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A NOTE ON VECTOR-VALUED DISCRETE SCHRÖDINGER OPERATORS

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Abstract: The main purpose of this paper is to extend some theory of Schrödinger operators from one dimension to higher dimension. In particular, we will give systematic operator theoretic analysis for the Schrödinger equations in multidimensional space. To this end, we will provide the detail proves of some basic results that are necessary for further studies in these areas. In addition, we will introduce Titchmarsh-Weyl $m-$ function of these equations and express $m-$ function in term of the resolvent operators.

Key Words: Discrete Schrödinger equation, Titchmarsh-Weyl m-function
AMS (MOS) Subject Classification. 39A70, 47A05, 34B20.

1. Introcuction

The Jacobi and Schrödinger equations are the basic equations in Mathematical physics. These equations are used to describe quantum mechanical particles. In one dimensional space, the theory of Schrödinger equations

$$-y'' + V(x)y = zy(x),\ x \in \mathbb{R}, z \in \mathbb{C}$$

and the Jacobi equations

$$a(n)y(n+1) + a(n-1)y(n-1) + b(n)y(n) = zy(n), z \in \mathbb{C},$$

where $a(n), b(n)$ are bounded sequences, are well developed. For a few references see [2, 8, 9, 11]. However, much less is known about the theory in multidimensional space. In this paper, we attempt to extend some theory of Schrödinger equations from one dimensional space to multidimensional space. So we consider discrete Schrödinger equations whose solutions are vector valued functions and extend some basic results of Jacobi and Schrödinger equations from one dimensional space. There has been some research work in higher dimension for example, see [3, 4, 7]. However, we did not find a detail presentations of the basic theory, required for further studies in this area, which we discussed in section 2.

In addition, we also discuss about the Titchmarsh-Weyl m function. These $m-$ functions play very important role in the spectral theory of Jacobi and Schrödinger operators.

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The spectrum of these operators can be described by these $m-$ function. Some of the analogous theory in one dimensional space can be found also in above mentioned references. We discuss about this in Section 3.

The extension of the theory to multidimensional space is not obvious in most of the cases. The situation becomes completely different and a careful analysis is required to study such phenomena.

We consider a multidimensional discrete Schrödinger equation of the form

$$y(n+1) + y(n-1) + B(n)y(n) = zy(n), \ z \in \mathbb{C} \tag{1.1}$$

where $y(n) = [y_1(n) \ y_2(n), \dots \ y_d(n)]^t$ (t stands for a transpose), is a vector valued sequence in $l^2(I, \mathbb{C}^d)$. Here $l^2(I, \mathbb{C}^d)$ is a Hilbert space of square summable vector valued sequences with the inner product

$$\langle u, v \rangle = \sum_{n \in I} u(n)^* v(n),$$

where " $*$ " stands for conjugate transpose and $B(n) \in \mathbb{C}^{d \times d}$ is a symmetric $d \times d$ matrix. Usually $I = \mathbb{Z}$ or $I = \mathbb{N}$. The equation (1.1) can be generalized to a multidimensional Jacobi equation of the form

$$A(n)y(n+1) + A(n-1)y(n-1) + B(n)y(n) = zy(n), \ z \in \mathbb{C} \tag{1.2}$$

with $A(n), B(n)$ are sequences of $d \times d$ matrices. If $I = N$ The equation (1.2) can be written in the form:

$$\begin{pmatrix} B(1) & A(1) & 0 & \\ A(1) & B(2) & A(2) & \ddots \\ 0 & A(2) & B(3) & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} y(1) \\ y(2) \\ \vdots \\ \vdots \\ \vdots \end{pmatrix} = z \begin{pmatrix} y(1) \\ y(2) \\ \vdots \\ \vdots \\ \vdots \end{pmatrix}.$$

The matrix

$$J = \begin{pmatrix} B(1) & A(1) & 0 & \\ A(1) & B(2) & A(2) & \ddots \\ 0 & A(2) & B(3) & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

is called a block Jacobi matrix. Some studies about the block Jacobi matrix can be found in the paper [7]. Equation (1.1) is a particular case of Jacobi equation with $A(n) \equiv 1$.

2. Some basic theory

In this section, we will give an operator theoretic analysis of the equation (1.1). The Equation (1.1) induces an operator J on $l^2(I, \mathbb{C}^d)$ as

$$Jy(n) = y(n+1) + y(n-1) + B(n)y(n).$$

If $I = \mathbb{N}$, then we need to slightly modify the definition of J as

$$Jy(1) = y(2) + B(1)y(1).$$

The matrix $B(n)$ is called the potential. We assume that

$$B(n)^* = B(n), \text{ and } \|B(n)\| \leq C$$

.

Proposition 2.1. If $B(n) \in \mathbb{C}^{d\times d}$, $B(n)^* = B(n)$, then J is a self-adjoint operator on $l^2(\mathbb{N}, \mathbb{C}^d)$.

Proof. It is clear by definition of J and Cauchy Schwartz inequality that J is a bounded linear operator on $l^2(\mathbb{N}, \mathbb{C}^d)$. In order to see self adjointness, suppose $u, v \in l^2(\mathbb{N}, \mathbb{C}^d)$ then

$$\begin{aligned}
\langle u, Jv\rangle =& \sum_{n=1}^{\infty} u(n)^* Jv(n) = u(1)^*(J(v)(1)) + \sum_{n=2}^{\infty} u(n)^*(v(n+1) + v(n-1) + B(n)v(n)) \\
=& u(1)^*(v(2) + B(1)v(1)) + \sum_{n=2}^{\infty} u(n)^* v(n+1) + \sum_{n=2}^{\infty} u(n)^* v(n-1) + \sum_{n=2}^{\infty} u(n)^* B(n)v(n)) \\
=& (u(2) + B(1)u(1))^* v(1) + \sum_{n=2}^{\infty}(u(n+1) + u(n-1) + B(n)u(n))^* v(n)) \\
=& \sum_{n=1}^{\infty}(Ju(n))^* v(n) \\
=& \langle Ju, v\rangle
\end{aligned}$$

□

Since J is a self adjoint operator, the spectrum of such operator is a set of real numbers: $\sigma(J) \subset \mathbb{R}$.

To get a solution of the equation (1.1), we may fix any two vectors $c_1, c_2 \in C^d$ at two consecutive sites, that is, we fix the values $u_k = c_1, u_{k+1} = c_2$ and evolve according to (1.1). Suppose τ is the difference expression in the left side of (1.1), then we have the following remark.

Remark 2.2. Let c_1, c_2 be any two vectors in $\mathbb{C}^d$ and $k \in \mathbb{N}_0 (= \mathbb{N} \cup \{0\})$. For any arbitrary sequence $f(n)$ there exists a unique solution $u(n)$ of $(\tau - z)u(n) = f(n)$ with $u(k) = c_1$ and $u(k+1) = c_2$.

Consequently the following preposition holds.

Proposition 2.3. The set of solutions u to $(\tau - z)u(n) = 0$ is a $2d$-dimensional vector space.

Proof. By above remark, for each $c_1, c_2 \in \mathbb{C}^d$ and $k \in \mathbb{N}_0$ there exists a unique solution u such that $u(k) = c_1$ and $u(k+1) = c_2$. Since $\mathbb{C}^d$ is a d dimensional space, the solution space is $2d$ dimensional vector space. □

In the following theorem we show that the number of linearly independent $l^2(\mathbb{N}, \mathbb{C}^d)$ solutions of $d-$ dimensional Schrödinger equations (1.1) is d.

Theorem 2.4. *Let $z \in \mathbb{C} - \mathbb{R}$. Then $(\tau - z)u(n) = 0$ has exactly d linearly independent solutions in $l^2(\mathbb{N}, \mathbb{C}^d)$.*

Proof. Since J is self adjoint the spectrum $\sigma(J) \subset \mathbb{R}$ and therefore for each $z \in \mathbb{C}^+$, $(J - z)$ is invertible in $B(l^2)$. Let

$$\delta_k = (e(k), 0, 0, , \ldots)$$

where for each $k, e(k)^t = (0, 0, \ldots, 1, \ldots, 0, 0)$ is a vector in C^d with 1 in the kth component and 0 otherwise. Let $u_k = (J - z)^{-1}\delta_k$ for each $k = 1, 2, ...d$. Clearly, u_k are linearly independent (being the images of linearly independent vector under bounded linear operator). Moreover,

$$(J - z)u_k = \delta_k.$$

For each $k, k = 1, 2, ..., d,$ $(\tau - z)u_k(n) = 0$ for $n \geq 2$. By suitably defining at $n = 0$ we can also achieve that $(\tau - z)u_k(1) = 0$. So this extended u_k is an l^2 solution. Thus there are d linearly independent solutions.

Suppose there is another solution v linearly independent to $u_1, u_2, ...u_d$. Then $v(0), u_1(0), ..., u_d(0)$ being $d + 1$ vectors in $\mathbb{C}^d$, are linearly dependent. So there exists constants $\alpha_1, \alpha_2, ...\alpha_d$ not all zero such that

$$v(0) = \alpha_1 u_1(0) + ... + \alpha_d u_d(0).$$

Define

$$f(n, z) = v(n, z) - (\alpha_1 u_1(n, z) + ... + \alpha_d u_d(n, z)).$$

Clearly $f(n, z)$ satisfies the difference equation and $f(n, z) \in l^2(\mathbb{N}, \mathbb{C}^d)$. Moreover, $f(0, z) = 0$. So $f(n, z)$ is an eigenvector for J, which contradicts that $\sigma(J) \subset \mathbb{R}$. So there are exactly d linearly independent l^2 solutions. □

As we know that the Wronskian of the solutions of a differential equations have close connection with the linearly independent solutions. It is also important in the difference equations as well. One of the applications of Wronskian can be found in [6]. We define the Wronskian of any two vector valued sequences in $l^2(\mathbb{N}, \mathbb{C}^d)$.

Definition 2.5. The Wronskian of any two sequences $f(n, z), g(n, z) \in l^2(\mathbb{N}, \mathbb{C}^d)$ is defined by

$$W_n(f, g) = [f^*(n + 1, \bar{z})g(n, z) - f^*(n, \bar{z})g(n + 1, z)].$$

This definition incorporate with the definition in one dimensional space and in the continuous case. Analogous to a result in ordinary differential equations, we have the following lemma.

Lemma 2.6. *If $f(n, z), g(n, z) \in l^2(\mathbb{N}, \mathbb{C}^d)$ are any two solutions of (1.1) then $W_n(f, g)$ is independent of n.*

Proof. Since $f(n,z), g(n,z)$ are solutions of (1.1) we have,

$$f(n+1,z)+f(n-1,z)+B(n)f(n,z)=zf(n,z)$$

and

$$g(n+1,z)+g(n-1,z)+B(n)g(n,z)=zg(n,z).$$

Multiply the complex conjugate of first equation by $g^*(n,z)$ from left and take the conjugate transpose we obtain,

$$f^*(n+1,\bar{z})g(n,z)+f^*(n-1,\bar{z})g(n,z)+f^*(n,\bar{z})B(n)g(n,z)=f^*(n,\bar{z})zg(n,z)$$

Similarly, multiplying second equation from left by $f^*(n,\bar{z})$ we get,

$$f^*(n,\bar{z})g(n+1,z)+f^*(n,\bar{z})g(n-1,z)+f^*(n,\bar{z})B(n)g(n,z)=f^*(n,\bar{z})zg(n,z).$$

On subtraction we get

$$f^*(n+1,\bar{z})g(n,z)-f^*(n,\bar{z})g(n+1,z)=f^*(n,\bar{z})g(n-1,z)-f^*(n-1,\bar{z})g(n,z)$$

so that $W_n(f,g)=W_{n-1}(f,g)$. □

Next we establish the Green's identity corresponding to equation (1.1).

Lemma 2.7 (Green's Identity)**.** *Let* $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. *For* $f(n,z), g(n,z)\in l^2(\mathbb{N}_0,\mathbb{C}^d)$

$$\sum_{j=0}^{n}\Big(f^*(\tau g)-(\tau f)^*g\Big)(j)=W_0(\bar{f},g)-W_n(\bar{f},g).$$

Proof.

$$\begin{aligned}
&\sum_{j=0}^{n}\Big(f^*(\tau g)-(\tau f)^*g\Big)(j)\\
&=\sum_{j=0}^{n}\Big[f^*(j)\Big(g(j+1)+g(j-1)+B(n)g(j)\Big)-\Big(f(j+1)+f(j-1)+B(n)f(j)\Big)^*g(j)\Big]\\
&=\sum_{j=0}^{n}\Big[\Big(f^*(j)g(j-1)-f^*(j-1)g(j)\Big)-\Big(f^*(j+1)g(j)-f^*(j)g(j+1)\Big)\Big]\\
&=\sum_{j=0}^{n}W_j(\bar{f},g)-W_{j-1}(\bar{f},g)\\
&=W_0(\bar{f},g)-W_n(\bar{f},g)
\end{aligned}$$

□

It is now convenient to fix a basis of the solution space. An easier way to choose a basis of the solution space of (1.1) is to prescribe a pair of initial conditions. For $z\in\mathbb{C}$, let

$$\begin{aligned}
U(n,z)&=[u_1(n),u_2(n),\dots,u_d(n)],\\
u_i(n)&=[u_{1,i}(n)\ u_{2,i}(n)\ \dots u_{d,i}(n)]^t\\
V(n,z)&=[v_1(n),v_2(n),\dots,v_d(n)]\\
v_i(n)&=[v_{1,i}(n)\ v_{2,i}(n)\ \dots v_{d,i}(n)]^t
\end{aligned}\tag{2.1}$$

be the set of solutions. Both of the sets $U(n,z)$ and $V(n,z)$ consists of d linearly independent solutions of $(\tau - z)u(n) = 0$. For a convenience, we may consider these sets as matrices in $M^{d\times d}(\mathbb{C})$. We further suppose that these solutions satisfy the following initial conditions

$$U(0,z) = -I, \qquad V(0,z) = O, \qquad U(1,z) = O, \qquad V(1,z) = I. \tag{2.2}$$

By iterating the difference equation, we see that for fixed $n \in \mathbb{N}, U(n,z), V(n,z)$ are polynomial of degree $n-2$ over $M^{d\times d}(\mathbb{C})$. So $\overline{U(n,z)} = U(n,\bar{z})$ and $\overline{V(n,z)} = V(n,\bar{z})$.

We extend the definition of Wronskian from above for the sets $U(n,z), V(n,z)$, each contains d linearly independent solutions of (1.1).

$$W_n(U,V) = \det[U^*(n+1,\bar{z})V(n,z) - U^*(n,\bar{z})V(n+1,z)].$$

We now extend the lemma 2.6 in the following proposition.

Proposition 2.8. $W_n(U,V)$ is independent of $n \in \mathbb{N}$

Proof. Since $U(n,z), V(n,z)$ are solutions of (1.1),

$$U(n+1,z) + U(n-1,z) + B(n)U(n,z) = zU(n,z)$$

and

$$V(n+1,z) + V(n-1,z) + B(n)V(n,z) = zV(n,z)$$

By multiplying the complex conjugate of first equation by $V^*(n,z)$ and taking the conjugate transpose we obtain,

$$U^*(n+1,\bar{z})V(n,z) + U^*(n-1,\bar{z})V(n,z) + U^*(n,\bar{z})B(n)V(n,z) = U^*(n,\bar{z})zV(n,z)$$

Similarly, multiplying second equation by $U^*(n,\bar{z})$ we get,

$$U^*(n,\bar{z})V(n+1,z) + U^*(n,\bar{z})V(n-1,z) + U^*(n,\bar{z})B(n)V(n,z) = U^*(n,\bar{z})zV(n,z).$$

On subtraction we get

$$U^*(n+1,\bar{z})V(n,z) - U^*(n,\bar{z})V(n+1,z) = U^*(n,\bar{z})V(n-1,z) - U^*(n-1,\bar{z})V(n,z)$$

so that

$$\det[U^*(n+1,\bar{z})V(n,z) - U^*(n,\bar{z})V(n+1,z)] = \det[U^*(n,\bar{z})V(n-1,z) - U^*(n-1,\bar{z})V(n,z)].$$

It follows that $W_n(U,V) = W_{n-1}(U,V)$. Continuing we get,

$$W_n(U,V) = W_{n-1}(U,V) = \ldots = W_0(U,V) = \det I = 1.$$

□

3. Titchmarsh-Weyl m function

Titchmarsh-Weyl m functions associated with the Schrödinger equations are very important objects in the direct and inverse spectral theory of the corresponding operator. These functions provides asymptotic behavior of the solutions of these equations. The history of these functions goes back to 1910 when H. Weyl introduce these functions in [13] for Sturn-Liouville differential equations. It was further studied by E. C. Titchmarsh in [12] and establish the connection between the analyticity of the solution and the spectrum of the operator of Sturn-Liouville differential equations. There has been tremendous work about the Weyl theory in one dimension which can be found in many literatures, please see [1, 5, 8, 10, 11] as a few references.

We introduce the Titchmarsh-Weyl m function for the vector-valued discrete Schrödinger operators and express in terms of resolvent operator.

Definition 3.1. Let $z \in \mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$. The Titchmarsh-Weyl m function is defined as the unique $M(z) \in \mathbb{C}^{d\times d}$ for which

$$F(n,z) = U(n,z) + M(z)V(n,z) \tag{3.1}$$

where $U(n,z), V(n,z)$ are the sets of d linearly independent solutions with initial values (2.2) and $F(n,z)$ is a set of d linearly independent solutions of (1.1) in $l^2(\mathbb{N}, \mathbb{C}^d)$.

This definition, is in fact well defined. As we mentioned above that there are only d linearly independent solutions in $l^2(\mathbb{N}_0, \mathbb{C}^d)$, if there is another $M(z)$ satisfying the above conditions then the solutions from both sets $U(n,z)$ and $V(n,z)$ will be in $l^2(\mathbb{N}_0, \mathbb{C}^d)$. The solution set $V(n,z)$ is such that $V(0,z) = 0$ which implies that $V(n,z)$ is the set of eigenfunctions for the self adjoint operator J. This contradicts that the spectrum of J is a set of real numbers.

The following is the main theorem in this section.

Theorem 3.2. $z \in \mathbb{C}^+$. *If* $(\tau - z)F = 0$ *and* F *consists of* d *solutions in* $l^2(\mathbb{N}, \mathbb{C}^d)$. *Then*

$$M(z) = -F(1,z)F(0,z)^{-1}. \tag{3.2}$$

Moreover,

$$M(z) = (m_{ij}(z))_{d\times d} \in \mathbb{C}^{d\times d},\ m_{ij}(z) = \langle \delta_j, (J - z)^{-1}\delta_i\rangle. \tag{3.3}$$

Proof. If F is specifically the set of l^2 solutions from (3.1). Then $F(0,z) = -I$ and $F(1,z) = M(z)$. So (3.2) holds. A set of arbitrary d solutions $G(n,z)$ is a constant (matrix) multiple of the solution set $F(n,z)$ from (3.1) because (3.1) is a set of d linearly independent solutions. That is,

$$G(n,z) = F(n,z)C$$
$$F(n,z) = G(n,z)C^{-1}$$

so that

$$\begin{aligned} -G(1,z)G(0,z)^{-1} =& - F(1,z)CC^{-1}F(0,z)^{-1} \\ =& - F(1,z)F(0,z)^{-1} \\ =& M(z). \end{aligned}$$

Let $F(n,z)$ as in (3.2) and let

$$g_i = (J-z)^{-1}\delta_i.$$

Then $(J-z)g_i = \delta_i$. So $(\tau - z)g_i(n) = 0$ for $n \geq 2$. Moreover $g_i \in l^2$ for all $i = 1, 2, ..., d$. Let

$$G(n,z) = [g_1, g_2, ..., g_d].$$

Then $G(n,z) = F(n,z)C,\ C \in \mathbb{C}^{d\times d}$. By comparing values at

$$n = 1,\ G(1,z) = [g_1(1), g_2(1), ..., g_d(1)].$$

Here

$$g_1(1) = (J-z)^{-1}\delta_1(1)$$

and

$$g_1 = [g_{11}, g_{21}, ..., ..., ..., g_{d1}]^t,\ \ g_{i1} = \langle \delta_i, g_1 \rangle, i = 1, 2, ..., d.$$

Then $M(z) = G(1,z)C^{-1}$ and

$$\begin{aligned} M(z) &= (m_{ij}(z)) \\ &= (\langle \delta_j, (J-z)^{-1}\delta_i \rangle)C^{-1}. \end{aligned}$$

To find the value of C, we compare values at $n = 2$.

First $(J-z)G(1,z) = (\delta_1, \delta_2, ..., \delta_d)$ so

$$(J-z)G(1,z) = \begin{pmatrix} 1 & 0\dots & 0 \\ 0 & 1\dots & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0\dots & 1 \end{pmatrix} = I$$

It follows that

$$\begin{aligned} G(2,z) + B(1)G(1,z) - zG(1,z) = I \\ G(2,z) = (z - B(1))G(1,z) + I............(i) \end{aligned}$$

Also,

$$\begin{aligned} F(2,z) = (z - B(1))F(1,z) - F(0,z)C \\ G(2,z) = (z - B(1))G(1,z) - G(0,z)...........(ii) \end{aligned}$$

Comparing (i) and (ii), we get $-F(0,z)C = I$ and so $I.C = I \implies C = I$. Hence (3.3) holds. That is

$$\begin{aligned} M(z) &= (m_{ij}(z)) \\ &= (\langle \delta_j, (J-z)^{-1}\delta_i \rangle). \end{aligned} \tag{3.4}$$

This result allows us to connect the m function with a matrix valued Borel measure using functional calculus for these resolvent operators $\langle \delta_j, (J-z)^{-1}\delta_i \rangle$. We aim to extend the Weyl theory and discuss the spectrum of vector-valued discrete Schrödinger operators in our next studies.

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□

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INVARIANCE OF DOMAIN FOR OPERATORS OF CLASS $\mathcal{A}_G(S_+)$

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Abstract: Let X be a real reflexive Banach space and X^* its dual space. Let $T : X \supset D(T) \to 2^{X^*}$ an operator of class $\mathcal{A}_G(S_+)$, where $G \subset X$ is open. An invariance of domain result for T is established. This result extends a similar result of Park for single-valued operators of type (S_+). The Skrypnik's topological degree theory is used, utilizing approximating schemes of mappings of class $\mathcal{A}_G(S_+)$, along with the methodology of a recent invariance of domain result by Kartsatos and the author.

Key Words: Browder and Skrypnik degree theory, invariance of domain, bounded demicontinuous operator of type (S_+)

AMS (MOS) Subject Classification. Primary 47H14; Secondary 47H05, 47H11

1. Introduction–Preliminaries

Let X be a real reflexive Banach space with X^* its dual space. The norms of X, X^* will be denoted by $\|\cdot\|$ and will be understood from the context in which the symbol is used. We denote by $\langle x^*, x\rangle$ the value of the functional $x^* \in X^*$ at $x \in X$. The symbols ∂D, $\overline{D}$ denote the strong boundary and closure of the set D, respectively. The symbol $B(x_0, r)$ denotes the open ball of radius r with center at x_0.

If $\{x_n\}$ is a sequence in X, we denote its strong convergence to x_0 in X by $x_n \to x_0$ and its weak convergence to x_0 in X by $x_n \rightharpoonup x_0$. An operator $T : X \supset D(T) \to Y$ is said to be "bounded" if it maps bounded subsets of the domain $D(T)$ onto bounded subsets of Y. The operator T is said to be "compact" if it maps bounded subsets of $D(T)$ onto relatively compact subsets in Y. It is said to be "demicontinuous" if it is strong-weak continuous on $D(T)$. The symbols $\mathbf{R}$ and $\mathbf{R}_+$ denote $(-\infty, \infty)$ and $[0, \infty)$, respectively. The normalized duality mapping $J : X \supset D(J) \to 2^{X^*}$ is defined by

$$Jx = \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2,\ \|x^*\| = \|x\|\},\ x \in X.$$

The Hahn-Banach theorem ensures that $D(J) = X$, and therefore $J : X \to 2^{X^*}$ is a multi-valued mapping defined on the whole space X. By a well-known renorming theorem due to Trojanski [15], one can always renorm the reflexive Banach space X with an equivalent norm with respect to which both X and X^* become locally uniformly convex (therefore strictly

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convex). Henceforth, we assume that X is a locally uniformly convex reflexive Banach space. With this setting, the normalized duality mapping J is single-valued homeomorphism from X onto X^*.

For a multivalued operator T from X to X^*, we write $T : X \supset D(T) \to 2^{X^*}$, where $D(T) = \{x \in X \ : \ Tx \neq \emptyset\}$ is the effective domain of T. We denote by $Gr(T)$ the graph of T, i.e., $Gr(T) = \{(x, y) \ : \ x \in D(T), \ y \in Tx\}$.

An operator $T : X \supset D(T) \to 2^{X^*}$ is said to be "monotone" if for every $x, \ y \in D(T)$ and every $u \in Tx, \ v \in Ty$ we have

$$\langle u - v, x - y \rangle \geq 0.$$

A monotone operator T is said to be "maximal monotone" if $Gr(T)$ is maximal in $X \times X^*$, when $X \times X^*$ is partially ordered by the set inclusion. In our setting, a monotone operator T is maximal if and only if $R(T + \lambda J) = X^*$ for all $\lambda \in (0, \infty)$.

Definition 1.1. An operator $C : X \supset D(C) \to X^*$ is said to be of type (S_+) if for every sequence $\{x_n\} \subset D(C)$ with $x_n \rightharpoonup x_0$ in X and

$$\limsup_{n\to\infty} \langle Cx_n, x_n - x_0 \rangle \leq 0,$$

we have $x_n \to x_0 \in \overline{D(C)}$ in X.

Definition 1.2. The family $C(t) : X \supset D \to X^*, t \in [0, 1]$, of operators is said to be a "homotopy of type (S_+)" if for any sequences $\{x_n\} \subset D$ with $x_n \rightharpoonup x_0$ in X and $\{t_n\} \subset [0, 1]$ with $t_n \to t_0$ and

$$\limsup_{n\to\infty} \langle C(t_n)x_n, x_n - x_0 \rangle \leq 0,$$

we have $x_n \to x_0$ in $X, \ x_0 \in D$ and $C(t_n)x_n \rightharpoonup C(t_0)x_0$ in X^*. A homotopy of type (S_+) is "bounded" if the set

$$\{C(t)x \mid t \in [0, 1], \ x \in D\}$$

is bounded.

We now define the class $\mathcal{A}_G(S_+)$ of multivalued operators, where is introduced by Kittila in [8].

Definition 1.3. Let G be an open subset of X. An operator $T : X \supset D(T) \to 2^{X^*}$ is of class $\mathcal{A}_G(S_+)$ if there exists a sequence (T_n) (called an approximating sequence of T) of bounded demicontinuous mappings of type (S_+) from $\overline{G}$ to X^* with the following properties.

- (A1) For each $C > 0$ there exists $K \geq 0$ such that $\langle T_n(x), x \rangle \geq -K$ for all $u \in \overline{G}$, $\|x\| \leq C$ and for all $n \in \mathbb{N}$.
- (A2) Let $\{t_n\} \subset [0, 1]$, $(u_n) \subset \overline{G}$ and let $\{T_{m_n}\}$ be any subsequence of $\{T_n\}$. If $t_n \to 0$, $x_n \to x$ in X and $t_n T_{m_n}(x_n) \rightharpoonup z$ in X^*, then $z = 0$.
- (A3) Let $\{x_n\} \subset \overline{G}$ and $\{T_{m_n}\}$ be any subsequence of $\{T_n\}$. If $x_n \rightharpoonup x$ in X, $T_{m_n}(x_n) \rightharpoonup w$ in X^* and

$$\limsup_{n\to\infty} \langle T_{m_n} x_n), x_n \rangle \leq \langle w, x \rangle,$$

then $x_n \to x$ in X, $x \in D(T)$ and $w \in T(x)$.

The main purpose of this paper is to give an invariance of domain result for operators of class $\mathcal{A}_G(S_+)$. These operators are generalizations of operators of type (S_+) and have an approximating scheme in terms of bounded demicontinuous (S_+)-operators as given in Definition 1.3, and therefore this paper generalizes results of Park in [10]. A multivalued degree for operators in $\mathcal{A}_G(S_+)$ is developed by Kittila in [8] via the Skyrpnik's degrees (cf. [14]). The methodologies of [8] and recent papers of the author and Kartsatos [1] and Kartsatos and Skrypnik [7] along with properties of the Skrypnik's degree in [14] have been utilized.

Invariance of domain results date as far back as Brouwer [3] for continuous injection in $\mathbf{R}^n$. Schauder [13] extended the Brouwer's invariance of domain result to infinite dimensional Banach spaces for compact displacements of the identity, i.e. for operators of the form $I+C$ with C compact. Tromba [16] extended the Schauder's result to Fredholm maps of index zero. For other results on invariance of domain under continuity or demicontinuity assumptions on the main operators, the reader is referred to Berkovits [2], Deimling [4], Kartsatos [5], Nagumo [9], Petryshyn [11, 12] (for A-proper mappings), Skrypnik [14, p.59] and the references therein. For the existence of pathwise connected set in the ranges of certain operators, the reader is referred to [6] and the references therein.

2. Invariance of Domain Result for Operators of Class $\mathcal{A}_G(S_+)$

In this section, we first prove a result that has to do with placing a pathwise connected set in the range of the operators of class $\mathcal{A}_G(S_+)$. As a consequence of this result, we then obtain an invariance of domain result for operators in class $\mathcal{A}_G(S_+)$. It is well-known (cf. Kittila [8, p.13]) that a densely defined maximal monotone operator $A : X \supset D(T) \to 2^{X^*}$, $0 \in D(A)$, $0 \in A(0)$ satisfies $A+T$ is in $\mathcal{A}_G(S_+)$ whenever $T \in \mathcal{A}_G(S_+)$. In particular, $T+J \in \mathcal{A}_G(S_+)$.

Proposition 2.1. Let $T : X \supset D(T) \to 2^{X^*}$ be of class $\mathcal{A}_G(S_+)$ with an approximating sequence T_n, where $G \subset X$ is open and bounded. Assume that $T + \epsilon J(\cdot - x_0)$ is injective on G for each $\epsilon \geq 0$ and for every $x_0 \in D(T)$. Moreover, assume that for each $x_0 \in D(T)$ and for each $r > 0$, there exists a bounded $\phi : \mathbf{R}^+ \to \mathbf{R}^+$ such that $\langle T_n x, x_0 \rangle \leq \phi(\|x\|)$ for all $x \in \overline{G} \cap \partial B(0,r)$ and for all large n. For a pathwise connected set $M \subset X^*$, assume that $T(D(T) \cap G) \cap M \neq \emptyset$ and $T(D(T) \cap \partial G) \cap M = \emptyset$. Then $M \subset T(D(T) \cap G)$.

Proof: Let $y_0 \in T(D(T) \cap G) \cap M$. Then there exists $x_0 \in D(T) \cap G$ such that $y_0 \in T(x_0)$. Let $p \in M$. Take $f : [0,1] \to M$ be a path in M such that $f(0) = y_0$ and $f(1) = p$. We now claim that there exist $n_0 \in \mathbb{N}$ such that

$$T_n x + \frac{1}{n} J(x - x_0) = f(t) \tag{2.1}$$

has no solution $x \in \partial G$ for any $t \in [0,1]$ and for all $n \geq n_0$. Assuming on the contrary and without loosing the generality, let $\{x_n\} \subset \partial G$ with $x_n \rightharpoonup x$ and $\{t_n\} \subset [0,1]$ with $t_n \to t_0$ be such that

$$T_n x_n + \frac{1}{n} J(x_n - x_0) = f(t_n).$$

This implies that $T_nx_n \to f(t_0)$. Since $x_n \rightharpoonup x$, we have

$$\limsup_{n\to\infty}\langle T_nx_n, x_n\rangle \le \langle f(t_0), x\rangle,$$

and then by the condition (A_3) of Definition 1.1, we have $x_n \to x$, $x \in D(T)$ and $f(t_0) \in Tx$. Since $f(t_0) \in M$ and $x \in D(T)\cap\partial G$, we have a contradiction to $T(D(T)\cap\partial G)\cap M = \emptyset$.

We now consider the following homotopy equation:

$$H_n(x,t) \equiv T_nx + \frac{1}{n}J(x-x_0) - f(t) = 0. \tag{2.2}$$

We have already established that this equation has no solution on ∂G for sufficiently large n and for any $t \in [0,1]$, and therefore this is an admissible homotopy of type (S_+).

We next consider the homotopy equation

$$G_n(x,t) \equiv (1-t)\left(T_nx + \frac{1}{n}J(x-x_0) - y_0\right) + tJ(x-x_0) = 0. \tag{2.3}$$

We show that (2.3) has no solution on ∂G for any $t \in [0,1]$ and for all $n \ge n_0$. If not, let $\{x_n\} \subset \partial B(x_0,r)$ with $x_n \rightharpoonup x$ and $\{t_n\} \subset [0,1]$ with $t_n \to t_0$ such that

$$(1-t_n)\left(T_nx_n + \frac{1}{n}J(x_n-x_0) - y_0\right) + t_nJ(x_n-x_0) = 0. \tag{2.4}$$

Since (2.1) has no solution on ∂G for any $n \ge n_0$ and $t \in [0,1]$, we have that $t_n = 0$ is impossible for all large n. Since J is injective, $t_n = 1$ is also imposssible. Suppose $t_0 = 1$. The equation (2.4) implies

$$(1-t_n)\langle T_nx_n, x_n-x_0\rangle + a_n\|x_n-x_0\|^2 - (1-t_n)\langle y_0, x_n-x_0\rangle) = 0, \tag{2.5}$$

where

$$a_n = \frac{1-t_n}{n} + t_n.$$

We may assume that $x_n \rightharpoonup x$ and let $C > 0$ be such that $\|x_n\| \le C$ for all n. By the condition (A_1), there exists a $K > 0$ such that $\langle T_nx_n, x_n\rangle \ge -K$ for all n. Also, by the hypothesis, there exists a bounded function $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ such that $\langle T_nx_n, x_0\rangle \le \phi(\|x_n\|)$ for all n. Then from (2.4), we have

$$-(1-t_n)K - (1-t_n)\phi(\|x_n\|) + a_n\|x_n-x_0\|^2 - (1-t_n)\langle y_0, x_n-x_0\rangle \le 0. \tag{2.6}$$

Since $t_n \to 1$, $a_n \to 1$ and ϕ is bounded, letting $n \to \infty$ in (2.6), we obtain $x_n \to x_0 \in \partial G$, which is a contradiction.

Next, we assume that $t_0 \in [0,1)$. If $t_0 = 0$, define $\alpha_n = \dfrac{t_n}{1-t_n}$. Then $\alpha_n \downarrow 0$ and

$$T_nx_n + \left(\frac{1}{n} + \alpha_n\right)J(x_n-x_0) = y_0.$$

This equation is like (2.1) for which we have already established the impossibility of solutions on ∂G with $f(t) \equiv y_0$. For the remaining case, $t_0 \in (0,1)$, we define

$$\beta_n = \frac{1}{n} + \frac{t_n}{1-t_n}.$$

Then $\beta_n \to \beta_0 := \dfrac{t_0}{1-t_0} > 0$. Then the equation becomes

$$T_nx_n + \beta_nJ(x_n-x_0) = y_0. \tag{2.7}$$

If

$$\limsup_{n\to\infty}\langle T_n x_n, x_n - x\rangle > 0,$$

then, by passing to a subsequence, let

$$q := \lim_{n\to\infty}\langle T_n x_n, x_n - x\rangle > 0.$$

In view of (2.7), this yields

$$\limsup_{n\to\infty}\langle \beta_n J(x_n - x_0), (x_n - x_0) - (x - x_0)\rangle = -q < 0.$$

Since $\beta_n \to \beta_0 > 0$ and J is of type (S_+), we obtain $x_n \to x \in \partial G$. From this and (2.7), we get $T_n x_n \to w := -\beta_0 J(x - x_0) + y_0$. By the condition (A_3), we obtain $x \in D(T)$ and $w \in T(x)$, i.e. $y_0 \in T(x) + \beta_0 J(x - x_0)$. This leads to a contradiction to the injectivity of $T + \epsilon J(\cdot - x_0)$ because $x \neq x_0$.

Thus, $H_n(x,t)$ and $G_n(x,t)$ are admissible homotopies for the Skrypnik's degree, $d_{(S_+)}$, for the mappings of type (S_+). By the invariance of the degree under these homotopies, we obtain

$$\begin{aligned} d_{(S_+)}(T_n + \frac{1}{n}J(\cdot - x_0) - p, G, 0) &= d_{(S_+)}(H_n(\cdot,1), G, 0) \\ &= d_{(S_+)}(H_n(\cdot,0), G, 0) \\ &= d_{(S_+)}(G_n(\cdot,1), G, 0) \\ &= d_{(S_+)}(G_n(\cdot,0), G, 0) \\ &= d_{(S_+)}(J(\cdot - x_0), G, 0) = 1. \end{aligned}$$

Here, the last equality follows by considering the (S_+)-homotopy

$$Q(x,t) = (1-t)J(x - x_0) + tJx$$

with a continuous curve $y(t) = tJx_0$ so that

$$\begin{aligned} d_{(S_+)}(J(\cdot - x_0), G, 0) &= d_{(S_+)}(Q(\cdot,0), G, 0) \\ &= d_{(S_+)}(Q(\cdot,1), G, Jx_0) \\ &= d_{(S_+)}(J, G, Jx_0) = 1. \end{aligned}$$

Therefore, for every n, there is $x_n \in G$ such that

$$H_n(x_n, 1) = 0,$$

i.e.

$$T_n x_n + \frac{1}{n}J(x_n - x_0) = p,$$

which implies $T_n x_n \to p$. By the condition (A_3), we deduce that $x_n \to x \in \overline{G}$, $x \in D(T)$, and $p \in T(x)$. Since $T(D(T) \cap \partial G) \cap M = \emptyset$, we can only have $x \in G$. Since p was an arbitrary point in M, we obtain $M \subset T(D(T) \cap G)$. $\square$

We now apply Proposition 2.1 to obtain the following invariance of domain result.

Theorem 2.2 (Invariance of Domain). *Let $T : X \supset D(T) \to 2^{X^*}$ be of class $\mathcal{A}_G(S_+)$ with an approximating sequence T_n, where $G \subset X$ is open. Assume that $T + \epsilon J(\cdot - x_0)$ is locally injective on G for each $\epsilon \geq 0$ and for every $x_0 \in D(T)$. Moreover, assume that for each $x_0 \in D(T)$ and for each $r > 0$, there exists a bounded $\phi : \mathbf{R}^+ \to \mathbf{R}^+$ such that $\langle T_n x, x_0\rangle \leq \phi(\|x\|)$ for all $x \in \overline{G} \cap \partial B(0, r)$ and for all large n. Then $T(D(T) \cap G)$ is open.*

Proof: Let $y_0 \in T(D(T) \cap G)$. Then there exists $x_0 \in G$ such that $y_0 \in T(x_0)$. Since T is locally injective on G, there is $r > 0$ such that T is injective on $\overline{B(x_0, r)} \cap D(T)$, where $\overline{B(x_0, r)} \subset G$. It is then clear that $y_0 \notin T(D(T) \cap \partial B(x_0, r))$. By Lemma 3.2 in [8], there exits $n_0 \in \mathbb{N}$ such that $y_0 \notin T_n(\partial B(x_0, r))$ for all $n \geq n_0$, where $\{T_n\}$ is an approximating sequence of T as in the hypothesis of the theorem.

We claim that there exists $\delta > 0$ such that $B(y_0, \delta) \cap T(D(T) \cap \partial B(x_0, r)) = \emptyset$. Assume the contrary, and let $w_n \in B(y_0, 1/n) \cap T(D(T) \cap \partial B(x_0, r))$. Then $y_n \to y_0$ and $w_n \in T(x_n)$ with $x_n \in \partial B(x_0, r)$. Now the condition (A3) applies since $x_n \rightharpoonup x$ (up to subsequence) and $y_n \to y_0$. We then obtain $x_n \to x \in \partial B(x_0, r)$, $x \in D(T)$ and $y_0 \in T(x)$. This contradicts $y_0 \notin T(D(T) \cap \partial B(x_0, r))$.

Since $B(y_0, \delta) \cap T(D(T) \cap \partial B(x_0, r)) = \emptyset$, $y_0 \in T(D(T) \cap B(x_0, r))$ and the ball $B(y_0, \delta)$ is pathwise connected, we can apply Proposition 2.1 to obtain $B(y_0, \delta) \subset T(D(T) \cap B(x_0, r))$.□

It would be interesting to establish analogous results via degree theory for operators of the form $A + T$, where $A : X \supset D(A) \to 2^{X^*}$ is maximal monotone and T is in $\mathcal{A}_G(S_+)$. Similar results are expected for the sum $L + A + T$ in the spirit of results in [1], where L is densely defined linear maximal monotone operator and T in class $A_G(S_+)$ with respect to $D(L)$.

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RELATION BETWEEN BMO AND A_2 WEIGHT FUNCTIONS

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Abstract: In this paper, we relate Bounded Mean Oscillation (BMO) function and A_2 weight function. We show that logarithm of any A_2 function is a BMO function and every BMO function is equal to a constant multiple of the logarithm of an A_2 weight function. Moreover, we show that logarithm of any A_p weight function for $1 < p < \infty$ is a BMO function.

Key Words: Weight function, BMO function, A_p weight function
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1. Introduction and main result

The space of bounded mean oscillation, abbreviated BMO, is the space of all functions whose deviations from their means over cubes is bounded. The BMO space is frequently used space in analysis. For example the BMO space comes into play in the characterization of L^2 boundedness of noncovolution singular integral operators having standard kernels. Careleson measures have natural relation with BMO functions such that the measures of functions are Carleson measures iff the functions are in BMO. Readers are suggested to refer [1] for more about the BMO space.

On the other hand, the theory of weights play an important role in various fields such as extrapolation theory, vector-valued inequalities and estimates for certain class of non linear differential equation. Moreover, they are very useful in the study of boundary value problems for Laplace's equation in Lipschitz domains. In 1970, Muckenhoupt characterized positive functions w for which the Hardy-Littlewood maximal operator M maps $L^p(\mathbb{R}^n, w(x)dx)$ to itself. Muckenhoupt's characterization actually gave the better understanding of theory of weighted inequalities which then led to the introduction of A_p class and consequently the development of weighted inequalities. For more about the weighted theory, please refer to [1],[2] and [3]. In this article, we relate the BMO function and A_2 weight function. Before this, some definitions are in order:

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Definition: let f be a locally integrable function on $\mathbb{R}^n$ and Q be a measurable set in $\mathbb{R}^n$. Then the mean oscillation of f over Q is

$$\frac{1}{|Q|}\int_Q |f(x) - \mathrm{Avg}_Q f| dx$$

where

$$\mathrm{Avg}_Q f = \frac{1}{|Q|}\int_Q f(x) dx$$

is the mean or average of f over Q.
The BMO norm of a complex valued function f on $\mathbb{R}^n$ is defined as

$$\| f \|_{BMO} = \sup_Q \frac{1}{|Q|}\int_Q |f(x) - \mathrm{Avg}_Q f| dx.$$

In the above definition the supremum is taken over all cubes Q in $\mathbb{R}^n$. Then the function f is said to be of bounded mean oscillation if $\| f \|_{BMO} < \infty$.

Definition: A locally integrable function on $\mathbb{R}^n$ that takes values in the interval $(0, \infty)$ almost everywhere is called a weight. So by definition a weight function can be zero or infinity only on a set whose Lebesgue measure is zero. We use the notation $w(E) = \int_E w(x)dx$ to denote the w-measure of the set E and we reserve the notation $L^n(\mathbb{R}^n, w)$ or $L^p(w)$ for the weighted L^p spaces. We note that $w(E) < \infty$ for all sets E contained in some ball since the weights are locally integrable functions.

Definition: Let $1 < p < \infty$. A weight w is said to be of class A_p if $[w]_{A_p}$ if finite where $[w]_{A_p}$ is defined as

$$[w]_{A_p} = \sup_{Q\text{cubes in}\mathbb{R}^n} \left(\frac{1}{|q|}|w(x)|dx\right)\left(|w(x)|^{\frac{-1}{p-1}}dx\right)^{p-1}.$$

We remark that in the above definition of A_p one can also use set of all balls in $\mathbb{R}^n$ instead of all cubes in $\mathbb{R}^n$. Readers are suggested to read for motivation, properties of A_P weights and much more about the A_P weights.

Theorem 1. *For all functions f in space of BMO defined on $\mathbb{R}^n$, for all cubes Q and $\alpha > 0$, we have*

$$\left|\left\{x : |f(x) - Avg_Q f| > \alpha\right\}\right| \leq e|Q|e^{-A\alpha/\|f\|_{BMO}}$$

with $A = (2^n e)^{-1}$.

This theorem is popularly known as John-Nirenberg theorem and for the proof, please refer to [4].

We first establish the following result:
Let ν be a real-valued locally integrable function on R^n and let $1 < p < \infty$. Then $e^\nu \in A_p$ if and only if the following two conditions are satisfied for some constant $c < \infty$:

(a) $\displaystyle\sup_{Q\text{cubes}} \frac{1}{|Q|}\int_Q e^{\nu(t)-\nu_Q} dt \leq C$
(b) $\displaystyle\sup_{Q\text{cubes}} \frac{1}{|Q|}\int_Q e^{-(\nu(t)-\nu_Q)\frac{1}{p-1}} dt \leq C$

Suppose $e^{\nu} \in A_p, 1 < p < \infty$..Then,

$$\begin{aligned} \frac{1}{|Q|}\int_Q e^{\nu(t)-\nu_Q}dt &= e^{-\nu_Q}\frac{1}{|Q|}\int_Q e^{\nu(t)}dt \\ &= \left(e^{\frac{-\nu_Q}{p-1}}\right)^{p-1}\frac{1}{|Q|}\int_Q e^{\nu(t)}dt. \\ &\leq \left(\frac{1}{|Q|}\int_Q e^{\frac{-\nu(t)}{p-1}}dt\right)^{p-1}\left(\frac{1}{|Q|}\int_Q e^{\nu(t)}dt\right) \\ &\leq [e^{\nu(t)}]_{A_p} < \infty. \end{aligned}$$

This proves (a). Again,

$$\begin{aligned} \frac{1}{|Q|}\int_Q e^{-(\nu(t)-\nu_Q)\frac{1}{p-1}}dt &= \frac{1}{|Q|}\int_Q e^{\frac{-\nu(t)}{p-1}}.e^{\frac{\nu_Q}{p-1}}dt. \\ &\leq \left(\frac{1}{|Q|}\int_Q e^{\frac{\nu(t)}{p-1}}dt\right)\left(\frac{1}{|Q|}\int_Q e^{\nu(t)}dt\right)^{\frac{1}{p-1}} \\ &\leq [e^{\nu}]_{A_p} \end{aligned}$$

This proves (b). Conversely, suppose that the conditions (a) and (b) hold. We need to show that $e^{\nu} \in A_p$. This follows because,

$$\begin{aligned} &\left(\frac{1}{|Q|}\int_Q e^{\nu(t)}dt\right)\left(\frac{1}{|Q|}\int_Q \left(e^{\nu(t)}\right)^{\frac{-1}{p-1}}dt\right)^{p-1} \\ &\quad= \left(\frac{1}{|Q|}\int_Q e^{\nu(t)-\nu_Q}dt\right)\left(\frac{1}{|Q|}\int_Q e^{\frac{-(\nu(t)-\nu_Q)}{p-1}}dt\right)^{p-1} \\ &\quad< C \end{aligned}$$

by (a) and (b). In particular for p=2, we have: $e^{v} \in A_2 \Longleftrightarrow$ for some constant$C < \infty$

$$\sup_Q \frac{1}{|Q|}\int_Q e^{V(t)-V_Q}dt \leq C$$

$$\sup_Q \frac{1}{|Q|}\int_Q e^{-(V(t)-V_Q)}dt \leq C$$

$$\Longleftrightarrow \sup_Q \frac{1}{|Q|}\int_Q e^{|V(t)-V_Q|}dt \leq C$$

if $\varphi \in A_2$, then

$$\left(\frac{1}{|Q|}\int_Q \varphi\right)\left(1/|Q|\int_Q \varphi^{-1}\right) \leq C$$

equivalently

$$\left(\frac{1}{|Q|}\int_Q \varphi/\varphi_Q\right)\left(1/|Q|\int_Q \varphi_Q/\varphi\right) \leq C.$$

By Jensen's inequality, each function is at least 1 and at most C, therefore

$$\frac{1}{|Q|}\int_Q e^{|\log\varphi - \log\varphi_Q|} \leq 2C$$

and so

$$\frac{1}{|Q|}\int_Q |\log\varphi - \log\varphi_Q| \le 2C.$$

This proves that $log\varphi \in BMO$.

Next we prove that if $\varphi \in BMO$, then $e^{C_\varphi \varphi} \in A_2$ for some constant C_φ. From the part (a), to show $e^{C_\varphi\varphi} \in A_2$ it suffices to prove $\sup_Q \frac{1}{|Q|}\int_Q e^{|f-f_Q|}dt \le C$ where $f = e^{C_\varphi\varphi}$. John-Nirenberg Theorem gives,

$$|x \in Q : |f(x) - avgf| > \alpha| \le C|Q|e^{-A\alpha/\|f\|_{BMO}}.$$

Let $Q_i = x \in Q : |f| \ge |f(x) - Avgf| > i$. Then

$$\frac{1}{|Q|}\int_Q e^{|f-f_Q|}dt = \frac{1}{|Q|}\sum_{i=1}^{\infty}\int_{Q_i} e^{|f-f_Q|}dt \le \frac{1}{|Q|}\sum_{i=1}^{\infty} e^{i+1}C|Q|e^{-Ai/\|f\|_{BMO}}$$

We take $C_\varphi, s.t.\|f\|_{BMO} = A/2$ so that

$$\frac{1}{|Q|}\int_Q e^{|f-f_Q|}dt \le Ce\sum_{i=1}^{\infty} e^{-i} < \infty.$$

This shows that $e^{C_\varphi\varphi} \in A_2$. This proves that every BMO function is equal to a constant multiple of the logarithm of an A_2 weight function.
Finally we show that logarithm of any A_p weight function for $1 < p < \infty$ is a BMO function. We already proved that $\|\log\varphi\|_{BMO} \le [\varphi]_{A_2}$ and when $1 < p \le 2$

$$\varphi_{A_Q} = \sup_Q(\frac{1}{|Q|}\int_Q \varphi(x)dx)(\frac{1}{|Q|}\int_Q \varphi(x)^{-1/p-1}dx)^{p-1}$$

$$\ge \sup_Q(\frac{1}{|Q|}\int_Q \varphi(x)dx)(\frac{1}{|Q|}\int_Q \varphi^{-1}dx) = [\varphi]_{A_2}.$$

Therefore $\|\log\varphi\|_{BMO} \le [\varphi]_{A_p}$ when $1 < p \le 2$

$$\|log\varphi^{-1/p-1}\|_{BMO} \le [\varphi^{-1/p-1}]_{A_{p'}}$$

when $p > \infty$. That is:

$$\sup_Q\left(\frac{1}{|Q|}\int_Q |\log\varphi^{-1/p-1} - \log\varphi_Q^{-1/p-1}|dt\right) \le \sup_Q\left(\frac{1}{|Q|}\int_Q p^{-1/p-1}\right)\left(\frac{1}{|Q|}\int_Q \varphi^{1/(p-1)(p'-1)}\right)^{p'-1}$$

$$\frac{1}{(p-1)}\sup_Q\left(\frac{1}{|Q|}\int |\log\varphi - \log\varphi_Q|dt\right) \le \sup\left(\frac{1}{|Q|}\int_Q \varphi^{-1/p-1}\right)\left(\frac{1}{|Q|}\int_Q \varphi\right)^{p-1}$$

$$\sup_Q\left(\frac{1}{|Q|}\int |\log\varphi - \log\varphi_Q|dt\right) \le (p-1)\sup\left(\frac{1}{|Q|}(1/|Q|\int_Q \varphi^{-1/p-1})^{p-1}\right)^{1/p-1}.$$

Consequently, when $p > 2$.

$$\|\log\varphi\|_{BMO} \le (p-1)[\varphi]_{A_p}^{1/p-1}.$$

This shows that the $\log\varphi$ is a BMO function.

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DIFFUSION APPROXIMATION TO RADIATION HEAT TRANSFER IN SEMITRANSPARENT MEDIUM

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Abstract: Conduction-radiation heat transfer problem is usually formulated with a highly nonlinear integro-differential equation. In this paper we describe a simple technique for modeling radiation heat transfer in a semitransparent medium with arbitrary varying spectral absorption coefficient and purpose a computational model (described by a coupled system of elliptic-parabolic PDEs) based on flux limited diffusion theory. Numerical simulations of a glass cooling problem in one dimension show that the proposed method is comparable to a higher order discrete ordinate ($\mathbf{S_8}$) method.

Key Words: conduction-radiation transfer, glass cooling, diffusion approximation
AMS (MOS) Subject Classification. 35Q35, 45K05, 65Z05.

1. Introduction

Radiation transfer in semitransparent medium plays an important role in many industrial applications. For instance, quality of the products in glass industry depends on the proper control of temperature during the various fabricating processes. In many such industrial processes, temperature is generally very high resulting a strong contribution from the radiative transfer to the overall transport of energy within the system. For semitransparent media such as glass with very good infrared transmission, contribution of radiation should not be neglected for temperature higher than $400°C$ [14]. To perform analysis of energy transfer in such situations, it is often necessary to consider the effects of convection, conduction as well as radiative transfer of energy. Since the coupling between the energy equation and the equation of radiative transfer is highly nonlinear, it is of utmost importance to utilize efficient and accurate solution procedures to the radiation part of the overall problem. Consequently, great efforts have been expended in developing such methods. An excellent survey on radiation transfer in participating media with major events in the development of engineering treatment can be found in [3, 5, 12]. Mathematical modeling of coupled radiation - conduction heat transfer in semitransparent media (glass) can be found

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in [2, 10, 11, 14, 15]. The detail mathematical descriptions of the flux limited diffusion theory can be found in [1, 4, 7, 8, 9, 13].

In this paper a simple technique for modeling radiation heat transfer in participating media with an arbitrary varying spectral absorption coefficient is presented and some computational models based on diffusion approximation are developed. Boundary condition is derived using asymptotic matching of slow scale interior solution and fast scale boundary layer solution. Some simulations of combined conduction-radiation transfer in a semitransparent material (glass) with specularly emitting and reflecting bounding walls are presented. Initial simulation results in one dimensional case shows that the diffusion approximation in glass cooling problem is comparable to the high order discrete ordinate ($\mathbf{S_8}$) method [10].

Rest of the paper is organized as follows. Section §2 describes the mathematical modeling for conduction-radiation transfer in semitransparent medium. The formal solution of the radiative transfer equation is presented in §3. The computational models based on the diffusion approximation of the radiative transfer equation are derived in section §4. Some numerical simulation results are presented in §5, with conclusions in §6.

2. Mathematical Model

Let a beam of radiation of intensity I travel in the direction $\mathbf{\Omega}$ along a path s through an absorbing, emitting and scattering medium. The angular distribution of radiation intensity $I(\mathbf{r}(x,y,z),\nu,\mathbf{\Omega}(\theta,\phi),t)$ which is defined as the radiative energy passing through an area per unit solid angle, per unit area projected normal to the direction of passage, and per unit time satisfies the following general equation of transfer governing the radiation field.

$$\begin{aligned}\frac{1}{c_g}\frac{\partial I(\mathbf{r},\nu,\mathbf{\Omega},t)}{\partial t} + \mathbf{\Omega}\cdot\nabla I(\mathbf{r},\nu,\mathbf{\Omega},t) &= -\left[k(\nu)+\sigma(\nu)\right] I(\mathbf{r},\nu,\mathbf{\Omega},t) + j^e(\nu,\mathbf{r},t)\\ &+ \frac{1}{4\pi}\sigma(\nu)\int\limits_{\mathbf{\Omega}'=4\pi} P(\mathbf{\Omega}',\mathbf{\Omega}) I(\mathbf{r},\nu,\mathbf{\Omega},t)d\Omega' \end{aligned} \tag{2.1}$$

where c_g is the speed of propagation of light in the medium; ν is wavelength; κ is spectral absorption coefficient; σ is spectral scattering coefficient; $j^e(\nu,\mathbf{r},t)$ is radiation emitted by the matter per unit time, volume, solid angle and frequency; $P(\mathbf{\Omega}',\mathbf{\Omega})$ is the probability that incident radiation at $\mathbf{\Omega}$ will be scattered into an element of solid angle $d\Omega$ about the direction $\mathbf{\Omega}'$.

To derive a model for the semitransparent medium (glass), we make the following simplification assumptions.

(1) Due to the large magnitude of the speed propagation c, for most of engineering applications the first term of left hand side of equation (2.1) i.e., $\frac{1}{c_g}\frac{\partial I}{\partial t}$ can be neglected in comparison to other terms
(2) Since we are concerned with *semitransparent* medium, we set $\sigma(\nu) = 0$. (Purely absorbing and emitting medium)
(3) However, in equation (2.1) the intensity depends on time. Therefore we can assume time variable s a parameter. We also assume that a local thermodynamic equilibrium (LTE) is established and the Kirchhoff's law is valid. Then the emission of radiation

$j^e(\nu, \mathbf{r}, t)$ in any direction is related to the spectral absorption coefficient and the blackbody radiation intensity at the temperature $T(\mathbf{r})$ of the medium is given by

$$j^e(\nu, \mathbf{r}) = k(\nu) I_b(\nu, T(\mathbf{r})),$$

where $k(\nu)$ = absorption (= emissivity) coefficient and the spectral intensity of black body radiation is given by the Planck's function. i.e.,

$$I_b(\nu, T) = \frac{2h_p\nu^3}{c_g^2\left(\exp\left(\frac{h_p\nu}{k_bT}-1\right)\right)} \tag{2.2}$$

(4) Finally we assume that the radiative properties changes with frequency of radiation and hence a nongray analysis of the problem is necessary. To this end, various models have been developed as a simplification of frequency dependent radiative properties [5]. In our present model we will adopt a multi-band model for absorption coefficient $k(\nu)$. i.e., we approximate the spectral absorption coefficient in each of the frequency band $\nu_k \leq \nu\nu_{k+1}$ by a piecewise constant function $k(\nu)$. Thus we have the following notational convention: For $\nu \in [\nu_k, \nu_{k+1}), k(\nu) := k(\nu_k) = k_k$ and

$$\mathbf{I^{(k)}}(\mathbf{r}, \mathbf{\Omega}) := \int_{v_k}^{v_{k+1}} I(\mathbf{r}, \nu, \mathbf{\Omega}) d\nu \text{ and } I_b^{(k)}(T(\mathbf{r})) := \int_{v_k}^{v_{k+1}} I_b(\nu, T(\mathbf{r})) d\nu$$

Radiative Transfer Equation

Under the above assumptions, the radiative heat transfer equation (2.1) reduces to

$$\mathbf{\Omega} \cdot \nabla I^{(k)}(\mathbf{r}, \mathbf{\Omega}) = k_k \left(I_b^{(k)}(T(\mathbf{r})) - I^{(k)}(\mathbf{r}, \mathbf{\Omega})\right) \tag{2.3}$$

The intensity of radiation $I^{(k)}$ leaving the boundary surface at $\boldsymbol{r_s}$ in the direction of $\mathbf{\Omega}$ is composed of two contributions - one is the transmitted into medium from the surrounding and other is the reflected internal intensity. We choose specularly emitting and specularly reflecting boundary model for our problem [5]. We assume that the surface is smooth and incident and reflected rays lie symmetrically with respect to the normal at the point of incidence and the reflected beam is contained within the solid angle $d\Omega$ equal to the solid angle of incidence $d\Omega'$. i.e., $d\Omega = d\Omega'$.

$$I^{(k)}(\boldsymbol{r_s}, \mathbf{\Omega}) = \underbrace{\epsilon\tau(\mathbf{\Omega}, \Omega'')I_b^{(k)}(T(\boldsymbol{r_s}))}_{\text{specular emitance}} + \underbrace{\rho(\mathbf{\Omega}, \mathbf{\Omega'})I^{(k)}(\boldsymbol{r_s}, \mathbf{\Omega}_s)}_{\text{specular reflectance}}, \quad \tau + \rho = 1 \tag{2.4}$$

where ϵ is the emissivity, $\rho(\mathbf{\Omega}, \mathbf{\Omega'})$ and $\tau(\mathbf{\Omega}, \mathbf{\Omega''})$ are the boundary reflecivities from direction $\mathbf{\Omega'}$ to $\mathbf{\Omega}$ and from $\mathbf{\Omega''}$ to $\mathbf{\Omega}$ respectively, where $\mathbf{\Omega'}$ and $\mathbf{\Omega''}$ are the $s$pecular directions with which the beam of rays hit the surface in order to travel into the direction $\mathbf{\Omega}$ after a specular reflection.

Energy Equation

The equation for conservation of energy in an isotropic, homogeneous medium which participates in the radiative transfer can be obtained by making an energy balance on an arbitrary volume of matter as

$$\rho_g C(T)\frac{\partial T}{\partial t} = \nabla \cdot (K_c(T)\nabla T - \boldsymbol{q^r}) + \dot{q} \tag{2.5}$$

where ρ_g is the density of the material, $C(T)$ is the specific heat capacity of the medium, $K_c(T)$ is the thermal conductivity, $\boldsymbol{q^r}$ is the radiation heat flux vector and $\dot{q}$ is the internal heat generation rate.

Considering only a diffusion external radiation source, energy balance on the boundary gives the boundary conditions for the temperature as

$$K_c(T)\frac{\partial T}{\partial n}(\boldsymbol{r_s}(x,y,z),t) + h[T_{out}(\boldsymbol{r_s}(x,y,z)) - T(\boldsymbol{r_s}(x,y,z))] + g = 0 \tag{2.6}$$

where

$$g(\boldsymbol{r_s}(x,y,z)) = \pi\epsilon\sum_k [I_{bo}^{(k)}(T_{out}(\boldsymbol{r_s}(x,y,z))) - I_{bo}^{(k)}(T(\boldsymbol{r_s}(x,y,z)))]$$

where h is the convection film coefficient, ϵ is the boundary emissivity in opaque spectral region, $\boldsymbol{r_s}$ is the point on the boundary surface, T_{out} is the surrounding temperature, K_c is the thermal conductivity of the material and $I_{bo}^{(k)}(T)$ is the blackbody radiation intensity at temperature T for an opaque spectral band k.

Divergence of Radiative Heat Flux

Integrating equation (2), first over all solid angles and then over the entire spectrum, we get the divergence of radiative heat flux which is required in the total energy balance equation (2.4) as

$$\nabla\cdot\boldsymbol{q^r} = \sum_k \nabla\cdot\boldsymbol{q^r}^{(k)} = \sum_k k_k\left[4\pi I_b^{(k)}(T) - G^{(k)}\right] \tag{2.7}$$

where radiative heat flux $\boldsymbol{q^r}$ and incident radiation G for the k^{th} spectral band are

$$\boldsymbol{q^r}^{(k)} = \int\limits_{\Omega=4\pi} I^{(k)}\,\boldsymbol{\Omega}\,d\Omega, \qquad G^{(k)} = \int\limits_{\Omega=4\pi} I^{(k)}\,d\Omega. \tag{2.8}$$

The term $4\pi k_k I_b^{(k)}(T)$ and $k_k G^{(k)}$ in (2.7) respectively represent local rate of emission and the local rate of absorption per unit volume and frequency.

3. Formal Solution of Radiative Transfer Equation

Choose a length s as the distance back along $\boldsymbol{\Omega}$ from the point $\boldsymbol{r}$. Then equation (2.3) can be rewritten as

$$-\frac{\partial I^{(k)}(\boldsymbol{r}-s\boldsymbol{\Omega},\boldsymbol{\Omega})}{\partial s} + k_k I^{(k)}(\boldsymbol{r}-s\boldsymbol{\Omega},\boldsymbol{\Omega}) = k_k I_b^{(k)}(T(\boldsymbol{r}-s\boldsymbol{\Omega})) \tag{3.1}$$

which is a first order differential equation in $I^{(k)}$ whose solution can be easily found by taking $e^{(s_0-s)k_k}$ as an integrating factor, where s_0 is an arbitrary point along s.

Integrating equation (3.1) from s to s_0, we get

$$\begin{aligned} I^{(k)}(\boldsymbol{r}-s\boldsymbol{\Omega},\boldsymbol{\Omega}) &= I^{(k)}(\boldsymbol{r}-s_0\Omega,\Omega)e^{(s-s_0)k_k} \\ &+\ k_k\int\limits_s^{s_0} I_b^{(k)}(T(\boldsymbol{r}-s'\boldsymbol{\Omega}))e^{(s-s')k_k}ds' \end{aligned} \tag{3.2}$$

Set, $s = 0$, we get the specific intensity at the point $\boldsymbol{r}$ in the direction of $\boldsymbol{\Omega}$ as

$$I^{(k)}(\boldsymbol{r},\boldsymbol{\Omega}) = I^{(k)}(\boldsymbol{r} - s_0\boldsymbol{\Omega},\boldsymbol{\Omega})e^{-s_0k_k} + k_k\int_0^{s_0} I_b^{(k)}(T(\boldsymbol{r} - s'\boldsymbol{\Omega}))e^{-s'k_k}ds' \tag{3.3}$$

The constant of integration $I^{(k)}(\boldsymbol{r} - s_0\boldsymbol{\Omega},\boldsymbol{\Omega})$ in equation (3.3) can be evaluated by using the formal boundary conditions.

$$I^{(k)}(\boldsymbol{r_s},\boldsymbol{\Omega}) = \Gamma^{(k)}(\boldsymbol{r_s},\boldsymbol{\Omega}), \boldsymbol{n}\cdot\Omega < 0 \tag{3.4}$$

where $\Gamma^{(k)}$ is the prescribed boundary data and $\boldsymbol{n}$ is a unit outward normal at the surface point $\boldsymbol{r_s}$.

Let s_0 be such that $\boldsymbol{r} - s_0\Omega = \boldsymbol{r_s}$ is a point on the boundary. i.e., $s_0 = |\boldsymbol{r} - \boldsymbol{r_s}|$ and from the boundary condition (3.4), we get

$$\Big(I^{(k)}(\boldsymbol{r} - s_0\boldsymbol{\Omega},\boldsymbol{\Omega})\Big)_{s_0=|\boldsymbol{r}-\boldsymbol{r_s}|} = \Gamma^{(k)}(\boldsymbol{r_s},\boldsymbol{\Omega}) \tag{3.5}$$

replacing s_0 by $|\boldsymbol{r} - \boldsymbol{r_s}|$ in (3.5) we get

$$I^{(k)}(\boldsymbol{r},\boldsymbol{\Omega}) = \Gamma^{(k)}(\boldsymbol{r_s},\boldsymbol{\Omega})e^{-|\boldsymbol{r}-\boldsymbol{r_s}|k_k} + \int_0^{|\boldsymbol{r}-\boldsymbol{r_s}|} I_b^{(k)}(T(\boldsymbol{r} - s'\boldsymbol{\Omega}))e^{-s'k_k}ds' \tag{3.6}$$

Equation (3.5) is an explicit expression for the radiation intensity if the temperature field is known and the function $\Gamma^{(k)}$ is a specified function of $\boldsymbol{r_s}$ and $\boldsymbol{\Omega}$. However, generally the temperature field is not known and must be found in conjunction with the overall conservation of energy and the prescribed boundary data involve the radiation intensity from the interior of the medium that is reflected by the boundary surface and hence depends on an unknown quantity.

4. Diffusion Approximation to Radiative Transfer Equation

One of the earliest diffusion type approximations for the radiation transfer was due to Rosseland (1936) which expresses the radiative heat flux in terms of gradient of radiative intensity.

$$\boldsymbol{q^r} = -K^r(T)\nabla T, \quad K^r(T) = \frac{4\pi}{3k(\nu)}\frac{\partial I_b}{\partial T} \tag{4.1}$$

In the classical diffusion or Eddington approximation specific intensity of radiation is represented by the first two terms in a spherical harmonic expansion. The resulting expression has the form of the Ficks Law of diffusion which is usually expressed as

$$\boldsymbol{q^r} = -D_F\nabla G, \quad D_F = \frac{1}{3k(\nu)}. \tag{4.2}$$

In the presence of steep spatial gradients equations (4.1) and (4.2) predict radiative fluxes which are nonphysically large. But from (2.8), we have

$$|\boldsymbol{q}^{r(k)}| \le G^{(k)} = c_gE^{(k)} \tag{4.3}$$

That is, the exact value of radiative heat flux cannot exceed the product of speed of propagation and energy density. Taking this into account, Levermore and Pomraning (1981)

proposed the flux limited diffusion theory [4] which was based on Chapman-Enskog Asymptotic method, originally developed for treating the nonlinear Boltzman equation, to the radiative transfer equation. Sanchez and Pomraning (1990) generalized this and presented as a family of flux limiting diffusion theory with improved accuracy over the Eddington approximation and the earlier flux limited diffusion theory [8], which was further generalized by Szilard and Pomraning (1993) [13].

Derivation of Flux-limiting Diffusion Approximation

This method assumes the separability condition of the specific intensity $I^{(k)}(\boldsymbol{r},\boldsymbol{\Omega})$ such that

$$I^{(k)}(\boldsymbol{r},\boldsymbol{\Omega}) = G^{(k)}(\boldsymbol{r})\psi^{(k)}(\boldsymbol{r},\boldsymbol{\Omega}) \tag{4.4}$$

where for consistency with an angular integration of (4.4) $\psi^{(k)}$ is normalized via

$$\int\limits_{\Omega=4\pi} \psi^{(k)}(\boldsymbol{r},\boldsymbol{\Omega})\, d\Omega = 1 \tag{4.5}$$

Using equation (4.4) in the transfer equation (2.3), we get

$$\left(\boldsymbol{\Omega}\cdot\nabla G^{(k)}(\boldsymbol{r}) + k_k G^{(k)}(\boldsymbol{r})\right)\psi^{(k)}(\boldsymbol{r},\boldsymbol{\Omega}) = k_k I_b^{(k)}(T(\boldsymbol{r})) \tag{4.6}$$

We assume that the space dependence of $\psi^{(k)}$ is sufficiently weak and, in the lowest order, can be neglected i.e., $\nabla\psi^{(k)} = 0$. This assumption is justified for two extreme limiting cases: isotropic limit given by Eddignton approximation and the asymptotic limit for the case of collimated irradiation [3].

In Eddington approximation, from (21) and (6) we have

$$\begin{aligned} I^{(k)}(\boldsymbol{r},\boldsymbol{\Omega}) &= \frac{1}{4\pi}\left(G^{(k)}(\boldsymbol{r}) - \boldsymbol{\Omega}\cdot\frac{1}{k_k}\nabla G^{(k)}(\boldsymbol{r})\right) \\ \text{i.e., } \psi^{(k)}(\boldsymbol{r},\boldsymbol{\Omega}) &= \frac{1}{4\pi}\left(-\boldsymbol{\Omega}\cdot\frac{1}{k_k G^{(k)}}\nabla G^{(k)}\right) \end{aligned}$$

where $\nabla G^{(k)}$ is assumed to be small. If $\nabla G^{(k)} = 0$ (i.e. $\nabla\psi^{(k)} = 0$) we get the equilibrium approximation. This corresponds to the case when radiation field is local Plankian, only temperature dependent.

In the case of collimated irradiation, the solution (3.5) to the radiative transfer equation has the following form [3]

$$I^{(k)}(\boldsymbol{r},\boldsymbol{\Omega}) = (1-\rho)q_c(\boldsymbol{r_s})\delta(\boldsymbol{\Omega} - \boldsymbol{\Omega_c}(\boldsymbol{r_s}))e^{-\tau_c} \tag{4.7}$$

Integrating (4.7) over all solid angle, we obtain

$$\begin{aligned} G^{(k)}(\boldsymbol{r}) &= (1-\rho(\mu_c,\phi_c))q_c(\boldsymbol{r_s})e^{-\tau_c} \qquad (4.8)\\ \text{i.e., } \psi^{(k)}(\boldsymbol{r},\boldsymbol{\Omega}) &= \delta(\boldsymbol{\Omega} - \boldsymbol{\Omega_c}(\boldsymbol{r_s})) \end{aligned}$$

Integrating (4.6) over all solid angle and using (4.5) we get the corresponding conservation equation as

$$\boldsymbol{f}^{(k)}\cdot\nabla G^{(k)}(\boldsymbol{r}) = k_k\left(4\pi I_b^{(k)}(T(\boldsymbol{r})) - G^{(k)}(\boldsymbol{r})\right) \tag{4.9}$$

where we have defined the normalized flux $\boldsymbol{f}^{(k)}(\boldsymbol{r})$

$$f^{(k)}(\boldsymbol{r}) = \frac{q^{(k)}(\boldsymbol{r})}{G^{(k)}(\boldsymbol{r})} = \int\limits_{\Omega=4\pi} \psi^{(k)}(\boldsymbol{r}, \boldsymbol{\Omega})\boldsymbol{\Omega} d\Omega \tag{4.10}$$

Subtracting (4.9) from (4.6), we get

$$k_k I_b^{(k)} = \left(\boldsymbol{\Omega}\cdot\nabla G^{(k)} - \boldsymbol{f}^{(k)}\cdot\nabla G^{(k)} + k_k 4\pi I_b^{(k)}\right)\psi^{(k)}$$
$$\Rightarrow \psi^{(k)} = \frac{1}{4\pi}\left(\frac{1}{1+\boldsymbol{f}^{(k)}\cdot\boldsymbol{R} - \boldsymbol{\Omega}\cdot\boldsymbol{R}}\right) \tag{4.11}$$

where

$$\boldsymbol{R} = -\frac{\nabla G^{(k)}}{4\pi k_k I_b^{(k)}} \tag{4.12}$$

The scalar quantity $R = |\boldsymbol{R}|$ is sometimes refered to the flux limiting parameter and it is a dimensionless measure of the strength of the spatial gradient of the incident intensity. We know from Fick's law of mass diffusion that heat flux vector and gradient of energy density are always (anti)parallel. From (4.10) and (4.12), we get $\boldsymbol{f}^{(k)}||\boldsymbol{q^{r(k)}}||$ and $\boldsymbol{R}||\nabla G^{(k)}||$. Therefore, we conclude that

$$\boldsymbol{f^{(k)}} = \lambda\boldsymbol{R} \tag{4.13}$$

$$\Rightarrow \psi^{(k)} = \frac{1}{4\pi}\left(\frac{1}{1+\lambda R^2 - \boldsymbol{\Omega}\cdot\boldsymbol{R}}\right) \tag{4.14}$$

To obtain the proportionality function $\lambda = \lambda(R)$ in (4.13) we use the normalization condition (4.5) in (4.14).

$$1 = \frac{1}{4\pi}\int\limits_{\Omega=4\pi}\frac{d\Omega}{1+\lambda R^2 - \boldsymbol{\Omega}\cdot\boldsymbol{R}} = \frac{1}{2\pi}\int\limits_0^{2\pi}\int\limits_0^{\pi}\frac{\sin\theta d\theta\, d\phi}{1+\lambda R^2 - \cos\theta R}$$
$$= \frac{1}{2R}\int\limits_{-R}^{R}\frac{dt}{1+\lambda R^2 + t} = \frac{1}{2R}\ln\left(\frac{1+\lambda R^2 + R}{1+\lambda R^2 - R}\right)$$
$$= \frac{1}{R}\tanh^{-1}\left(\frac{R}{1+\lambda R^2}\right) \tag{4.15}$$

Solving this for λ, we get

$$\lambda(R) = \frac{1}{R}\left(\coth(R) - \frac{1}{R}\right) \tag{4.16}$$

$$\Rightarrow \psi^{(k)} = \frac{1}{4\pi}\left(\frac{1}{R\coth(R) - \boldsymbol{\Omega}\cdot\boldsymbol{R}}\right) \tag{4.17}$$

From (4.5), (4.14), (4.15), and (4.17) we can derive the Fick's Law of diffusion as

$$\boldsymbol{q^{r(k)}} = -D_F^{(k)}\nabla G^{(k)} \tag{4.18}$$

where the dimensionless diffusion coefficient D_F is given by

$$D_F^{(k)}(\omega, R) = \frac{\lambda(R)}{\omega k_k}, \omega = \frac{4\pi I_b^{(k)}}{G^{(k)}} \tag{4.19}$$

Thus we get the diffusion equation for the incident radiation as

$$-\nabla \frac{\lambda(R)}{\omega k_k} \nabla G^{(k)} = k_k \left(4\pi I_b^{(k)} - G^{(k)} \right) \tag{4.20}$$

We derive an appropriate boundary conditions for the diffusion equation (4.20) based on the asymptotic matching between the slow scale interior solution governed by the flux limited diffusion equation and the fast scale boundary layer solution. The following description parallels to Pomraning [7] where he has used the case discrete normal mode to include more complicated transport phenomenon. Consider the equation of transfer (2.3) and the boundary condition (2.4)

$$\boldsymbol{\Omega} \cdot \nabla I^{(k)}(\boldsymbol{r}, \boldsymbol{\Omega}) = k_k \left(I_b^{(k)}(T(\boldsymbol{r})) - I^{(k)}(\boldsymbol{r}, \boldsymbol{\Omega}) \right) \tag{4.21}$$

$$I^{(k)}(\boldsymbol{r_s}, \Omega) = \epsilon\tau(\boldsymbol{\Omega}, \boldsymbol{\Omega}'') I_b^{(k)}(T(\boldsymbol{r_s})) + \rho(\boldsymbol{\Omega}, \boldsymbol{\Omega}') I^{(k)}(\boldsymbol{r_s}, \boldsymbol{\Omega}) \tag{4.22}$$

We decompose the specific intensity of radiation $I^{(k)}$ into the sum of two intensities $I_{in}^{(k)}$ and $I_{out}^{(k)}$ as

$$I^{(k)}(\boldsymbol{r}, \boldsymbol{\Omega}) = \underbrace{I_{in}^{(k)}(\boldsymbol{r}, \boldsymbol{\Omega})}_{\text{LP diffusion sol.}} + \underbrace{I_{out}^{(k)}(\boldsymbol{r}, \boldsymbol{\Omega})}_{\text{Boundary Layer sol.}} \tag{4.23}$$

where $I_{in}^{(k)}$ is the interior solution presumed to be an accurate approximation of $I^{(k)}$ away from the boundary layer and $I_{out}^{(k)}$ is the contribution to the boundary (near surface) layer which is presumed to decay rapidly in space in a direction normal to the surface. Both the intensities $I_{in}^{(k)}$ and $I_{out}^{(k)}$ satisfy the transfer equation (4.21) i.e.,

$$\boldsymbol{\Omega} \cdot \nabla I_{in}^{(k)} + k_k I_{in}^{(k)} = k_k I_b^{(k)} \tag{4.24}$$

(4.25)

where $\mu = \boldsymbol{n} \cdot \boldsymbol{\Omega}$ (cosine of angle between the incident ray and the z-axis), $\boldsymbol{n}$ is unit outward normal at the surface point $\boldsymbol{r_s}$. Due to the assumed dominance of the normal spatial derivative, we have neglected the tangential derivative in equation (4.25). Since $I_{out}^{(k)}$ is assumed to decay rapidly with z, in particular faster than the exponential mean free path characteristic distance i.e., faster than $e^{-k_k z}$, this equation holds for $0 \le z < \infty$. Thus, the boundary condition for (4.25) can be given by combining (4.22) and (4.23) as

$$I_{in}^{(k)}(\boldsymbol{r_s}, \boldsymbol{\Omega}) + I_{out}^{(k)}((\boldsymbol{r_s})_{z=0}, \boldsymbol{\Omega}) = I^{(k)}(\boldsymbol{r_s}, \boldsymbol{\Omega}), \mu > 0 \tag{4.26}$$

To remove the ϕ dependence from equation (4.25) and (4.26), we define $\bar{h}(\mu)$ as

$$\bar{h}(\mu) = \int_0^{2\pi} h(\boldsymbol{\Omega}) d\Omega$$

for any function $h(\boldsymbol{\Omega})$ and integrate (4.25) and (4.26) over ϕ. This gives

$$\mu \frac{\partial \bar{I}_{out}^{(k)}}{\partial z} + k_k \bar{I}_{out}^{(k)} = 0, 0 \le z < \infty \tag{4.27}$$

$$\bar{I}_{out}^{(k)}((\boldsymbol{r_s})_{z=0}, \mu) = \bar{I}^{(k)}((\boldsymbol{r_s}), \mu) - \bar{I}_{in}^{(k)}(\boldsymbol{r_s}, \mu), \mu > 0 \tag{4.28}$$

(4.27) being a linear ordinary differential equation of first order, we can write its solution as

$$\bar{I}^{(k)}_{out} = \bar{I}^{(k)}_{out}((\boldsymbol{r_s})_{z=0}, \mu) e^{\frac{-k_k z}{\mu}} \tag{4.29}$$

Clearly, (4.29) vanished at $z = \infty$, but since $\mu > 0$, it must satisfy

$$\bar{I}^{(k)}_{out}((\boldsymbol{r_s})_{z=0}, \mu) = 0. \tag{4.30}$$

If it is to decay faster than $e^{-k_k z}$ i.e., for a small layer near the boundary, the incoming intensity right hand side of (4.28) can be neglected. Now, multiply (4.28) by some weight function $W(\mu)$ and integrating w.r. to μ and using (4.30) we get

$$\int_0^1 \left(\bar{I}^{(k)}((\boldsymbol{r_s}), \mu) - \bar{I}^{(k)}_{in}((\boldsymbol{r_s}), \mu \right) W(\mu) d\mu = 0 \tag{4.31}$$

Since $I^{(k)}_{in}$ is supposed to be accurately described by the LP flux limited diffusion theory, we have from (4.30)

$$\bar{I}^{(k)}(\boldsymbol{r_s}, \mu) = G^{(k)}(\boldsymbol{r_s}) \bar{\psi}^{(k)}(\mu)$$

and from (4.13)

$$\begin{aligned}\bar{\psi}^{(k)}(\mu) &= \int_0^{2\pi} \psi^{(k)}(\boldsymbol{\Omega}) d\phi = \int_0^{2\pi} \frac{1}{4\pi} \left(\frac{1}{R \coth{(R)} - \boldsymbol{\Omega} \cdot \boldsymbol{R}} \right) d\phi \\ &= \frac{1}{2R^2} \left(\frac{R \coth{(R)} + \mu R}{\coth^2{(R)} - \mu^2} \right) = \frac{1}{2R^2} \left(\frac{R \coth(R) - \mu \frac{\partial G^{(k)}}{\partial z} / k_k \omega}{\coth^2{(R)} - \mu^2} \right)\end{aligned}$$

we get

$$\bar{I}^{(k)}_{in}(\boldsymbol{r_s}, \mu) = \frac{1}{2R^2} \left(\frac{R \coth R G^{(k)}(\boldsymbol{r_s}) + \frac{\mu}{k_k \omega} (\boldsymbol{n} \cdot \nabla G^{(k)}(\boldsymbol{r_s}))}{\coth^2{(R)} - \mu^2} \right) \tag{4.32}$$

Using (4.32) in (4.31) and choosing $W(\mu) = \mu$ (as usual Marshack's choice [5]) we obtain

$$\frac{1}{2} \int_0^1 \left(\frac{\coth R G^{(k)}(\boldsymbol{r_s})}{R(\coth^2 R - \mu^2)} + \frac{\boldsymbol{n} \cdot \nabla G^{(k)}(\boldsymbol{r_s})}{R^2 (\coth^2 R - \mu^2) k_k \omega} \right) \mu d\mu = \int_0^1 \bar{I}^{(k)}(\boldsymbol{r_s}, \mu) \mu d\mu$$

$$\text{i.e., } \frac{1}{2} \left(\alpha G^{(k)}(\boldsymbol{r}) \boldsymbol{s}) + \frac{\beta}{k_k \omega} \boldsymbol{n} \cdot \nabla G^{(k)}(\boldsymbol{r_s}) \right) = \int_{\boldsymbol{n} \cdot \boldsymbol{\Omega} < 0} |\boldsymbol{n} \cdot \boldsymbol{\Omega}| I^{(k)}(\boldsymbol{r_s}, \boldsymbol{\Omega}) d\Omega \tag{4.33}$$

where $\boldsymbol{n}$ is a unit outward normal vector at the surface point $\boldsymbol{r_s}$,

$$\alpha = \int_0^1 \frac{\coth R \mu d\mu}{R(\coth^2 R - \mu^2)} = \frac{\coth R}{R} \ln |\coth^2 R / (\coth^2 R - 1)| \tag{4.34}$$

and

$$\beta = \int_0^1 \frac{\mu d\mu}{R(\coth^2 R - \mu^2)} = \frac{1}{R} \ln |\coth^2 R / (\coth^2 R - 1)| \tag{4.35}$$

5. Numerical Computations

Now we present some numerical simulations employing a semi-implicit finite volume scheme for a glass cooling problem in one dimension. The glass has an initial temperature of $600°C$ uniformly distributed and start cooling down through radiation on the boundary surface which is directly exposed to the surrounding temperature of $T_{out} = 26.85°C$. The convection heat transfer coefficient (h) and the boundary emissivity (ϵ) both are chosen to be 0.89. But effect of convection in the total energy transport is found to be almost non-influencing. The refractive index for the glass is assumed to be constant $Rn_g = 1.46$. Thermal conductivity $K(T)$, specific heat capacity $C(T)$ and density of glass ρ_g are also taken to be constant throughout the computations. Absorption coefficient of glass k as a function of wave number for various spectral band widths is shown in the table 1. The

Table 1. Absorption coefficient for glass

η	1000-2000	2000-2500	2500-3500	3500-6000	6000-10000	10000-200000
k	80.0	35.0	10.0	1.0	0.01	0.02

boundary reflectivity ρ, τ are direction dependent and cannot be considered as constants. We assume that radiation on the boundary is unpolarized, obeys the laws of refraction and ρ is expressed in terms of incident angle θ as [5]

$$\rho = \frac{1}{2}\left\{\left(\frac{\sqrt{N-1+\mu^2}-\mu}{\sqrt{N-1+\mu^2}+\mu}\right)^2 + \left(\frac{N\mu-\sqrt{N-1+\mu^2}}{N\mu-\sqrt{N+1+\mu^2}}\right)^2\right\} \tag{5.1}$$

where

$$\mu = \cos\theta, N = 1/Rn_g^2.$$

The integrals involving blackbody radiation function (2.2) and the boundary reflectivity function (5.1) are numerically approximated using the 6-point Gaussian quadrature formula. Step sizes for spatial and temporal direction are taken to be 0.01 and 0.02 respectively and the simulations up to 100 seconds are reported. Since there is no exact solution, we use the results obtained by a higher order discrete ordinates ($\mathbf{S_8}$) method [15] as the basis for our comparison.

Temperature profiles at various times obtained from pure conduction and the Rosseland approximation are shown in figure 1. Here, by the pure conduction, we mean only the radiation term in the energy equation is neglected but only the surface radiation given by the boundary conditions is taken into account. Solution of the energy equation without consideration of all the radiation effect, being constant, is of no interest. Temperature distribution predicted by Eddington approximation and Levermore-Pomraning flux-limited diffusion approximation are presented in figure 2. In figure 3 the temperature profiles from Flux-limited diffusion and discrete ordinate methods are plotted together for comparison.

From figures 1 - 3, we see that the Rosseland approximation underestimates temperature profile where as the pure conduction overestimates the same. It is natural to expect higher temperature profile from the pure conduction for the case of glass which has a very

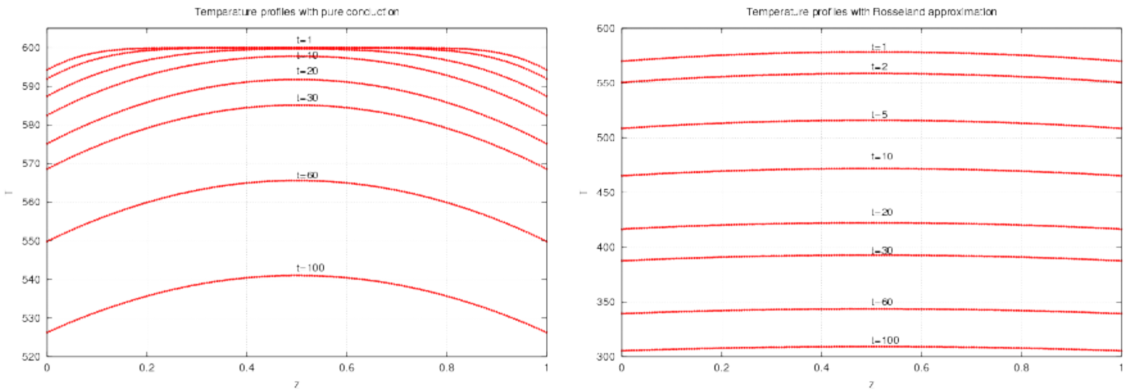

FIGURE 1. Temperature at t=1, 2, 5, 10, 20, 30, 60, 100s using pure conduction (left), Rosseland aproximation (right)

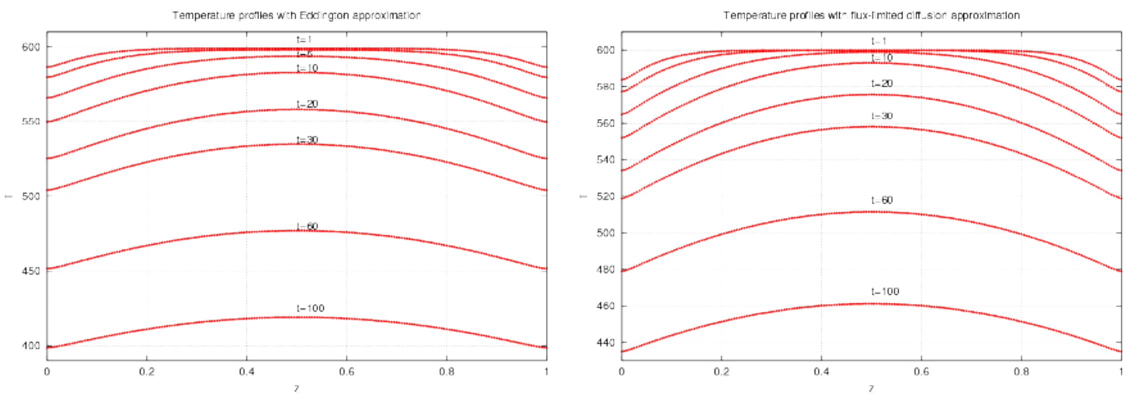

FIGURE 2. Temperature at t=1, 2, 5, 10, 20, 30, 60, 100s using Eddington approximation (left), flux-limited diffusion approximation (right)

good infrared transmission property for which at temperatures higher than $400°C$ radiation is a major influencing component in the energy transport. If we compare the Rosseland approximation with the Flux limited diffusion approximation, we observe an increasing discrepancies with the increase in time. This is because results Rosseland approximation is valid only for optically thick medium ($\tau_{l,k} >> 1$). For the characteristic length $l = 1$ cm and the absorption coefficient $\kappa = 0.01$, the optical dimension $\tau_{(l,k)} = 0.01 << 1$. Figure 3 shows that the Levermore-Pomraning flux limited diffusion approximation with appropriately chosen boundary condition is comparable with the higher order discrete ordinate ($\mathbf{S_8}$) method.

6. Conclusion

In this paper, a straightforward coupling of conduction and radiation using the diffusion approximation for the radiative transfer equation is presented. Based on the one dimensional simulations of a glass cooling problem, the flux limited diffusion descriptions with appropriately chosen boundary conditions is found to be comparable with the sophisticated discrete ordinate $\mathbf{S_8}$ method. The diffusion approximation method has several advantages.

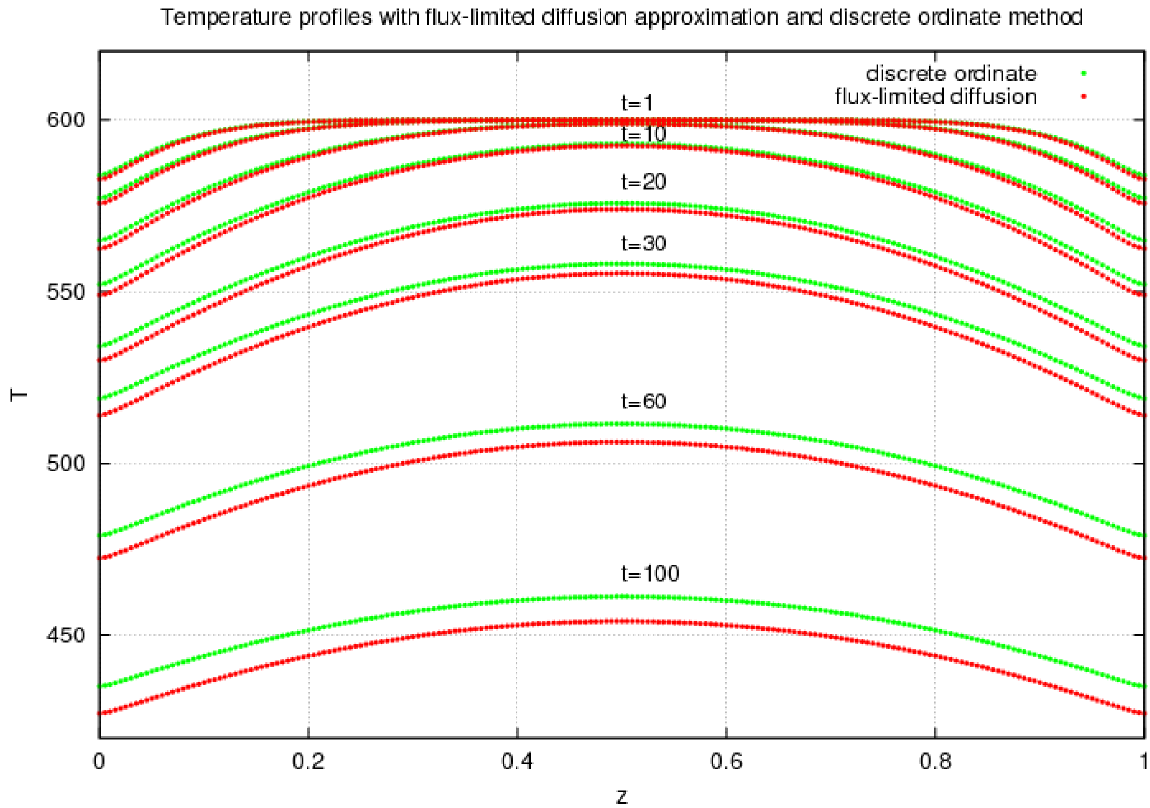

FIGURE 3. Temperature profiles at different times as predicted by discrete ordinate ($\mathbf{S_8}$) method (green) and flux-limited diffusion approximation (red)

It is easy to implement for any spectral variation of the absorption coefficients and for multidimensional cases with arbitrary geometry.

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TRANSSHIPMENT CONTRAFLOW ON MULTI-TERMINAL NETWORKS

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Abstract: Contraflow technique is the widely accepted model on network optimization. It allows arc reversal that increases the arc capacities. The earliest arrival transshipment contraflow is an important model that transship the given flow value by sending the maximum amount at each time point from the beginning within given time period by reversing the direction arcs from the sources to the sinks at time zero. This problem has not been solved polynomially on complex networks, i.e., multi-terminal networks yet. However, its 2-value-approximation solution has been found by Pyakurel and Dhamala [13] in pseudo-polynomial time complexity. Moreover, they have claimed that for the special case of zero transit time on each arc, the 2-value-approximation solution can be computed in polynomial time complexity. In this paper, we solve their claim presenting an efficient algorithm.

Key Words: Network optimization, value-approximation, contraflow, complexity.

AMS (MOS) Subject Classification. Primary: 90B10, 90C27, 68Q25; Secondary: 90B06, 90B20.

1. Introduction

Contraflow increases the outbound capacities of arcs with the capabillity of arc reversals in the required direction. The obtained network with increased arcs capacities is the auxiliary network. On auxiliary network, contraflow problem not only maximizes the flow value but also minimizes the time to transship the given flow value. From the practice in evacuation planning, the evacuation time is reduced at least 40 percent with at most 30 percent of the total arc reversals, [8]. For the various mathematical models, heuristics, optimization and simulation techniques with contraflow configuration, we refer to Dhamala [1].

From the analytical point of view, we can find that the flow values obtained by contraflow models increase significantly that may be doubled for given time horizon. Moreover, the contraflow model is two times faster than the models without contraflow to transship the given flow value. Some contraflow problems with efficient solution algorithms in particular networks have been solved in [2, 9, 10, 11, 12, 13].

Moreover, using the natural transformation of [4], we have computed some contraflow solutions in different particular network by reversing the direction of arcs at time zero in continuous time model with the same complexity as in discrete time model [15, 3].

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The earliest arrival contraflow problem maximizes flow at every point of time from the beginning with arc reversal capability. If the supplies/demands are given and we have to transship given supplies to satisfied the demands within fixed time horizon, then the problem turns into the earliest arrival transshipment contraflow problem. So far, to the best of the author's knowledge, the earliest arrival transshipment contraflow problem on multi-terminal networks has been studied by Pyakurel and Dhamala [13]. They have presented an pseudo-polynomial time approximation algorithm to solve the problem for arbitrary transit time on each arc of the network. Moreover, they have claimed that an approximate earliest arrival transshipment contraflow problem can be solved on multi-terminal networks with zero transit time on each arc in polynomial time complexity. In zero transit time, arc capacities of networks restrict the quantity of flow that can be sent at any one time with arc reversal capability. In this paper, we solve their claim in detail. We present a polynomial time approximation algorithm to solve the problem accordingly.

The organization of the paper is as follows. In Section 2, we model the earliest arrival transshipment contraflow problem with a short description of required concepts and denotations. In section 3, we present our main results on the earliest arrival transshipment contraflow problem on multi-terminal networks. Section 4 concludes the paper.

2. Preliminaries

Let $G = (V, A)$ be a directed graph with a finite set of nodes V and a finite set of arcs A. We assume that $|V| = n$ and $|A| = m$. As the case is of contraflow, two way network configuration is allowed. Let $S \subset V$ and $D \subset V$ be a set of source nodes which are the starting points of flow and a set of sink nodes with enough capacity, i.e., the final destination of flow. Nodes s and d represent a single source and a single sink.

The network consists of nonnegative functions of arc capacities $b_A : A \to \mathcal{Z}^+$, node capacities $b_V : V \to \mathcal{Z}^+$ and arc transit times $\tau : A \to \mathcal{Z}^+$. The arc capacities $b_A(e),\ e \in A$ represent the maximum units of evacuees that may enter the initial node of arc e per time period. The node capacities $b_V(v),\ v \in V$ bound the amount of evacuees allowed to hold at node v. The time needed to travel one unit of evacuees on the arc $e = (v, w)$ from node v to node w is the transit time $\tau(e)$. The vectors $\mu(s)$ and $\nu(d)$ represent the given supply and demand at each source and sink, respectively. We assume that $A_d^{out} = A_s^{in} = \emptyset$, where $A_v^{out} = \{(v, w) \in A\}$ and $A_v^{in} = \{(w, v) \in A\}$ for the node $v \in V$.

The transportation network $\mathcal{N} = (V, A, b_A, \tau, S, D, \mu(s), \nu(d), T)$ is represented by the collection of all data in the evacuation scenario with predetermine time T. We assume a finite time horizon T that means everything must happen before time T. Time can increase in discrete increments or continuously. We consider the discrete time with a suitable time unit like at times $t = 0, 1, \ldots, T$ and all time related parameters are integers. The choice of time unit effects the problem directly i.e., if the time unit is shorter then the problem is more complex. Let $\mathbf{T}$ be the domain of time i.e., $\mathbf{T} = \{0, 1, \ldots, T\}$.

Let the reversal of an arc $e = (v, w)$ be $e^{-1} = (w, v)$. For a contraflow configuration of a network $\mathcal{N}$ with symmetric travel times, the auxiliary network $\overline{\mathcal{N}} = (V, E, b_E, b_V, \tau, S, D, T)$

consists of the modified arc capacities and travel times as

$$b_E(\overline{e}) = b_A(e) + b_A(e^{-1}) \text{ , and } \tau(\overline{e}) = \begin{cases} \tau(e) & \text{if } e \in A \\ \tau(e^{-1}) & \text{otherwise} \end{cases}$$

where, an edge $\overline{e} \in E$ in $\overline{\mathcal{N}}$ if $e \vee e^{-1} \in A$ in $\mathcal{N}$. The remaining graph structure and data are unaltered.

Let a non-negative function $x_s : A \rightarrow \mathcal{R}^+$ represents the static flow and let the value of a static s-d flow x_s be $val(x_s)$. Similarly, let the non-negative function $x_d : A \times \mathbf{T} \rightarrow \mathcal{R}^+$ represents the dynamic flow and its value is $val(x_d)$.

Let $\mathcal{N}^R_{x_s} = (V, \overrightarrow{A} \cup \overleftarrow{A})$ be the residual network of $\mathcal{N}$ where $\overrightarrow{A} = \{\overrightarrow{e} = e \mid x_s(e) < b_A(e)\}$ with capacity $b_A(e) - x_s(e)$ and transit time $\tau(e)$, and $\overleftarrow{A} = \{\overleftarrow{e} = (\text{head}(e), \text{tail}(e)) \mid x_s(e) > 0\}$ with capacity $x_s(e)$ and a transit time $-\tau(e)$.

A dynamic s-d flow x_d for given time T with arc reversal capability satisfies the flow conservation and capacity constraints (2.1-2.3). The inequality flow conservation constraints allow to wait flow at intermediate nodes, however, the equality flow conservation constraints force that flow entering an intermediate node must leave it again immediately.

$$\sum_{\sigma=\tau(e)}^{T} \sum_{e \in A_v^{in}} x_d(e, \sigma - \tau(e)) - \sum_{\sigma=0}^{T} \sum_{e \in A_v^{out}} x_d(e, \sigma) \quad = \quad 0, \ \forall \ v \notin \{s, d\} \tag{2.1}$$

$$\sum_{\sigma=\tau(e)}^{t} \sum_{e \in A_v^{in}} x_d(e, \sigma - \tau(e)) - \sum_{\sigma=0}^{t} \sum_{e \in A_v^{out}} x_d(e, \sigma) \quad \geq \quad 0, \forall v \notin \{s, d\}, t \in \mathbf{T} \tag{2.2}$$

$$0 \ \leq \ x_d(e, t) \ \leq \ b_A(e, t), \qquad \forall \ e \in A, \ t \in \mathbf{T} \tag{2.3}$$

The earliest arrival flow (EAF) problem with arc reversal capability maximizes the $val(x_d, t)$ in (2.4) for all $t \in \mathbf{T}$ satisfying the constraints (2.1-2.3). We denote the maximum flow value by $val_{max}(x_d, t)$.

$$val(x_d, t) = \sum_{\sigma=0}^{t} \sum_{e \in A_s^{out}} x_d(e, \sigma) = \sum_{\sigma=\tau(e)}^{t} \sum_{e \in A_d^{in}} x_d(e, \sigma - \tau(e)) \tag{2.4}$$

If the supplies $\mu(s)$ and demands $\nu(d)$ are given and we have to transship given supplies within time T, then the EAF is turns into the earliest arrival transshipment (EAT) problem. For contraflow network, the problem is the earliest arrival transshipment contraflow (EATCF).

For a network $\mathcal{N}$, the time expanded network $\mathcal{N}(T) = (V_T, A_M \cup A_H)$ is defined is defined by copying the network for each time step as presented for static network in [5] as follows:

$$\begin{aligned} V_T &= \{v(t) | v \in V, t \in \{0, 1, ..., T\}\} \,. \\ A_M &= \{e(t) = (v(t), w(t + \tau_e) | e = (v, w) \in A, \theta \in \{0, 1, ..., T - \tau_e\}\} \text{ , for movement arcs.} \\ A_H &= \{(v(t), v(t + 1)) | v \in V, t \in \{0, 1, ..., T - 1\}\} \text{ , for holdover arcs.} \end{aligned}$$

Let $\mathcal{N}^\star$ be the two-terminal extended network of multi-terminal network $\mathcal{N}$ obtained by adding a super-terminal node ($\star$) and introducing arcs $(\star, s_i)$ to each $s_i \in S$ with infinite

capacity and zero transit time, and arcs $(d_i, \star)$ to each $d_i \in D$ with infinite capacity and transit time -$(T+1)$ for given time period T.

Authors in [6, 7] introduced a $\beta-$value-approximate EAT that is a dynamic flow x_d that achieves at every point in time $t \in \{1, ..., T\}$ and at least a $\beta-$ fraction of the maximum flow value can be sent at time t. They proved that proved that $\beta = 2$ is the best possible approximation factor. They presented a 2-value-approximate algorithm that gives 2-value approximate EAT in pseudo-polynomial time complexity. It works on time expanded network $\mathcal{N}(t)$. For each source $s \in S$, there is a source s_0 with t arcs (s_0, s_i) for the t copies of s. Similarly, for each sink $d \in D$, there is a sink d_0 with t arcs (d_i, d_0). Let s^* be a super-source with $|S|$ arcs (s^*, s_0) having capacity $b_A(s_0)$, and d^* a super-sink with $|D|$ edges (d_0, d^*) having capacity $-b_A(d_0)$. Then, maximum flow with time horizon t is equivalent to a maximum static flow from s^* to d^* in $\mathcal{N}(t)$.

For the special networks with zero transit time, authors in [6, 7] has presented Algorithm 2.1 that computes a maximum flow. All the arcs are of the form $e = (v(t), w(t))$ for some time t. The maximum flow is independently computed from previously computed flow in earlier time layers. The same static flow can be sent repeatedly in each time using original network until the supplies at sources shift into sinks. This algorithm has polynomial time complexity.

Algorithm 2.1. *Zero time 2-value-approximate EAT algorithm*

Input: *Given a dynamic network* $\mathcal{N} = (V, A, b_A, \tau, S, D, \nu)$ *with zero transit times* $\tau(e) = 0$.

1. *Define the supplies* $\nu' = \nu$ *and set* $t = 1$.
2. *Respecting the supplies* ν', *a maximum static transshipment* x_{stat_t} *is obtained.*
3. *Let the maximum time needed to transship* x_{stat_t} *flow be* $a(t)$ *until a source or a sink becomes empty:*
$$a(t) = min\left\{\left\lfloor\frac{\nu'(s)}{val(x_{stat_t})}\right\rfloor \mid s \in S, \nu'(s) > 0\right\} \cup \left\{\left\lfloor\frac{-\nu'(d)}{val(x_{stat_t})}\right\rfloor \mid d \in D, \nu'(d) < 0\right\}$$
4. *Depending of the sending flow* x_{stat_t}, *update the supplies:*
$\nu'(s) = \nu'(s) - a(t).ex(x_{stat_t})$ *for* $s \in S$ *with* $\nu'(s) > 0$,
$\nu'(d) = \nu'(d) + a(t).ex(x_{stat_t})$ *for* $d \in D$ *with* $\nu'(d) < 0$.
5. *If* $\nu' \neq 0$, *set* $t = t + 1$ *and continue with Step 2.*

Output: *The dynamic flow that sends* x_{stat_t} *units starting at* $\sum_{j=1}^{t-1} a(t)$ *for* $a(t)$ *time units.*

Theorem 2.2. [6, 7] *Algorithm 2.1 computes a 2-value-approximate EAT in a dynamic network* $\mathcal{N}$ *with zero transit time in polynomial time complexity.*

3. Approximate EATCF on multi-terminal networks

Pyakurel and Dhamala [13] have solved the earliest arrival transshipment contraflow (EATCF) problem on multi-terminal networks. Their algorithm is based on the MDCF algorithm of [16] for MDCF problem and value-approximate algorithm of [6, 7] for an approximate EAT problem for the arbitrary transit time on each arc of the network. As

the solution depends directly upon the size of time expanded network, there exists no polynomial approximation algorithm to solve the 2-value-approximate EATCF problem on multi-terminal networks with arbitrary transit times. Thus, they have claimed that an approximate solution for the EATCF problem can be computed with the modification of their algorithm assuming the transit times zero in each arc of the multi-terminal network $\mathcal{N} = (V, A, b_A, \tau, S, D, \nu(v))$. We extend their claim, in this section, with detail algorithm and proofs.

For the special case of zero transit times, a 2-value-approximate EAT solution is computed in the auxiliary network $\overline{\mathcal{N}} = (V, E, b_E, S, D, \nu(v))$ using the algorithm, Algorithm 3.1. Our algorithm is based on the MDCF algorithm of [16] for the MDCF problem and Algorithm 2.1 of [6, 7] for zero time 2-value approximate EAT problem. In the procedure, the maximum static flow is computed in the auxiliary network $\overline{\mathcal{N}}$ using Algorithm 2.1 and repeated them until a terminal runs out of demand/supply in polynomial time complexity. For more details, we refer to [14].

Algorithm 3.1. *Zero time 2-value-approximate EATCF algorithm*

(1) *Given a network* $\mathcal{N} = (V, A, b_A, \tau, S, D, \nu(v))$.
(2) *Construct the auxiliary network* $\overline{\mathcal{N}} = (V, E, b_E, \tau, S, D, \nu(v))$ *of* $\mathcal{N}$
(3) *Construct the extended network* $\overline{\mathcal{N}}^\star$ *of* $\overline{\mathcal{N}}$
(4) *Solve the EAT problem on network* $\overline{\mathcal{N}}^\star$ *using Algorithm 2.1 of* [6, 7].
(5) *Arc* $(w, v) \in A$ *is reversed, if and only if the flow along arc* (v, w) *is greater than* $b_A(v, w)$ *or if there is a nonnegative flow along arc* $(v, w) \notin A$.
(6) *Obtain 2-value approximate EATCF solution for the network* $\mathcal{N}$ *with zero transit time.*

To solve the 2-value-approximate EATCF problem with zero transit time with time horizon T, we construct the auxiliary network $\overline{\mathcal{N}}$ first. Then the extended auxiliary network $\overline{\mathcal{N}}^\star$ of auxiliary network $\overline{\mathcal{N}}$ is constructed. In Step 4 of Algorithm 3.1, we use Algorithm 2.1 of [6, 7] in extended auxiliary network $\overline{\mathcal{N}}^\star$ that contains T copies of each arc and whose supplies/demands are shifted to newly introduced super terminals. This network has a one to one correspondence between an arc copy $e(t)$ and the copy of the arc $(v(t), w(t))$ on time layer t in the time expanded network. In order to prove that the computed flow is a 2-value-approximate EAT on $\overline{\mathcal{N}}^\star$, the flow is considered in the residual network of $\overline{\mathcal{N}}^\star$ with respect to static flow x_{stat} in which the reverse arcs of the super terminal arcs are deleted. The algorithm performs one MSF calculation per step. The choice of maximal time $a(t)$ guarantees that at least one source or sink runs empty in every iteration and obtains $\nu' = 0$ after δ iterations. Thus a 2-value-approximate EAT on $\overline{\mathcal{N}}^\star$ can be computed with at most $\delta \log \nu_{max}$ MSF computations where δ is the number of terminals and $\nu_{max} = \max\{|\nu| \mid \nu \in S \cup D\}$ is the largest supply/demand.

First, we prove that the best possible factor computed by the 2-value approximate EATCF Algorithm 3.1 is 2. For this we give the proof of Lemma 3.2 on the auxiliary network $\overline{\mathcal{N}} = (V, E, b_E, \tau, S, D, \nu(v))$ for the sake of completeness similar to the results in [6, 7] on $\mathcal{N}$.

Lemma 3.2. *Let x_{stat_t} be a maximum flow for time horizon t and let the computed flow by 2-value approximate EATCF algorithm of* [13] *in $\overline{\mathcal{N}}$ be x'_{stat_t}. Then it holds that*

$$val(x_{stat_t}) \leq 2.val(x'_{stat_t}).$$

Proof: First we convert the given network $\mathcal{N}$ into auxiliary network $\overline{\mathcal{N}}$ according to the contraflow configuration. On auxiliary network $\overline{\mathcal{N}}$, we use the 2-value approximate algorithm of [6, 7]. In a step of the algorithm, we compute a difference flow that obtained from subtracting the flow $val(x'_{stat_t})$ from the maximum flow $val(x_{stat_t})$. Let us define the difference flow $x^*_{stat} = (x_{stat_t} - x'_{stat_t})$ that is obtained by sending $(x_{stat_t} - x'_{stat_t})$ on forward arc e, if the value is positive, and sending $-(x_{stat_t} - x'_{stat_t})$ on the backward arc e if the value is negative.

From the difference flow values, we obtain $val(x_{stat_t}) = val(x'_{stat_t}) + val(x^*_{stat})$. The flow x^*_{stat} is valid in $\overline{\mathcal{N}_t^R}$ but not necessarily in $\overline{\mathcal{N}_t^{R'}}$. Let P be any path in the path decomposition of x^*_{stat} which sends an additional unit of flow that is not sent by x'_{stat_t}. As x'_{stat_t} is a maximum flow and the path augmenting algorithm has not found another path, P must be an $s^* - d^*$-path using one of the deleted edges.

However, the total flow value sent through these paths is bounded by the sum of the capacities of the deleted backward edges. This sum is at most $val(x'_{stat_t})$. Thus, we have $val(x^*_{stat}) \leq val(x'_{stat_t})$ and $val(x_{stat_t}) \leq val(x'_{stat_t}) + val(x'_{stat_t})$ and thus, $val(x_{stat_t}) \leq 2.val(x'_{stat_t})$.□

Theorem 3.3. *Algorithm 3.1 computes a 2-value-approximate EATCF solution on $\mathcal{N}$ with zero transit time.*

Proof: Algorithm 3.1 is feasible because of the feasibility of Step 4. Recall that the any approximation solution to an EAT problem with arc reversal on network $\mathcal{N}$ is also a feasible solution to the approximation EAT problem on the auxiliary network $\overline{\mathcal{N}}$. Algorithm 2.1 of [6, 7] and Theorem 2.2 induced a 2-value-approximative EAT solution on $\overline{\mathcal{N}}$. As the amount of flow sent from sources S to sinks D induced from Step 4 is not changed in Step 5, an efficient solution to the 2-value-approximative EATCF problem on $\mathcal{N}$ is obtained. □

Corollary 3.4. *Algorithm 3.1 computes a 2-value-approximate EATCF solution with zero transit time in polynomial time complexity.*

Proof: As a MSF solution is computed in each time period in Step 4 of Algorithm 3.1, the complexity is dominated by the complexity of MSF computation. It simply concludes that the complexity of the algorithm is bounded by polynomial time. □

4. Conclusions

We solved the approximate earliest arrival transshipment contraflow problem on multi-terminal networks in polynomial time complexity that had been claimed in [13]. Although the problem was solved in the same complexity as without contraflow, the flow value computed by contraflow model increases significantly. From the analytical point of view, it

has been realized that flow value may be doubled for given time horizon. Moreover, the time needed to transship given amount of flow value will be at most half with contraflow configuration.

To the best of our knowledge, the problem we solved is for the first time on complex contraflow networks using discrete time setting. However, it is still unsolved problem whether the earliest arrival transshipment contraflow problem on multi-terminal networks is polynomially solvable.

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ERMAKOV EQUATION AND CAMASSA-HOLM WAVES

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Abstract: Since the works of [1] and [2], it is known that the solution of the Ermakov equation is an important ingredient in the spectral problem of the Camassa-Holm equation. Here, we review this interesting issue and consider in addition more features of the Ermakov equation which have an impact on the behavior of the shallow water waves as described by the Camassa-Holm equation.

Key Words: Ermakov equation, Camassa-Holm equation, nonlinear waves
AMS (MOS) Subject Classification. 34A05, 34A34, 35Q35.

1. Introduction

The Camassa-Holm equation ([3])

$$u_t - u_{xxt} + 2ku_x + 3uu_x = 2u_xu_{xx} + uu_{xxx} \tag{1.1}$$

can be factored ([4]) as

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + 2u_x\right) q(x,t) = 0 \ , \qquad q(x,t) = (u - u_{xx} + k) \ , \tag{1.2}$$

where $q(x,t)$ is known as momentum. There are two conservation laws

$$\frac{\partial}{\partial t} q(x,t) + \frac{\partial}{\partial x}\left[2ku + \frac{3}{2}u^2 - uu_{xx} - \frac{1}{2}(u_x)^2\right] = 0, \tag{1.3}$$

and

$$(\sqrt{q})_t + (u\sqrt{q})_x = 0 \tag{1.4}$$

respectively. Using the momentum, Eq. (1.1) is equivalent to

$$q_t + uq_x + 2u_xq = 0. \tag{1.5}$$

The Lax pairs for CH are (Constantin 2001)

$$\begin{cases} \psi_{xx} = \left(\frac{1}{4} + \lambda q\right)\psi \\ \psi_t = \left(\frac{1}{2\lambda} - u\right)\psi_x + \frac{1}{2}u_x\psi \end{cases} \tag{1.6}$$

and the compatibility condition $\psi_{xxt} = \psi_{txx}$ leads back to Eq. (1.1).

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2. Liouville transformation on CH

The common procedure is to use $q = \left(\frac{dy}{dx}\right)^2$ followed by Liouville's transformation $\phi = q^{\frac{1}{4}}\psi$ in the first equation of the Lax pair system (1.6) to obtain

$$\phi_{yy} - \left[\frac{4q(1+q_{yy}) - 3(q_y)^2}{16q^2}\right]\phi = \lambda\phi. \tag{2.1}$$

In addition, since $q(x,t) = u - u_{xx} + k$ with $\frac{dy}{dx} = \sqrt{q}$, the momentum equation becomes

$$qu_{yy} + \frac{1}{2}q_y u_y - u = k - q \tag{2.2}$$

Therefore, if we know q then one can try to solve CH by finding u in Eq. (2.2). As a scattering Schrödinger problem, we can rewrite Eq. (2.1) as

$$\phi_{yy} - \left[Q + \frac{1}{4k}\right]\phi = \lambda\phi \tag{2.3}$$

with potential Q defined in terms of the momentum by

$$q_{yy} - \frac{3}{4q}(q_y)^2 - 4\left[Q + \frac{1}{4k}\right]q + 1 = 0 \tag{2.4}$$

Let $q = E^4$, then we obtain the Ermakov equation

$$E_{yy} - \left[Q + \frac{1}{4k}\right]E + \frac{1}{4}E^{-3} = 0 \tag{2.5}$$

The solution of Eq. (2.5) is given by Pinney (1950)

$$E = \sqrt{F_1^2 - \frac{1}{4}\left(\frac{F_2}{W}\right)^2} \tag{2.6}$$

where F_1 and F_2 are the two independent solutions of the linear ODE

$$F_{yy} - \left[Q + \frac{1}{4k}\right]F = 0 \tag{2.7}$$

The Ermakov-Lewis invariant is

$$\mathcal{I} = \frac{1}{2}\left[(EF_y - FE_y)^2 - \frac{1}{4}\left(\frac{E}{F}\right)^{-2}\right]. \tag{2.8}$$

By using E from Eq. (2.5) and F as a superposition of homogenous solutions $F = aF_1 + bF_2$, the invariant is

$$\mathcal{I} = \frac{1}{2}\left(-\frac{a^2}{4} + b^2W^2\right) = const. \tag{2.9}$$

It is easy to show that Eq. (2.7) and (2.5) are related to the linear third order ODE

$$\phi_{yyy} - 4\left[Q + \frac{1}{4k}\right]\phi_y - 2Q_y\phi = 0\ , \tag{2.10}$$

which is of maximal symmetry algebra $sp(5)$ ([5]) and has ϕ itself as an integrating factor; thus Eq. (2.10) becomes

$$\phi\phi_{yy} - \frac{1}{2}(\phi_y)^2 - 2\left[Q + \frac{1}{4k}\right]\phi^2 + \frac{1}{2} = 0\ . \tag{2.11}$$

3. Solutions

Let us use the solitonic potential

$$Q = -2K^2 \operatorname{sech}^2(\theta)\ , \qquad \theta = Ky + \Omega t - \alpha \tag{3.1}$$

where $K = \frac{\mu}{\sqrt{k}}$, $\Omega = \frac{\mu}{2\lambda}$, $\mu = \frac{1}{2}\sqrt{1+4\lambda k}$, and α, an arbitrary phase. Thus, Eq. (2.10) becomes

$$\phi_{\theta\theta\theta} + 4\left[2\ \operatorname{sech}^2\theta - \frac{1}{1-\frac{2k}{c}}\right]\phi_\theta - 8\ \operatorname{sech}^2\theta\ \tanh\theta\ \phi = 0 \tag{3.2}$$

and has solution

$$\phi(\theta) = \frac{c}{2\sqrt{k}}\left(\operatorname{sech}^2\theta + \frac{2k}{c}\ \tanh^2\theta\right)\ , \tag{3.3}$$

where $c = -\frac{1}{2\lambda}$. Since $q = \phi^2$, we get

$$q(\theta) = \frac{c^2}{4k}\left(\operatorname{sech}^2\theta + \frac{2k}{c}\ \tanh^2\theta\right)^2 \tag{3.4}$$

In the θ variable Eq. (2.2) becomes

$$\frac{1}{4k}\left(1 - \frac{2k}{c}\right)\left(q u_{\theta\theta} + \frac{1}{2} q_\theta u_\theta\right) - u = k - q \tag{3.5}$$

and after we substitute q, q_θ in Eq. (3.5) it becomes

$$\left[c - k + k\ \cosh^2(2\theta)\right]^2 u_{\theta\theta} - 2(c-2k)\left[c - k + k\ \cosh^2(2\theta)\right]\ \tanh\theta\ u_\theta - 16ck^2 u = f(\theta) \tag{3.6}$$

with nonhomogenous function given by $f(\theta) = -(c-2k)\ \operatorname{sech}^4\theta\,[4ck\,(c + 2k\ \cosh(2\theta)]$. For the particular case of $k \to 0$ Eq. (3.6) simplifies to

$$u_{\theta\theta} - 2\ \tanh\theta\ u_\theta = 0 \tag{3.7}$$

with solution

$$u(\theta) = \mathcal{C}_1 + \mathcal{C}_2\left(\frac{\theta}{2} + \frac{1}{4}\ \sinh 2\theta\right). \tag{3.8}$$

Solving Eq. (3.6) yields to general solution $u(\theta) = u_p(\theta) + \mathcal{C}_1 u_1 + i\mathcal{C}_2 u_2$. Denoting $k_c = \frac{2k}{c}$ the particular solution is

$$u_p(\theta) = \frac{c(1-k_c)}{1 - k_c + k_c\ \cosh^2\theta} = \frac{c(1-k_c)}{1 + k_c\ \sinh^2\theta} \tag{3.9}$$

while the general solutions are

$$\begin{cases} u_1(\theta) = \cosh\left\{\frac{2}{\sqrt{1-k_c}}\theta - 2\operatorname{arctanh}\left[\sqrt{1-k_c}\ \tanh\ \theta\right]\right\}, & 0 < k_c < 1 \\ u_2(\theta) = \sinh\left\{\frac{2}{\sqrt{1-k_c}}\theta - 2\operatorname{arctanh}\left[\sqrt{1-k_c}\ \tanh\ \theta\right]\right\}, & 0 < k_c < 1 \end{cases} \tag{3.10}$$

and

$$\begin{cases} u_1(\theta) = \cos\left\{\frac{2}{\sqrt{k_c-1}}\theta + 2\arctan\left[\sqrt{k_c-1}\ \tanh\ \theta\right]\right\}, & k_c > 1 \\ u_2(\theta) = -i\ \sin\left\{\frac{2}{\sqrt{k_c-1}}\theta + 2\arctan\left[\sqrt{k_c-1}\ \tanh\ \theta\right]\right\}, & k_c > 1. \end{cases} \tag{3.11}$$

See Fig. 1 (top) for the particular solution u_p if $k_c < 1$ which shows that the soliton profiles are tending to a compacton when k_c is small, and on (bottom) when $k_c > 1$ which shows that soliton profiles are more and more peakon-like. In Figs. 2 and 3 we show the

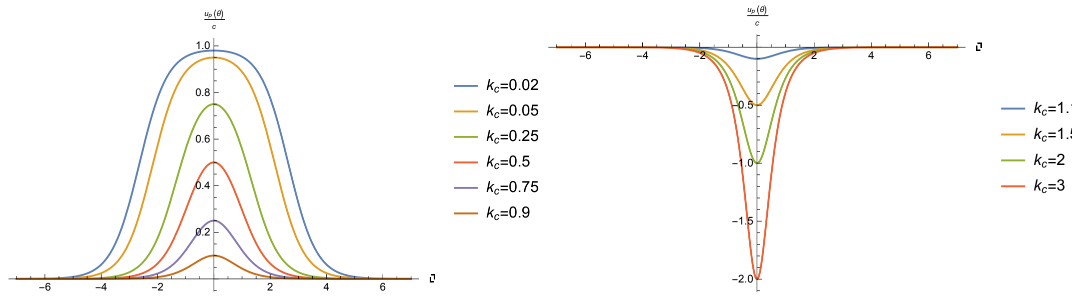

FIGURE 1. The particular solution $\frac{u_p}{c}$ for $k_c = \frac{2k}{c} < 1$ (top) and $k_c = \frac{2k}{c} > 1$ (bottom).

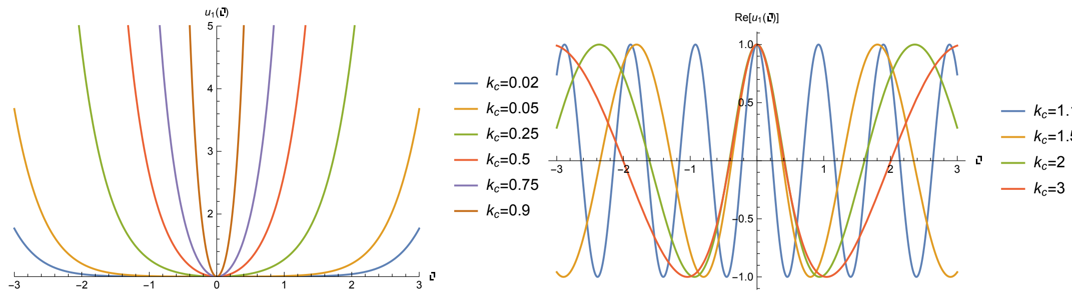

FIGURE 2. The particular solution u_1 for $k_c = \frac{2k}{c} < 1$ (top) and $k_c = \frac{2k}{c} > 1$ (bottom).

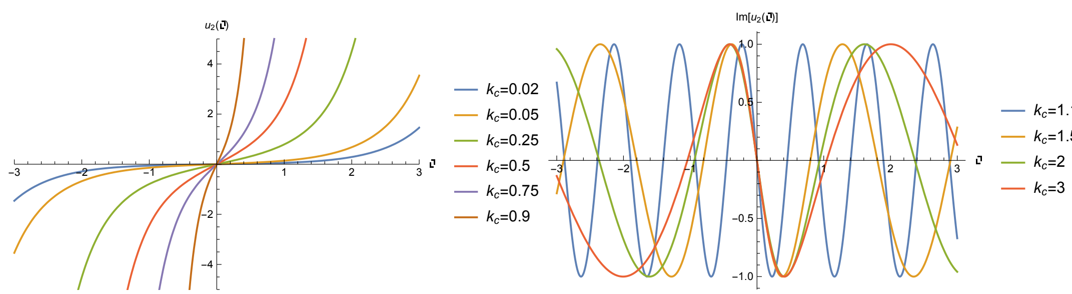

FIGURE 3. Same as in the previous figure for particular solution u_2.

particular solution u_1 and u_2 for both cases.

To find the solution in terms of x and t, we need to use the relation between θ and x. Since $\sqrt{q(\theta)} = \frac{dy}{dx}$, we have

$$\int \frac{d\theta}{\operatorname{sech}^2\theta + k_c \tanh^2\theta} = \frac{\sqrt{1-k_c}}{2k_c}(x - ct - x_0). \tag{3.12}$$

To find this integral we notice that if we differentiate the argument of the homogenous solutions u_1 and u_2, we have

$$\frac{d}{d\theta}\left\{\frac{2}{\sqrt{1-k_c}}\theta - 2\operatorname{arctanh}\left[\sqrt{1-k_c}\,\tanh\,\theta\right]\right\} = \frac{2k_c}{\sqrt{1-k_c}}\frac{1}{\operatorname{sech}^2\theta + k_c \tanh^2\theta} \tag{3.13}$$

Using Eqs. (3.12) and (3.13) we conclude that

$$x - ct - x_0 = \begin{cases} \frac{2}{\sqrt{1-k_c}}\theta - 2\operatorname{arctanh}\left[\sqrt{1-k_c}\,\tanh\,\theta\right], & 0 < k_c < 1 \\ \frac{2}{\sqrt{k_c-1}}\theta + 2\arctan\left[\sqrt{k_c-1}\,\tanh\,\theta\right], & k_c > 1 \end{cases} \tag{3.14}$$

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MODELING OF INDOOR AIR FLOW DISTRIBUTION IN A NATURALLY VENTILATED KITCHEN

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Abstract: This paper focuses on the modeling of indoor air pollution in a naturally ventilated kitchen based on the computational fluid dynamics (CFD) approach to assess its ventilation effectiveness. The 3D incompressible Navier-Stokes equations with conservation of total energy are solved numerically using ANSYS-Fluent software and the pollutant paths are investigated from the profiles of velocity, pressure, turbulent kinetic energy and temperature throughout different sections of the kitchen. Experimental verification is made through the measurement of indoor air contaminant in the same kitchen. The simulation results agrees well with the on-site measured data.

Key Words: Indoor air pollution, Navier-Stokes equation, numerical modeling, natural ventilation
AMS (MOS) Subject Classification. Primary: 65N08; Secondary: 76F60, 476D05.

1. Introduction

Biomass is one of the most important source of indoor air pollution(IAP). Existing studies demonstrate quite clearly that the increased risk of contracting cancer and respiratory symptoms for non-smoking Asian women appears to be associated with certain cooking practices instead of cigarette smoking[4]. The incomplete combustion of biomass releases complex mixture of organic compounds, which include suspended particulate matter, carbon monoxide, carbon dioxide, volatile organic compounds, fine particulate matters and ultrafine particles, poly-organic material, poly-aromatic hydrocarbons, formaldehyde etc.[12]. Carbon Monoxide (CO) is one of the most deadly pollutant and has adverse health effect on human health.

Ventilation of the kitchen needs to be healthy or well-functioning so that the people inside get comfortable environment to breath. It should not allow contaminated air to accumulate and pollutant concentrations to increase which is injurious to the human health

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especially a large number of women living inside the kitchen most of the time. There are mainly two types of ventilations which are mechanical and natural. Natural ventilation of buildings is an important approach towards a sustainable and energy-efficient built environment. Natural ventilation can be driven by wind-induced pressure differences or by thermally-induced pressure differences, or by a combination of both (e.g. Linden 1999, Hunt and Linden 1999, Li and Delsante 2001, Heiselberg et al. 2004, Larsen and Heiselberg 2008, Chen 2009, Van Hooff and Blocken[10]. Apart from these driving forces, natural ventilation requires the presence of sufficiently large ventilation openings between the outdoor and indoor environment[1].

Considerable progress has been made recently in developing mathematical models for predicting pollutant concentration in air[5]. Numerical methods involves replacing the partial differential equations with discretized algebraic equations. These equations are then numerically solved to obtain flow field values at the discrete points in space and time. Compared to the full scale experiments, the computational fluid dynamics method is inexpensive and alternative in indoor design, optimum and pollutant dispersion for health and safety reasons [4] [9]. Turk purposed a general equation for calculating concentration in a chamber that included both exterior and interior source and the removal effect of pollutants by air treatment system in 1963. Jones and Fagon used Turkes equation to calculate carbon monoxide (CO) concentration from cigarette smoke in 1974. In 1974 Nielson used numerical predictions of indoor airflow and worked mainly on two-dimensional, steady and isothermal flows. Even though his two-dimensional results are not very useful for engineering applications, the methods he used showed a very strong potential for solving practical air flow problems in a room. Many researchers validated their computational results with experiments, including Nielson, Sakamoto and Matsuo, and Gosman et al. Lu et al. used computational fluid dynamics(CFD) to simulate airflow/temperature in a room to track pollutant dynamics and found that the CFD results correlated reasonably well with measured experiments. Sinha et al. also used finite volume CFD to compare discrete vent configuration cases, but also ran simulations for different fixed Reynolds number Re and Grashof number Gr. Freire et al. studied the problem of optimizing thermal comfort and energy savings using model-based predictive control where their controlled input was to apply power to the heating, ventilation and air conditioning (HVAC) device. Ishizu examined experimentally the inclusion of mixing factor into these models and Repace and Lawrey also developed a modification of the Turk equation incorporating a mixing factor. Trynor G. W. et al.[13] studied the effects of ventilation on residential air pollution, due to emission from a gas-fired range in 1982 and showed that the range hood is the effective means of the removing pollutant from gas fired range; removal rates varied from 60 to 87[13]. In 1996 Ott et al. showed theoretically that a mathematical trend correlation term should be incorporated into the time averaged version of the model to make it exact. Neil E. Klepeis[6] studied on validity of the ‘uniform mixing’ assumption: determining human exposure to environmental tobacco smoke, environmental health perspectives.

Hensen, J.L.M et al.[3] demonstrated as a merit and drawbacks of various computer modeling approaches for HVAC design and performance prediction. He pointed out some

of the future works including integration of CFD in general building energy simulation for third approach, prediction of thermal comfort as affected by the flow and temperature field within a room. Ott Wayne R.[7] developed a mathematical model for predicting indoor air quality from smoking activity. Chen Q. and Srebric J. in 2000 [2] has developed a new model to assess building shape design, to evaluate effectiveness of natural ventilation in buildings to model volatile organic compound (VOC) emission from building materials and calculated indoor air environment parameter. Fernanda Carmen Fuoco et. al. [10] concluded that diffusion is an important transport mechanism in cross-ventilation of buildings, and that special care is needed to select the right amount of physical diffusion and to reduce the numerical diffusion, by using high-resolution grids and by using at least second-order accurate discretization schemes.

In this paper, we develop a CFD model for indoor pollution in a typical Nepali kitchen using ANSYS-Fluent software and explore on the distribution pattern of the velocity, pressure, contaminants inside a kitchen, and the analysis of flow driven by wind and the buoyancy. The study investigates the dispersion of indoor pollutants in a kitchen under the conditions with change of window positions in the wall near and far from the stove. Measurements of carbon monoxide levels at different locations in the kitchen with different conditions are compared with the simulated results.

Rest of the paper is organized as follows. The mathematical model is described in section §2 and numerical methods are presented in section §3. After giving a detailed description of the experimental study in section §4, the numerical simulation results are presented and discussed in §5, with conclusions in §6.

2. Mathematical Model

The study of three-dimensional incompressible flow of air as a multi-component fluid includes dry air and contaminants. The fluid properties vary according to ideal gas model and therefore, it accounts for the buoyancy associated to the natural convection of the heated fluids. The mathematical model is based on the Reynolds-Averaged Navier-Stokes equations and associated boundary conditions.

2.1. Geometry of the Model.

We consider a typical Nepali kitchen with one door, two windows and a vent with the dimensions as shown in figure 1. The proposed model kitchen has dimensions of $4.2 \times 3.0 \times 2.7m^3$. Here the positions of the windows of similar size $0.4m \times 1.6m$ are situated at a distance of $0.8m$ from each other. The vent of dimension $0.5 \times 0.5m^2$ is situated in the side adjacent to windows.

2.2. Governing Equations.

The fluid flow is incompressible and turbulent. The following mass, momentum, energy conservation equations with 2-equation k-ϵ turbulence model with wall functions are used as the governing equations.

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_i)}{\partial x_i} = 0 \tag{2.1}$$

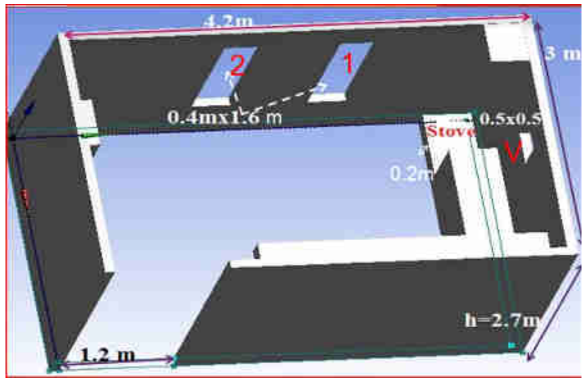

FIGURE 1. Description of a naturally ventilated kitchen

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] + \rho g_i \tag{2.2}$$

$$\frac{\partial(\rho H)}{\partial t} + \frac{\partial(\rho u_i H)}{\partial x_i} = \frac{\partial}{\partial x_i}\left[\frac{k}{c_p}\frac{\partial H}{\partial x_i}\right] + S_H \tag{2.3}$$

where, u_i is the velocity (m/s) component (u, v, w), p (Pa) is the pressure, H $(W/m^2.K)$the enthalpy and S_H a source term. The diffusion term is indicated by the kinematic viscosity μ $(kg/m.s)$, the thermal conductivity k $(W/m.K)$ and the specific heat c_p $(J/kg.K)$. The time is indicated with t, x_i is the coordinate axis (x, y, z), ρ (kg/m^3) is the density and g_i (m/s^2) is the gravitational acceleration.

For the standard k-ϵ equation, the transport equations for turbulent quantities are given by:

$$\frac{\partial(\rho k)}{\partial t} + \frac{\partial}{\partial x_i}(\rho k u_i) = \frac{\partial}{\partial x_j}\left[\frac{\mu_t}{\sigma_k}\frac{\partial k}{\partial x_j}\right] + 2\mu_t S_{ij}.S_{ij} - \rho\epsilon \tag{2.4}$$

$$\frac{\partial(\rho \epsilon)}{\partial t} + \frac{\partial}{\partial x_i}(\rho \epsilon u_i) = \frac{\partial}{\partial x_j}\left[\frac{\mu_t}{\sigma_\epsilon}\frac{\partial \epsilon}{\partial x_j}\right] + C_{1\epsilon}\frac{\epsilon}{k}2\mu_t S_{ij}.S_{ij} - C_{2\epsilon}\rho\frac{\epsilon^2}{k} \tag{2.5}$$

where eddy viscosity $\mu_t = \rho C_\mu \frac{k^2}{\epsilon}$, component of rate of deformation $S_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]$ with $C_\mu = 0.09$, Prandtl numbers σ_k=1 and $\sigma_{1\epsilon}$=1.3, $C_{1\epsilon}$=1.44, $C_{2\epsilon}$=1.92.

2.3. **Boundary Conditions.** The governing equations are closed with appropriate thermo-fluid boundary conditions at all the boundaries such as air inlets, outlets, heat flux and wall surfaces. Window-1 and window-2 are taken as the velocity inlets and door is taken as the pressure outlet as per the wind direction. At the inlet an inlet velocity is specified. Pressure outlet is used in the outlets, where the pressure at the outlet is taken as ambient pressure. As a fluid particle grows in proximity to a rigid wall, the greater will be the influence of shear forces from the wall so that in the limit, the velocity will theoretically be zero. The boundary condition on the surface is assumed to have zero relative velocity between the surface and gas, which is the no-slip condition. Since the surface is stationary, with the flow passed it, $u = u_{wall}$=0 i.e. $u = v = w = 0$ at the solid walls. Wall functions are used

at walls. The temperature T of the fluid layer immediately in contact with the surface is equal to the material temperature T_w at the surface i.e. $T = T_{wall}$ (at wall). Since the wall is non-porous, there is no mass flow into and out of the wall. So temperature flux at the solid wall is taken as zero.

3. Numerical Method

3.1. **Discretization.** The computational domain (geometry of the kitchen described in figure 1) is generated with ANSYS software using the inbuilt design and mesh modeling section. The indoor space of the model is discretized into non-uniform computational cells with unstructured hexahedral mesh. We have avoided tetrahedral and pyramid cells, which can have negative effects in terms of numerical diffusion and convergence with higher-order discretization schemes[1].

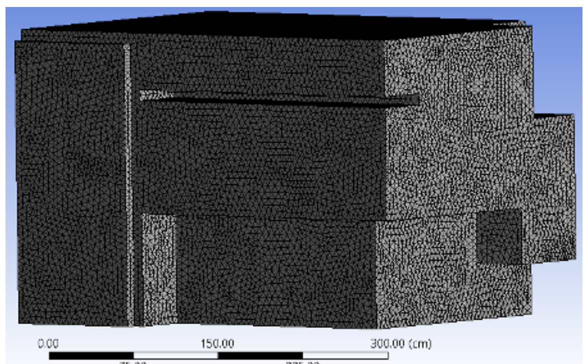

FIGURE 2. Discretization of the kitchen (interior/fluid part)

The governing equations (2.1) - (**??**) are discretized using the finite volume method (FVM) on a spatially rectangular computational mesh refined locally at the specified fluid regions where high gradients are expected. The FVM grants a conservative discretization of the governing equations, with spatial derivatives (fluxes) approximated with second-order upwind scheme. Pressure based solver with Semi-Implicit Pressure Linked Equations (SIMPLE) algorithm [8] is used for pressure velocity coupling. Resulting system of linear algebraic equations is solved by iterative method.

The values of velocity, temperature, kinetic energy, dissipation rate of kinetic energy are set at the boundaries. Least square cell-based method is used for the interpolation of field variables stored at cell centers to the faces of control volumes. Second order schemes are used for the pressure, momentum, turbulent kinetic energy and turbulent dissipation of kinetic energy. Default values are taken as the under relaxation factors. Gauss-Seidel method is used for smoothing the grids discretized.

3.2. **Simulation Setup.** In the simulation reported here, we employ a fairly fine non-uniform grid consisting of 117662 nodes and 641942 elements. Finer grids were tested, with no discernible effect (to at least 3 significant digits). Inlet velocity of $0.2 m/s$ is taken as the velocity inlet boundary conditions. No slip conditions for momentum and zero flux for

thermal boundary condition are used. The room temperature of $300K$ is used in the room as temperature boundary. The operating pressure of $101325Pa$ is used with air density $1.225kg/m^3$. Using the time step size of 0.1 seconds with maximum 20 iterations per time step for the iterative solvers, the simulations were carried out up to time $t_{max} = 822$ seconds. It took approximately 30 hours to complete the simulation in Intel i7-4600U CPU @ 2.10GHz × 4, 8GB RAM computer.

4. Experimental Study

The measurement of indoor air contaminants in a room was done in Chitwan, Nepal. Figure 3 shows description of the kitchen for experimental setup. Continued data for two and half days was recorded. The main instruments used for the study were 1) Micro-Aeth-AE51 for measuring Black Carbon, 2) IAQ Probe to measure CO_2, CO, relative humidity, temperature, VOC and 3) Aerocet-831 to measure the particulate matters PM_1, $PM_{2.5}$, PM_4, PM_{10} and total suspended particles (TSP).

Figure 3. Experimental setup in the kitchen

Five different sampling locations in the room, as shown in figure 4, were used for the measurement of these pollutants. Three sets were kept in each of the five locations A, B, C, D and E. Set A was near the vent at the level of $1.7m$ from the ground, set-B was adjacent to stove 0.9 meter above and right of the stove level, two of the sets C and D were in the breathing zone/dining area and the fifth set E was in the corner of the room at a height of $2.4m$ in the opposite side of the location of stove.

History of carbon monoxide concentration at various sample locations inside the room under different conditions (opening and closing of windows 1, 2 and the vent) are plotted and displayed in the figures 5 - 9.

Figure 5 shows the distribution of CO when window-1 is open and vent is open. The concentration at position A is the highest. The concentration at location B is lower than concentration of C and D. The average of CO concentration at B, C and D is 16ppm whereas the concentration at A is found to be 91 ppm during the time period of 7:30 to 9:10.

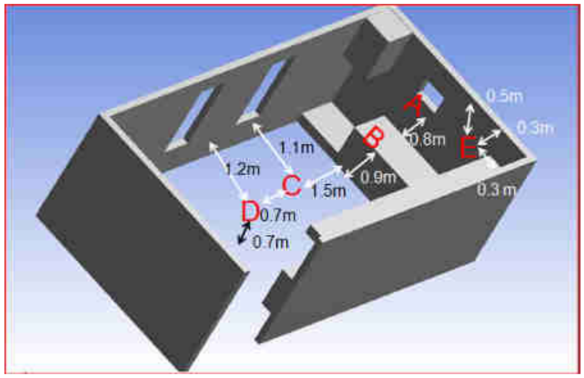

FIGURE 4. Sampling Locations

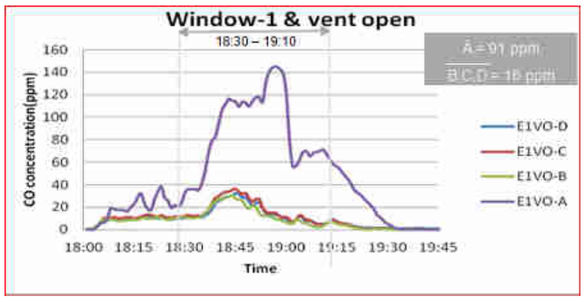

FIGURE 5. CO distribution at A, B, C, D when window-1 and vent is open

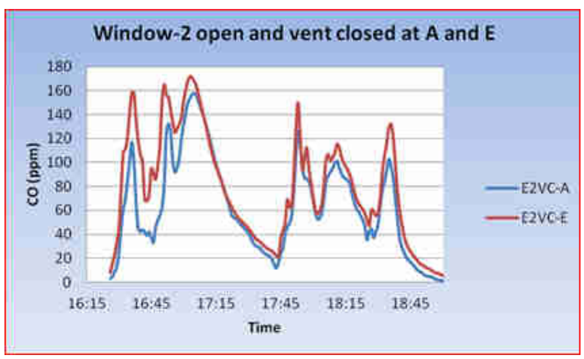

FIGURE 6. CO distribution at A and E when window-1 and vent is closed

Figure 6 shows that the concentration at E is slightly higher than at A. Opening of window-1 significantly decreases the concentration at B than that of opening of window-2.

When the window-2 is open the concentration goes higher at B than at C and D as can be seen in figure 8. When vent and window-2 both are open as in figure 9, the concentration at B increases and the concentrations at D is decreased significantly. The concentration of dining region C also increases. This condition gets very less volume for mixing of indoor air pollution with the fresh air from window-2 as a result, more region in the room gets higher concentration.

From the experimental study we found that the concentration of carbon monoxide at location A is about 3-5 times higher than at B, C, D for all cooking events. Exposure is

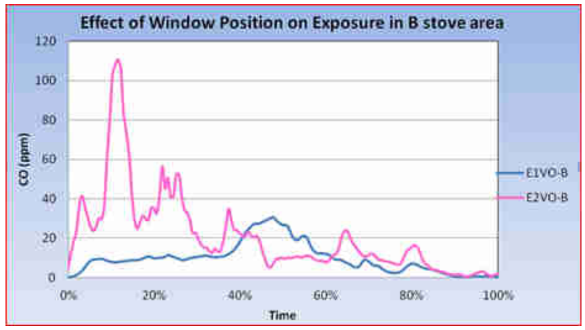

FIGURE 7. Opening of window-1 decreases the concentration at B

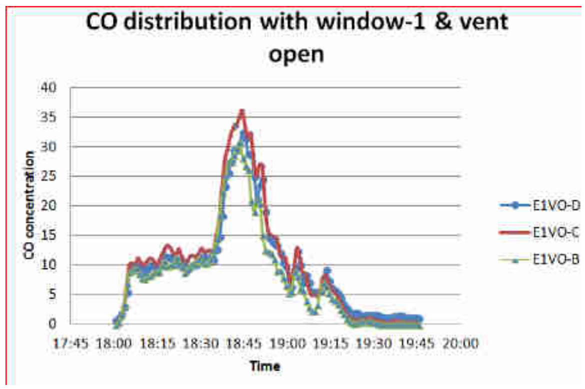

FIGURE 8. Concentration when window-1 and vent open

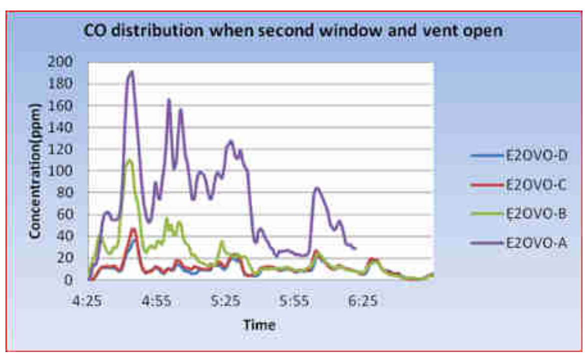

FIGURE 9. Concentration when window-2 and vent open

highest for occupants standing at A near the stove. Opening window-1 (instead of window-2) affects the CO distribution in two ways. It reduces concentrations near the stove at B; increases concentration in the dining area C and D. Opening or closing of vent does not have much effect on the concentration perhaps because vent is very small compared to the window. Above conclusions also apply to BC, $PM_{2.5}$, and total volatile organic compounds(TVOCs).

5. Results and Discussions

The numerical simulations are used to analyze the fluid flow parameters such as velocity, pressure, temperature turbulence with the window positions in the kitchen. Additional

planes are created at different locations especially at $1.1m$, $2m$ and $2.4m$. The first plane is at the height of occupant's breathing zone, second plane is at the height of 2m which is the height of the occupant. Measuring instruments are placed in the level of these planes. Volume renderings and surface flow distribution analysis on different planes are generated and studied to find the flow distribution patterns.

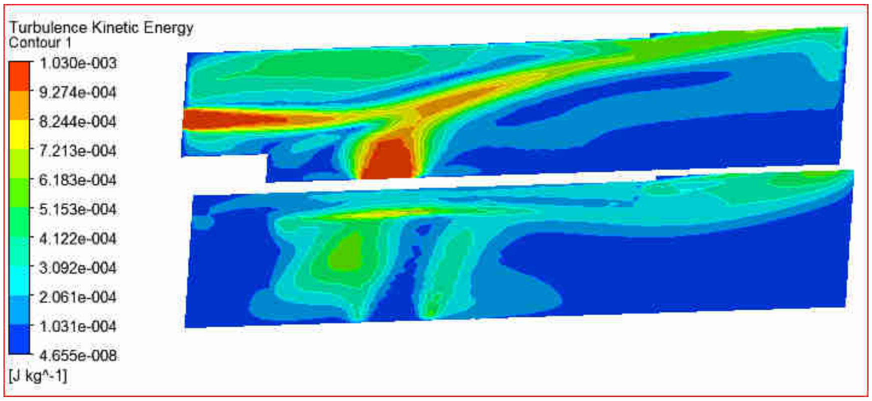

FIGURE 10. Trubulent kinetic energy above $1.1m$ when window-1 and vent are open

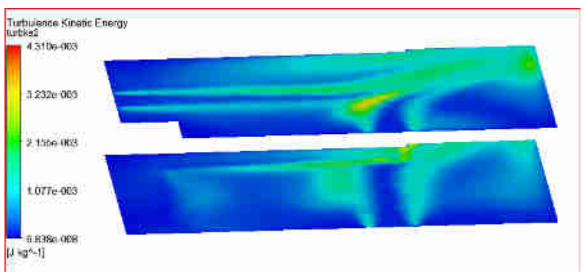

FIGURE 11. Turbulent kinetic energy at $1.1m$ and $2m$ when window-2 and vent open

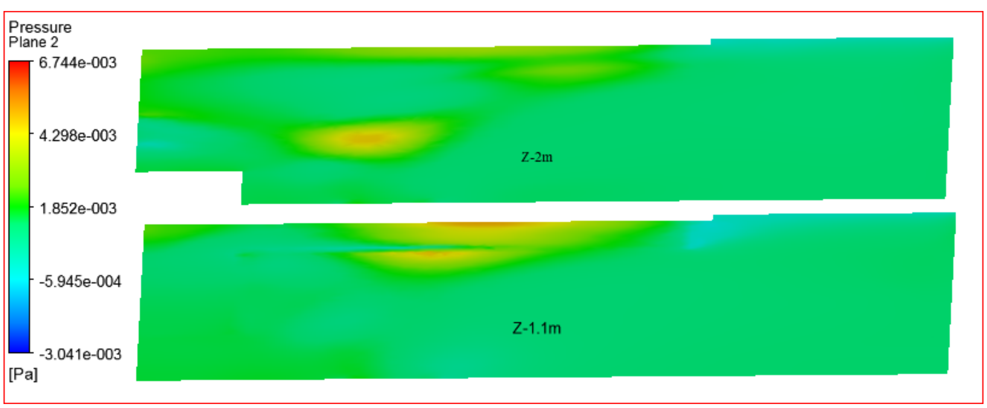

FIGURE 12. Pressure distribution at $1.1m$, $2m$ from ground with Window-1 and vent open

Figures 10 and 11 respectively show turbulence of the fluid flow when window-1 and window-2 are open. The flow is turbulent in two sides of the flow path from window-2

towards door, turbulence is observed at opposite to wall and door. Small turbulence is also observed in the slabs just above the level of stove and in wall. Figure 11 indicates the turbulence of air flow at $1.1m$ at $2m$ above the ground which shows greater turbulence contributed from the vent and the window. The plane at 2m has larger turbulence in fluid flow than that of the plane at $1.1m$. Figure 12 shows the pressure distribution in the kitchen. Distribution of pressure in the room is normal. Slightly higher pressure in the opposite side of the kitchen.

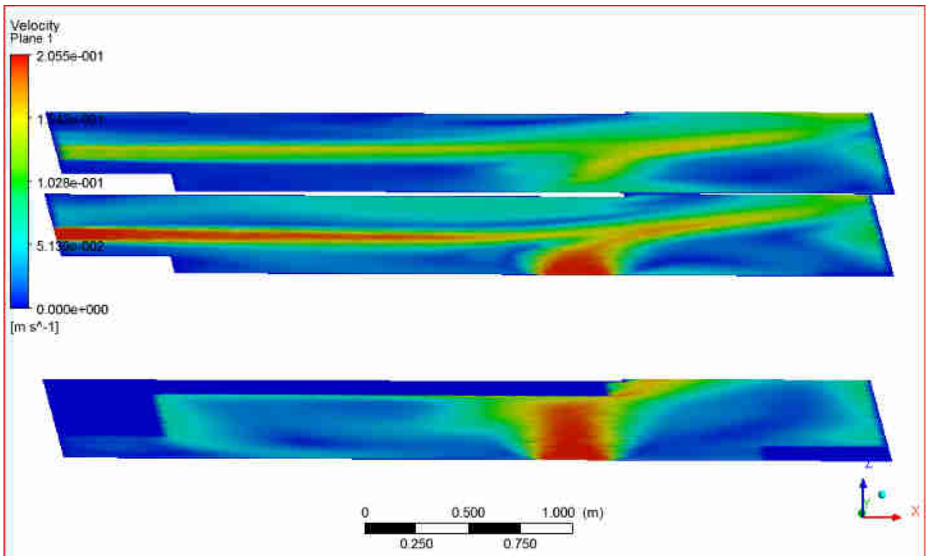

FIGURE 13. Velocity distribution at $1.1m$, $2m$ and $2.4m$ from ground with window-2 and vent open

Figure 13 shows the distribution of the velocity at $1.1m$, $2m$ and $2.4m$ above the ground. The plane at $1m$ shows the higher velocity in path of the fluid from window-2 and moves towards the opposite wall and door. Slightly higher velocities are observed in the corners of the slabs. In the plane at height of $2m$ shows the higher velocity in the flow path from vent and window-2 and near the opposite wall. In the top plane the velocity gradually decreases and velocity near wall are also low.

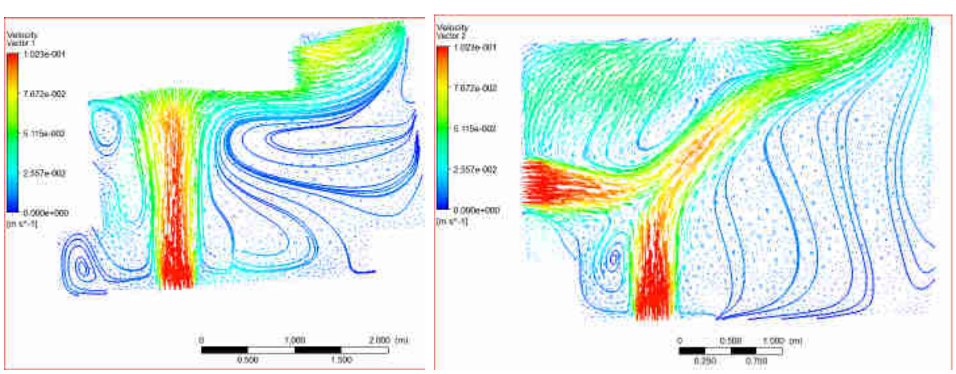

FIGURE 14. Velocity streamlines at $1.1m$ (left) and at $2m$ (right) with window-1 and vent open

Velocity streamlines on the planes at $1.1m$ and $2m$ are shown in figure 15 and figure 14 for the case of window 1 and window 2 open respectively. The formation of vortex at the height of $1m$ in the side of stove can be seen in figure 15 which increases the concentration of pollutants in the location B and also results in lowering the concentration in D area as also observed in the measured results plotted in figure 9. Thus, opening of window-2 instead of window-1 increases the concentration of pollutants in the area where occupant stays which is near to location B.

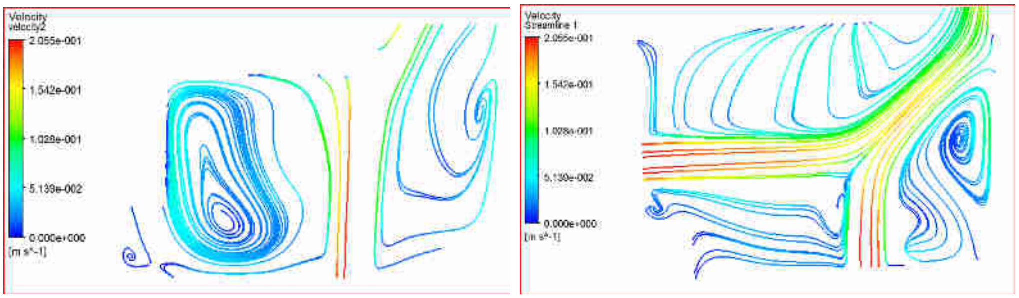

FIGURE 15. Velocity streamlines at $1.1m$ (left) and at $2m$ (right) with window-2 and vent open

It is explored that the concentrations at the occupant location is higher when the window in open position is located far from the stove. Simulation and experimental results verify that in such cases the concentration increases significantly near the position of stove. The fresh air from the window dilutes pollutants in the dining region. It is found that the velocity and turbulence are higher near the door and near windows. The vortices are formed in both sides of the path of fluid flow in room from door. Such areas where the air could be trapped should be avoided for the occupant's position in the kitchen. The pressure is observed higher in wall opposite to the door. As seen in the results of the fluid flow, cross ventilated rooms are effective and efficient for indoor air quality.

6. Conclusions

We developed a three dimensional time-dependent computational model for indoor air pollution in a kitchen using ANSYS-Fluent software and investigated the dispersion of pollutants under the conditions with change of window positions in the wall near and far from the stove. The computational model is validated comparing the numerical simulations with the on-site measured data of carbon monoxide levels at different location in the kitchen under various conditions. Simulations of up to 15 minutes shows a good agreement with the experimental data. Full-fledged simulations of the model will be very useful for the proper design of an efficient ventilation system in the kitchen which provides better quality of air to the occupants of buildings through effective removal of the pollutants. Due to the intricate geometry of the kitchen, the problem requires very intensive computations demanding high performance computing.

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A REVIEW ON THE STRUCTURE AND PROPERTIES OF THE ESCAPING SET OF TRANSCENDENTAL ENTIRE FUNCTIONS

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Abstract: For a transcendental entire function f, we study the structure and properties of the escaping set $I(f)$ which consists of points whose iterates under f escape to infinity. We concentrate on Eremenko's conjecture and we review some attempts of its proofs. A significant amount of progress in Eremenko's conjecture has been made possible via fast escaping set $A(f)$ which consists points that escape to infinity as fast as possible. This set can be written as union of closed sets, called levels of $A(f)$. We review classes of functions for which $A(f)$ and each of its levels has the structure of infinite spider's web. In general, we study classes of entire functions for which the escaping set $I(f)$ is a spider's web. Spider's web is a recently investigated structure of $I(f)$ that gives new results in the direction of Eremenko's conjecture.

Key Words: Escaping set, Eremenko's conjecture, Fast escaping set, Spider's web, Regularity condition etc.

AMS (MOS) Subject Classification. 37F10, 30D05.

1. Introduction

The subject *complex dynamics* formally originated by the independent work of Fatou [20, 21] and Julia [25] during 1917-1926. All these early papers of Fatou and Julia considered the iteration of rational functions. In this article, we consider only the iteration of transcendental entire function (TEF) which was initiated by Fatou [20] in 1926 and developed much more in the work of Baker [2, 3, 4, 5, 6]. Later, solid body of knowledge in transcendental iteration theory has developed in the work of Eremenko [15], Eremenko and Lyubich [16, 17, 18, 19], Bergweiler [7, 8, 9, 10, 11, 12, 13], Rippon and Stallard [37, 38, 39, 40, 41, 42, 44], Schleicher [48, 49, 50], Rempe [34, 35, 36], Sixsmith [51, 52, 53, 54, 55] and Osborne [28, 29, 30, 31, 32, 33].

We denote the *complex plane* by $\mathbb{C}$ and set of integers greater than zero by $\mathbb{N}$. We assume the function $f : \mathbb{C} \to \mathbb{C}$ is *transcendental entire function* (TEF) unless otherwise stated. For any $n \in \mathbb{N}$, f^n always denotes the nth iterates of f. The *order* $\rho(f)$ and *lower order* $\lambda(f)$ of TEF f are defined respectively by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r} \quad \text{and} \quad \lambda(f) = \liminf_{r\to\infty} \frac{\log\log m(r, f)}{\log r}.$$

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where $M(r, f) = \max_{|z|=r} |f(z)|, r > 0$ and $m(r, f) = \min_{|z|=r} |f(z)|, r > 0$ denote respectively the maximum and minimum modulus of the function f. We will see in section 3 that these terms are important in the study of functions for which $A_R(f)$ is a spider's web.

A family $\mathcal{F} = \{f^n : n \in \mathbb{N}\}$ of the iterates of TEF f forms a *normal family* if every sequence $(f^n)_{n\in\mathbb{N}}$ of functions contains a subsequence which converges uniformly to a limit $f \neq \infty$ or converges to ∞ on every compact subset D of $\mathbb{C}$. The *Fatou set* of f denoted by $F(f)$ is the set of point's $z \in \mathbb{C}$ such that sequence $(f^n)_{n\in\mathbb{N}}$ forms a normal family in some neighborhood of z in the sense of Montel. A connected component of the Fatou set $F(f)$ is called *Fatou component*. The complement of Fatou set is called *Julia set* and it is denoted by $J(f)$. The basic properties and structures of these sets can be found in [7, 14, 24, 26, 27].

For a TEF f, if $f'(z) = 0$, we say z is a *critical point* and $w = f(z)$ is a *critical value*. For a TEF f, a curve $\Gamma : [0, \infty) \to \mathbb{C}$ is an *asymptotic curve* with *asymptotic value* α if $\Gamma(t) \to \infty$ and $f(\Gamma(t)) \to \alpha$ as $t \to \infty\ \forall\ \ t \in [0, \infty)$. The set $SV(f) = \overline{(CV(f) \cup AV(f))}$ is called set of *singular values*, where $CV(f)$ and $AV(f)$ respectively denote the set of critical values and asymptotic values. Note that the set $SV(f)$ coincides with the set of singularities of the inverse function f^{-1} of f, and so this set is also denoted by $Sing(f^{-1})$. If $SV(f)$ has only finitely many elements, then f is said to be of *finite type*. If $SV(f)$ is a bounded set, then f is said to be of *bounded type*. The sets $\mathcal{S} = \{f : f \ \text{ is of finite type}\}$ and $\mathcal{B} = \{f : f \ \text{ is of bounded type}\}$ are respectively called *Speiser class* and *Eremenko-Lyubich class*. Note that $\rho(f) \geq \frac{1}{2}$ for any bounded type transcendental map and $\rho(f) \geq 1$ for any finite type map. The class $\mathcal{B}$ introduced in complex dynamics by Eremenko and Lyubich [16]. The most important result of this paper[16] is $F(f) \cap I(f) = \emptyset$ if $f \in \mathcal{B}$. The most familiar functions in this class are the functions in the *exponential family* $\{f : f(z) = \lambda \exp(z), \lambda \neq 0\}$ and the functions in the *cosine family* $\{f : f(z) = \cos(\alpha z + \beta), \alpha \neq 0\}$

2. Escaping Set and Eremenko's Conjecture

In recent years, much interest and more effort have been devoted to understanding the structure and properties of the escaping set $I(f)$ of f which is defined as follows:

Definition 2.1 (**Escaping Set**). *: For a TEF f, the set of the form*

$$I(f) = \{z : f^n(z) \to \infty \ \ as \ \ n \to \infty\}$$

is called escaping set.

For a TEF f, the escaping set $I(f)$ was first studied by A. Eremenko [15]. The fundamental properties of the escaping set $I(f)$ are as follows.

Theorem 2.1. *For a TEF f, the following statements are hold.*

(1) $I(f) = I(f^n)$ *for* $n \geq 2$.
(2) $I(f)$ *is completely invariant.*
(3) $I(f) \neq \emptyset$.
(4) $J(f) \cap I(f) \neq \emptyset$.
(5) $J(f) = \partial I(f)$.

(6) *$\overline{I(f)}$ has no bounded components.*

The first two statements (1) and (2) of this theorem 2.1 follows from the definition of $I(f)$ and the rest (3), (4), (5) and (6) are proved in [15]. If U is a Fatou component such that $U \cap I(f) \neq \emptyset$, then by normality $U \subset I(f)$. We say that such a Fatou component U is *escaping Fatou component*. However, the boundary of such escaping Fatou component may not be in $I(f)$. For example, the function $f(z) = e^{-z} + z + 1$ has escaping Fatou component but its boundary contains periodic points which are not in $I(f)$. On the basis of statement (6) of above theorem 2.1 and Ninty years old Fatou's original question [20] initiating from the functions such as $f(z) = e^{-z} + z + 1$ and $f_\lambda = \lambda \sin z$ concerning whether there are infinitely many curves $\gamma_k, k \in \mathbb{N}$ such that $z \in \gamma_k$, $f^n(z) \to \infty$ as $n \to \infty$, Eremenko made the following conjectures in more precise form in [15].

Conjecture 2.1 (**Normal (weak) version**). Each component of $I(f)$ is unbounded.

Conjecture 2.2 (**Strong version**). Each escaping point can be connected to infinity along a unique curve within $I(f)$.

2.1. **Attempt of Proving Normal Version of Eremenko's Conjecture.** This conjecture in normal form in general case has been proved by using the fast escaping set $A(f)$, which consists of points whose iterates tends to infinity as fast as possible. This set is introduced first time by Bergweiler and Hinkkanen [13] and now plays a key role in transcendental dynamics. We have used here the definition given by Rippon and Stallard in [40] as follows.

Definition 2.1.1 (**Fast escaping set**). *For a TEF f, the fast escaping set is a set of the form:*

$$A(f) = \{z : \exists L \in \mathbb{N} \text{ such that } |f^{n+L}(z)| \geq M^n(R, f) \;\; \forall \;\; n \in \mathbb{N}\}$$

where $M(r, f) = \max_{|z|=r} |f(z)|, r > 0$ and $M^n(r, f)$ denotes iteration of $M(r, f)$ with respect to r, and $R > 0$ can be taken any value such that $M(r, f) > r$ for $r \geq R$.

The set $A(f)$ has many strong properties that can be used in the study of $I(f)$ and $J(f)$. Different properties of $A(f)$ and even different definitions of $A(f)$ were found in [11, 13, 37, 40]. Significant progress in Eremenko's conjecture has been made possible by studying properties and structure of fast escaping set $A(f)$. The fundamental properties of this set are as follows.

Theorem 2.1.1. For a TEF f, the following statements are hold.

(1) $A(f) = A(f^n)$ for $n \geq 2$.
(2) $A(f) \neq \emptyset$.
(3) $A(f)$ is completely invariant.
(4) $A(f)$ is independent of R.
(5) $J(f) \cap A(f) \neq \emptyset$.
(6) $J(f) = \partial A(f)$.
(7) $A(f)$ has no bounded components.

The proof of the statement (1) is given in [37], the statements (2), (3) and (4) are stated in [13] and proved in [37], statements (5) and (6) are proved in [13, 37] and statement (7) is proved in [40]. This result (7) is an important one that provides a partial answer to Eremenko conjecture which is obtained on the basis of certain subsets of $A(f)$ based on above definition 2.1.1.

Definition 2.1.2 (**Level of Fast Escaping Set**). *Let f be a TEF. Let $L \in \mathbb{Z}$ and $R > 0$ be such that $M(r, f) > r$ for $r \geq R$. The L^{th} level of $A(f)$ with respect to R is the set*

$$A_R^L(f) = \{z : |f^n(z)| \geq M^{n+L}(R, f) \text{ for } n \in \mathbb{N}, n + L \geq 0\}$$

In particular

$$A_R(f) = A_R^0(f) = \{z : |f^n(z)| \geq M^n(R, f) \text{ for } n \in \mathbb{N}\} \text{ and}$$

$$A(f) = \bigcup_{n \geq 0} f^{-n}(A_R(f))$$

Note that each of the level of $A(f)$ is a closed set. Since

$$M^{n+1}(R, f) > M^n(R, f) \;\; \forall \;\; n \geq 0$$

So we have

$$A_R^L(f) \subset A_R^{L-1}(f) \;\;\; \forall \;\; L \in \mathbb{N}$$

Also

$$A(f) = \bigcup_{L \in \mathbb{N}} A_R^{-L}(f) \;\text{ and }\; A_R^{-L}(f) \subset A_R^{-(L+1)}(f), L \in \mathbb{N}$$

The concept of level as defined in the definition 2.1.2 provides a new understanding of the structure of $A(f)$ as a countable union of closed sets. On the basis of this definition, Rippon and Stallard [40] have obtained the strongest result for general TEF in the direction of Eremenko's conjecture which is nothing other than the statement (7) of above theorem 2.1.1.

Theorem 2.1.2. Let f be a TEF and $R > 0$ be such that $M(r, f) > r$ for $r \geq R$. Then for each $L \in \mathbb{Z}$, each component of $A_R^L(f)$ is closed and unbounded. In particular, each component of $A(f)$ is unbounded.

The proof of this theorem 2.1.2 is given in [40]. Since $A(f) \subset I(f)$, so this theorem provides partial answer to the Eremenko's conjecture 2.1 that $I(f)$ has at least one unbounded component. If U is a Fatou component such that $U \cap A(f) \neq \emptyset$, then by normality $U \subset A(f)$. We say that such a Fatou component U is *fast escaping Fatou component.* The following theorem due to Rippon and Stallard [37] gives important properties of the fast escaping Fatou component that provides us a contrasting feature of $A(f)$

Theorem 2.1.3. Let f be a TEF and $R > 0$ be such that $M(r, f) > r$ for $r \geq R$ and let $L \in \mathbb{Z}$. If U is a Fatou component that meet $A_R^L(f)$, then

(1) $\overline{U} \subset A_R^{L-1}(f)$
(2) If, in addition U is simply connected then $\overline{U} \subset A_R^L(f)$.

This Theorem 2.1.3 implies that Fatou component U of $A(f)$ has boundary in $A(f)$ but we have already mentioned that this does not happen in $I(f)$. In this context, Sixsmith [54] raised the question: Is there a TEF that can have simply connected fast escaping Fatou components without having multiply connected Fatou components? His affirmative answer is as follows:

Theorem 2.1.4. [54] There is a TEF with simply connected fast escaping Fatou component and no multiply connected Fatou components.

The levels $A_R^L(f)$ of $A(f)$ are also useful for the establishment of relationship between $A(f)$ and $J(f)$. In [40], this relation is shown in the following theorem.

Theorem 2.1.5. Let f be transcendental entire function. Let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$ and $L \in \mathbb{Z}$. Then all components of $A_R^L(f) \cap J(f)$ are unbounded if and only if f has no multiply connected Fatou components.

This theorem 2.1.5 is the main result that provides an alternative condition for the partial solution of the normal version of Eremenko's conjecture. It says if f has no multiply connected Fatou component, then all components of $A(f) \cap J(f)$ are unbounded. On the basis of theorem 2.1.4, there is a TEF that have simply connected fast escaping Fatou components. So we conclude that there is TEF in which $A_R^L(f) \cap J(f)$ are unbounded and hence all components of $A(f) \cap J(f)$ are unbounded. Which is a strong partial answer to the Eremenko's conjecture 2.1.

In recent years, active research in the field of escaping set has been devoted mostly to see the structure that has number of strong dynamical properties as well as able to establish the connection between the conjecture of Baker and the conjecture of Eremenko. The new research in this direction has become possible by the introduction of infinite spider's web. The first example of functions for which $A(f)$ has this structure have been given in [37]. Many TEF f for which $A_R(f)$ and hence $A(f)$ has this structure has been given in [40]. This new set structure is defined as follows:

Definition 2.1.3 (**Spider's Web**). *A set E is an (infinite) spider's web if E is connected and there exists a sequence of bounded simply connected domains G_n with $G_n \subset G_{n+1}$ for $n \in \mathbb{N}, \partial G_n \subset E$ for $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} G_n = \mathbb{C}$.*

We begin with basic properties of spider's web structure which are useful in proving the theorems 2.1.7, 2.2.1, 2.2.2.

Theorem 2.1.6. Let f be a transcendental entire function and let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$ and $L \in \mathbb{Z}$.

(1) If G is a bounded components of $A_R^L(f)^C$, then $\partial G \subset A_R^L(f)$ and f^n is a proper map of G onto the bounded component of $A_R^{n+L}(f)^C$, for each $n \in \mathbb{N}$.
(2) If $A_R^L(f)^C$ has bounded component, then $A_R^L(f)$ is a spider's web and hence every component of $A_R^L(f)^C$ is bounded.
(3) $A_R(f)$ is a spiders web if and only if $A_R^L(f)$ is a spider's web.

(4) For $R' > R$, then $A_R(f)$ is a spider's web if and only if $A_{R'}(f)$ is a spiders web.
(5) If $I(f), J(f), I(f) \cap J(f)$ contain spider's web, then each of set is a spider's web.

Note that if $I(f)$ is a spider's web then $I(f)$ is connected and unbounded and so Eremenko's conjecture holds. In [37], Rippon and Stallard have proved that $A_R(f)$, $A(f)$ and $I(f)$ are spider's web for a TEF f whenever f has multiply connected Faou component. In [38, 39], there are many TEF f of sufficiently small growth such that f has no multiply connected Fatou components and $A_R(f)$ is a spider's web. Rippon and Stallard proved the following strong results in [40, 42]

Theorem 2.1.7. Let f be a transcendental entire function and let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$.

(1) If $A_R(f)^C$ has a bounded component, then each of $A_R(f), A(f)$ and $I(f)$ is a spider's web.
(2) If $A_R(f)$ is a spider's web, then $A(f)^C$ has uncountably many components each of which is compact.
(3) If $A_R(f)$ is a spider's web, then $A(f)^C$ has singleton periodic components which are dense in $J(f)$.
(4) If $A_R(f)$ is a spider's web and f has no multiply connected Fatou component, then each of $A_R(f) \cap J(f), A(f) \cap J(f)$, $I(f) \cap J(f)$ and $J(f)$ is a spider's web.
(5) The function f has no unbounded Fatou component.

This theorem 2.1.7 is a good example which shows a number of strong dynamical properties of $A_R(f)$ in the sense that when $A_R(f)$ is a spider's web, then so are $A(f)$ and $I(f)$. When $I(f)$ is a spider's web, then $I(f)$ is connected and unbounded, it follows that Eremenko's conjecture holds whenever $A_R(f)$ is a spider's web. The part (2) of this theorem 2.1.7 demonstrates the fact that spider's web structure of $A(f)$ is connected with uncountably many complimentary components, each of which is closed and bounded. This contrasts with the fact that all components of $A_R^L(f)^C$ are open. This is a good example that $A(f)$ has very intricate structure, if $A_R(f)$ is a spider's web. Further results about the intricate structure of $A(f)$ are obtained by Osborne [30]. Part (5) of this theorem 2.1.7 provides a connection between the existence of an $A_R(f)$ spider's web and conjecture of Baker (Baker's conjecture is that if the order of TEF f is less than $\frac{1}{2}$, then f has unbounded Fatou components and it is dealt nicely in [4] and[5] and survey of the advances of this conjecture is found in [22]). Note that in [43], it is shown that if f is a TEF of order less than $\frac{1}{2}$ and with all zeros in the negative real axis, then all components of $F(f)$ are bounded. From the above theorem 2.1.7 and all examples given in [40], we conclude that either $A(f)$ or $I(f)$ is a spider's web if $A_R(f)$ is a spider's web. However, the following statements are remained open.

Open Problem: Can $A(f)$ be a spider's web when $A_R(f)$ is not spider's wed?

Open Problem: Can $I(f)$ be a spider's web when $A_R(f)$ is not a spider's web?

2.2. **Attempt of Proving Strong Version of Eremenko's Conjecture.** The dynamical study of transcendental entire function was initiated by Fatou in 1926. In his memoire

[21], Fatou observed from the function $f(z) = e^{-z} + z + 1$ that there are infinitely many curves $\gamma_k, k \in \mathbb{N}$ such that $z \in \gamma_k$, $f^n(z) \to \infty$ as $n \to \infty$. Fatou then posed the question: Is this always true in the case of all transcendental entire functions? Sixty years after the Fatou's original question, Eremenko's made a precise study of the escaping set of transcendental entire functions and he posed his conjecture (strong version) 2.2.

The first family of transcendental entire functions whose escaping set has been investigated was the family of exponential functions:

$$E_\lambda = \lambda e^z, \ \ \lambda \in \mathbb{C}$$

It was shown in [50] that every escaping point of this function can be connected to infinity along a curve consisting of escaping points. In the same paper [50], they provided complete classification of such escaping points and they organized in the form of differentiable curves called rays which are diffeomorphic to open intervals together with endpoints (landing points) of the ray. In fact, the most significant results in the direction of Eremenko's conjecture (strong version) given by Rottenfusser, Ruckert, Rempe and Schleicher in[47]. First of all, they provided an example of TEF $f \in \mathcal{B}$ such that every path connected components of $J(f)$ is bounded, together with the fact that $F(f) \cap I(f) = \emptyset$. This provides the answer for the question of Eremenko's conjecture in special case.

The feature of spider's web (definition 2.1.3) has become very important instrument of checking intricate structure of the set $A(f)$. The intricate nature (structure) of $A(f)$, where $A_R(f)$ is a spider's web has been investigated by Osborne in [28]. The following theorem of Rippon and Stallard [40] provides certain nature of $A(f)$ and $I(f)$ that helps to prove strong version of Eremenko's conjecture:

Theorem 2.2.1. Let f be a transcendental entire function. Let $R > 0$ be such that $M(R, f) > r$ for $r \geq R$ and $A_R(f)$ be a spider's web:

(1) Each point in $I(f)$ belongs to the unbounded continuum in $I(f)$ on which all points escape to infinity uniformly.
(2) If K is a component of $A(f)^c$, then either $K \cap I(f) = \emptyset$ or all points in K escape to infinity uniformly.

Part(1) of the Theorem 2.2.1 answers the question raised by Rempe in [34]. The same thing also holds for many functions in the Eremenko-Lyubich class $\mathcal{B}$ which consists of transcendental entire functions whose set of singular values is bounded. Rempe [36] also recognized a TEF in the class $\mathcal{B}$ such that every path connected components of $J(f)$ is bounded for which theorem 2.2.1(1) does not hold. Together with this fact and similar fact shown by Eremenko and Lyubich in [16] that $I(f) \subset J(f)$ (in particular $A(f) \subset J(f)$) if $f \in \mathcal{B}$. In such a case, Eremenko conjecture 2.2 does not hold in general.

On the other hand, for many transcendental entire functions in the class $\mathcal{B}$, the escaping set consists of family of curves tends to ∞. This situation occurs if f is expressed as the finite composition of functions of finite order that are belonged to the class $\mathcal{B}$. In [36], this statement is proved for large class of TEF f and also in [45] for the class $\mathcal{H}$ of functions

satisfying *head-start* condition. This is the main result in the direction of Eremenko's conjecture (strong version) which is stated as follows:

Theorem 2.2.2. Suppose that $f : \mathbb{C} \to \mathbb{C}$ can be written as the finite composition $f = f_1 \circ f_2 \circ f_3 \circ \cdots \circ f_n$, where each $f_i, (i = 1, 2, 3 \ldots, n)$ is of bounded type (that is, $f_i \in \mathcal{B}$) and finite order, then $I(f) \cup \{\infty\}$ is path connected.

We discussed some cases where the strong version of Eremenko's conjecture holds. There are some transcendental entire functions that disprove this conjecture. In [45], Rottenfusser constructed a function F in logarithmic coordinate such that there is an escaping point which can not connected to ∞ by a curve consisting of escaping points. As stated in the following theorem, there are many entire functions which do not have $A_R(f)$ spider's web. In particular, if $f \in \mathcal{B}$, then $A_R(f)$ is not a spider's web.

Theorem 2.2.3. Let f be transcendental function. Let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$ and let $A_R(f)$ be a spider's web. Then there is no path to ∞ on which f is bounded and so,

(1) f does not belongs to the class $\mathcal{B}$.
(2) f has no exceptional points (that is, points with finite backward orbits).

We have seen that for a given TEF f, if $A_R(f)$ is a spider's web, then each of sets $A(f)$ and $I(f)$ is spider's web. For such a function, the sets $A_R(f))$, $A(f)$ and $I(f)$ are connected and f has no bounded components and so both Eremenko's and Baker's conjecture hold. With these strong dynamical properties, it is better to ask: Which function f that gives $A_R(f)$ a structure of spider's web? Several classes of functions that gives $A_R(f)$ a structure of spider's web are derived using the idea of following theorem.

Theorem 2.2.4. [40] Let f be transcendental function. Let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$. Then $A_R(f)$ is a spider's web if one of the following holds:

(1) f has a multiply connected Fatou component.
(2) f has very small growth.
(3) f has order less than $\frac{1}{2}$ and regular growth.
(4) f has finite order, Fabry gaps and regular growth.
(5) f has a sufficiently strong version of pits effects and has regular growth.

Parts $(1)-(3)$ of this theorem 2.2.4 are given in [38] and [39], and the class of functions that are belong to $(4), (5)$ are defined and described in [40, 54]. This theorem 2.2.4 is important one for determining a function f for which $A_R(f)$ is a spider's web. Next, we give a criterion which allows us to construct many more functions if a function f is known which gives $A_R(f)$ a structure of spider's web.

Theorem 2.2.5. Let f be a TEF. Let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$.Then for $n \in \mathbb{N}$, $A_R(f^n)$ is a spider's web if and only if $A_R(f)$ is a spider's web.

3. Classes of Functions giving the Structure of Spider's Web

In this section we try to elaborate the idea for the classes of functions given in theorem 2.2.4. This theorem tells us there are large classes of functions that give $A_R(f)$, a structure of escaping set whenever $M(r, f) > r$ for $r \geq R > 0$. Part (1) of this theorem is nothing other than the following result which is a corollary of theorem 2.1.7(1).This theorem is proved in [40].

Theorem 3.1. *Let f be transcendental function. Let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$. If f is multiply connected Fatou component, then each of sets $A_R(f)$, $A(f)$ and $I(f)$ is a spider's web.*

The classes of functions given in the rest parts (2)-(5) of theorem 2.2.4 are obtained by using the following general results of Rippon and Stallard [40].

Theorem 3.2. *Let f be transcendental function. Let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$. Then $A_R(f)$ is a spider's web if and only if there exists a sequence of bounded simply connected domains $(G_n)_{n\geq 0}$ such that*

(1) $\{z : |z| < M^n(R)\} \subset G_n$, *for all* $n \geq 0$
(2) G_{n+1} *is contained in the bounded component of* $\mathbb{C} \setminus f(\partial G_n)$ *for all* $n \geq 0$.

This result is very abstract and general. The essence of this theorem 3.2 holds if domains G_n are replaced by discs. So the following result is considered a corollary of this theorem 3.2 and this will be more applicable in order to construct examples.

Theorem 3.3. *Let f be transcendental function. Let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$. Then $A_R(f)$ is a spider's web if there exists a sequence (ρ_n) such that $\rho_n > M^n(R)$ and $m(\rho_n) \geq \rho_{n+1}$, for all $n \geq 0$. Where $m(\rho_n)$ is a minimum modulus function with respect to ρ.*

We refer [22] for more detailed survey of this problem. The more strong results on this problem are given in [23] and [39]. In these papers, it is shown that Baker conjecture holds for all functions of small growth that have no unbounded Fatou components whenever the condition of this theorem 3.3 is satisfied. It is also shown in [39] that the conditions of this theorem are satisfied if f is a TEF and and there exists $n \geq 2$ and $r_0 > 0$ such that

$$\log\log M(r, f) < \frac{\log r}{\log^n r} \quad \text{for} \quad r > r_0$$

where $\log^n r$ denotes nth iteration of logarithm function $\log r$. A TEF f that satisfies this condition is called a function of *arbitrarily small growth*.

The following theorem gives a general result for other classes of functions which are not discussed above. For such classes of functions, the conclusion of theorem 3.2 holds. So, the following theorem is also a corollary of theorem 3.2 and this will also be more applicable in order to construct concrete examples.

Theorem 3.4. *Let f be transcendental function. Let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$. Then $A_R(f)$ is a spider's web if for some $m > 1$*

(1) *there exists* $\rho \in (r, r^n)$ *with* $m(\rho) \geq M(r, f)$, *for all* $r \geq R_0 > 0$, *and*
(2) *there exists a sequence* (r_n) *such that* $r_n > M^n(R, f)$ *and* $M(r_n, f) \geq r_{n+1}^m$, *for* $n \geq 0$.

Note that function f that satisfies the part (2) of this theorem 3.4 is called the function of *regular growth* and the condition is known as *regularity condition.* Note that there are more stronger regularity conditions than the condition given in this theorem 3.4(2). They are *ψ-regularity* and *log-regularity* where *log*-regularity is more stronger than ψ-regularity. We refer [53, 54], for more detailed study of both regularity conditions. In fact, if f has order less than $\frac{1}{2}$, then f satisfies part (1) of this theorem 3.4 for all sufficiently large values of m. If function f has finite order and positive lower order, then f satisfies part (2) of this theorem 3.4. For more detailed study of the existence of both part of this theorem 3.4 we refer [1, 39, 40, 54].

Now we have arrived in the position of to be more specific. From theorem 3.4, [22], [40] and [54], we have arrived the following conclusions.

Theorem 3.5. (1) *If f is a TEF of finite order and positive lower order, then f is log-regular.*
(2) *If f is a TEF of order less than $\frac{1}{2}$ and positive lower order, then $A_R(f)$ is a spider's web, where $R > 0$ is such that $M(r, f) > r$ for $r \geq R$.*

How to to produce classes of functions that satisfy theorem 3.5(2) ? As suggested in [54], the following operator will be quite helpful

$$T_{m,n}(f(z)) = \frac{1}{n}\sum_{k=1}^{n} f(e^{\frac{2\pi ik}{n}} z^{\frac{m}{n}})$$

for all $m, n \in \mathbb{N}$, and f is entire function. We choose a consistent branch of the nth root for each term in the sum. Note that this operator has the following elementary properties:

$$T_{1,m} \circ T_{1,n} = T_{1,nm} \text{ and } \ T_{m,n}(f(z^n)) = f(z^n)$$

. The most important Property of this operator appears in the order and lower order of a function as shown in the following results.

Theorem 3.6. *If f is a TEF of order $\rho(f)$, then $T_{m,n}(f)$ is well defined entire function (for all $m, n \in \mathbb{N}$) of order at most $\frac{m}{n}\rho(f)$.*

Theorem 3.7. *Let $f(z) = \sum_{p=0}^{\infty} a_p z^p$ be a TEF and let $m, n \in \mathbb{N}$.*

(1) *If* $\liminf_{p\to\infty} \frac{P \log p}{\log |a_{pn}|^{-1}} > 0$, *then $T_{m,n}(f)$ has positive lower order.*
(2) *If $T_{m,n}(f)$ has positive lower order and $g(z) = \sum_{p=0}^{\infty} b_p z^p$ be a TEF with $|b_p| \geq |a_p|$ for sufficiently large p, then $T_{m,n}(g)$ has a positive lower order.*

For the proof of these both theorems 3.6, 3.7, we refer [54]. Finally, we examine some explicit classes of functions g for which $A_R(g)$ is a spider's web.

Example 3.1. Let $f(z) = e^z$ and $g = T_{m,n}(f(z))$, where $n > 2m$. Then $A_R(g)$ is a spider's web, where $R > 0$ is such that $M(r, g) > r$ for $r \geq R$.

Solution: Since $\rho(f) = 1$ and satisfies inequality of theorem 3.7(1) for $n > 1$. g has order less than $\frac{1}{2}$ by theorem 3.6 and positive lower order by 3.7(1).Thus by theorem 3.5(2), $A_R(g)$ is a spider's web, where $R > 0$ is such that $M(r, f) > r$ for $r \geq R$.

Example 3.2. Let $f(z) = ze^{z^2} + e^z$ and $g = T_{m,n}(f(z))$, where $n > 4m$ and n is odd. Then $A_R(g)$ is a spider's web, where $R > 0$ is such that $M(r, g) > r$ for $r \geq R$.

Solution: Since $\rho(f) = 1$ and satisfies inequality of theorem 3.7(1) for odd n $(n > 1)$. By theorem 3.6, g has order less than $\frac{1}{2}$ and positive lower order by 3.7(2). So by theorem 3.5(2), $A_R(g)$ is a spider's web, where $R > 0$ is such that $M(r, f) > r$ for $r \geq R$.

As a particular case of example 3.1, the following is a famous TEF g that gives $A_R(g)$, a structure of spider's web.

Example 3.3. Let $f(z) = e^z$ and $g = T_{2,4}(f(z))$, where $n \geq 2m$. Then $A_R(g)$ is a spider's web, where $R > 0$ is such that $M(r, g) > r$ for $r \geq R$.

Solution: From example 3.1, $A_R(g)$ is a spider's web, where $R > 0$ is such that $M(r, f) > r$ for $r \geq R$. In this case, $g(z) = T_{2,4}(f(z)) = \sum_{n=0}^{\infty} \frac{z^n}{(4n)!} = \frac{1}{2}(\cos z^{\frac{1}{4}} + \cosh z^{\frac{1}{4}})$. This function has order $\frac{1}{4}$ and positive lower order. In particular, for the function $h(z) = 2g(z^4) = \cos z + \cosh z$, for which $A_R(h)$ is a spider's web.

For further explicit examples of classes of functions indicated by the theorem 2.2.4 that give $A_R(f)$, a structure of spider's web, we refer [38, 40, 54].

At the end of this section, we introduce some open problems arising from very familiar classes of transcendental entire functions about the structure of escaping set.

Let $\omega_n^k = e^{\frac{2\pi ik}{n}}$ be nth roots of unity for some $n \in \mathbb{N}$ with $k = 1, 2, \ldots, n$. Let

$$E_n = \{f : f(z) = \sum_{k=1}^{n} a_k e^{(\omega_n^k z)}, a_k \neq 0 \ \text{ for } \ k = 1, 2, \ldots n\}$$

In particular,

$$E_1 = \{f : f(z) = \lambda e^z, \ \ \lambda \in \mathbb{C}\}$$

is a well known exponential family and

$$E_2 = \{f : f(z) = \alpha e^z + \beta e^{-z}, \alpha \neq 0, \beta \neq 0, \ \text{ and } \ \alpha, \beta \in \mathbb{C}\}$$

is well known cosine family upto conjugacy. As we indicated in introduction section, E_1 and E_2 are most familiar classes of TEF f in Eremenko-Lyubich class $\mathcal{B}$.

From example 3.3, we observed that function $g(z) = \cos z + \cosh z$ gives $A_R(g)$, a structure of spider's web. We easily deduce that $g \in E_4$. In [54], Sixsmith drew the following statement as a open problem.

Open Problem: Is is true that if $f \in E_n$, for $n \geq 3$, then $A_R(f)$ is a spider's web?

Note that all functions in E_n are log-regular and so from [41] that all functions in E_n do not have multiply connected Fatou components. Therefore, if this question is solved, then by theorem 2.1.7(4), each of $A_R(f) \cap J(f), A(f) \cap J(f)$, $I(f) \cap J(f)$ and $J(f)$ is a spider's web.

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