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"The Nepali Mathematical Sciences Report" will publish the original research papers, short notes and survey articles in different branches of Mathematics and Mathematical Sciences.

Contributors are required to submit two copies of the manuscripts to the Chief Editor. Manuscripts should be in English and typed in double space. The text of the paper should be preceded by a brief abstract.

References should be listed at the end of the paper in alphabetical order and each reference is to be arranged in the following form:

1. Erdelyi, A., (1954), Tables of Integral Transforms, Vol. II, McGraw Hill.
2. Stokes, G.G., On the effect of the internal friction on the motion of pendulums, Trans. Camb. Phil. Soc., 9 (1851), 8-106.

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Magnetohydrodynamic Boundary Layer along a Porous Wall in the presence of Transverse Magnetic Field

R.C. Choudhary

Abstract

The momentum and the kinetic energy integral equations have been derived for the two dimensional laminar incompressible boundary layers along porous walls of a fluid of small electrical conductivity in the presence of transverse magnetic field.

The equations have been numerically solved with the aid of a family of velocity profiles to study the hydromagnetic boundary layer along a semi-infinite porous flat plate.

1. Introduction

Rossow [7] studied by the method of series expansion the flow of an electrically conducting fluid past a flat plate in the presence of transverse magnetic field. Meksyn [4] considered the flow of an electrically conducting fluid past a semi-infinite plate in an aligned magnetic field and obtained a solution by an approximation method. Regirer [6] investigated the flow of a a conducting fluid over a porous flat plate in the presence of a uniform magnetic field.

Iglisch [3] investigated the non-conducting boundary layer with suction along a semi-infinite porous flat plate and found that the asymptotic state suction profile was reached after a distance given by $\overline{V}_s^2\,\overline{x} = 4$. M.R. Head [2] gave in dimensionless forms the momentum and the kinetic energy integral equations for two dimensional boundary layers with suction of non-conducting fluids and suggested a method based on the joint use of the two.

In the present paper, following M.R. Head [2], the momentum and the kinetic energy integral equations for two dimensional MHD boundary layers along porous walls have been derived for the flow of a fluid of small electrical conductivity in the presence of transverse magnetic field. The equations have been used with aid of a family of velocity profiles to study how the MHD boundary layer develops from the leading edge of a semi-infinite porous flat plate to the asymptotic state profile which is reached after a certain distance depending upon the rate of suction and the strength of the magnetic field.

2. Momentum Integral Equation

With x as the coordinate along the wall and y the coordinate perpendicular to it, the two dimensional boundary layer equation [5] for the steady laminar flow of a viscous incompressible fluid of electrical conductivity σ over a porous wall in the presence of an externally applied transverse magnetic field of uniform strength H_0 is

$\overline{U} = \frac{U(x)}{U_o}$ where U_o is the free stream velocity,

$$l = \frac{\theta}{U} \left(\frac{\partial u}{\partial y}\right)_{y=o}; \quad \Lambda = \frac{\theta^2}{\nu} \frac{dU}{dx},$$

$$\lambda = \frac{v_s \theta}{\nu}, \quad t^* = \left(\frac{\theta}{a}\right)^2 \cdot \frac{U_o a}{\nu}, \quad H = \frac{\delta^*}{\theta},$$

and

$R_M^2 = \frac{\sigma}{\rho} \frac{\mu_e^2 H_o^2 a}{U}$, we have

$$(4) \quad \frac{d}{dx}\left(\frac{\theta^2}{\nu}\right) = \frac{2}{U}\left[l - (2 + H)\Lambda + \lambda - H R_M^2 t^*\right]$$

The momentum integral equation in dimensionless form becomes

$$(5) \quad \frac{dt^*}{d\overline{x}} = \frac{2}{\overline{U}}\left[l - (2 + H)\Lambda + \lambda - H R_M^2 t^*\right]$$

3. Kinetic Energy Integral Equation

Adding $\frac{u}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$ to the left hand side of equation (1) multiplying through by u, using $v = v_s - \int_o^y \frac{\partial u}{\partial x} dy$ and integrating with respect to y from y = 0 to y = ∞, we get

$$(6) \quad \frac{d}{dx}\left(\frac{\epsilon^2}{\nu}\right) = \frac{2H}{U}\left[2D - 3\Lambda H_\epsilon + \lambda - 2 R_M^2 t^*\right]$$

where

$$\epsilon = \int_o^\infty \frac{u}{U}\left(1 - \frac{u^2}{U^2}\right) dy; \text{ energy loss thickness,}$$

$$D = \int_o^\infty \left(\frac{\theta}{U}\right)^2 \left(\frac{\partial u}{\partial y}\right)^2 d(y/\theta) \text{ ; dissipation integral}$$

and

$H_\epsilon = \frac{\epsilon}{\theta}$ is a dimensionless parameter.

The variation of H_ϵ is given by

$$\frac{dH_\epsilon}{dx} = \frac{d}{dx}\left(\frac{\epsilon}{\theta}\right) = \frac{1}{\theta}\frac{d\epsilon}{dx} - \frac{\epsilon}{\theta^2}\frac{d\theta}{dx}$$

Substituting for $\frac{d\theta}{dx}$ and $\frac{d}{dx}$ from equations (4) and (6) the kinetic energy integral equation in dimensionless form is

(7) $$\frac{dH_\epsilon}{d\bar{x}} = \frac{1}{\bar{U}t^*}\left[2D - H_\epsilon \left\{ l - (H-1)\Lambda + \lambda - HR_M^2 t^* \right\} + \lambda - 2R_M^2 t^*\right]$$

4. Wall Compatibility Condition

At the wall where $y = 0$: $u = 0$, $v = v_s$ equation (1) gives

(8) $$m = -\Lambda + l \ \lambda - R_M^2 t^*$$

where

$$m = \frac{\theta^2}{U} \left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0}$$

5. Family of Velocity Profiles

The one parameter family of velocity profiles given by H. Schlichting [9] may be used for the approximate calculation of MHD boundary layers along porous walls in the presence of transverse magnetic field.

The velocity profile is

(9) $$\frac{u}{U} = F_1(\eta) + K F_2(\eta), \quad \eta = \frac{y}{\delta(x, R_M^2, \lambda)}$$

where

$$F_1(\eta) = 1 - e^{-\eta}$$
$$F_2(\eta) = F_1 - \sin\frac{\eta\pi}{6}, \quad 0 \leq \eta \leq 3$$
$$F_3(\eta) = F_1 - 1, \qquad \eta \geq 3$$

and $\delta(x, R_m^2, \lambda)$ is the local boundary layer thickness.

For an infinite porous flat plate $\frac{\partial u}{\delta x} = 0$, $U(x) = U_o$, and the solution of equation (1) under conditions (3) gives the asymptotic Magnetohydrodynamic suction velocity profile

(10) $$\frac{u}{U_o} = 1 - e^{-\frac{1}{2}\left(\sqrt{\bar{\lambda}^2 + 4R_h^2} - \bar{\lambda}\right)\bar{\eta}}$$

where

$$R_h = \sqrt{\frac{\sigma \mu_e^2 H_o^2 a^2}{\rho}}, \quad \text{the Hartmann number,}$$

$$\bar{\lambda} = \frac{v_s a}{\nu},$$

and

$$\bar{\eta} = \frac{y}{a}$$

It is seen that an asymptotic profile exists for all values of $\bar{\lambda}$ ($\bar{\lambda} \gtreqless 0$) with $R_h \neq 0$ and for only negative values of $\bar{\lambda}$ when $R_h = 0$. For this asymptotic profile $H = \frac{\delta^*}{\theta} = 2$ and the asymptotic profile (10) can be written as

$$\frac{u}{U_o} = 1 - e^{-\frac{y}{\delta^*}}$$

The asymptotic profile (10) is represented by the parameter $K = 0$ of Schlichiting's profiles (9) for which $\frac{\delta^*}{\delta} = 1$ when $K = 0$. The Blasius profile at the leading edge is given by $K = -1$.

For this system of velocity profiles the variations of the boundary layer parameters θ/δ , H, H_ϵ , l , D against the profile parameter K are shown in Table 1.

Table 1

Boundary Layer Characteristics for Various Values of the Parameter K of the Schlighting's Profiles (eqn.9)

K	θ/δ	H	H_ϵ	l	D
-0.0	0.5000	2.0000	1.6666	0.5000	0.2500
-0.1	0.4931	2.0462	1.6572	0.4696	0.2392
-0.2	0.4857	2.0960	1.6474	0.4394	0.2289
-0.3	0.4779	2.1490	1.6371	0.4096	0.2194
-0.4	0.4696	2.2062	1.6265	0.3801	0.2105
-0.5	0.4608	2.2679	1.6155	0.3510	0.2021
-0.6	0.4515	2.3345	1.6043	0.3224	0.1943
-0.7	0.4418	2.4062	1.5924	0.2944	0.1871
-0.8	0.4316	2.4839	1.5799	0.2671	0.1804
-0.9	0.4209	2.5685	1.5671	0.2405	0.1742
-1.0	0.4098	2.6600	1.5532	0.2145	0.1685
-1.1	0.3981	2.7608	1.5396	0.1894	0.1631
-1.2	0.3859	2.8715	1.5251	0.1652	0.1581
-1.3	0.3735	2.0909	1.5089	0.1421	0.1525
-1.4	0.3605	3.1238	1.4921	0.1201	0.1491
-1.5	0.3470	3.2713	1.4745	0.0990	0.1449
-1.6	0.3330	3.4359	1.4558	0.0791	0.1405

-1.7	0.3185	3.6206	1.4358	0.0605	0.1369
-1.8	0.3042	3.8204	1.4136	0.0432	0.1331
-1.9	0.2882	4.0638	1.3901	0.0273	0.1289
-2.00	0.2724	4.3325	1.3638	0.0128	0.1248
-2.099	0.2562	4.6413	1.3353	0.0000	0.1204

(Separation)

The compatibility condition (8) takes the form

$$(11) \qquad (K+1)\ (\theta/\delta)^2 + \left\{ 1 + (1 - \frac{\pi}{6})\ K \right\} \frac{\theta}{\delta} \lambda - R_M^2 t^* - \Lambda = 0$$

6. Semi-Infinite Porous Flat Plate

It is proposed to investigate the MHD boundary layer with suction along the initial length of a semi-infinite porous flat plate in the presence of transverse magnetic field.

For a semi-infinite flat plate we have

$U(x) = U_o$, free stream velocity,

i.e. $\overline{U} = 1$

$$\Lambda = -\frac{\theta^2}{\nu} \frac{dU}{dx} = 0$$

and

$$\lambda = \frac{v_s \theta}{\nu} = t^{*\frac{1}{2}}\ \overline{v}_s; \quad \text{where } \overline{v}_s = \frac{v_s}{U_o} \sqrt{\frac{U_o a}{\nu}}$$

The momentum integral equation (5), the kinetic energy integral equation (7) and the wall compatibility condition (11) take the forms.

$$(12) \qquad \frac{dt^*}{d\overline{x}} = 2(l + \overline{v}_s t^{*\frac{1}{2}} - H\ R_M^2 t^*)$$

$$(13) \qquad \frac{dH_\epsilon}{d\overline{x}} = \frac{1}{t^*} \left[2D - H_\epsilon\ (l + \overline{v}_s t^{*\frac{1}{2}} - H\ R_M^2 t^*) + \overline{v}_s t^{*\frac{1}{2}} - 2R_M^2 t^*\right]$$

and

$$(14) \qquad (K + 1)\ (\theta/\delta)^2 + (1 + 0.4764K)\ \frac{\theta}{\delta}\ \overline{v}_s t^{*\frac{1}{2}} - R_M^2 t^* = 0$$

At the leading edge ($\overline{x} = 0$) the momentum thickness parameter $t^* = 0$. Therefore, for all values of $\overline{v}_s$ and R_m^2 the compatibility condition (14) gives $K = -1$.

Hence

$$\left.\begin{array}{lcl} l & = & 0.2145 \\ H_\epsilon & = & 1.5532 \\ H & = & 2.6600 \\ D & = & 0.1685 \\ \theta/\delta & = & 0.4100 \end{array}\right\} \quad \text{from Table 1}$$

As the kinetic energy integral equation (13) has a singularity at the leading edge ($\overline{x} = 0$) where $t^* = 0$, the momentum integral equation (12) only has been solved by Runge-Kutta method [8] over a short distance from the leading edge with the satisfaction of the compatibility condition (14). At a few subsequent steps the momentum and the kinetic energy integral equations (12) and (13) have been integrated by Runge-Kutta method [8]. Integrations further down-stream have been carried on by Adam's method [1] using a quadrature formula.

The asymptotic state is reached when $\frac{dt^*}{d\overline{x}} = 0$ and $\frac{dH_\epsilon}{d\overline{x}} = 0$.

i.e. when $K = 0$

and

(15)
$$\left.\begin{array}{lcl} l & = & 0.500 \\ D & = & 0.250 \\ H & = & 2.000 \end{array}\right\} \quad \text{from Table 1}$$

Then from equation (12) we get the asymptotic state value of t^* given by

(16)
$$t^{*\frac{1}{2}} = \frac{\overline{v}_s + \sqrt{\overline{v}_s^{\,2} + 4\, R_M^2}}{4\, R_M^2}$$

For different fixed values of R_m^2 and $\overline{v}_s$ calculations have been made step-by-step with the aid of Table 1 till the asymptotic state values given by equation (15) are approximately reached.

7. Results

The results of calculations have been shown by curves in figures 1 and 2. The following table illustrates the effect of suction and transverse magnetic field on the MHD boundary layer over a semi-infinite porous flat plate.

Table 2

Magnetic parameter	Suction parameter	Point where asymptotic state is approximately reached	Asymptotic value of t^*_A as calculated	Asymptotic value of t^*_A as estimated from equation (16)
R_M^2	$\overline{v}_s$	$\overline{X}_A$	t^*_A	t^*_A
0	-1.0	4.00	0.2280	0.250
1	0.0	2.40	0.2430	0.250
1	-1.0	1.12	0.0899	0.095

The curves in Figure 1 illustrate how the boundary layer develops from zero thickness at the leading edge to the asymptotic state value at a certain distance from the leading edge.

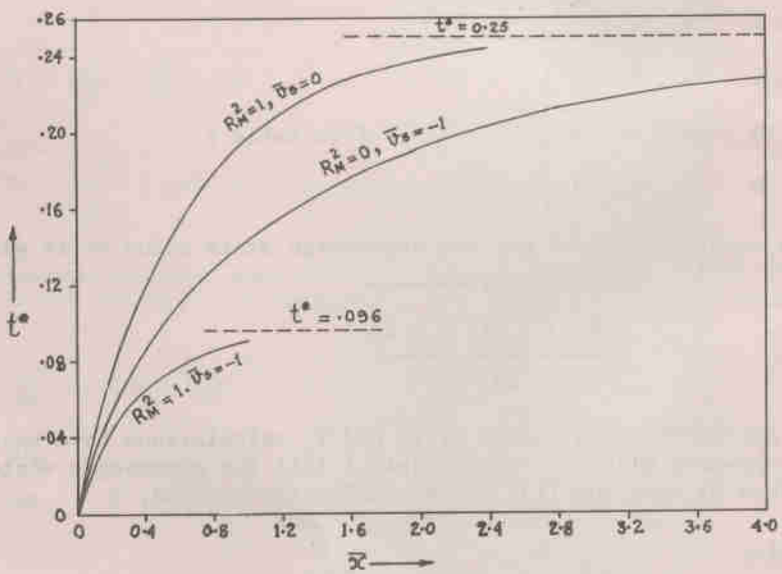

FIGURE-1

MAGNETOHYDRODYNAMIC BOUNDARY LAYER ALONG A SEMI-INFINITE POROUS FLAT PLATE IN THE PRESENCE OF TRANSVERSE MAGNETIC FIELD. VARIATION OF t^* AGAINST $\overline{x}$ FOR DIFFERENT VALUES OF R_M^2 AND $\overline{v}_s$.

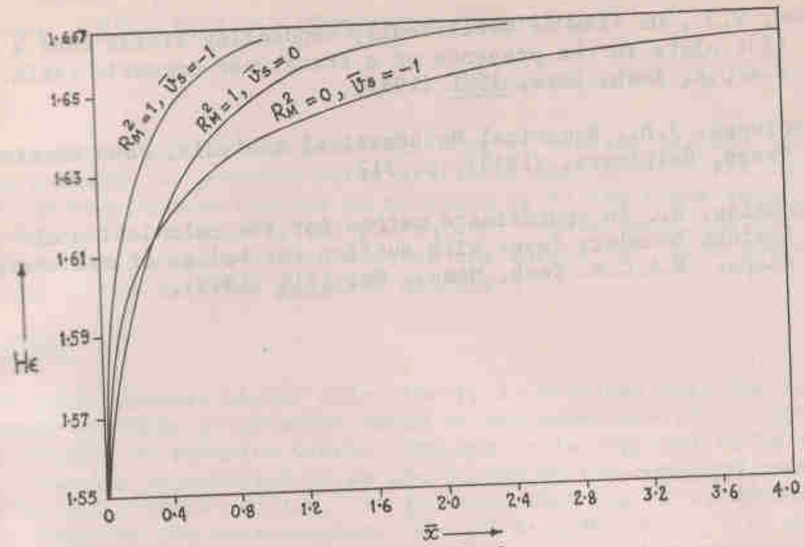

FIGURE-2

MAGNETOHYDRODYNAMIC BOUNDARY LAYER ALONG A SEMI-INFINITE POROUS FLAT PLATE IN THE PRESENCE OF TRANSVERSE MAGNETIC FIELD. VARIATION OF H_ϵ AGAINST $\bar{x}$ FOR DIFFERENT VALUES OF R_M^2 AND $\bar{U}_s$.

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A higher order theory of Interfacial Periodic Waves affected by Surface Tansion

Madan Pathak & S.R.P. Sinha

Abstract

The solution of the non-linear problem of finding surface tension affected periodic progressive waves of finite amplitude on the interface of two homogeneous liquids is obtained up to the first three approximations. The motion is two-dimensional irrotational and the liquids are unbounded on either side of the interface. Some physical properties of the solution are also discussed.

Introduction

The infinitesimal higher order theory is obtained here for two-dimensional periodic progressive waves on the interface of two homogeneous liquids of infinite width. The motion is supposed to be irrotational and the interfacial waves are caused by the combined action of gravity and surface tension. It is found that the solution has singularities for the wave-lengths, $\lambda = \sqrt{\nu}\cdot\lambda_0$, ($\nu = 2, 3, \ldots$), where λ_0 corresponds to the minimum value of c^2, as obtained in the linear theory, by the formula,

$$c^2 = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} \cdot \frac{g\lambda}{2\pi} + \frac{2\pi T}{(\rho_2 + \rho_1)\lambda}$$

in usual notations ([5], pp. 373-374).

It is noted that $\lambda = \sqrt{2}.\ \lambda_0$ is the bifurcation point of the solution when the second order terms are included, and the elevation $\gtreqless$ depression according as $\lambda \lesseqgtr \sqrt{2}\,\lambda_0$. Hence for $\lambda < \sqrt{2}\,\lambda_0$, the waves exhibit a purely capillary like character, and for $\lambda > \sqrt{2}.\lambda_0$, they have a pure gravity like nature.

Formulation of n the B.V.-problem

Let the wave profile ℓ with space period λ be progressing without change of form with constant velocity c horizontally from the right to the left direction. Let us have a coordinate frame OXY with OX coinciding with the horizontal line of separation of the liquids while at rest and directed opposite to that of the progressing waves. The coordinate axis OY is taken vertically upwards. Let the elements associated with the upper liquid be denoted by the subscript 1 and those with the lower liquid by 2. In particular, the density of the upper liquid is taken as ρ_1 and that of the lower by ρ_2 . Let us suppose that the motion starts from rest, so that there is rest with regard to OXY at an infinite distance from the wave-profile. The interfacial profile, in this case, has equation $Y = \eta(X + ct)$ and therefore the velocity potential of the flow of the upper and lower

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liquid will be like ϕ_1 (X+ct, Y) and ϕ_2 (X+ct, Y). The motion will then be steady with regard to a frame of reference oxy moving with the wave profile and will be determined by ϕ_1 (x,y), ϕ_2 (x,y) and $y=\eta(x)$; x=X +ct, y= Y . We take the ox- axis coincident with the OX- axis passing through a crest of the wave.

The wave motion will be determined from the following B.V.-problem: To determine ϕ_1 (x,y), ϕ_2 (x,y), $\eta(x)$ and c satisfying the Laplace's equation,

$$(1) \qquad \nabla^2\phi_1 = 0 \quad \text{in the region above the line l.}$$

$$\nabla^2\phi_2 = 0 \quad \text{in the region below the line l.}$$

and the boundary conditions,

$$(2) \qquad \left[\begin{array}{l} \eta_x \left(\frac{\partial\phi_1}{\partial x} + c\right) - \frac{\partial\phi_1}{\partial y} = 0 \\ \eta_x \left(\frac{\partial\phi_2}{\partial x} + c\right) - \frac{\partial\phi_2}{\partial y} = 0 \end{array}\right. \qquad \text{for } y = \eta$$

$$(3) \qquad \frac{\partial\phi_1}{\partial x} = \frac{\partial\phi_1}{\partial y} = 0 \qquad \text{for } y = +\infty$$

$$(4) \qquad \frac{\partial\phi_2}{\partial x} = \frac{\partial\phi_2}{\partial y} = 0 \qquad \text{for } y = -\infty$$

$$(5) \qquad c\left(\rho_2 \frac{\partial\phi_2}{\partial x} - \rho_1 \frac{\partial\phi_1}{\partial x}\right) + 1/2\, (\rho_2 q_2^2 - \rho_1 q_1^2) +$$

$$+ g\,(\rho_2 - \rho_1)\,\eta = T.\, \frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} \quad \text{for } y = \eta$$

where $$q_i^2 = \left(\frac{\partial\phi_i}{\partial x}\right)^2 + \left(\frac{\partial\phi_i}{\partial y}\right)^2, \qquad (i = 1,2).$$

T is the constant surface tension at the interface l and g is the acceleration due to gravity. ϕ_1, ϕ_2 and η are supposed to be periodic of x of a given period λ and the wave velocity c is to be determined in terms of λ .

Approximate solution of the B.V.-problem

We shall determine the solution of the B.V.-problem (1)-(5) by perturbation method, supposing the motion to be of perturbation from the state of rest. Following Stoke's method, we write

$$\phi_1 = \varepsilon.\phi_1^{(1)} + \varepsilon^2 \phi_1^{(2)} + \varepsilon^3 \phi_1^{(3)} + \ldots\ldots$$

$$\phi_1 = \varepsilon.\phi_2^{(1)} + \varepsilon^2 \phi_2^{(2)} + \varepsilon^3.\phi_2^{(3)} + \ldots\ldots \tag{6}$$

$$\eta = \varepsilon.\ \eta^{(1)} + \varepsilon^2\ \eta^{(2)} + \varepsilon^3\ \eta^{(3)} + \ldots\ldots$$

$$c = c_o + \varepsilon.\ c_1 + \varepsilon^2\ c_2 + \ldots\ldots$$

and determine $\phi_1^{(i)}$, $\phi_2^{(i)}$, $\eta^{(i)}$ and c_i after substituting the series (6) in the B.V.-problem (1)-(5), and equating the coefficients of like powers of ε. ε is a small parameter of qualitative nature of amplitude/wave length. In fact, we shall get a set of B.V.-problems for the successive approximations of the unknown functions and the constant c. In the first approximation we get,

$$\begin{aligned}
&\nabla^2\phi_1^{(1)} = 0 \quad \text{in } y \geqslant 0 \quad \text{and} \quad \nabla^2\phi_2^{(1)} = 0 \text{ in } y \leqq 0. \\
&c_o\eta_x^{(1)} = \phi_{1y}^{(1)} = \phi_{2y}^{(1)} \qquad , \text{ for } y = 0 \\
&c_o\ (\rho_2\ \phi_{2x}^{(1)} - \rho_1\ \phi_{1x}^{(1)}) + g\ (\rho_2 - \rho_1)\ \eta^{(1)} = T\,\eta_{xx}^{(1)}, \text{ for } y=0 \\
&\phi_{1x}^{(1)} = \phi_{1y}^{(1)} = 0 \qquad \text{for } y = +\infty \\
&\phi_{2x}^{(1)} = \phi_{2y}^{(1)} = 0 \qquad \text{for } y = -\infty
\end{aligned} \tag{7}$$

In the second approximation we find the following B.V.-problem:

$$\begin{aligned}
&\nabla^2\phi_1^{(2)} = 0 \quad \text{in } y \geqslant 0 \quad \text{and} \quad \nabla^2\phi_2^{(2)} = 0 \quad \text{in } y \leqq 0 \\
&c_o\eta_x^{(2)} = \phi_{1y}^{(2)} + \phi_{1yy}^{(1)}\ \eta^{(1)} - (\phi_{1x}^{(1)} + c_1)\ \eta_x^{(i)} \\
&\qquad = \phi_{2y}^{(2)} + \phi_{2yy}^{(1)}\ .\ \eta^{(1)} - (\phi_{2x}^{(1)} + c_1)\ \eta_x^{(1)} \quad \text{for } y = 0 \\
&\phi_{1x}^{(2)} = \phi_{1y}^{(2)} = 0 \qquad \text{for } y = +\infty \\
&\phi_{2x}^{(2)} = \phi_{2y}^{(2)} = 0 \qquad \text{for } y = -\infty
\end{aligned} \tag{8}$$

$$c_o(\rho_2\phi_{2x}^{(2)} - \rho_1\phi_{1x}^{(2)}) + g\,(\rho_2 - \rho_1)\,\eta^{(2)} - T\eta_{xx}^{(2)}$$

$$= -c_1\,(\rho_2\,\phi_{2x}^{(1)} - \rho_1\,\phi_{1x}^{(1)}) - c_o\,(\rho_2\,\phi_{2yx}^{(1)} - \rho_1\,\phi_{1yx}^{(1)})\,\eta^{(1)}$$

$$- \tfrac{1}{2}\left\{\rho_2\,(\phi_{2x}^{(1)2} + \phi_{2y}^{(1)2}) - {}_1\,(\phi_{1x}^{(1)2} + \phi_{1y}^{(1)2})\right\} \quad \text{for } y = 0.$$

For the third approximation we get the following equations and the boundary conditions:

$$\nabla^2\phi_1^{(3)} = 0 \text{ in } y \geqslant 0\,, \quad \nabla^2\phi_2^{(3)} = 0 \quad \text{in} \quad y \leqslant 0\,,$$

$$c_o\eta_x^{(3)} - \phi_{1y}^{(3)} = \phi_{1yy}^{(1)}\,\eta^{(2)} + \tfrac{1}{2}\,\phi_{1yyy}^{(1)}\,\eta^{(1)2} + \phi_{1yy}^{(2)}\,\eta^{(1)}$$

$$- (\phi_{1x}^{(1)} + c_1)\,\eta_x^{(2)} - (\phi_{1yx}^{(1)}\,\eta^{(1)} + \phi_{1x}^{(2)} + c_2)\,\eta_x^{(1)} \quad \text{for } y = 0$$

$$c_o\,{}_x^{(3)} - \phi_{2y}^{(3)} = \phi_{2yy}^{(1)}\,\eta^{(2)} + \tfrac{1}{2}\,\phi_{2yyy}^{(1)}\cdot\eta^{(1)2} + \phi_{2yy}^{(2)}\,\eta^{(1)}$$

$$- (\phi_{2x}^{(1)} + c_1)\,\eta_x^{(2)} - (\phi_{2yx}^{(1)}\,\eta^{(1)} + \phi_{2x}^{(2)} + c_2)\,\eta_x^{(1)} \quad \text{for } y = 0$$

(9) $$c_o\,(\rho_2\,\phi_{2x}^{(3)} - \rho_1\,\phi_{1x}^{(3)}) + g\,(\rho_2 - \rho_1)\,\eta^{(3)} - T\,\eta_{xx}^{(3)} = \tfrac{3}{2}\cdot T\cdot\eta_x^{(1)2}\cdot\eta_{xx}^{(1)}$$

$$- c_o[\rho_2\left\{\phi_{2yx}^{(1)}\,\eta^{(2)} + \tfrac{1}{2}\,\phi_{2yyx}^{(1)}\cdot\eta^{(1)2} + \phi_{2yx}^{(2)}\,\eta^{(1)}\right\}$$

$$- \rho_1\left\{\phi_{1yx}^{(1)}\,\eta^{(2)} + \tfrac{1}{2}\,\phi_{1yyx}^{(1)}\cdot\eta^{(1)2} + \phi_{1yx}^{(2)}\,\eta^{(1)}\right\}]$$

$$- c_1[\rho_2\,(\phi_{2yx}^{(1)}\,\eta^{(1)} + \phi_{2x}^{(2)}) - \rho_1\,(\phi_{1yx}^{(1)}\cdot\eta^{(1)} + \phi_{1x}^{(2)})]$$

$$- c_2\,(\rho_2\,\phi_{2x}^{(1)} - \rho_1\,\phi_{1x}^{(1)})$$

$$- [\rho_2\left\{\phi_{2x}^{(1)}\,(\phi_{2yx}^{(1)}\cdot\eta^{(1)} + \phi_{2x}^{(2)}) + \phi_{2y}^{(1)}\,(\phi_{2yy}^{(1)}\cdot\eta^{(1)} + \phi_{2y}^{(2)})\right\}$$

$$- \rho_1\left\{\phi_{1x}^{(1)}\,(\phi_{1yx}^{(1)}\cdot\eta^{(1)} + \phi_{1x}^{(2)}) + \phi_{1y}^{(1)}\,(\phi_{1yy}^{(1)}\cdot\eta^{(1)} + \phi_{1y}^{(2)})\right\}]$$

$$\ldots \quad \text{for } y = 0$$

$$\phi_{1x}^{(3)} = \phi_{1y}^{(3)} = 0 \qquad \text{for } y = +\infty$$

$$\phi_{2x}^{(3)} = \phi_{2y}^{(3)} = 0 \qquad \text{for } y = -\infty$$

In the first approximation we find the following periodic solution with period $\lambda = 2\pi/_k$:

$$\text{(10)} \quad \begin{aligned} \phi_1^{(1)} &= \frac{c_o}{k} \,.\, e^{-ky} \sin kx \\ \phi_2^{(1)} &= -\frac{c_o}{k} \cdot e^{ky} \,.\, \sin kx \\ \eta^{(1)} &= \frac{1}{k} \cdot \cos kx \\ c_o^2 &= \frac{T.k}{\rho_2+\rho_1} + \frac{(\rho_2-\rho_1)}{(\rho_2+\rho_1)} \cdot \frac{g}{k} \,. \end{aligned}$$

We find that (c_o^2, λ) - graph is a hyperbola with one of its asymptotes. $\lambda = 0$. c_o^2 is minimum for

$$\lambda = \lambda_o = 2\pi \sqrt{\frac{T}{g(\rho_2-\rho_1)}}$$

$$(c_o^2)_{min} = \frac{\sqrt{4gT\ (\rho_2-\rho_1)}}{(\rho_2+\rho_1)} \,.$$

The unknowns in the second approximation can now be obtained from (8). The unknowns in the third approximation can also be determined when they are known in the first and second approximations. We present below the final solution without presenting any details of the calculation.

$$\text{(11)} \quad \begin{aligned} \phi_1(x,y) &= \varepsilon \,.\, \frac{c_o}{k} \,.\, e^{-ky} \,.\, \sin kx + \varepsilon^2 \,.\, B_2^{(2)} \,.\, e^{-2ky} \,.\, \sin 2kx \\ &\quad + \varepsilon^3 \,.\, Q_3^{(3)} \,.\, e^{-3ky} \,.\, \sin 3kx + \cdots \cdots \\ \phi_2(x,y) &= -\varepsilon \,.\, \frac{c_o}{k} \,.\, e^{ky} \,.\, \sin kx + \varepsilon^2 \,.\, D_2^{(2)} \,.\, e^{2ky} \,.\, \sin 2kx \\ &\quad + \varepsilon^3 \,.\, (N_1^{(3)} \,.\, e^{ky} \,.\, \sin kx + N_3^{(3)} \,.\, e^{3ky} \,.\, \sin 3ky) + \cdots \cdots \end{aligned}$$

$$\eta^{(x)} = \varepsilon \,.\, \frac{1}{k} \,.\, \cos kx + \varepsilon^2 \,.\, \left(\frac{s}{d} - \frac{1}{2k}\right) \cos 2kx$$

$$+ \varepsilon^3 \,.\, \left[\left(\frac{5}{8k} - \frac{s}{2d} - \frac{c_2}{c_o k}\right) \cos kx \right.$$

$$\left. + \left(\frac{3}{8k} + \frac{1}{c_o} \,.\, Q_3^{(3)} - \frac{3s}{2d}\right) \cos 3kx \right] + \ldots$$

and

$$c = c_o \left[1 + \varepsilon^2 \,.\, \frac{c_2}{c_o} + \ldots \right],$$

where

$$s = 2\rho_2 (\rho_2 - \rho_1) \,.\, \frac{g}{k} - Tk (\rho_2 + 3\rho_1)$$

$$d = \left\{ 2g (\rho_2 - \rho_1) - 4k^2 T \right\} (\rho_2 + \rho_1)$$

$$c_o^2 = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} \,.\, \frac{g}{k} + \frac{Tk}{\rho_2 + \rho_1} \quad , \; A = Tk^2 + g (\rho_2 - \rho_1)$$

$$c_2 = \frac{c_o}{4A (\rho_2 + \rho_1)} \,.\, \left[Tk^2 \left\{ \frac{8ks}{d} \,.\, (2\rho_2 - \rho_1) - \frac{1}{4} (47\rho_2 - 3\rho_1) \right\} \right.$$

$$\left. + g \left\{ \frac{2ks}{d} \,.\, (7\rho_2 - 5\rho_1) - \frac{1}{4} (34\rho_2 - 16\rho_1) (\rho_2 - \rho_1) \right\} \right]$$

$$E = gk (\rho_2 - \rho_1) - 3Tk^3$$

$$Q_3^{(3)} = \frac{c_o}{8E} \left[\frac{Tk2}{4(\rho_2 + \rho_1)} \,.\, \left\{ \frac{ks}{d} \,.\, (1015\rho_2 - 41\rho_1) - 2 (19\rho_2 + \rho_1) \right\} \right.$$

$$\left. + g \left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}\right) \,.\, \left\{ 4ks (67\rho_2 + \rho_1) - (8\rho_2 - \rho_1) \right\} \right]$$

$$D_2^{(2)} = \frac{c_o}{d} \left\{ 2\rho_1 (\rho_2 - \rho_1) \,.\, \frac{g}{k} - Tk (3\rho_2 + \rho_1) \right\}$$

$$N_1^{(3)} = \frac{3c_o}{2k} \left(\frac{2ks}{d} - 1\right)$$

$$N_3^{(3)} = \frac{3c_o}{2k} \left(18 \frac{ks}{d} - 1\right) - \frac{c_o}{8E} \left[\frac{Tk^2}{4(\rho_2 + \rho_1)} \,.\, (1015\rho_2 - 41\rho_1) \frac{ks}{d} \right.$$

$$- 2\ (19\rho_2 + \rho_1) \} + \frac{g(\rho_2-\rho_1)}{(\rho_2+\rho_1)} \left\{ \frac{4ks}{d}.\ (67\rho_2+\rho_1) - (8\rho_2-\rho_1) \right\}]$$

and

$$B_2^{(2)} = \frac{c_o s}{d}$$

Properties of the wave motion

The elevation 'α' and the depression 'δ' of the wave profile above and below the ox- axis are given by

$$(12)\quad \alpha = \eta(x)\Big|_{x=o} = \varepsilon.\frac{1}{k} + \varepsilon^2.\ (\frac{s}{d} - \frac{1}{2k})$$
$$+ \varepsilon^3 \left\{ \frac{1}{k} - \frac{2s}{d} - (c_2 - k\ Q_3^{(3)}).\ \frac{1}{c_o k} \right\} + \ldots$$

$$(13)\quad - \delta = - \eta(x)\Big|_{x=\pi/k} = \varepsilon.\frac{1}{k} - \varepsilon^2.\ (\frac{s}{d} - \frac{1}{2k})$$
$$+ \varepsilon^3 \left\{ \frac{1}{k} - \frac{2s}{d} - (c_2 - k\ Q_3^{(3)}).\ \frac{1}{c_o k} \right\} + \ldots$$

From (12) we easily

$$\varepsilon = \frac{2\pi\alpha}{\lambda} - (\frac{2\pi\alpha}{\lambda})^2\ (\frac{ks}{d} - \frac{1}{2})$$
$$(14)\qquad + (\frac{2\pi\alpha}{\lambda})^3\ .\ [2(\frac{ks}{d} - \frac{1}{2})^2 - \left\{ 1 - \frac{2ks}{d} - (c_2 - k\ Q_3^{(3)}).\frac{1}{c_o} \right\}] + \ldots$$

From (11) and (14) we obtain,

$$= (\frac{2\pi\alpha}{\lambda}).\ \frac{\cos kx}{k} + (\frac{2\pi\alpha}{\lambda})^2 \left\{ - (\frac{s}{d} - \frac{1}{2k})\ \cos kx + (\frac{s}{d} - \frac{1}{2k})\ \cos 2kx \right\}$$
$$+ (\frac{2\pi\alpha}{\lambda})^2\ [\left\{ - \frac{3}{8} + \frac{3ks}{2d} + 2\ (\frac{ks}{d} - \frac{1}{2})^2 - \frac{k\ Q_3^{(3)}}{c_o} \right\} \frac{\cos kx}{k}$$

$$-2.\ \left(\frac{ks}{d} - \frac{1}{2}\right)^2 .\ \frac{\cos kx}{k} + \left(\frac{3}{8} - \frac{3ks}{2d} + \frac{k.Q_3^{(3)}}{c_o}\right).\ \frac{\cos 3kx}{k}] + \dots$$

Next we find the position of a particle at time t whose coordinates in the state of rest were (x_o, y_o).

For $y_o > 0$, they are given by

$$x(t) = x_o + \mathcal{E}.\ \frac{e^{-ky_o}}{k}.\ \sin k\ (x_o + c_o t)$$

$$= \mathcal{E}^2 . \left\{ - c_o t.\ e^{-2ky_o} + \frac{s}{d}\ .e^{-2ky_o}.\ \sin 2k\ (x_o + c_o t) \right\} + \dots$$

$$y(t) = y_o + \mathcal{E}.\ \frac{e^{-ky_o}}{k}.\ \cos k\ (x_o + c_o t)$$

$$+ \mathcal{E}^2 .\ \frac{s}{d}.\ e^{-2ky_o}.\ \cos 2k\ (x_o + c_o t) + \dots$$

and for $y_o < 0$ i.e., for the particles of the lower liquid,

$$x(t) = x_o - \mathcal{E}.\ \frac{e^{ky_o}}{k}\ .\ \sin k\ (x_o + c_o t)$$

$$+ \mathcal{E}^2\ - c_o t\ .\ e^{2ky_o} + \left(\frac{1}{k} - \frac{s}{d}\right)\ e^{2ky_o}.\ \sin 2k\ (x_o + c_o t) + \dots$$

$$y(t) = y_o + \mathcal{E}\ .\ \frac{e^{ky}}{k}\ .\ \cos k\ (x_o + c_o t)$$

$$- \mathcal{E}^2 .\ \left(\frac{1}{k} - \frac{s}{d}\right)\ e^{2ky_o}.\ \cos 2k\ (x_o + c_o t) + \dots$$

We find that the inclusion of second order term in the expressions for x and y amounts to a kinematical mass drift with velocity

$c_o.\ e^{-2ky_o}$, $(y_o > 0)$ or $c_o.\ e^{2ky_o}$, $(y_o < 0)$ in the direction of the progressive interfacial waves. The mass transport vanishes as $y_o \to \infty$ for particles of the upper liquid and as $y_o \to -\infty$ for those of the lower liquid.

After simple computations we find that the coefficient of $\mathcal{E}^2$ in the expansion (12) of 'α' is given by

$$\frac{s}{d} - \frac{1}{2k} = \frac{A\ (\rho_2 - \rho_1)}{4\pi\ (\rho_2 + \rho_1)}\ .\ \frac{\lambda^3\ .\ \lambda_o^2}{(\lambda^2 - 2\ \lambda_o^2)\ T}$$

where λ_0 is the value of λ for which c_0^2 is minimum. We find that upto the s second approximation, $\alpha < -\bar{c}$ when $\lambda < \sqrt{2}\,\lambda_c$. This means that the effect of inclusion of higher order terms is to lower the through and to broaden and lower the crest. This happens in the case of pure capillary waves ([2]; [5], pp. 653-667). But for $\lambda > \sqrt{2}\,\lambda_0$ this effect is just the reverse and is similar to that in the case of pure gravity waves.

For $\lambda = \sqrt{2}.\ \lambda_0$ there should be a reversal of curvature at the middle of a flat portion of the waves. Proceeding on the lines of Wilton (161) one may see the possibility of such a reversal.

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Invariant Submanifolds of f(3,-1)-Structure Manifolds

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1. Introduction

Let V^m be an m-dimensional C^∞ Riemannian manifold imbedded in an n-dimensional C^∞ Riemannian manifold M^n, where $m < n$; the imbedding being denoted by $\emptyset : V \longrightarrow M$. Let B be the mapping induced by $\emptyset$, that is, $B = d\emptyset : T(V) \longrightarrow T(M)$, where T(V) and T(M) are tangent bundles of V and M respectively. On putting T(V,M) as the set of all null vectors tangent to $\emptyset(V)$, we have [2], B: $T(V) \longrightarrow T(V,M)$ to be an isomorphism. The set of all vectors normal to $\emptyset(V)$ forms a vector bundle N(V,M) over $\emptyset(V)$ which is called the normal bundle of V. The vector bundle induced by $\emptyset$ from N(V,M) is denoted by N(V). We denote by $C: N(V) \longrightarrow N(V,M)$ the natural isomorphism.

Let us now introduce the following notations:

Let $\mathcal{T}^r_s(V)$ be the space of all C^∞ tensor fields of the type (r,s), that is of contravariant order r and covariant order s, associated with T(V). $\mathfrak{U}^r_s(V)$ is the space of all differentiable functions defined on V^m. An element of $\mathcal{T}^1_o(V)$ is a vector field on V^m. An element of $\mathfrak{U}^1_o(V)$ is a vector field normal to V^m.

Let $\overline{X}$ and $\overline{Y}$ be vector fields defined along $\emptyset(V)$ and tangential to $\emptyset(V)$. Let $\tilde{X}$ and $\tilde{Y}$ be the local extensions of $\overline{X}$ and $\overline{Y}$. Then $[\tilde{X},\tilde{Y}]$ is a vector field tangential to M^n and its restriction $[\tilde{X},\tilde{Y}]/\emptyset(V)$ to $\emptyset(V)$ is determined independently from the choice of these local extensions $\tilde{X}$ and $\tilde{Y}$. Therefore, we can define $[\overline{X},\overline{Y}]$ by

$$(1.1) \qquad [\overline{X},\overline{Y}] = [\tilde{X},\tilde{Y}]/\emptyset(V).$$

Since B is an isomorphism,

$$(1.2) \qquad [BX, BY] = B[X,Y],$$

holds for all X, Y $\in \mathcal{T}^1_o(V)$.

Let $\tilde{G}$ be the Riemannian metric tensor of M^n and we define g and g* on V^m and N(V) respectively as follows:

$$g(X_1, X_2) = \tilde{G}(BX_1, BX_2)o\emptyset$$

(1.3) and

$$g^*(N_1, N_2) = \tilde{G}(CN_1, CN_2),$$

for all $X_1, X_2 \in \mathcal{T}^1_o(V)$ and $N_1, N_2 \in \mathcal{U}^1_o(V)$.

It can be easily verified that g is a Riemannian metric tensor in V^m which is called the induced metric tensor of V^m and g^* is a tensor field defining an inner product in N(V). The tensor g^* is called the induced metric of N(V).

Let $\widetilde{\nabla}$ be the Riemannian connexion determined by G in M^n, then $\widetilde{\nabla}$ induces a connexion $\widetilde{\nabla}$ in $\emptyset(V)$ defined by [3]:

$$\widetilde{\nabla}_{\overline{X}} \overline{Y} = (\widetilde{\nabla}_{\widetilde{X}} \widetilde{Y})/ \emptyset(V), \tag{1.4}$$

where $\overline{X}$ and $\overline{Y}$ are arbitrary C^∞ vector fields defined along $\emptyset(V)$ and tangential to $\emptyset(V)$. Thus taking account of (1.1), we have

$$\widetilde{\nabla}_{\overline{X}} \overline{Y} - \widetilde{\nabla}_{\overline{Y}} \overline{X} = [\overline{X},\overline{Y}]. \tag{1.5}$$

In this paper we shall assume that M^n is a C^∞ f(3,-1)-structure manifold with structure tensor $\widetilde{f}$ of type (1,1). Let $\widetilde{L}$ and $\widetilde{M}$ be the complementary distributions corresponding to the projection operations $\widetilde{l}$ and $\widetilde{m}$ respectively, where

$$l \overset{\text{def}}{=\!=\!=} \widetilde{f}^{\,2}, \quad m \overset{\text{def}}{=\!=\!=} I - \widetilde{f}^{\,2}, \tag{1.6}$$

I denotes the identity operator. These projection operators satisfy the following relations [4]:

$$\begin{aligned} &(a) \quad \widetilde{f}\,\widetilde{l} = \widetilde{l}\,\widetilde{f} = \widetilde{f}\,, \qquad (b)\ \widetilde{f}\,\widetilde{m} = \widetilde{m}\,\widetilde{f} = 0 \\ (1.7) \qquad &(c) \quad f^2 = \widetilde{l}\,\widetilde{f}^{\,2} = \\ \text{and} \qquad &(d) \quad \widetilde{f}^{\,2}\,\widetilde{m} = \widetilde{m}\,\widetilde{f}^{\,2} = 0\,, \ \widetilde{l}\,\widetilde{m} = \widetilde{m}\,\widetilde{l} = 0. \end{aligned}$$

Such a manifold M^n always admits a Riemannian metric say $\widetilde{G}$ which satisfies the following relations:

(i) the distribution $\widetilde{L}$ and $\widetilde{M}$ are $\widetilde{G}$-orthogonal,

(ii) $\widetilde{G}(\widetilde{X},\widetilde{Y}) = \widetilde{G}(\widetilde{f}\,\widetilde{X}, \widetilde{f}\,\widetilde{Y}) + \widetilde{G}(\widetilde{m}\,\widetilde{X}, \widetilde{Y})$,

(iii) $\widetilde{G}(\widetilde{X}, \widetilde{f}\,\widetilde{Y}) = \widetilde{G}(\widetilde{f}\,\widetilde{X}, \widetilde{f}^{\,2}\,\widetilde{Y})$,

(iv) $\tilde{G}(\tilde{f}\,\tilde{X}, \tilde{Y}) = \tilde{G}(\tilde{f}^{\,2}\,\tilde{X}, \tilde{f}\,\tilde{Y})$,

for all $\tilde{X}, \tilde{Y} \in \mathcal{T}^1_o(M)$.

2. Invariant submanifolds in $f(3,-1)$-structure manifold

Let V^m be a C^∞, m-dimensional manifold, imbedded as a submanifold in a C^∞ $f(3,-1)$-structure manifold M^n with (1,1) structure tensor $\tilde{f}$. V^m is defined to be an invariant submanifold of M^n if the tangent space $T_p(\emptyset(V))$ of $\emptyset(V)$ is invariant by the linear mapping $\tilde{f}$ at each point p of $\emptyset(V)$. Throughout this paper we shall assume V^m to be an invariant submanifold of M^n, so that, for $X \in \mathcal{T}^1_o(V)$, we have;

$$(2.1)\qquad \tilde{f}BX = BX^o, \text{ for some } X^o \in \mathcal{T}^1_o(V).$$

Thus we define a (1,1) tensor field f in V^m, that is a mapping $f:\mathcal{X}(V) \longrightarrow \mathcal{X}(V)$ by $fX=X^o$.

From (2.1) we obtain

$$(2.2)\qquad \tilde{f}\,(BX) = Bf(X).$$

Let $\tilde{N}$ and N be the Nijenhuis tensor of M^n and V^m determined by the (1,1) tensor field $\tilde{f}$ and f respectively.

Thoerem 2.1. The Nijenhuis tensor $\tilde{N}$ and N of M^n and V^m respectively are related as follows:

$$(2.3)\qquad \tilde{N}\,(BX,\ BY) = BN(X,Y).$$

Proof: By means of (1.2) and (2.2) we get

$$\tilde{N}(BX,BY) = [\tilde{f}BX,\ \tilde{f}BY] - \tilde{f}\,[BX,\ \tilde{f}\,BY] - \tilde{f}\,[\tilde{f}\,BX,\ BY] + \tilde{f}^{\,2}\,[BX,\ BY],$$

$$= \overline{[BfX,\ BfY]} - \tilde{f}\,[BX,\ BfY] - \tilde{f}\,[BfX,\ BY] + \tilde{f}^{\,2}\,[BX,\ BY],$$

$$= B\left\{[fX,\ fY] - f\,[X,\ fY] - f\,[fX,\ Y] + f^2\,[X,\ Y]\right\},$$

$$= BN(X,\ Y).$$

Cases: For any invariant submanifold V in an $f(3,-1)$-structure manifold M, we consider the following two cases:

(i) The distribution $\tilde{M}$ is never tangential to $\emptyset(V)$, that is, to any vector field of the type $\tilde{m}\,\tilde{X}$ where $\tilde{X}$ is a vector field tangential to $\emptyset(V)$. We shall show that in this case V is necessarily odd-dimensional.

(ii) The distribution $\widetilde{M}$ is always tangential to $\emptyset(V)$.

Let us first consider the case I.

The distribution $\widetilde{M}$ is never tangential to the invariant submanifold $\emptyset(V)$, implies that any vector field of the type $\widetilde{m}\,\widetilde{X}$ is independent of any vector field of the same frame BX, $X \in \mathcal{J}^1_o(V)$.

Applying $\widetilde{f}$ to (2.2), we obtain

$$B\, f^2\, X = \widetilde{f}^{\,2}\, BX.$$

Since vector fields of the type BX, $X \in \mathcal{J}^1_o(V)$ are in the distribution $\widetilde{L}$, we get

$$Bf^2X = BX,$$

by virtue of (1.7)c, which implies

$$f^2\, X = X. \tag{2.4}$$

Consequently (1,1) tensor field f in V is almost product structure, called the induced almost product structure on the invariant submanifold V.

Let us define a tensor field $\widetilde{S}$ of type (1,2) in M as follows:

$$\widetilde{S}(\widetilde{X}, \widetilde{Y}) = \widetilde{N}(\widetilde{X}, \widetilde{Y}) + \widetilde{\nabla}_{\widetilde{X}}\, (\widetilde{m}\,\widetilde{Y}) - \widetilde{\nabla}_{\widetilde{Y}}\, (\widetilde{m}\,\widetilde{X}) - \widetilde{m}\,[\widetilde{X}, \widetilde{Y}]. \tag{2.5}$$

for any vector fields X, $Y \in \mathcal{J}^1_o(M)$.

Theorem 2.2. The (1,2) tensor field $\widetilde{S}$ defined in M is given by

$$\widetilde{S}(BX, BY) = \widetilde{N}(BX, BY) = BN(X, Y) \tag{2.6}$$

for $X, Y \in \mathcal{J}^1_o(V)$

Proof. Since any vector field tangential to $\emptyset(V)$ is not contained in the the distribution $\widetilde{M}$, we have therefore, for any $X \in \mathcal{J}^1_o(V)$

$$\widetilde{m}\ (BX) = 0,$$

which in view of (2.3) and (2.5) yields

$$\widetilde{S}\ (BX, BY) = \widetilde{N}(BX, BY) = BN(X, Y).$$

Theorem 2.3. An invariant submanifold V is imbedded in an f(3,-1)-structure manifold, such that, the distribution $\widetilde{M}$ is never tangential

to $\phi(V)$ is an almost product manifold with induced almost product structure f. If the f(3,-1)-structure is normal, the invariant submanifold is almost product manifold, consequently the dimension of V is odd.

Proof: The proof follows easily by means of the theorem (2.2) and the equation (2.4).

Now we consider the Case II.

The distribution $\tilde{M}$ is always tangential to the invariant submanifold $\phi(V)$, implies that

$$\tilde{m}\, BX = BX^{o}, \quad X \in \mathcal{T}^{1}_{o}(V), \tag{2.7}$$

where X^{o} is some vector field in V.

Now we define (1,1) tensor field m in V such that $mX = X^{o}$.

Then (2.7) can be written as

$$\tilde{m}\, BX = BmX. \tag{2.8}$$

Also, we can define a (1,1) tensor field l on V by

$$l\, BX = B\, l X. \tag{2.8}$$

Thus (1,1) tensor field l in V is well defined, since in M the following relation holds

$$l + m = \tilde{I},$$

where I is an identity.

Theorem 2.4. The (1,1) tensor field m and l in V defined by (2.8) and (2.9) satisfy the following:

$$l + m = I,\ l\, m = m\, l = 0,\ \ l^{2} = l\ ,\ m^{2} = m. \tag{2.10}$$

Proof: We have

$$l + \tilde{m} = \tilde{I}.$$

Operating the above relation with $BX\,(X \in \mathcal{T}^{1}_{o}(V))$, we obtain

$$l\ BX + \tilde{m}\ BX = \tilde{I}\ BX \tag{2.11}$$

Thus by virtue of (2.8), (2.9) and (2.11), we get

$$B\,l\,X + BmX = BIX,$$

which yields,

$$l\,X + mX = X$$

or $\qquad l + m = I.$

Now operating $\tilde{l}\,\tilde{m} = \tilde{m}\,\tilde{l} = 0$ by BX, $X \in \mathcal{T}^1_o(V)$ and using (2.8) and (2.9), we obtain

$$B\,l\,mX = 0,$$

which shows that $l\,m = 0$.

Now by means of the relation $\tilde{l}^2 = \tilde{l}$, $\tilde{m}^2 = \tilde{m}$ and the equations (2.8) and (2.9), we obtain

$$l^2 = l,\ m^2 = m.$$

These relations show that (2.10) holds in V. Thus l and m are complementary projection operators in V, given by

$$l = f^2 \quad \text{and} \quad m = I - f^2.$$

This proves the theorem.

Now we have in view of (2.2) for $X \in \mathcal{T}^1_o(V)$,

$$B\,f^3\,X = \tilde{f}^3\,BX,$$

$$= \tilde{f}\,BX,$$

$$= BfX,$$

which yields [1].

(2.12) $\qquad f^3 - f = 0.$

Hence f acts as on f(3,-1)-structure on V, called induced f-structure on V. The Riemannian metric given by [4].

(2.13) $\qquad g(X, Y) = g(fX, fY) + g(mX, Y)$

also holds for V.

Let ∇ be an operator in V defined by

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On N-Ring

Nirmesh Kumar

In this paper, our purpose is to define a new type of algebraic structure. We shall call it N-Ring.

An N-ring is a mathematical system $(R, \oplus, *)$ consisting of a non-empty set R and two binary operations $\oplus$ and $*$ defined on R such that

(1) $(R, \oplus)$ is a B-monoid, i.e., $(R, \oplus)$ is a groupoid and $(a) \oplus (b \oplus c) = (c) \oplus (b \oplus a)$

(2) $(R, *)$ is a groupoid

(3) $(a * b) \oplus (c * d) = (a \oplus c) * (b \oplus d)$ for all a, b, c and $d \in R$. We consider the following example.

Example 1 : We consider the set

$$Q = \left\{ \frac{p}{q} \;\middle|\; p \text{ and } q \text{ are non zero positive integers} \right\}$$

We define two binary operations $\oplus$ and $*$ on R as follows:

$$\frac{p_1}{q_1} \oplus \frac{p_2}{q_2} = \frac{p_1 p_2}{q_1 q_2}$$

and

$$\frac{p_1}{q_1} \oplus \frac{p_2}{q_2} = \frac{p_1 q_2}{q_1 p_2}$$

Q, obviously, satisfies postulates (1), (2) and (3) So, $(Q, \oplus, *)$ is an N-Ring.

N-ring with zero element: If there exists an element $0 \in R$ such that

$$a \oplus 0 = a = 0 \oplus a, \text{ for all } a \in R$$

we say that $(R, \oplus, *)$ is an N-ring with zero element.

N-ring with identity: If there exists an element $1 \in R$ such that

$$1 * a = a * 1 = a \text{ for all } a \in R$$

we say that $(R, \oplus, *)$ is an N-ring with identity.

N.B. (1) In example (1), $1/1 = 1$ is the zero element of $(Q, \oplus, *)$.

(2) The identity element is also $1/1 = 1$.

(3) The zero element as well as the identity element of an N-ring, if it exists, is unique.

(4) In case of N-rings $a * 0 = 0$ is not necessarily true, but $(a * 0) \oplus (0 * a) = a^2$ always holds good.

(5) An N-ring $(R, \oplus, *)$ with zero element is a ring if the elements of R are idempotent, $*$ is associative in R and every element of R has a unique inverse under $\oplus$.

Evidently the inclusion of zero element in R ensures the commutativity and associativity of $\oplus$ and idempotency together with axiom (3) ensures that the distributive laws hold good.

Homomorphism of N-Rings

Let $(R, \oplus, *)$ and $(S, \oplus', *')$ be two N-rings and $f : R \longrightarrow S$, then f is said to be a homomorphism iff

$$f(a \oplus b) = f(a) \oplus' f(b) \text{ for all } a, b \in R$$

$$f(a * b) = f(a) *' f(b) \text{ for all } a, b \in R$$

Now, we shall prove some results relating to homomorphism of N-rings. Before proving these results, we prove the following lemmas:

Lemma 1: For any two elements a, b of an N-ring $(R, \oplus, *)$, $(na) * (nb) = n(a * b)$ for every integer $n > 0$.

Proof: We prove this result by mathematical induction. For $n=2$, the result is obviously true.

We suppose that the result is true for $n=r$, i.e.

$$(ra) * (rb) = r(a * b)$$

We shall prove that the result holds for $n=r + 1$

$$[(r+1)a] * [(r+1)b] = [ra * rb] \oplus [a * b] \text{ by axiom (3) } \ldots\ldots$$
$$= (r+1)\ (a * b)$$

Thus, by mathematical induction the result is true for all values of n.

Lemma 2: Under the condition of lemma 1,

$$n(a \oplus b) = na \oplus nb$$

This can also be proved by mathematical induction.

Theorem 1: Let $(R, \oplus, *)$ be an N-ring. Then the complex

$$n\,M = \{\, na \mid a \in R \text{ and } n \text{ is a fixed positive integer} \,\}$$

forms an N-ring under the operations $\oplus$ and $*$. Consequently the mapping $a \rightarrow na$ is a homomorphism

Proof: (1) For all $na, nb \in nM$

$$na \oplus nb = n(a \oplus b) \in nM$$

Also, $na \oplus [nb \oplus nc] = n\,[a \oplus (b \oplus c)] = (nc) \oplus [(nb) \oplus (na)]$

So, $(nM, \oplus)$ is a B-monoid.

(2) For all $na, nb \in nM$

$$(na) * (nb) = n(a * b) \in nM$$

So, $(nM, *)$ is a groupoid

(3) $[(na) * (nb)] \oplus [(nc) * (nd)]$

$$= n\,[(a*b) \oplus (c*d)] = [n(a \oplus c)] * [n(b \oplus d)]$$

$$= [(na) \oplus (nc)] * [(nb) \oplus (nd)]$$

Thus, $(nM, \oplus, *)$ is an N-ring.

We now consider the mapping

$$\emptyset : a \in R \longrightarrow na \in nM$$

It is easy to see that

$$\emptyset(a \oplus b) = n(a \oplus b) = na \oplus nb = \emptyset(a) \oplus \emptyset(b)$$

and,

$$\emptyset(a * b) = n(a * b) = \emptyset(a) * \emptyset(b)$$

This completes the proof of the theorem.

Theorem 2: Let $(R, \oplus, *)$ and $(S, \oplus', *')$ be any two N-rings and f and g be two homomorphisms from $R \longrightarrow S$, then $f \oplus g$ is also a homomorphism where $f \oplus g$ is defined by

$$(f \oplus g)\,x = f(x) \oplus' g(x) \text{ for all } x \in R$$

Proof: For any $x, y \in R$

$$(f \oplus g)\ (x \oplus y)$$

$$= (f(x \oplus y) \oplus' g(x \oplus y) = [f(x) \oplus' f((y)] \oplus' [g(x) \oplus' g(y)]$$

$$= [f \oplus g]\ (x) \oplus' [f \oplus g]\ (y)$$

Further,

$$(f \oplus g)\ (x * y) = f(x * y) \oplus' g(x * y)$$

$$= [f(x) *' f(y)] \oplus' [g(x) *' g(y)]$$

$$= [f(x) \oplus' g(x)] *' [f(y) \oplus' g(y)]$$

$$= [f \oplus g]\ (x) *' [f \oplus g]\ (y)$$

Theorem 3: The set F of all endomorphisms of an N-ring $(R, \oplus, *)$ is closed and consequently, it is an N-ring if $*$ is symmetric in R.

Proof: For all $x, y \in R$ and for all $f, g \in F$

$$(f \oplus g)\ (x \oplus y)$$

$$= f(x \oplus y) \oplus g(x \oplus y) = [f(x) \oplus f(y)] \oplus [g(x) \oplus g(y)]$$

$$= [f \oplus g]\ (x) \oplus [f \oplus g]\ (y)$$

Also,

$$[f \oplus g]\ (x * y) = [f(x * y)] \oplus [g(x * y)]$$

$$= [f(x) * f(y)] \oplus [g(x) * g(y)] = [f(x) \oplus g(x)] * [f(y) * g(y)]$$

$$= [f \oplus g]\ (x) * [f \oplus g]\ (y)$$

It may be verified that the set of all endomorphisms is a B-monoid. We, now, define the other operation in R as follows:

$$(f * g)\ (x) = f(x) * g(x) \text{ for all } x \in R$$

Evidently, $(F, *)$ is a groupoid.

Further, for any $f_1, f_2, f_3, f_4 \in F$ and for all $x \in R$

$$[(f_1 * f_2) \oplus (f_3 * f_4)]\ (x) = (f_1 * f_2)x \oplus (f_3 * f_4)\ x$$

$$= [f_1(x) * f_2(x)] \oplus [f_3(x) * f_4(x)]$$

$= \quad [f_1(x) \oplus f_3(x)] * [f_2(x) \oplus f_4(x)]$

$= \quad (f_1 \oplus f_3)(x) * (f_2 \oplus f_4(x)$

$= \quad [(f_1 \oplus f_3) * (f_2 \oplus f_4)](x)$

So, $(F, \oplus, *)$ is an N-ring.

This completes the proof of the theorem.

Now, we define 'Sub-N-ring' of an N-ring.

Definition: Let $(R, \oplus, *)$ be an N-ring and $S \subseteq R$ be a non empty subset of R. If the system $(S, \oplus, *)$ is itself an N-ring, then $(S, \oplus, *)$ is said to be a sub-N-ring of the N-ring $(R, \oplus, *)$.

Example 1: We have seen that $(Q, \oplus, *)$ is an N-ring. Consider, now, the set

$$Q^* = \left\{ \frac{p^2}{q^2} \;\middle|\; p \text{ and } q \text{ are non zero positive integers} \right\}$$

Evidently Q^* is a subset of Q. It may be easily verified that $(Q^*, \oplus, *)$ is a sub-N-ring of the N-ring $(Q, \oplus, *)$.

Example 2: If Z is the set of integers, then $(Z, +, -)$, where + and - are ordinary addition and subtraction of integers, is an N-ring. Also Z^*, the set of even integer including 0, form an N-ring under + and -. Consequently $(Z^*, +, -)$ is a sub-N-ring of the N-ring $(Z, +, -)$.

Now we enumerate a theorem which gives a set of necessary and sufficient condition for a non-empty subset S to be a sub-N-ring of an N-ring.

Theorem 4: Let R be an N-ring and $S \subseteq R$. Then, S is a sub-N-ring of R iff

(1) $a, b \in S \Longrightarrow a \oplus b \in S$

(2) $a, b \in S \Longrightarrow a * b \in S$.

The proof is immediate.

Every N-ring R (without zero element) has one obvious sub-N-ring, namely, the set R itself. This sub-N-ring is usually referred as trivials sub-N-ring of R; all other sub-N-rings (if any exist) are called non trivial.

Now we define 'N-ideal' of an N-ring.

Definition: A sub-N-ring K of the N-ring R is said to be a left N-ideal of R if $r \in R$ and $k \in K$ imply $r * k \in K$.

In a similar manner, we may define right N-ideal. When $r \in R$ and $k \in K$ imply both $r * k$ and $k * r \in K$, then K is said to be a two-sided N-ideal.

Example 1: Consider the set

$$R = \{ (p, q) \mid p \text{ and } q \text{ are non-zero positive integers} \}$$

We define the two binary operations $\oplus$ and $*$ in R as follows:

$$(p_1, q_1) \oplus (p_2, q_2) = (p_1 p_2, q_1 q_2)$$

and $$(p_1, q_1) * (p_2, q_2) = (p_1 q_2, q_1 p_2)$$

Evidently, $(R, \oplus, *)$ is an N-ring.

We, now, consider the set

$$K = \{ (2p, q) \mid p \text{ and } q \text{ are non-zero positive integers} \}$$

The system $(K, \oplus, *)$ is a right N-ideal of the N-ring $(R, \oplus, *)$ as for all $(2p, q) \in K$ and $(r, s) \in R$ imply

$$(2p, q) * (r, s) = (2ps, qr) \in K$$

Further, if we consider the set

$$K^* = \{ (2p, 2q) \mid p \text{ and } q \text{ are non-zero positive integers} \}$$

then $(K^*, \oplus, *)$ is a two sided N-ideal as

$$(r, s) * (2p, 2q) = (2rq, 2sp) \in K^*$$

and $(2p, 2q) * (r, s) = (2ps, 2qr) \in K^*$ for every $(r, s) \in R$.

Taking stock of Theorem 4, our current definition of a two sided N-ideal may be reformulated as follows:

Definition: Let K be a non-empty subset of an N-ring R. Then K is a two sided N-ideal iff

(1) $a, b \in K$ imply $a \oplus b \in K$

(2) $r \in R$, $k \in K$ imply both $r * k$ and $k * r \in K$

Theorem 5: Let $\{K_i\}$ be an arbitrary collection of N-ideal of the N-ring R, where i ranges over some index set. Then $\cap K_i$ is also an N-ideal of R.

The proof is immediate.

Theorem 6: If U and V are any two N-ideals of the N-ring R. Let

$$U \oplus V = \{ u \oplus v \mid u \in U, v \in V \}$$

then $U \oplus V$ is an N-ideal if $r \oplus r = r$ for every $r \in R$.

Proof: For any $u_1 \oplus v_1$ and $u_2 \oplus v_2 \in U \oplus V$

$$(u_1 \oplus v_1) \oplus (u_2 \oplus v_2)$$

$$= \quad (u_1 \oplus u_2) \oplus (v_1 \oplus v_2) \in U \oplus V$$

Also, for every $r \in R$ and $u \oplus v \in U \oplus V$

$$r * (u \oplus v)$$

$$= (r \oplus r) * (u \oplus v) = (r * u) \oplus (r * v) \in U \oplus V$$

Similarly, $(u \oplus v) * r \in U \oplus V$

So, $U \oplus V$ is an N-ideal of R.

This completes the proof of the theorem.

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Pythagoras and his Theorem

Girard Tarr

The name Pythagoras means different things to different people, for he has been known as "... the philosopher, the astronomer, the mathematician, ..., the saint, the prophet, the performer of miracles, the magician, and the charlatan." [1] Most people, however, remember him as "... the first genius and founder of Greek Mathematics." [2]

Pythagoras was born in Tyre on the island of Samos around 570 B.C. He lived on Samos with his family for about eighteen years. He then moved to Lesbos to live with his uncle. While in Lesbos he studied with Ferekid and after two years he moved on to Miletus where he studied physics and mathematics under Anaksimander and Thales. The latter, being near ninety years of age, directed Pythagoras to Egypt where, in 547 B.C., he entered the priest caste and studied under Sonchis. His studies in Egypt came to an abrupt end in 526 B.C. when the priests were captured by the Persian King Kambis and Pythagoras was sent as a captive to Babylon. At this time, Babylon was the center of world commerce and so also the crossroads of intellectual activity. Here Pythagoras had the opportunity to learn from the Bactrians, the Indians, the Jews, and the Chinese. He was freed after nine years of captivity in Babylon and returned to Samos. Pythagoras moved around for a few years looking for students and finally settled in Croton around 510 B.C. In Croton, Pythagoras found people eager to listen as he spoke of high morals and the condemnation of luxury. A faction of his followers in Croton became known as the Pythagoreans.

The Pythagoreans were a cultist, politically conservative group. They were vegetarians, and much of their work sought to find the relationships between mathematics and the real world around them. Their motto was "All is Number." The symbol of the Pythagoreans was the star pentagon whose diagonals have been shown to intersect each other in the ratio of the golden section. (It is not known if the Pythagoreans realized this relationship).

Much of what has been credited to Pythagoras was actually due to his followers, the Pythagoreans. They recognized the relationship between numbers and music in that tones that are harmonious (to the western ear) are created by integral divisions of a vibrating string. They realized some if not all of the regular solids and found constructions for them. The Pythagoreans studied proportions, such as the proportions that relate the arithmetic, geometric, and harmonic means. (For example: for any two non-zero numbers, a and b: a is to their arithmetic mean as their harmonic mean is to b). The Pythagoreans preceded Copernicus with the heliocentric (sun-centered) view of our solar system. The Pythagorean definition of the right angle is related to the real world in that it was defined as the angle formed by the intersection of vertical (in line with the pull of gravity) and hori-

zontal (the horizon). Pythagoras is even said to have coined the words "philosophy--the love of wisdom" and "mathematics--that which is learned," to describe his intellectual activities.

The school of Pythagoras reached its peak around 490 B.C. It was during this time that Hypasos, a former student who had been expelled, came to power in the democratic party. Hypasos had the school dissolved and Pythagoras was exiled to Tarentum. After fourteen years in Tarentum, the democratic party came into power there and again Pythagoras was forced to move. He then went to Metapontus where the process repeated itself yet again. This time, he narrowly escaped with his life only to die a short time later. Pythagoras was well over ninety years of age at the time of his death.

In speaking of Pythagoras and his philosophy, the Journal of the Royal Society of Canada says: "He was the Newton, the Galileo, perhaps the Edison and Marconi of his Epoch,..., but he committed none of his views to writing and forbid his followers to do so, insisting that they listen and hold their tongues." [3]

Of all the discoveries and conjectures made by Pythagoras, the one that is most important and the one for which he is most often remembered is the theorem that bears his name. It states that in a right-angled triangle, the square on the longest side (the hypotenuse) is equal to the sum of the squares on the two shorter sides (the legs). This theorem has also been known as The Forty-seventh Proposition of Euclid (from Euclid's book The Elements), The Carpenter's Theorem, and The Hectatomb Proposition. The latter is due to the claim that Pythagoras sacrificed one hundred oxen upon discovery of the theorem. This claim contradicts the fact that Pythagoras was a vegetarian due to his belief in transmigration of the soul.

It is not known whether Pythagoras discovered the theorem on his own or learned of it in his studies, but he is alleged to have been the first to demonstrate (prove) the theorem (540 B.C.). That demonstration (if it ever existed) has been lost but Pythagoras is nonetheless credited with taking the proposition from empirical fact to mathematical theorem. Properties of these special (right) triangles were certainly known long before Pythagoras's time. The Babylonians knew about "builders' triangles" by 2000 B.C. The three, four, five right triangle was known to Tschou-Gun in China around 1100 B.C. and also to the Egyptians long before Pythagoras studied there. Whether any of these early peoples generalized from these special cases to the theorem is not known.

The theorem of Pythagoras was probably the most important discovery in all of early mathematics. Proving the Pythagorean Theorem is a challenge that has been accepted by many people over the past two thousand years, ranging from accomplished and famous mathematicians, to political figures, to school children. There was even a time when an original proof of the theorem was required to obtain a master's

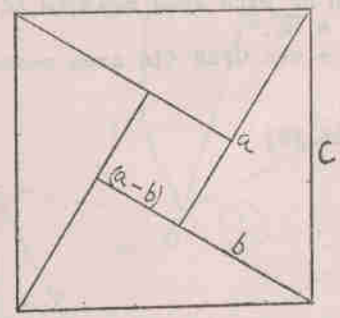
degree in mathematics. [4] As many and varied as these proofs are, they all fall into one of four categories: [5].

1. Algebraic: Those based upon linear relationships.
2. Geometric: those based upon comparison of areas.
3. Quaternionic: those based upon vector operations.
4. Dynamic: those based upon mass and velocity.

One might ponder the question: "Why are there no trigonometric, analytic, or calculus proofs of the Pythagorean Theorem?" The reason is that these more modern areas of mathematics are based on the truths of elementary plane geometry. Without the Pythagorean Theorem there would be no trigonometry of the right triangle no analytic geometry, and possibly, no calculus. So trying to prove the theorem by any of these methods would require using the theorem.

On the following pages there can be found several intriguing proofs of the Pythagorean theorem. They have been chosen for their unique value, whether it be mathematical, historical or intrinsic.

Bhaskara's One Word Proof



Behold!

Proof (Obvious to Bhaskara):
area of large square = area of four triangles + area of small square

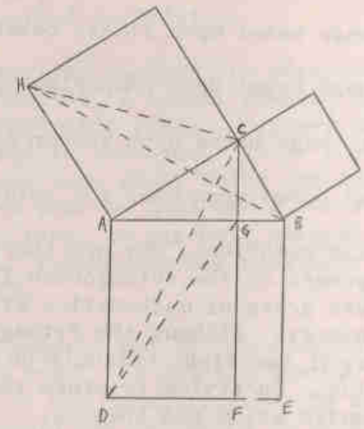
$$c^2=4\ [(1/2)ab] + (a-b)^2$$

$$c^2=2ab + a^2-2ab + b^2$$

$$c^2=a^2 + b^2$$

(Note:- Bhaskara was a famous Indian mathematician who wrote "Lilavati")

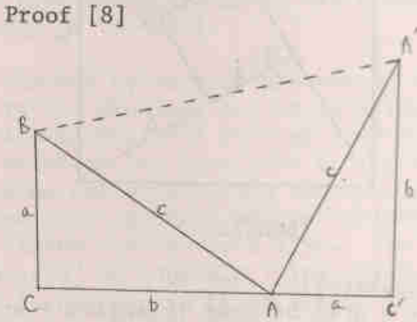
Euclid's Proof [7]



Proof:

1. Draw CF perpendicular to DE through C.
2. Rectangle ADFG = 2 triangles ADG.
3. Tri ADG ≡ tri ADC (same base and height) (in area measure).
4. Tri ADC = tri ABH (rotation through 90°).
5. Tri HAB = tri HAC (same base and height) (in area measure).
6. 2 tri HAC = square on AC.
7. Rec ADFG = square on $\overline{AC}$ with area measure $\overline{AC}^2$ (substitution in ≠ 2).
8. Similarly, rec GFEB = $\overline{BC}^2$.
9. Sq. ADEB = rec ADFG + rec GFEB (in area measure).
10. $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$.

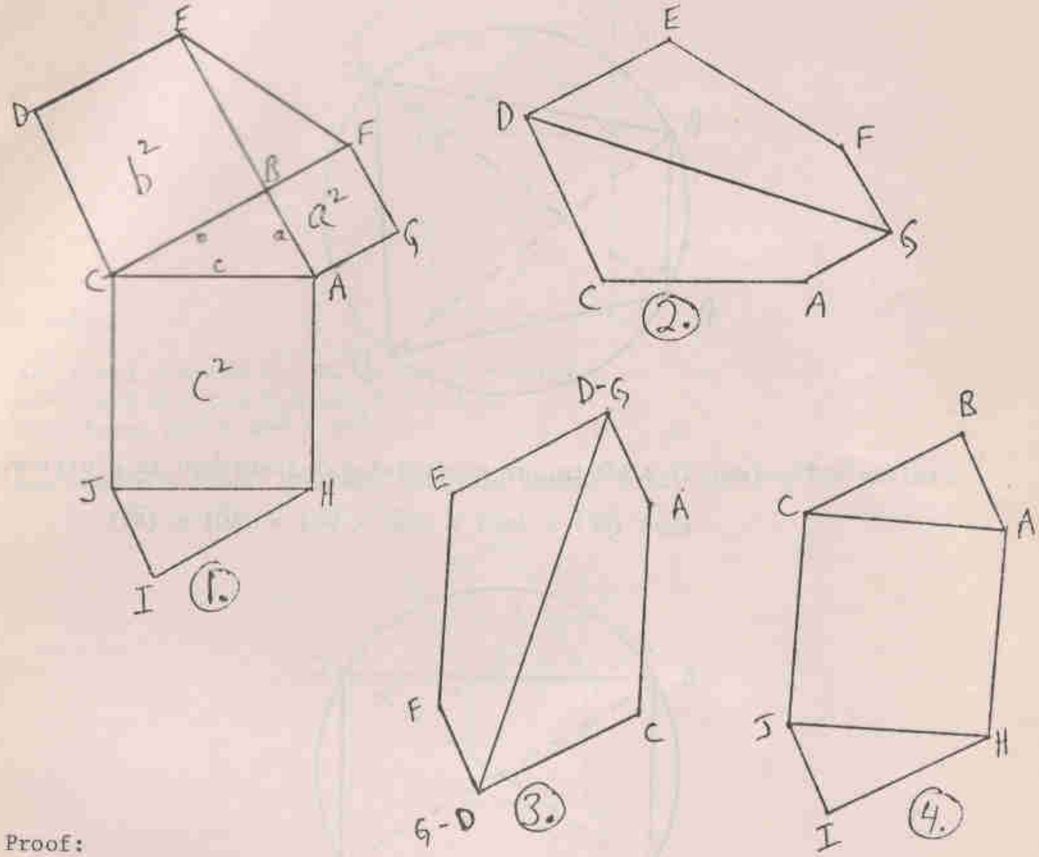
President Garfield's Proof [8]



1. Construct tri A'AC' = tri ABC by 90° rotation and parallel shift.
2. Measure of Angle A'AB' = 90° (by rotation).
3. $\overline{A'C'}$ $\overline{BC}$.
4. Trapezoid CC'A'B = tri ABC + tri A'AC' + tri AA'B'.
5. $1/2\ (a + b)\ (a + b) = 1/2\ ab + 1/2\ ab + 1/2\ c^2$.
6. $a^2 + 2ab + b^2 = 2ab + c^2$.
7. $a^2 + b^2 = c^2$.

(Note: James A. Garfield was the 20th President of the United States of America. He lived from 1831-1881).

Leonardo da Vinci's Proof [9]



Proof:

Figure 2 is the top portion of Figure 1 with the diagonala removed.

The area of Figure 2 is $a^2 + b^2 + 1/2\ ab + 1/2\ ab$, as is seen in Figure 1.

Figure 3 is formed from figure 2 by removing quad. DEFG, flipping it over, and rejoining it with quad. DCAG.

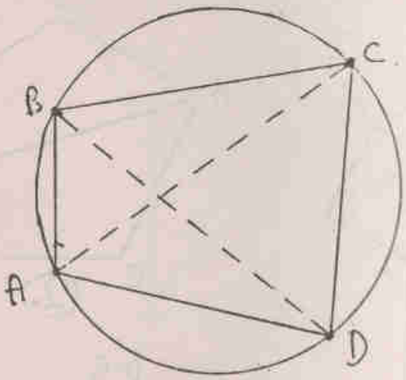
So Figure 3 also has area $a^2 + b^2 + 1/2\ ab + 1/2\ ab$.

Figure 4 has area $c^2 + 1/2\ ab + 1/2\ ab$, as seen in Figure 1.

Figure 3 = Figure 4, so $c^2 + 1/2\ ab + 1/2\ ab = a^2 + b^2 + 1/2\ ab + 1/2\ ab$.

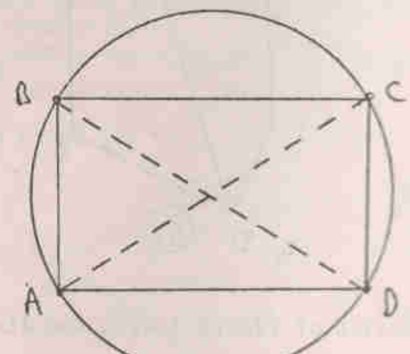
Therefore $c^2 = a^2 + b^2$.

Proof Using Ptolemy's Theorem (By Elisha Scott Loomis) [10]



Ptolemy's Theorem: If ABCD is any quadralateral inscribed in a cirlce,

then $(\overline{BC}) \times (\overline{AD}) + (\overline{AB}) \times \overline{DC}) = (\overline{AC}) \times (\overline{BD})$



Proof:

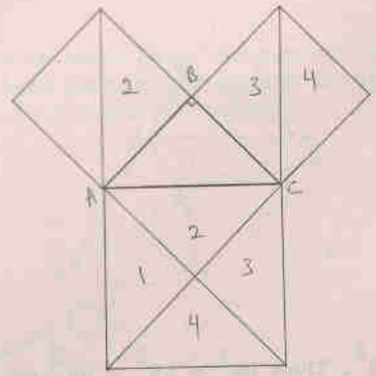
If quad. ABCD is a rectangle, then tri ABC is a rt. triangle.

By Ptolemy's theorem; $(AB \times CD) + (BC \times AD) = AC \times BD$.

Since ABCD is a rectangle, $AB = CD$, $BC = AD$, and $AC = BD$.

So by substitution, $AB^2 + BC^2 = AC^2$.

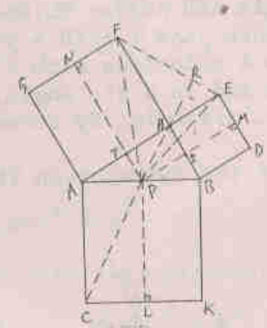
Socrates' Proof (for Isosceles Right Triangles) [11]



Proof:

AB^2 = tri 1 + tri 2. BC^2 = tri 3 + tri 4.
AC^2 = tri 1 + tri 2 + tri 3 + tri 4.
Therefore, $AC^2 = AB^2 + BC^2$.

Ann Condit's Proof (Sixteen Year Old School Girl, 1938) [12]

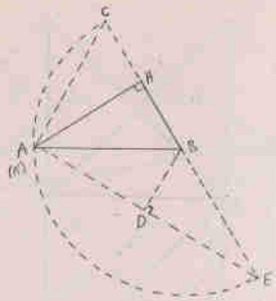


Proof:

$\overline{PN}$, $\overline{PM}$, and $\overline{PL}$ all bisect their respective squares into equal rectangles, tri BHA = tri EHF, so EF = AB = AC. Since PH = PA, tri's PAC, HPE, and PHF have equal bases. Therefore, tri EHP (= 1/4 sq. HD): tri PHF (= 1/4 sq. HG) = ER:FR. Therefore, tri HPE + tri PHF : tri PHF = (ER + FR = AC): FR. Therefore, 1/4 sq. HD + 1/4 sq. HG: tri PHF = AC: FR. But, tri PAC (= 1/4 sq. AK): tri PHF = AC : FR. Therefore, 1/4 sq. HD + 1/4 sq. HG: 1/4 sq. AK = tri PHE: tri PHF. Therefore, 1/4 sq. HD + 1/4 sq. HG = 1/4 sq. AK. So sq. HD + sq. HG = sq. AK. Therefore, $AH^2 + HB^2 = AB^2$.

Note: This proof is significant because it is the only one ever devised where all of the auxiliary lines are emitted from the midpoint of the hypotenuse.

Liebniz's Indirect Proof [13]

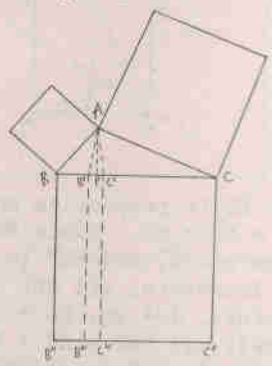


Proof:

Suppose $HA^2 + HB^2 = AB^2$. Then $HA^2 = AB^2 - HB^2$ so $HA^2 = (AB + HB)\ (AB - HB)$.
Construct $\overline{BE}$ and $\overline{BC}$ each equal to $\overline{AB}$ and form semicircle CA'E.
Construct $\overline{AE}$ and $\overline{AC}$ and construct $\overline{BD}$ perpendicular to $\overline{AE}$.
Now $HE = AB + HB$ and $HC = AB - HB$.
$HE \times HC = AB^2 - HB^2 = HA^2$, which is only true when triangles.
AHC and EHA are similar.

Therefore angle CAH = angle AEH and so HC:HA = HA:HE; since /HAC = /E, then /CAH = /EAH. Therefore /EAH + /AEH = 90° and /CAH + /EAH = 90°. Therefore /EAC = 90°. So A coincides with A'. Therefore /EAC is inscribed in a semicircle and is a rt. angle. So /AHB is a rt. angle. Therefore tri AHB is a rt. triangle. By reversing the steps $h^2 = a^2 + b^2$.

Thabit's Generalization of the Pythagorean Theorem [14]

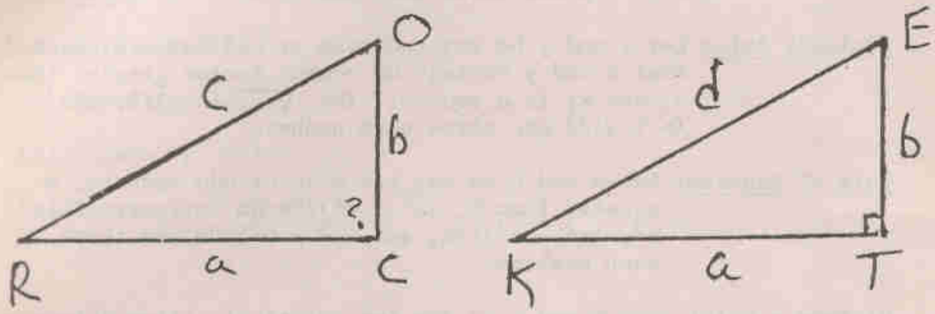


Proof:

Place B' and C' such that angles AB'B and AC'C both equal angle A.
Therefore tri's ABC, B'BA, and C'AC are similar triangles.
So BC:BA = AB:BB' or $AB^2 = (BC\ (BB')$ Therefore, $\underline{AB^2 + AC^2}$
And AC:CC' = BC:AC or $AC^2 = (BC)\ (CC')$ $= \underline{BC\ (BB' + CC')}$
Clearly, when angle A is a rt. angle, B' coincides with C',
In which case, BB' + CC' = BC and $\underline{AB^2 + AC^2 = BC^2}$.

Proof of the Converse of the Pythagorean Theorem [15]

Statement: If a triangle is such that the sum of the squares on the two shorter legs equals the square on the longest leg, then the triangle is a right triangle.



Proof:

Let tri ROC be such that $c^2 = a^2 + b^2$.

Construct right triangle KET with legs of length a and b.

Since tri KET is a rt. triangle, $d^2 = a^2 + b^2$.

Since $d^2 = a^2 + b^2 = c^2 \Longrightarrow d^2 = c^2$.

So $d = c$.

Therefore tri ROC $\cong$ tri KET (by SSS).

Therefore angle C = angle T.

Since angle T is a right angle, angle C is a right angle.

Therefore tri ROC is a right triangle.

Not only have proofs of the Pythagorean Theorem interested many people, but methods for finding integral lengths for the sides of right triangles have also been considered. Below are five such different but related methods:

1. Rule of Pythagoras: Let n be odd; then n, $(n^2-1)/2$, and $(n^2+1)/2$ are three such numbers. (Example n=5, $\frac{n^2-1}{2} = 12$, $\frac{n^2+1}{2} = 13$).

2. Plato's Rule: Let m be any multiple of 4; then m, $m^2/4 - 1$, and $m^2/4 + 1$ are three such numbers.

3. Euclid's Rule: Let x and y be any two even or odd numbers, such that x and y contain no common factor greater than 2, and xy is a square. The $\sqrt{xy}$, $(x-y)/2$, and $(x + y)/2$ are three such numbers.

4. Rule of Maseres: Let m and n be any two even or odd numbers, m greater than n, $(m^2 + n^2)/2n$ an integer. Then m^2, $(m^2 - n^2)/2n$, and $(m^2 + n^2)/2n$ are three such numbers.

5. Dickson's Rule: Let m and n be any two relatively prime integers, one even and the other odd, m greater than n, and 2mn a square. The $m + \sqrt{2mn}$, $n + \sqrt{2mn}$. and $m + n + \sqrt{2mn}$ are three such numbers.[16]

The reader is invited to try to find yet other proofs of the Theorem and other means for generating Pythagorian triplets.

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[11] Ibid., p. 86.

[12] Ibid., p. 140.

[13] Ibid., p. 59.

[14] Boyer, p. 259.

[15] Jacobs, p. 353.

[16] Loomis, pp. 19-20.

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GLOSSARY OF MATHEMATICAL TERMS
(Proposed)

(D)

Data तथ्यांक, आँकडा, दत्तन्यास

Decagon दशभुज

Decagram डेकाग्राम

Decametre डेकामिटर

Deceleration अवत्वरण

Decigram डेसिग्राम

Decile दशमक

Decimal दशमलव

Decimalise दशमलव करण गर्नु

Decimeter डेसिमीटर

Declination अवनति

Decompose विघटन गर्नु

Decomposition विघटन

Decrease घट्नु, ह्रास हुनु

Decreasing ह्रासोन्मुख

Deduce निगमन गर्नु

Deduction निगमन

Deducible निगम्य

Deductive निगमनिक

Define परिभाषा दिनु, परिभाषित गर्नु

Definite निश्चित, सीमित

Definition परिभाषा

Deflation अपस्फीति

Deflection अपचलन, विस्थापन

Deform विकृत गर्नु

Deformation विरूपता, विरूपीकरण

Degree डिग्री, मात्रा

Delete छोड्नु, हटाउनु, उच्छेद गर्नु

Delta डेल्टा

Demand माग

Demography जनविद्या

Demonstrate प्रदर्शन गर्नु, निरूपण गर्नु

Demonstration प्रदर्शन, निरूपण

Denominator हर

Density घनत्व

Dependency पराश्रयता, परनिर्भरता

Dependent आश्रित, अवलम्बी

Depend निर्भर हुनु, आधारीत हुनु, अवलम्बित हुनु

Derangement क्रमभङ्गण, व्युत्क्रम

Derivation व्युत्पत्ति

Derivative अवकलन, व्युत्पत्ति मूलक

Derive व्युत्पन्न गर्नु, निर्गमन गर्नु

Derived व्युत्पन्न, निर्गमित

Descend अवरोहण, ओर्लनु

Descending अवरोही

Describe वर्णन गर्नु, रचना गर्नु, विवरण दिनु

Design नमूना, डिजाइन

Designate पदनाम दिनु, उपलक्षण दिनु

Determinant सारणिक, निर्धारक, परिच्छेदक

Determinate परिमित, परिच्छिन्न

Determine निर्धारण गर्नु

Determining निर्धारणशील

Developable विकासनीय

Deviance विचल, विचलितता

यो शब्दावली त्रि० वि०, कीर्तिपुर बहुमुखी क्याम्पस, गणित तथा नेपाली शिक्षण समितिले संयुक्त रूपमा तयार गरिएको हो ।

Deviate विचलित हुनु

Deviation विचलन

Diagonal विकर्ण

Diagram रेखाचित्र; आरेख

Diameter व्यास

Diametrically व्यासत:

Dichotomize आधीकरण

Dichotomous द्विधात्मक

Difference भिन्नता, अन्तर

Different विषम; विभिन्न; असमान

Differentiable अवकलनिय, अन्तरणिय· विभेदनिय

Differentiability अवकलनियता, अन्तर-णियता

Differential अवकलन, अवकल

Differentiate अवकलन गर्नु

Differentiated अवकलित: विभेदित

Differentiation अवकलन

Diffuse विसरण गर्नु; विसृत हुनु

Diffusion विसरण

Digit अङ्क

Dihedral द्वितल

Dimension आयाम; परिमाप; परिमित

Dimensional आयामिक

Diminish घटनु; ह्रास हुनु

Diminishing ह्रासमान; ह्रासशील

Dipole द्विध्रुव; युग्म ध्रुव

Dipolar द्विध्रुवी; युग्मध्रुवी

Direct प्रत्यक्ष; सिधा

Directed निर्दिष्ट

Direction दिशा

Directional दिशागत

Directly प्रत्यक्षत:; प्रत्यक्षतरिकाले

Director (circle) नियामी (वृत)

Directrix नियामिका

Disc चक्र; मण्डल; डिस्क

Discharge निस्सरण; क्षेपण

Discontinuity असातत्य

Discontinuous असतत्

Discrete अलग; असंबद्ध

Discriminant विवेचक; परिच्छेदक

Discriminating विवेचलशील

Disjoint वियोजन

Displace विस्थापित गर्नु

Displacement विस्थापन; प्रतिस्थापन

Disprove असिद्ध गर्नु

Disproved असिद्ध

Dissimilar असदृत; असमरूप

Dissipation विक्षेप; विकिरण

Dissipative आचयी

Distance दूरी

Distinct भिन्न; अलग,

Distinction विशिष्ट: भिन्नता; विशिष्टता

Distinguish छुट्याउनु; भेददेखाउनु

Distribution वितरण; बाँडफाँड

Distributive वितरणशील

Diurnal दैनिक

Div. (Divergence) अपसरण

Diverge अपसृत हुनु

Divergent अपसारी

Divide भाग गर्नु

Divided विभाजीत

Dividend भाग; लब्धि

Divisibility भाज्यता

Divisible विभाज्य; भाज्य

Division विभाजन; भाग

Divisor भाजक

Dodecagon द्वादशभूज

Dodecahedron द्वादशपन्चभूजक

Domain क्षेत्र

Dot बिन्दु

Double दोब्बर

Doublet उभयक; द्विक

Downward अधोमुखी

Drag कर्षी

Draw आकर्षी

Drawing रेखा चित्र; आलेख

Drop त्याग, पात; थोपा; बिन्दु

Dual द्वैत; द्वैध

Duality द्वैतता; द्वैधता

Dummy मूक; निश्चल

Duplicate अनुलिपि; दुपर्ती; द्विक

Duplication द्विगुणन

Dyad द्वयक

Dyadic द्वयकीय

Dynamic गतिज; गतिमय; गतिक

Dynamical गतिमयक

Dynamics गतिविज्ञान

Dyne डाइन

(E)

Eccentric उत्केन्द्र

Eccentricity उत्केन्द्री, उत्केन्द्रता

Echelon सोपानक

Eclipse ग्रहण

Effect असर, प्रभाव, परिणाम

Effective प्रभावकारी, फलकारी

Efficiency दक्षता, निपुणता

Efflux बाहिस्राव, निस्राव

Effort प्रयास, कोशिश

Eight आठ

Elapse व्यतितहुनु

Elastic लचकदार, प्रत्यास्थ

Elasticity लचक, प्रत्यास्थता

Element तत्व

Elementary प्रारम्भिक; प्राथमिक

Elevate उठाउनु, उत्थापन

Elevated उन्नत, उठेको

Elevation उठान, उन्नतता

Eleven एघार

Eliminant निरसन फल

Elimination लोपगर्नु, विलोपन गर्नु

Ellipse दीर्घवृत्त, इलिप्स

Ellipsoid दीर्घवृत्तज, इलिप्स्वाइड

Ellipsoidal दीर्घवृत्तीय

Elliptical दीर्घवृत्ताकारीय

Elliptic दीर्घवृत्ताकार

Elongate आयत तुल्याउनु

Elongated आयत; वितत

Elongation आयतीकरण, दीर्घीकरण
Elucidate स्पष्टपार्नु
Elucidation स्पष्टीकरण
Empirical अनुमानिक, प्रयोगसिद्ध
Empty खालि, रित्तो, शून्य
Enclose घेर्नु
End अन्त; सीमा
Energy उर्जा
Entire संपूर्ण, सकल समस्त
Entropy एन्ट्रोपी, अपचयन
Enumerable गणनिय, गणनागर्नसकिने
Enumerate गणनागर्नु
Enumerated प्रगणित
Enumeration गणन, परिगणन
Emunerator गणनाकार; गणक
Enunciation प्रतिज्ञा
Envelope अन्वालोप, आवेष्टन
Enveloping अन्वालोपींग, आवेष्टनशील
Epicentre अधिकेन्द्र.
Epicycloid वाह्यचक्रज
Epsilon इपसीलन (ϵ)
Equal तुल्य, समान, वरावर
Equality समानता
Equally तुल्यता पूर्वक; वरावरी
Equate समीकृतगर्नु, तुल्यपार्नु
Equation समीकरण
Equator विषुवत रेखा; मूमध्य रेखा
Equatorial विषुवतीय, मूमध्य
Equiangular समकोणिय
Equiconjugate समयुग्मी
Equidistant समदुरस्थ
Equilateral सममूजाकार; समवाहु
Equilibrant सन्तुलक
Equilibrium सन्तुलन
Equipotential तुल्यविमव
Equivalent समरूप, तुल्य, समकक्ष
Erg अर्ग
Error त्रुटी, गल्ती
Essential सारमूत; आधारमूत
Estimate अनुमानगर्नु
Estimated अनुमानित
Estimation आगणन; अनुमान
Eta इटा (η)
Evaluate मूल्यनिकाल्नु; मान वताउनु
Evaluation मूल्याङ्कन, मानाङ्कन
Even सम; जोर
Everywhere सर्वत्र
Evolute केन्द्रज
Evolution विकसन
Exact यथार्थ; तात्विक; यथातथ
Example उदाहरण
Excentre वहिरकेन्द्र
Exercise अभ्यास; प्रश्नावली
Exhaust निरसन, निशेषगर्नु
Existence अस्तित्व
Exit निष्क्रमण; निर्गय
Expand प्रसारगर्नु; विस्तारगर्नु
Expanded प्रसारित; विस्तारित

Expansion प्रसार, विस्तार

Expectation प्रत्याशा

Expected प्रत्याशित

Explicit व्यक्त; स्पष्ट

Exponent घात; घाताङ्क

Exponential घातीय

Express अभिव्यक्त गर्नु

Expression व्यन्जक

Extend बढाउनु, विस्तारगर्नु

Extension विस्तार

Exterior वाह्यभाग; वहिर्भाग

External वाह्य

Externally वाह्यता

Extrema परममान, चरममान

Exremities चरमता; चरमविन्दु

(F)

Factor गुणनखण्ड

Foctorial गुणनखण्डीय; खण्डगुणित

Factorisation खण्डगुणन

Factorise खण्डगुणन गर्नु; गुणनखण्ड गर्नु

Fallacy हेत्वाभास

Family कूल; परिवार

Fathom तल, पिंध

Field क्षेत्र

Fifth पाँचौं

Figure अङ्क, चित्र

Filter प्रशोधन; शोधन; फिल्टर

Final अन्तिम; पछिल्लो

Finite परिमित; ससीम

Five पाँच

Fixed स्थिर, तोकिएको, निश्चित

Flat चेप्टो, चाक्लो

Flow वहनु, प्रवाह

Fluctuation घटवढ, विचलन

Fluid तरल

Fluidity तरलता

Flux फ्लक्स; अभिवाह

Focal नाभीय

Foci नाभीहरू

Focus नाभी, केन्द्रविन्दु

Fold गुणा, पत्र, पट्याउनु

Following निम्न; निम्नलिखित

Foot फूट

Force वल

Forced वलमूलक; जवर्जस्ती

Form रूप

Formula सूत्र

Formulate प्रस्थापन; सूत्रितगर्नु

Forward अग्रसर, अग्र

Four चार

Fraction भिन्न

Fractional भिन्नात्मक

Frame ढाँचा

Free स्वतन्त्र, मुक्त

Frequency आवृत्ति, वारंवारता; सातत्य

Friction घर्षण

Frictional घर्षणात्मक

Front सामुन्ने, अग्रभाग, पार्श्वभाग

Frontier सीमा; सीमान्त

Fulcrum आलम्ब

Function फल; फलन

Functional फलनीय

Fundamental मूल, मूलभूत, मौलिक

Registered No.

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