac{\partial b^{ik}}{\partial t} + v^m b^{lm}_{,m} - b^{im} v^k_{,m} - b^{mk} v^i_{,m} .$$

Formulation of the Problem

In cylindrical polar co-ordinates (r, θ, z) with z - axis along the axis of the cylinder it can be shown that the flow of Walters liquid B' with short memories, satisfies the following equations of continuity and motion

$$(1) \qquad \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = o,$$

$$(2) \qquad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + u\frac{\partial w}{\partial z} = \frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 w}{\partial z^2} - \frac{u}{r^2}\right] -$$

$$- K_o\left[\frac{\partial}{\partial t}\left\{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 w}{\partial z^2} - \frac{u}{r^2}\right\} + \right.$$

$$+ \frac{\partial^2 u}{\partial r^2}\left\{\frac{2u}{r} - 6\frac{\partial u}{\partial r}\right\} + \frac{\partial u}{\partial r}\left\{- \frac{4}{r}\frac{\partial u}{\partial r} - \frac{2u}{r^2} -\right.$$

$$\left. - 3\frac{\partial^2 w}{\partial r\partial z} + 2u\frac{\partial^3 u}{\partial r^3} + 6\frac{u^2}{r^3}\right],$$

$$(3) \qquad \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right] - K_o\left[\frac{\partial}{\partial t}\left\{\frac{\partial^2 w}{\partial r^2} + \right.\right.$$

$$\left. + \frac{1}{r}\frac{\partial w}{\partial r}\right\} + \frac{\partial w}{\partial r}\left\{\left(\frac{\partial^2 w}{\partial r^2} - 2\frac{\partial^2 w}{\partial r^2} + 2\frac{\partial^2 w}{\partial r\partial z}\right) -\right.$$

$$\left. - \frac{1}{r}\left(\frac{\partial w}{\partial z} - 3\frac{\partial u}{\partial r}\right)\right\} - 3\frac{\partial^2 w}{\partial r^2}\frac{\partial u}{\partial r} + \frac{u}{r}\frac{\partial^2 w}{\partial r^2} +$$

$$\left. + w\left(\frac{\partial^3 w}{\partial r^2\partial z} + \frac{1}{r}\frac{\partial^2 w}{\partial r^2}\right) + u\frac{\partial^3 w}{\partial r^3}\right];$$

where u is the radial velocity, w the axial velocity and transverse velocity is assumed zero.

The contraction velocity is function of time only as radius of the cylinder varies with time and is equal to $\frac{da}{dt} = \dot{a}$, where 'a' is radius of the cylinder at time $t = o$.

Now, the boundary conditions, are,

$$(4) \qquad \begin{cases} u = \dot{a},\ w = o \text{ at } r = a(t), \\ u = o,\ \frac{\partial w}{\partial r} = o \text{ at } r = o,\ w = o \text{ at } z = o \text{ and } u \text{ is left free.} \end{cases}$$

In present case, the velocity components (u,w) satisfying the equation of continuity can be taken as

$$(5) \qquad u = -\frac{\nu}{a}\frac{f}{\eta}, \quad w = \frac{\nu z}{a^2}\frac{f_\eta}{\eta},$$

where $\eta = \frac{r}{a}$ and $f_\eta = \frac{\partial f}{\partial \eta}$

Now, differentiating (3) with respect to r and putting $\frac{\partial^2 p}{\partial r \partial z} = o$, using (5) in it and integrating it with respect to η, we get following differential equation for f

$$(6) \quad \frac{Kf}{\eta}\frac{\partial^3}{\partial\eta^3}\left(\frac{1}{\eta}\frac{\partial f}{\partial\eta}\right) + \frac{\partial^2}{\partial\eta^2}\left(\frac{1}{\eta}\frac{\partial f}{\partial\eta}\right)\left[1 - K\left\{-4\alpha - \frac{f}{\eta^2} + \frac{1}{\eta}\frac{\partial f}{\partial\eta} + 3\frac{\partial}{\partial\eta}\left(\frac{f}{\eta}\right)\right]\right. +$$
$$+ \frac{\partial}{\partial\eta}\left(\frac{1}{\eta}\frac{\partial f}{\partial\eta}\right)\left[\frac{1}{\eta} + \frac{f}{\eta} + \alpha\eta - k\left\{-\frac{4\alpha}{\eta} + 4\frac{\partial^2}{\partial\eta^2}\left(\frac{f}{\eta}\right)\frac{\partial}{\partial\eta}\left(\frac{1}{\eta}\frac{\partial f}{\partial\eta}\right)\right.\right. -$$
$$\left.\left. - \frac{1}{\eta}\frac{\partial f}{\partial\eta}\left(\frac{1}{\eta}\frac{\partial f}{\partial\eta} - 2\alpha\right)\right\} - \frac{a^2}{\nu}\left\{\frac{1}{\eta}\frac{\partial^2 f}{\partial\eta\partial t} + K\left(\frac{1}{\eta}\frac{\partial^2 f}{\partial\eta\partial t} + \frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\frac{1}{\eta}\frac{\partial f}{\partial\eta}\right)\right\}\right] = c,$$

c being constant of integration and $\alpha = a\dot{a}/\nu$, $K = K_o/a^2$.

The above equation is to be solved under following transformed boundary conditions

$$(7) \quad \begin{cases} \frac{f}{\eta} = o, \quad \left(\frac{f_\eta}{\eta}\right)_\eta = o \text{ at } \eta = o, \\ \frac{f}{\eta} = -\alpha, \quad \frac{f_\eta}{\eta} = o \text{ at } \eta = 1. \end{cases}$$

<u>Similar Solutions With Respect to Space and Time</u>

On assuming α = constant and $f_{\eta t} = o$, the solution so obtained still incorporates the nonlinear characteristic of the problem. Here we obtain a series solution for small values of α which is seen to be a parameter representing wall Reynolds number, by letting

$$(8) \qquad f(\eta) = \alpha f_1(\eta) + \alpha^2 f_2(\eta) + \ldots\ldots + \alpha^n f_n(\eta) + \ldots\ldots,$$

$$(9) \qquad c = \alpha c_1 + \alpha^2 c_2 + \ldots\ldots\ldots\ldots + \alpha^n c_n + \ldots\ldots\ldots;$$

And as in [1] separating coefficients of different powers of α , we get

$$\frac{\partial^2}{\partial\eta^2}\left(\frac{1}{\eta}\,\frac{f_1}{\partial\eta}\right) + \frac{1}{\eta}\,\frac{\partial}{\partial\eta}\left(\frac{1}{\eta}\,\frac{\partial f_1}{\partial\eta}\right) = c_1\ , \tag{10}$$

$$\begin{aligned}
&\frac{K}{\eta} f_1 \frac{\partial^3}{\partial\eta^3}\left(\frac{1}{\eta}\,\frac{\partial f_1}{\partial\eta}\right) + \frac{\partial^2}{\partial\eta^2}\left(\frac{1}{\eta}\,\frac{\partial f_2}{\partial\eta}\right) + \frac{1}{\eta}\,\frac{\partial}{\partial\eta}\left(\frac{1}{\eta}\,\frac{\partial f_2}{\partial\eta}\right) + \frac{\partial}{\partial\eta}\left(\frac{1}{\eta}\,\frac{\partial f_1}{\partial\eta}\right)\left(\frac{f_1}{\eta} + \eta\right) - \\
&- \frac{1}{\eta}\,\frac{\partial f_1}{\partial\eta}\left(\frac{1}{\eta}\,\frac{\partial f_1}{\partial\eta} - 2\right) - K\Big[\frac{\partial^2}{\partial\eta^2}\left(\frac{1}{\eta}\,\frac{\partial f_1}{\partial\eta}\right)\Big\{-4 - \frac{f_1}{\eta^2} + 3\,\frac{\partial}{\partial\eta} + (f_1/\eta)\Big\} + \\
&+ \frac{\partial}{\partial\eta}\left(\frac{1}{\eta}\,\frac{\partial f_1}{\partial\eta}\right)\Big\{-\frac{4}{\eta} + 4\,\frac{\partial^2}{\partial\eta^2}(f_1/\eta) - \frac{3}{\eta}\,\frac{\partial}{\partial\eta}\left(\frac{f_1}{\eta}\right)\Big\}\Big] = c_2\ ,
\end{aligned} \tag{11}$$

$$\begin{aligned}
&\frac{K}{\eta} f_1 \frac{\partial^3}{\partial\eta^3}(f_2/\eta) + \frac{K}{\eta} f_2 \frac{\partial^3}{\partial\eta^3}(f_2/\eta) - K\,\frac{\partial^2}{\partial\eta^2}\left(\frac{1}{\eta}\,\frac{\partial f_2}{\partial\eta}\right)\Big\{-4 - \frac{f_1}{\eta} + \\
&+ 3\,\frac{\partial}{\partial\eta}(f_2/\eta) - \frac{\partial}{\partial\eta}(f_2/\eta) - \frac{3}{\eta}\,\frac{\partial f_1}{\partial\eta}\Big\} + \frac{\partial^2}{\partial\eta^2}\left(\frac{1}{\eta}\,\frac{\partial f_1}{\partial\eta}\right)\Big\{-\frac{f_1}{\eta^2} + \\
&+ 3\,\frac{\partial}{\partial\eta}(f_2/\eta) - \frac{\partial}{\partial\eta}(f_2/\eta) - \frac{3}{\eta}\,\frac{\partial f_2}{\partial\eta}\Big\} + 4\,\frac{\partial}{\partial\eta}\left(\frac{1}{\eta}\,\frac{\partial f_1}{\partial\eta}\right)\frac{\partial^2}{\partial\eta^2}(f_2/\eta) - \\
&- \frac{4}{\eta}\,\frac{\partial f_2}{\partial\eta}\Big\{\frac{1}{\eta}\,\frac{\partial}{\partial\eta}(f_2/\eta)\Big\} + 4\,\frac{\partial^2}{\partial\eta^2}(f_2/\eta)\,\frac{\partial}{\partial\eta}\left(\frac{1}{\eta}\,\frac{\partial f_2}{\partial\eta}\right) = c_3
\end{aligned} \tag{12}$$

The above equations are to be solved under following boundary conditions

$$\begin{cases} \dfrac{f_1}{\eta} = o, & \dfrac{f_1'}{\eta} = o \text{ at } \eta = o, \\[2ex] f_1 = -1, & f_1' = o \text{ at } \eta = 1; \end{cases} \tag{13}$$

$$\begin{cases} \dfrac{f_n}{\eta} = o = \dfrac{f_n'}{\eta} \text{ at } \eta = o, \\[2ex] f_n = o = f_n' \text{ at } \eta = 1. \end{cases} \qquad \text{for } \eta \geqslant 2 \tag{14}$$

The integration of equations from (10) to (12) under boundary conditions (13) and (14) gives

$$
(15)\quad \left[\begin{aligned}
f_1 &= \left(-2\eta^2 + 4\eta^4\right). \\
f_2 &= \left(-\tfrac{5}{18}\eta^2 + \tfrac{7}{12}\eta^4 - \tfrac{1}{3}\eta^6 + \tfrac{1}{36}\eta^8\right) + K\left(\tfrac{5}{4}\eta^6 - \tfrac{25}{12}\eta^4 + 16\eta^2\right); \\
f_3 &= \left(-\tfrac{1057}{10800}\eta^2 + \tfrac{271}{1080}\eta^4 - \tfrac{47}{216}\eta^6 + \tfrac{2}{27}\eta^8 - \tfrac{7}{120}\eta^8 + \tfrac{1}{32}\eta^{10}\right) + \\
&\quad + K\left(\tfrac{10615}{432}\eta^2 + \tfrac{22111}{864}\eta^4 + \tfrac{25}{24}\eta^6 - \tfrac{125}{864}\eta^8 + \tfrac{1}{32}\eta^{10}\right) + \\
&\quad + \tfrac{10}{3}K^2\left(-\tfrac{15567}{326}\eta^2 + \tfrac{2329}{144}\eta^4 - \tfrac{845}{96}\eta^6 + \tfrac{7}{9}\eta^8\right), \\
c_1 &= 16,\ c_2 = -\tfrac{44}{3}(1-K) \text{ and} \\
&\qquad c_3 = -\tfrac{208}{135} + \tfrac{167111}{135}K + \tfrac{2599}{9}K^2.
\end{aligned}\right.
$$

Now the velocity components are calculated as follows:

$$
(16)\quad \left[\begin{aligned}
\frac{u}{u_w} &= -\left(\frac{f_1}{\eta}\right) - \alpha\left(\frac{f_2}{\eta}\right) - \alpha^3\left(\frac{f_\eta}{\eta}\right) - \ldots.. \\
&= \Big[(-2\eta + \eta^3) + \alpha\left(-\tfrac{5}{18}\eta + \tfrac{7}{12}\eta^3 - \tfrac{1}{3}\eta^5 + \tfrac{1}{30}\eta^7\right) + K\left(\tfrac{5}{4}\eta^5 - \right. \\
&\quad \left. - \tfrac{25}{12}\eta^3 + 10\eta\right) + \alpha^2\Big\{ -\tfrac{1057}{10800}\eta + \tfrac{271}{1080}\eta^3 - \tfrac{47}{216}\eta^5 + \tfrac{2}{27}\eta^7 - \tfrac{7}{210}\eta^9 + \\
&\quad + \tfrac{1}{5400}\eta^{11}) + K\left(\tfrac{10615}{432}\eta + \tfrac{22111}{864}\eta^3 + \tfrac{25}{24}\eta^5 - \tfrac{125}{864}\eta^7 - \tfrac{1}{32}\eta^9\right) \\
&\quad + \tfrac{10}{3}K^2\left(-\tfrac{15567}{320}\eta + \tfrac{2329}{144}\eta^3 - \tfrac{845}{96}\eta^5 + \tfrac{7}{9}\eta^7\right)\Big\}\Big], \\
&\text{and} \\
\frac{w}{u_m} &= -\tfrac{1}{2}\Big[(-4+4\eta^2) + \alpha\Big\{-\tfrac{5}{9}\eta^2 - 2\eta^4 + \tfrac{2}{9}\eta^6 + \\
&\quad + K\left(\tfrac{15}{2}\eta^9 - \tfrac{25}{3}\eta^2 + 20\right)\Big\} + \alpha^2\Big\{-\tfrac{1057}{5400} + \tfrac{271}{270}\eta^2 -
\end{aligned}\right.
$$

$$- \frac{47}{36}\eta^4 + \frac{16}{27}\eta^6 - \frac{7}{72}\eta^8 + \frac{1}{450}\eta^{10} + K\left(\frac{10615}{216} - \frac{22111}{216}\eta^2 + \frac{25}{4}\eta^4 - \right.$$

$$\left. - \frac{125}{8}\eta^6 + \frac{5}{16}\eta^8\right) + \frac{10}{3}K^2\left(- \frac{15567}{160} + \frac{2329}{36}\eta^2 - \frac{845}{16}\eta^4 + \frac{56}{9}\eta^6\right)\Big\}\Big]$$

where $u_m\ (z,t) = 2z\dot{a}/_a$, the mean flow velocity and u_w the radial velocity of the fluid at the wall.

Substitution for u and w from equations (5) and (15) in (2) and its integration with respect to η gives the following expression for the pressure.

$$(17)\qquad \frac{p-p_c}{\rho\nu/a_o^2} = \Big[\alpha\{-4\eta^2\} + \alpha^2\{(-\frac{7}{3}\eta^2 + 3\eta^4 - \frac{13}{18}\eta^6) - K(\frac{328}{3}\eta^2 - \frac{25}{2}\eta^4) -$$

$$- K^2(-\frac{75}{4}\eta^2 - 15\eta^4)\} + \alpha^3\{(-\frac{173}{135}\eta^2 + \frac{13}{6}\eta^4 - \frac{163}{108}\eta^6 +$$

$$+ \frac{11}{24}\eta^8 - \frac{3}{100}\eta^{10}) - K(-\frac{18245}{108}\eta^2 + \frac{23824}{24}\eta^4 - \frac{2433636}{952560}\eta^6 +$$

$$+ \frac{41339}{720}\eta^8) - K^2\ (-\frac{4225}{192}\eta^2 + \frac{3019992}{1728}\eta^4 - \frac{229970}{972}\eta^6)$$

$$- \frac{10}{3}K^3\ (\frac{4225}{192}\eta^2 - \frac{23}{19}\eta^4)\}\Big]\left[\frac{a}{a_o}\right]^{-2},$$

where p_c is the value of p at the centre line $\eta = o$. Taking $\eta = 1$ the above equation yields the pressure difference between the wall and the centre line.

$$(18)\qquad \frac{p_w - p_c}{\rho\nu/a^2} = \Big[-4\alpha + \alpha^2\Big\{-\frac{1}{18} - 101.83333\,K + 33.75\,K^2\Big\} +$$

$$+ \alpha^3\Big\{-\frac{1057}{5400} - 878.6051\,K + 83.63202\,K^2 -$$

$$- 69.31333\,K^3\Big\}\Big]\left[\frac{a}{a_o}\right]^{-2}$$

The expression for shearing stress at the wall for elastico-viscous Walters liquid B' corresponding to equation (18) in [2] can be given as

$$(19)\quad -\frac{\tau' a^3}{\rho \nu^2 z} = \left[\left(\frac{\partial w}{\partial r}\right) - k\left\{\frac{\partial^2 w}{\partial t \partial r} + u\frac{\partial^2 w}{\partial r^2} + w\frac{\partial^2 w}{\partial r \partial z} - \frac{\partial u}{\partial r}\cdot\frac{\partial w}{\partial r}\right\}\right]_{r=a}$$

$$= \left(\frac{f''}{\eta}\right)_{\eta=1} - k\left[-2\alpha\frac{\partial^2 f}{\partial \eta^2} - \left(\frac{f}{\eta}\right)\frac{\partial^2}{\partial \eta^2}\left(\frac{f_\eta}{\eta}\right) - \frac{f_\eta}{\eta}\frac{\partial^2 f}{\partial \eta^2} + \frac{\partial}{\partial \eta}\left(\frac{f}{\eta}\right)\frac{\partial^2 f}{\partial \eta^2}\right]_{\eta=1}$$

$$= \left[8\alpha + \alpha^2(-2 + 8K + 24.5K^2) + K^3\left(-\frac{56}{135} - 14.364975K + 2911.3502K^2 - 412.3333K^3\right)\right].$$

Conclusions

Table 1 shows variation of shearing stress with respect to α at $\eta = 1$ for different values of K,

And Table 2 shows variation of pressure difference between the wall and center line of the cylinder.

Table 1

S.No.	α	K = 0.01	K = 0.1	K = 0.25
1.	-0.5	-4.4584855	-7.04336	-17.05607
2.	-0.4	-3.496102	-4.718198	-13.93160
3.	-0.3	-2.568065	-3.01236	- 6.89297
4.	-0.2	-1.675364	-1.7647748	- 2.91083
5.	-0.1	-0.819008	-0.814347	- 0.819008
6.	0.00	0.00	0.00	0.00
7.	0.1	0.780657	0.839350	0.986822
8.	0.2	1.52196	1.86477	3.03333
9.	0.3	2.2229057	3.23736	7.168599
10.	0.4	2.8824902	5.118198	14.42168
11.	0.5	3.4997105	7.668356	25.82164

Table 2

S.No.	α	K = 0.0	K = 0.01	K = 0.1	K = 0.25
1.	-0.5	2.010546	2.8540611	10.435821	23.111787
2.	-0.4	1.603639	2.003023	5.6022963	11.660291
3.	-0.3	1.200286	1.3459389	2.6656877	4.917584
4.	-0.2	0.7993439	0.8289677	1.1022594	1.589446
5.	-0.1	0.3996402	0.4022685	0.5882762	0.6016558
6.	0.00	0.00	0.00	0.00	0.00
7.	0.1	-0.000751	-0.0089735	-0.5863031	-0.849748
8.	0.2	-0.803788	-0.9146083	-1.8943698	-3.4618064
9.	0.3	-1.210284	-1.5386303	-4.447936	-9.1303941
10.	0.4	-1.621415	-2.3455855	-8.7707379	-19.149731
11.	0.5	-2.03836	-3.3893207	-15.386511	-34.814037

It is seen that both for contracting ($\alpha < 0$) and expanding ($\alpha > 0$) pipe, the numerical value of shearing stress increases on account of elasticity and same holds for variation of pressure difference between the wall and the centre line.

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Convergence & Inversion of an Integral Transform

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1. Introduction

An integral of the form

$$(1.1)\quad \int_0^R e^{-st}\,(st)^{\mu}\,L_n^{(\mu)}(st)\ d\,\alpha(t)\ ,$$

where $L_n^{(\mu)}(st)$ is the generalised Laguerre polynomial [1] defined by

$$(1.2)\quad L_n^{(\mu)}(st) = \sum_{k=0}^{n} \frac{(-1)^k\,(1+\mu)_n\,(st)^k}{k!\,(n-k)!\,(1+\mu)_k}\ ,\quad \mathrm{Re}\,(\mu) > -1$$

and $\alpha(t)$ is a function of bounded variation in $0 \leq t \leq R$ for any positive R, is known to exist for a given value of $s = \sigma + i\tau$. We now define an improper integral by

$$(1.3)\quad f(s) = \int_0^{\infty} e^{-st}\,(st)^{\mu}\,L_n^{(\mu)}(st)\ d\,\alpha(t)$$

$$= \lim_{R\to\infty} \int_0^R e^{-st}\,(st)^{\mu}\,L_n^{(\mu)}(st)\ d\,\alpha(t).$$

In particular, if $\alpha(t)$ is absolutely continuous,

$$(1.4)\quad \alpha(t) = \int_0^t \phi(u)\ du,$$

then (1.3) reduces to the form

$$(1.5)\quad f(s) = \int_0^{\infty} e^{-st}\,(st)^{\mu}\,L_n^{(\mu)}(st)\,\phi(t)\ dt.$$

We shall refer to (1.3) or (1.5) as a generalised Laguerre transform and for brevity, call it the $L_n^{(\mu)}$- transform. The $L_n^{(\mu)}$- transform reduces to the classical Laplace transform [4]

$$(1.6) \qquad f(s) = \int_0^\infty e^{-st}\, d\,\alpha(t) \,,$$

when $n = 0 = \mu$.

Two useful $L_n^{(\mu)}$- transforms [1] are as follows:

$$\text{a)} \qquad \int_0^\infty e^{-st}\,(st)^\mu\, L_n^{(\mu)}(st)\; t^{b-\mu-1}\, dt = \frac{\Gamma(\mu-b+n+1)\,\Gamma(b)}{n!\,\Gamma(\mu-b+1)\; s^{b-\mu}} \,,$$

$$\mathrm{Re}(\mu) > -1, \ \mathrm{Re}\ (b) > 0$$

and

$$\int_0^\infty e^{-st}\,(st)^\mu\, L_n^{(\mu)}(st)\; e^{-t}\, L_n^{(\mu)}(t)\, dt$$

$$= \frac{\Gamma(1+\mu+n)\ (-1)^n}{(1+s)^{1+\mu+n}\ n!}\; P_n^{(\mu,0)}\left(\frac{1-3s}{1+s}\right), \ \mathrm{Re}(\mu) > -1, \ \mathrm{Re}(s+1) > 0.$$

2. Convergence

We shall now show that the integral transform defined by (1.3) is convergent under certain conditions; and then see the nature of the region of convergence of the integral transform.

Theorem. Convergence of

$$\int_0^\infty e^{-s_o t}\,(s_o t)^\mu\, L_n^{(\mu)}(s_o t)\; d\ \alpha(t) \quad , \quad \mathrm{Re}\ (\mu) > -1$$

$$\Longrightarrow \ \text{Convergence of } \int_0^\infty e^{-st}\,(st)^\mu\, L_n^{(\mu)}(st)\; d\ \alpha(t), \ \text{for } \mathrm{Re}(s) > \mathrm{Re}(s_o)$$

To prove this we need the following lemma:

Lemma.
(2.1)
$$\underset{0 \le u < \infty}{\text{u.b.}} \left| \int_b^u e^{-s_o t}\,(s_o t)^\mu\, L_n^{(\mu)}(s_o t)\ \ d\ \alpha(t) \right| \le M$$

$$(0 \le b \le u < \infty, \ \mathrm{Re}(\mu) > -1)$$

$$\Longrightarrow \quad \int_0^\infty e^{-st}\,(st)^\mu\, L_n^{(\mu)}(st)\ \ d\ \alpha(t)$$

$$\leq M \left|(s/s_o)^{\mu}\right| \left|L_n(s/s_o)\right| \left|\frac{s-s_o}{\sigma-\sigma_o}\right| e^{-b(\sigma-\sigma_o)}, \quad \sigma > \sigma_o \ \mathrm{Re}(\mu) > -1$$

$$(s = \sigma + i\tau \ , \ s_o = \sigma_o + i\,\tau_o)$$

Proof. Let us define $\beta(t)$ by

$$(2.2) \qquad \beta(t) = \int_b^u e^{-s_o t} (s_o t)^{\mu + k}\, d\,\alpha(t), \quad 0 \leq b \leq u < \infty \ ,$$

$$\mathrm{Re}\ (\mu) > -1$$

where k is a non-negative integer less than n.

Then, it is obvious that (2.1) implies

$$(2.3) \qquad \underset{0 \leq u < \infty}{\mathrm{u.b.}} |\beta(t)| \leq M.$$

Now

$$\int_b^R e^{-st} (st)^{\mu} L_n^{(\mu)}(st)\ d\,\alpha(t)$$

$$= L_n^{(\mu)}(s/s_o)\ (s/s_o)^{\mu} \int_b^R e^{-(s-s_o)t}\ d\,\beta(t)$$

$$= L_n^{(\mu)}(s/s_o)\ (s/s_o)^{\mu} \Big[\beta(R)\ e^{-(s-s_o)R}$$

$$+ (s-s_o) \int_b^R e^{-(s-s_o)t}\ \beta(t)\ dt\Big].$$

Since (2.3) implies the boundedness of $\beta(R)$, the integrated portion vanishes as $R \to \infty$. Moreover,

$$(2.4) \qquad \int_0^{\infty} e^{-st} (st)^{\mu} L_n^{(\mu)}(st)\ d\,\alpha(t)$$

$$= (s-s_o)\ (s/s_o)^{\mu}\ L_n^{(\mu)}(s/s_o) \int_b^{\infty} e^{-(s-s_o)t}\ \beta(t)\ dt$$

provided that the integral is convergent. But

$$\left| \int_b^{\infty} e^{-(s-s_o)t} \beta(t)\, dt \right| \leq M \, \frac{e^{-b(\sigma - \sigma_0)}}{\sigma - \sigma_0} . \quad (\sigma > \sigma_0)$$

The conclusion of the lemma is thus evident.

As regards the theorem, it is now obvious if we take $b = 0$.

As an immediate consequence of the above theorem, we may derive the following conclusion. Since convergence of the $L_n^{(\mu)}$- transform at a point $s=s_o$ implies its convergence at all points for which $\sigma > \sigma_0$. The $L_n^{(\mu)}$- transform may be convergent for all values of s, or for none of them or for some only. In the last case, there exists a number σ_c, such that the $L_n^{(\mu)}$- transform is convergent for $\sigma > \sigma_c$ and divergent or oscillatory for $\sigma > \sigma_c$. The number σ_c is called the abscissa of convergence; and it determines a right-half plane. This right-half plane is the region of convergence of the $L_n^{(\mu)}$- transform. The first two cases correspond to the values $\sigma_c = -\infty$ and $\sigma_c = +\infty$ respectively. Furthermore, the theorem is true if the convergence is replaced by absolute convergence, and therefore we can think of an abscissa of absolute convergence σ_a.

3. Inversion

For a complete theory of integral transform, it is always necessary to consider the inversion problem. We shall now see how the $L_n^{(\mu)}$ - transform may be inverted with the help of Mellin transform. The Mellin transform and its inversion formula are defined by

$$(3.1) \qquad f(s) = \int_0^{\infty} t^{s-1}\, \phi(t)\, dt,$$

where s is a complex variable, and

$$(3.2) \qquad \phi(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-s}\, f(s)\, ds.$$

Inversion formula for the $L_n^{(\mu)}$ - transform.

To obtain the inversion formula we shall consider (2.5) rather than (1.3). Let us multiply both sides of (1.5) by s^{-k} and integrate from 0 to ∞ with respect to s. Thus, if we write

$$g(k) = \int_0^{\infty} s^{-k} f(s)\, ds \tag{3.3}$$

and assume the convergence of the integrals concerned and the validity of the change of the order of integration, we have

$$g(k) = \int_0^{\infty} s^{-k}\, ds \int_0^{\infty} e^{-st} (st)^{\mu} L_n^{(\mu)}(st)\, \phi(t)\, dt$$

$$= \int_0^{\infty} t^{k-1} \phi(t)\, dt \int_0^{\infty} e^{-y} y^{\mu-k} L_n^{(\mu)}(y)\, dy$$

Integrating the inner integral with the help of a known result [2;46.6 (1)], we get

$$g(k) = \frac{\Gamma(n+k)\,\Gamma(\mu-k+1)}{n!\,\Gamma(k)} \int_0^{\infty} t^{k-1} \phi(t)\, dt \quad \mathrm{Re}(\mu-k+1) > 0$$

Since the integral on the right side is the Mellin transform of $\phi(t)$, we may apply Mellin's inversion formula (3.2) to obtain $\phi(t)$ in terms of $g(k)$ and hence in terms of $f(s)$. Thus

$$\phi(t) = \frac{n!}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(k)\, t^{-k}}{\Gamma(n+k)\,\Gamma(\mu-k+1)}\, g(k)\, dk \quad , \quad t > 0 \tag{3.4}$$

where c (k=c+i w) is so chosen that

(i) $\int_0^{\infty} t^{k-1} \phi(t)\, dt$ converges absolutely

(ii) $\int_0^{\infty} t^{-k} f(t)\, dt$ converges absolutely

and (iii) $\phi(t)$ is of bounded variation in the neighbourhood of the point $t = t_o$. .

The change of order of integration is valid under these assumptions.

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Generalized Logarithmic Mean of an Entire Dirichlet Series

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1. Consider a Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$, ($s = \sigma + it$, $\lambda_{n+1} > \lambda_n$, $\lambda_1 \geqslant 0$, $\lambda_n \to \infty$ with n), which we assume to be absolutely convergent for all finite s, and hence it defines an entire function. The Ritt order ρ ($0 \leqslant \rho < \infty$) of $f(s)$ is defined [3, p. 38] as the limit superior of $(\log \log M(\sigma))/\sigma$, as $\sigma \to \infty$, with $M(\sigma) = \sup \{ |f(\sigma + it)| : -\infty < t < \infty \}$.

The logarithmic mean L of an entire function $f(s)$ is defined [1, p. 13] as :

$$(1.1) \qquad L(\sigma) = \lim_{T\to\infty} \frac{1}{2T} \left\{ \int_{-T}^{T} \log |f(\sigma + it)| \right\} dt$$

Also, for $\delta > 0$, we consider the following generalized logarithmic mean of $f(s)$:

$$(1.2) \qquad L_\delta(\sigma) = \lim_{T\to\infty} \frac{1}{2T\, e^{\delta\sigma}} \int_0^{\sigma} \int_{-T}^{T} e^{\delta x} \log |f(x + it)| \, dx \, dt$$

Our aim in this paper is to study some of the growth properties of $L(\sigma)$ and $L_\delta(\sigma)$ relative to an auxiliary function. Various constants have been defined and a number of relations involving these constants have been obtained. In theorem 3, we establish that if either $L(\sigma)$ or $L_\delta(\sigma)$ is asymptotic to $K e^{\delta\sigma} \phi(e^{\sigma})$, then the other is also asymptotic to the same function, only K, the constant depending on 'a' and ρ or 'c' and ρ, differs.

2. Let us set, for $0 < \rho < \infty$,

$$(2.1) \qquad \lim_{\sigma\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{L_\delta(\sigma)}{e^{\rho\sigma} \phi(e)^{\sigma}} = \begin{matrix} a \\ b \end{matrix}, \quad 0 < b \leqslant a < \infty ,$$

and

$$(2.2) \qquad \lim_{\sigma\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{L(o)}{e^{\rho\sigma} \phi(e^{\sigma})} = \begin{matrix} c \\ d \end{matrix}, \quad 0 < d \leqslant c < \infty ,$$

where $\phi(e^{\sigma})$ satisfies the following two conditions:

(i) $\phi(x) > 0$ is continuous for $x > e^{\sigma_0}$,

and

(ii) $\phi(\ell x) \approx \phi(x)$, as $x \to \infty$, for every constant $\ell > 0$.

Now, we prove:

Theorem 1. The constants a,b,c,d as defined by (2.1) and (2.2) satisfy the following relations:

$$(2.3) \qquad a \geq \frac{c^{1+\rho/\delta}\,\rho^{\rho/\delta}}{(\rho+\delta)\,\{(\rho+\delta)c-\delta d\}^{\rho/\delta}},$$

$$(2.4) \qquad b \geq \frac{d}{\rho+\delta},$$

$$(2.5) \qquad a \leq \frac{c}{\rho+\delta},$$

and

$$(2.6) \qquad b \leq \frac{d\,\{(\rho+\delta)\,c^{\delta/\sigma} - \rho\, d^{\delta/\sigma}\}}{\delta(\rho+\delta)\,c^{\delta/\rho}}.$$

Proof. From (1.1) and (1.2), we have

$$(2.7) \qquad L_\delta(\sigma) = e^{-\delta\sigma}\int_0^\sigma e^{\delta x}\, L(x)\, dx.$$

Let $h > 0$. Then, for any $\epsilon > 0$ and $\sigma > \sigma_0$,

$$L_\delta(\sigma + \tfrac{h}{\rho}) = \frac{1}{e^{\delta(\sigma+\frac{h}{\rho})}} \int_0^{\sigma+\frac{h}{\rho}} e^{\delta x}\, L(x)\, dx$$

$$= \frac{1}{e^{\delta(\sigma+\frac{h}{\rho})}} \left\{ \int_0^{\sigma_0} + \int_{\sigma_0}^{\sigma} + \int_{\sigma}^{\sigma+\frac{h}{\rho}} \right\} e^{\delta x}\, L(x)\, dx$$

$$> \frac{A}{e^{\delta\sigma}} + \frac{(d-\epsilon)}{e^{\delta(\sigma+\frac{h}{\rho})}} \int_{\sigma_0}^{\sigma} e^{(\rho+\delta)x}\, \phi(e^x)\, dx +$$

$$+ \frac{L(\sigma)}{e^{\delta(\sigma + \frac{h}{\rho})}} \int_{\sigma}^{\sigma \pm \frac{h}{\rho}} e^{\delta x}\, dx$$

(2.2)
$$+ \frac{A}{e^{\delta\sigma}} + \frac{d - \epsilon}{e^{\delta(\sigma + \frac{h}{\rho})}} \int_{e^{\sigma_0}}^{e^{\sigma}} x^{\rho+\delta-1}\, \phi(x)\,dx + \frac{L(\sigma)}{\delta e^{\delta(\sigma + \frac{h}{\rho})}} \left\{ e^{\delta(\sigma + \frac{h}{\rho})} - e^{\delta\sigma} \right\}.$$

Now, by lemma 5 [2, p. 54] :

$$\int_{\sigma_0}^{\sigma} u^{\beta-1}\, \phi(u)\, du \approx \frac{\sigma^{\beta}\phi(\sigma)}{\beta},$$

for every positive β, and so we get

$$L_{\delta}(\sigma + \frac{h}{\rho}) > \frac{A}{e^{\delta\sigma}} + \frac{(d-\epsilon)\phi(e^{\sigma})}{e^{\delta(\sigma + \frac{h}{\rho})}} \frac{e^{(\rho+\delta)\sigma}}{(\rho+\delta)} + L(\sigma) \left\{ \frac{e^{\delta h/\sigma} - 1}{\delta e^{\delta h/\sigma}} \right\}$$

Therefore,

$$\frac{L_{\delta}(\sigma + \frac{h}{\rho})}{e^{\rho(\sigma + \frac{h}{\rho})}\, \phi(e^{\sigma})} > \frac{1}{e^{h}} \left\{ \frac{A}{e^{(\rho+\delta)\sigma}\phi(e^{\sigma})} + \frac{(d-\epsilon)}{(\rho+\delta)e^{\delta h/\rho}} + \frac{L(\sigma)}{\delta e^{\rho\sigma}\phi(e^{\sigma})} \frac{e^{\delta h/\rho} - 1}{e^{\delta h/\rho}} \right\},$$

where A denotes a constant not necessarily the same at each occurrence. Hence, taking limits and using (2.1) and (2.2), we get

(2.8)
$$a \geqslant \left\{ \frac{1}{e^{h}} \quad \frac{d}{(\rho+\delta)\ e^{\delta h/\rho}} + \frac{c}{\delta} \left(1 - \frac{1}{e^{\delta h/\rho}}\right) \right\}.$$

and

(2.9)
$$b \geqslant \frac{1}{e^{h}} \left\{ \frac{d}{(\rho+\delta)e^{\delta h/\rho}} + \frac{d}{\delta} \left(1 - \frac{1}{e^{\delta h/\rho}}\right) \right\}.$$

It can be seen after a long calculation that the maxima of the right-hand side expressions in (2.8) and (2.9) occur at

(2.10)
$$h = \frac{\rho}{\delta} \log \left\{ \frac{(\rho+\delta)\ c - d\,\delta}{\rho c} \right\},$$

and

$$(2.11) \qquad h = 0,$$

respectively. Substituting these values of h from (2.10) and (2.11) in (2.8) and (2.9), respectively, we get (2.3) and (2.4). Again,

$$L_{\delta}\left(\sigma + \frac{h}{\rho}\right) < \frac{A}{e^{\delta\sigma}} + \frac{(c+\epsilon)}{e^{\delta\left(\sigma+\frac{h}{\rho}\right)}} \int_{\sigma_0}^{\sigma} e^{(\rho+\delta)x}\, \phi(e^{x})\,dx + \frac{L\left(\sigma + \frac{h}{\rho}\right)}{e^{\delta\left(\sigma+\frac{h}{\rho}\right)}} \int_{\sigma}^{\sigma+\frac{h}{\rho}} e^{\delta x}\,dx$$

$$= \frac{A}{e^{\delta\sigma}} + \frac{(c+\epsilon)}{e^{\delta\left(\sigma+\frac{h}{\rho}\right)}} \int_{e^{\sigma_0}}^{e^{\sigma}} x^{\rho+\delta-1}\, \phi(x)\,dx + \frac{L\left(\sigma+\frac{h}{\rho}\right)}{\delta e^{\left(\sigma+\frac{h}{\rho}\right)}} \left\{ e^{\delta\left(\sigma+\frac{h}{\rho}\right)} - e^{\delta\sigma} \right\}$$

$$\approx \frac{A}{e^{\delta\sigma}} + \frac{(c+\epsilon)\,\phi(e^{\sigma})}{e^{\delta\left(\sigma+\frac{h}{\rho}\right)}}\, \frac{e^{(\rho+\delta)\sigma}}{\rho+\delta} + L\left(\sigma+\frac{h}{\rho}\right) \left\{ \frac{e^{\delta h/\rho} - 1}{e^{\delta h/\rho}} \right\},$$

and so we have

$$(2.12) \qquad a \leq \frac{1}{e^{h}} \left\{ \frac{c}{(\rho+\delta)\, e^{\delta h/\rho}} + \frac{c\, e^{h}}{\delta} \left(1 - \frac{1}{e^{\delta h/\rho}}\right) \right\},$$

and

$$(2.13) \qquad b \leq \frac{1}{e^{h}} \left\{ \frac{c}{(\rho+\delta)\, e^{\delta h/\rho}} + \frac{d\, e^{h}}{\delta} \left(1 - \frac{1}{e^{\delta h/\rho}}\right) \right\}.$$

It can also be seen that the minima of the right-hand side expressions in (2.12) and (2.13) occur at

$$(2.14) \qquad h = 0,$$

and

$$(2.15) \qquad h = \log \frac{c}{d},$$

respectively. Substituting these values of h from (2.14) and (2.15) in (2.12) and (2.13), respectively, we get (2.5) and (2.6). This proves the theorem 1.

<u>Theorem 2.</u> If $L(\sigma)$ and $L_\delta(\sigma)$ are, respectively, the logarithmic and the generalized logarithmic means of f(s) of Ritt- order $\rho(0<\rho<\infty)$, then

$$(2.16)\qquad \frac{1}{(\rho+\delta)}\ \frac{d}{c} \leq \lim_{\sigma\to\infty}\ \inf \frac{L_\delta(\sigma)}{L(\sigma)} \leq \lim_{\sigma\to\infty}\ \sup \frac{L_\delta(\sigma)}{L(\sigma)} \leq \frac{1}{(\rho+\delta)}\ \frac{c}{d},$$

where c and d are given by (2.2).

<u>Proof.</u> From (2.1) and (2.2), we have, for any $\epsilon>0$ and $\sigma>\sigma_0$,

$$\frac{b-\epsilon}{c+\epsilon} < \frac{L_\delta(\sigma)}{L\ (\sigma)} < \frac{a+\epsilon}{d-\epsilon}\ .$$

Taking limits and using (2.4) and (2.3), we get

$$\frac{1}{\rho+\delta}\ \frac{d}{c} \leq \lim_{\sigma\to\infty}\ \inf \frac{L_\delta(\sigma)}{L(\sigma)}\quad \lim_{\sigma\to\infty}\ \sup \frac{L_\delta(\sigma)}{L(\sigma)} \leq \frac{1}{\sigma+\delta}\ \frac{c}{d}\ .$$

<u>Corollary.</u> If c = d

$(\rho+\delta)\ L_\delta(\sigma) \approx L(\sigma)$.

<u>Theorem 3.</u> If $L_\delta(\sigma) \approx a\ e^{\rho\sigma}\phi(e^\sigma)$, then

$L(\sigma) \approx a(\rho+\delta)\ e^{\rho\sigma}\ \phi(e^\sigma)$ and conversely.

<u>Proof.</u> From (2.4) and (2.5), if c = d, then $a = b = \frac{c}{\rho+\delta}$.
Suppose now a = b. We shall show that c = d. If $0<\eta<1$, we have from (2.7).

$$\eta e^{\delta\sigma} L(\sigma) < \int_\sigma^{\sigma+\eta} e^{\delta x}\ L(x)\ dx$$

$$= \int_0^{\sigma+\eta} e^{\delta x}\ L(x)\ dx - \int_0^\sigma e^{\delta x}\ L(x)\ dx$$

$$= e^{\delta(\sigma+\eta)}\ L_\delta(\sigma+\eta) - e^{\delta\sigma}\ L_\delta(\sigma).$$

Therefore

$$\eta L(\sigma) < a \; e^{\rho\sigma + (\rho+\delta)\eta} \phi(e^{\sigma+\eta}) - a\, e^{\rho\sigma} \phi(e^{\sigma}) + o(e^{\rho\sigma} \phi(e^{\sigma}))$$

$$= a\, e^{\rho\sigma} \left\{ 1 + (\rho+\delta)\eta + O(\eta^2) \right\} (1+o(1))\, \phi(e^{\sigma})$$

$$- a\, e^{\rho\sigma} \phi(e^{\sigma}) + o(e^{\rho\sigma} \phi(e^{\sigma})).$$

Hence

$$\lim_{\sigma \to \infty} \sup \frac{L(\sigma)}{e^{\rho\sigma} \phi(e^{\sigma})} \leq a \left\{ \rho + \delta + H\eta \right\},$$

where H is a constant. Since η is arbitrary and so making $\eta \to 0$, we find that

$$\lim_{\sigma \to \infty} \sup \frac{L(\sigma)}{e^{\rho\sigma} \phi(e^{\sigma})} \leq a\, (\rho + \delta).$$

By considering $\left\{ e^{\delta\sigma} L_{\delta}(\sigma) - e^{\delta(\sigma-\delta)} L_{\delta}(\sigma - \eta) \right\}$ and proceeding as above, we get

$$\lim_{\sigma \to \infty} \inf \frac{L(\sigma)}{e^{\rho\sigma} \phi(e^{\sigma})} \geq a\, (\rho + \delta).$$

and hence

$$L(\sigma) \approx a\, (\rho + \delta)\, e^{\rho\sigma} \phi(e^{\sigma}).$$

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Some Deterministic Population Models and Ultimate Population Size for Nepal

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1. Background

One of the main problems in any population studies is the formation of a 'model' which can be effectively used to predict the population sizes at different time periods and to determine the ultimacy of such size. Population size of a country is expected to tend to a finite limit because it is checked either by Malthus's positive checks such as wars, famines etc. or by preventive checks such as the usage of contraceptives and social upliftments of the population. In demographic studies, population growth rates and subsequently the population sizes at different time periods are predicted first by projecting the fertility, mortality and migration rates in keeping view of expected socio-economic developments. Theoretically, population models based on above demographic variables can be easily constructed+, but in practice it is not so, because the degree of migration and the levels of socio-economic developments are very hard to measure.

A popular mathematical tool to study 'Population phenomenon' is the stable-population model where a constraint that the population concerned must be a closed one is imposed. In case of Nepal whose almost all southern boarder with India is open, it is hardly plausible to make the assumption that its population is closed one. Even in the case the population is considered closed one, the present available demographic data which are greatly influenced by mis-statements of the Ages and Under-enumeration [5] hardly makes it possible to use the model for determining the population trend of Nepal. Also, though at present Nepal has series of estimates of CBR, CDR (mostly made by using U.N's model life tables), the projections of these rates without making any assessment of the social developments would be only a guess work.

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+Population size at time 't' is measured as $P(t) = P(o) + (B-D) + (M-m)$,

where, $P(t)$ = Estimates of the population size at time 't'
$P(o)$ = Population size recorded at base period 'o'
B and D = Number of births and deaths recorded between 'o' and 't'
M and m = Number of In-migration and out-migration recorded between 'O' and 't'

Therefore, in this paper, simple population models based on the relationships between the population size and the time period 't', which do not need to deal with the complexity of problems of measuring 'Correct values' of the demographic variables are workout on the basis of the population sizes observed at different censuses so far taken in Nepal.

2. Census Population Sizes

Though there is evidence that Nepal's population had been counted as a part of census since ancient time, the first population figure whose official record is available is that counted in 1911. From 1911 to 1971, so far seven censuses have been taken in Nepal; of which last three are considered modern censuses. The population sizes of Nepal as recorded at various censuses are:

Table 1
(Population of Nepal, 1911-1971)

In 10'ooo's

Census	Population
1911	56.58
1920	55.73
1930	55.32
1941	62.83
1952-54	82.56
1961	94.13
1971	115.55

Source: Census reports of CBS, Kathmandu, Nepal.

Singh [6] has pointed out that the census dates in Nepal are not uniform. Therefore, the population sizes reported in the censuses have to be adjusted to the same date preferably the mid-year which turns to be 1st Aswin according to Nepalese calendar. Also the census years are to be adjusted to equal intervals of 10 years so that the comparisons of the population sizes at different time points will be greatly simplified.

3. Models Worked Out

Population size $P(t)$ at time 't' is defined as a function of time 't' i.e. $P(t) = f\ (t)$ .... (1)
Three possibilities of $f(t)$ are considered for the present study and two are discussed in the present section.

i) Simple linear growth model: If only two informations about population sizes recorded at base year '0' and last year 'T' are to be used, the population at time 't' ($<$T) is estimated as

$$P(t) = P(0) + k.t \quad \ldots\ldots\ldots\ldots \quad (2)$$

where,

$P(t)$ = Expected population size at time 't'

k = a constant equal to $r.P(0)$

r = Annual growth rate = $\frac{P(T) - P(0)}{T.P(o)}$

On the other hand, if all the informations about the population sizes recorded at times t's between 0 and T are to be used, P(t) is estimated as

$$P(t) = \hat{a} + \hat{b}\, t \quad \ldots\ldots\ldots\ldots \quad (2.1)$$

where $\hat{a}$ and $\hat{b}$ are estimated by fitting $P(t) = a + bt + e$ to the given set of population data, (e being the error of fitting).

Comparison of (2) and (2.1) shows that a and b are the estimates of P(0) and k respectively.

Also the relationship (2.1) shows that

$$P(t) \rightarrow +^{\infty} \text{ as } t \rightarrow +^{\infty} \text{ and } P(t) \rightarrow \mp^{\infty} \text{ as } t \rightarrow +-\infty$$

These limits are unreasonable for a population with limited resources neither tends to $+\infty$ nor to $-\infty$ because it will be checked by either Malthus positive checks or modern preventive checks.

ii) Simple exponential growth model: In this model, the growth of the population from P(0) to P(t) is related by

$$P(t) = P(0).\, e^{rdt} \quad \ldots\ldots\ldots\ldots \quad (3)$$

$$\text{or } \; P(t) = P(0)\; (1+r)^{t} \quad \ldots\ldots\ldots\ldots \quad (3.1)$$

The relationship is also called log linear. In above model, only two observations of the population sizes are to be utilised to estimate the annual growth rate of the population. In case, where more than two observations are to be considered, the model

$$P(t) = \hat{a}\, e^{\hat{b}\, t} \quad \ldots\ldots\ldots\ldots \quad (3.2)$$

where, $\hat{a}$ and $\hat{b}$ are estimated by fitting the relationship (3.2) to the given set of population data, is used.

The relationship (3.2) shows that

$$P(t) \rightarrow +\infty \text{ as } t \rightarrow +\infty \text{ and } P(t) \rightarrow -\infty \text{ as } t \rightarrow -\infty$$

As discussed above, these limits are contrary to Malthus's proposition that the population will be ultimated checked by positive checks as vices.

Application to Nepal's data: Though the limits shown by the models discussed are unreasonable to describe the ultimate population growth phenomenon, it is assumed that the models are good to project the population sizes of Nepal for short time periods say 5 to 10 years of time.

Under above assumptions, the models fitted to Nepal's data (Table 1) gives the following results:

In 10,000's

	Population data 1911-1971		Population data 1930-1971	
Model	$P(t)$	R^2	$P(t)$	R^2
Linear	$45.1426+0.9725t$	63.91%	$50.548+1.4733t$	96.91%
Exponential	$49.14e^{0.0124t}$	86.96%	$53.51e^{0.0183t}$	98.92%

The above results show that the coefficient of determination R^2 which is often used in regression models to determine the lack of fits have increased in the case where the population data of Nepal from 1911 to 1920 are deleted as against the case where all data through 1911 to 1971 are used.

In the first case the population data used are the census population figures through 1930 to 1971 and in the second case the figures are those through 1911 to 1971 (Table 1). Thus for Nepal's case the choosing of 1930 rather than from 1911 gives the reliable fitting of the population models. Also in above result R^2 for exponential models are found to be greater than R^2 for linear model in both cases, one is likely to conclude that former is better than later for population projections of Nepal. However, it is to be noted that the comparisons of models cannot be made on the basis of R^2 unless the dependent variables in both models are converted into same scale.

In such situation, the comparisons are made by using the technique given in Applied Econometric [4]. In this technique sum of square of errors for both models are estimated by converting the dependent variables of both models by weighting the value of $P(t)$ in the linear model

by its geometric mean. The model which gives minimum sum of square of errors rather distance function is considered best one.

As for illustration, using population data of Table 1, for Nepal's case, it is found that

$$\log_e G = \frac{1}{n} \log_e P(i) = \frac{1}{7}.62.11 = 8.6728 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

where, G = Geometric mean of the population sizes

$P(i)$ = Population size at time (i)

so that $G = 7135.63$.

Defining $\overset{*}{P}$ as $\frac{P}{G}$; C as $\frac{1}{G}$; denoting the sum of squares of errors of linear model before and after transforming of P to G $\overset{*}{P}$ by $\sum e^2$ and $\sum \overset{*}{e}{}^2$ respectively and sum of squares of exponential model by $\sum e'^2$, it is found that

$$C^2 = (1.401 \times 10^{-4})^2; \; \sum e^2 = 6048 \times 10^4 \text{ and } \sum e'^2 = 0.0668$$

so that $\sum \overset{*}{e}{}^2 = C^2 \sum e^2 = 1.18$.

Since $\sum \overset{*}{e}{}^2 > \sum e'^2$, it is concluded that loglinear model i.e. exponential model is definitely better than simple linear model in describing the population growth trend of Nepal.

4. Ultimate Population Size

The models discussed above leads the population size of Nepal ultimately to infinity according to their limits discussed in the previous sections. Therefore these models are not useful to estimate the finite ultimate population size of Nepal. The ultimate population size may be attained in two ways. In the first way, it is attained by drastically reducing the present fertility rate such that the replacement index R_o becomes zero by intensive use of the contraceptive methods. In the second a way, the ultimate size is attained in the course of natural phenomenon of the population. The model to be used in the first way is that of Nathan Keyfitze [3]. The model to be used in the second way is the logistic model where the population size P(t) is linked to the time 't' by imposing the restriction that population growth rate diminishes in the proportion to the population size attained.

Application of these models to estimate the ultimate population size for Nepal are illustrated herein:

i) Nathan Keyfitze's model: The model suggested by him is as follows:

$$\frac{Q_o}{P} = \frac{b e_o^o}{r\,\mu} \frac{(R_o - 1)}{R_o} \quad \text{.......................................} \quad (5)$$

where,

Q_o = Estimate of ultimate population size

P = Present population size

b_o = Present crude birth rate

e_o^o = Present expectation of life at birth

μ = Female mean age at child bearing period

R_o = Replacement index

r = Annual growth rate.

Using the present officially recognised demographic values for Nepal i.e. e_o^0 = 45 years, r = 2.09%, u = 27.5 years, b = 43 per thousand.

R_o = 1.778 and P(1975) = 13.20 millions, the estimate of ultimate population size for Nepal is found to be 17.40 millions.

ii) Logistic model: The logistic model [1] is given as

$$P(t) = \frac{L}{1 + e^{r(B-t)}} \quad \text{.......................................} \quad (6)$$

where, P(t) = Population size at time 't'

L = Ultimate population size

r = Population growth rate

B = The value of 't' for which P becomes L/2

An attempt to fit (6) to the population data through 1911 to 1971 shown in Table 1 could not succeed.* So ignoring the first two population figures i.e. of 1911 and 1920 in Table 1, the model (6) is fitted as

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*Growth rate 'r' is found to be negative, which is contrary to present growth trend of Nepal.

$$P(t) = \frac{166.66 \text{ millions*}}{1 + e^{0.223(15.46-t)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.1)$$

This shows the ultimate population size of Nepal will be 166.66 millions, if preventive checks are not applied to the present growth rate of the population. This value is about nine times more than that estimated by (5).

The efficiency of the model is compared with that of linear and exponential models in Section 6.

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*The fitting of the model is shown as below:

Rewriting the relation (6) as

$$\frac{1}{P_i} = \frac{1 - e^{-r}}{L} + e^{-r}. \frac{1}{P_{i-1}}$$

we get $y_i = a + b\ x_i$ .... (6.2)

where,

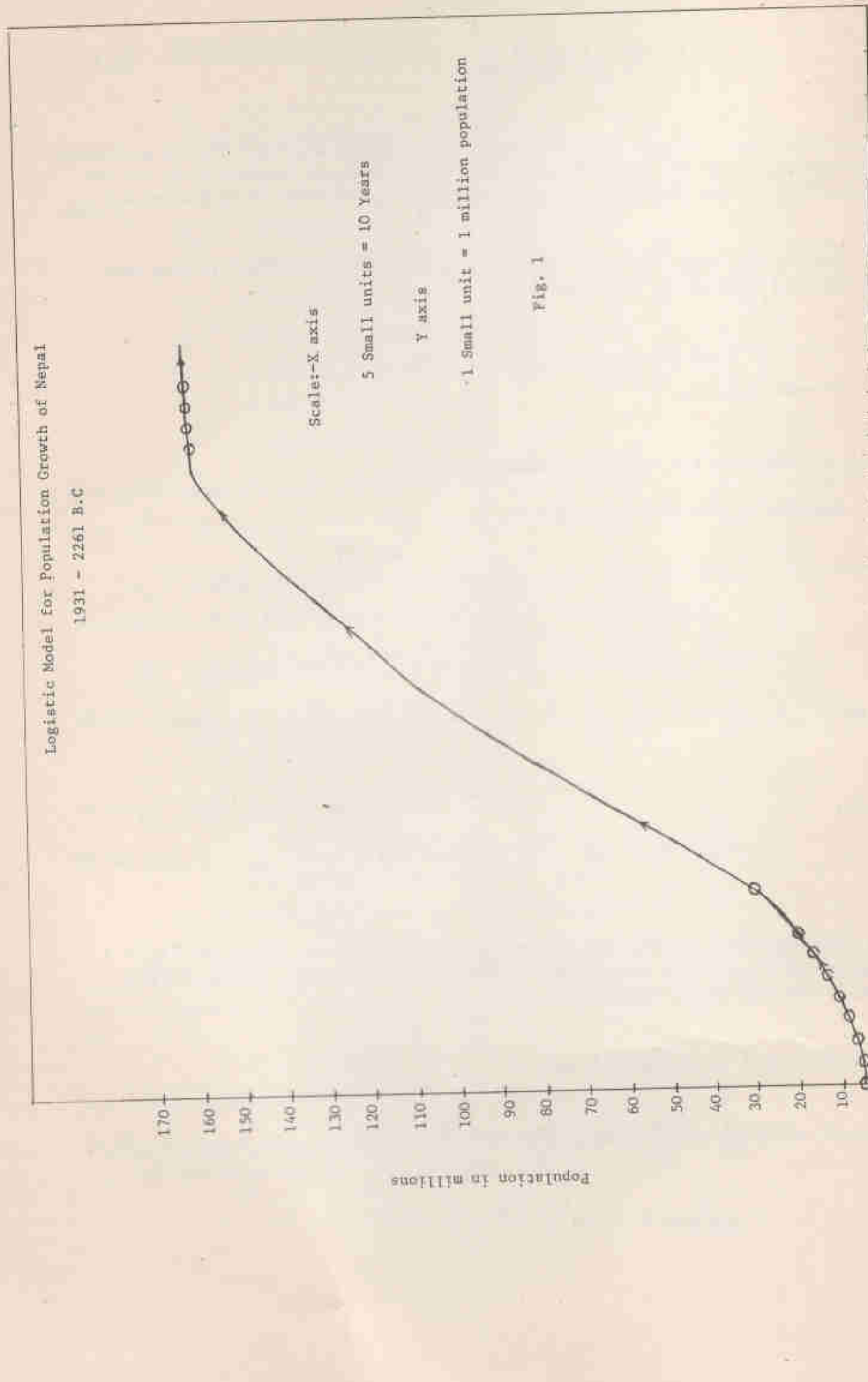
$$y_i = \frac{1}{P_i},\ x_i = \frac{1}{P_{i-1}},\ a = \frac{1-e^{-r}}{L} \text{ and } b = e^{-r} \ \ldots\ldots\ldots\ldots (6.3)$$

Now fitting (6.2) to the inverted population figures shown in Table 1 through 1930 to 1971, we get b = 0.80 and a = 0.0012. These values give r = 0.223 and L = 166.66.

The value of B is estimated as

$$B = \frac{1}{nr} \sum_{o}^{n-1} \log_e \left(\frac{1}{P_i} - 1\right) + \frac{n-1}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.4)$$

$$= \frac{1}{5x0.223} \times 15.01 + \frac{5-1}{2} = 15.46.$$



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5. Population Projection for Nepal by Logistic Model

Giving different values for 't' in (6.1) the expected population size at different time periods are found as shown below:

Table 2
(Population projections for Nepal under logistic model)

In millions

Years	Population
1931	5.14
1941	6.37
1951	7.89
1961	9.74
1971	12.00
1981	14.74
1991	18.03
2001	21.94
2011	26.54
2021	31.90
....	.....
....	.....
2221	160.25
2231	161.25
....	......

The graph of the logistic model fitted to Nepal's data is shown in Figure 1.

6. Tests of Goodness of Fits of the Models

How good is the given deterministic model fitted to a given set of data is checked by any of the following tests.

a. Test of lack of fit
b. Test of randomness of error variate
c. Test of normality
d. Test by determination coefficient R^2.

In this paper three population models viz, linear, exponential and logistic models based on regression analysis and one based on demographic parameters viz, Keyfitz's model are dealt. Of these models, only last two models provide the ultimacy of the population size. However, according to Duncan and Hauser [2], it is to be noted that there does not exists any population theory which is invariant. The population models fitted in this paper may therefore be assumed to be workable only if present population trend of Nepal continues in future also.

Since, as discussed in Section 1 of the paper, the main objective of the paper is to deal with models of the form $P(t) = f(t)$, the test of goodness of fits of the linear, exponential and logistic models are dealt in this section.

Also since all above models are based on regression, the lack of fits of these models are usually to be tested by R^2 or F-tests. But the application of these tests depend on the context and situation. R^2 or F-tests are applicable only if same model is applied to different sets of data. In case of different models with different scales of dependent variable applied to the same set of data, as is the case of the models fitted in the paper, there is a practical difficulty of estimating comparable R^2 and F-values.

In Section 3, it was already suggested that in such situation the comparisons are to be made by comparing the sum of squares of error of the models after converting the dependent variables of the models to the same scale. Since it is not easy to convert the dependent variable of logistics model to the same scale as that of linear and exponential models, an alternative test, which is easily understood and convenient in application is to be searched. Also, since the basic criteria of the comparison of the models are based on the minimum values of sum of squares of errors i.e.

$\sum(o_i - e_{ij})^2$, X^2 values which is usually defined as $\sum\frac{(o_i - e_{ij})^2}{e_{ij}}$ appears to be pluasible, for $X^2 \longrightarrow 0$ as $\sum(o_i - e_{ij})^2 \longrightarrow 0$.

In view of above facts, X^2 values are estimated on the basis of observed population sizes and expected population sizes of Nepal according to linear, exponential and logistic models. The estimated values are 1.068, 0.3647 and 0.75 for linear, exponential and logistic models respectively.

The distance function i.e. sum of squares of errors, $\sum(O_i - e_{ij})^2$ of these models are found to be 6048×10^4 for linear model, 0.0668 for exponential model and 28.44 for logistic model. Comparison of these values with corresponding values of X^2 shows that heighest distance function has heighest X^2 value and lowest distance function has lowest X^2 value. Therefore use of X^2 values is quite consonance with theoretical requirement for the validity of the comparisons.

In keeping view of X^2 values estimated, it appears that exponential model is best one to project the population growth trend of the country. Logistic model appears to be second best one and linear model is the worst one. But if we consider the limits of exponential model discussed in Section 3, we find a snag that it is not suit-

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Reference

[1] Goon

[2] Haus

[3] Keyf

[4] Rao,

[5] Singh

[6] Ibid,

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able for long time projection, for ultimately it leads the population to infinity which is however not true according to Malthus prepositions. Therefore, it is concluded that exponential model is best for explaining short-time population phenomenon and Logistic model is best for long-time population phenomenon of Nepal.

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On Geometric Nearfields

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Abstract

We have given the definition of a Pasch geometry and its concept in [2]. Here we define a geometric nearfield and develop some structure theory for it. We point out that the incidence group and the nearfield of [5] are particular cases of geometric nearfields.

1. Preliminary

Definition (1.1): Let (A, Δ_A, e) be a geometry [2] and Γ a monoid with identity element 1. We say Γ acts on A if $\gamma \in \Gamma$, $a \in A \Longrightarrow \gamma.a \in A$ subject to:

(i) $\gamma_1.(\gamma_2.a) = (\gamma_1 \gamma_2).a$ (ii) $\gamma.e = e$ and $1.a = a$, $\gamma \in \Gamma$, $a \in A$ and (iii) $(a,b,c) \in \Delta_A \Longrightarrow (\gamma a, \gamma b, \gamma c) \in \Delta_A$ for $\gamma \in \Gamma$. Suppose further that Γ is a group. For $a_1, a_2 \in A$, let $a_1 \sim a_2$ if $\exists \gamma \in \Gamma$ such that $a_1 = \gamma a_2$. Then $\sim$ is an equivalence relation. Let A/Γ denote the set of equivalence classes. Let $([a_1], [a_2], [a_3]) \in \Delta A/\Gamma$ if and only if $(a_1, \gamma_1 a_2, \gamma_2 a_3) \in \Delta_A$ for some $\gamma_1, \gamma_2 \in \Gamma$. Then it can be easily checked that $(A_{/\Gamma}, \Delta A_{/\Gamma}, \{e\})$ is a geometry. In particular, if V is a vector space over a field F, then $F^* = F - \{0\}$ acts on V and the geometry $(V_{/F^*}, \Delta V_{/F^*}, \{0\})$ gives the structure e of the classical projective space.

Definition (1.2): Suppose A is a geometric ring and V is an abelian geometry. Then V is called a geometric module over A if A acts on V as in (1.1) subject to : (a) $0_A.v = a.0_V = 0_V$, $v \in V$, $a \in A$ (b) $(\alpha, \beta, \gamma)_A$, $v \in V \Longrightarrow (\alpha v, \beta v, \gamma v) \in \Delta_V$ (c) $(\alpha v, \beta v, \gamma v) \in \Delta$, $v \neq 0 \Longrightarrow (\alpha, \beta, \gamma) \in \Delta_A$ (d) $(\alpha_1 v, \alpha_2 v, w) \in \Delta_V \Longrightarrow w = \alpha_3 v$ for some $\alpha_3 \in A$. Note that if A is sharp and therefore a ring, then geometric modules over A correspond naturally to ring modules over A.

Let A be a geometric sfield and V a geometric A-module. Let $\Gamma \triangleq A^*$. As Γ acts on A, $A_{/\Gamma}$ is naturally a geometric sfield. Also,

since Γ acts on V, $V/_{\Gamma}$ is a geometry. For $\alpha \in A$, Let $\bar{\alpha} \in A_{/\Gamma}$. For $\bar{\alpha} \in A_{/\Gamma}$, $\bar{v} \in V_{/\Gamma}$, let $\bar{\alpha}.\bar{v} = \overline{\alpha v}$. This makes $V/_{\Gamma}$ into a geometric $A_{/\Gamma}$ - module.

Definition (1.3): Let A, B be geometries. A map f: $A \longrightarrow B$ is called a geometry homomorphism if $f(e_A) = e_B$, $(a_1, a_2, a_3) \in \Delta_A \Longrightarrow (f(a_1), f(a_2), f(a_3)) \in \Delta_B$ and $(f(a_1), f(a_2), b) \in \Delta_B \Longrightarrow b = f(a_3)$ for some $a_3 \in A$. A bijective homomorphism is an isomorphism. All the isomorphism theorems similar to those of group theory hold for geometries [3]. A geometric ring homomorphism is defined obviously.

Let V_1 be a geometric A_1-module and V_2 and A_2-module. Then a pair of maps $(f,\hat{f})$: $(V_1, A_1) \longrightarrow (V_2, A_2)$ is a semi-isomorphism if $\hat{f}$: $A_1 \longrightarrow A_2$ is an isomorphism of geometric rings, f: $V_1 \longrightarrow V_2$ is an isomorphism of geometries and $f(\alpha v) = \hat{f}(\alpha)f(v) \; \forall \; \alpha \in A$, $v \in V$. In particular, if $A_1 = A_2 = A$ and $\hat{f}$ is identity on A, then f is an isomorphism of geometric modules.

Let $(f, \hat{f})$: $(V_1, A_1) \longrightarrow (V_2, A_2)$ be a semi-isomorphism were A_1, A_2 are geometric sfields. Let $\Gamma_1 \trianglelefteq A_1^*$ and $\Gamma_2 = f(\Gamma_1) \trianglelefteq A_2^*$. Define $(T(f), T(\hat{f}))$: $(V_{1/\Gamma_1}, A_{1/\Gamma_1}) \longrightarrow (V_{2/\Gamma_2}, A_{2/\Gamma_2})$ by $T(f)(\bar{v}) = \overline{f(v)}$ and $(T(\hat{f}))(\bar{\alpha}) = \overline{f(\alpha)}$ $v \in V_1$, $\alpha \in A_1$. It can be shown that $(T(f), T(\hat{f}))$ is a semi-isomorphism. Thus, in particular, if V is a vector space over a skew field A, every semilinear transformation of V induces a geometric automorphism of $V_{/A^*}$ and hence a collineation of the projective space [1].

Let V be a geometric module over geometric sfield A. For $X \subseteq V$, let Sp(x) denote the geometric submodule of V generated by X. Then propositions 2.2 and 2.3 of [2] are easily seen to be true when $\langle X \rangle$ is replaced by Sp(x) and subgeometry by geometric submodule. Using these it can be shown easily that Sp(X) satisfies the axioms of formal dependence theory [6]. Therefore its consequences hold in V. So every such V contains a basis over A and the dimension formulas hold. In particular, if v_1, v_2 are independent in V, then $Sp(v_1, v_2) = \{u: \exists$ unique $\alpha_1, \alpha_2 \in A$ such that $(u, \alpha_1 v_1, \alpha_2 v_2) \in \Delta_V\}$.

2. Geometric Nearfield

Definition (2.1): A nearfield is defined in [4] or [5]. We call $(a, \Delta_A, .)$ a geometric nearfield if $(A, \Delta_A, 0)$ is an abelian geometry, $(A^*, .)$ is a group with unit element 1 such that $a.0 = 0.a = 0 \ \forall\ a \in A$, and $(a_1, a_2, a_3) \in \Delta_A$, $a \in A \Longrightarrow (a.a_1, a.a_2, a.a_3) \in \Delta$. If A is sharp then it is naturally a nearfield. If A is a geometric nearfield and $(a_1, a_2, a_3) \in \Delta_A$, $a \in A \Longrightarrow (a_1.a, a_2.a, a_3.a) \in \Delta_A$, then note that A is a geometric sfield. We write ab for a.b in A.

We call (A, R) a normal pair of geometric nearfields if (i) A is a geometric nearfield (ii) R is a geometric subnearfield of A with $R^* \trianglelefteq A^*$ (iii) $(r_1, r_2, r_3) \in \Delta$, $a \in A$, $r_1, r_2, r_3 \in R \Longrightarrow (r_1a, r_2a, r_3a) \in \Delta$. From (iii) R is a geometric sfield and for $r \in R$, $a \in A$, the multiplication ra makes A into a geometric R-module.

Let (A, R) be a normal pair of geometric nearfields. For $0 \neq a \in A$, define $a_1: A \longrightarrow A$ by $a_1(x) = ax$ and $\hat{a}_1: R \longrightarrow R$ by $\hat{a}_1(r) = ara^{-1}$. The following can be easily checked.

Proposition (2.1): $(a_1, \hat{a}_1): (A, R) \longrightarrow (A, R)$ is a semi-isomorphism of geometric R-modules.

Now let A be a geometric nearfield, $\Gamma \trianglelefteq A^*$. Then A/Γ is a geometric nearfield. Also if (A, R) is a normal pair of geometric nearfields and $\Gamma \trianglelefteq R^*$, $\Gamma \trianglelefteq A^*$, then $(A/\Gamma, R/\Gamma)$ is a normal pair of geometric nearfields.

We call $\emptyset : (A_1, R_1) \longrightarrow (A_2, R_2)$ an isomorphism of normal pair of geometric nearfields if $\emptyset : A_1 \longrightarrow A_2$ is an isomorphism of geometric nearfields with $\emptyset(R_1) = R_2$. So if $\hat{\emptyset}$ denotes $\emptyset$ restricted to R_1, then $(\emptyset, \hat{\emptyset}) : (A_1; R_1) \longrightarrow (A_2, R_2)$ is a semi-isomorphism of geometric modules, since $\emptyset(ra) = \emptyset(r)\ \emptyset(a) = \hat{\emptyset}(r)\ \emptyset(a)$ for $r \in R_1$, $a \in A_1$.

We state the following theorem the proof of which is too long to be given here. [See Ph.D. thesis by H.N. Bhattarai, UO, Eugene, USA].

Theorem (2.1): Let V_1, V_2 be vector spaces over skewfields F_1, F_2 respectively, $\dim_{F_1} V_1 \geqslant 3$. Let $\Gamma_1 \trianglelefteq F_1^*$, $\Gamma_2 \trianglelefteq F_2^*$ such that

$(\sigma, \hat{\sigma}) : (V_{1/\Gamma_1}, F_{1/\Gamma_1}) \longrightarrow (V_{2/\Gamma_2}, F_{2/\Gamma_2})$ is a semi-isomorphism of geometric modules. Then, there exists a semi-linear transformation $(\psi, \hat{\psi}) : (V_1, F_1) \longrightarrow (V_2, F_2)$ such that $\hat{\psi}(\Gamma_1) = \Gamma_2$ and $(T(\psi), T(\hat{\psi})) = (\sigma, \hat{\sigma})$.

Now let (F_1, K_1) and (F_2, K_2) be normal pairs of nearfields. Let $\Gamma_1 \trianglelefteq K_1^*$ with $\Gamma_1 \trianglelefteq F_1^*$ and $\Gamma_2 \trianglelefteq K_2^*$ with $\Gamma_2 \trianglelefteq F_2^*$. Then $(F_{1/\Gamma_1}, K_{1/\Gamma_1})$, $(F_{2/\Gamma_2}, K_{2/\Gamma_2})$ are normal pairs of geometric nearfields. Suppose also $\dim_{K_1} F_1 \geqslant 3$, $\dim_{K_2} F_2 \geqslant 3$, Then

<u>Theorem (2.2):</u> There exists an isomorphism of normal pairs of geometric nearfields $\phi : (F_{1/\Gamma_1}, K_{1/\Gamma_1}) \longrightarrow (F_{2/\Gamma_2}, K_{2/\Gamma_2})$ if and only if there exists an isomorphism $g : F_1 \longrightarrow F_2$ of nearfields with $g(K_1) = K_2$ and $g(\Gamma_1) = \Gamma_2$.

<u>Proof:</u> If g exists as stated, define $\phi : F_{1/\Gamma_1} \longrightarrow F_{2/\Gamma_2}$ by $\phi(\bar{x}) = \overline{g(x)}$. Then as in (1.3), ϕ is well defined and is an isomorphism of geometric nearfields. Since $g(K_1) = K_2$, $\phi(K_{1/\Gamma_1}) = K_{2/\Gamma_2})$ and ϕ is an isomorphism of normalpair of geometric nearfields.

Conversely, let $\phi : (F_{1/\Gamma_1}, K_{1/\Gamma_1}) \longrightarrow (F_{2/\Gamma_2}, K_{2/\Gamma_2})$ be an isomorphism. Then if $\hat{\phi}$ denotes ϕ restricted to K_{1/Γ_1}, then $(\phi, \hat{\phi})$: $(F_{1/\Gamma_1}, K_{1/\Gamma_1}) \longrightarrow (F_{2/\Gamma_2}, K_{2/\Gamma_2})$ is also a semi-isomorphism of geometric modules. So by theorem (2.1) stated above there exists semi-linear transformation $(f, \hat{f}) : (F_1, K_1) \longrightarrow (F_2, K_2)$ with $\hat{f}(\Gamma_1) = \Gamma_2$, $(T(f), T(\hat{f})) = (\phi, \hat{\phi})$. Let e_1, e_2 be identities of F_1^*, F_2^* respectively, so that $\bar{e}_1$, $\bar{e}_2$ are those of F_{1/Γ_1} and F_{2/Γ_2} respectively. So $\phi(\bar{e}_1) = \bar{e}_2$ i.e. $\overline{f(e_1)} = \bar{e}_2$. So there exists $\gamma \in \Gamma_2$ with $f(e_1) = \gamma e_2$. Define $g : F_1 \longrightarrow F_2$ by $g(x) = \gamma^{-1} f(x)$. Then $g(\alpha x) = \gamma^{-1} f(\alpha x) = \gamma^{-1} \hat{f}(\alpha) f(x) = \gamma^{-1} \hat{f}(\alpha) \gamma . \gamma^{-1} f(x) = \hat{g}(\alpha) g(x)$ where $\hat{g}(\alpha) = \gamma^{-1} \hat{f}(\alpha) \gamma$ is clearly an isomorphism of skewfield K_1 onto K_2. So $(g, \hat{g}) : (F_1, K_1) \longrightarrow (F_2, K_2)$ is a semi-isomorphism with $g(e_1) = e_2$. For $x \in F_1$, $\overline{g(x)} = \gamma^{-1}\overline{f(x)} = \overline{f(x)}$,

so $T(f) = T(g) = \emptyset$. For $\alpha \in \Gamma_1$, $\hat{g}(\alpha) = \gamma^{-1}\hat{f}(\alpha)\gamma \in \Gamma_2$, so $\hat{g}(\Gamma_1) = \Gamma_2$. For $\alpha \in K_1 \subseteq F_1$, $g(\alpha) = g(\alpha . e_1) = \hat{g}(\alpha)g(e_1) = \hat{g}(\alpha)e_2 = \hat{g}(\alpha)$, so $\hat{g}$ is the restriction of g to K_1.

Let $0 \neq x \in F_1$. Then $\forall\, y \neq 0$ in F_1, $\overline{g(xy)} = \emptyset\overline{(xy)} = \emptyset(\bar{x})\emptyset(\bar{y}) = g(x)g(y) = \overline{g(x)g(y)}$, so there exists $\gamma_y \in \Gamma_2$ with $g(xy) = \gamma_y g(x)g(y)$ i.e. $(g0x_1)(y) = \gamma_y(g(x)0_1 g)(y)$. Thus $G0x_1$ and $g(x)_1 0g$ are two semi-linear transformations with $(g0x_1)(y) = \gamma_y(g(x)_1 0g)(y)\ \forall\, y \in F_1$. So since $\dim F_1 \geqslant 3$, it can be shown easily that $\gamma_y = \gamma\ \forall\, y$ (i.e. γ is independent of y) [1]. In particular, $(g0x_1)(e_1) = g(x) = (g(x)e_2) = g(x)$, so $\gamma = 1$. Thus we get $g(xy) = g(x)g(y)$. Hence g is a homomorphism of multiplication. Therefore, $g : F_1 \longrightarrow F_2$ is an isomorphism of nearfields with $g(K_1) = g(K_2)$ and $g(\Gamma_1) = \hat{g}(\Gamma_1) = \Gamma_2$.

The structure of a geometric nearfield (A, R) will appear later.

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GLOSSARY OF MATHEMATICAL TERMS
(Proposed)

(J)

Jacobian ज्याकोवियन

Joint संयुक्त

Jump उफ्रनु, उफ्राई, प्लुति

Juxtaposition सन्निधि, गाँसिएको

(K)

Kernel अन्तस्थ, मिश्री, कर्नेल

Kilometer किलोमिटर, (एकहजार मिटर)

Kinematics शुद्ध गति विज्ञान

Kinetic गतिज, गति मूलक

Kinetics वल गति विज्ञान

Known ज्ञात, थाहाभएको

Kurtosis ककुदता, जुरो उठेको

(L)

Lamina पटल, पाता

Laplacian लाप्लासियन

Latent अन्तर्निहित, गुप्त, अभिलाक्षणिक

Lateral पार्श्व, पार्श्वीय, पार्श्ववर्ती

Lattice ल्याटिस, जालक

Latus rectum नाभिलंव, लेटस रेक्टम

Least न्यूनतम, अल्पतम, लघुतम

L.C.M. (Least common multiple) लघुतम, समापवर्त्य

L.U.B. (Least Upper Bound) न्यूनतम उच्च परिबन्ध

Left बायाँ, वाम

Length लम्बाई

Leptokurtic तुंगक कुदी

Leptokurtosis तुंगक कुदता

Leveler समतलक

Level समतल, स्तर

Lever लिवर

Lexicographically कोशक्रमानुसार

Lie-algebra ली वीजावली

Lie-group ली समूह

Light हलुंगो

Like उस्तै, सदृश, तुल्य, समान

Limit सीमा, सीमान्त

Limiting सीमान्त, (चरम)

Limiting friction चरम घर्षण

Limit-point सीमा-विन्दु, गुच्छ विन्दु

Line रेखा, धर्सो

Linear रैखिक, रेखाकार, एकघाती

Lineaority एकघातता, रैखिकता

Linearly एकघाततः, रैखिकत

Litre लिटर

Lobe अंश, पिंडिका

Local स्थानीय

Localized स्थानगत, स्थानीयकृत

Locally स्थानतः, स्थानीय रूपमा, स्थानीयत:

Locus विन्दुपथ

यो शब्दावली त्रि.वि., कीर्तिपुर वहुमुखी क्याम्पस, गणित तथा नेपाली शिक्षण शिक्षण समितिले संयुक्त रूपमा तयार गरिएको हो ।

Logarithm (log) लघुगणक, लागेरिथ्म
Logarithmic लघुगणकीय
Logic तर्क
Logical तर्क संगत
Lognormal लघुगणक प्रासामान्य
Long दीर्घ, लम्बा, लामो
Longitude रेखांश, देशान्तर
Longitudinal देशांतरीय, अनुदैर्घ्य
Loop पाश, फन्दा, लूप
Lottery प्रचयन, प्रतिचयन
Lunar चन्द्र

(M)

Magnetic चुंवकीय
Magnetodynamics चुम्बक-गतिकी
Magnetofluiddynamics चुम्बक-तरल-गतिकी
Magnetosphere चुम्बक मंडल
Magnetostatic स्थिर चुम्बकीय
Magnetostatics स्थिर चुम्बकिकी
Major मुख्य, प्रमुख
Majorant माथिल्लो सीमा (उच्च परिबन्ध)
Malfunction कुसंक्रिया
Manifold बहुमुख, बहुरूप
Many-one बहुएक
Map प्रतिचित्र, प्रतिफलन
Mapping प्रतिचित्रण, प्रतिफलन
Margin सीमा, उपांत
Marginal सीमान्त, उपांतस्थ
Mark अंक, चिन्ह, निशाना
Mass द्रव्यमान, संहति
Match सुमेल, जोडा
Matching सुमेलन
Material पदार्थ, भौतिक, द्रव्यात्मक
Mathematical गणितीय
Mathematics गणित
Matrix आव्यूह, मेट्रिक्स
Matter द्रव्य, पदार्थ
Maxima उच्चमान, उच्चिष्ठ
Maximality अधिकतमता
Maximization अधिकतमीकरण
Maximum महन्तम, उच्चतम, अधिकतम
Maxmin महालिपस्थ
Mean वीचको, माध्य, औसत
Means मध्यपदहरू
Meanvalue मध्यमान
Measurable मेय, मापन्योग्य
Measurability मेयता
Measure माप
Measurement नाप, मापन
Measuring मापी, मापक
Mechanical यांत्रिक
Mechanics यन्त्रविज्ञान, यांत्रिकी
Median मध्य, माध्यिका, मिडियन

Medium माध्यम, मध्यम
Meet संधि, अवसंधि
Member सदस्य
Merge विलय गर्नु वा हुनु
Meridian ध्रुववन्त, याम्योन्तर
Meromorphic अनंतकी
Mesokurtic मध्यककुदी
Metaphysical अधि भौतिकीय
Meteorite उल्कापिंड
Meter मीटर, मापक
Method रीति, विधि
Metric दूरीक, मेट्रिक
Metrization दूरीकरण, दूरिकीकरण
Microwave सूक्ष्म तरंग
Middle वीचको, मध्य
Mile मील आधाकोश
Millilitre मिलिलिटर
Millimeter मिलिलिटर
Million दश लाख
Minimal अल्पतम, न्यूनतम, अल्पिष्ठ
Minimax अल्पमहिष्ठ
Minimum निम्नतम, अल्पतम, न्यूनतम
Minor उपसारणिक, लघु, गौण सानो
Minus ऋण, घटाउ
Minute मिनट, कला (मिनुट)
Missile प्रक्षेपणास्त्र
Mixed मिश्रित, मिश्र
Mode मोड, वहुलक
Model आदर्श, मोडेल
Modified रूपांतरित, आपरिवर्तित
Modular प्रतिरूपक, मापांक
Module प्रतिरूपक, माड्यूल
Modulo सापेक्ष
Modulus मापांक
Moment आघूर्ण, क्षण
Momentum संवेग
Money मुद्रा
Monotone एकदिष्ट
Most व्यापक, दक्षतम
Motion गति, चाल
Move चल
Movement गति, संचलन
Moving गतिमान
Mu म्यू (μ) ग्रीक वर्णमालाको बाह्रौं एक अक्षर
Multi-dimensional (Manifold) वहुविम (प्रसमष्टि)
Multilinear वहु एकघाती, वहरैखिक
Multinomial वहुपदी
Multinormal वहुप्रासामान्य
Multiple गुणज, वहुगुण, वहुगुणित
Multiplicate गुन्य
Multiplication गुणन
Multiplicative गुणात्मक

Multiplicator गुणक

Multiplicity बहुकता, गुणकता

Multiplier गुणक

Multiply गुन्नु

Multivariate बहुचर

Mutual पारस्परिक, अन्योन्य

Mutually परस्पर