THE NEPALI MATHEMATICAL SCIENCES REPORT

INSTITUTE OF SCIENCE
DEAN'S OFFICE
TRIBHUVAN UNIVERSITY
KIRTIPUR
NEPAL

VOLUME 5 NO 1

JANUARY 1980

Instructions to Contributors

"The Nepali Mathematical Sciences Report" will publish the original research papers, short notes and survey articles in different branches of Mathematics and Mathematical Sciences.

Contributors are required to submit two copies of the manuscripts to the Chief Editor. Manuscripts should be in English and typed in double space. The text of the paper should be preceded by a brief abstract.

References should be listed at the end of the paper in alphabetical order and each reference is to be arranged in the following form:

1. Erdelyi, A., Tables of Integral Transforms, Vol. II, McGraw Hill, (1954
2. Stokes, G.G., On the effect of the internal friction on the motion of pendulums, Trans. Camb. Phil. Soc., 9 (1851), 8-106.

ALL COMMUNICATIONS REGARDING THIS JOURNAL SHOULD BE MADE TO THE CHAIRMAN, MATHEMATICS INSTRUCTION COMMITTEE, TRIBHUVAN UNIVERSITY, KIRTIPUR, NEPAL.

THE NEPALI MATHEMATICAL SCIENCES REPORT

INSTITUTE OF SCIENCE
DEAN'S OFFICE

TRIBHUVAN UNIVERSITY
KIRTIPUR
NEPAL

Contents

Nep. Math. Sc. Rep.
Vol. 5, No. 1 (1980), 1-4

A Jacobi Polynomial Integral

R.M. Shrestha#

The modified moments of the distribution $d\sigma(x) = (1-x)^a x^\alpha \log(1/x)$ on $[0,1]$ with respect to the shifted Jacobi polynomials, are explicitly evaluated.

1. Introduction

As recently as 1979, Blue [1] proved that

$$(1.1) \qquad \int_0^1 P_n(2x-1)\log(1/x)\,dx = \frac{(-1)^n}{n(n+1)}, \text{ for } n \geqslant 1.$$

This formula has been generalised by Gautschi [3] in the form

$$(1.2) \qquad \int_0^1 x^\alpha \log(1/x)\, P_n(2x-1)\,dx \quad \alpha > -1,\ n = 0,1,2,3,\ldots$$

$$= \begin{cases} (-1)^{n-m}\dfrac{(m!)^2\,(n-m-1)!}{(n+m+1)!}, & \alpha = m < n, \quad m \geqslant 0 \text{ an integer}, \\[2ex] \dfrac{1}{\alpha+1}\left\{\dfrac{1}{\alpha+1} + \displaystyle\sum_{k=1}^{n}\left(\dfrac{1}{\alpha+1+k} - \dfrac{1}{\alpha+1-k}\right)\right\} & \displaystyle\prod_{k=1}^{n}\dfrac{\alpha+1-k}{\alpha+1+k}, \\ \text{otherwise.} \end{cases}$$

When $\alpha = 0$, the result (1.1) follows from (1.2).

Further extension of the above formula to an integral involving Jacobi polynomial is given in the following section. Our method of proof is similar to that of Gautschi.

2. A Jacobi Polynomial Integral

Theorem. Let the modified moments of the distribution

$$d\sigma(x) = (1-x)^a x^\alpha \log(1/x)$$

on $[0,1]$ with respect to the shifted Jacobi polynomial

$$P_n^{(a,b)*}(x) = P_n^{(a,b)}(2x-1)$$

The author is at present a Humboldt Fellow.

be defined by

$$(2.1)\qquad \nu_n(\alpha,a,b) = \int_0^1 (1-x)^a\, x^{\alpha} \log(1/x)\, P_n^{(a,b)*}(x)\, dx, \quad \alpha>-1,\ n,a=0,1,2,3,\ldots$$

Then

$$(2.2)\qquad \nu_n(\alpha,a,b)= \begin{cases} (-1)^{n-m} \dfrac{(m!)^2\,(n-m-)!\,(a+n)!}{(n+m+1)!\,n!}, & \alpha-b=m<n,\ m\geqslant 0, \text{an integer} \\[2ex] \dfrac{(\alpha+2)_a}{\alpha+1}\left\{\dfrac{1}{\alpha+1} - s_a + S_n\right\} \displaystyle\prod_{k=1}^{n} \dfrac{\alpha-b+1-k}{\alpha+1+k}, & \end{cases}$$

where $s_o = 0,\ s_a = \sum_{k=1}^{a} \frac{1}{\alpha+n+k+1},\quad S_n = \sum_{k=1}^{n} \left(\frac{1}{\alpha+1+k} - \frac{1}{\alpha-b+1-k}\right)$

Proof. We note that

$$(2.3)\qquad \nu_n(\alpha,a,b) = -2^{-(\alpha+a+1)} \int_{-1}^{1} (1-t)^a (1+t)^{\alpha} \log\left(\frac{1+t}{2}\right) P_n^{(a,b)}(t)\, dt$$

$$= -2^{-(\alpha+a+1)} \lim_{\nu\to n} \left\{ \int_{-1}^{1} (1-t)^a (1+t)^{\alpha} \log(1+t)\, P_{\nu}^{(a,b)}(t)\, dt - \log 2 \int_{-1}^{1} (1-t)^a (1+t)^{\alpha} P_{\nu}^{(a,b)}(t)\, dt \right\}$$

where $P_{\nu}^{(a,b)}(t)$ is the Jacobi function of degree ν. It is well known [2,p284,eq.(1)] that

$$(2.4)\qquad \int_{-1}^{1} (1-t)^a (1+t)^{\alpha} P_n^{(a,b)}(t)\, dt = \frac{2^{a+\alpha+1}\,\Gamma(\alpha+1)\,\Gamma(a+n+1)\,\Gamma(\alpha-b+1)}{n!\,\Gamma(\ -b-n+1)\,\Gamma(a+\alpha+n+2)}.$$

Differentiate (2.4) with respect to α to find

$$(2.5)\qquad \int_{-1}^{1} (1-t)^a (1+t)^{\alpha} \log(1+t)\, P_n^{(a,b)}(t)\, dt$$

$$= \frac{2^{a+\alpha+1}\,\Gamma(\alpha+1)\,\Gamma(a+n+1)\,\Gamma(\alpha-b+1)}{n!\,\Gamma(\alpha-b-n+1)\,\Gamma(a+\alpha+n+2)} \Big[\log 2 + \psi(\alpha+1) + \psi(\alpha-b+1) = \psi(\alpha-b-n+1) - \psi(a+\alpha+n+2)\Big]$$

with $\psi(x) = \Gamma'(x) / \Gamma(x)$ the logarithmic derivative of the gamma function. Using (2.4) and (2.5) in (2.3), we arrive at (2.2) by virtue of the difference equations $\Gamma(x+1) = x\,\Gamma(x)$ and $\psi(x+1) = \psi(x) + 1/x$, together with the fact that for any integer $r \geqslant 0$

$$\frac{\psi(-r+\varepsilon)}{\Gamma(-r+\varepsilon)} \longrightarrow (-1)^{r-1}\, r! \quad \text{as} \quad \varepsilon \longrightarrow 0.$$

In particular, if we put $a = 0$ and $b = 0$, we arrive at (1.2). An interesting particular case of (2.2) involving Gegenbauer polynomial $C_n^{\nu}(x)$, may be obtained by putting $a = b = \nu - \frac{1}{2}$.

Repeated differentiation of (2.5) yields the value of the following integral:

$$\nu_{n,k}(\alpha,a,b) = \int_0^1 (1-x)^a\, x^{\alpha} [\log(1/x)]^k \; P_n^{(a,b)*}(x)\, dx.$$

References

[1] Blue, J.L., A Lgendre polynomial integral, Math. Comp. 33 (1979), 739-741.

[2] Erdelyi, A. (Ed.), Tables of Integral Transforms, Vol. II, McGraw Hill, N.Y., 1954.

[3] Gautschi, W., On the Preceding Paper 'A Legendre polynomial integral' by J.L. Blue, Math. Comp. 33 (1979), 742-743.

Mathematisches Institut
der Universitat
8700 Wurzburg
Am Hubland
West Germany

ger

dt

Nep. Math. Sc. Rep.
Vol. 5, No. 1 (1980), 5-6

Hydromagnetic Pulsating Flow Between Parallel Surfaces

Y.R. Sthapit

The present note seeks an exact solution for unsteady flow of viscous conducting fluid between two parallel surfaces with oscillating pressure gradient in time in the presence of uniform transverse magnetic field.

Consider the flow of an incompressible viscous conducting fluid between two non-conducting parallel surfaces which are placed at $y = \pm a$ and consider the pressure-gradient in the direction of x-axis to oscillate in time. The velocity will be in the direction of x-axis only and will also oscillate in time. That is the only non-zero component of velocity will be $u(y,t)$. Since the motion is affected by a transverse magnetic field of uniform strength H_0 perpendicular to the surfaces, the governing equation describing the fluid flow is

$$(1) \qquad \frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma \mu_e^2 H_o^2 u}{\rho},$$

where the notations have their usual meanings.

The boundary conditions are

$$(2) \qquad u(a,t) = u(-a,t) = 0, \quad t > 0.$$

The pressure-gradient is assumed to oscillate in time such that $\partial p/\partial x$ will be taken to be of the form

$$\frac{\partial p}{\partial x} = P_x \cos nt,$$

where P_x is a constant, which represents the magnitude of pressure-gradient oscillations.

Assume that the pressure gradient and tne velocity may be of the form

$$(3) \qquad \frac{\partial p}{\partial x} = \mathrm{Re}\left[P_x \exp(int)\right],$$

$$(4) \qquad u(y,t) = \mathrm{Re}\left[w(y) \exp(int)\right].$$

Substituting (3) and (4) in (1) we obtain the following non-homogeneous differential equation

$$(5) \qquad \frac{d^2 w}{dy^2} - (m+k^2)w = \frac{P_x}{\rho\nu},$$

where $m = \dfrac{\sigma \mu_e^2 H_o^2}{\rho\nu}$ and $k^2 = \dfrac{in}{\nu}$

The general solution of (5) is

$$(6) \qquad w(y) = A \operatorname{ch} (m+k^2)^{\frac{1}{2}} y + B \operatorname{sh} (m+k^2)^{\frac{1}{2}} y - \frac{P_x}{\rho\nu(m+k^2)} ;$$

where A and B are constants of integration. Applying the boundary conditions (2) we get

$$A = \frac{P_x}{\rho\nu(m+k^2) \operatorname{ch} (m+k^2)^{\frac{1}{2}} a}, \quad B = 0.$$

Thus the solution for w(y) is

$$w(y) = \frac{P_x}{\rho\nu(m+k^2)} \left[\frac{\operatorname{ch} (m+k^2)^{\frac{1}{2}} y}{\operatorname{ch} (m+k^2)^{\frac{1}{2}} a} - 1 \right].$$

Hence the expression for the velocity in the fluid becomes

$$u(y,t) = \operatorname{Re} \left(\frac{P_x}{\rho\nu(m+k^2)} \left[\frac{\operatorname{ch} (m+k^2)^{\frac{1}{2}} y}{\operatorname{ch} (m+k^2)^{\frac{1}{2}} a} - 1 \right] \exp(int) \right).$$

For m = 0, we get

$$u(y,t) = \operatorname{Re} \left(\frac{P_x}{\rho\nu k} \left[\frac{\operatorname{ch} ky}{\operatorname{ch} ka} - 1 \right] \exp(int) \right)$$
$$= \operatorname{Re} \left(\frac{iP_x}{n} \left[1 - \frac{\operatorname{ch}(in/\nu)y}{\operatorname{ch}(in/\nu)a} \right] \exp(int) \right)$$

which is the solution for the flow of viscous incompressible non-conducting fluid between two parallel surfaces with oscillating pressure-gradient in time ([1] , pp. 237).

References

1 Currie, I.G., Fundamental mechanics of fluids, McGraw-Hill, Inc., 1974.

Mathematics Instruction Committee
Tribhuvan University
Kirtipur Campus
Kathmandu, Nepal.

Nep. Math. Sc. Rep.
Vol. 5, No. 1 (1980), 7-14

A Class of Hypergeometric Polynomials

Geeta Bhakta Joshi

1. Introduction

A number of formulas involving differential operators were furnished by Carlitz [1] quite recently. Two of them are

$$(1.1)\qquad (DxD)^n = \sum_{s=0}^{n} \binom{n}{s} \frac{n!}{s!} x^s D^{n+s}$$

and

$$(1.2)\qquad (D(xD)^m)^n = (xD+1)^m (xD+2)^m \dots\dots\dots\dots (xD+n)^m D^n ,$$

where $\quad D = d/dx.$

Shrestha [5] using (1.2) considered the exponential differential operators

$$e^{[D(xD)^m]^p}$$

and introduced a set of polynomials

$$(1.3)\qquad H_n(x:m;p) = e^{[D(xD)^m]^p} x^n.$$

This set $H_n(x:m:p)$ is a generalisation of the Gould-Hopper polynomials [3] defined by

$$(1.4)\qquad g_n(x;p) = e^{-D^p} x^n.$$

In the present paper, we intend to study some properties of the special case $H_n(x:m:2) = g_{n,m}(x)$.

The set of polynomials of our interest is given by

$$(1.5)\qquad g_{n,m}(x) = e^{-[D(xD)^m]^2} x^n,$$

where m and n are non-negative integers,

i.e. $$g_{n,m}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k [D(xD)^m]^{2k}}{k!} x^n,$$

or,

$$(1.6) \qquad g_{n,m}(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (n!)^{m+1} x^{n-2k}}{k!\,[(n-2k)!]^{m+1}}\,.$$

2. Nature of the Generalised Hypergeometric Polynomial

Replacing x by -x in equation (1.6), we have

$$(2.1) \qquad g_{n,m}(-x) = (-1)^n g_{n,m}(x).$$

This shows that $g_{n,m}(x)$ is an even function of x for even n and an odd function of x for odd n.

If n is an even function of x for even n, it follows from relation (1.6) that

$$g_{2n,m}(-x) = \sum_{k=0}^{n} \frac{(-1)^k [(2n)!]^{m+1} x^{2n-2k}}{k!\,[(2n-2k)!]^{m+1}}$$

and consequently

$$(2.2) \qquad g_{2n,m}(0) = \frac{(-1)^k [2^{2n}\, n!\, (\tfrac{1}{2})_n]^{m+1}}{n!}$$

similarly, replacing n by 2n+1,

$$(2.3) \qquad g_{2n+1,m}(0) = 0.$$

Now differentiating the relation (1.6) w.r.t. x, we get

$$(2.4) \qquad g^1_{n,m}(x) = \sum_{k=0}^{[(n-1)/2]} \frac{(-1)^{k+1} (n!)^{m+1} (n-2k) x^{n-2k-1}}{k!\,(n-2k)!^{\,m+1}}$$

In particular,

$$g^1_{n,m}(0) = (-1)^{(n-1)/2}\, 2!\, n (n!)^m\,.$$

3. Generating Functions

We shall now obtain a generating function of the hypergeometric polynomial,

$$(3.1) \qquad g_{n,m}(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (n!)^{m+1} x^{n-2k}}{k!\,[(n-2k)!]^{m+1}}$$

Now multiplying both sides by $\frac{t^n}{(n!)^{m+1}}$ and summing from 0 to ∞, we find

$$\sum_{n=0}^{\infty} \frac{g_{n,m}(x)\; t^n}{(n!)^{m+1}} = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \frac{(-1)^k x^{n-2k} t^n}{k!\,[(n-2k)!]^{m+1}}$$

$$= \Big(\sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{k!}\Big)\Big(\sum_{n=0}^{\infty} \frac{t^n x^n}{n!\,(n!)^m}\Big)$$

i.e.

$$\exp(-t^2)\; {}_0F_m(-;1,\ldots\ldots,1;xt) = \sum_{n=0}^{\infty} \frac{g_{n,m}(x)\; t^n}{(n!)^{m+1}} \tag{3.2}$$

4. Hypergeometric Form

The generalised hypergeometric polynomial may be expressed in the simple hypergeometric form. From (1.6), we have

$$g_{n,m}(x) = x^n \sum_{k=0}^{[n/2]} \frac{(-1)^k \left[\left(-\frac{n}{2}\right)_k\right]^{m+1} \left[\left(\frac{-n+1}{2}\right)_k\right]^{m+1} (2^{2k})^{m+1}\; x^{-2k}}{k!} \tag{4.1}$$

or,

$$g_{n,m}(x) = x^n\; {}_{2m+2}F_0 \left[\begin{matrix} -\frac{n}{2},\ldots\ldots,-\frac{n}{2},\frac{-n+1}{2},\ldots\ldots\frac{-n+1}{2}; \\ -; \end{matrix} \; -\left(\frac{2^{m+1}}{x}\right)^2 \right] \tag{4.2}$$

5. Operation with $[D(xD)^m]^s$

Operating both sides of the equation (1.6) with $D(xD)^m$ once, we get

$$[D(xD)^m] g_{n,m}(x) = [D(xD)^m] \sum_{k=0}^{[n/2]} \frac{(-1)^k\; (n!)^{m+1}\; x^{n-2k}}{k!\,[(n-2k)!]^{m+1}} \tag{5.1}$$

$$= n^{m+1} \sum_{k=0}^{\left[\frac{n-2}{2}\right]} \frac{(-1)^k [(n-1)!]^{m+1} x^{n-2k-1}}{k!\,[(n-2k-1)!]^{m+1}}$$

i.e.

$$[D(xD)^m] g_{n,m}(x) = n^{m+1} g_{n-1,m}(x). \tag{5.2}$$

Repeated application of the operator s times, we have

$$(5.3)\qquad [D(xD)^m]^s\, g_{n,m}(x) = \frac{(n!)^{m+1} g_{n-s}(x)}{[(n-s)!]^{m+1}}\ .$$

6. Pure Recurrence Relation

Consider the polynomial

$$(6.1)\qquad g_{n,m}(x) = x^n\ {}_{2m+2}F_0\left[\begin{matrix} -\frac{n}{2},\ldots,-\frac{n}{2},\frac{-n+1}{2},\ldots,\frac{-n+1}{2}; \\ \quad\quad -\left(\frac{2}{x}\right)^{m+1} 2 \\ -\ ; \end{matrix}\right]$$

or,

$$(6.2)\qquad g_{n,m}(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k\ (n!)^{m+1} x^{n-2k}}{k!\ [(n-2k)!]^{m+1}}$$

To obtain a pure recurrence relation, we employ Sister Celine's technique [4]

$$(6.3)\qquad \text{Now put}\quad \lambda_n(x) = \frac{g_{n,m}(x)}{(n!)^{m+1}}\ .$$

Then

$$(6.4)\qquad \lambda_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{n-2k}}{k!\ [(n-2k)!]^{m+1}} = \sum_{k=0}^{\infty} \epsilon(k,n)$$

Thus

$$(6.5)\qquad x\,\lambda_{n-1}(x) = \sum_{k=0}^{\infty} (n-2k)^{m+1}\,\epsilon(k,n)$$

$$(6.6)\qquad \lambda_{n-2}(x) = \sum_{k=0}^{\infty} -k\,\epsilon(k,n)$$

$$(6.7)\qquad x^2\lambda_{n-2}(x) = \sum_{k=0}^{\infty} (n-2k)^{m+1}(n-2k-1)^{m+1}\,\epsilon(k,n)$$

and

$$(6.8)\qquad \lambda_{n-3}(x) = \sum_{k=0}^{\infty} -k(n-2k)^{m+1}\,\epsilon(k,n)$$

We shall investigate whether the series (6.4) to (6.8) do have some linear relation of the form

$$(6.9)\qquad \lambda_n(x) + Ax\,\lambda_{n-1}(x) + (B + Cx^2)\,\lambda_{n-2}(x) + Dx\,\lambda_{n-3}(x) = 0$$

in which the constants A, B, C, D are determined in the following way

$$(6.10)\qquad \sum_{k=0}^{\infty} \epsilon(k,n) + A\sum_{k=0}^{\infty} (n-2k)^{m+1}\epsilon(k,n) + B\sum_{k=0}^{\infty} -k\,\epsilon(k,n) +$$

$$\sum_{k=0}^{\infty} (n-2k)^{m+1}(n-2k-1)^{m+1}\epsilon(k,n) + D\sum_{k=0}^{\infty} -k(n-2k)^{m+1}\epsilon(k,n) = 0$$

This gives

$$(6.11)\qquad 1 + A(n-2k)^{m+1} - Bk + C(n-2k)^{m+1}(n-2k-1)^{m+1} - Dk(n-2k)^{m+1} = 0$$

The identity (6.11) yields

$$A = -1/n^{m+1}, \qquad B = 2/n,$$

$$C = 0 \quad \& \quad D = 2/n^{m+1}.$$

Hence the polynomial $\lambda_n(x)$ satisfies the relation

$$(6.12)\qquad n^{m+1}\lambda_n(x) - x\,\lambda_{n-1}(x) + 2n^m\lambda_{n-2}(x) + 2x\,\lambda_{n-3}(x) = 0\,.$$

But from (6.3) $\lambda_n(x) = \dfrac{g_{n,m}(x)}{(n!)^{m+1}}$,

Hence the polynomials $g_{n,m}(x)$ satisfy the recurrence relation.

$$(6.13)\qquad n^{m+1}g_{n,m}(x) - x n^{m+1}g_{n-1,m}(x) + 2n^{2m+1}(n-1)^{m+1}g_{n-2,m}(x)$$

$$+\, 2n^{m+1}(n-1)^{m+1}(n-2)^{m+1}x\,g_{n-3,m}(x) = 0.$$

7. Integral Involving $g_{n,m}(x)$

We start evaluating the integral

$$(7.1) \qquad \frac{2}{(n!)^{m+1}\sqrt{\pi}} \int_0^\infty e^{-t^2} t^n g_{n,m}(x)\, dt = I \text{ (say)}$$

put $\quad t^2 = sy, \quad \therefore \quad 2t\, dt = s\, dy$

$$\& \; x = 1/ \quad , \quad dt = s\, dy/2\sqrt{sy}$$

The equation (7.1) can be written as the Laplace transform

$$I = \frac{1}{(n!)^{m+1}\sqrt{\pi}} \int_0^\infty e^{-sy}(sy)^{n/2}\, g_{n,m}(\sqrt{y})\, \frac{s}{\sqrt{sy}}\, dy$$

Substituting the value of $\frac{g_{n,m}(\sqrt{y})}{(n!)^{m+1}}$, from the relation (1.6) and changing the order of integration which is obviously valid, we have

$$I = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{[n/2]} \frac{(-1)^k y^{\frac{1}{2}n-k}}{k!\,[(n-2k)!]^{m+1}} \int_0^\infty e^{-sy}(sy)^{n/2}\, \frac{s}{\sqrt{sy}}\, dy$$

$$= \frac{1}{\sqrt{\pi}} \sum_{k=0}^{[n/2]} \frac{(-1)^k s^{\frac{n}{2}+\frac{1}{2}}}{k!\,[(n-2k)!]^{m+1}} \int_0^\infty e^{-sy} y^{n-k-\frac{1}{2}}\, dy .$$

By the relation of gamma function [2],

$$(7.2) \qquad \int_0^\infty e^{-sy} y^{a-1}\, dy = \Gamma(a)\, s^{-a}, \quad \text{if } \begin{array}{l} \mathrm{Re}(s) > 0 \\ \mathrm{Re}(a) > 0 \end{array}$$

Therefore, for $\mathrm{Re}(s) > 0$, $\mathrm{Re}(n-k-\frac{1}{2}) > 0$, we get

$$I = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{[n/2]} \frac{(-1)^k s^{\frac{n}{2}+\frac{1}{2}}}{k!\,[(n-2k)!]^{m+1}} \;\Gamma(n-k+\tfrac{1}{2})\, s^{k-n-\frac{1}{2}}$$

$$= \sum_{k=0}^{[n/2]} \frac{(-1)^k \, (\frac{1}{2})_{n-k}}{k! \, [(n-2k)!]^{m+1}} \; x^{n-2k} \qquad (\because x = \frac{1}{\sqrt{s}})$$

This gives a new set of polynomials defined by

$$(7.3) \qquad P_{n,m}(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k \, (\frac{1}{2})_{n-k}}{k! \, [(n-2k)!]^{m+1}} \cdot x^{n-2k}$$

An interesting result of this set is

$$(7.4) \qquad P_{n,m}(x) = \frac{2}{(n!)^{m+1}\sqrt{\pi}} \int_0^{\infty} e^{-t^2} \, t^n g_{n,m} \, dt,$$

where the integral on the right is the Laplace transformation of $g_{n,m}(x)$

This may be written in the form

$$(7.5) \qquad P_{n,m}(x) = \frac{(\frac{1}{2})_n \, x^n}{(n!)^{m+1}} \sum_{k=0}^{[n/2]} \frac{[(-n)_{2k}]^{m+1} \, x^{-2k}}{k! \, (\frac{1}{2}+n)_k}$$

This polynomial has an interesting hypergeometric representation, possesses generating function, and a three term recurrence relation. We now quote them without proof:

(a) Hypergeometric form:

$$(7.6) \qquad P_{n,m}(x) = \frac{(\frac{1}{2})_n \, x^n}{(n!)^{m+1}} \; {}_{2m+2}F_1 \left[\begin{matrix} -\frac{n}{2}, \dots, -\frac{n}{2}, \frac{-n+1}{2}, \dots, \frac{-n+1}{2}; \\ \frac{1}{2}+n\,; \end{matrix} \; \left(\frac{2^{m+1}}{x}\right)^2 \right]$$

(b) Generating function:

$$(7.7) \qquad \sum_{n=0}^{\infty} P_{n,m}(x) \, t^n = (1+t^2)^{-\frac{1}{2}} \; {}_mF_1 \left[\begin{matrix} \frac{1}{2}; \\ 1, \dots 1; \end{matrix} \; \frac{xt}{1+t} \right]$$

(c) Pure recurrence relation:

$$(7.8) \qquad P_{n,m}(x) - \frac{(2n-1)}{2n^{m+1}} \, x \; P_{n-1,m}(x) + \frac{(n-1)}{n} \; P_{n-2,m}(x) = 0$$

known result for Legendre Polynomials may be obtained from the above results by putting $m = 0$ and replacing x by $2x$.

Thanks are due to my teacher Dr. Ram Man Shrestha for his kind help in the preparation of this paper.

References

[1] Carlitz, L., A Theorem on differential operators; Amer. Monthly, 83, No. 5 (1975), 351-354.

[2] Erdelyi, A., Higher Transcendental functions, McGraw Hill & Co. (1953).

[3] Gould, H.W. and Hopper, A.J., Operational formulas connected with two generalisations of Hermite polynomials, Duke, Math. J. 29 (1962), 51-53.

[4] Rainville, R.D., Special functions, Mcmillan and Co., N.Y. (1960).

[5] Shrestha, R.M., On polynomials generalised by exponential differential operator, Nep. Month. Sc. Report, 2, No. 2 (1977), 6-9.

Institute of Engineering
Pulchowk Campus,
Pulchowk, Kathmandu.

Nepa. Math. Sc. Rep.
Vol. 5, No. 1 (1980), 15-22

Horizontal Lift of F_λ - Structure in Cotangent Bundle

S.C. Gupta & A.K. Agrawal*

1. Introduction

Upadhyaya and Gupta [2] defined F_λ -structure. Awasthi and Gupta [1] studied certain theorems on complete lift of F_λ -structure in the tangent bundle. In this paper, we have considered horizontal lift of F_λ -structure in the cotangent bundle and studied some of its properties.

Let M be an n-dimensional differentiable manifold of class C^∞. If there exists a (1,1) tensor field F on M of class C^∞ satisfying [2] such that

$$(1.1) \qquad F^3 = \lambda^2 F$$

λ being a non zero complex number,

let us take

$$(1.2) \qquad s = (F/\lambda)^2 \ , \ t = I - (F/\lambda)^2$$

where I is an identity operator. Then we have [2]

$$(1.3) \qquad s + t = I \ , \ st = ts = 0,$$

$$s^2 = s \text{ and } t^2 = t$$

Thus we have two distributions S and T in M given by (1.3), corresponding to the projection operators s and t respectively. When the rank of F is constant and equal to r everywhere, then S is r-dimensional and T is (n-r) dimensional.

Such a structure is called an F - structure of rank r 2 .

We have the relations

$$(1.4) \qquad Fs = sF = F \ , \ Ft = tF = 0,$$

$$F^2 s = \lambda^2 s \text{ and } F^2 t = tF^2 = 0$$

where F acts on S as a π- structure opeator and on T as a null operator. We can prove that F_λ - structure of maximal rank is a π- structure.

---

*At present working on an assignment in Lucknow University under Faculty Improvement Programme.

2. Horizontal lift of F_λ -structure in cotangent bundle

Let J^r_s (M) be the set of tensor fields of class C^∞ and of the type (r,s) in M and let T(M) be the cotangent bundle over M. Let $F \in J^1_1$ (M) and F have local components F^h_i in a coordinate neighbourhood U of M. Then the horizontal lift F^H of F will have the components of the form $\tilde{F}^A_B$ defined in [3] with respect to induced coordinates in T*(M). Here Γ^h_{ji} are the components of ∇ in M and $\Gamma_{ji} = p_a \Gamma^a_{ji}$

$$\tilde{F}^A_B : \begin{pmatrix} F^h_i & 0 \\ -\Gamma_{ia}F^a_h + \Gamma_{ha}F^a_i & F^i_h \end{pmatrix}$$

Theorem (2.1)

Let $F \in J^1_1$ (M), then the horizontal lift F^H of F is an F_λ -structure in T*(M) iff so is F. Thus F is of the rank r iff F^H is of rank 2r.

Proof

Since $F^H G^H + G^H F^H = (FG + GF)^H$ for F = G, $(F^H)^2 = (F^2)^H$

(2.2) Similarly $(F^H)^3 = (F^3)^H = \lambda^2 F^H$

which shows that F^H has a structure in T*(M) similar to that of F in M.

Let F be an F_λ -structure of rank r in M, then the horizontal lift s^H of s and t^H of t are complementary projection tensors in T*(M). Thus two complementary distributions S^H and T^H exist in T*(M) determined by s^H and t^H respectively.

3. Integrability conditions of F_λ -structure in cotangent bundle

Let F be an F_λ -structure of rank r in M, then in view of (1.2), the Nijenhuis tensor N of F is given by [3]

(3.1) $N(X,Y) = [FX,FY] - F[FX,Y] - F[X,FY] + \lambda^2 s[X,Y]$

For any $X,Y \in J^1_0(M)$ and $F \in J^1_1(M)$, we have [3]

(3.2) $F^H X^H = (FX)^H$; $[X^H, Y^H] = [X,Y]^H + \gamma R(X,Y)$

and $(X+Y)^H = X^H + Y^H$

From (1.4) and (3.2), we obtain

(3.3) $F^H t^H = (Ft)^H = 0$

Theorem (3.1)

The horizontal lift T^H of a distribution T in M is integrable iff T is so in a locally flat manifold M.

Proof

Distribution T is integrable in M iff [3]

$$(3.4)\qquad s\,[tX, tY] = 0 \quad \text{for } X, Y \in J_0^1(M)$$

Taking horizontal lift of both the sides, we get

$$(3.5)\ (a)\quad s^H\,[t^H X^H,\ t^H Y^H] - \gamma R(tX, tY) = 0$$

where $s^H = I - t^H$ is the projection tensor complementary to t^H.

Since $R(X,Y) = 0$ if M is a locally flat manifold, then from (3.5)(a), we get

$$(3.5)\ (b)\quad s^H\,[t^H X^H,\ t^H Y^H] = 0.$$

(3.4) and (3.5)(b) are equivalent equations. Hence, the result follows.

Theorem (3.2)

For any $X, Y \in J_0^1(M)$, let the distribution T be integrable in M i.e., $N(tX, tY) = 0$, then for M being locally flat, the distribution T^H is integrable in $T^*(M)$ if $N^H(t^H X^H, t^H Y^H) = 0$.

Proof

Let N^H be the Nijenhuis tensor of F^H in $T^*(M)$ of F in M. Then, we have

$$N^H(X^H, Y^H) = [F^H X^H, F^H Y^H] - F^H[F^H X^H, Y^H] - F^H[X^H, F^H Y^H] + (F^H)^2[X^H, Y^H]$$

or

$$(3.6)\quad N^H(X^H, Y^H) = [FX, FY]^H - F^H[FX, Y]^H - F^H[X, FY]^H + (F^H)^2[X, Y]^H$$
$$+ \gamma R(FX, FY) - F^H \gamma R(FX, Y) - F^H \gamma R(X, FY)$$
$$+ (F^H)^2 \gamma R(X, Y)$$

On putting tX and tY for X and Y respectively in (3.6) and using the fact that M is loclly flat, we obtain

$$(3.7)\qquad N^H(t^H X^H,\ t^H Y^H) = [FtX,\ FtY]^H - F^H[FtX,\ tY]^H - F^H[tX,\ FtY]^H + (F^H)^2\ [tX, tY]^H$$

which in view of (1.4) and (3.5) (b) yields

$$N^H(t^H X^H,\ t^H Y^H) = 0$$

This proves the theorem.

Theorem (3.3)

For any $X, Y \in J_o^1(M)$, let the distribution S be integrable in M i.e., $tN(X,Y) = 0$, then for M being locally flat, the distribution S^H is integrable in $T^*(M)$ iff $t^H N^H(X^H, Y^H) = 0$.

Proof

In consequence of (3.2), (3.3) and (3.6), we obtain

$$t^H N^H(X^H, Y^H) = t^H[F^H X^H,\ F^H Y^H]$$
$$= t^H[FX, FY]^H + t^H \gamma R(FX, FY)$$
$$= \{t[FX, FY]\}^H + t^H \gamma R(FX, FY)$$

Using the fact that M is locally flat and the distribution S is integrable in M, we obtain

$$t^H\ N^H(X^H, Y^H) = 0$$

This proves the theorem.

When the distribution S is integrable, then F operates on each integral manifold of S as a π-structure operator F such as $F_* X_1 = FX_1$, where X_1 is an arbitrary vector field tangent to the integral manifold of S.

When both the distribution S and the π-structure F_* induced from F on each integral manifold of S are integrable, the F_λ-structure is said to be partially integrable.

Theorem (3.4)

For any $X, Y \in J_o^1(M)$, let the F_λ-structure be partially integrable in M i.e. $N(sX, sY) = 0$, then the F_λ-structure is partially integrable in $T^*(M)$ iff $N^H(s^H X^H,\ s^H Y^H) = 0$, M being locally flat.

Proof

From (3.6), we have

$$N^H(s^H X^H, s^H Y^H) = [FsX, FsY]^H - F^H[FsX, Y]^H - F^H[X, Fsy]^H$$
$$+ [F^H]^2[sX, sY]^H + \gamma R(FsX, FsY) - F^H \gamma R(FsX, sY)$$
$$- F^H \gamma R(sX, FsY) + (F^H)^2 \gamma R(sX, sY)$$

Due to M being locally flat, the above equation reduces to
$N^H(s^H X^H, s^H Y^H) = FsX, FsY - F\, FsX, Y - F\, X, FsY + F^2\, sX, sY$

Since the F_λ -structure is partially integrable, we obtain

$$N^H(s^H X^H, s^H Y^H) = \{N(sX, sY)\}^H = 0$$

which proves the theorem.

Theorem (3.5)

For any $X, Y \in J_o^1(M)$, let the F_λ -structure be partially integrable in M i.e. $N(sX, sY) = 0$, then the F_λ -structure is integrable in $T^*(M)$ iff $N^H(X^H, Y^H) = 0$, M being locally flat.

Proof

From (3.1), we have

$$N^H(X^H, Y^H) = [F^H X^H, F^H Y^H] - F^H[F^H X^H, Y^H] - F^H[X^H, F^H Y^H]$$
$$+ (\lambda^2 s)^H [X^H, Y^H]$$

This reduces to

$$N^H(X^H, Y^H) = \{N(X, Y)\}^H + \gamma R(FX, FY) - F^H \gamma R(FX, Y) - F^H \gamma R(X, FY)$$
$$+ (F^2)^H \gamma R(X, Y)$$

In a locally flat M, we get

$$N^H(X^H, Y^H) = \{N(X, Y)\}^H = 0$$

since the F_λ - structure is partially integrable in M.

Theorem (3.6)

Let F be an F_λ -structure in M, N the Nijenhuis tensor of F and T the torsion of F and F^2, then

$$F^H + \gamma \frac{FN - 2T}{\lambda^2 + 1} \text{ is an } F_\lambda \text{ -structure in } T^*(M)$$

where $\quad \Gamma = \dfrac{FN - 2T}{\lambda^2 + 1}$

Proof

Since $F^H X^H = F^H X^H$

AND $\gamma P\, X^H = \gamma P_X$

on adding, we get

$$(F^H + \gamma P)X^H = (F^H X^H + \gamma P_X)$$

On multiplication with $(F^H + \gamma P)$, we get

$$(F^H + \gamma P)^2\, X^H = F^H\, (F^H X^H + \gamma P_X) + \gamma P (F^H X^H + \gamma P_X)$$

Using $(\gamma S)F^H = \gamma(SF) = (\gamma S)F^C$

and $(\gamma S)\,(\gamma T) = 0$, we obtain

$$(F^H + \gamma P)^2 X^H = (F^H)^2 + \gamma(P_X F + \gamma P_{FX})$$

Similarly

$$(F^H + \gamma P)^3 X^H = (F^H)^3\, X^H + \gamma(P_X F^2 + \gamma P_{FX} F + P_{F^2 X})$$

$$(3.8) \qquad (F^H + \gamma P)^3 X^H = \lambda^2 (F^H + \gamma P) X^H$$

provided

$$(3.9) \qquad \gamma(P_X F^2 + \gamma P_{FX} F + P_{F^2 X}) = \lambda^2 (\gamma P) X^H$$

(3.8) shows that $(F^H + \gamma P)$ admits F_λ -structure.

For any $X \in J_0^1(M)$, (3.9) is equivalent to

$$P(X, F^2 Y) + P(F^2 X, Y) + P(FX, FY) = \lambda^2 \gamma P_X = \lambda^2 P(X,Y)$$

or $\quad P(X,Y) + P(X, F^2 Y) + P(F^2 X, Y) + P(FX, FY) = (\lambda^2 + 1)\, P(X,Y)$

In view of equation (8.10) of Chapter VIII of [3] , we get

$$(FN - 2T)\,(X,Y) = (\lambda^2 + 1)\, P(X,Y)$$

which gives that P is equivalent to $\dfrac{FN-2T}{\lambda^2 + 1}$. hence, the theorem follows:

Corollary 1

If $X,Y,Z \in M$ and $X^H, Y^H, Z^H \in T^*(M)$

we have

$$(3.10)\quad \{[\gamma R(X,Y)Z^H] + [\gamma R(Y,Z),X^H] + [\gamma R(Z,X),Y^H]\} = -\{\gamma R([X,Y],Z) + \gamma R([Y,Z],X) + \gamma R([Z,X],Y)\}$$

Proof

Since $[X^H,Y^H] = [X,Y]^H + \gamma R(X,Y)$

we have

$$\{[X^H,Y^H],Z^H\} = \{[X,Y],Z\}^H + [\gamma R(X,Y),Z^H] + \gamma R([X,Y],Z)$$

Taking their cyclic sum on X,Y,Z and using the fact that

$$\Big[[X,Y],Z\Big] + \Big[[Y,Z],X\Big] + \Big[[Z,X],Y\Big] = 0$$

we get

$$(3.11)\quad \Big[[X^H,Y^H],Z^H\Big] + \Big[[Y^H,Z^H],X^H\Big] + \Big[[Z^H,X^H],Y^H\Big]$$

$$= [\gamma R(X,Y),Z^H] + [\gamma R(Y,Z),X^H] + [\gamma R(Z,X),Y^H]$$

$$+ [\gamma R([X,Y],Z)] + \gamma R([Y,Z],X) + \gamma R([Z,X],Y)$$

If the horizontal lift to X,Y,Z in contangent bundle follows Jacobi identity, we obtain (3.10). hence, the result follows.

The authors are grateful to Sri M.K. Jain, Principal for his encouragement.

References

[1] Awasthi, C.P. and Gupta, V.C., Complete lift of F -structure in tangent bundles - Comptes redus del'Academie bulgare des Sciences Tome 30, N°3, (1977). 335-338.

[2] Upadhyaya, M.D. and Gupta, V.C., Integrability Conditions of a Structure F Satisfying $F^3-F^2 = 0$ $(F\neq 0, \neq 0)$. Under publication.

[3] Yano, K and Ishihava, S - Tangent and Cotangent bundles - Marcel Dekker, New York, 1973.

Department of Mathematics
Sahu Jain College
Najibabad (U.P.), India

follows:

Rep. Math. Sc. Rep.
Vol. 5, No. 1 (1980), 23-30

Certain Binomial Identities and A Differential Operator

D.P. Shukla

1. Introduction

Making use of the operator $T_k = x\,(k+xD)$ where k is a constant earlier Mittal [4] obtained the generating relations

$$(1.1) \qquad \sum_{n=0}^{\infty} \frac{t^n}{n!}\, T^n_{a+1}\, \{x^b f(x)\} = x^b (1-xt)^{-a-b-1} f\left[\frac{x}{1-xt}\right]$$

$$(1.2) \qquad \sum_{n=0}^{\infty} \frac{t^n}{n!}\, T^n_{a-n}\, \{x^b f(x)\} = [x(1+xt)]^b\,(1+xt)^{a-1} f\,[x(1+xt)]$$

where $f(x)$ admits a formal power series in x.

Mittal [5,6] also proved that

$$(1.3) \qquad \sum_{n=0}^{\infty} \frac{t^n}{n!}\, T^n_{(m-1)n+a+1} \left\{x^b f(x)\right\} = \frac{x^b (1+\vartheta)^{a+b+1}}{1-(m-1)} f\,[x(1+\vartheta)]$$

where $\vartheta = xt(1+\vartheta)^m$, m being constant and

$$(1.4) \qquad \sum_{n=0}^{\infty} \frac{t^n}{n!}\, T^{n-1}_{a+mn+1} \left\{af(x)+xf'(x)\right\} = \frac{(1+\vartheta)^a}{x} f\,[x(1+\vartheta)]$$

where $\vartheta = at(1+\vartheta)^{m+1}$, a and m being constant and $f(x)$ admits a formal power series in x.

Besides these results Mittal [7] proved that

$$(1.5) \qquad \sum_{n=0}^{\infty} \frac{t^n}{n!}\, {}_aT^n_{mn+k} \{f(x)\} = \frac{(1+\vartheta)^{k/a}}{1-\frac{m}{a}\vartheta} f\,\left[x(1+\vartheta)^{1/a}\right]$$

where $\vartheta = ax^a t(1+\vartheta)^{\frac{m+a}{a}}$, a and m being constant.

Putting m=0 in (1.5) we immediately get

$$(1.6) \qquad \sum_{n=0}^{\infty} \frac{t^n}{n!}\, {}_aT^n_k \{f(x)\} = (1-ax^a t)^{-k/a} f\left[x(1-ax^a t)^{-1/a}\right]$$

where ${}_aT_k = x^a(k+xD)$, $D \equiv \frac{d}{dx}$, a and k are constants and

$$ {}_aT_k^n \{x^b\} = x^{b+an} \, a^n \left(\frac{b+k}{a}\right)_n $$

Putting a=1 above we get corresponding results for T_k operator and

(1.7) $$\sum_{n=o}^{\infty} \frac{t^n}{(2n)!} T_b^{2n} \{f(x)\} = \tfrac{1}{2}(1+(x^2t)^{\frac{1}{2}})^{-b} f\left[\frac{x}{1+(x^2t)^{1/2}}\right] + \tfrac{1}{2}(1-(x^2t)^{\frac{1}{2}})^{-b} f\left[\frac{x}{1-(x^2t)^{1/2}}\right].$$

Gould [1,2,3] in a series of papers considered convolution identities and Mittal [8] recently generalized the Jensen's formula and Engelberg identities. We [10] also generalized Gould [1] convolution identities.

In this paper we propose to derive some new identities by direct applications of the operator formula (1.1) to (1.7). Our identity (2.8) is a generalization of Engelberg identity and Mittal 8,5,3 identity.

2. Certain Identities

We now proceed to derive following identities

(2.1) $$\sum_{k=o}^{n} \binom{b/a+k-1}{k}\binom{c/a+n-k-1}{n-k} = \binom{(b+c)/a+n-1}{n}$$

(2.2) $$\sum_{k=o}^{n} \frac{c}{c-n+k} \binom{c-1}{n-k}\binom{-b/a}{k} = \binom{-b/a+c}{n}$$

(2.3) $$\sum_{k=o}^{n} \binom{b/a+n-k-1}{n-k}\binom{c+2k-1}{2k} = \tfrac{1}{2} \sum_{k=o}^{2n} \binom{-c}{k}\binom{b/a+n-k/2-1}{n-k/2} + \tfrac{1}{2} \sum_{k=o}^{2n} \binom{c+k-1}{k}\binom{b/a+n-k/2-1}{n-k/2}$$

(2.4) $$\sum_{n,k=o}^{\infty} \binom{b+d-1}{n}\binom{(c+b)/a-1+(m/a+1)k}{k} a^k \, t^{n+k} = \frac{(1+t)^{b+d-1}(1+\mathcal{R})^{(b+c)/a}}{1-\frac{m}{a}\mathcal{R}}$$

where $\mathcal{R} = at(1+\mathcal{R})^{m/a+1}$

$$(2.5)\qquad \sum_{n,k=0}^{\infty} \binom{c+b-1}{n}\binom{a+b-(m-1)k}{k} t^{n+k} = \frac{(1+t)^{b+c-1}(1+\vartheta)^{a+b+1}}{1+m\vartheta}$$

where $\vartheta = t(1+\vartheta)^{-m+1}$

$$(2.6)\qquad \sum_{k=0}^{n} \binom{a+b+k-1}{k}\binom{b-c+n-k-1}{n-k} = \binom{a+2b-c+n-1}{n}$$

$$(2.7)\qquad \sum_{k=0}^{n} \binom{b+c-1}{n-k}\binom{a+b}{k} = (-1)^n \binom{-a-2b-c+n}{n}$$

$$(2.8)\qquad \sum_{k=0}^{n} \frac{a}{a+bk} \binom{a-1+(b+1)k}{k}\binom{b+\alpha-1+(b+1)n-k}{n-k}$$

$$= \frac{n+1}{a+\alpha+b+(b+1)n} \binom{a+\alpha-1+(b+1)(n+1)}{n+1}$$

$$(2.9)\qquad \sum_{k=0}^{n} \binom{a-n+2k-1}{k}\binom{b+\quad +2n-2k}{n-k} - \binom{b-n+2k-k}{k-1}\binom{a+\alpha+2n-2k}{n-k}$$

$$= \binom{a+b+\alpha+n}{n}$$

<u>Proof</u>. We consider

$$(2.10)\qquad \sum_{k=0}^{\infty} \frac{t^k}{k!}\; {}_aT_b^k \{f(x)\} \sum_{n=0}^{\infty} \frac{t^n}{n!}\; {}_aT_c^n \{g(x)\}$$

Putting $f(x) = g(x) = 1$ in (2.10) and using (1.6) we have

$$(2.11)\qquad \sum_{k=0} \frac{t^k}{k!}\; {}_aT_b^k \{1\} \sum_{n=0} \frac{t^n}{n!}\; {}_aT_c^n \{1\} = (1-ax^a t)^{-(b+c)/a}$$

Equating the coefficients of $a^n t^n x^{an}$ on both sides of (2.11) we get (2.1).

Now consider

$$(2.12)\qquad \sum_{k=0}^{\infty} \frac{t^k}{k!} (-1)^k x^{-ak} a^{-k}\; {}_aT_b^k \{f(x)\} \times$$

$$\times \sum_{n=o}^{\infty} x.\frac{t^n}{n!} \frac{x^{-n}}{(1+t)^{(m+1)n}} T^{n-1}_{c+mn+1} \{ cg(x)+xg'(x)\}$$

Putting $f(x) = g(x) = 1$, $m=-1$ in (2.12) and using (1.6) and (1.4), we get

$$(2.13) \quad \sum_{k=o}^{\infty} (-1)^k \frac{t^n}{k!} x^{-ak} a^{-k} {}_aT_b \{1\} \sum_{n=o}^{\infty} \frac{t^n}{n!} x^{-n} T^{n-1}_{c-n+1} \{c\}$$

$$= (1+t)^{-(b/a-c)}$$

Equating the coefficients of t^n on both sides of (2.13) after little simplification we get (2.2).

Next consider

$$(2.14) \quad \underset{k=o}{} x^{2n} \frac{t^n}{k!} x^{-an} a^{-n} {}_aT^n_b \; f(x) \quad \underset{n=o}{} \frac{t^k}{(2k)!} T^{2k}_c \; g(x)$$

Putting $f(x) = g(x) = 1$ in (2.14) and using (1.6) and (1.7), we get

$$(2.15) \quad \underset{k=o}{} x^{2n} \frac{t^n}{k!} x^{-an} a^{-n} {}_aT^n_b \; 1 \quad \underset{n=o}{} \frac{t^k}{(2k)!} T^{2k}_c \; 1$$

$$= \tfrac{1}{2}(1-x^2t)^{-b/a} \left[(1+(x^2t)^{\frac{1}{2}})^{-c} + (1-(x^2t)^{\frac{1}{2}})^{-c} \right]$$

$$= \tfrac{1}{2} \sum_{n,k=o}^{\infty} \frac{(b/a)_n}{n!\,k!} (c)_k \, (x^2t)^{n+k/2}$$

$$= \tfrac{1}{2} \sum_{n,k=o}^{\infty} \frac{(b/a)_n (c)_k (x^2t)^{n+k/2} (-1)^k}{n!\,k!}$$

Equating the coefficients of $x^{2n}t^n$ in (2.15) after little simplification we get (2.3).

Consider the repeated operation

$$(2.16) \quad \sum_{k=o}^{\infty} \frac{x^{-ak}t^k}{k!} {}_aT^k_{mk+c} \left\{ \sum_{n=o}^{\infty} \frac{t^n}{n!} x^{-n} T^n_{d-n} \{x^b\} \right\}$$

using (1.5) and in (2.16) we get

$$(2.17) \quad \sum_{k=o}^{\infty} \frac{x^{-ak} t^{k}}{k!} \, {}_{a}T^{k}_{mk+c} \sum_{n=o}^{\infty} \frac{t^{n}}{n!} \, x^{-n} \, T^{n}_{d-n} \left\{ x^{b} \right\}$$

$$= (1+t)^{b+d-1} \sum_{k=o}^{\infty} \frac{x^{-ak} t^{k}}{k!} \, {}_{a}T_{mk+c} \left\{ x^{b} \right\}$$

$$= \frac{(1+t)^{b+d-1} (1+\vartheta)^{c/a} \, x^{b} (1+\vartheta)^{b/a}}{1-(m/a)}$$

where $\vartheta = at \, (1+\vartheta)^{m/a+1}$

(2.17) reduces to (2.4) after a little simplification. Again consider the repeated operation

$$(2.18) \quad \sum_{k=o}^{\infty} \frac{t^{k}}{k!} \, x^{-b-k} \, T^{k}_{a-mk+1} \left\{ \sum_{n=o}^{\infty} \frac{t^{n}}{n!} \, x^{-n} \, T^{n}_{c-n} \left\{ x^{b} \right\} \right\}$$

Using (1.2) and (1.4) in (2.18), we get

$$\sum_{k=o}^{\infty} \frac{t^{k}}{k!} \, x^{-b-k} \, T^{k}_{a-mk+1} \left\{ \sum_{n=o}^{\infty} \frac{t^{n} x^{-n}}{n!} \, T^{n}_{c-n} \left\{ x^{b} \right\} \right\}$$

$$(2.19) \qquad = (1+t)^{b+c-1} \, \frac{(1+\vartheta)^{a+b+1}}{1+m\vartheta}$$

where $\vartheta = t(1+\vartheta)^{-m+1}$,

which after simplification reduces to (2.5).

Consider the repeated operation

$$(2.20) \quad \sum_{k=o}^{\infty} \frac{t^{k}}{k!} \, x^{-b-k} \, T^{k}_{a} \left\{ \sum_{n=o}^{\infty} \frac{t^{n}}{n!} \, x^{-n} \, T^{n}_{-c} \left\{ x^{b} \right\} \right\}$$

using (1.1) and the property $T^{n}_{c} \left\{ x^{b} \right\} = x^{b} \, T_{c+b} \left\{ 1 \right\}$ of the operator in (2.20), we get

$$(2.21) \quad \sum_{k=o}^{\infty} \frac{t^{k} x^{-b-k}}{k!} \, T^{k}_{a} \left\{ \sum_{n=o}^{\infty} \frac{t^{n} x^{-n}}{n!} \, T^{n}_{-c} \left\{ x^{b} \right\} \right\} = (1-t)^{c-a-2b}$$

Equating the coefficients of t^{n} in (2.21) we easily get (2.6).

Now consider the repeated operation

$$(2.22)\quad \sum_{k=o}^{\infty} (-1)^k \frac{t^k}{k!} x^{-b-k} T^k_{a-k+1} \left\{ \sum_{n=o}^{\infty} (-1)^n \frac{t^n x^{-n}}{n!} T^n_{c-n} \{x^b\} \right\}$$

using (1.2) and the property of the operator used in (2.21), we get

$$(2.23)\quad \sum_{k=o}^{\infty} (-1)^k \frac{t^k}{k!} x^{-b-k} T^k_{a-k+1} \left\{ \sum_{n=o}^{\infty} (-1)^n \frac{t^n}{n!} x^{-n} T^n_{c-n} \{x^b\} \right\}$$

$$= (1-t)^{a+2b+c-1}$$

Equating the coefficients of t^n in (2.23) we easily get (2.7).

Considering

$$(2.24)\quad \sum_{n=o}^{\infty} \frac{t^n}{n!} T^n_{b+bn} \{f(x)\} \sum_{k=o}^{\infty} \frac{t^k}{k!} x\, T^{k-1}_{a+1+mk} \{ag(x)+ag'(x)\}$$

Putting $g(x) = 1$, $f(x) = x^{\alpha}$ in (2.24) and using (1.3), (1.4) and after little adjustment, we get

$$(2.25)\quad \sum_{n=o}^{\infty} \frac{t^n}{n!} T^n_{b+bn} \{x^{\alpha}\} \sum_{k=o}^{\infty} \frac{t^k}{k!} x\, T^{k-1}_{a+1+mk} \{a1^{\alpha} + x\delta^{\alpha}\}$$

$$= \frac{(1+\vartheta)^{a+b+}}{1-b\vartheta} x^{\alpha}$$

Equating the coefficients of $t^n x^n$ in (2.25) we easily get (2.8) which is a generalization of Mittal 8,5

Following the method of Mittal 7 we get the operator formula

$$(2.26)\quad \sum_{k=o}^{\infty} \frac{t^k}{k!} x^{-k} T^k_a \left\{ \sum_{n=o}^{\infty} \frac{t^n}{n!} x^{-n} T^n_{b+n+1} \{f(x)\} \right\}$$

$$- t \sum_{k=o}^{\infty} \frac{t^k}{k!} x^{-k} T^k_{b+1} \left\{ \sum_{n=o}^{\infty} \frac{t^n}{n!} x^{-n} T^n_{a+n+1} \{f(x)\} \right\}$$

$$= \sum_{n=o}^{\infty} \frac{t^n}{n!} x^{-n} T_{a+b+1} \{f(x)\} .$$

Putting $f(x) = x^{\alpha}$ in (2.26) and equating the coefficients of t^n on both sides of (2.26), we get after little simplification (2.9), which for = 0 reduces to Mittal 8,5,3 .

Acknowledgement

I am thankful to Dr. R.P. Agrawal for valuable and helpful discussions during the preparation of the paper.

References

[1] H.W., Gould, A series transformation of finding convolution Identities. Duke Math. J., 28, No. 2, June (1961), 193-202.

[2] ___ Generalization of a theorem of Jensen concerning convolution. Duke Math. J., 27, (1960), 71-76.

[3] ___ A new series transformation with Application to Bessel, Legendre and Tchebycheff polynomials. Duke Math. J., 31, (1964), 325-334.

[4] H.B., Mittal, Some Generating Function, Rev. Fac. Ci. Univ. Lisboa, 13, (1970), 43-45.

[5] ___ Polynomials defined by generating relations, Trans. Amer. Math. Soc., 168, (1972), 73-84.

[6] ___ Unusual generating relations for polynomial sets, J. fur die reine und angewandte Mathematik, Band 271 Seite 122, (1974).

[7] ___ Bilinear and Bilateral Generating Relation, Amer. J. Math., 99, (1976).

[8] ___ Combinatorial Identities of Engelberg and Jensen's formula, Accepted for publication in J. Math. Analysis and Application.

[9] D.P., Shukla, Certain Binomial Identities, Glasnik Mathematik, 11, No. 2 (1976).

[10] ___ A note on Certain Identities of Gould, (Communicated for publication).

Department of Mathematics
and Astronomy
Lucknow University
Lucknow (INDIA).

Nep. Math. Sc. Rep.
Vol. 5, No. 1 (1980), 31-36

Comparison of Some Sampling Schemes Under Superpopulation structure

A.V. Kharshikar

1. Introduction

In the theory of sampling from finite populations there has always been search for an appropriate sampling design and an appropriate estimator of the population total, which takes into account prior knowledge if there is any. The elegant unified approach introduced by Godambe [2] and the use of sufficiency and likelihood principles put forth by Basu[1] have helped in developing a logical and compact treatment. The two estimates which are most often discussed in the literature are the Horvitz-Thompson estimator [3] and estimator based on sample mean.

Here in this note we make comparison of some sampling schemes, using the above mentioned estimators and some other estimators, under a superpopulation structure.

We have a population $U = \{1,2,\ldots,N\}$ of distinct and distinguishable elements and with i^{th} element there is associated value of y-characteristic (the characteristic under study) to be denoted by y_i, $i = 1,2\ldots,N$. We regard $y_1, y_2,\ldots, y_N$ as observations on $Y_1, Y_2,\ldots, Y_N$ which are independent random variables with $E(Y_i) = c_i\mu$ and $\text{var}(Y_i) = c_i^2\sigma^2$, $1 \leq i \leq N$, where (i) $Y_1, Y_2,\ldots, Y_N$ are almost surely non-negative random variables (ii) $c_1, c_2,\ldots, c_N$ are known positive constants with $\Sigma c_i = n$ and each $c_i < 1$. (iii) μ and σ^2 are unknown parameters. We shall denote the parameter space $\{(\mu\,\sigma^2) \mid 0<\mu<\infty, 0<\sigma^2<\infty\}$ by Ω. We wish to make a guess about $n\mu$, which is $E\sum_i Y_i$ on the basis of a sample, which is obtained through a sampling design. A sample s is a non-empty subset of U and a sampling design p is a function from $\mathcal{S}$, the class of all non-empty subsets of U to $[0,1]$ with $\sum_s p(s) = 1$. The single and double inclusion probabilities corresponding to p are defined as

$$\pi_i(p) = \sum_{s \ni i} p(s), \qquad 1 \leq i \leq N$$

$$\pi_{ij}(p) = \sum_{s \ni i,\ s \ni j} p(s), \qquad 1 \leq i,j \leq N$$

In this note we shall consider only those sampling designs p for which each π_i is positive and $\sum_i \pi_i(p) = n$. The number of elements in s is called size of s and is denoted by $n(s)$. Average sample size for a sampling design p is $\sum_s n(s)p(s)$, which is also equal to $\sum_i \pi_i(p)$. A sampling design p is called a fixed size sampling design with size n if $n(s) \neq n \Longrightarrow p(s) = 0$. An estimator based on sample S is a function of S and only those Y_i's with i in S. A sampling scheme is a pair of a sampling design and an estimator. Now we define four estimators which are considered in Section 2:

$$T_1 = t_1(S, \underline{Y}) = \sum_{i \in s} \frac{Y_i}{\pi_i}$$

where π_i's are the single inclusion probabilities of the sampling design of the scheme. This is the welknown Horvitz-Thompson estimator.

$$T_2 = t_2(S, \underline{Y}) = \sum_{i \in s} \frac{Y_i}{c_i}$$

$$T_3 = t_3(S, \underline{Y}) = \frac{n}{n(s)} \sum_{i \in s} \frac{Y_i}{c_i}$$

and $$T_4 = t_4(S, \underline{Y}) = n \frac{\sum_{i \in s} Y_i}{\sum_{i \in s} c_i}$$

T_4 is a ratio estimator.

2. Comparison of some sampling schemes

Let us consider the sampling schemes (p_1, T_1), (p_2, T_2), (p_3, T_3) and (p_4, T_4) where p_1, p_2, p_3 and p_4 are sampling designs with $\pi_i(p_\alpha) > 0$, $1 \le i \le N$, $1 \le \alpha \le 4$ and average size n. We may note here that

$$E(T_1) = E(T_2) = E(T_3) = E(T_4) = n\mu .$$

$$\text{var}(T_1) \quad \sigma^2 \sum_i \frac{c_i^2}{\pi_i(p_1)} + \mu^2 \sum_{i,j} \left\{ \pi_{ij}(p_1) - \pi_i(p_1)\pi_j(p_1) \right\} \frac{c_i \; c_j}{\pi_i(p_1)\; \pi_j(p_1)}$$

$$\text{var}(T_2) = n\sigma^2 + \mu^2 \psi(p_2)$$

where $$\psi(p_2) = \sum_{i,j} \left\{ \pi_{ij}(p_2) - \pi_i(p_2)\; \pi_j(p_2) \right\}$$

$$\text{var}(T_3) = n\sigma^2 + \sigma^2 \phi(p_3)$$

where $$\phi(P_3) = n\, E\left\{ \frac{n}{n(s)} - 1 \right\}$$

$$\text{var}(T_4) = n\sigma^2 \; \}_c (p_4)$$

where $\xi_c(p_4) = E\left\{ \frac{\sum_{i \in s} c_i^2}{(\sum_{i \in s} c_i)^2}\, n \right\}$.

As all the sampling schemes are unbiased, we shall compare these schemes by comparing the variances. Writing $v_\alpha(\mu, \sigma^2)$ for var (T_α) we shall use the following terms:

(i) (p_α, T_α) is said to be at least as good as (p_β, T_β)

if $\quad v_\alpha(\mu, \sigma^2) \le v_\beta(\mu, \sigma^2)$ for all $(\mu, \sigma^2) \in \Omega$.

(ii) (p_α, T_α) and (p_β, T_β) are said to be equally good

if $\quad v_\alpha(\mu, \sigma^2) = v_\beta(\mu, \sigma^2)$ for all $(\mu, \sigma^2) \in \Omega$.

(iii) (p_α, T_α) is said to be preferable to (p_β, T_β) if for all $(\mu, \sigma^2) \in \Omega \quad v_\alpha(\mu, \sigma^2) \le v_\beta(\mu, \sigma^2)$ and there exists at least one point in Ω at which strict inequality holds.

The following lemma is useful while comparing the above schemes.

Lemma

For any sampling design p with average size n

(i) $\psi(p) \geq o$ with equality iff p is a fixed size design;

(ii) $\phi(p) \geq o$ with equality iff p is a fixed size design;

(iii) the N x N matrix $\left(\pi_{ij}(p) - \pi_i(p)\ \pi_j(p)\right)$ is positive semidefinite;

(iv) $\sum_i \frac{c_i^2}{\pi_i(p)} \geq n$, equality holding iff $c_i = \pi_i(p)$ for all i;

(v) $\xi_c(p) \geq 1$.

Proof. We note that $\psi(p) = \text{var}\left[n(S)\right]$ and hence we have (i). Further since n(S) is an almost surely positive random variable, $E\frac{1}{n(S)} \geq \frac{1}{E\, n(S)}$, with equality iff n(S) has one point distribution. This gives us (ii) Next define $x_i(S) =$ if $i \in S$ and o otherwise, $1 \le i \le N$. Then the variance-covariance matrix of the random vector $\left((x_1(S), x_2(S), \ldots, x_N(S)\right)$ is $\left(\pi_{ij}(p) - \pi_i(p)\ \pi_j(p)\right)$ and hence (iii). Writing $c_i = \pi_i(p) + d_i$ and noting that $\sum_i d_i = o$ we get $\sum_i \frac{c_i^2}{\pi_i(p)} = \sum_i (p) + \sum_i \frac{d_i^2}{\pi_i(p)} \geq n$ which gives us (iv) Lastly we note that

$$\xi_c(p) = E\left\{ \frac{\sum_{i \in s} c_i^2}{\left(\sum_{i \in s} c_i\right)^2}\, n \right\} = E\left\{ \frac{\sum_{i \in s} c_i^2}{n(S)\, \bar{c}\,(S)^2}\, \frac{n}{n(S)} \right\}$$

$$\geqslant E\left\{ \frac{n}{n(S)} \right\} \geqslant 1, \text{ and hence (v).}$$

Result 1

If p_1, p_2, p_3 and p_4 are all fixed size designs then

(i) (p_2, T_2) is at least as good as (p_1, T_1), (p_2, T_2) and (p_1, T_1) are equally good iff $c_i = \quad_i (p_1)$ for all i.

(ii) (p_2, T_2) and (p_3, T_3) are equally good.

(iii) (p_2, T_2) is at least as good as (p_4, T_4).

Result 2

Suppose p_1, p_2 and p_3 are not fixed size designs. Then

(i) given that $c_i = \pi_i(p_1)$ for all i, (p_1, T_1) is preferable to (p_2, T_2) whenever $\psi(p_1) \leq \psi(p_2)$; (p_2, T_2) is preferable to (p_1, T_1) whenever $\psi(p_1) > \psi(p_2)$ and they are equally good whenever $\psi(p_1) = \psi(p_2)$.

(ii) $\nu_2(\mu, \sigma^2) < \nu_3(\mu, \sigma^2)$ whenever $\dfrac{\psi(p_2)}{\phi(p_3)} < \dfrac{\sigma^2}{\mu^2}$

$\nu_2(\mu, \sigma^2) > \nu_3(\mu, \sigma^2)$ whenever $\dfrac{\psi(p_2)}{\phi(p_3)} > \dfrac{\sigma^2}{\mu^2}$ and

$\nu_2(\mu, \sigma^2) = \nu_3(\mu, \sigma^2)$ whenever $\dfrac{\psi(p_2)}{\phi(p_3)} = \dfrac{\sigma^2}{\mu^2}$.

Proofs of results 1 and 2 follow immediately with the use of the lemma.

References

[1] Basu, D., Sufficiency and likelihood principle in sample survey theory, Sankhya, Series A, 31, (1969), 441-454.

[2] Godambe, V.P., A unified theory of sampling from finite populations, J. Roy. Statist. Soc., Series B, 17, (1955), 268-278.

[3] Horvitz, D.G. and Thompson, D.J., A generalization of sampling without replacement from a finite universe, J. Amer. Statist. Assoc. 47, (1952), 663-685.

A.V. Kharshikar
University of Poona

Presently working in Indian Cooperation Mission, Kathmandu and Tribhuvan University, Kathmandu, on leave from Poona University.

Nep. Math. Sc. Rep.
Vol. 5, No. 1 (1980), 37-44

On Compatibility of Order with Some Algebras

Dr. Yuri A. Selivanov

1. A partially ordered algebra is an algebraic system where composition laws and relation of order are defined on the same set, say A, and the order is compatible [7] with all the composition laws. Starting with a given algebra and applying well-known methods we can find necessary and sufficient conditions to make this algebra partially ordered, see [6]-[8].

In the problem section of [6] there is the following problem: to find necessary and sufficient conditions for a given partially ordered set A to make it a partially ordered algebra with the given order. In other words: how to define a composition law starting with a given partially ordered set A so as to make this set a partially ordered algebra. In the present article this problem is solved in terms of properties of the group of order morphisms of a given partially ordered set A.

2. Let us recall some definitions.

Let A & A' be two given non-empty partially ordered (p.o.) sets with the orders $\leq_1$ & $\leq_2$ correspondingly. The mapping $\alpha: A \dashrightarrow A^1$ is called an isotonic mapping if $a \dashrightarrow a^1$ & $b \dashrightarrow b^1$ ($a, b \in A$; $a^1, b^1 \in A^1$) and $a \leq_1 b \Longrightarrow a^1 \leq_2 b^1$.

We shall say that the isotomic mapping

1) α - is an o-morphism, if $\operatorname{Im} \alpha \subset A^1$;

2) α - is an o-endomorphism if $\alpha: A \dashrightarrow A$ & $\operatorname{Im} \alpha \subset A$;

3) α - is an o-epimorphism if $\operatorname{Im} \alpha = A^1$;

4) α - is an o-isomorphism if α & $\exists\ \alpha^{-1}$ - are o-epimorphisms.

5) α - is an o-automorphism if $\alpha: A \rightarrow A$ & α is o-isomorphism.

In cases [1] - [3] we can say about injective and bijective o-morphisms that means injective and bijective mappings but one-sided isotomic. If $\alpha: A \leftrightarrow A^1$ & $a \leq_1 b \Longleftrightarrow a^1 \geq_2 b^1$, then α is called a dual isomorphism.

In this article we shall consider partially ordered sets with arbitrary elements and shall study only the properties of the order, therefore we shall consider p.o. set, say M, as a model $\mathfrak{M} = \langle M, \leq \rangle$.

We shall say that the algebra $\mathfrak{A} = \langle M, \Sigma \rangle$, where M is the main set & Σ is the set of all composition laws on M, is compatible (one-sided compatible) with the model $\mathfrak{M} = \langle M, \leq \rangle$ if the algebraic system $\mathfrak{A} = \langle M, \Sigma, \leq \rangle$ is a partially ordered algebra (one-sided ordered algebra). When we say,

that the model $\mathfrak{M} = \langle M, \leq \rangle$ is compatible with the algebra $\mathfrak{A} = \langle M, \Sigma \rangle$ it means that they have the common main set M and that the model can be transformed into p.o. algebra $\mathfrak{A} = \langle M, \Sigma, \leq \rangle$.

By an o-morphism of a p.o. algebra $\mathfrak{A} = \langle \ , \Sigma, \leq \rangle$ will mean the o-morphism of the model $\mathfrak{M} = \langle M, \leq \rangle$, which is not, in general, a morphisms of the algebra $\langle M, \Sigma \rangle$.

A group G(M) of one-to-one mappings of the set M into itself if called transitive group if each element of the set M can be transformed into any other element of M using an element of G(M). If $\forall x, y \in M \ \exists \ f \in G(M)$ such that fx=y, the group G(M) is called a simply transitive group.

3. Let $\langle M, \leq \rangle$ be a model in which the group $G_o(M)$ of all o-automorphisms is transitive. It is easy to prove the following properties of this model:

1. Each element of M belongs to some infinite chain of M.

2. There are no maximal and minimal elements in M.

3. An element $x \in M$ is called a point of bifurcation in M if $\exists \ a, b \in M$ such that $x \leq a$ & $x \leq b$ ($x \geqslant a$ & $x \geqslant b$) but $a \parallel b$ (a & b are non-comparable).

   Either there are no points of bifurcation in M or each element of M is a point of bifurcation.

4. It m has no points of bifurcation, then M is a union of all its maximal chains.

5. It M has no points of bifurcation then all the maximal chains of M are o-isomorphic, in other words $G_o(M)$ is imprimitive group and all the maximal chains of M are the systems of imprimitivity of $G_o(M)$.

4. Let $\mathfrak{M}^* = \langle M, \cdot, \leq \rangle$ be any p.o. groupoid. It is clear that the families of translations $\{R_a\} = \{R_a: M \to M: x \to xa\}$ & $\{L_a\} = \{L_a: M \to M: x \to ax\}$ are the families of o-endomorphisms of the model $\mathfrak{M} = \langle M, \leq \rangle$. These families generate subsemigroups $\mathcal{S}_o^R(M)$ & $\mathcal{S}_o^L(M)$ of the semigroup $\mathcal{S}_o(M)$ of all o-endomorphisms of the model $\langle M, \leq \rangle$. It is evident that $\mathcal{S}_o(M)$ is a semigroup with identity E, because the identity mapping E of the set M is o-automorphism of $\langle M, \leq \rangle$.

Let us correspond to any element $a \in M$ to the o-endomorphism $L_a \in \mathcal{S}_o(M)$. This correspondence is a single valued and therefore we have made mapping, say f; $\mathfrak{M} \to \mathcal{S}_o(M)$. The mapping f is an o-morphism; in fact the partial order in M naturally induces a partial order in $\mathcal{S}_o(M)$: $T_1 \preceq T_2$ ($T_1, T_2 \in \mathcal{S}_o(M)$ iff $T_1 x \leq T_2 x \ \forall x \in M$. So, we have $a \leq b \Longrightarrow L_a \preceq L_b$ i.e. $\forall x \in M$; $L_a x = ax \leq bx = L_b x$ and therefore $a \leq b \Longrightarrow fa \preceq fb$, i.e. f is an o-morphism $\langle M, \leq \rangle \to \mathcal{S}_o(M)$.

On the other hand, let there exist a mapping f: $\mathfrak{M} \to \mathcal{S}_o(M)$ so that to each element $a \in M$ corresponds a unique element (o-endomorphism) $T_a \in \mathcal{S}_o(M)$.

Let us define on M a binary composition law $(\cdot)$, say multiplication, by the equality:

(1) $a.b = T_a b \ \forall\, a,b \in M.$

This definition transforms the model $\mathfrak{M} = \langle M, \leq \rangle$ into left-ordered groupoid $\langle M, \cdot, \leq \rangle$.

The fact that T_a is an o-endomorphism of $\mathfrak{M}$ implies the uniqueness of the product and left monotonic law; so we get $x \leq y \Longrightarrow T_a x \leq T_a y$ i.e. $ax \leq a\cdot y \ \forall\, a \in M$. But as it was shown earlier $\mathcal{S}_o(M)$ is a p.o. semigroup in which the order is induced by the order on M; therefore $T_1 \preceq T_2 \Longrightarrow T_1 x \leq T_2 x \ \forall\, x \in M$ and so f is an o-morphism $\mathfrak{M} \dashrightarrow {}_o(M)$ i.e.: $a \leq b \Longrightarrow fa = T_a \preceq T_b = fb$, $T_a x \leq T_b x \ \forall\, x \in M \Longrightarrow a\cdot x \leq b\cdot x \ \forall\, x \in M$ and so $\langle M, \cdot, \leq \rangle$ is also a right ordered groupoid and therefore it is a p.o. groupoid. All the above discussion implies the following:

Theorem 1. The model $\mathfrak{M} = \langle M, \leq \rangle$ is compatible with the groupoid $\mathfrak{M}^* = \langle M, \cdot \rangle$ iff there exists o-morphism $f: \mathfrak{M} \longrightarrow \mathcal{S}_o(M)$.

Cor. If o-morphism $f: \mathfrak{M} \dashrightarrow {}_o(M)$ is such that Imf contains the identity mapping E then p.o. groupoid $\langle M, \cdot, \leq \rangle$ has a left identity e. If, moreover, $T_a \in \mathrm{Imf} \Longrightarrow T_a e = a$, then e is identity.

Proof: $E \in \mathrm{Imf} \Longrightarrow \exists\, e \in M$ such that $fe = E \Longrightarrow \forall\, x \in M$ $ex = Ex = x$ and e is a left identity. If $\forall\, T_a \in \mathrm{Imf}$: $T_a e = a \Longrightarrow a\cdot e = a \ \forall\, a \in M$ and e is identity. Let us show the uniqueness of e. In fact, if $\exists\, e' \in M$ such that $fe' = E$ & $T_a e' = a \ \forall\, T_a \in \mathrm{Imf}$, then $T_e e = e'$ & $T_e e' = e$ $\Longrightarrow e' = e$.

Let now $\mathfrak{M}^* = \langle M, \cdot, \geqslant \rangle$ be a p.o. semigroup, then the families $\{R_a\}$ & $\{L_a\}$ form subsemigroups of ${}_o(M)$. In this case the mentioned o-morphism f is not only the morphism of the model $\langle M, \geqslant \rangle$ but also is a homomorphism of the semigroup $\langle M, \cdot \rangle$ into semigroup ${}_o(M)$. Really, let $fa = L_a$ as before, then $f(ab) = L_{ab}$, but $L_{ab}\, x = (ab)\, x = a(bx) = L_a\, (L_b x) = (L_a \circ L_b)\, x$, i.e. $L_{ab} = L_a \circ L_b$, but $L_a = fa$ & $L_b = fb$, so $f(ab) = fa \circ fb$, it shows that $\{L_a\}$ is a semigroup and moreover it is a homomorphic image of the semigroup $\mathfrak{M}^* = \langle M, \cdot \rangle$ under the o-morphism $f: \mathfrak{M}^* \longrightarrow \mathcal{S}_o(M)$. The equality $L_a \circ L_b = L_{a.b}$ can be written in another form

(2) $$L_a \circ L_b = L_{L_a b}$$

So, $\mathrm{Imf} = \{L_a\}$ is a subsemigroup of the semigroup $\mathcal{S}_o(M)$ which satisfies the condition (2).

On the other hand, let f be o-morphism $\mathfrak{M} = \langle M, \leq \rangle \dashrightarrow \mathcal{S}_o(M)$ such that $\mathrm{Imf} = \{T_a\}$ is a semigroup satisfying the condition (2). Using the definition (1) we shall get a semigroup $\mathfrak{M}^* = \langle M, \cdot \rangle$, which is compatible with the model $\mathfrak{M} = \langle M, \leq \rangle$, so $\mathfrak{M}^* = \langle M, \cdot, \leq \rangle$ is a p.o. semigroup. To prove it we need to check only the associativity

of multiplication $(\cdot)$. In fact, $\forall\, a,b,c \in M$ we have $a\cdot(b\cdot c) = T_a(T_b c)$ $= (T_a \circ T_b)c = T_{T_a b}\, C = (T_a b).\, C = (a.b).c$. This implies

Theorem 2. Model $\mathfrak{M} = \langle M, \leq \rangle$ is compatible with a semigroup $\mathfrak{M}^* = \langle M, \cdot \rangle$ iff there exists o-morphism $f: \mathfrak{M} \rightarrow \mathcal{S}_o(M)$ such that Imf is a subsemigroup of $\mathcal{S}_o(M)$ satisfying the condition (2).

Note: The semigroup $\mathcal{S}_o(M)$ of any model $\langle M, \leq \rangle$ contains the identical mapping E. It means that there always exists o-morphism $i: \langle M, \leq \rangle \rightarrow E$ which satisfies the condition (2). As $\{E\}$ is a subgroup of $\mathcal{S}_o(M)$ we can say that using the definition (1) any model $\langle M, \leq \rangle$ can be transformed into idempotent p.o. semigroup. But in general case (if $f \neq i$) this method gives general result.

5. Let now $\mathfrak{M}^* = \langle M, \cdot, \backslash, /, \leq \rangle$ be primitive p.o. quasigroup. The family of translations $\{L_a\}$ of p.o. quasigroup $\mathfrak{M}^*$ is a family of o-automorphisms of the model $\langle M, \leq \rangle$. It should be mentioned that the given family is simply transitive on M, and therefore the mapping $f: a \rightarrow L_a$ of $\mathfrak{M}^* \rightarrow \{L_a\}$ is a one-to-one mapping. As before $G_o(M)$ is a group of all o-automorphisms of the model $\mathfrak{M}$. This group is ordered by the induced order i.e. $g_1 \preccurlyeq g_2$ $(g_1, g_2 \in G_o(M))$, iff $g_1(x) \leq g_2(x)\ \forall x \in M$. Under these conditions we have: $a \leq b \Rightarrow L_a \preccurlyeq L_b$ i.e. $a \leq b \Rightarrow ax \leq bx\ \forall x \in M$. Let now $L_a x \leq L_b x$ for some element x of M, then $ax \leq bx \Rightarrow a \leq b \Rightarrow L_a \preccurlyeq L_b$. The mapping f which is defined above $(f: a \rightarrow L_a)$ is an o-isomorphism and o-automorphisms $L_a \in \{L_a\}$ satisfy the condition:

(*) It $L_a x \leq L_b x$ for some element $x \in M$, then $L_a \preccurlyeq L_b$.

On the other hand, let there exist o-injection $f: \mathfrak{M} \rightarrow G_o(M)$ such that Imf is simply transitive om M, and satisfies the condition (*). So, we have: $\forall a \in M\ \exists!\ fa = T_a \in$ Imf i.e. the mapping $f: M \leftrightarrow$ Imf is an o-isomorphism.

Let us define the multiplication (o) on M using (1) i.e.

$a \circ b = T_a b\ \forall a,b \in M$, where $T_a = fa \in$ Imf.

Consider the equations

(3) $\quad a \circ x = b$

(4) $\quad y \circ a = b.$

The equation (3) $a \circ x = T_a x = b$ has a unique solution in M because T_a is a substitution of M.

The equation (4) $y \circ a = T_y a = b$ has also unique solution in M, because Imf is simply transitive on M, i.e. $\forall a,b \in M\ \exists!\ T_y \in$ Imf: $T_y a = b$.

We shall check now the monotony of multiplication and cancellation laws for inequalities:

a) $a \leq b \Longrightarrow T_x a \leq T_x b \equiv xoa \leq xob$, i.e. T_x is o-automorphism. Then $a \leq b \Longrightarrow T_a \preccurlyeq T_b$, i.e. $T_a x \leq T_b x \equiv aox \leq box$ because f is an isomorphism.

b) $aox \leq box \equiv T_a x \leq T_b x \Longrightarrow T_a \preccurlyeq T_b$ (condition (*)) and therefore $f^{-1}(T_a) \leq f^{-1}(T_b) \equiv a \leq b$ as f is an o-isomorphism $M \longleftrightarrow Imf$. Then $xoa \leq xob \equiv T_x a \leq T_x b \Longrightarrow T_x^{-1}(T_x a) \leq T_x^{-1}(T_x b)$ $\equiv a \leq b$ as T_x is an o-automorphism of the model $\mathfrak{M} = \langle M, \leq \rangle$.

Applying proposition 1 from [7] we have proved

Theorem 3. Model $\mathfrak{M} = \langle M, \leq \rangle$ is compatible with a primitive quasigroup $\mathfrak{M}^* = \langle M, \cdot, \backslash, / \rangle$ iff the group $G_o(M)$ contains a simply transitive family of o-automorphisms which is o-isomorphic to $\mathfrak{M}$.

Applying Theorem 3 and Theorem about the L - P isotopy from [1] we shall get

Theorem 4. Model $\mathfrak{M} = \langle M, \leq \rangle$ is compatible with a primitive loop $\mathfrak{M}^{**} = \langle M, \cdot, \backslash, / \rangle$ iff it is compatible with a primitive quasigroup $\mathfrak{M}^* = \langle M, \cdot, \backslash, / \rangle$.

Taking into consideration some geometrical applications there is a reason to study one-sided quasigroups and loops with one-sided order. Moreover, if quasigroup is left, then it should be right-ordered and if quasigroup is right, then it is left-ordered. For such kind of quasigroups Theorem 3 implies.

Theorem 5. Model $\mathfrak{M} = \langle M, \leq \rangle$ is compatible with a right left-ordered quasigroup $\mathfrak{M}^* = \langle M, \cdot, / \rangle$ iff there exist the mapping f of this model into the group $G_o(M)$.

It should be mentioned, that the quasigroup $\mathfrak{M} = \langle M, \cdot \rangle$ from the next theorem doesn't suppose to be primitive.

Theorem 6. Model $\mathfrak{M} = \langle M, \leq \rangle$ is compatible with a quasigroup $\mathfrak{M}^* = \langle M, \cdot \rangle$ iff there exist o-morphism $f: \mathfrak{M} \rightarrow G_o(M)$ such that Imf is a simply transitive on M and moreover the narrowing of f on Imf should be bijective o-morphism.

6.° In this section we shall find necessary and sufficient conditions to make the model $\mathfrak{M} = \langle M, \leq \rangle$ partially ordered group.

Considering the p.o. group $\mathfrak{M}^* = \langle M \cdot, \leq \rangle$ we can say that the family of translatiions $\{L_a\}$ of a group $\mathfrak{M}^*$ forms a subgroup of $G_o(M)$, where $G_o(M)$ is a group of all automorphisms of the model $\langle M, \leq \rangle$. Moreover these translations satisfy the conditions (2) and

(5) $\quad L_a^{-1} = L_{a^{-1}}$

Condition (2) is a corollary of the associativity and condition (5) means that the inverse mapping of the translation in the group is also a translation. Corresponding to any element $a \in M$ the translation L_a from $G_o(M)$ we get o-injection f: $\mathfrak{M}^* \to G_o(M)$ such that the narrowing of f on Imf is an o-isomorphism $\mathfrak{M}^* \longleftrightarrow \mathrm{Imf} = \{L_a\}$. Let us mention that the group $\{L_a\}$ is a simply transitive group on M.

On the other hand, let the group $G_o(M)$ of the model $\langle M, \leq \rangle$ be transitive, and let f: $\mathfrak{M} \to G_o(M)$ be o-injection such that Imf is a simply transitive subgroup of $G_o(M)$ which is o-isomorphic to $\mathfrak{M} = \langle M, \leq \rangle$, and satisfies the condition (2). Using definition (1) as before, we shall transform the model $\langle M, \leq \rangle$ into p.o. group. So the following theorem is period.

Theorem 7. Model $\mathfrak{M} = \langle M, \leq \rangle$ is compatible with a group $\mathfrak{M}^* = \langle M, \cdot \rangle$ if the group $G_o(M)$ of o-automorphisms of the model $\mathfrak{M}$ contains simply transitive subgroup o-isomorphic to $\mathfrak{M}$, which satisfies the condition (2).

From the above consideration it follows that the transitivity of the group $G_o(M)$ is very important for the definition of group and quasigroup composition laws which are compatible with a given order on M.

So if the group of o-automorphisms of the model $\langle M, \leq \rangle$ is intransitive, then this model can't be transformed into p.o. group or p.o. quasigroup with the given order on M. The totally ordered sets and the sets with a minimum condition are the examples of such kind of the models.

Let $V = \{(-n,o); (o,o); (n,o); (o,n)\}$, where n is a natural number. Let us define the order on V by the following way: $(a,b) \leq (c,d)$ iff $a \leq c$ & $b \leq 1$. Then the model $\langle V, \leq \rangle$ has intransitive group $G_o(V)$ as a pair (o,o) is a point of bifuraction of V.

It is known that there exist such models $\langle M, \leq \rangle$ which induce only trivial order on $G_o(M)$. These models can't also be transformed into p.o. quasigroup or p.o. group in the mentioned sense. The examples of such models can be found.[2] - [5] .

References

[1] A. Albert, Quasigroups I, Trans. Amer. Math. Soc., 54 (1943), 507-519.

[2] P. Cohu, Groups of order automorphisms of ordered sets, Mathematika, 4, Part 1, No. 7 (1957), 41-50.

[3] P. Conrad, The group of order preserving automorphisms of an ordered Abelian Group, Proc. Amer. Math. Soc., 9 (1958), 382-389.

[4] P. Conrad, Non-abelian ordered groups, Pacif. J. Math., 9 (1959), 25-41.

[5] P. Conrad, Right-ordered groups, Michigan Math. J., 6 (1959), 267-275.

[6] L. Fuchs, Partially ordered algebraic systems, Pergamon Press, Oxford-London-New York-Paris, 1963.

[7] Y. Selivanov, A survey of ordered loops I & II, Nep. Math. Sc. Rep. 3, No. 1 (1978), 36-45; 3, No. 2 (1978), 87-91.

[8] D. Zelinsky, On ordered loops, Amer. J. Math., 70 (1948), 681-697.

Nep. Math. Sc. Rep.
Vol. 5, No. 1 (1980), 45-50

A Theorem on a Generalised Laplace Tranafsrm*

G.B. Thapa

1. Introduction

The Laplace transform is defined by the relation [5].

$$(1.1) \qquad f(s) = \int_0^\infty e^{-st}\, \phi(t)\, dt, \qquad (\mathrm{Re}\,(s) > 0).$$

Various generalisations of the Laplace transform are known. The generalisation which is considered here is [3].

$$(1.2) \qquad f(s) = \int_0^\infty e^{-\frac{1}{2}st}\, (st)^{-m-\frac{1}{2}}\, M_{k,m}\,(st)\, \phi(t)\, dt,$$

where $M_{k,m}(x)$ is one of Whittaker's confluent hypergeometric functions. The parameters k,m may be real or complex with ($2m \neq -1, -2, \ldots$). The transform (1.2) reduces to (1.1) when $k = m + \frac{1}{2}$ by virtue of the relation 4 .

$$M_{m+\frac{1}{2},m}\,(x) = e^{-\frac{1}{2}x}\, x^{m+\frac{1}{2}}\,.$$

For brevity, (1.2) will be written in the form

$$(1.3) \qquad f_{k,m}\,(s) = M\,(\phi(t)).$$

2. Theorem

Suppose

(i) $G(t) = O(t^q) \qquad (t \to 0),\ (q > -1);$

(ii) $G(t) = O(t^p) \qquad (t \to \infty);$

then the integrals

$$(2.1) \qquad g(s) = \int_0^\infty e^{\frac{1}{2}st}\, (st)^{m-\frac{1}{2}}\, W_{k,m}(st)\, t^{-k-m-3/2}\, G(1/t)\, dt,$$

and

$$(2.2) \qquad f_{-\frac{1}{2}(k+3m),\ \frac{1}{2}(m-k)}\,(s) = M\,(G(t))$$

*An extract from Degree Thesis, 1978.

exist for all s, $Re(s) > 0$, provided that $k + m + \frac{1}{2} + p < 0$; moreover for $Re(s) > 0$, and $k - m < \frac{1}{2}$.

$$g(s) = \frac{s^{2m}}{\Gamma(1-2k+2m)} H(\sqrt{s}), \tag{2.3}$$

where $H(s)$ is the Laplace transform of

$$y^{2m-2k} f_{-\frac{1}{2}(k+3m),\, \frac{1}{2}(m-k)} (y^2/4).$$

Proof. In order to establish the convergence of (2.1), it suffices to show that for some $R > d > 0$ the two integrals.

$$I_1 = \int_0^d \left| e^{\frac{1}{2}st} (st)^{m-\frac{1}{2}} W_{k,m}(st)\, t^{-k-m-3/2} G(1/t) \right| dt,$$

$$I_2 \int_R^\infty \left| e^{\frac{1}{2}st} (st)^{m-\frac{1}{2}} W_{k,m}(st)\, t^{-k-m-3/2} G(1/t) \right| dt$$

are finite.

The asymptotic behaviour at the $W_{k,m}$- function is given by [4, p 61].

$$W_{k,m}(x) = 0\,(x^{-m+\frac{1}{2}}) \qquad (x \to 0), \tag{2.4}$$

$$W_{k,m}(x) = 0\,(x^{k} e^{-\frac{1}{2}x}) \qquad (x \to \infty). \tag{2.5}$$

The hypothesis (ii) and (2.4) imply the existence of a $d > 0$, and a constant K_1 such that for $Re(s) > 0$.

$$\int_\epsilon^d \left| e^{\frac{1}{2}st} (st)^{m-\frac{1}{2}} W_{k,m}(st)\, t^{-k-m-3/2} G(1/t) \right| dt$$
$$\leq K_1\, e^{\frac{1}{2}d\, Re(s)} \int_\epsilon^d t^{-k-m-p-3/2}\, dt.$$

This shows that I_1 is finite if $k+m+\frac{1}{2}+p < 0$.

The hypbthesis (i) and (2.5) imply the existence of a $R > d > 0$, and a constant K_2 such that for $Re(s) > 0$.

$$\int_R^x \left| e^{\frac{1}{2}st} (st)^{m-\frac{1}{2}} W_{k,m}(st)\, t^{-k-m-3/2} G(1t) \right| dt$$

$$\leq K_2 \; |s|^{k+m-\frac{1}{2}} \int_R^x t^{-q-2} \, dt.$$

since $q > -1$, this shows that I_2 is finite.

The convergence of the integral (2.2) can be shown by following a similar procedure; the asymptotic behaviour of the $M_{k,m}$ - function that will have to be taken into account is given by [4].

$$(2.6) \qquad M_{k,m}(x) = 0(x^{-k}\, e^{\frac{1}{2}x}) \qquad (x \longrightarrow \infty),$$

$$(2.7) \qquad M_{k,m}(x) = 0(x^{m+\frac{1}{2}}) \qquad (x \longrightarrow 0).$$

To proceed further, the relation [2(I), 5.20(11)] is used to obtain

$$W_{k,m}(st) = s^{m+\frac{1}{2}}\, e^{-\frac{1}{2}st} \frac{2^{1-k+m}\, t^{\frac{1}{2}(m+k+1)}}{\Gamma(1-2k+2m)} \int_0^\infty e^{-\sqrt{s}\, y} \times$$

$$\times\, y^{m-k-1}\, e^{-y^2/(8t)}\, M_{-\frac{1}{2}(k+3m),\frac{1}{2}(m-k)}\left(\frac{y^2}{4t}\right) dy$$

for $t > 0$, $k-m < \frac{1}{2}$. Substitution of this value of $W_{k,m}(st)$ in (2.1) gives for $Re(s) > 0$, and $k-m < \frac{1}{2}$

$$g(s) = s^{2m} \frac{2^{1-k+m}}{\Gamma(1-2k+2m)} \int_0^\infty t^{\frac{1}{2}m-\frac{1}{2}k-3/2} \left[\int_0^\infty e^{-\sqrt{s}y}\, y^{m-k-1} \times \right.$$

$$\left. \times\, e^{-y^2/(8t)}\, M_{-\frac{1}{2}(k+3m)\,\frac{1}{2}(m-k)}\left(\frac{y^2}{4t}\right) dy \right] G(1/t)\; dt$$

$$= \frac{-s^{2m}}{\Gamma(1-2k+2m)} \int_0^\infty e^{-\sqrt{s}y}\, y^{2m-2k}\, dy \int_0^\infty e^{-\frac{1}{2}\left(\frac{y^2}{4t}\right)} \times$$

$$\times \left(\frac{y^2}{4t}\right)^{-\frac{1}{2}(m-k)-\frac{1}{2}} M_{-\frac{1}{2}(k+3m),\frac{1}{2}(m-k)}\left(\frac{y^2}{4t}\right) G(1/t)\; d\,(1/t)$$

$$= \frac{s^{2m}}{\Gamma(1-2k+2m)} \int_0^\infty e^{-\sqrt{s}y}\, y^{2m-2k}\, t_{-\frac{1}{2}(k+3m),\frac{1}{2}(m-k)}\left(\frac{y^2}{4}\right) dy.$$

This implies the result. The change in the order of integration is obviously possible by virtue of the preceding arguments.

Example 1. Suppose

$$G(t) = e^{-\frac{1}{2}\beta t} W_{\lambda,\nu}(\beta t) \qquad (\mathrm{Re}(\beta) > 0). \tag{2.8}$$

It is clear that this function satisfies hypotheses (i), (ii) of the theorem. Now, the use of the relation [2 (II), 20.3 (55)] gives

$$g(s) = \frac{s^{m-\frac{1}{2}}\,\beta^{-k-1}}{\Gamma(\frac{1}{2}-k+m)\,\Gamma(\frac{1}{2}-k-m)} \times \tag{2.9}$$

$$\times\, G^{41}_{42}\left(\beta s \;\middle|\; \begin{matrix} 1+k,\ 2-\lambda+k \\ \frac{1}{2}+m,\ \frac{1}{2}-m,\ 3/2+\nu+k,\ 3/2-\nu+k \end{matrix}\right)$$

for $|\arg(1/s)| < 3\pi/2$, $|\nu| < \frac{1}{2}$. The use of the relation [2(II), 20.3(43)] gives

$$f_{-\frac{1}{2}(k+3m),\frac{1}{2}(m-k)}(s) = \frac{\Gamma(3/2+\nu)\,\Gamma(3/2-\nu)}{\Gamma(2-\lambda)}\,\beta^{-1} \times \tag{2.10}$$

$$\times\, {}_3F_2\left(\tfrac{1}{2}-k-m,\ 3/2+\nu,\ 3/2-\nu;\ m-k+1,\ 2-\lambda;\ -s/\beta\right)$$

for $\mathrm{Re}(s) > 0$, $|\nu| < 3/2$.

Suppose further that

$$k+m+\tfrac{1}{2}=0, \qquad \lambda = 3/2-m+k = 1-2m, \tag{2.11}$$

so that the formula [2(I), 4.23 (12)] may be applicable to find the Laplace transform of (2.10). Then by use of the formula [1(I), 5.6 (55)], and the identity [1(I), 5.31 (8)].

$$H(\sqrt{s}) = \frac{\Gamma(2m-2k+1)}{s^{2m}\,\Gamma(2m+1)}\, G^{31}_{13}\left(\beta s \;\middle|\; \begin{matrix} 0 \\ 0,\ \frac{1}{2}+\nu,\ \frac{1}{2}-\nu \end{matrix}\right) \tag{2.12}$$

for $k-m < \frac{1}{2}$. Now, (2.9) under the assumption (2.11) takes the form

$$g(s) = \frac{1}{\Gamma(2m+1)}\, G^{31}_{13}\left(\beta s \;\middle|\; \begin{matrix} 0 \\ 0,\ \frac{1}{2}+\nu,\ \frac{1}{2}-\nu \end{matrix}\right).$$

Comparison with (2.12) finally yields

$$g(s) = \frac{s^{2m}}{\Gamma(2m-2k+1)} \; H(\sqrt{s})$$

for $Re(s) > 0$, $k-m < \frac{1}{2}$, $|\arg(1/s)| < 3\pi/2$, and $|\nu| < \frac{1}{2}$.

Example 2. Suppose

$$G(t) = t^{p} \qquad (p > -1).$$

Clearly, hypotheses (i) and (ii) of the theorem are satisfied by this function. The use of the relation [4, 3.6.2] gives

$$f_{-\frac{1}{2}(k+3m),\frac{1}{2}(m-k)}(s) = \frac{\Gamma(p+1)\,\Gamma(-k-m-\frac{1}{2}-p)\,\Gamma(1+m-k)}{\Gamma(-k-m+\frac{1}{2})\,\Gamma(m-k-p)} \; s^{-p-1}$$

for $p < -k-m-\frac{1}{2}$. Hence

$$H(\sqrt{s}) = \frac{4^{p+1}\Gamma(p+1)\,\Gamma(-k-m-\frac{1}{2}-p)\,\Gamma(1+m-k)\,\Gamma(2m-2k-2p-1)}{\Gamma(-k-m+\frac{1}{2})\;\Gamma(m-k-p)} \times s^{-m+k+p+\frac{1}{2}}$$

for $p < m-k-\frac{1}{2}$, $p < -k-m-\frac{1}{2}$. Finally, by use of the theorem

$$(2.13) \qquad \int_0^{\infty} e^{\frac{1}{2}st}\,(st)^{m-\frac{1}{2}}\,W_{k,m}(st)\; t^{-k-m-p-3/2}\,dt$$

$$= \frac{4^{p+1}\Gamma(p+1)\,\Gamma(-k-m-\frac{1}{2}-p)\,\Gamma(1+m-k)\,\Gamma(2m-2k-2p-1)}{\Gamma(-k-m+\frac{1}{2})\,\Gamma(m-k-p)\,\Gamma(1-2k+2m)} \times s^{m+k+p+\frac{1}{2}}$$

for $Re(s) > 0$, $k-m < \frac{1}{2}$, $p > -1$, $p < -k-m-\frac{1}{2}$, $p < m-k-\frac{1}{2}$.

I wish to express my indebtedness to Dr. S.R. Pant, and to Dr. R.M. Shrestha who have been so kind and helpful in the preparation of this paper.

References

[1] Erdelyi, A., Higher Transcendental Functions, Vol. I, McGraw-Hill Co., (1953).

[2] ____ Tables of Integral Transforms, Vols. I & II, McGraw-Hill Co., (1954).

[3] Pant, S.R., A Study of a Generalised Laplace Transform and its Cesaro Mean (Ph.D. Thesis), Ranchi Univ., (1967).

[4] Slater, L.J., Confluent Hypergeometric Function, Cambridge, (1960).

[5] Widder, D.V., The Laplace Transform, Princeton, (1963).

Mathematics Instruction Committee
Kirtipur Campus
Tribhuvan University

Registered No.

५३।०३३-०३४, जि.का.का.

Contents

Printed by University Press

Tribhuvan University, Kathmandu.