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Contents

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On Methods of Generating Random Permutations

Jerrold Grossman

1. Introduction

It is desired in many situations to shuffle a set of objects - to randomize their order. In card games, for example, the deck of cards is put into a random order before being distributed to the players, so that each person has an equal chance of receiving every possible combination of cards. In the United States a few years ago, the birthdates of young men were randomly ordered to determine the sequence in which the men would be conscripted into the army. Physical processes can often be simulated on a computer, and again the ability to randomly order a set may be required.

To put our discussion into concrete terms, let us suppose that we have a set of cards numbered $1,2,3,\ldots,N$ arranged in numerical order in a row. We are seeking a process (called a <u>shuffle</u>) by which the cards are put into a possibly) different order, or <u>permutation.</u> The process must involve some random choices, for we surely want to impose the requirement:

<u>Condition 1.</u> The probability that card s ends up in position t after the shuffle is completed is $\frac{1}{N}$, for all $1 \leq s \leq N$, $1 \leq t \leq N$.

A moment's reflection, however, shows this condition to be inadequate for most purposes. Using only the N cyclic permutations $(1,2,\ldots,N)$, $(2,3,\ldots,N,1),\ldots,(N,1,2,\ldots,N-1)$ could satisfy it. Instead we shall impose the stronger requirement:

<u>Condition 2.</u> Each distinct permutation is equally likely to result from the shuffle.

Our problem, then, is to find efficient algorithms for shuffles satisfying Condition 2. We shall discuss several such algorithms in this paper, as well as an interesting variant which fails both conditions.

Note that Condition 2 implies Condition 1. Indeed, since there are $N!$ distinct permutations of which exactly $(N-1)!$ have card s in position t, the probabilities in Condition 1 are all $\frac{(N-1)!}{N!} = \frac{1}{N}$.

2. The Algorithms Which Work

Before we can describe any algorithms, we need a randomization device. For our purposes it is enough to have a way of choosing an integer from the set $\{1,2,\ldots,M\}$ in such a way that the probability of choosing integer i is $\frac{1}{M}$ for all $1 \leq i \leq M$. Physical devices could

be used (e.g. a spinning wheel divided into M sections or an electronic clock that rapidly counted from 1 to M repeatedly), with a consultation of the device for each "random number" needed. On a computer various arithmetic algorithms, while not truly random number generators, seem to work reasonably well. To generate a random integer from 1 to 100, for example, one could take a given 10-digit number, square it, retain the 11^{th} through 20^{th} digits of the answer as the next 10-digit number to be used, and take the 11^{th} and 12^{th} digits of the answer, plus one, as the desired random integer. We shall assume that some randomization device is available and proceed to discuss the algorithms for shuffling the N cards.

> Algorithm I [1]. Number the permutations from 1 to N!. Choose a random number i from the set $\{1,2,\ldots N!\}$ and put the cards into the order numbered i.

Condition 2 is automatically satisfied. Unfortunately, for N at all large, N! is much too large a number to deal with, so this algorithm, while theoretically sound, is impractical.

> Algorithm II. Set up a new row for the cards. Successively, for $i = 1,2,\ldots,N$, choose a random number r_i from the set $\{1,2,\ldots,N-i + 1\}$ and put the r_ith largest card remaining in the original row into the i^{th} position in the new row.

To see that this algorithm satisfies Condition 2, consider the sequence $(r_1,r_2,\ldots,r_N)$ of random numbers chosen during the algorithm. Since $1 \leq r_i \leq N-i + 1$, there are N! possible sequences, each equally likely. If two such sequences, $(r_1,r_2,\ldots,r_N)$ and $(r_1',r_2',\ldots,r_N')$ differ, let j be the first subscript for which $r_j \neq r_j'$. Then the cards in position j of the two resulting permutations differ. Thus there is a one-to-one correspondence between permutations and sequences $(r_1,r_2,\ldots,r_N)$ with $1 \leq r_i \leq N-i + 1$, so Condition 2 holds.

This algorithm is reasonably easy to implement. The main drawbacks are the need for a second row for inserting the new permutation and the need to search for the r_ith largest remaining card at each stage.

> Algorithm III [2,3]. Successively for $i = 1,2,\ldots,N$, choose a random number r_i from the set $\{1,2,\ldots,N-i + 1\}$ and interchange the cards in position i and position $k = N - r_i + 1$. (Note that $k \geq i$. If $k = i$, then the interchange for that value of i leaves the card in position i fixed.)

The proof of Condition 2 is similar to the proof for Algorithm II and is left to the reader. Note the improvement over Algorithm II in eliminating the two drawbacks.

3. An Algorithm Which Fails

Suppose, in Algorithm III, we allowed the interchange at step i not only between the cards in positin i and position k where $i \leq k \leq N$,

but between the cards in position i and position k where $1 \leq k \leq N$. We then get the simpler.

Algorithm IV [1]. Successively for $i = 1,2,\ldots,N$, choose a random number r_i from the set $\{1,2,\ldots,N\}$ and interchange the cards in position i and position r_i.

De Balbine [1] gives the following simple proof that Algorithm IV fails to satisfy Condition 2 for $N \geq 3$. There are N^N possible sequences $(r_1,r_2,\ldots,r_N)$, each of which yields one of the N! possible permutations. If the permutations are to be equally likely, then the same number C of sequences must yield each permutation, with $CN! = N^N$. But for $N \geq 3$, N! contains a prime factor distinct from those in N (namely a prime factor of $N - 1$), so N! is not a divisor of N^N. Thus the permutations cannot be equally likely.

In the remainder of this paper, we show that Algorithm IV also fails Condition 1. The techniques involved in the proof are typical of those encountered in discrete probability computations.

THEOREM. *The probability that card* s *ends up in position* t *after an Algorithm IV shuffle is*

$$\frac{1}{N}\left[\left(\frac{N-1}{N}\right)^{N-t} + \left(\frac{N-1}{N}\right)^{s-1}\right] \quad \textit{if} \quad s > t$$

$$\frac{1}{N}\left[\left(\frac{N-1}{N}\right)^{N-t} + \left(\frac{N-1}{N}\right)^{s-1} - \left(\frac{N-1}{N}\right)^{N+s-t-1}\right] \quad \textit{if} \quad s \leq t.$$

In particular if we take $s = N$, $t = 1$, then the probability is $\frac{2}{N}\left(\frac{N-1}{N}\right)^{N-1} \neq \frac{1}{N}$ if $N \geq 3$.

To prove the theorem, we first prove a

LEMMA. *Suppose* $i - 1$ *interchanges have been completed and* r_i *is about to be chosen. Then the probability that the card currently in position* i *ends up in position* j *is* $\frac{1}{N}$ *for each* j.

Proof: We proceed by backwards induction on i. The statement is clearly true for $i = N$. Assume it is true for $i+1,\ldots,N$. If $j \leq i$, then for the card currently in position i to end up in position j, either $r_i = j$ and $r_{i+1},\ldots,r_N$ are different from j, or $r_i = l$ with $i + 1 \leq l \leq N$ and $r_{i+1},\ldots,r_{l-1}$ are different from e, and then the induction hypothesis applies. The probability is therefore

$$(\frac{1}{N})\ (\frac{N-1}{N})^{N-i} + \sum_{l=i+1}^{N} (\frac{1}{N})\ (\frac{N-1}{N})^{l-i-1}\ (\frac{1}{N}) = \frac{1}{N}\ .$$

If $j > i$, a lengthy but similar analysis shows that the probability is

$$(\frac{i}{N})\ (\frac{1}{N})\ (\frac{N-1}{N})^{N-j} + \sum_{l=i+1}^{j} (\frac{1}{N})\,[(\frac{N-1}{N})^{l-i-1}(\frac{1}{N})$$

$$+ (1 - (\frac{N-1}{N})^{l-i-1})\ (\frac{1}{N})\ (\frac{N-1}{N})^{N-j}] +$$

$$\sum_{l=j+1}^{N} (\frac{1}{N})\ [(\frac{1}{N})\ (\frac{N-1}{N})^{N-j} + (\frac{N-1}{N})^{l-i-1}(\frac{1}{N})] = \frac{1}{N}\ .$$

Proof of the theorem: If $s > t$, then card s ends up in position t if the t^{th} interchange puts card s into position t (probability $\frac{1}{N}$) and $r_{t+1},\ldots,r_N$ are different from t (probability $(\frac{N-1}{N})^{N-t}$), or if $r_1,\ldots,r_{s-1}$ are different from s (probability $(\frac{N-1}{N})^{s-1}$) and then card s ends up in position t (probability $\frac{1}{N}$ by the lemma). If $s \leq t$, then card s ends up in position t if $r_1,\ldots,r_{s-1}$ are different from s (probability $(\frac{N-1}{N})^{s-1}$) and then card s ends up in position t (probability $\frac{1}{N}$ by the lemma), or if one or more of $r_1,\ldots,r_{s-1}$ are equal to s (probability $1 - (\frac{N-1}{N})^{s-1}$) and the t^{th} interchange puts card s into position t (probability $\frac{1}{N}$) and $r_{t+1},\ldots,r_N$ are different from t (probability $(\frac{N-1}{N})^{N-t}$).

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Heat Transfer From a Sphere With a Fluid Source at its Centre in a Slow Uniform Stream With Slip

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Abstract

The effect of velocity slip and temperature jump on the heat transfer from a sphere with a fluid source at its centre which is placed in a slow uniform stream of viscous incompressible fluid is investigated at small Reynolds number and Prandtl number of order unity. An expression for the average Nusselt number is calculated up to the term of order Re. It is found that the velocity slip does not affect the Nusselt number to this order where as the temperature jump affects it.

1. Introduction

Acrivos and Taylor [2] has considered the problem of heat transfer from a sphere assuming that the velocity field to be given by Stokes flow [10] and using the method of inner and outer expansions the solution for small Peclet number Pe which is the product of Prandtl number σ and the Reynolds number Re has been obtained. An expression for the average Nusselt number N as a function of Peclet number Pe has also been calculated. This work has been extended by Rimmer [6], [7] using the velocity field obtained by Proudman and Pearson [5]. Recently, the author [9] has studied the problem of heat transfer from a sphere with a fluid source at its centre which is placed in a slow uniform stream using the velocity field obtained by Datta [4] for Stokes flow past a sphere with a source at its centre, and concluded that the effect of the source is to reduce the average Nusselt number. Taylor [11] has studied the effect of velocity slip and temperature jump on the heat transfer from a sphere in a low Reynolds number slip flow using the velocity field obtained by Basset [3] and found that the velocity slip does not affect the average Nusselt number to the first approximation of small Peclet number where as the temperature jump affect.

The purpose of the present analysis is to study the effect of velocity slip and temperature jump on the heat transfer from a sphere with a fluid source at its centre which is placed in a slow uniform stream, the velocity field for which is taken as that obtained by the author [8] for Stokes flow with slip past a sphere with a source at its centre.

The method of solution is similar to that given by the author [9] therefore some of the details of the solution method will be omitted since they are given in detail in aforementioned paper. The results obtained in this analysis are, however, different since they incorporate both the effects of velocity slip and temperature jump.

2. Formulation of the Problem

Consider the uniform flow U of a viscous incompressible fluid of kinematic viscosity ν past a sphere of radius a with a fluid source of strength Q at its centre. Let v_r, v_θ, v_ϕ denote the velocity components in the spherical polar coordinate system (r, θ, ϕ) with the axis along the direction of uniform stream. Since the flow is axisymmetric, the circumferential component v_ϕ vanishes identically and v_r, v_θ are those obtained by the author [8]. The temperature field is governed by the equation

$$(1) \qquad \nu \nabla^2 t^* = \sigma\left(v_r \frac{\partial t^*}{\partial r^*} + v_\theta \frac{\partial t^*}{r^* \partial \theta}\right)$$

with boundary conditions [11]

$$(2) \qquad \left|\begin{array}{ll} \dfrac{\partial t^*}{\partial r^*} = \lambda(t^* - t_s) \text{ at } r^* = a, & \text{(over the surface)} \\ t^* \longrightarrow t_\infty \text{ as } r^* \longrightarrow \infty, & \text{(far away from the sphere)} \end{array}\right.$$

where λ is the parameter determining the effect of the temperature jump.

As in [9], introducing the following non-dimensional quantities

$$r^* = ar, \quad t = (t^* - t_\infty)/(t_s - t_\infty),$$

$$v_r = Q/a^2r^2 + U\{u(r) + 1\}\cos\theta, \quad v_\theta = U\{v(r) - 1\}\sin\theta,$$

where $t_s - t_\infty$ is the temperature difference between the surface and the fluid far away from the sphere, the above equation reduces to

$$(3) \qquad \nabla_r^2 t = \sigma \mathrm{Re}\left[(u+1)\xi \frac{\partial t}{\partial r} - (v-1)(1-\xi^2)\frac{\partial t}{r\partial \xi}\right] + \frac{\sigma s}{r^2}\frac{\partial t}{\partial r},$$

where with $\xi = \cos\theta$,

$$\nabla_r^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \xi}\left\{(1-\xi^2)\frac{\partial}{\partial \xi}\right\},$$

and $s = Q/\nu a$ is the parameter determining the effect of the source. Here us and v as obtained by the author [8] for no surface injection are given by

$$(4)\qquad \begin{cases} u = \dfrac{2A}{s^3}\left\{r^2(1 + s/r)e^{-s/r} - s^2E_4(s/r) - r^2 + 5s^2/6\right\} + 2B/r^3, \\[2ex] v = -\dfrac{A}{s^3}\left\{4r^2(1 + s/r)e^{-s/r} + s^2E_4(s/r) - 4r^2 + 5s^2/3\right\} + B/r^3, \end{cases}$$

where

$$A = -\frac{3s^3}{10C}(2f + 1),$$

$$B = \frac{1}{10C}[2 - 2(1 + s)e^{-s} - 3s^2E_4(s) + f\{(6 - 6s + 5s^2)e^{-s} - 6s^2E_4(s) - 6\}$$

$$C = (1 + s)e^{-s} - 1 + s^2/2 + fs^2(1 - e^{-s}),$$

$$E_n(s) = \int_1^{\infty} \frac{e^{sx}}{x^n}dx$$

and f is the slip coefficient. In terms of non-dimensional variables the boundary conditions (2) reduce to

$$(5)\qquad \begin{cases} \dfrac{\partial t}{\partial r} = \lambda(t - 1) \quad \text{at} \quad r = 1, \\[2ex] t \longrightarrow 0 \quad \text{as} \quad r \longrightarrow \infty. \end{cases}$$

Now as in [9] we solve equation (3) with the boundary condition at the surface in the inner region. In the outer region, we introduce new variables by setting $\rho = \sigma r\mathrm{Re}$ and $T(\rho, \xi) = t(r, \xi)$; we thus get from (3).

$$(6)\qquad \nabla_\rho^2 T = [(u + 1)\,\xi\,\frac{\partial T}{\partial \rho} - (v - 1)(1 - \xi^2)\,\frac{\partial T}{\rho\partial \xi}] + \frac{s\sigma^2\mathrm{Re}}{\rho^2}\frac{\partial T}{\partial \rho},$$

where ∇_ρ^2 is the same operator as in (3) but with r replaced by ρ, and u and v are also expressed in terms of ρ. We then solve equation (6) for T with the boundary condition at far away from the sphere (i.e., $T \longrightarrow 0$ as $\rho \longrightarrow \infty$). Further the remaining undetermined constants in the inner and outer solutions are to be determined from the matching requirement [2]

$$(7)\qquad t(r \longrightarrow \infty, \xi) = T(\rho \longrightarrow 0, \xi).$$

3. Method of Solution

We assume as in [9] that the inner and outer expansions are represented respectively by

$$(8) \qquad t(r,\xi) = t_o(r,\xi) + \sigma \mathrm{Re}\, t_1(r,\xi) + \dots \; \dots$$

$$(9) \qquad T(\rho,\xi) = \sigma \mathrm{Re}\, T_o\,(\rho,\xi) + (\sigma \mathrm{Re})^2\, T_1(\rho,\xi) + \dots \; \dots$$

Substituting the expansion (8) in (3) and the expansion (9) in (6), the terms independent of σRe give the following differential equations

$$(10) \qquad \nabla_r^2 t_o = \frac{\sigma s}{r^2}\frac{\partial t_o}{\partial r}$$

and

$$(11) \qquad \nabla_\rho^2 T_o = \xi \frac{\partial T_o}{\partial \rho} + (1-\xi^2)\frac{\partial T_o}{\rho \partial \xi}.$$

The required solution of (10) and (11) satisfying the relevant conditions

$$(12) \qquad \begin{aligned} &\frac{\partial t_o}{\partial r} = \lambda(t_o - 1) \text{ and } T_o(\infty, \xi) = 0, \\ &t_o(r \longrightarrow \infty, \xi) = \sigma \mathrm{Re}\, T_o(\rho \longrightarrow 0, \xi) \end{aligned}$$

are respectively

$$(13) \qquad t_o = (1 - e^{-\sigma s/r})/(1 - \beta e^{-\sigma s})$$

and

$$(14) \qquad T_o = \frac{\sigma s}{(1 - \beta e^{-\sigma s})}\,\frac{e^{-\frac{1}{2}\rho(1-\xi)}}{\rho},$$

where $\beta = (1 - \frac{\sigma s}{\lambda})$.

Now substituting the inner expansion (8) in (3), the terms linear in σRe show that t_1 satisfies the following inhomogeneous equation

$$(15) \qquad \nabla_r^2 t_1 - \frac{\sigma s}{r^2}\frac{\partial t_1}{\partial r^2} = \frac{\sigma s}{(e^{-\sigma s} - 1)}\,\frac{e^{-\sigma s/r}}{r^2}\Big[1 + \frac{2A}{s^3}\big\{r^2 e^{-s/r} + 4rs\, E_5(s/r) - r^2 + 5s^2/6\big\} + 2B/r^3\Big]\xi .$$

Following [9], the particular solution of the above equation is

$$(16) \qquad t_{1p} = \Big\langle (M - C_1)\big(\tfrac{r}{s} - \tfrac{\sigma}{2}\big) + C_1\big(\tfrac{r}{s} + \tfrac{\sigma}{2}\big) e^{-\sigma s/r} + \frac{8\sigma s}{(e^{-\sigma s} - 1)}\Big[\frac{r e^{-\sigma s/r}}{8\sigma s}$$

$$+ \frac{A}{s}\Big\{\Big(\frac{r^2}{16s^2} - \Big(\frac{\sigma^4 - \sigma^3 + 9\sigma^2 + 24\sigma + 12}{48\sigma^3 s}\Big)r - \frac{1}{48\sigma} + \frac{s}{48}\frac{1}{r}\Big)e^{-(1+\sigma)s/r}$$

$$+\Big(\frac{10\sigma^2 + 15\sigma + 6}{24}\Big)\Big(\frac{r}{\sigma^4 s} + \frac{1}{2\sigma^3}\Big)e^{-\sigma s/r}\, E_1(s/r) - \frac{s^2}{48\sigma}\frac{e^{-\sigma s/r}}{r^2}\, E_1(s/r)$$

$$- \Big(\frac{\sigma^5 + 10\sigma^2 + 15\sigma + 6}{24\sigma^4}\Big)\Big(\frac{r}{s} - \frac{\sigma}{2}\Big)\, E_1\Big(\frac{(1+\sigma)s}{r}\Big) + \frac{\sigma}{24}\Big(\frac{r}{s} - \frac{\sigma}{2}\Big)\, E_1(\sigma s/r)$$

$$+ \Big(\frac{\sigma^2 + 10}{48\sigma s}\, r - \frac{r^2}{16s^2}\Big)\, e^{-\sigma s/r}\Big\} + \frac{B}{4s^3}\Big(\frac{6r}{\sigma^4 s} + \frac{3}{\sigma^3} - \frac{s^2}{2\sigma}\frac{1}{r^2}\Big)\, e^{-\sigma s/r}\Big]\Big\rangle\xi\,,$$

where

$$M = \frac{8\sigma s}{(e^{-\sigma s} - 1)}\Big\{\frac{1}{8\sigma} + \frac{A}{s}\Big(\frac{-2\sigma^3 + \sigma^2 - 24\sigma - 12}{48\sigma^3} + \frac{\sigma^5 + 10\sigma^2 + 15\sigma + 6}{24\sigma^4}\ \ln(1+\sigma)$$

$$- \frac{\sigma \ln\sigma}{24}\Big) + \frac{3B}{2\sigma^4 s^3}\Big\};$$

the integrals occurring have been evaluated by integrating by parts and the recurrence relation [1]

$$E_{n+1}(x) = \frac{1}{n}\,[e^{-x} - x\,E_n(x)]$$

have also been used above. The unknown constant C_1 in equation (16) is given by the equation

$$\Big[\frac{\partial t_{1p}}{\partial r} - \lambda(t_{1p} - 1)\Big]_{r=1} = 0.$$

Hence the general solution of equation (15) is

$$(17) \qquad t_1 = B_1\,(e^{-\sigma s/r} - e^{-\sigma s}) + t_{1p}.$$

From the matching requirement

$$(18) \qquad t_o(r \longrightarrow \infty, \xi) + \sigma \mathrm{Re}\ t_1(r \longrightarrow \infty, \xi) = \sigma \mathrm{Re}\ T_o(\rho \longrightarrow 0, \xi),$$

we find that

$$B_1 = -\frac{1}{2}\frac{\sigma s}{(1-\beta e^{-\sigma s})^2}.$$

Therefore the inner expansion is

$$t = t_o + \sigma \mathrm{Re}\; t_1 \tag{19}$$

$$= \frac{1 - e^{-\sigma s/r}}{1-\beta e^{-\sigma s}} + \sigma \mathrm{Re}\; \left[\frac{\sigma s}{2}\frac{(e^{-\sigma s} - e^{-\sigma s/r})}{(1-\beta e^{-\sigma s})^2} + t_{1p}\right].$$

Nusselt Number

The expression for the average Nusselt number in non-dimensional variable is given by

$$N = -\int_{-1}^{+1} \left(\frac{\partial t}{\partial r}\right)_{r=1} d\xi \tag{20}$$

$$= \frac{2\sigma s\; e^{-\sigma s}}{(1-\beta e^{-\sigma s})}\left[1 + \frac{s\sigma^2 \mathrm{Re}}{2(1-\beta e^{-\sigma s})}\right].$$

This result shows that to order Re the velocity slip does not affect the average Nusselt number. When there is no temperature jump i.e., when $\beta = 1$ or $\lambda \longrightarrow \infty$, this result agrees with that obtained by the author [9]. When the source is absent ($s = 0$) the above expression reduce to

$$N = \frac{2\lambda}{1+\lambda}\left(1 + \frac{1}{2}\sigma \mathrm{Re}\frac{\lambda}{1+\lambda}\right).$$

This agrees with the first two terms of the result obtained by Taylor [11].

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Creep Transitions Around a Circular Hole in an Infinite Plate

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Abstract

Seth's transition theory is applied to study the creep phenomenon around a circular hole in an infinite plate under various conditions. The assumptions about the creep laws is not necessary and the results obtained include the classical results as a special case. The reason for the results to be more general than the classical ones stems from the fact that the incompressibility of the materials in creep is not assumed.

1. Introduction

The creep deformations around a circular hole in an infinite plate has been a subject of extensive study by various research workers [1,3,4,5]. In general, the method consists of assuming some creep law and then deriving the creep stresses from it. Naturally, there are different solutions, depending upon the creep law, given for the same problem. Here we take the transition theory approach (developed by Seth [6]) to the above-mentioned problems. Neither, we assume the creep law nor the incompressibility of the materials in creep. This is usual with the transition approach to the creep phenomenon. Our results include the effect of compressibility and as a special case reduce to the classical results derived from the creep law with von Mises rule as given by Nadai [4].

2. Transition Points

Consider an infinite plate of constant thickness with a circular hole of radius a. We shall consider the plate under uniform tension or uniform pressure at the hole. We assume that the plate is thick enough to consider it as a problem in three dimensions. Let the plate be at some constant temperature and in a steady-state creep conditions. We can treat the problem as one of plane strain. Take the centre of the circular hole as origin and z-axis as the axis of the hole assumed to be normal to the plate. The displacements in cylindrical coordinates (r,θ,z) can be taken, due to the symmetry in the problem, as

$$(2.1) \qquad u = r\,(1-\beta), \quad v = o, \quad w = o,$$

where $\beta = \beta(r)$.

The generalised strains, from equation (2.1), are [7]

$$e_{rr} = \frac{1}{n^m} \left\{ 1 - (r\beta' + \beta)^n \right\}^m ,$$

(2.2) $$e_{\theta\theta} = \frac{1}{n^m} (1 - \beta^n)^m ,$$

$$e_{zz} = e_{rz} = e_{\theta z} = e_{r\theta} = 0 ,$$

where $\beta' = \frac{d\beta}{dr}$.

The stress-strain relations are

(2.3) $$T_{ij} = \lambda \Delta \delta_{ij} + 2\mu e_{ij} ,$$

with usual conventions.

The stresses, from equations (2.2) and (2.3), are given by

$$T_{rr} = \lambda\Delta + \frac{2\mu}{n^m} \left\{ 1 - (r\beta' + \beta)^n \right\}^m ,$$

$$T_{\theta\theta} = \lambda\Delta + \frac{2\mu}{n^m} (1 - \beta^n)^m ,$$

(2.4)

$$T_{zz} = \lambda\Delta ,$$

$$T_{r\theta} = T_{\theta z} = T_{rz} = 0 ,$$

where

$$\Delta = \frac{1}{n^m} \left[\left\{ 1 - (r\beta' + \beta)^n \right\}^m + (1 - \beta^n)^m \right] .$$

All the equations of equilibrium are satisfied identically, except

(2.5) $$\frac{\partial T_{rr}}{\partial r} + \frac{T_{rr} - T_{\theta\theta}}{r} = 0 ,$$

which, after using the stresses from equation (2.4), reduces to

$$\{1-(r\beta'+\beta)^n\}^{m-1}(r\beta'+\beta)^{n-1}(r\beta''+2\beta') + (1-c)(1-\beta^n)^{m-1}\beta^{n-1}\beta' -$$

$$-\frac{c}{mnr}[\{1-(r\beta'+\beta)^n\}^m - (1-\beta^n)^m] = 0\ ,$$

where $c = 2\mu/(\lambda+2\mu)$.

Putting $r\beta' = \beta P$ in the above equation, we get

$$(2.6)\quad P(P+1)^{n-1}\ \beta\{1-\beta^n(P+1)^n\}^{m-1}\frac{dP}{d\beta} + P(P+1)^n\{1-\beta^n(P+1)^n\}^{m-1} +$$

$$+ P(1-c)(1-\beta^n)^{m-1} - \frac{c}{mn\beta^n}[\{1-\beta^n(P+1)^n\}^m - (1-\beta^n)^m] = 0.$$

This shows that the transition points of β are

$$P = -1\ ,\ P = \pm\infty\ .$$

We are interested in the transition point $P = -1$. The other transition points lead to plastic stresses.

3. Transition Function

The appropriate transition function is R_2 given by

$$(3.1)\qquad R_2 = T_{\theta\theta} - T_{rr}$$

$$= \frac{2\mu}{n^m}[(1-\beta^n)^m - \{1-\beta^n(P+1)^n\}^m]\ ,$$

after using equation (2.4) and the substitution $r\beta' = \beta P$.

Logarithamic differentiation of equation (3.1) leads to

$$\frac{d\log R_2}{d\log\beta} = -c\frac{d\log r}{d\log\beta} - \frac{mn(2-c)\beta^n(1-\beta^n)^{m-1}}{\{1-\beta^n(P+1)^n\}^m - (1-\beta^n)^m}\ ,$$

which gives

$$(3.2) \qquad \log (R_2/A_\circ) = - c \log r + (2-c) \log \left\{ 1-(1 - \beta^n)^m \right\},$$

as $P \longrightarrow -1$, where $A_\circ$ is a constant of integration.

Since $\beta \longrightarrow (\beta_\circ/r)$ when $P \longrightarrow -1$, equation (3.1) gives

$$(3.3) \qquad T_{\theta\theta} - T_{rr} = A_\circ \, r^{-c} \left\{ 1-(1-B_\circ r^{-n})^m \right\}^{2-c},$$

where $A_\circ$ and $B_\circ$ are constants.

The equation (3.3) is very general in character. This form can be used to describe all the stages of creep provided we take product of two such forms [7]. However, we shall restrict ourselves to a particular case given by m=1. The reason is that the resulting equations have a compact appearance. Therefore, the equation (3.3) for m=1 gives

$$(3.4) \qquad T_{\theta\theta} - T_{rr} = A r^{-2n+c(n-1)},$$

where A is a constant.

Further, the equations (2.5) and (3.4) give

$$(3.5) \qquad T_{rr} = \frac{A}{-2n+c(n-1)} \, r^{-2n+c(n-1)} + B,$$

where B is a constant of integration.

It can be shown, by following the method given in [2], that the stress-strain rate relation for the transition function R_2 is

$$(3.6) \qquad \dot{\varepsilon}_{ij} = k \, (T_{\theta\theta} - T_{rr})^{\frac{1}{n} - 1} \, T'_{ij}$$

where $\dot{\varepsilon}_{ij}$ is the Cauchy strain rate with respect to some suitable flow parameter, T'_{ij} is the stress deviator, k is creep constant and (1/n) is creep index.

4. Hole under Uniform Pressure

We first consider the situation where the hole is under uniform pressure p. The boundary conditions for this are

$$(4.1)\qquad \begin{aligned} T_{rr} &= -\,p \quad \text{at} \quad r = a\,, \\ T_{rr} &= \;\;0 \quad \text{at} \quad r = b \quad \text{where } b \longrightarrow \infty\,. \end{aligned}$$

The stress T_{rr} given by equation (3.5) subjected to the above boundary conditions gives

$$(4.2)\qquad \begin{aligned} T_{rr} &= -\,p\;(a/r)^{2n-c(n-1)}\,, \\ T_{\theta\theta} &= p\left\{(2n-1)-c(n-1)\right\}(a/r)^{2n-c(n-1)}\,, \\ T_{zz} &= p\;(n-1)(1-c)(a/r)^{2n-c(n-1)}\,, \end{aligned}$$

where we have used equations (2.4) and (3.4). These are transitional stresses in creep and take into account the compressibility effect because of the presence of c. Now the classical results can be obtained by letting $c \longrightarrow 0$. They are

$$(4.3)\qquad \begin{aligned} T_{rr} &= -\,p\;(a/r)^{2n}\,, \\ T_{\theta\theta} &= p\;(2n-1)(a/r)^{2n}\,, \\ T_{zz} &= p\;(n-1)(a/r)^{2n}\,. \end{aligned}$$

5. <u>Plate under Uniform Tension</u>

Now we consider the plate to be under all round uniform tension T_o. The boundary conditions in this case are

$$(5.1)\qquad \begin{aligned} T_{rr} &= 0 \quad \text{at} \quad r = a\,, \\ T_{rr} &= T_o \;\text{at} \quad r = b \text{ when } b \longrightarrow \infty\,. \end{aligned}$$

The stress T_{rr} given by equation (3.5) which satisfies the above boundary conditions leads to

$$T_{rr} = T_o\left\{1-(a/r)^{2n-c(n-1)}\right\},$$

$$T_{\theta\theta} = T_o \; [\, \{(2n-1)-c(n-1)\} \; (a/r)^{2n-c(n-1)} + 1] \; ,$$

(5.2)

$$T_{zz} = T_o \; (\tfrac{1-c}{2-c}) \; [\{(2n-2)-c(n-1)\}(a/r)^{2n-c(n-1)} + 2] \; ,$$

after recalling equations (2.4) and (3.4).

These are again the transitional stresses in creep and account for the compressibility effect in creep through c. The classical results, given by Nadai [4], can be obtained by letting $c \longrightarrow 0$. Hence, we get

$$T_{rr} = T_o \{ 1-(a/r)^{2n} \} ,$$

(5.3)
$$T_{\theta\theta} = T_o \{ (2n-1)(a/r)^{2n} + 1 \} ,$$

$$T_{zz} = T_o \{ (n-1)(a/r)^{2n} + 1 \} .$$

It should be noted that the creep index is the inverse of our strain measure index n. The stress concentration at the hole can be easily found.

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On Problems Relating the Horizontal and Complete Lifts of ϕ_μ – Structure

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1. Preliminaries

The ϕ_μ-structure has been defined and studied by Upadhyay and Gupta in [1]. The purpose of the present paper is to obtain certain results on horizontal and complete lifts of ϕ_μ-structure.

Let M be a differentiable manifold of differentiability class C^∞ and of dimension n and let T*(M) denote its cotangent bundle. Then T* (M) is also a differentiable manifold of class C^∞ and of dimension 2n [2]. Throughout this paper, we use the following notations and conventions:

(i) The map $\pi : T^*(M) \longrightarrow M$ is the projection map of T*(M) onto M.

(ii) Suffixes a, b, c, ... ; h, i, j, ... take the values 1 to n and $\bar{i} = i+n$ etc. Suffixes A, B, C, D ... etc. take the values 1 to 2n.

(iii) $\mathcal{J}^r_s(M)$ denotes the set of tensor fields of class C^∞ and of type (r, s) in M. Similarly $\mathcal{J}^r_s(T^*(M))$ denotes the corresponding set of tensor fields in T*(M).

(iv) Vector fields in M are denoted by X, Y, Z ... etc. The Lie product of X and Y is denoted by [X, Y] and the Lie derivative with respect to X and L_X.

If A is a point in M, $\pi^{-1}(A)$ is the fibre over A. Any point $P \in \pi^{-1}(A)$ is an ordered pair (A, p_A) where p is 1-form in M and p_A is the value of p at A. Let U be a coordinate neighbourhood in M such that $A \in U$. Then U induces a coordinate neighbourhood $\pi^{-1}(U)$ in T*(M) and $P \in \pi^{-1}(U)$.

2. Complete lift of ϕ_μ-structure

Let ϕ be a tensor field of type (1, 1) and of class C^∞ in M such that,

$$\phi^4 - \mu^2 \phi^2 = 0 , \tag{2.1}$$

where μ is a complex number not equal to zero. Then M is equipped with ϕ_μ-structure [1].

Let ϕ^h_i be the components of ϕ at a point A in the coordinate neighbourhood U. Then the complete lift ϕ^C of ϕ is also a tensor field of type (1, 1) in T*(M) whose components $\widetilde{\phi}^A_B$ in $\pi^{-1}(U)$ are given by [3],

$$
\tilde{\phi}^{h}{}_{i} = \phi^{h}{}_{i} \; ; \qquad \tilde{\phi}^{h}{}_{\bar{i}} = 0 \; ;
$$

(2.2)

$$
\tilde{\phi}^{\bar{h}}{}_{i} = p_a \quad \left(\frac{\partial \phi^{a}{}_{h}}{\partial x^{i}} - \frac{\partial \phi^{a}{}_{i}}{\partial x^{h}}\right) \; ; \; \phi^{\bar{h}}{}_{\bar{i}} = \phi^{i}{}_{h} \quad ,
$$

where $(x^1, x^2, \ldots, x^n)$ are the coordinates of A relative to U and p_A has the components $(p_1, p_2, \ldots, p_n)$. Therefore, we have,

$$
(2.3) \qquad \phi^{C} = (\tilde{\phi}^{A}{}_{B}) = \begin{pmatrix} \phi^{h}{}_{i} & 0 \\ p_a \left(\partial_i \phi^{a}{}_{h} - \partial_h \phi^{a}{}_{i} \right) & \phi^{i}{}_{h} \end{pmatrix} ,
$$

where $\quad \partial_i = \partial / \partial x^i$.

Thus, in consequence of (2.3), we have,

$$
(\phi^{C})^2 = \begin{pmatrix} \phi^{h}{}_{i} & 0 \\ p_a \left(\partial_i \phi^{a}{}_{h} - \partial_h \phi^{a}{}_{i} \right) & \phi^{i}{}_{h} \end{pmatrix} \times
$$

$$
\times \begin{pmatrix} \phi^{i}{}_{j} & 0 \\ p_t \left(\partial_j \phi^{t}{}_{i} - \partial_i \phi^{t}{}_{j} \right) & \phi^{j}{}_{i} \end{pmatrix}
$$

or,

$$
(2.4) \qquad (\phi^{C})^2 = \tilde{\phi}^{A}{}_{B} \tilde{\phi}^{B}{}_{C} = \begin{pmatrix} \phi^{h}{}_{i} \phi^{i}{}_{j} & 0 \\ 2\phi^{i}{}_{j} \, p_a \, \partial [i \; \phi^{a}{}_{h}] + 2\phi^{i}{}_{h} \, p_t \, \partial [j \; \phi^{t}{}_{i}] & \phi^{i}{}_{h} \phi^{j}{}_{i} \end{pmatrix}
$$

where,

$$(2.5)\qquad 2\,\partial_{[i}\,\phi^a{}_{h]} \;\underline{\underline{\mathrm{def}}}\; (\partial_i\,\phi^a{}_h - \partial_h\,\phi^a{}_i)\,.$$

If we now put,

$$(2.6)\qquad L_{hj} \;\underline{\underline{\mathrm{def}}}\; 2(\phi^i{}_j\,P_a\;\partial_{[i}\,\phi^a{}_{h]} + \phi^i{}_h\,P_t\;\partial_{[j}\,\phi^t{}_{i]})\,,$$

then in view of (2.6), the equation (2.4) takes the form,

$$(2.7)\qquad (\phi^C)^2 = \tilde{\phi}^A{}_B\,\tilde{\phi}^B{}_C = \begin{pmatrix} \phi^h{}_i\,\phi^i{}_j & 0 \\ L_{hj} & \phi^i{}_h\,\phi^j{}_i \end{pmatrix}$$

Thus, we have,

$$(\phi^C)^4 = \tilde{\phi}^A{}_B\,\tilde{\phi}^B{}_C\,\tilde{\phi}^C{}_D\,\tilde{\phi}^D{}_E =$$

$$\begin{pmatrix} \phi^h{}_i\,\phi^i{}_j & 0 \\ L_{hj} & \phi^i{}_h\,\phi^j{}_i \end{pmatrix} \begin{pmatrix} \phi^j{}_k\,\phi^k{}_l & 0 \\ L_{jl} & \phi^k{}_j\,\phi^l{}_k \end{pmatrix},$$

or,

$$(2.8)\qquad (\phi^C)^4 = \tilde{\phi}^A{}_B\,\tilde{\phi}^B{}_C\,\tilde{\phi}^C{}_D\,\tilde{\phi}^D{}_E =$$

$$\begin{pmatrix} \phi^h{}_i\,\phi^i{}_j\,\phi^j{}_k\,\phi^k{}_l & 0 \\ L_{hj}\,\phi^j{}_k\,\phi^k{}_l + L_{jl}\,\phi^i{}_h\,\phi^j{}_i & \phi^i{}_h\,\phi^j{}_i\,\phi^k{}_j\,\phi^l{}_k \end{pmatrix}.$$

We now prove the following theorem:

Theorem (2.1)

In order that the complete lift ϕ^C of a tensor field ϕ having ϕ_μ -structure in M may have the same structure in T*(M), it is necessary and sufficient that,

$$L_{hj}\, \phi^j_{\ k}\, \phi^k_{\ l} + L_{jl}\, \phi^i_{\ h}\, \phi^j_{\ i} = \mu^2 L_{hl}. \tag{2.9}$$

Proof

On considering the structure given by (2.1), we have

$$\phi^4 - \mu^2 \phi^2 = 0 ;$$

or,

$$\phi^h_{\ i}\, \phi^i_{\ j}\, \phi^j_{\ k}\, \phi^k_{\ l} = \mu^2\, \phi^h_{\ r}\, \phi^r_{\ l} \tag{2.10}$$

By virtue of (2.10), the equation (2.8) becomes,

$$(\phi^C)^4 = \tilde{\phi}^A_{\ B}\, \tilde{\phi}^B_{\ C}\, \tilde{\phi}^C_{\ D}\, \tilde{\phi}^D_{\ E} = \tag{2.11}$$

$$\begin{pmatrix} \mu^2\, \phi^h_{\ r}\, \phi^r_{\ l} & 0 \\ L_{hj}\, \phi^j_{\ k}\, \phi^k_{\ l} + L_{jl}\, \phi^i_{\ h}\, \phi^j_{\ i} & \mu^2\, \phi^r_{\ h}\, \phi^l_{\ r} \end{pmatrix}.$$

Let us now assume that the condition (2.9) is also satisfied. Thus, from (2.11), we have,

$$(\phi^C)^4 = \tilde{\phi}^A_{\ B}\, \phi^B_{\ C}\, \phi^C_{\ D}\, \phi^D_{\ E} =$$

$$\begin{pmatrix} \mu^2\, \phi^h_{\ r}\, \phi^r_{\ l} & 0 \\ \mu^2\, L_{hl} & \mu^2\, \phi^r_{\ h}\, \phi^l_{\ r} \end{pmatrix},$$

which, by virtue of (2.7) yields,

$$\tilde{\phi}^{A}{}_{B}\,\tilde{\phi}^{B}{}_{C}\,\tilde{\phi}^{C}{}_{D}\,\tilde{\phi}^{D}{}_{E} = \mu^{2}\,\tilde{\phi}^{A}{}_{F}\,\tilde{\phi}^{F}{}_{E} .$$

Thus the tensor field ϕ^C in T*(M) satisfies,

$$(\phi^C)^4 - \mu^2(\phi^C)^2 = 0 .$$

The necessary part can be proved in a straight forward manner.

<u>Theorem</u> (2.2)

In order that the complete lift ϕ^C of a tensor field ϕ having ϕ_μ -structure in M may have the same structure in T*(M) it is necessary and sufficient that,

$$\phi^{i}{}_{j}\phi^{j}{}_{k}\phi^{k}{}_{l}\, p_a \;\partial[i\, \phi^{a}{}_{h}] + \phi^{i}{}_{h}\phi^{j}{}_{k}\phi^{k}{}_{l}\, p_t \;\partial[j\, \phi^{t}{}_{i}] +$$

$$+ \phi^{i}{}_{h}\phi^{j}{}_{i}\phi^{s}{}_{l}\, p_r \;\partial[s\, \phi^{r}{}_{j}] + \phi^{i}{}_{h}\phi^{j}{}_{i}\phi^{s}{}_{j}\, p_r \;\partial[l\, \phi^{r}{}_{s}] =$$

$$= \mu^2\, (\phi^{t}{}_{l}\, p_a \;\partial[t\, \phi^{a}{}_{h}] + \phi^{t}{}_{h}\, p_a \;\partial[l\, \phi^{a}{}_{t}]) .$$

<u>Proof</u>

From theorem (2.1), it follows that the complete lift ϕ^C of the tensor field ϕ with ϕ_μ -structure in M will have the same structure in T*(M) if and only if (2.9) is satisfied.

The condition (2.9) in view of (2.6) can be expressed as,

$$\left\{\phi^{i}{}_{j}\, p_a \;\partial[i\, \phi^{a}{}_{h}] + \phi^{i}{}_{h}\, p_t \;\partial[j\, \phi^{t}{}_{i}]\right\} \phi^{j}{}_{k}\, \phi^{k}{}_{e} +$$

$$\left\{\phi^{s}{}_{l}\, p_r \;\partial[s\, \phi^{r}{}_{j}] + \phi^{s}{}_{j}\, p_r \;\partial[l\, \phi^{r}{}_{s}]\right\} \phi^{i}{}_{h}\, \phi^{j}{}_{i} =$$

$$= \mu^2 \left\{\phi^{t}{}_{l}\, p_a \;\partial[t\, \phi^{a}{}_{h}] + \phi^{t}{}_{h}\, p_a \;\partial[l\, \phi^{a}{}_{t}]\right\}$$

$$\phi^i_{\ j}\, \phi^j_{\ k}\, \phi^k_{\ l}\, p_a \ \ \partial\,[i\ \phi^a_{\ h}] + \phi^i_{\ h}\, \phi^j_{\ k}\, \phi^k_{\ l}\, p_t \ \ \partial\,[j\ \phi^t_{\ i}] \ +$$

$$\phi^i_{\ h}\, \phi^j_{\ i}\, \phi^s_{\ l}\, p_r \ \ \partial\,[s\ \phi^r_{\ j}] + \phi^i_{\ h}\, \phi^j_{\ i}\, \phi^s_{\ j}\, p_r \ \ \partial\,[l\ \phi^r_{\ s}] \ =$$

$$= \mu^2\, \phi^t_{\ l}\, p_a \ \ \partial\,[t\ \phi^a_{\ h}] + \mu^2\, \phi^t_{\ h}\, p_a \ \ \partial\,[l\ \phi^a_{\ t}]\ ,$$

which proves the theorem.

3. Nijenhuis Tensor of the complete lift of ϕ^4

Since ϕ is a (1, 1) tensor field with ϕ_μ-structure in M, the Nijenhuis tensor of ϕ is given by [3],

$$(3.1) \qquad N_{\phi,\phi}\,(X,Y) = [\phi X,\ \phi Y] - \phi\,[\phi X,\ Y] - \phi\,[x,\ \phi Y] + \phi^2\,[X,\ Y]\ .$$

Now, we prove the following theorem:

Theorem (3.1)

The Nijenhuis tensor of the complete lift of ϕ^4 vanishes if the Lie derivatives of the tensor field ϕ^2 with respect to X and Y are both zero and ϕ admits π-structure on M.

Proof

In view of (3.1) the Nijenhuis tensor of $(\phi^4)^C$ is given by,

$$(3.2) \qquad N_{(\phi^4)^C,(\phi^4)^C}(X^C,Y^C) = [(\phi^4)^C X^C,\ (\phi^4)^C Y^C] \ -$$

$$- (\phi^4)^C\ \ [(\phi^4)^C X^C,\ Y^C] - (\phi^4)^C\ [X^C, (\phi^4)^C Y^C] \ +$$

$$+ (\phi^4)^C (\phi^4)^C\ \ [X^C,\ Y^C]\ ,$$

which by virtue of (2.1) becomes,

$$N_{(\phi^4)^C,(\phi^4)^C}(X^C, Y^C) = [(\mu^2\phi^2)^C X^C, (\mu^2\phi^2)^C Y^C] -$$

$$- (\mu^2\phi^2)^C [(\mu^2\phi^2)^C X^C, Y^C] - (\mu^2\phi^2)^C [X^C, (\mu^2\phi^2)^C Y^C]$$

$$+ (\mu^2\phi^2)^C (\mu^2\phi^2)^C [X^C, Y^C] ,$$

or,

$$(3.3) \quad N_{(\phi^4)^C,(\phi^4)^C}(X^C, Y^C) = \mu^4 \{ [(\phi^2)^C X^C, (\phi^2)^C Y^C] -$$

$$(\phi^2)^C [(\phi^2)^C X^C, Y^C] - (\phi^2)^C [X^C, (\phi^2)^C Y^C] +$$

$$+ (\phi^2)^C (\phi^2)^C [X^C, Y^C] \} .$$

But we know that ([2], p. 243)

$$(3.4) \quad (\phi^2)^C X^C = (\phi^2 X)^C + (L_X \phi^2)^V .$$

Therefore, by virtue of (3.4), the equation (3.3) becomes,

$$(3.5) \quad N_{(\phi^4)^C,(\phi^4)^C}(X^C, Y^C) = \mu^4 \{ [(\phi^2 X)^C, (\phi^2 Y)^C] +$$

$$+ [(L_X\phi^2)^V, (\phi^2 Y)^C] + [(\phi^2 X)^C, (L_Y\phi^2)^V] +$$

$$+ [(L_X\phi^2)^V, (L_Y\phi^2)^V] - (\phi^2)^C [(\phi^2 X)^C, Y^C] -$$

$$- (\phi^2)^C [(L_X\phi^2)^V, Y^C] - (\phi^2)^C [X^C, (\phi^2 Y)^C] -$$

$$- (\phi^2)^C [X^C, (L_Y\phi^2)^V] + (\phi^2)^C (\phi^2)^C [X^C, Y^C] \} .$$

Let the Lie derivatives of ϕ^2 with respect to X and Y be both zero i.e.

$$L_X \phi^2 = L_Y \phi^2 = 0 ,$$

we have, from (3.5),

$$(3.6)\qquad N_{(\phi^4)^C,(\phi^4)^C}(X^C,Y^C) = \mu^4\Big\{[(\phi^2X)^C, (\phi^2Y)^C]$$

$$- (\phi^2)^C\,[(\phi^2X)^C Y^C] - (\phi^2)^C\,[X^C,(\phi^2Y)^C]$$

$$+ (\phi^2)^C(\phi^2)^C\,[X^C, Y^C]\Big\}.$$

We also know that for arbitrary vector fields X, Y in M ([2], p. 238),

$$(3.7)\qquad [X^C, Y^C] = [X, Y]^C .$$

By virtue of (3.7) the equation (3.6) becomes,

$$(3.8)\qquad N_{(\phi^4)^C,(\phi^4)^C}(X^C, Y^C) = \mu^4\Big\{[\phi^2X, \phi^2Y]^C -$$

$$- (\phi^2)^C\,[\phi^2X, Y]^C - (\phi^2)^C\,[X, \phi^2Y]^C +$$

$$+ (\phi^2)^C(\phi^2)^C\,[X, Y]^C\Big\}.$$

Further, if ϕ is a π-structure on M, then [1],

$$(3.9)\qquad \phi^2 = \mu^2 I ,$$

where I denotes the unit tensor field. Thus we have,

$$N_{(\phi^4)^C,(\phi^4)^C}(X^C, Y^C) = \mu^8\Big\{[X, Y]^C - [X,Y]^C - [X,Y]^C + [X,Y]^C\Big\},$$

$$= 0 .$$

This proves the theorem.

<u>Theorem (3.2)</u>

The Nijenhuis tensor of the complete lift of ϕ^4 is equal to the complete lift of the Nijenhuis tensor of ϕ^2 multiplied by μ^4 if,

$$\text{(i)} \quad L_X \phi^2 = L_Y \phi^2 = 0 ,$$

(3.9) and

$$\text{(ii)} \quad [X,Y]^C = 0 , \quad \tilde{K}^V = 0 ,$$

where,

$$(3.10) \qquad \tilde{K}^V \ \underline{\underline{\mathrm{def}}} \ L_{[\phi^2 X,Y]} \phi^2 + L_{[X,\phi^2 Y]} \phi^2 - L_{[X,Y]} \phi^4 .$$

Proof. We have

$$\left(N_{\phi^2,\phi^2}(X,Y)\right)^C = [\phi^2 X,\phi^2 Y]^C - (\phi^2\ [\phi^2 X,Y])^C$$
$$- (\phi^2 [X,\phi^2 Y])^C + (\phi^4 [X,Y])^C ,$$

which, in view of (3.4) yields,

$$(3.11) \quad \left(N_{\phi^2,\phi^2}(X,Y)\right)^C = [\phi^2 X,\phi^2 Y]^C - \left\{(\phi^2)^C [\phi^2 X,Y]^C - \right.$$
$$\left. -\left(L_{[\phi^2 X,Y]} \phi^2\right)^V \right\} - \left\{ (\phi^2)^C [X,\phi^2 Y]^C - \right.$$
$$\left. -\left(L_{[X,\phi^2 Y]} \phi^2\right)^V \right\} + \left\{ (\phi^4)^C\ [X,Y]^C - (L_{[X,Y]} \phi^4)^V \right\} .$$

But we know that ([2], p. 253),

$$(3.12) \qquad (\phi^2)^C (\phi^2)^C = (\phi^4)^C + (N_{\phi^2,\phi^2})^V .$$

Thus the equation (3.11) by virtue of (3.12) becomes,

$$(3.13) \qquad \left(N_{\phi^2,\phi^2}(X,Y)\right)^C = [\phi^2 X,\phi^2 Y]^C - (\phi^2)^C [\phi^2 X,Y]^C -$$

$$- (\phi^2)^C [X,\phi^2 Y]^C + (\phi^2)^C(\phi^2)^C [X,Y]^C - (N_{\phi^2,\phi^2})^V [X,Y]^C$$

$$+ \left(L_{[\phi^2 X,Y]} \phi^2 + L_{[X,\phi^2 Y]} \phi^2 - L_{[X,Y]} \phi^4 \right)^V ,$$

which with the help of (3.8) and (3.10) becomes,

$$\mu^4 (N_{\phi^2,\phi^2}(X,Y))^C = N_{(\phi^4)^C,(\phi^4)^C}(X^C,Y^C) \tag{3.14}$$

$$- \mu^4 (N_{\phi^2,\phi^2})^V [X,Y]^C + \widetilde{K}^V .$$

If $[X,Y]^C = 0$ and $\widetilde{K}^V = 0$, the equation (3.14) reduces to,

$$N_{(\phi^4)^C,(\phi^4)^C}(X^C,Y^C) = \mu^4 (N_{\phi^2,\phi^2}(X,Y)^C ,$$

which proves the theorem.

4. Horizontal Lift of ϕ_μ -structure

Let us prove the following theorem.

Theorem (4.1)

Let ϕ be a (1,1) tensor field admitting ϕ_μ -structure in M, then the horizontal lift ϕ^H of ϕ also admits the same structure in the cotangent bundle T*(M).

Proof

For ϕ and ψ such that $\phi, \psi \in \mathfrak{J}^1_1(M)$, we have [4],

$$\phi^H \psi^H + \psi^H \phi^H = (\phi\psi + \psi\phi)^H . \tag{4.1}$$

Taking the particular case when ϕ and ψ are identical, we get,

(4.2) $$(\phi^H)^2 = (\phi^2)^H .$$

Similarly, giving values ϕ^2 and ϕ^3 to ψ in (4.1), we get,

(4.3) $$(\phi^H)^3 = (\phi^3)^H$$

and,

(4.4) $$(\phi^H)^4 = (\phi^4)^H$$ respectively.

Since ϕ admits ϕ_μ -structure in M, hence ϕ satisfies,

$$\phi^4 - \mu^2 \phi^2 = 0 .$$

Thus the euqation (4.4) becomes,

$$(\phi^H)^4 = (\mu^2 \phi^2)^H ,$$

which again, in view of (4.2) becomes,

$$(\phi^H)^4 - \mu^2 (\phi^H)^2 = 0 .$$

Thus the horizontal lift ϕ^H of ϕ also admits ϕ_μ -structure in the cotangent bundle T*(M).

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A Formal Solution of Qadruple Integral Equations

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1. Introduction

Recently Saxena and Sethi [4, p. 516] have obtained a solution of triple integral equations by applying the technique used by Cooke [1], Erdelyi [3] and Sneddon [5].

In this note by using Saxena and Seth's operators and following the technique given by Cooke [2], we have obtained a solution of the following quadruple integral equations:

$$(1.1) \qquad \int_0^\infty P^{\beta}_{-\frac{1}{2}+i\xi} (\cosh x)\ [1 + r(\xi)]\ \psi(\xi) d\xi = f_1(x),\ 0 < x < a.$$

$$(1.2) \qquad \int_0^\infty P^{\beta}_{-\frac{1}{2}+i\xi} (\cosh x)\ \psi(\xi)\ d\xi = g_2(x), \quad a < x < b$$

$$(1.3) \qquad \int_0^\infty P^{\beta}_{-\frac{1}{2}+i\xi} (\cosh x)\ [1 + \bar{r}(\xi)]\ \psi(\xi) d\xi = \bar{f}_3(x), \quad b < x < c$$

$$(1.4) \qquad \int_0^\infty P^{\beta}_{-\frac{1}{2}+i\xi} (\cosh x)\ \psi(\xi)\ d\xi = 0\ , \quad c < x < \infty\ ,$$

where f's and g's are known functions and ψ is an unknown function.

The method is purely formal and no conditions are given under inversion of order of integrations.

2. Known Results

The following known results will be used in our proposed work. We know that Saxena and Sethi [4, p. 514],

$$(2.1)\qquad I^{\beta}(f) = \frac{1}{\Gamma(\frac{1}{2}-\beta)} \int_{0}^{x} (\cosh x - \cosh\alpha)^{-\frac{1}{2}-\beta} f(\alpha)\,d\alpha\ , \qquad \operatorname{Re}\beta < \tfrac{1}{2}\ ,$$

$$(2.2)\qquad K^{\beta}(f) = \frac{1}{\Gamma(\frac{1}{2}+\beta)} \int_{x}^{\infty} (\cosh\alpha - \cosh x)^{\beta-\frac{1}{2}} f(\alpha)\,d\alpha\ , \qquad \operatorname{Re}\beta < \tfrac{1}{2}\ ,$$

$$(2.3)\qquad K^{\beta^{-1}}(f) = \frac{1}{\Gamma(\frac{1}{2}+\beta)} \frac{d}{dx} \int_{0}^{x} (\cosh x - \cosh\alpha)^{\beta-\frac{1}{2}} \times$$

$$\times \sinh\alpha\ f(\alpha)\,d\alpha\ , \qquad \operatorname{Re}\beta < -\tfrac{1}{2}$$

$$(2.4)\qquad K^{\beta^{-1}}(f) = \frac{1}{\Gamma(\frac{1}{2}-\beta)} \frac{d}{dx} \int_{x}^{\infty} (\cosh\alpha - \cosh x)^{-\frac{1}{2}-\beta} \times$$

$$\times \sinh\alpha\ f(\alpha)\,d\alpha\ , \qquad \operatorname{Re}\beta < \tfrac{1}{2}$$

$$(2.5)\qquad T_{c}^{\beta}(f) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} (\sinh z)^{\beta} \int_{0}^{\infty} f(\alpha)\cos(\alpha x)\,d\alpha$$

$$(2.6)\qquad T_{s}^{\beta}(f) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} (\sinh z)^{-\beta} \int_{0}^{\infty} f(\alpha)\sin(\alpha x)\,d\alpha$$

$$(2.7)\qquad T_{p}^{\beta}(f) = \int_{0}^{\infty} f(\nu)\, P^{\beta}_{-\frac{1}{2}+i\nu}(\cosh x)\,d\nu$$

$$(2.8)\qquad T_{p}^{\beta^{-1}}(f) = \int_{0}^{\infty} \frac{\nu}{\pi}\sinh(\pi\nu)\,\Gamma(\tfrac{1}{2}-\beta+i\nu)\,\Gamma(\tfrac{1}{2}+\beta-i\nu)\,f(x) \times$$

$$\times P_{-\frac{1}{2}+i\nu}(\cosh x)\sinh x\,dx.$$

As is well known that the operators

$$\binom{x}{a} \; I^{\beta}(f) \quad \text{and} \quad \binom{b}{a} K^{\beta}(f)$$

denote the original ones where the new limits are a to x and a to b respectively.

(2.9) $$I^{\beta}[T_c^{\beta}(f)] = T_p^{\beta}(f).$$

(2.10) $$K^{\beta}[T_s^{\beta}(f)] = T_p^{\beta}[\tfrac{1}{\pi}\Gamma(\tfrac{1}{2}-\beta+i\nu)\,\Gamma(\tfrac{1}{2}+\beta-i\nu)\sinh(\pi\nu)f(\nu)]$$

(2.11) $$T_p^{\beta}[T_c^{\beta}(f)] = I^{\beta}(f)$$

(2.12) $$I^{\beta^{-1}}[T_p^{\beta}(f)] = T_c^{\beta}(f)$$

(2.13) $$K^{\beta^{-1}}[T_p^{\beta}\{\tfrac{1}{\pi}\;\Gamma(\tfrac{1}{2}-\beta+i\nu)\;\Gamma(\tfrac{1}{2}+\beta-i\nu)\sinh(\pi\nu)f(\nu)\}] = T_s^{\beta}(f)$$

(2.14) $$T_p^{\beta^{-1}}[I^{\beta}(f)] = T_c^{\beta}(f)$$

The operator L^{β} and M^{β} are given by

(2.15) $$\binom{x}{d} \; I^{\beta^{-1}} \binom{f}{e} I^{\beta}(f) = \begin{pmatrix} x & f \\ d & e \end{pmatrix} L^{\beta}(f)$$

$$= \frac{\sin\,\pi(\tfrac{1}{2}+\beta)\sinh x}{\pi(\cosh x - \cosh d)^{\frac{1}{2}-\beta}} \int_e^f \frac{(\cosh d - \cosh s)^{\frac{1}{2}-\beta} f(s)ds}{(\cosh x - \cosh s)},$$

(2.16) $$\binom{d}{x} \; K^{\beta^{-1}} \binom{f}{e} K^{\beta}(f) = \begin{pmatrix} d & f \\ x & e \end{pmatrix} M^{\beta}(f)$$

$$= \frac{\sin \pi(\frac{1}{2} + \beta)\sinh x}{\pi(\cosh x - \cosh d)^{\frac{1}{2} - \beta}} \int_e^f \frac{(\cosh s - \cosh d)^{\frac{1}{2} - \beta} f(s)ds}{(\cosh s - \cosh x)},$$

provided that $x > d \geqslant f > e$ and $x < d \leqslant e < f$ respectively.

3. Solution of the Quadruple Integral Equations

Let us suppose that I_1, I_2, I_3 and I_4 denote the intervals $(0 < x < a)$, $(a < x < b)$, $(b < x < c)$ and $(c < x < \infty)$ respectively and write any function $f(x)$ for $x \geqslant 0$ as

$$f = f_1 + f_2 + f_3 + f_4$$

where

$$f_i(x) = \begin{cases} f(x); \; x \in I_i \\ 0; \; x \notin I_i, \; i=1,2,3,4 \end{cases}$$

Application of (2.7) on the quadruple integral equations (1.1), (1.2), (1.3) and (1.4) yield

$$(3.1) \qquad T_p^{\beta} [\{1 + r(x)\} \psi(x)] = f(x)$$

$$(3.2) \qquad T_p^{\beta} [\{1 + \bar{r}(x)\} \psi(x)] = \bar{f}(x)$$

$$(3.3) \qquad T_p^{\beta} [\psi(x)] = g(x)$$

Let

$$(3.4) \qquad g(x) = I^{\beta} \{\phi(x)\}$$

where $\phi(x)$ is some unknown function.

Then from (3.3), (3.4) and (2.14), we have

$$\psi(x) = T_c^{\beta} \{ \phi(x) \}$$

Substitute this value of $\psi(x)$ in (3.1) and use (2.11), we get

$$= \frac{\sin \pi(\frac{1}{2} + \beta)\sinh x}{\pi(\cosh x - \cosh d)^{\frac{1}{2} - \beta}} \int_e^f \frac{(\cosh s - \cosh d)^{\frac{1}{2} - \beta} f(s)ds}{(\cosh s - \cosh x)},$$

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Then from (3.3), (3.4) and (2.14), we have

$$\psi(x) = T_c^{\beta}\,\{\,\phi(x)\}$$

Substitute this value of $\psi(x)$ in (3.1) and use (2.11), we get

$$(3.5) \qquad I^{\beta}\phi(x) + \int_0^{\infty} \phi(y)K(x,y)dy = f(x)$$

where

$$(3.6) \qquad K(x,y) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} (\sinh z)^{\beta} \int_0^{\infty} r(\nu)\cos(\nu y)P^{\beta}_{-\frac{1}{2}+i\nu}(\cosh x)d\nu .$$

A similar expression with bars over r and f also exists. Now evaluating (3.5) on I_4, we find that (since $g_4 = 0$) $\phi_4 = 0$, whilst g_2, f_1 and $\bar{f}_3$ are known. Hence the upper limit of y-integral in (3.5) is c. Evaluating (3.5) on I_1 and solving for ϕ_1, we get

$$(3.7) \qquad \phi_1(x) = \begin{pmatrix} x \\ o \end{pmatrix} I^{\beta^{-1}} f_1(x) - \int_0^{c} \phi(y)K_1(x,y)dy \quad ,$$

where $K_1(x,y)$ is equal to

$$(3.8) \qquad \frac{2}{\pi}(\sinh z)^{2\beta} \int_0^{\infty} r(\nu)\cos(\nu x)\cos(\nu y)dy .$$

Equation (3.4) when evaluated on I_2, gives

$$g_2 = \begin{pmatrix} b \\ x \end{pmatrix} I^{\beta}(\phi_2) + \begin{pmatrix} c \\ b \end{pmatrix} I^{\beta}(\phi_3),$$

which on solving for ϕ_2, yields

$$(3.9) \qquad \phi_2 = \begin{pmatrix} b \\ x \end{pmatrix} I^{\beta^{-1}}(g_2) + \begin{pmatrix} x & c \\ b & b \end{pmatrix} L^{\beta}(\phi_3).$$

Finally evaluating (3.5) with bars over r and f on I_3, we find that

$$\begin{pmatrix} a \\ o \end{pmatrix} I^{\beta}(\phi_1) + \begin{pmatrix} b \\ a \end{pmatrix} I^{\beta}(\phi_2) + \begin{pmatrix} x \\ b \end{pmatrix} I^{\beta}(\phi_3) +$$

$$+ \int_0^{c} \phi(y)\bar{K}(x,y)dy = \bar{f}_3.$$

Soving it for ϕ_3, we get

$$(3.10) \qquad \phi_3 = \begin{pmatrix} x \\ b \end{pmatrix} I^{\beta^{-1}} (\bar{f}_3) - \begin{pmatrix} x & a \\ b & o \end{pmatrix} L^{\beta}(\phi_1) - \begin{pmatrix} x & b \\ b & a \end{pmatrix} L^{\beta}(\phi_2) -$$

$$- \int_0^c \phi(y) K_2(x,y)\,dy \ ,$$

where

$$(3.11) \qquad K_2(x,y) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} (\sinh z)^{\beta} \begin{pmatrix} x \\ b \end{pmatrix} I^{\beta^{-1}} \int_0^{\infty} \bar{r}(\nu) \cos(\nu y) \times$$

$$\times P^{\beta}_{-\frac{1}{2}+i\nu} (\cosh x)\, d\nu \ .$$

Thus (3.7), (3.9) and (3.10) are three equations for three unknown functions ϕ_1, ϕ_2 and ϕ_3.

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The Characteristic Geometrical Property of First Order Linear Ordinary Differential Equations

S.R.P. Sinha

The equation

(1) $$\frac{dy}{dx} + P\,y = Q\ ,$$

where P and Q are continuous functions of x in the segment (x_1, x_2) of the x- axis, is called a linear ordinary differential equation of first order. The solution of this equation may be found for any $y_1 < y < y_2$. In particular, the domain of the solution may be the whole of the xoy-plane $(-\infty < x < +\infty,\ -\infty < y < +\infty)$. The integral curve of (1) through (x_0, y_0) is given by,

(2) $$y\, e^{\int P dx} - y_0\, e^{\int P_0 dx_0} = \int_{x_0}^{x} e^{\int P dx}\, Q.dx$$

It should be noted that the solution may be put in the general form,

(3) $$y = C\,.\,\phi(x) + \psi(x)$$

where C is an arbitrary constant.

The integral curves of (1) in $(-\infty < x < +\infty,\ -\infty < y < +\infty)$ have an interesting geometrical property. The tangents to the integral curves at points of a line x= k (constant) meet at a point (X (k), Y(k)):

Proof

The tangent to the integral curve of (1) at (k,y) is given by,

$$Y - y = \{Q(k) - y.\ P(k)\}\ (X-x)$$

i.e., $$Y + k\,Q(k) - X.\ Q(k) - y\ \{P(k).\ X - 1 - k.\ P(k)\} = 0$$

This is satisfied by

$$X = k + \frac{1}{P(k)}\ , \quad Y = \frac{Q(k)}{P(k)}\ , \quad \text{for all } y.$$

This property may be understood from the following figure.

It is to be noted that when $P(k)=0$, $X = Y = \infty$, and the tangents are all parallel.

This geometrical property is characteristic of a first order linear ordinary differential equation. In fact, we can show that a first order differential equation,

$$\frac{dy}{dx} = F(x,y) \tag{4}$$

with F and F_y continuous in the whole of the xoy- plane, is linear if the tangents to the integral curves at all points of a line, $x=k$, are concurrent. For, the tangent to the integral curve of (4) at (k,y) is given by

$$Y - y = F(k,y)\,(X-k)$$

If it passes through a point $(X(k), Y(k))$ for all y, then

$$-1 = F_y \cdot (X-k)$$

$$\text{i.e.,} \qquad F_y = -\frac{1}{X-k}$$

This, on integration, gives

$$F(k,Y) = -\frac{y}{X(k)-k} + Q(k) \ ,$$

where $Q(k)$ is an arbitrary function of k.

Thus (4) is reduced to

$$\frac{dy}{dx} + \frac{1}{X(x)-x} \cdot y = Q(x)$$

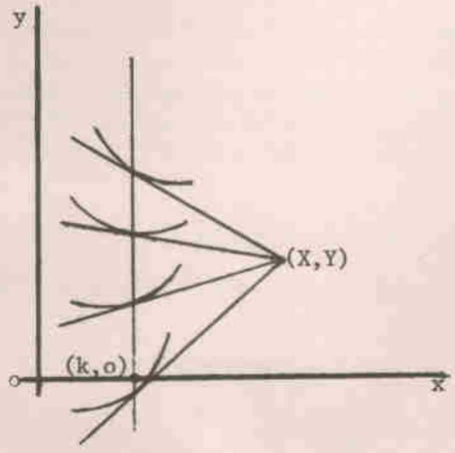
which is linear. Substituting $P(x) = \frac{1}{X(x)-x}$ in the above equation we get

$$\frac{dy}{dx} + P(x) \cdot y = Q(x)$$

N.B.: (i) If $P(x)$ has no zero in $-\infty < x < +\infty$, then the above geometrical characterization will hold in the segment

$$\text{glb}\left\{\frac{Q(x)}{P(x)}\right\} < y < \text{lub}\left\{\frac{Q(x)}{P(x)}\right\}.$$

(ii) One can verify that the family (3) has the same geometrical property as that of (1).



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A Survey of Ordered Loops II

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4. A partial order P_1 is called an extension of the partial order P if $P \subseteq P_1$. If $P \subset P_1$, P_1 is called a proper extension of P, otherwise P_1 is improper extension. If the partial order P has no proper extensions it is called a maximal partial order.

Propsition 5 [1]. If P_1 & P_2 are two different partial orders of p.o. loop $\mathfrak{Q}$ such that $P_1 \cap P_2 = e$ and $P_1 \cap N_2 = e$, where N_2 is negative cone of the second partial order, then P_1P_2 defines a partial order P' which is an extension of the first and the second partial orders.

Really, from Lemma 1, it follows that $P' = P_1P_2$ is an invariant groupoid, because $e \in P_1$ & $e \in P_2 \Longrightarrow P' \supset P_1$, $P' \supset P_2$ and $e \in P'$. So, we have to show that $x, y \in P'$ & $xy = e \Longrightarrow x = y = e$. Let $x \in P_1$, $y \in P_2$ and $xy = e$, then $x \in N_2$, but $P_1 \cap N_2 = e \Longrightarrow x = e$ and $y = x = e$.

Lemma 3. If $a \in Q$, $a \neq e$, $e \notin S(R_ae)$, then $S_e(R_ae)$ defines some partial order on $\mathfrak{Q}$.

The condition $e \notin S(R_ae)$ and invariance of the groupoid $S(R_ae)$ imply the following properties of $S_e(R_ae)$:

1) $e \in S_e(R_ae)$;

2) $S_e \cdot S_e = S_e$;

3) $xy = e \ (x, y \in S_e(R_ae)) \Longrightarrow x = y = e$;

4) $S_e \cdot (xy) = (S_ex).y = x.(S_ey) \ \forall \ x, y \in Q$.

So, all the conditions of the Theorem 1 are satisfied and $S_e(R_ae)$ can be considered as a positive cone of some partial order on $\mathfrak{Q}$.

Theorem 2. If the partial order P at p.o. loop $\mathfrak{Q}$ has the property: for some element $a \in Q$, $a /\!/ e$ either $P \cap S(R_ae) = \emptyset$ or $P \cap S(R_a^{-1}e) = \emptyset$, then correspondently either $P' = P \cdot S(R_a^{-1}e)$ or $P' = P \cdot S(R_ae)$ defines partial order P' which is an extension of P.

Really, if $P \cap S(R_ae) = \emptyset$ then $e \notin S(R_ae) \Longrightarrow e \notin S(R_a^{-1}e)$ and so $S_e(R_a^{-1}e)$ defines a partial order on Q satisfying proposition 5 and $P' = PS_e(R_a^{-1}e)$ defines an extension of partial order P on Q.

Theorem 3. The partial order P of p.o. loop $\mathfrak{Q}$ is a maximal partial order iff for any element $a \in Q (a /\!/ e)$

*This survey is in continuation of "A survey of the ordered loops I" which appeared in the previous issue of this journal.

$$\text{(i) } P \cap S(R_a e) \neq \emptyset \ \& \ \text{(ii) } P \cap S(R_a^{-1} e) \neq \emptyset .$$

Let P be a maximal partial order on Q and $a \in Q$, $a // e$. If $P \cap S(R_a e) = \emptyset \Longrightarrow$ there exists a proper extension $P' = PS_e(R_a^{-1}e)$ contrary to the maximality of P.

And conversely, let some partial order P satisfy the conditions (i) & (ii) of the Theorem, then P is a maximal partial order. Really, if there exists an extension P' then there exists at least one element $a \in Q$ $(a \neq e)$ such that either $P \cap S(R_a e) = \emptyset$ or $P \cap S(R_a^{-1} e) = \emptyset$, but then a is comparable with e and so either $P' = P \cdot S_e(R_a^{-1} e) = P$ or $P' = PS_e(R_a e) = P$ and P is a maximal partial order.

This theorem can also be stated in the following way:

<u>Theorem 3a.</u> The partial order P of p.o. loop $\mathfrak{Q}$ has no proper extensions iff any other partial order P' on Q satisfies the condition: $P \cap N'$ contains at least one element $a \neq e$.

<u>Def.</u> An element $a \in Q$ is called a generalized periodic element is $S(R_a e) \cap T(R_a e) \neq \emptyset$.

This implies that the loop whose elements are generalized periodic doesn't allow any non-trivial partial order.

If for some partial order P of p.o. loop $\mathfrak{Q}$ all the elements non-comparable with e are generalized periodic then P is a maximal partial order.

Let $\{S\}$ be the set of groupoids in Q such that:

i) $e \in S \ \forall S \in \{S\}$; ii) all the $S \in \{S\}$ are invariant in Q;

iii) $xy = e \ (x, y \in S) \Longrightarrow x = y = e \ \forall S \in \{S\}$.

It is clear that any of these groupoids defines on Q some partial order. But the set $\{S\}$ is a partially ordered set with respect to inclusion. But the union of any chain of groupoids $S \in \{S\}$ also satisfies the conditions i) $\div$ iii) and moreover it is an upper bound of this chain. So by the Kuratowsky-Zorn's Theorem any groupoid with properties i) $\div$ iii) is included in some maximal groupoids with the same properties.

So, any partial order of p.o. loops can be extended to a maximal one. Surely any total order on Q is a maximal order.

5. In this section, we shall find the necessary and sufficient conditions for the partial order to be extended to the total order.

Lemma 4. If the partial order P of p.o. loop $\mathfrak{O}\!\!\!\!\!\!\mathcal{L}$ satisfies the condition: (*) for every finite collection of elements $a_1, a_2, \ldots, a_n \in Q$ $(a_i \neq e)$ it is possible to choose signs $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ $(\varepsilon_i = 1$ or $-1)$ such that

$$P \cap S(R_a^{\varepsilon_1} e), \ldots, R_{a_n}^{\varepsilon_n} e) = \emptyset$$

then for any element a in Q either $P \cdot S_e(R_a e)$ or $P \cdot S_e(R_a^{-1} e)$ defines partial order P′ on Q which also satisfied the condition (*).

Proof. Suppose that neither $P \cdot S_e(R_a e)$ nor $P \cdot S_e(R_a e)$ satisfies the condition (*), then there exist elements $a_1, a_2, \ldots, a_n$ & $b_1 b_2, \ldots, b_m$ such that for any choice of signs ε_i & η_j they satisfy:

1) $PS_e^*\ (R_a e) \cap S(R_a^{\varepsilon_1} e, \ldots, R_{a_n}^{\varepsilon_n} e) \neq \emptyset$

2) $PS_e\ (R_a e) \cap S(R_{b_1}^{\eta_1} e, \ldots, R_{b_m}^{\eta_m} e) \neq \emptyset$

From (1) it follows that there exists an element ps* (where $p \in P$, $s^* \in S_e^*(R_a e)$), such that $ps^* \in S(R_{a_1}^{\varepsilon_1} e, \ldots, R_{a_n}^{\varepsilon_n} e)$.

Let $(S(R_a e, R_{a_1}^{\varepsilon_1} e, \ldots, R_{a_n}^{\varepsilon_n} e);\ T\ (R_a e, R_{a_1}^{\varepsilon_1} e, \ldots, R_{a_n}^{\varepsilon_n} e))$,

$(S^*(R_a e);\ T^*(R_a e))$, $(S(R_a e);\ T(R_a e))$ be the pairs generated by the collections $\binom{a,\ a_1, \ldots, a_n}{1, \varepsilon_1, \ldots, \varepsilon_n}$, $\binom{a}{-1}$ & $\binom{a}{1}$ respectively.

The definition of pairs implies:

$$S(R_a e, R_{a_1}^{\varepsilon_1} e, \ldots, R_{a_n}^{\varepsilon_n} e) \supset S(R_a e) \ \& \qquad \text{(I)}$$

$$S(R_a e, R_{a_1}^{\varepsilon_1} e, \ldots, R_{a_n}^{\varepsilon_n} e) \supset S(R_{a_1}^{\varepsilon_1} e, \ldots, R_{a_n}^{\varepsilon_n} e) \qquad \text{(II)}$$

From lemma 2 if follows that

$$S(R_a e, R_{a_1}^{\varepsilon_1} e, \ldots, R_{a_n}^{\varepsilon_n} e) \supset T^*(R_a e). \qquad \text{(III)}$$

(II) implies $ps^* \in S(R_a e, R_{a_1}^{\varepsilon_1} e, \ldots, R_{a_n}^{\varepsilon_n} e)$.

Moreover, $s^* \in S^*(R_a e) \Longrightarrow R_{s^*}^{-1} e \in T^*(R_a e)$ and so from (III)

it follows $R_{s^*}^{-1} e \in S(R_a e, R_{a_1}^{\varepsilon_1} e, \ldots, R_{a_n}^{\varepsilon_n} e)$. So groupoid

$S(R_a e, R_{a_1}^{\varepsilon_n} e, \ldots, R_{a_n}^{\varepsilon_n} e)$ contains elements ps^* and $R_{s^*}^{-1\varepsilon_n} e$.

Proposition 3 and definition of pairs imply.

$R_{s^*}^{-1}(ps^*) = R_{s^*}^{-1}(R_{s^*}p) = b\, R_{s^*}^{-1} e = p \in S(R_a e, R_{a_1}^{\varepsilon_1} e, \ldots, R_{a_n}^{\varepsilon_n} e)$.

So, we have:

$$P \cap S(R_a e, R_{a_1}^{\varepsilon_1} e, \ldots, R_{a_n}^{\varepsilon_n} e) \neq \emptyset.$$

Similarly for 2) we can get

$$P \cap S(R_a^{-1} e, R_{b_1}^{\eta_1} e, \ldots, R_{b_m}^{\eta_m} e) \neq \emptyset$$

But in this case $P \cap S\,(R_a^{\varepsilon} e, R_{a_1}^{\varepsilon_1} e, \ldots, R_{a_n}^{\varepsilon_n} e, R_{b_1}^{\eta_1} e, \ldots, R_{b_m}^{\eta_m} e) \neq \emptyset$

for any choice of signs E_i, η_j $(i=1,\ldots,n; j=1,\ldots,m)$ which is contradiction with the conditions of Lemma:

So, we have the following alternative: for any finite collection of elements $a_1, \ldots, a_n$ $(\neq e) \in Q$ it is possible to choose signs $E_1, \ldots, E_n$ such that either

$$\text{(i)} \quad P \cap S(R_a^{-1} e, R_{a_1}^{\varepsilon_1} e, \ldots, R_{a_n}^{\varepsilon_n} e) = \emptyset,$$

then we put $p' = PS_e^*\ (R_a e)$

$$\text{or (ii)} \quad P \cap S(R_a e, R_{a_1}^{\varepsilon_1} e, \ldots, R_{a_n}^{\varepsilon_n} e) = \emptyset$$

then we put $p' = PS_e\ (R_a e)$.

If both (i) and (ii) hold, we can take either of them to define P'.

To complete the proof we need to show that P' defines a partial order on Q. From the condition (*) it follows that neither $S(R_a e)$ nor

$S^*(R_a e)$ contains identity. Let, for instance, $P' = PS_e(R_a e)$, then $S(R_a e)$ doesn't contain any negative element, because of (i). So Lemma 3 implies that $P' = PS_e(R_a e)$ defines some partial order on Q, because from the condition (*) it follows that $\forall b \in Q$, $P' \cap S(R_b^{\varepsilon} e) = \emptyset$, so either $b \in P'$ or $R_b^{-1} e \notin P'$ and the lemma is proved.

Theorem 4. The partial order P of p.o. loop $\mathcal{Q}$ can be extended to the total order iff it satisfies the condition (*) of Lemma 4.

Necessity. If P can be extended to the total order L, choose ε_i such $R_{a_i}^{-\varepsilon_i} e$ L.

Then $S^-(R_{a_1}^{\varepsilon_1} e, \ldots, R_{a_n}^{\varepsilon_n} e) = S^*(R_{a_1}^{\varepsilon_1} e, \ldots, R_{a_n}^{\varepsilon_n} e) \subseteq L$, so

$$P \cap S(R_{a_1}^{\varepsilon_1} e, \ldots, R_{a_n}^{\varepsilon_n} e) \subseteq L \cap S(R_{a_1}^{\varepsilon_1} e, \ldots, R_{a_n}^{\varepsilon_n} e) = \emptyset.$$

Sufficiency. Yet us consider the set $\{P_\alpha\}$ of all the partial orders of loop $\mathcal{Q}$, which are the extensions of partial order P and satisfy the condition (*). Let L be a maximal element of $\{P_\alpha\}$, then L exists (see at the end of section 4 of this article). From Lemma 4 it follows that $\forall a \in Q$ either L $S_e(R_a e)$ or L $S_e(R_a^{-1} e)$ defines a partial order $P' \in \{P_\alpha\}$. So from the maximality of L it follows that $P' \subseteq L$. But P' is a proper extension of L so, $L \subseteq P' \Longrightarrow P' = L$. It means that either $a \in L$ or $R_a^{-1} e \in L$ and L is a total order.

Reference

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A Classroom Experience in the Discovery Method

George F. Feeman

Quite often one reads about teaching mathematics by the discovery method. This method is time consuming and requires a lot of skill on the part of the teacher. It also requires a cooperative questioning attitude on the part of the students in the class. They must play along, so to speak. Consequently, it is a method that is talked about a lot, but that is used only infrequently.

In this paper, we will give an account of a classroom experience in the discovery method. The scene is a class in vector geometry. The topic under discussion is translations on the number line and compositions of these motions.

Let $T_a(x) = x + a$, for x an arbitrary integer and a a fixed integer. Let $T_b(x) = x + b$ be a second such function, for b a fixed integer different from a. For this exercise, suppose that a and b are positive integers. Then T_a and T_b are translations of the number line to the right. For example, if a = 2, then $T_2(x) = x + 2$ adds 2 to any number. Hence it translates the number line 2 units to the right.

The composition of T_a and T_b, denoted by $T_b \circ T_a$ and defined by $T_b \circ T_a\ (x) = T_b[T_a(x)]$ yields

$$T_b[T_a(x)] = T_b(x + a) = x + (a + b)$$

Thus composition corresponds to addition of the integers a and b. It is easy to show that the translations $\{T_a\}$ are in a one-to-one correspondence with the integers $\{a\}$ and that the set of translations forms a commutative group under composition. The identity translation is $(T_0(x) = x$, and the inverse of a translation T_a is T_{-a} since $T_{-a}[T_a(x)] = x = T_a[T_{-a}(x)]$.

Since T_a and T_b correspond to translations of the number line by amounts a and b, respectively, applications of T_a, T_b, $T_b \circ T_a$, and so on, can be pictured intuitively as jumps on the line by the respective amounts. For example,

$$T_b \circ T_a \circ T_a \circ T_b \circ T_a \circ T_b \circ T_b\ (x) = x + 3a + 4b.$$

If we start at x, we end at x + 3a + 4b after making 3 jumps of size a and 4 jumps of size b. Thus, in particular, if we start at 0, we end at 3a + 4b.

With this background a **student** asked which points on the number line could be reached for a particular pair of values a and b, with different combinations and with a non-negative number of uses of each. We decided to assume that a and b are relatively prime, for if a and b are not relatively prime, an unlimited number of values cannot be reached. Another student suggested that we start with values of a and b which are 1 unit apart. That is, $b = a + 1$. So we did this and constructed the following table of numbers which could or could not be expressed as combinations of a and b of the form $xa + yb$, with x and y non-negative integers.

Case 1.

a	$b = a + 1$	Numbers not expressible as $xa + yb$	Numbers expressible as $xa + yb$
2	3	1	any no. ≥ 2
3	4	1,2,5	3,4,any no. ≥ 6
4	5	1,2,3,6,7,11	4,5,8,9,10,any no. ≥ 12
5	6	1,2,3,4,7,8,9,13,14,19	5,6,10,11,12,15,16,17,18, any no. ≥ 20

From this it was conjectured that for any two numbers a and $a + 1$, from $(a - 1)a$ on any number could be reached. This was guessed since 2,6,12, and 20 are (1) (2), (2) (3), (3) (4), and (4) (5) respectively. It also appeared that the number of nonattainable numbers was $\frac{1}{2}(a - 1)a$ since, in the various cases, these numbers are 1,3,6, and 10, just half of the limiting values of 2,6,12 and 20.

The students then suggested that we try again with numbers which are 2 units apart, so that $b = a + 2$. We took this suggestion and obtained the following table. Note that both numbers are odd because of the condition of relative primeness.

Case 2.

a	$b = a + 2$	Numbers not expressible as $xa + yb$	Numbers expressible as $xa + yb$
1	3	None	All
3	5	1,2,4,7	3,5,any no. ≥ 8
5	7	1,2,3,4,6,8,9,11,13, 16,18,23	5,7,10,12,14,15,17,19,20, 21,22,any no. ≥ 24
7	9	1,2,3,4,5,6,8,10,11,12, 13,15,17 ,19,20,22,24, 26,29,31,33,38,40,47	7,9,14,16,18,21,23,25,27, 28,30,32,34,35,36,37,39, 41,42,43,44,45,46, any no. ≥ 48

Since 8 = (2) (4), 24 = (4) (6), and 48 = (6) (8), the class again observed that from (a + 1) (a - 1) on, any number can be reached. Again the number of non-reachable numbers turned out to be $\frac{1}{2}(a + 1)(a - 1)$, a pattern similar to the previous case.

The students then suggested that we try a couple of cases in which the numbers were three units apart, so that b = a + 3. This time one number is even and one is odd. Our results were as follows:

Case 3.

a	b = a + 3	Numbers not expressible as xa + yb	Numbers expressible as xa + yb
2	5	1,3	2, any no. ≥ 4
4	7	1,2,3,5,6,9,10,13, 17	4,7,8,11,12,14,15,16, any no. ≥ 18

With these cases our previous results continued. From (a - 1) (a + 2) on, any number can be reached, since 18 = (3) (6) and 4 = (1) (4). The number of nonattainable results is $\frac{1}{2}(a - 1)(a + 2)$.

The students then noticed that if b = a + 1 as in Case 1, the form (a - 1)(a) is actually (a - 1)(b - 1). In Case 2, (a - 1)(a + 1) is equal to (a - 1)(b - 1), since b = a + 2. In Case 3, (a - 1)(a + 2) equals (a - 1)(b - 1), since b = a + 3. Thus, they conjectured, the patterns in all three cases are the same. From the observed data, they then conjectured the following results.

Let a,b be two relatively prime positive integers. Then

1. Any natural number N ≥ (a - 1)(b - 1) can be expressed as a combination xa + yb, where x and y are non-negative integers.
2. There are $\frac{1}{2}(a - 1)(b - 1)$ numbers less than (a - 1)(b - 1) which cannot be expressed in the form xa + yb, where x and y are non-negative integers.
3. The greatest number which cannot be expressed in the form xa + yb, where x and y are non-negative integers, is (a - 1) (b - 1) - 1 = ab - (a + b); that is, the difference between the product and the sum of the numbers.

Of course, once these results are conjectured, it still remains to prove them for all possible pairs a, b. We invite the reader to attempt this, utilizing an argument on lattice points in the first quadrant. The proofs will be provided in a subsequent article.

To close, we cite one more possible application of these results other than the geometric one already considered. One might think of a game being played by one person. After each play, the player receives either a or b points, and his score accumulates from play to play. The question asked would be to find those scores which are attainable.

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GLOSSARY OF MATHEMATICAL TERMS

(Proposed)

(B)

Balance: तराजु; संतुलन
Ball: बल; गोला; पिण्ड
Ballistic: प्राक्षेपिक
Ballistics: प्राक्षेप गणित, प्राक्षेपिकी
Band: ब्याण्ड, पट्टी, समूह
Bar: डण्डी, छड
Barograph: वायुदबाब, लेखाचित्र
Barometer: वायुचाप मापक
Barometric: वायुचाप मापी
Base: तल, आधार
Basic: मूल, आधारभूत
Basis: आधार
Bead: दाना
Beam: सतरी, बीम
Beat: स्पन्दन
Belt: मेखला
Bend: मोड
Beta: बीटा
Bi: द्वि
Biannual: द्विवार्षिक, दुइवर्षे
Bias: पूर्वाग्रह, अभिनति
Biased: अभिनत
Biaxial: द्वि-अक्षीय, द्व-अक्षा
Bicentenary: द्वि-शत वार्षिक, दुइसयवर्षे
Biconcave: उभयावतल
Bicone: द्विकोण, द्विशङ्कु०क
Biconvex: उभयोत्तल
Biennial: द्विवर्षी
Bifocal: द्वि-नाभिय
Bifurcate: द्वि-शाखित हुनु, दुइफ्याँके हुनु
Bifurcation: द्वि शाखन
Bilateral: द्वि पार्श्विय
Bilinear: द्वि-घाती
Billion: अरब
Binary: द्वि-गुण, द्वि-वर्गी
Binode: द्वि-पात
Binomial: द्वि-पद, द्वि-पदीय
Binormal: उभयलम्बी, द्विगुण अभिलम्ब
Biplanar: द्वि समतल, द्विसमतलीय
Biquadratic: चतुर्घातात्मक
Bisect: समद्वि भाजन गर्नु
Bisection: समद्वि भाजन
Bisector: समद्वि भाजक
Bivariate: द्विचर
Blank: खाली, रित्तो
Block: प्रखण्ड
Bob: गोलक
Body: पिण्ड
Border: किनारा, सीमा
Bound: परिबद्ध, अभिमुख
Boundary: सीमा, हद
Bounded: परिसीमित
Braces: धनु:कोष्ठ
Bracket: कोष्ठ
Branch: शाखा
Breadth: चौडाइ
Bulk: परिमाण, आयतन
Buoy: बोय, प्लाव
Buoyancy: प्लवन शिलता, प्लावकता

(C)

Calculate: गणना गर्नु, हिसाब गर्नु
Calculated: गणना गरिएको
Calculation: गणना
Calculator: गणक
Calculus: कलन, कलन कलन
Calorie: क्यालरी, उस्माङ्क०
Cancel: काट्नु, रद्दगर्नै, खारेज गर्नु
Cancellation: खारेज, परित्याग
Canonical: विहित
Capacity: क्षमता

यो शब्दावली त्रि०वि०, कीर्तिपुर बहुमुखी क्याम्पस, गणित तथा नेपाली शिक्षण समितिले संयुक्त रूपमा तयार गरिएको हो ।

Cardinal: कार्डिनल, प्रमुख
Cardiod: हृदयाभ
Carried: नीत, लगिएको
Cascade: प्रपात
Case: स्थिति
Category: वर्ग, श्रेणी
Catastrophe: विपत, आपत
Catenary: कैटेनरी, रज्जुका
Catenoid: रज्जुकाभ
Celestial: खगोलिय
Cent: शत
Centennial: शताब्दी
Centesimal: सेण्टिसिमल
Centi : शतांश
Centigrade: सेन्टीग्रेड, शताङ्क
Centigram: सेन्टीग्राम
Central: केन्द्रीय, केन्द्री
Centre: केन्द्र
Centrifugal: अपकेन्द्री
Centripetal: अभिकेन्द्री
Centrode: केन्द्रपथ
Centroid: परि-केन्द्र
Century: शताब्दी
Chainette: शृङ्खलिका
Chance: संभाविता, संयोग
Change: परिवर्तन
Characteristic: लाक्षणिक
Chart: चार्ट, लेखाचित्र
Choice: विकल्प, छनौट
Chord: जीवा
Chrono-graph: समय लेखी
Cipher: शून्य
Circle: वृत्त
Circuit: परिपथ
Circulant: चक्रक, वृत्तक
Circular: वृत्ताकार, वृत्तीय
Circulation: परिवहन, संचार
Circumference: परिधी
Circumscribed: परिगत
Cissoid: तोरणाभ
Class: वर्ग, श्रेणी
Classical: क्लासिकल, निरप्रतिष्ठित
Classification: वर्गीकरण
Classified: वर्गीकृत
Classify: वर्गीकरण गर्नु
Closed: बन्द
Closure: क्लोजर
Cluster: गुच्छा, झुप्पपरेको
Coalesce: सम्मिलित हुनु
Co-axial: समअक्षिय
Co-efficient: गुणाङ्क
Cofactor: सहगुणन खण्ड
Cogradient: सहरूपान्तरी
Cohesive: संसजक
Coincidence: संपात
Coincide: मिल्नजानु, एकसाथहुनु
Coincident: संपाती
Collect: सङ्ग्रहगर्नु, एकत्रगर्नु
Collection: सङ्ग्रह
Collinear: समरेखिय, एकरेखस्थ
Collinearity: समरेखता
Collision: ठक्कर
Column: स्तंभ, पंक्ती
Columnwise: स्तंभानुसार, प्रपङ्क्ति
Combination: संचय
Commence: शुरुगर्नु, आरम्भगर्नु
Commencement: आरम्भ, शुरुवात
Commensurable: सम्मेय, सापवर्ती
Common: साझा, सामान्य
Commutative: विपरिणामी, विपर्ययी
Commute: विपर्ययगर्नु
Compact: संहत, ठोस, सघन

Compactness: सघनता, ठोसपन,

Companion: सहचर

Compare: तुलनागर्नु

Comparison: तुलना

Compass: कम्पास, दिशासूचक यन्त्र

Compatibility: सह०गतता

Compile: सह०कलन गर्नु, बटुल्नु

Compilation: सह्०कलन

Complement: कोटिपूरण

Complementary: कोटिपूरक

Complete: पूर्ण

Completion: समाप्ती

Complex: सम्मिश्र, जटील

Component: अङ्०ग, अवयव

Composite: समिश्र, संयुक्त,

Composition: बनौट, रचना, संघटन

Compound: यौगिक,

Compress: संपीडन, दबाव

Compression: संपीडन

Compressibility: संपिड्यता

Compressible: संपिड्य

Compressor: संपीडक

Computation: अभिगणाना

Compute: अभिगणानागर्नु

Concave: अवतल, उत्तान, उत्तानो

Concavity: अवतलता, उत्तानता

Concentric: एककेन्द्रीय, सकेन्द्रीय

Concept: धारणा

Concurrence: सङ्०गमन

Concurrency: सङ्०गमनीयता

Concurrent: सङ्०गामी

Concyclic: एकवृत्तीय

Condensation: संहनन्

Condition: शर्त, स्थिति

Conditional: सशर्त

Cone: कोन, शङ्०कु

Configuration: संस्थिति, समाकृति

Confluence: सम्मिलिन, सङ्०गम

Confluent: संमेलक, सङ्०गमक

Confocal: संनाभि

Conformable: संवादी, समविन्यासी

Confounding: सङ्०कुल, कङ्०कुलन

Congruence: सर्वाङ्०गसमता, सुसङ्०गतता

Congruent: सर्वाङ्०गसम, सुसङ्०गी

Conic: शाङ्०कव

Conical: शाङ्०कवीय

Conicoid: शाङ्०कवज

Conics: शाङ्०कव-गणित

Conjugate: संयुग्मी

Conjunction: संयोजक

Connect: सम्बन्ध गर्नु, जोड्नु

Connection: सम्बन्ध

Conoid: शंङ्०कुवत, शङ्०कुम

Consecutive: क्रमागत

Consequence: परिणाम, फल

Consequent: परपद, अनुवर्ती

Conservative: परिरक्षक

Conservation: परिरक्षण

Conserve: परिरक्षण गर्नु

Consistency: सामन्जस्य, संवादिता

Consistent: समन्जक, संवादी

Constant: अचर, स्थिरांक

Constituent: घटक, रचक

Constrained: निरुद्ध

Construct: रचना गर्नु

Construction: रचना

Contact: स्पर्श

Contents: अन्तर्वस्तु

Contiguous: संनिकट, संलग्न, निकटस्थी

Contingency: संनिकटता, आसङ्०गकता

Continuation: निरन्तरता, सततता

Continue क्रमगत हुनु

Continued: क्रमागत

Continuity: नैरन्तर्य, सातत्य

Continuous: निरन्तर, सतत, अविच्छिन्न

Contour: परिणाह, परिधि

Contract: खुम्चनु, सिकुडिनु

Contraction: खुम्चावट, सङ्कोचन, सिकुडन

Contravariant: प्रतिचर

Control: नियन्त्रण

Converge: अभिसृत हुनु

Convergence: अभिसरण, समवायन

Convergent: अभिसारी, समवामी

Converse: विलोम, विपरीत

Conversely: विलोमत

Conversion: व्यक्तिक्रम

Convex: उत्तल

Convolution: आहनन्

Coordinate: निदेशाङ्क

Comprime: सह-अभाज्य

Corollary: उप-प्रमेय

Correct: शुद्ध

Corrected: संशोधित

Correction: शुद्धि

Correlation: सह-सम्बन्ध

Correspondence: अनुरूपता, सहगामिता

Corresponding: सङ्गत, सहगामी

Cos (Cosine): कस, (कसाइनको संक्षिप्त रूप)

Cosecant: कसेकेण्ट

Coset: कोसेट, सहकुलक

Cosmos: ब्रह्माण्ड

Cot: कट

Cotangent: कोटान्जेण्ट

Coterminus: सहावसानी

Count: गणना गर्नु, गन्नु

Countable : गणानिय, गणनायोग्य

Counterclockwise: वामावर्त

Couple: युग्म, बलयुग्म

Covariance: सहचरता

Covariant: सहचर

Covariation: सहविचरण

Cover: आवरण, ढक्कन

Coverage: व्याप्ति

Covering: व्यापी

Covers: कोभर्स

Critical: सिमान्तीक, क्रान्तिक

Crore: करोड

Cube: घन

Cubic: घन, घनात्मक

Cubical: घनाकार, घनाकृतिक

Cubit: हात, हस्त

Cumulative: सन्चयी

Curl: कुन्तल

Curvature: वक्रता

Curve: वक्र

Curved: वक्रीय, वक्रक

Cusp: उभयोग्रीय

Cuspidal: कटाव

Cut: कटाव

Cyclic: चक्र

Cyclic : चक्रीय

Cyclical: चक्रात्मक

Cyclically: चक्रीयत

Cycloid: चक्रज

Cylinder: बेलन

Cylindrical: बेलनाकार

Cylindroid: बेलनाभ