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Solvability of Perturbed Maximal Monotone Operator Inclusions in Banach Spaces

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Abstract

Let X be a real reflexive Banach space and G_1, G_2 two nonempty, open and bounded subsets of X such that $0 \in G_2$ and $\overline{G_2} \subset G_1$. Let$T: X \supset D(T) \to 2^{X^*}$ be a positively homogeneous maximal monotone operator of degree$\alpha \in (0,1]$, $C: X \supset D(C) \to X^*$bounded demicontinuous of type (S_+), and $G: X \supset D(G) \to 2^{X^*}$ of class (P) as introduced by Hu and Papageorgiou in [9]. The problem of existence of nonzero solutions for $Tx + Cx + Gx \ni 0$is solved by utilizing the Browder and Skrypnik degree theories. The result in this papergeneralizes and extends similar results of the author and Kartsatos [1] for $\alpha = 1$ and G=0 and has applications in elliptic and parabolic boundary value problems involving p–Laplacian with $p \in (1,2]$.

Introduction–Preliminaries

Let X be a real reflexive Banach space with norm $\|\cdot\|$ with the dual space X^*. The norm of the space X^* will also be denoted by $\|\cdot\|$ and will be understood from the context. The symbol $B_\varepsilon(x)$ denotes the open ball of radius ε with center at x. We denote by $\langle x^*, x\rangle$ the value of the functional$x^* \in X^*$ at $x \in X^*$. If$\{x_n\}$ is a sequence in X, we

denote its strong convergence to x_0 in X by $x_n \to x_0$ and its weak convergence in X by $x_n \rightharpoonup x_0$. An operator $T: X \supset D(T) \to Y$, with Y another Banach space, is said to be "bounded" if it maps bounded subsets of the domain $D(T)$ onto bounded subsets of Y. The operator T is said to be "compact" if it maps bounded subsets of $D(T)$ onto relatively compact subsets in Y. It is called "demicontinuous" if it is strong-weak continuous on $D(T)$. The symbols $\boldsymbol{R}$ and $\boldsymbol{R}_+$ denote $(-\infty,\infty)$ and $[0,\infty)$ respectively. The normalized duality mapping $J: X \supset D(J) \to 2^{X^*}$ is defined by

$$Jx = \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2, \quad \|x^*\| = \|x\|\}, \quad x \in X.$$

The Hahn-Banach theorem ensures that $D(J)=X$ and therefore $J: X \to 2^{X^*}$.

By a well-known renorming theorem due to Trojanski [18] one can always renorm a reflexive Banach space X with an equivalent norm so that both X and X^* are locally uniformly convex (therefore strictly convex). Henceforth we assume that X is a locally uniformly convex reflexive Banach space. In this setting, the normalized duality mapping J is single-valued homeomorphism from X onto X^* and satisfies

$$J(\alpha x)=\alpha J(x), \quad (\alpha,x)\in \boldsymbol{R}_+\times X.$$

For a multi-valued operator T from X to X^* we write $T: X \supset D(T) \to 2^{X^*}$, where $D(T) = \{x \in X : Tx \neq \emptyset\}$ is the effective domain of T. We denote by $Gr(T)$ the graph of T, i.e., $Gr(T) = \{(x,y): x \in D(T), y \in Tx\}$.

Definition 1. *An operator $T: X \supset D(T) \to 2^{X^*}$ is said to be monotone if for every $x, y \in D(T)$ and every $u \in Tx$, $v \in Ty$ we have*

$$\langle u - v, x - y\rangle \geq 0.$$

A monotone operator T is said to be maximal monotone if $Gr(T)$ is maximal in $X \times X^*$ when $X \times X^*$ is partially ordered by inclusion.

For a reflexive Banach space X, it is well known that a monotone operator T is maximal if and only if $R(T + \lambda J) = X^*$ for all $\lambda\in(0,\infty)$ (equivaleintly for some $\lambda>0$). If T is maximal monotone, then the operator $T_t = (T^{-1} + tJ^{-1})^{-1} : X \to X^*$ is bounded, demicontinuous, maximal monotone and such that $T_t x \rightharpoonup T^{\{0\}}x$ as $t \to 0^+$ for every $x\in D(T)$, where $T^{\{0\}}x$ denotes the element $y^* \in Tx$ of minimum norm, i.e., $\|T^{\{0\}}x\| = \inf\{\|y^*\|: y^* \in Tx\}$. In our setting, this infimum is always attained and $D(T^{\{0\}}) = D(T)$. Also, $T_t x \in TJ_t x$ where $J_t \equiv I - tJ^{-1}T_t : X \to X$ and satisfies $\lim_{t\to 0} J_t x = x$ for all $x \in \overline{co D(T)}$ where coA denotes the convex hull of the set A. In addition, $x\in D(T)$ and

$t_0 > 0$ imply $\lim_{t \to t_0} T_t x = T_{t_0} x$. The operators T_t, J_t were introduced by Brézis, Crandall and Pazy in [8]. For their basic properties, we refer the reader to [1] as well as Pascali and Sburlan [14, pp. 128-130].

Definition 2. *Let $C: X \supset D(C) \to X^*$ be bounded and demicontinuous. We say that C is of type (S_+) if for every sequence $\{x_n\} \subset D(C)$ with $x_n \rightharpoonup x_0$ in X and*

$$\limsup_{n \to \infty} \langle Cx_n , x_n - x_0 \rangle \leq 0,$$

we have $x_n \to x_0$ in X.

The normalized duality mapping J is known to be bounded maximal monotone and of class (S_+).

The following lemma about maximal monotone operators can be found in Zeidler [19, p. 915] and will be needed for the main result in Section 2.

Lemma A. Let $T: X \supset D(T) \to 2^{X^*}$ be maximal monotone. Then the following are true:

(i) $\{x_n\} \subset D(T), x_n \to x_0$ and $Tx_n \ni y_n \rightharpoonup y_0$ imply $x_0 \in D(T)$ and $y_0 \in Tx_0$.
(ii) $\{x_n\} \subset D(T), x_n \rightharpoonup x_0$ and $Tx_n \ni y_n \to y_0$ imply $x_0 \in D(T)$ and $y_0 \in Tx_0$.

The following lemma is essentially due to Brézis, Crandall and Pazy [8], and a proof of it can be found in [1].

Lemma 1. *Assume that the operators $T: X \supset D(T) \to 2^{X^*}$ and $S: X \supset D(S) \to 2^{X^*}$ are maximal monotone, with $0 \in D(T) \cap D(S)$ and $0 \in S(0) \cap T(0)$. Assume, further, that $T+S$ is maximal monotone and that there is a sequence $\{t_n\} \subset (0,\infty)$ such that $t_n \downarrow 0$, and a sequence $\{x_n\} \subset D(S)$ such that $x_n \rightharpoonup x_0 \in X$ and $T_{t_n} x_n + w_n^* \rightharpoonup y_0 \in X^*$ where $w_n^* \in Sx_n$. Then the following statements hold true:*

(i) *the inequality*

$$\lim_{n \to \infty} \langle T_{t_n} x_n + w_n^*, x_n - x_0 \rangle < 0 \; (1)$$

is impossible;

(ii) *if*

$$\lim_{n \to \infty} \langle T_{t_n} x_n + w_n^*, x_n - x_0 \rangle = 0$$

then $x_0 \in D(T+S)$ and $y_0^* \in (T+S)x_0$. (2)

The existence of nonzero solutions in Section 2 will be established by utilizing the topological degree theories developed by Browder [7] and Skrypnik [17]. This theory generalizes and extends similar results of the author and [1] for $\alpha = 1$ and $G = 0$ and has applications in elliptic and parabolic boundary value problems involving p-Laplacian.

For additional information and applications on various degree theories related to the subject of this paper, the reader is referred to the author and Kartsatos [2] Kartsatos and Lin [10] and Kartsatos and Skrypnik [12], [11]. For information on various concepts and ideas of Nonlinear Analysis used herein, the reader is referred to Barbu [4], Browder [5], Pascali and Sburlan [14], Simons [15], Skrypnik [16], [17], and Zeidler [19].

Existence of Nonzero Solutions of $Tx + Cx + Gx \ni 0$

Hu and Papageorgiou [9] generalized the degree theory of Browder [6] to the mappings of the form$T + C + G$, where T is maximal monotone, C bounded demicontinuous of type (S_+) and G belongs to class (P) to be made precise below. In this section, an existence of nonzero solutions of $Tx + Cx + Gx \ni 0$has been established when T is a positively homogeneous maximal monotone operator as defined below.

Definition 3. *An operator $T: X \supset D(T) \to 2^{X^*}$ is said to be positively homogeneous of degree α>0 if, for a fixed α>0, $x \in D(T)$implies $tx \in D(T)$ for all $t \in R_+$ and $T(tx) = t^\alpha Tx$.*

Definition 4. *An operator $G: X \supset D(G) \to 2^{X^*}$ is said to belong to class (P) if it maps bounded sets to relatively compact sets, for every $x \in D(G), G(x)$ is closed and convex subsets of X^* and $G(\cdot)$ is upper-semicontinuous (usc), i.e., for every closed set$F \subset X^*$, the set$G^-(F) = \{x \in D(G): G(x) \cap F \neq \emptyset\}$ is closed in X.*

An important fact about a compact-set valued usc operator G is that it is closed. Furthermore, for every sequence $\{[x_n, y_n]\} \subset Gr(G)$ such that $x_n \to x \in D(G)$, the sequence $\{y_n\}$ has a cluster point in $G(x)$.

The following lemma, which plays an important role in the proof of the existence theorem of this section, shows that the Yosida approximants of a positively homogeneous maximal monotone operator are not so unless α=1.

Lemma 2. *Let $T: X \supset D(T) \to 2^{X^*}$ is maximal monotone and positively homogeneous of degree $\alpha \in (0,1]$. Then, for each t>0, the Yosida approximant $T_t$satisfies*

$$T_t(sx) = s^\alpha T_{ts^{\alpha-1}}(x) \text{ for all } (s,x) \in (0,+\infty) \times X. \tag{3}$$

Proof. Let

$$y = T_t(sx) = (T^{-1} + tJ^{-1})^{-1}(sx),$$

for $t, s > 0$, $x \in X$. The positive homogeneity of the duality mapping J implies

$$\begin{aligned} y \in T(-tJ^{-1}y + sx) &= T\left(s\left(-\frac{t}{s}J^{-1}y + x\right)\right) \\ &= s^{\alpha}\, T\left(-\frac{t}{s}J^{-1}y + x\right) \\ &= s^{\alpha}\, T\left(-\frac{t}{s^{1-\alpha}}J^{-1}\left(\frac{y}{s^{\alpha}}\right) + x\right). \end{aligned}$$

This is equivalent to

$$x \in T^{-1}\left(\frac{y}{s^{\alpha}}\right) + t\, s^{\alpha-1}J^{-1}\left(\frac{y}{s^{\alpha}}\right),$$

and therefore

$$y = s^{\alpha}(T^{-1} + ts^{\alpha-1}J^{-1})^{-1}x = s^{\alpha}T_{ts^{\alpha-1}}(x).$$

This completes the proof.

We next prove the existence of nonzero solutions of the operator inclusion $Tx + Cx + Gx \ni 0$ with an application of the Browder and Skrypnik degree theories. This extends and generalizes the results by the author and Kartsatos (cf. [1], Theorem 6, p.1246, for α=1 and G=0) to multivalued T with $\alpha \in (0,1)$ and $G \neq 0$. The result in this paper is new for $\alpha \in (0,1)$ and therefore applies to partial differential equations involving p–Laplacian with $p \in (1,2]$.

Theorem 1. *Assume that $G_1, G_2 \subset X$ are open, bounded with $0 \in G_2$ and $\overline{G_2} \subset G_1$. Let $T: X \supset D(T) \to 2^{X^*}$ be maximal monotone, and positively homogeneous of degree $\alpha \in (0,1]$, $C: \overline{G_1} \to X^*$ bounded, demicontinuous and of type (S_+) and $G: \overline{G_1} \to 2^{X^*}$ of class (P). Moreover, assume the following:*

(H1) There exists $v_0^* \in X^* \setminus \{0\}$ such that $\lambda v_0^* \notin Tx + Cx + Gx$ for any $(\lambda, x) \in \boldsymbol{R}_+ \times (D(T) \cap \partial G_1)$;

(H2) $0 \notin Tx + Cx + Gx + \lambda J$ for any $(\lambda, x) \in \boldsymbol{R}_+ \times (D(T) \cap \partial G_2)$.

Then the inclusion $Tx + Cx + Gx \ni 0$ has a nonzero solution $x \in D(T) \cap (G_1 \setminus G_2)$.

Proof. We consider the equation

$$Tx + Cx + Gx \ni 0$$

and then the associated equation

$$T_t x + Cx + g_{\varepsilon} x = 0. \tag{4}$$

Here, ε>0 and $g_\varepsilon : \overline{G_1} \to X^*$ is an approximate continuous Cellinaselection (cf. [9], p. 236, Lemma 6, [3]) satisfying

$$g_\varepsilon x \in G(B_\varepsilon(x) \cap \overline{G_1}) + B_\varepsilon(0)$$

for all $x \in \overline{G_1}$ and$g_\varepsilon(\overline{G_1}) \subset \overline{co}\ G(\overline{G_1})$.

We show that equation (4) has a solution $x_{t,\varepsilon}$for all sufficiently small t and ε. To this end, we first show that there exist $\tau_0 > 0,\ t_0 > 0$ and $\varepsilon_0 > 0$ such that the equation

$$T_t x + Cx + g_\varepsilon x = \tau v_0^* (5)$$

has no solution in $G_1$for every $\tau \geq \tau_0,\ \ t \in (0, t_0]$ and $\varepsilon \in (0, \varepsilon_0]$.
Assuming the contrary, let $\{\tau_n\} \subset (0,\infty),\ \{t_n\} \subset (0,\infty),\ \{\varepsilon_n\} \subset (0,\infty)$, and $\{x_n\} \subset G_1$ be such that $\tau_n \to \infty,\ t_n \downarrow 0,\ \varepsilon \downarrow 0$ and

$$T_{t_n} x_n + Cx_n + g_{\varepsilon_n} x_n = \tau_n v_0^*. \tag{6}$$

We may assume that$g_{\varepsilon_n} x_n \to g^* \in X^*$in view of the properties of G. Then $\|T_{t_n} x_n\| \to \infty$ because $\|\tau_n v_0^*\| \to \infty$ and $\{Cx_n\}$ is bounded. Thus, from (6), we get

$$\frac{T_{t_n} x_n}{\|T_{t_n} x_n\|} + \frac{Cx_n}{\|T_{t_n} x_n\|} + \frac{g_{\varepsilon_n} x_n}{\|T_{t_n} x_n\|} = \frac{\tau_n v_0^*}{\|T_{t_n} x_n\|}. \tag{7}$$

From (3), we obtain

$$\frac{T_{t_n} x_n}{\|T_{t_n} x_n\|} = T_{t_n \lambda_n}\left(\frac{x_n}{\|T_{t_n} x_n\|^{\frac{1}{\alpha}}}\right),$$

where$\lambda_n = \|T_{t_n} x_n\|^{(\alpha-1)/\alpha}$.

It clear that $\lambda_n \to 0$ for $\alpha \in (0,1)$ and $\lambda_n = 1$ for $\alpha = 1$. Then (7) implies

$$1 - \left\|\frac{Cx_n}{\|T_{t_n} x_n\|} + \frac{g_{\varepsilon_n} x_n}{\|T_{t_n} x_n\|}\right\| \leq \frac{\tau_n \|v_0^*\|}{\|T_{t_n} x_n\|} \leq 1 + \left\|\frac{Cx_n}{\|T_{t_n} x_n\|} + \frac{g_{\varepsilon_n} x_n}{\|T_{t_n} x_n\|}\right\|.$$

Thus,

$$\frac{\tau_n \|v_0^*\|}{\|T_{t_n} x_n\|} \to 1 \text{ so that} \frac{\tau_n}{\|T_{t_n} x_n\|} \to \frac{1}{\|v_0^*\|} \text{as } n \to \infty. \tag{9}$$

Let

$$u_n = \frac{x_n}{\|T_{t_n}x_n\|^{\frac{1}{\alpha}}}.$$

We have $u_n \to 0$.By (7), (8) and (9), we obtain $T_{t_n\lambda_n}u_n \to h$ with $h = v_0^*/\|v_0^*\|$. Therefore,

$$\lim_{n\to\infty}\langle T_{t_n\lambda_n}u_n, u_n\rangle = \langle h, 0\rangle = 0.$$

Since$t_n\lambda_n \to 0$, by (*ii*) of Lemma with *S*=0 we obtain, 0∈*D*(*T*) and $h = T(0)$. Since $T(0)=0$, this is a contradiction to $\|h\| = 1$.

Fix $t \in (0, t_0], \varepsilon \in (0, \varepsilon_0]$ and consider the homotopy mapping

$$H_1(s, x, t, \varepsilon) = T_t x + Cx + g_\varepsilon x - s\tau_0 v_0^*, \quad s \in [0, 1], \qquad x \in \overline{G_1}. \tag{10}$$

For every $s\in[0,1]$ the operator$x \mapsto Cx - s\tau_0 v_0^*$ is demicontinuous and bounded on $\overline{G_1}$. In order to see that it is of type (S_+), assume that $\{x_n\} \subset \overline{G_1}$ is such that $x_n \rightharpoonup x_0 \in X$ and

$$\limsup_{n\to\infty}\langle Cx_n - s\tau_0 v_0^*,\ x_n - x_0\rangle \le 0.$$

This implies

$$\limsup_{n\to\infty}\langle Cx_n,\ x_n - x_0\rangle \le 0,$$

which by the (S_+)-property of C, implies $x_n \to x_0 \in \overline{G_1}$. Before we consider the Skrypnik degree of this homotopy on the setG_1, we show that the equation$H_1(s, x, t, \varepsilon) = 0$ has no solution on ∂G_1 for all sufficiently small $t \in (0\ t_0]$, $\varepsilon \in (0, \varepsilon_0]$ and all $s\in[0,1]$. To this end, assume the contrary and let $\{x_n\} \subset \partial G_1$, $\{t_n\} \subset (0, t_0]$, $\{s_n\} \subset [0, 1]$ and $\{\varepsilon_n\} \subset (0, \varepsilon_0]$ such that$t_n \downarrow 0$, $s_n \to s_0$ for some $s_0 \in [0, 1]$,$\varepsilon_n \downarrow 0$ and

$$T_{t_n}x_n + Cx_n + g_{\varepsilon_n}x_n = s_n\tau_0 v_0^*.$$

We may assume that $x_n \rightharpoonup x_0 \in X$. Since $\{Cx_n\}$ is bounded, we may assume that $Cx_n \rightharpoonup y_0^* \in X^*$ and $g_{\varepsilon_n} x_n \to g^* \in X^*$. Then we have $T_{t_n} x_n \rightharpoonup -y_0^* - g^* + s_0 \tau_0 v_0^*$. From

$$\langle T_{t_n} x_n, x_n - x_0 \rangle + \langle Cx_n, x_n - x_0 \rangle + \langle g_{\varepsilon_n} x_n, x_n - x_0 \rangle = \langle s_n \tau_0 v_0^*, x_n - x_0 \rangle,$$

we obtain

$$\limsup_{n\to\infty} [\langle T_{t_n} x_n, \ x_n - \ x_0 \rangle + \langle Cx_n, \ x_n - \ x_0 \rangle] = 0. \tag{11}$$

Let us assume that

$$\limsup_{n\to\infty} \langle Cx_n, x_n - x_0 \rangle > 0. \tag{12}$$

Then there exists a subsequence of$\{x_n\}$, which we again denote by $\{x_n\}$, such that

$$\lim_{n\to\infty} \langle Cx_n, \ x_n - \ x_0 \rangle = q > 0. \tag{13}$$

for some constant q>0. By (11) and (13, we obtain

$$\lim_{n\to\infty} \langle T_{t_n} x_n, \ x_n - \ x_0 \rangle = -q < 0.$$

Applying (i) of Lemma with S=0, we obtain a contradiction., Therefore, (12) is false, and we now only have

$$\limsup_{n\to\infty} \langle Cx_n, \ x_n - \ x_0 \rangle \le 0.$$

Since C is of type (S_+), we have$x_n \to x_0 \in \partial G_1$. Since C is also demicontinuous, we obtain $Cx_n \rightharpoonup Cx_0$. This implies

$$T_{t_n} x_n \rightharpoonup -Cx_0 - g^* + s_0 \tau_0 v_0^*.$$

Applying (ii) of Lemma with S=0, we obtain$x_0 \in D(T) \cap \partial G_1$ and

$$Tx_0 + Cx_0 + \ Gx_0 \ni s_0 \tau_0 v_0^*,$$

which leads to a contradiction to our hypothesis $(H1)$. Thus, we may now chooset_0 and ε_0 further so that we also have that $H_1(s, x, t, \varepsilon) = 0$has no solution$x \in \partial G_1$ for any

$t \in (0\ t_0]$, $\varepsilon \in (0, \varepsilon_0]$ and any$s \in [0, 1]$. It is clear that the mapping $H_1(s, x, t, \varepsilon)$ is an admissible homotopy for Skrypnik's degree and the Skrypnik degree $d_S(H_1(s,\cdot, t, \varepsilon), G_1, 0)$ well-defined and is constant for all $s \in [0,1]$ and for all $t \in (0, t_0]$, $\varepsilon \in (0, \varepsilon_0]$. Consequently, the Browder's degree generalized by Hu and Papageorgiou [9], d_{HP},is well-defined and satisfies

$$d_{HP}(T + C + G - \tau_0 v_0^*, G_1, 0) = d_S(T_t + C + g_\varepsilon - \tau_0 v_0^*, G_1, 0) \tag{14}$$

for all $t \in (0\ , t_0]$, $\varepsilon \in (0, \varepsilon_0]$.

Assume that $d_S(H_1(1,\cdot, t_1, \varepsilon_1), G_1, 0) \neq 0$ for some sufficiently small $t_1 \in (0, t_0]$ and and $\varepsilon_1 \in (0, \varepsilon_0]$. Then the equation

$$T_{t_1}x + Cx + g_{\varepsilon_1}x = \tau_0 v_0^*$$

has a solution in the set G_1. However, this contradicts our choice of the number $\tau_0$in (5).Consequently, by homotopy invariance property of the degree mapping,

$$d_s(T_t + C + g_\varepsilon, G_1, 0) = d_S(H_1(0,\cdot, t_1, \varepsilon_1), G_1, 0) = 0,\ t \in (0\ t_0], \varepsilon \in (0, \varepsilon_0].$$

We next consider the homophony mapping

$$H_2(s, x, t, \varepsilon) = s(T_t x + Cx + g_\varepsilon x) + (1 - s)Jx,\ \ (s, x) \in [0,1] \times \overline{G_2}\,. \tag{15}$$

We first show that there exist $t_1 \in (0\ , t_0]$, $\varepsilon_1 \in (0, \varepsilon_0]$ such that the equation $H_2(s, x, t, \varepsilon) = 0$ has no solution on ∂G_2 for any $s \in [0,1]$, $t \in (0\ t_1]$, and any $\varepsilon \in (0, \varepsilon_1]$. Let us assume the contrary. Then there exist sequences$t_n \in (0, t_0]$, $\varepsilon_n \in (0, \varepsilon_1]$, $s_n \in [0, 1]$ and$x_n \in \partial G_2$ such that $t_n \downarrow 0$, $s_n \to s_0$ for some $s_0 \in [0, 1]$. $\varepsilon_n \downarrow 0$, $x_n \rightharpoonup x_0 \in X$, $Cx_n \rightharpoonup y_0^* \in X^*$, $g_{\varepsilon_n}x_n \to g^* \in X^*$, $Jx_n \rightharpoonup z_0^* \in X^*$and

$$s_n\big(T_{t_n}x_n + Cx_n + g_{\varepsilon_n}x_n\big) + (1 - s_n)Jx_n = 0. \tag{16}$$

Note that $s_n = 0$ is impossible because $J(0) = 0$ and J is injective, and therefore we may assume that $s_n > 0$ for all n. If $s_n \to 0$, then

$$\langle T_{t_n}x_n + Cx_n, x_n\rangle = -\left(\frac{1}{s_n} - 1\right)\langle Jx_n, x_n\rangle - \langle g_{\varepsilon_n}x_n, x_n\rangle \to -\infty (17)$$

Because $\{\|x_n\|\}$ is bounded below away from zero. Since $\langle T_{t_n}x_n, x_n\rangle \geq 0$ and $\{\langle Cx_n, x_n\rangle\}$ is bounded, we see that (17) is impossible. Thus, $s_0 \in (0,1]$, and then (16) implies

$$T_{t_n}x_n \rightharpoonup -y_0^* - g^* - \left(\frac{1}{s_0} - 1\right) z_0^*.$$

Also, from (16),

$$\begin{aligned}
\langle T_{t_n}x_n + Cx_n, x_n - x_0\rangle &= -\left(\frac{1}{s_n} - 1\right)\langle g_{\varepsilon_n}x_n + Jx_n,\, x_n - x_0\rangle \\
&= -\left(\frac{1}{s_n} - 1\right)[\langle Jx_n - Jx_0,\, x_n - x_0\rangle + \langle\, g_{\varepsilon_n}x_n + Jx_0, x_n - x_0\rangle] \\
&\leq -\left(\frac{1}{s_n} - 1\right)\langle\, g_{\varepsilon_n}x_n + Jx_0, x_n - x_0\rangle, \qquad (18)
\end{aligned}$$

by the monotonicity of the duality mapping J. Since $s_0 \in (0,1]$, $g_{\varepsilon_n}x_n \to g^*$ and $x_n \rightharpoonup x_0$, we see from (18) that

$$\limsup_{n\to\infty}\{\mathrm{q_n} \coloneqq \langle T_{t_n}x_n + Cx_n, x_n - x_0\rangle\} \leq 0.$$

Let

$$\limsup_{n\to\infty}\langle Cx_n, x_n - x_0\rangle > 0. \qquad (19)$$

Then, for some subsequence of $\{n\}$ denoted by $\{n\}$ again, we have

$$\lim_{n\to\infty}\langle Cx_n, x_n - x_0\rangle = q > 0. \qquad (20)$$

From

$$\langle T_{t_n}x_n, x_n - x_0\rangle = q_n - \langle Cx_n, x_n - x_0\rangle$$

we see that

$$\limsup_{n\to\infty}\langle T_{t_n}x_n, x_n - x_0\rangle \leq \limsup_{n\to\infty} q_n + \lim_{n\to\infty}[-\langle Cx_n, x_n - x_0\rangle] \leq -q < 0.$$

Using (i) of Lemma , we conclude that (19) is impossible. Thus, (19) holds with "$\leq$" in place of ">". Since C is of type (S_+), we have $x_n \to x_0 \in \partial G_2$. This implies $Cx_n \to Cx_0$, $Jx_n \to Jx_0 = z_0^*$, and

$$T_{t_n}x_n \rightharpoonup -Cx_0 - g^* - \left(\frac{1}{s_0} - 1\right)Jx_0.$$

Since $x_n \to x_0$,

$$\lim_{n\to\infty} \langle T_{t_n}x_n, x_n - x_0\rangle = 0.$$

Using (*ii*) of Lemma, we have that $x_0 \in D(T)$ and

$$-Cx_0 - g^* - \left(\frac{1}{s_0} - 1\right)Jx_0 \in Tx_0.$$

By a property of the selection $g_{\varepsilon_n}x_n$ (cf. [9] p. 238), we have $g^* \in G(x_0)$. This implies

$$Tx_0 + Cx_0 + Gx_0 + \left(\frac{1}{s_0} - 1\right)Jx_0 \ni 0,\ \ x_0 \in D(T) \cap \partial G_2.$$

Thus we arrived at a contradiction to our hypothesis (H2). For the sake of convenience, we assume that t_0 and ε_0 are sufficiently small so that we may take $t_1 = t_0$ and $\varepsilon_1 = \varepsilon_0$.

It is therefore clear that the mapping $H_2(s, x, t, \varepsilon)$ is an admissible homotopy for Skrypnik's degree, and so the Skrypnik degree, $d_S(H_2(s,\cdot,t,\varepsilon), G_2, 0)$, is well-defined and constant for all $s \in [0, 1]$, $t \in (0, t_0]$ and $\varepsilon \in (0, \varepsilon_0]$. By the homotopy invariance of the degree mapping, for all$t \in (0, t_0]$, $\varepsilon \in (0, \varepsilon_0]$, we have

$$d_S(H_2(1,\cdot,t,\varepsilon), G_2, 0) = d_S(T_t + C + g_\varepsilon, G_2, 0) = d_S(H_2(0,\cdot,t,\varepsilon), G_2, 0) = d_S(J, G_2, 0) \\ = 1.$$

Thus, for all $t \in (0, t_0]$, $\varepsilon \in (0, \varepsilon_0]$, we have

$$d_S(T_t + C + g_\varepsilon, G_1, 0) \neq d_S(T_t + C + g_\varepsilon, G_2, 0).$$

From the excision property of the Skrypnik degree, which is an easy consequence of its finite-dimensional appriximations, we obtain a solution $x_{t,\varepsilon} \in G_1 \setminus G_2$of $T_tx + Cx + g_\varepsilon x = 0$ for every $t \in (0, t_0]$ and every $\varepsilon \in (0, \varepsilon_0]$. We pick $t_n \in (0, t_0]$ and$\varepsilon_n \in (0, \varepsilon_0]$ be such that $t_n \downarrow 0$, $\varepsilon_n \downarrow 0$ and let $x_n \in G_1 \setminus G_2$be the corresponding solutions of $T_tx + Cx + g_\varepsilon x = 0$, i.e. $T_{t_n}x_n + Cx_n + g_{\varepsilon_n}x_n = 0$.We may assume that $x_n \rightharpoonup x_0$ and $g_{\varepsilon_n}x_n \to g^* \in X^*$.

If

$$\limsup_{n\to\infty}\langle Cx_n + g_{\varepsilon_n}x_n,\ x_n - x_0\rangle > 0,$$

then we obtain a contradiction from (i) of Lemma . Consequently

$$\limsup_{n\to\infty}\langle Cx_n + g_{\varepsilon_n}x_n,\ x_n - x_0\rangle \le 0,$$

and hence

$$\limsup_{n\to\infty}\langle Cx_n,\ x_n - x_0\rangle \le 0.$$

By the (S_+)-property of C, we obtain $x_n \to x_0 \in \overline{G_1 \setminus G_2}$. Then$Cx_n \rightharpoonup Cx_0$ and $T_{t_n}x_n \rightharpoonup -Cx_0 - g^*$. Using this in (ii) of Lemma A, we get $x_0 \in D(T)$ and $T_{t_n}x_n \rightharpoonup -Cx_0 - g^* \in Tx_0$. By a property of the selection $g_{\varepsilon_n}x_n$ (cf. [9], p. 238), we have $g^* \in G(x_0)$, and therefore $Tx_0 + Cx_0 + Gx_0 \ni 0$. We also have

$$x_0 \in \overline{G_1 \setminus G_2} = (G_1 \setminus G_2) \cup \partial(G_1 \setminus G_2) \subset (G_1 \setminus G_2) \cup \partial G_1 \cup \partial G_2.$$

But, by conditions (H1) and (H2), $x_0 \notin \partial G_1 \cup \partial G_2$. Thus, $x_0 \in D(T) \cap (G_1 \setminus G_2)$ and the proof is complete.

Application

We consider the space $X = W_0^{m,p}(\Omega)$ with the integer $m\ge 1$, the number $p\in(1,\infty)$, and the domain $\Omega \subset R^N$with smooth boundary. We let N_0 denote the number of all multi-indices $\alpha = (\alpha_1, \dots, \alpha_N)$ such that$|\alpha| = \alpha_1 + \cdots + \alpha_N \le m$. For,$\xi = (\xi_\alpha)_{|\alpha|\le m} \in \boldsymbol{R}^{N_0}$, we have a representation $\xi = (\eta, \zeta)$, where $\eta = (\eta_\alpha)_{|\alpha|\le m-1} \in \boldsymbol{R}^{N_1}$, $\zeta = (\zeta_\alpha)_{|\alpha|=m} \in \boldsymbol{R}^{N_2}$ and$N_1 + N_2 = N_0$. We let

$$\xi(u) = (D^\alpha u)_{|\alpha|\le m}, \qquad \eta(u) = (D^\alpha u)_{|\alpha|\le m-1}, \qquad \zeta(u) = (D^\alpha u)_{|\alpha|=m},$$

where

$$D^\alpha u = \prod_{i=1}^{N}\left(\frac{\partial}{\partial x_i}\right)^{\alpha_i}$$

Also, let $q = p/(p-1)$.

We now consider the partial differential operator in divergence form

$$(Au)(x) = \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^{|\alpha|} A_\alpha\big(x, u(x), \dots, D^m u(x)\big), \qquad x \in \Omega.$$

The coefficients $A_\alpha \colon \Omega \times \boldsymbol{R}^{N_0} \to \boldsymbol{R}$ are assumed to be Carathéodory functions, i.e., each $A_\alpha(x,\xi)$ is measurable in x for fixed $\xi \in \boldsymbol{R}^{N_0}$ and continuous in ξ for almost all $x \in \Omega$. We consider the following conditions:

(A1) There exist $p \in (1,\infty)$, $c_1 > 0$ and $\kappa_1 \in L^q(\Omega)$ such that

$$|A_\alpha(x,\xi) \le c_1 |\xi|^{p-1} + \kappa_1(x), \quad x \in \Omega, \qquad \xi \in \boldsymbol{R}^{N_0}, \qquad |\alpha| \le m.$$

(A2) The Leray-Lions Condition

$$\sum_{|\alpha|=m} [A_\alpha(x,\eta,\zeta_1) - A_\alpha(x,\eta,\zeta_2)](\zeta_{1_\alpha} - \zeta_{2_\alpha}) > 0\,,$$

is satisfied for every $x \in \Omega, \eta \in \boldsymbol{R}^{N_1}, \zeta_1, \zeta_2 \in \boldsymbol{R}^{N_2}$ with $\zeta_1 \neq \zeta_2$.

(A3) The momotonicity condition

$$\sum_{|\alpha|\le m} [A_\alpha(x,\xi_1) - A_\alpha(x,\xi_2)](\xi_{1_\alpha} - \xi_{2_\alpha}) \ge 0$$

is satisfied for every every $x \in \Omega, \xi_1, \xi_2 \in \boldsymbol{R}^{N_0}$.

(A4) There exist every $c_2 > 0$, $\kappa_2 \in L^1(\Omega)$ such that

$$\sum_{|\alpha|\le m} A_\alpha(x,\xi)\xi_\alpha \ge c_2|\xi|^p - \kappa_2(x), \qquad x \in \Omega, \qquad \xi \in \boldsymbol{R}^{N_0}.$$

If an operator $T \colon W_0^{m,p}(\Omega) \to W^{-m,q}(\Omega)$ is given by

$$\langle Tu, v\rangle = \int_\Omega \sum_{|\alpha|\le m} A_\alpha(x, \xi(u)) D^\alpha v\,, \qquad u, v \in W_0^{m,p}(\Omega),$$

then conditions $(A1)$, $(A3)$ imply that it is bounded, continuous and monotone (cf. e.g. Kittila [13, pp. 25-26], Pascali and Sburlan [14, pp. 274-275]. Since it is continuous, it is maximal monotone. Similarly, condition $(A1)$, with A replaced by B, implies that the operator $C \colon W_0^{m,p}(\Omega) \to W^{-m,q}(\Omega)$ defined by

$$\langle Cu, v\rangle = \int_\Omega \sum_{|\alpha|\le m} B_\alpha(x, \xi(u)) D^\alpha v\,, \qquad u, v \in W_0^{m,p}(\Omega),$$

is a bounded continuous mapping. We also know that conditions $(A1)$, $(A2)$ and $(A4)$, with B in place of A everywhere, imply that the operator C is of type (S_+) (cf. Kittila [13, p. 27]).

We also consider a multifunction$H: \Omega \times \boldsymbol{R}^{N_1} \to 2^{\boldsymbol{R}}$such that

$(A5)$ $H(x,r) = [\phi(x,r),\ \psi(x,r)]$is measurable in x and u.s.c. in r,where $\phi, \psi: \Omega \times \boldsymbol{R}^{N_1} \to \boldsymbol{R}$ are measurable functions;

$(A6)$ $|H(x,r)| = \max[|\phi(x,r)|, |\psi(x,r)|] \le a(x) + c_2|r|$a. e. on $\Omega \times \boldsymbol{R}^{N_1}$ and $a(\cdot) \in L^q(\Omega),\ c_2 > 0$.

Define $G: W_0^{m,p}(\Omega) \to 2^{W^{-m,q}(\Omega)}$ by

$$Gu = \Big\{h \in W^{-m,q}(\Omega) : \exists\ w \in L^q(\Omega) \text{such that } w(x) \in H\big(x, u(x)\big) \text{and} \langle h, v\rangle = \int_\Omega w(x)\ v(x) \text{for all } v \in W_0^{m,p}(\Omega)\Big\}.$$

It is well-known that G is u.s.c and compact with closed and convex values (cf. [9], p.254), and therefore is of class (P). We can now state the following theorem.

Theorem 2. *Assume that the operators T, C and G defined as above with T(0)=0, C(0)=0. Assume, further, that the rest of the conditions of Theorem 1 are satisfied for two balls $G_1 = B_r(0)$ and $G_2 = B_q(0)$, where $0 < q < r$.Then the Dirichlet boundary value problem*

$$\begin{aligned}&(Au)(x) + (Bu)(x) + (Hu)(x) \ni 0\,,\ \ x \in \Omega,\\ &(D^\alpha u)(x) = 0, \qquad x \in \partial\Omega, \qquad |\alpha| \le m-1,\end{aligned}$$

has a "weak" nonzero solution $u \in B_r(0) \setminus B_q(0) \subset W_0^{m,p}(\Omega)$,which satisfies the equation $Tu + Cu + Gu \ni 0$.

As noted earilier, the author and Kartsatos [1] have established similar results for densely defined operators T and C in the context of the degree theories by Kartsatos and Skrypnik [11]. In the light of recent degree theories for more general combinations of operators, such as ones in [2], the results of the paper may be extended. However, for the triplet $T + C + G$

in Theorem 1, the existence of nonzero solutions when the homogeneity condition for degree $\alpha>1$ ($p>2$ for p–Laplacian) still remains open.

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A Study on Fixed Point Theorems of Asymptotic Contractions

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Abstract
The purpose of this paper is to study some fixed point theorems of asymptotic contractions in metric space.

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Keywords: Asymptotic contraction, common fixed point, iterates, rateof convergence.

1. Introduction

Asymptotic fixed point theory deals with conditions describing a behavior of iterates of a mapping. In 1922, S. Banach [17] established a contraction mapping theorem in metric space. In 1930, R. Caccioppoli -[16] introduced the concept of asymptotic contraction based on Banach Contraction Principle. In 1962, E. Rakotch [6] probably was the first for the extension of milder form of Banach Contraction Principle for the contraction constant. In 1962, D. W. Boyd and J. S. Wong[4] obtained a more general condition. In 1986, M. R. Taskovic [13] established some fixed point theorems on topological space. In 2003, W.A Kirk [23] introduced an asymptotic version of Boyd - Wong Contraction. Since then, many extensions of weaker forms including fixed point theorems have been established.

The propose of this paper is to present a brief survey work on some fixed point results of asymptotic contraction in metric space.

2. Preliminaries

Now, we start with the following definitions.

Definition 1. *[13] A metric space (X, d) satisfies the condition of TCS- convergence iff $x \in X$ and if $d(T^n x, T^{n+1} x) \to 0$ as $n \to \infty$ implies that $\{T^n x\}_{n \in \mathbb{N}}$ has a convergent subsequence.*

Definition 2. *[13] The set $O_T(x) = \{x, Tx, T^2x, ...\}$ is called the orbit of x.*

Definition 3. *[13] A function $f : X \to \mathbb{R}$ is T-orbitally lower semicontinuous at the point p iff for all sequences $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \to p$ implies that $f(p) \le \liminf_{n\to\infty} f(x_n)$*

Definition 4. *[13] A mapping $T : X \to X$ is said to be orbitally continuous if $\xi, x \in X$ are such that ξ is a cluster point of O_T then $T(\xi)$ is a cluster point of $T(O_T(x))$.*

Definition 5. *[23] Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be asymptotic contraction if,*

$d(T^n x, T^n y) \le \phi_n(d(x, y))$ for all $x, y \in X$ where

$\phi_n : [0,\infty) \to [0,\infty)$ *and* $\phi_n \to \phi \in \Phi(1)$

uniformly on the range of d, where Φ is the class of functions.

Definition 6. *[15] A function $T : X \to X$ on a metric space (X, d)is called an asymptotic contraction if for each $b > 0$ there exists a moduli $\eta^b : (0,b] \to (0,1)$ and $\beta^b : (0,b] \times (0,\infty) \to \mathbb{N}$ and the following hold :*

1. there exists a sequence of functions $\phi_n : (0,\infty) \to (0,\infty)$ such that for all $x, y \in X$, for all $\varepsilon > 0$ and for all $n \in \mathbb{N}$ we have
$$b \ge d(x,y) \ge \varepsilon \to d(T^n x, T^n y) \le \phi_n(\varepsilon).d(x,y)$$
2. for each $0 < l \le b$ the function $\beta_l^b := \beta^b(l,.)$ is a modulus of uniform convergence for ϕ_n on [l, b], i.e.,
$\forall \varepsilon > 0 \quad \forall s \in [l,b] \quad \forall m, n \ge \beta_l^b(\varepsilon)(|\phi_m(s) - \phi_n(s)| \le \varepsilon)$, and
3. defining $\phi := \lim_{n\to\infty} \phi_n$, then for each $\varepsilon > 0$ we have that $\eta^b(\varepsilon) > 0$ and $\phi(s) + \eta^b(\varepsilon) \le s$ for each $s \in [\varepsilon, b]$,

where there is no ambiguity, superscript b from the moduli η^b, β^b are removed.

Definition 7. *[18] A function ϕ from $[0,\infty)$ into itself is called an L-function if $\phi(0) = 0$, $\phi(s) > 0$ for $s \in (0,\infty)$, and for every $s \in (0,\infty)$ there exists $\delta > 0$ such that $\phi(t) \le s$ for all $t \in [s, s+\delta]$.*

Definition 8. *[19] Let (X, d) be a metric space. Then, a mapping T on X is said to be an asymptotic contraction of Meir-Keeler type (ACMK, for short) if there exists a sequence $\{\phi_n\}$ of functions from $[0,\infty)$ into itself satisfying the following:*

1. $\limsup_n \phi_n(\varepsilon) \le \varepsilon$ for all $\varepsilon \ge 0$,

2. for each $\varepsilon > 0$, there exists $\delta > 0$ and $\nu \in \mathrm{N}$ such that $\phi_\nu \leq \varepsilon$ for all $t \in [\varepsilon, \varepsilon + \delta]$, and
3. $d(T^n x, T^n y) < \phi_n(d(x,y))$ for all $n \in \mathrm{N}$ and $x, y \in X$ with $x \neq y$.

Definition 9. *[20] Let (X, d) be a metric space. Then, a mapping T on X is said to be an asymptotic contraction of final type (ACF, for short) if the following hold:*

1. $lim_{\delta \to +0} sup\{lim_{n \to \infty} sup d(T^n x, T^n y : d(x,y)) < \delta\} = 0$,
2. for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$ with $\varepsilon < d(x,y) < \varepsilon + \delta$, there exists $\nu \in \mathrm{N}$ such that $d(T^\nu x, T^\nu y) \leq \varepsilon$.
3. for x, y∈ X, with $x \neq y$,there exits $\nu \in \mathrm{N}$ such that $d(T^\nu x, T^\nu y) < d(x,y)$.
4. for $x \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ and $\nu \in \mathrm{N}$ such that

   $\varepsilon < d(T^i x, T^j x) < \varepsilon + \delta$ implies $d(T^\nu \circ T^i x, T^\nu \circ T^j x) \leq \varepsilon$ for all $i, j \in \mathrm{N}$.

Definition 10. *[5] Let Φ be the class of functions $\phi : [0,\infty) \to [0,\infty)$ with the properties*

(i) ϕ is the Lebesque - integrable on each interval [0, a) with $a > 0$, and

(ii) $\int_0^\varepsilon \phi(t) > 0$ for each $\varepsilon > 0$.

Let (X, d) be a metric space. Then, a mapping T on X is said to be an asymptotic contraction of integral Meir - Keeler type (ACIMK, for short) if there exists a sequence $\{\phi_n\}$ of functions from $[0,\infty)$ into itself satisfying the following,

1. $limsup_{n \to \infty} \phi_n(\varepsilon) \leq \varepsilon$ for all $\varepsilon > 0$,
2. for each $\varepsilon > 0$ there exists a $\delta > 0$ and $s \in \mathrm{N}$ such that $\phi_s(t) \leq \varepsilon$ for all $t \in [\varepsilon, \varepsilon + \delta]$, and
3. $\int_0^{d(T^n x, T^n y)} \psi(t)dt < \phi_n(\int_0^{d(x,y)} \psi(t)dt)$ for all $n \in \mathrm{N}$ and $x, y \in X$ with $x \neq y$ where $\psi \in \Psi$.

3 Main Theorems

In 1962, E. Rakotch obtained the following theorem.

Theorem 1. *[6] Let X be a complete metric space and suppose $T : X \to X$ satisfies $d(T(x), T(y)) \leq \alpha(d(x,y))d(x,y)$ for each $x, y \in X$ where $\alpha : [0,\infty) \to [0,1)$ is*

monotonically decreasing then T has a unique fixed point x_* and $\{T^n x\}$ converges to x_* for each $x \in X$.

In 1969, D.W. Boyd and J.S. Wong established a more general result on contraction mapping theorem in metric space which is as follows.

Theorem 2. *[4] Let (X, d) be a complete metric space. Let $T: X \to X$ be a function satisfying $d(Tx,Ty) \le \phi(d(x,y))$ for each $x, y \in X$ where $\phi:[0,\infty) \to [0,\infty)$ such that $\phi(t) < t$ for all $t > 0$ and ϕ is uppersemicontinuous from the right, then T has a unique fixed point x_* for each $x \in X$ and $\{T^n x\}$ converges to x_* for each $x \in X$.*

In this theorem, it is assumed that $\phi:[0,\infty) \to [0,\infty)$ is uppersemicontinuous from the right(i.e $r_j \downarrow r \ge 0 \Rightarrow limsup_{n\to\infty}\phi(r_j) \le \phi(r)$).

In 1986, M. R. Taskovic established the following results in topological space.

Theorem 3. *[13] Let T be a mapping of topological space X:= (X, d) into itself, where X satisfies the condition of TCS- convergence. Suppose that there exists a sequence of nonnegative real functions $\{\alpha_n(x,y)\}_{n\in\mathbb{N}}$ such that $\alpha_n(x,y) \to 0$ and a positive integer $m(x,y)$ such that*

$$d(T^n x, T^n y) \le \alpha_n(x,y) \quad forall \quad n \ge m(x,y) \tag{2}$$

and for all $x, y \in X$ where $d: X \times X \to \mathbb{R}^+$. If $x \mapsto d(x,Tx)$ is a T-orbitally continuous and $d(a,b) = 0$ implies $a = b$, then T has a unique fixed point $\xi \in X$ and $T^n x \to \xi$ for each $x \in X$.

As a localization of condition (2) of above theorem, we have the following theorem.

Theorem 4. *[13] Let T be a mapping of topological space X:= (X, d) into itself, where X satisfies the condition of TCS- convergence. Suppose that there exists a sequence of nonnegative real functions $\{\alpha_n(x,y)\}_{n\in\mathbb{N}}$ such that $\alpha_n(x,Tx) \to 0$ and a positive integer $m(x)$ such that*

$$d(T^n x, T^{n+1} y) \le \alpha_n(x,Tx) \quad forall \quad n \ge m(x)$$

and for every $x \in X$ where $d: X \times X \to \mathbb{R}^0_+$. If $x \mapsto d(x,Tx)$ is a T-orbitally lower semicontinuous or T is orbitally continuous and $d(a,b) = 0$ implies $a = b$, then T has at least one fixed point in X.

In 2003, W.A. Kirk obtained a result which is asymptotic version of the Boyd and Wong.The concept of asymptotic contractions is suggested by one of the earliest

version of Banach's contraction principle attributed to Cacciopoli [16] whose result asserts that if X is a complete metric space then the Picard iterates of a mapping $T: X \to X$ converges to the unique fixed point of T provided for each $n \geq 1$ there exits a constant c_n such that,

$$d(T^n x, T^n y) \leq c_n d(x,y) \quad x, y \in X \quad with \quad \sum_{n=1}^{\infty} c_n < \infty . \qquad (3)$$

Theorem 5. *[23] Let (X, d) be a complete metric space and suppose $T: X \to X$ is an asymptotic contraction for which the mappings ϕ_n in (1) are continuous. Assume also that some orbit of T is bounded. Then T has a unique fixed point $z \in X$ and moreover the picard sequence $\{T^n x\}_{n=1}^{\infty}$ converges to z for some $x \in X$.*

In 2004, J. Jachymski and I. Jozwik [10] extended and gave a constructive proof of Kirk. They obtained a complete characterisation of asymptotic contraction on a compact metric space. As a by-product, they have established a separation theorem for upper semicontinuous functions satisfying some limit conditions with suitable example.

Theorem 6. *[10] Asume that (X,d) is complete metric space and T is a continuous selfmap of X. Then, the following statements are equivalent:*

1. T is an asymptotic contraction;
2. the core $Y := \cap_{n \in N} T^n(X)$ is a sigleton;
3. T is an asymptotic ϕ_0 contraction, where $\phi_0(t) := 0$ for all $t \in \mathsf{R}$
4. T is a Banach contraction under some metric equivalent to d.

In 2004, Y-Z Chen proved the theorem of Kirk under weaker assumptions without the use of ultrafilter methods. Kirk's paper assumes the continuity for ϕ and all ϕ_n, but Chen assumes the upper semicontinuity of ϕ and one of the $\phi_n{}'s$ which is weaker and easier to check.

Theorem 7. *[24] Suppose that (X, d) is a complete metric space and suppose $T: X \to X$ such that,*

$$d(T^n x, T^n y) \leq \phi_n(d(x,y)) \quad for\ all\ x, y \in X\ where\ \phi_n : [0,\infty) \to [0,\infty),$$

and $\phi_n \to \phi$ uniformly on any bounded interval [0, b]. Suppose that ϕ is an uppersemicontinuous and $\phi(t) < t$ for $t > 0$. Furthermore, suppose there exists a positive integer n_* such that ϕ_{n_*} is uppersemicontinuous and $\phi_{n_*}(0) = 0$. If there exists $x_0 \in X$ which has a bounded orbit $O(x_0) = \{x_0, Tx_0, T^2 x_0, ...\}$ then T has a unique fixed point $x_* \in X$ such that

$$lim_{n \to \infty} T^n x = x_* \quad \forall x \in X .$$

In 2004, P. Gerhardy [15] using techniques from proof mining, developed a variant of the notion of asymptotic contraction and established a quantitative version of the corresponding fixed point theorem. Using techniques from proof mining as developed in [22, 21], he first derived a suitable generalization of the notion of asymptotic contractivity and subsequently established an elementary proof of Kirk's fixed point theorem providing an explicit rate of convergence (to the unique fixed point) for sequences $\{T^n x\}$.

In detail, he has shown that

1. the rate of convergence only depends on the starting point x via a bound on the iteration sequence $\{T^n x\}$,
2. the rate of convergence only depends on the function T via suitable moduli expressing its asymptotic contractivity, and
3. only the continuity of T is necessaryto prove the existence of a unique fixed point, while the convergence to such a fixed point can be proved without the continuity of T

Theorem 8. *[15] Let (X, d) be a metric space, let T be an asymptotic contraction and let $b>0$ and η, β are given. Assume that T has a unique fixed point z. Then for every $\varepsilon>0$ and every $x_0 \in X$ s.t. $\{x_n\}$ is bounded by b and $d(x_n, z) \le b \; \forall n \; \exists$ an $m \le M$ s.t.*

$d(x_m, z) \le \varepsilon$, where

$$M(\eta,\beta,\varepsilon,b) = k\lceil (\frac{lg(\delta)-lg(b)}{lg(1-\frac{n(\delta)}{2})})\rceil, \quad k=\beta_\delta(\frac{\eta(\delta)}{2}), \quad \delta = \frac{\eta(\varepsilon).\varepsilon}{4}$$

Theorem 9. *[15] Let (X, d) be a complete metric space, let T be a continuous asymptotic contraction and let $b>0$ and η, β be given. If for some $x_0 \in X$ the sequence $\{x_n\}$ is bounded by b then T has a unique fixed point z, $\{x_n\}$ converges to z and for every $\varepsilon>0$ there exists an $m \le M$ s.t. $d(x_m, z) \le \varepsilon$, where M is as in above Theorem*

In 2006, T. Suzuki [19] introduced the notion of asymptotic contraction of Meir-Keeler type and established a fixed point theorem for such contractions which is generalization of fixed point theorems of Meir-Keleer [1] and Kirk [23]. Also, T. Suzuki has used the characterization of Meir- Keeler contraction proved by Lim [18].

Theorem 10. *[19] Let (X, d) be a complete metric space. Let T be an ACMK on X. Assume that T^k is continuous for some $k \in \mathbb{N}$. Then, there exists a unique fixed-point $z \in X$. Moreover, we have $lim_n T^n x = z$ for all $x \in X$*

In 2007, T. Suzuki introduced a more generalized notion of asymptotic contraction of final type (ACF,for short) and established fixed point theorems for such contractions.

Theorem 11. *[20] Let T be an ACMK on a metric space (X, d). Then, T is an ACF.*

Theorem 12. *[20] Let (X, d) be a complete metric space and let T be an ACF on X. Assume that the following holds*

if $u \in X$ and $lim_n T^n u = v$, then $\exists \ell \in \mathbb{N}$ such that $T^\ell v = v$.

Then, there exists a unique fixed point $z \in X$ of T. Moreover, we have $lim_n T^n x = z$ holds for every $x \in X$.

Theorem 13. *[20] Let (X, d) be a complete metric space and let T be an ACF on X. Assume that T^ℓ is continuous for some $\ell \in \mathbb{N}$. Then, there exists a unique fixed point $z \in X$ of T. Moreover $lim_n T^n x = z$ holds for every $x \in X$.*

In 2007, M. Arav, F. E. Castillo Santos, S. Reich, and A. J.Zaslavski provided sufficient condition for the iterates of an asymptotic contraction on a complete metric space X to converge to its unique fixed point uniformly on each bounded subset of X. They improved the theorem of Chen [24] and established more general result.

Theorem 14. *[2] Let $x_* \in X$ be a fixed point of $T : X \to X$. Assume thàt*

$d(T^n x, x_*) \le \phi_n(d(x, x_*))$ $\forall x \in X$ and all natural numbers n, where $\phi_n : [0,\infty) \to [0,\infty)$ and $\phi_n \to \phi$ uniformly on any bounded interval [0, b]. Suppose that ϕ is an uppersemicontinuous and $\phi(t) < t$ for $t > 0$ then $lim_{n\to\infty} T^n x = x_*$, uniformly on each bounded subset of X.

Theorem 15. *Let $T : X \to X$ such that*

$$d(T^n x, T^n y) \le \phi_n(d(x, y))$$

for all $x, y \in X$ and all the natural numbers n, where $\phi_n : [0,\infty) \to [0,\infty)$ and $\lim_{n\to\infty} \phi_n = \phi$, uniformly on any bounded interval [0, b]. Suppose that ϕ is uppersemicontinuous and that $\phi(t) < t$ for all $t > 0$. Furthermore, suppose that there exists a positive integer n_* such that ϕ_{n_*} is uppersemicontinuous and $\phi_{n_*}(0) = 0$. If there exists $x_0 \in X$ at which it has a bounded orbit, then T has a unique fixed point $x_* \in X$ and $\lim_{n\to\infty} T^n x = x_*$, uniformly on each bounded subset of X.

In 2007, I. D. Arandelovic [9] established a fixed point theorem of Kirk's type unifying and generalizing the results of Jachymski and Jozwik [10], W.A. Kirk [23], and Chen [24].

Theorem 16. *[9] Let (X,d) be a complete metric space, $T: X \to X$ continuous function and (ϕ_i) sequence of functions such that $\phi_i : [0,\infty) \to [0,\infty)$ and for each $x, y \in X$*

$$d(T^i x, T^i y) < \phi_i(d(x,y)).$$

Assume also that there exists uppersemicontinuous function $\phi : [0,\infty) \to [0,\infty)$ such that for any $r > 0$, we have $\phi(r) < r, \psi(0) = 0$ and $\phi_i \to \psi$ uniformly on any bounded interval [0, b]. If one of the following conditions is satisfied:

1. there exists $x \in X$ such that the orbit of T at x is bounded; or
2. $\underline{lim}_{t\to\infty}(t - \phi(t)) > 0$, or;
3. $\overline{lim}_{t\to\infty}\frac{\phi(t)}{t} < 1$

then T has a unique fixed point $y \in X$ and all sequences of Picard iterates defined by T converges to y, uniformly on each bounded subset of X.

In 2007, the results established by E. M. Briseid [7] are based on the analysis of Kirk's fixed point theorem for asymptotic contractions given by Gerhardy. He had proved fixed point theorems on asymptotic contractions which give an explicit rate of convergence to the fixed point for any sequence $\{T^n x\}$ without assuming that T is nonexpansive. This amounts to a fully effective version of Kirk's theorem on asymptotic contractions with an elementary proof. The rate of convergence depends on the space, the mapping and the starting point through a bound on the iteration sequence and some moduli for the mapping appearing as parameters, but is otherwise fully uniform. A weaker result is guaranteed by novel application of the logical metatheorem due to Kohlenbach [22] in the case where function is also non expansive through a rate of proximity. As a by product of the uniformity feature of the analysis, he also obtained characterization of asymptotic contractions in the sense of Kirk on nonempty, bounded, complete metric spaces, obtaining that they are exactly the mappings for which every Picard iteration sequence converges to the same point with a rate of convergence which is uniform in the starting point.

In 2007, K.P.R. Sastry, G.V.R. Babu, S. Ismail and M. Balaiah [11] established a fixed point theorem with hypothesis slightly different from that of Chen [[24], Theorem 2.2].

In 2011, B. D. Rauhani and J. Love [3] introduced the weaker condition $liminf_{n\to\infty} d(x, T^n x) = 0$ for some x in X, and proved that this condition implies the existence of a fixed point and the convergence of the Picard iterates to this fixed point.

Theorem 19. *Let (X, d) be a complete metric space. Let $T: X \to X$ such that $d(T^n x, T^n y) \le \phi_n(d(x,y))$ for all $x, y \in X$ where $\phi_n : [0,\infty) \to [0,\infty)$ and $\phi_n \to \psi$*

uniformly on any bounded interval [0, b]. Suppose that ϕ *is uppersemicontinuous and* $\phi(t) < t$ *for* $t > 0$ *and assume that there is a positive integer* n^* *such that* ϕ_{n^*} *is uppersemicontinuous and* $\phi_{n^*}(0) = 0$. *If* $liminf_{n\to\infty} d(x, T^n x) = 0$, *then T has a unique fixed point* $x \in X$ *,and we have* $lim_{n\to\infty} T^n y = x$ *for all* $y \in X$.

In 2012, E. C.anzoneri and P. Vetro introduced the notion of asymptotic contraction of integral Meir-Keeler type on a metric space and proved a theorem which ensures existence and uniqueness of fixed points for such contractions.

Theorem 20. *[5] Let (X, d) be a complete metric space and T be an ACIMK on X. Assume that* T^m *is continuous for some* $m \in \mathbb{N}$. *Then there exists a unique fixed point* $z \in X$. *Moreover, we have* $lim_{n\to\infty} T^n x = z$ *for all* $x \in X$.

Remark: On the basis of the above theorems, we can observe that weaker forms of contractive conditions for the existence and uniqueness of fixed point are rapidly being developed. The condition of mappings to be continuous is necessary for the existence of fixed point but not for the convergence to such a fixed point. The rate of convergence depends upon the space, mapping and the starting point through a bound on iteration sequence. The notion of asymptotic contraction has been developed towards Boyd- Wong type and Meir -Keeler type conditions with suitable applications to rate of convergence.

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On The Regularity of Weak Solution to Three Dimensional Boussinesq Equations

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Abstract
The global regularity problem concerning three dimensional Boussinesq equations remains an outstanding open problem in fluid dynamics. The regularity of the weak solution for three dimensional Boussinesq equations is studied in this paper. We prove that if $\int_0^T \| \partial_i u_3(s) \|_\alpha^\beta ds < \infty$, then the solution to three dimensional Boussinesq equations can be extended in $[0, T+\in)$.

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Key words and phrases: . 3D Boussinesq equations, global, regularity.

1. Introduction

The study of fluid dynamics is of great interest from both the mathematical as well as the physical point of view. A number of mathematical models have been proposed to describe the natural phenomena like atmospheric & oceanic flows, geophysical flows, and electrically conducting flows. Boussinesq system of equations is widely used to model large scale atmospheric & oceanic flows such as tornadoes, cyclones, and hurricanes. It describes the dynamics of fluid under the influence of gravitational force. This system is one of the well known models that describe the geophysical flows as well as other astrophysical situations where the stratification of the medium and the rotation of the earth plays a dominant role.

The paper is concerned with a regularity criterion for the weak solution to the three dimensional Boussinesq equations. The three dimensional Boussinesq equations can be written as:

$$\begin{cases} \partial_t u + (u.\nabla)u = \mu\Delta u - \nabla p + \theta\overrightarrow{e_3}, \\ \partial_t\theta + (u.\nabla)\theta = K\Delta\theta, \\ \nabla.\ u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^3$, $t > 0$, $u = (u_1, u_2, u_3)$ is the velocity field, $\theta = \theta(x,t)$ the scalar temperature, $p(x,t)$ the pressure, $\mu > 0$ is the kinematic viscosity, $k > 0$ the thermal diffusivity, and $\overrightarrow{e_3} = (0.0,1)^T$. For simplicity, we consider $\mu = k = 1$.

When $\theta = 0$ in (1.1), then this system reduces to the Navier-Stokes Equations. The global regularity or finite time singularity for the three-dimensional Navier-Stokes equations is the most challenging open problem.

In fluid dynamics [22]. In fact, this is one million dollar prize problem announced by Clay Mathematics Institute [14]. Various efforts have been made by mathematicians, physicists, and engineers, but the mystery is still there. For two dimensional Boussinesq equations with full dissipation and thermal diffusion, the global (in time) regularity have been established. There are numerous papers regarding the global regularity of two-dimensional boussinesq Equations ([3, 7, 9, 10, 11, 12, 13, 15, 17, 18, 20, 21] and references therein). However, the global regularity issue for three dimensional Boussineq Equations is an outstanding open problem. There have been numerous papers related to 3D Navier-Strokes equations about the regularity criterion. Recently Cao and Titi found the regularity criteria in terms the one component of the velocity gradient [8]. The natural question is the extension of this result to the 3D Boussinesq equations. With this motivation, we study the regularity issues of the three dimensional Boussisesq Equations.

For three dimensional Boussinesq Equations, Ishimura and Morimoto [19] proved the following Bale-Kato-Majda type regularity criterion.

$\nabla u \in L^1(0,T;L^\infty)$

In [23], the authors proved the regularity criterion for weak solution to the 3D Boussinesq equations. More precisely the solution to 1.1 can be extended beyond t=T provided that $\int_0^T \| u_z \|_\alpha^\beta$, where $\frac{3}{\alpha} + \frac{2}{\beta} \leq 1$ and $\alpha \geq 3$.

In this paper, we follow the method of Cao and Titi [8]. They proved the regularity criteria for 3D Navier-Stokes equations. Here we apply their results to 3D Boussinesq equations and prove the same regularity condition. In fact, this is a simple observation of their result. Since the Boussinesq Equations have the thermal diffusivity, which is absent in Navier-Stokes equations. In order to apply the result in [8] for 3D boussinesq Equations, we need to bound the terms involving thermal diffusivity. More precisely, we prove the following theorem.

Theorem1.1 *Let* $(u_0,\theta_0)\in H^1(\mathbb{R}^3), \nabla.\ u_0=0$. *Let* (u,θ) *be the weak solution to the 3D MHD equations. Let* $T>0$, *if*

$$\Theta=\int_0^T \|\partial_1 u_3(s)\|_\alpha^\beta\, ds<\infty, \tag{1.2}$$

where $\alpha>3,\ 1\le\beta<\infty$ *and* $\dfrac{3}{\alpha}+\dfrac{2}{\beta}\le\dfrac{\alpha+3}{2\alpha}$

then (u,θ) *can be extended to the time interval* $[0,T+\in)$ *for some* $\in>0$.

2. Preliminary

Throughout this paper, the following notations will be used.

- C is a harmless constant which may have different values in different steps.
- For every $p\in[1,\infty], \|\cdot\|_{L^p}$ or $\|\cdot\|_p$ denotes the norm in the Lebesgue Space L^p.
- $\partial_i f=\dfrac{\partial f}{\partial x_i}, \partial_{ij}^2 f=\dfrac{\partial^2 f}{\partial x_i \partial x_j}.\ i=1,2,3\ and\ j=1,2,3.$
- $\nabla_h u=(\partial_1 u,\partial_2 u,0)\ and\ \nabla_h u=\partial_{11}^2 u+\partial_{22}^2 u.$

Definition 2.1 (L^p space). For $1\le p\le\infty$, $L^p(\mathbb{R}^n)$ is the space of functions such that

$$\|u\|_{L^p}=\|u\|_p=\begin{cases}\left(\int_{\mathbb{R}^n}|u(x,\mathrm{T})|^p\,dx\right)^{\frac{1}{p}} & if \quad 1\le p<\infty,\\ ess\sup_{x\in\mathbb{R}^n}|u|(x,\mathrm{T}), & if \quad p=\infty,\end{cases}$$

is finite.

Definition 2.2 (Sobolev Space). *Let $\Omega \subset \mathbb{R}^d$ be an open set, for integer $k \geq 0$ and $1 \leq p \geq \infty$. The Sobolev space $W^{k,p}(\Omega)$ consists of the function $f \in L^p(\Omega)$ that have weak derivatives $D^\alpha f \in L^p(\Omega)$ of all orders $|\alpha| \leq k$. The norm is defined as*

$$\| f \|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha| \leq k} \| D^\alpha f \|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} & if \quad 1 \leq p < \infty, \\ \max_{|\alpha| \leq k} \| D^\alpha f \|_{L^\infty(\Omega)} & if \quad p = \infty \end{cases}$$

When $p = 2$ we write $W^{k,2} = H^k$.

Definition 2.3 *($H^k -norm$) For any $k \in \mathbb{R}$, the $H^k -norm$ is equivalently defined as*

$$\| f \|_{H^k} = \int_{\mathbb{R}^d} \left(1 + |\xi|\right)^2 \left| \hat{f}(\xi) \right|^2 d\xi$$

Definition 2.4 *(Weak Solution). A pair (u, θ) is called a weak solution of (1.1) with $u^0 \in L^2(\mathbb{R}^3), \theta_0 \in L^1 \cap L^\infty(\mathbb{R}^3)$ $\nabla . u_0 = 0$ for $T > 0$ provided that (u, θ) satisfies*

(1) $(u, \theta) \in L^\infty(0, T; L^2(\mathbb{R}^3) \cap L^2(0, T; H^1(\mathbb{R}^3)), \theta \in L^\infty(0, T; L^1 \cap L^\infty(\mathbb{R}^2))$ with $\nabla . u = 0$ in the sense of distribution.

(2) u and θ satisfy equation (1.1) in the sense of distribution.

(3) the energy inequality

$$\| u(t) \|_2^2 + \int_0^t \| \nabla_u(\tau) \|_2^2 \, d\tau \leq \| u_0 \|_2^2 + \int_0^t \int_{\mathbb{R}^3} \theta \, \overrightarrow{e_3} dx d\tau$$

$$\| \theta(t) \|_2^2 + \int_0^t \| \nabla \theta(\tau) \|_2^2 \, d\tau \leq \| \theta_0 \|_2^2 \text{ for all } t \leq T.$$

We would like to recall some lemmas (see [8] and [6] for details)

Lemma 2.5. For every $2 < r < 3$, we have the following inequalities

$$\int_{\mathbb{R}3} \phi f \psi dx \leq C \| \phi \|_2^{\frac{r-1}{r}} \| f \|_2^{\frac{r-2}{r}} \| \psi \|_2 \| \partial_3 \phi \|_{\frac{r}{3-r}}^{\frac{1}{2}} \| \partial_1 f \|_2^{\frac{1}{r}} \| \partial_2 f \|_2^{\frac{1}{2}} \tag{2.1}$$

$$\int_{\mathbb{R}3} \phi f \psi dx \leq C \| \phi \|_2^{\frac{r-1}{r}} \| f \|_2^{\frac{r-2}{r}} \| \psi \|_2 \| \partial_2 \phi \|_{\frac{r}{3-r}}^{\frac{1}{2}} \| \partial_1 f \|_2^{\frac{1}{r}} \| \partial_2 f \|_2^{\frac{1}{2}} \tag{2.2}$$

$$\int_{\mathbb{R}3} \phi f \psi dx \leq C \| \phi \|_2^{\frac{r-1}{r}} \| f \|_2^{\frac{r-2}{r}} \| \psi \|_2 \| \partial_1 \phi \|_{\frac{r}{3-r}}^{\frac{1}{2}} \| \partial_1 f \|_2^{\frac{1}{r}} \| \partial_2 f \|_2^{\frac{1}{2}} \tag{2.3}$$

We will apply the first inequality for $r = \dfrac{3\alpha - 2}{\alpha}$.

In order to prove the main theorem, we need the global L^2-bound. More precisely, we prove the following lemma.

Lemma 2.6. *Let* (u,θ) *be a smooth solution of (1.1) then*

$$\|\theta\| L^p \le \|\theta_0\| p, \quad \text{for any } p \in [1,\infty] \tag{2.4}$$

$$\|u\|_2^2 + 2\int_0^t \|\nabla u\|_2^2 \le (\|u_0\|_2 + \|\theta_0\|_2 T)^2 \tag{2.5}$$

for any $t \ge 0$.

Proof. Multiplying the second equation of (1.1) by $\theta|\theta|^{2p-2}$, integrating in space and integrating by parts, we obtain

$$\|\theta(t)\|_p^p + p(p-1)\int_0^t \|\nabla \theta|\theta|^{\frac{p-2}{2}}(\tau)\|_2^2\, d\tau = \|\theta_0\|_p^p$$

Since, $\int_{\mathbb{R}^3} (u.\nabla)\theta.\theta|\theta|^{2p-2} d\tau = 0$ by divergence free condition. We immediately get

$$\|\theta(t)\|_p \le \|\theta_0\|_p \qquad t \in [0,T],\ p \in [1,\infty)$$

At $p = \infty$ by maximum principle $\|\theta(t)\|_\infty \le \|\theta_0\|_\infty$.

To prove the second inequality, multiplying the first equation by u, integrating in space, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \|\nabla u\|_2^2 &\le \|\theta\|_2 \|u\|_2 \\ &\le (\|\theta_0\|_2^2 + \|u\|_2^2) \end{aligned}$$

Therefore, by Gronwall's inequality we obtain the desired bound,

$$\|u\|_2^2 + \int_0^t \|\nabla u\|_2^2\, ds \le K(t).$$

3. Proof of the Theorem

It is well known that there exists a unique local strong solution to 3D Boussinesq equations. For $(u_0, \theta_0) \in H^1$ with $\nabla . u_0 = 0$, the week solution is the same as the strong solution in short interval *(0,T)*. If we can find a priori uniform H^1- bound in *(0,T)* for the strong solution with the regularity condition of our main theorem then the solution be extended by a standard process. Thus the main theorem is reduced to establish the uniform H^1- bound for such strong solution.

Proof. Multiplying the first equation of (1.1) by $-\Delta_h u$ and the second equation of (1.1) by $-\Delta_h \theta$ and integrating with respect to space variable yields.

$$\frac{1}{2}\frac{d}{dt}(\|\nabla_h u\|_2^2+\|\nabla_h\theta\|_2^2)+(\|\nabla_h\nabla_u\|_2^2+\|\nabla_h\nabla\theta\|_2^2)=$$
$$\int(u.\nabla)u\ .\ \Delta h^u dx-\int\theta\ \overrightarrow{e_3}\ .\ \Delta_h u dx$$
$$+\int(u\ .\ \nabla)\theta\ .\ \Delta_h\theta dx\ = I_1+I_2+I_3$$

The following bound for (I_1) can be found in [8].

$$I_1=\int(u.\nabla)u.\Delta_h u dx=\sum_{j,k=1}^{3}\sum_{i=1}^{2}\int_{\mathbb{R}^3}\partial_i u_j\partial_j u_k\partial_i u_k dx$$
$$=\sum_{k=1}^{3}\sum_{i=1}^{2}\int_{\mathbb{R}^3}[\partial_i u_1\partial_1 u_k\partial_i u_k+\partial_i u_2\partial_2 u_k\partial_i u_k+\partial_i u_3\partial_3 u_k\partial_i u_k]dx$$
$$=\sum_{k=1}^{2}\sum_{i=1}^{2}\int_{\mathbb{R}^3}[\partial_i u_1\partial_1 u_k\partial_i u_k+\partial_i u_2\partial_2 u_k\partial_i u_k]dx+\int_{\mathbb{R}^3}[\partial_i u_1\partial_1 u_3\partial_i u_3+\partial_i u_2\partial_2 u_3\partial_i u_3]dx+$$
$$\sum_{k=1}^{3}\sum_{i=1}^{2}\int_{\mathbb{R}^3}[\partial_i u_3\partial_3 u_k\partial_i u_k]dx$$
$$=I_{11}+I_{12}+I_{13}$$

Clearly

$$|I_{12}|+|I_{13}|\le C\int_{\mathbb{R}^3}|u_3||\nabla\nabla_h u||\nabla u|dx$$
$$I_{11}=\sum_{k=1}^{2}\sum_{i=1}^{2}\int_{\mathbb{R}^3}[\partial_i u_1\partial_1 u_k\partial_i u_k+\partial_i u_2\partial_2 u_k\partial_i u_k]dx$$
$$=\int_{\mathbb{R}^3}[(\partial_1 u_1)^3+(\partial_2 u_2)^3+\left((\partial_1 u_2)^2+(\partial_2 u_1)^2\right)(\partial_1 u_1+\partial_2 u_2)dx$$
$$\le\int_{\mathbb{R}^3}|u_3||\nabla\nabla_h u||\nabla u|dx$$

In above inequality we apply divergence free condition $\partial_1 u_1+\partial_2 u_2=-\partial_3 u_3$.

Collecting all inequalities, we obtain

$$I_1\le C\int_{\mathbb{R}^3}|u^3||\nabla\nabla_h u||\nabla|dx$$

Since , $\alpha>3$, so $2<\frac{3\alpha-2}{\alpha}<3$. We can use (2.1) with $\phi=|u_3|, f=|\nabla u|, \psi=|\nabla_h\nabla_u|$ and $r=\frac{3\alpha-2}{\alpha}$ and Holder's inequality, we obtain

$$I_1\quad\le C\|u_3\|_2^{\frac{2(\alpha-1)}{3\alpha-2}}\|\partial_1 u_3\|_\alpha^{\frac{\alpha}{3\alpha-2}}\|\nabla u\|_2^{\frac{(\alpha-2)}{(3\alpha-2)}}\|\partial_2\nabla u\|_2^{\frac{\alpha}{(3\alpha-2)}}\|\partial_3\nabla u\|_2^{\frac{\alpha}{(3\alpha-2)}}\|\nabla\nabla_h u\|_2$$
$$\le C\|u_3\|_2^2\|\partial_1 u_3\|_\alpha^{\frac{\alpha}{\alpha-1}}\|\nabla u\|_2^{\frac{\alpha-2}{\alpha-1}}\|\partial_3\nabla u\|_2^{\frac{\alpha}{\alpha-1}}+\frac{1}{4}\|\nabla\nabla_h u\|_2^2 .$$

$$\leq C\|\partial_1 u_3\|_{\alpha}^{\frac{\alpha}{\alpha-1}}\|\nabla u\|_2^{\frac{\alpha-2}{\alpha-1}}\|\partial_3\nabla u\|_2^{\frac{\alpha}{\alpha-1}}+\frac{1}{4}\|\nabla\nabla_h u\|_2^2$$

We can bound I_2 and I_3 in the following way.

$$I_2=\int\theta\overline{e_3}.\Delta_h u=-\int\nabla_h\theta.\nabla_h u\leq\|\nabla_h\theta\|_2^2+\|\nabla_h u\|_2^2$$

From θ equation we get

$$\begin{aligned}I_3&=\int_{\mathbb{R}^3}[(u.\nabla)\theta]\Delta_h\theta dx=-\sum_{i=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}\partial_k u_i\partial_j\theta\partial_k\theta\\&=\sum_{i=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}\partial_{jk}^2 u_i\theta\,\partial_k\theta+\sum_{i=1}^{3}\sum_{k=1}^{2}\int_{\mathbb{R}^3}\partial_k u_i\theta\partial_{ik}^2\theta\\&\leq\|\theta\|_{L^\infty}\|\nabla\nabla_h u\|_{L^2}\|\nabla_h\theta\|_2+\|\theta\|_{L^\infty}\|\nabla_h u\|_{L^2}\|\nabla\nabla_h\theta\|_{L^2}\\&\leq C\|\nabla_h\theta\|_{L^2}^2+C\|\nabla_h u\|_{L^2}^2+\frac{1}{4}\|\nabla\nabla_h u\|_{L^2}^2+\frac{1}{4}\|\nabla\nabla_h\theta\|_{L^2}^2.\end{aligned}$$

Combining above estimates we get

$$\frac{1}{2}\frac{d}{dt}\left[\|\nabla_h u\|_2^2+\|\nabla_h\theta\|_2^2\right]+\|\nabla_h\nabla u\|_2^2+\|\nabla_h\nabla\theta\|_2^2$$

$$\leq C\|\partial_1 u_3\|_{\alpha}^{\frac{\alpha}{\alpha-1}}\|\nabla u\|_2^{\frac{\alpha-2}{\alpha-1}}\|\partial_3\nabla u\|_2^{\frac{\alpha}{\alpha-1}}+C\left(\|\nabla_h\theta\|_2^2+\nabla_h u\|_2^2\right)$$

Integrating and applying Holder's inequality

$$\left[\|\nabla_h u\|_2^2+\|\nabla_h\theta\|_2^2\right]+\int_0^t\left(\|\nabla_h\nabla u\|_2^2+\|\nabla_h\nabla\theta\|_2^2\right)$$

$$\leq\|\nabla_h u_0\|_2^2+\|\nabla_h\theta\|_2^2+C\left(\int_0^t\|\partial_1 u_3\|_{\alpha}^{\frac{2\alpha}{\alpha-2}}\|\nabla u\|_2^2\right)^{\frac{\alpha-2}{2(\alpha-1)}}\left(\int_0^t\|\Delta u\|_2^2\right)^{\frac{\alpha}{2(\alpha-1)}}$$

Multiply the first equation of 1.1 by $-\Delta u$ and the second equation by $-\Delta\theta$ and integration by parts, $X(t)=\|\nabla u(t)\|_2^2+\|\nabla\theta\|_2^2$ and $Y(t)=\|\Delta u\|_2^2+\|\Delta\theta\|_2^2$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}X(t)+Y(t)&=\int_{\mathbb{R}^3}u.\nabla u.\Delta u dx-\int_{\mathbb{R}^3}\theta e_3.\Delta u dx+\int_{\mathbb{R}^3}u.\nabla\theta.\Delta\theta dx\\&=J_1+J_2+J_3\end{aligned}$$

J_1 can be bounded as (see [8]),

$$|J_1|\quad\leq C\|\nabla_h u\|_2\|\nabla u\|_2^{\frac{1}{2}}\|\nabla_h\nabla u\|_2\|\Delta u\|_2^{\frac{1}{2}}+C\|u_3\|_2^{\frac{4(\alpha-1)}{\alpha-2}}\|\partial_3 u_3\|_{\alpha}^{\frac{2\alpha}{\alpha-2}}\|\nabla u\|_2^2+\epsilon\|\Delta u\|_2^2$$

J_2 *and* J_3 can be bonded as :

$$|J_2| \quad \leq C \| \nabla u \|_2^2 + \| \theta \|_2^2$$

$$|J_3| \quad = \int u_j \partial_j \theta \theta_{kk}^2 \theta dx \quad = -\int_{\mathbb{R}^3} \partial_k u_j \partial_j \theta \theta_k \theta = \int_{\mathbb{R}^3} \partial_k u_j \theta \theta_j \partial_k \theta$$

$$\leq \| \theta \| \infty \| \nabla u \|_2 \| \Delta \theta \|_2 \leq C \| \nabla u \|_2^2 + \in \| \Delta \theta \|_2^2 \leq C \| \nabla u \|_2^2 + \in Y(t)$$

Combining all above inequalities, we obtain

$$\frac{1}{2}\frac{d}{dt} X(t) + Y(t) \quad \leq \| \nabla_h u \|_2 \| \nabla u \|_2^{\frac{1}{2}} + C \| u_3 \|_2^{\frac{4(\alpha-1)}{\alpha-2}} \| \partial_1 u_3 \|_\alpha^{\frac{2\alpha}{\alpha-2}} \| \nabla u \|_2^2 +$$

$$C(\| \nabla u \|_2^2 + \| \nabla \theta \|_2^2) + C \| \nabla u \|_2^2 + \in Y(t).$$

$$\frac{1}{2}\frac{d}{dt} X(t) + Y(t) \leq C \| \nabla_h u \|_2 \| \nabla u \|_2^{\frac{1}{2}} \| \nabla_h \nabla u \|_2 \| \nabla u \|_2^{\frac{1}{2}} + C \| \partial_1 u_3 \|_\alpha^{\frac{2\alpha}{\alpha-2}} \| \nabla u \|_2^2 + C(\| \nabla u \|_2^2 + \| \nabla \theta \|_2^2$$

Integrating

$$X(t) + \int_0^t Y(t) dt$$

$$\leq C \int_0^t \| \nabla_h u \|_2 \| \nabla u \|_2^{\frac{1}{2}} \| \nabla_h \nabla u \|_2 \| \Delta u \|_2^{\frac{1}{2}} ds + C \int_0^t \| \partial_1 u_3 \|_\alpha^{\frac{2\alpha}{\alpha-2}} \| \nabla u \|_2^2 ds + C(\| \nabla u_0 \|_2^2 + \| \nabla \theta_0 \|_2^2$$

$$= R_1 + R_2 + R_3.$$

The rest of the calculation is exactly the same as in the paper of Cao and Titi (see [8]). The difference here is the term involving θ, which we have shown in J_1, and J_2. Finally we get (detail see [8]).

$$X(t) + \int_0^t Y(t) ds \leq C(\| \nabla u_0 \|_2^2 + \| \nabla \theta_0 \|_2^2 + C \int_0^t \partial_1 u_3 \|_\alpha^{\frac{4\alpha}{\alpha-3}} \| \nabla u \|_2^2 ds$$

Let $\beta = \dfrac{4\alpha}{\alpha - 3}$. Clearly β satisfies the condition of the main theorem. Thus $\int_0^t \| \partial_1 u_3 \|_\alpha^\beta ds < M$ implies X $(t) < \infty$. Thus (u, θ) can be extended smoothly beyond $T > 0$. This completes the proof.

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Some Topological Properties of Certain Normed Space Valued Function Space Defined by Orlicz Function

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Abstract

The aim of this paper is to introduce and study a new class$S(X, (Y, \|\,.\,\|), \Phi, \gamma, u)$ of normed space Y- valued functions using Orlicz functionΦas a generalization of some of the wellknown sequence spaces and function spaces. Besides the investigation of linear space structures of the class S $(X, (Y, \|\,.\,\|), \Phi, \gamma, u, M)$, our primarily interest is to explore the conditions pertaining the containment relation of the class S $(X, (Y, \|\,.\,\|), \Phi, \gamma, u)$ in terms of different $\gamma$and$u$so that such a class of functions is contained in or equal to another class of similar nature.

Key words: Orlicz Function, Orlicz Sequence Space, Solid Space

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1. Introduction

Lindenstrauss and Tzafriri [9] used the idea of Orlicz function Φ to construct the sequence space ℓ_Φ of scalars (x_k) such that

$$\ell_\Phi = \left\{ x = (x_k) \in \omega \ : \ \sum_{k=1}^{\infty} \Phi\left(\frac{|x_k|}{r}\right) < \infty \text{ for some } r > 0 \right\}.$$

The space ℓ_Φ with the norm

$$\|x\|_\Phi = \inf\left\{ r > 0 : \sum_{k=1}^{\infty} \Phi\left(\frac{|x_k|}{r}\right) \leq 1 \right\}$$

becomes a Banach space which is called an *Orlicz-sequence space*.The space ℓ_Φ is closely related to the space ℓ_p which is an Orlicz sequence space with

$$\Phi(x) = x^p : 1 \leq p < \infty.$$

We recall [8] that an Orlicz function is a function $\Phi : [0,\infty) \to [0,\infty)$ which is continuous, non decreasing and convex with

$$\Phi(0) = 0,\ \Phi(x) > 0 \text{ for } x > 0, \text{ and } \Phi(x) \to \infty \text{ as } x \to \infty.$$

An Orlicz function Φ can be represented in the following integral form

$$\Phi(x) = \int_0^x q(t)\, dt$$

where q, known as the kernel of Φ, is right-differentiable for $t \geq 0$, $q(0) = 0$, $q(t) > 0$ for $t > 0$, q is non decreasing, and $q(t) \to \infty$ as $t \to \infty$ (see, Krasnosel'skiî and Rutickiî, [8]). Note that an Orlicz function is always unbounded.

An Orlicz function Φ is said to satisfy Δ_2 - condition for all values of t, if there exists a constant $K > 0$ such that

$$\Phi(2t) \leq K\, \Phi(t), \text{ for all } t \geq 0.$$

The Δ_2-condition is equivalent to the satisfaction of inequality

$$\Phi(Lt) \leq K\, L\, \Phi(t)$$

for all values of t for which $L > 1$, (see, Krasnosel'skiî and Rutickiî , [8]).

A simple example of an Orlicz function which satisfies the Δ_2 - condition for all values of t is

$$\Phi(t) = a|t|^\alpha\ (\alpha > 1), \text{ since } \Phi(2t) = a2^\alpha |t|^\alpha = 2^\alpha \Phi(t).$$

Subsequently, Basariv and Altundag [1], Bhardwaj and Bala [2], Chen [3] , Ghosh and Srivastava [4], Kamthan and Gupta [5], Khan [6], Kolk [7], Parashar and Choudhary [18], Rao and Subremanina [19],Rao and Ren [20], Savas and Patterson [21] , Srivastava and Pahari [22,23,24], and many othershave been introduced and studied the algebraic and topological properties of various sequence spaces using Orlicz function as a generalization of several well known sequence spaces.

Corresponding to the definition on vector valued sequences, see [5] , we now introduce the following definitions on normed space valued function spaces.

Let $(Y, \|.\|)$ be a normed space and $S(Y) = \{\phi : X \to Y\}$ with topology $\Im$. Then $S(Y)$ is called *solid* if $\phi \in S(Y)$ and scalars $\alpha(x)$, $x \in X$ such that $|\alpha(x)| \leq 1$, $x \in X$ implies

$$\alpha(x)\,\phi(x) \in S(Y).$$

2. The Classes $S(X, (Y, \|.\|), \Phi, \gamma, u)$ and $S(X, (Y, \|.\|), \Phi, \gamma, u, M)$ of Normed Space Valued Functions

Let X be an arbitrary non empty set (not necessarily countable) and $\mathcal{F}(X)$ be the collection of all finite subsets of X directed by inclusion relation. Let $(Y, \|.\|)$ be a normed space over the field of complex number C. Let u and v be any functions on $X \to R^+$, the set of positive real numbers, and

$$\ell_\infty(X, R^+) = \{ u : X \to R^+ \text{ such that } \sup_x u(x) < \infty \}.$$

Further, we write γ, μ for functions on $X \to C \setminus \{0\}$, and the collection of all such functions will be denoted by $s(X, C \setminus \{0\})$. For $u \in \ell_\infty(X, R^+)$, we denote $M = \max\{1, \sup_x u(x)\}$.

We now introduce the following new class of Banach space Y – valued functions using Orliczfunction Φ:

$$S(X, (Y, \|.\|), \Phi, \gamma, u, M) = \{ \phi : X \to Y : \text{for some } r > 0, \sup_{x \in X} \Phi\left(\frac{\| \gamma(x)\, \phi(x) \|^{u(x)/M}}{r} \right) < \infty \} \quad (2.1)$$

and its subclass

$$\bar{S}(X, (Y, \|.\|), \Phi, \gamma, u, M) = \{ \phi : X \to Y : \text{for every } r > 0, \sup_{x \in X} \Phi\left(\frac{\| \gamma(x)\, \phi(x) \|^{u(x)/M}}{r} \right) < \infty \} \quad (2.2)$$

Besides studying the classes (2.1) and (2.2), we also deal the following class of normed space - Y valued functions

$$S(X, (Y, \|.\|), \Phi, \gamma, u) = \{ \phi : X \to Y : \text{for some } r > 0, \sup_{x \in X} \Phi\left(\frac{\| \gamma(x)\, \phi(x) \|^{u(x)}}{r} \right) < \infty \} \quad (2.3)$$

Further when $\gamma : X \to C \setminus \{0\}$ is a function such that $\gamma(x) = 1$ for all x, then $S(X, (Y, \|.\|), \Phi, \gamma, u)$ will be denoted by $S(X, (Y, \|.\|), \Phi, u)$ and when $u : X \to R^+$ is a function such that $u(x) = 1$ for all x, then

$S(X, (Y, \|.\|), \Phi, \gamma, u)$ will be denoted by $S(X, (Y, \|.\|), \Phi, \gamma)$.

Actually, these classes are the generalizations of the familiar sequence and function spaces, studied in Pahari [10,11,12,13,14,15,16,17], Srivastava and Pahari [22,23,24] and Tiwari *et al.* [25] using norm.

3. Main Results

In this section, we shall investigate some results that characterize the linear space structure of $S(X, (Y, \|.\|), \Phi, \gamma, u, M)$ of normed space Y-valued functions . Beside this, we shall explore the conditions in terms of different u and γ so that a class $S(X, (Y, \|.\|), \Phi, \gamma, u)$ is contained in or equal to another similar class and thereby derive the conditions of their equality.

As far as the linear space structure of the class over the field C of complex numbers is concerned, we throughout take pointwise operations i.e., for functions ϕ, ψ and scalar α,

$$(\phi + \psi)(x) = \phi(x) + \psi(x)$$

and

$$(\alpha\phi)(x) = \alpha\phi(x),\ x \in X.$$

Moreover, we shall denote the zero element of this space by θ by which we shall mean the function

$$\theta : \lambda \to Y \text{ such that } \theta(x) = \theta, \text{ for all } x \in X.$$

We shall also frequently use the notations

$$M = \sup_x u(x) \text{ and for scalar } \alpha,\ A[\alpha] = \max(1, |\alpha|).$$

But when the functions $u(x)$ and $v(x)$ occur, then to distinguish M we use the notations $M(u)$ and $M(v)$ respectively.

Theorem 3.1: $S(X, (Y, \|.\|), \Phi, \gamma, u, M)$ forms a linear space over the field complex numbers C with respect to the pointwise vector operations.

Proof: Suppose $\phi, \psi \in S(X, (Y, \|.\|), \Phi, \gamma, u, M)$, $r_1 > 0$ and $r_2 > 0$ are associated with ϕ and ψ respectively and $\alpha, \beta \in C$. Then

$$\sup_{x\in X} \Phi\left(\frac{\|\gamma(x)\,\phi(x)\|^{u(x)/M}}{r_1}\right) < \infty$$

and $$\sup_{x\in X} \Phi\left(\frac{\|\gamma(x)\,\psi(x)\|^{u(x)/M}}{r_2}\right) < \infty.$$

We now choose r such that $2r_1 A[\alpha] \le r$ and $2r_2 A[\beta] \le r$. For such r, using non decreasing and convex properties of Φ we have

$$\sup_{x\in X}\Phi\left(\frac{\|\gamma(x)\,(\alpha\phi(x)+\beta\psi(x))\|^{u(x)/M}}{r}\right)$$

$$\leq \sup_{x\in X}\Phi\left(\frac{[\,\|\gamma(x)\,\alpha\phi(x)\|+\|\gamma(x)\,\beta\psi(x)\|]^{u(x)/M}}{r}\right)$$

$$\leq \sup_{x\in X}\Phi\left(\frac{\|\gamma(x)\,\alpha\phi(x)\|^{u(x)/M}}{r}+\frac{\|\gamma(x)\,\beta\psi(x)\|^{u(x)/M}}{r}\right)$$

$$\leq \sup_{x\in X}\Phi\left[\frac{|\alpha|^{u(x)/M}}{r}\,\|\gamma(x)\,\phi(x)\|^{u(x)/M}+\frac{|\beta|^{u(x)/M}}{r}\,\|\gamma(x)\,\psi(x)\|^{u(x)/M}\right]$$

$$\leq \sup_{x\in X}\Phi\left[\frac{A\,[\alpha]}{r}\,\|\gamma(x)\,\phi(x)\|^{u(x)/M}+\frac{A\,[\beta]}{r}\,\|\gamma(x)\,\psi(x)\|^{u(x)/M}\right]$$

$$\leq \sup_{x\in X}\Phi\left(\frac{1}{2\,r_1}\,\|\gamma(x)\,\phi(x)\|^{u(x)/M}+\frac{1}{2\,r_2}\,\|\gamma(x)\,\psi(x)\|^{u(x)/M}\right)$$

$$\leq \frac{1}{2}\sup_{x\in X}\Phi\left(\frac{\|\gamma(x)\,\phi(x)\|^{u(x)/M}}{r_1}\right)+\frac{1}{2}\sup_{x\in X}\Phi\left(\frac{\|\gamma(x)\,\psi(x)\|^{u(x)/M}}{r_2}\right)<\infty.$$

This implies that $\alpha\phi+\beta\psi\in S\,(X,(Y,\|.\|),\Phi,\gamma,u)$ and so $S\,(X,(Y,\|.\|),\Phi,\gamma,u,M)$ forms a linear space over C.

Theorem 3.2: If Φ satisfies the Δ_2-condition, then

$S\,(X,(Y,\|.\|),\Phi,\gamma,u,M) = \overline{S}\,(X,(Y,\|.\|),\Phi,\gamma,u,M)$.

Proof:It suffices to show that $S\,(X,(Y,\|.\|),\Phi,\gamma,u,M)$ is a subspace of $\overline{S}\,(X,(Y,\|.\|),\Phi,\gamma,u,M)$, since the reverse inclusion is always true.

Let $\phi\in S\,(X,(Y,\|.\|),\Phi,\gamma,u,M)$, $r>0$ be associated with ϕ ,then we have

$$\sup_{x\in X}\Phi\left(\frac{\|\gamma(x)\,\phi(x)\|^{u(x)/M}}{r}\right)<\infty \text{ and hence } \Phi\left(\frac{\|\gamma(x)\,\phi(x)\|^{u(x)/M}}{r}\right)<\infty.$$

Let us consider an arbitrary $r_1>0$.

If $r\leq r_1$, then by non decreasing property of Φ, we have

$$\Phi\left(\frac{\|\gamma(x)\,\phi(x)\|^{u(x)/M}}{r_1}\right)\leq\Phi\left(\frac{\|\gamma(x)\,\phi(x)\|^{u(x)/M}}{r}\right)$$

and hence $\displaystyle\sup_{x\in X}\Phi\left(\frac{\|\gamma(x)\,\phi(x)\|^{u(x)/M}}{r_1}\right)\leq\sup_{x\in X}\Phi\left(\frac{\|\gamma(x)\,\phi(x)\|^{u(x)/M}}{r}\right)<\infty,$

shows that $\phi\in\overline{S}\,(X,(Y,\|.\|),\Phi,\gamma,u,M)$.

On the other hand, if $r>r_1$, then put $s=\frac{r}{r_1}>1$. Since Φ satisfies Δ_2-condition. There exists a constant $K>0$ such that

$$\Phi\left(\frac{\|\gamma(x)\,\phi(x)\|^{u(x)/M}}{r_1}\right)=\Phi\left(\frac{s\,\|\gamma(x)\,\phi(x)\|^{u(x)/M}}{r}\right)$$
$$\leq K.s.\Phi\left(\frac{\|\gamma(x)\,\phi(x)\|^{u(x)/M}}{r}\right)$$

Therefore,

$$\sup_{x\in X}\Phi\left(\frac{\|\gamma(x)\,\phi(x)\|^{u(x)/M}}{r_1}\right)\leq K.\frac{r}{r_1}\sup_{x\in X}\Phi\left(\frac{\|\gamma(x)\,\phi(x)\|^{u(x)/M}}{r}\right)<\infty,$$

and so $\phi\in \overline{S}\,(X,(Y,\|.\|),\Phi,\gamma,u,M)$.

Corollary 3.3: If Φ satisfies the Δ_2 -condition, then $\overline{S}\,(X,(Y,\|.\|),\Phi,\gamma,u,M)$, forms a linear space over C.

Proof: The proof immediately follows from the consequence of Theorems 3.1, 3.2.

Theorem 3.4: $S\,(X,(Y,\|.\|),\Phi,\gamma,u,M)$ is solid.

Proof:Let $\phi\in S\,(X,(Y,\|.\|),\Phi,\gamma,u,M)$, $r>0$ be associated with ϕ .Then we have

$$\sup_{x\in X}\Phi\left(\frac{\|\gamma(x)\phi(x)\|^{u(x)/M}}{r}\right)<\infty.$$

Now, if we take scalars $\alpha(x)$, $x\in X$ such that $|\alpha(x)|\leq 1$, then

$$\sup_{x\in X}\Phi\left(\frac{\|\alpha(x)\,\gamma(x)\,\phi(x)\|^{u(x)/M}}{r}\right)\leq\sup_{x\in X}\Phi\left(\frac{|\alpha(x)|^{u(x)/M}\|\gamma(x)\,\phi(x)\|^{u(x)/M}}{r}\right)$$
$$\leq\sup_{x\in X}\Phi\left(\frac{\|\gamma(x)\,\phi(x)\|^{u(x)/M}}{r}\right)<\infty.$$

This shows that $\alpha\phi\in S(X,(Y,\|.\|),\Phi,\gamma,u,M)$ and hence $S(X,(Y,\|.\|),\Phi,\gamma,u,M)$ is solid.

In the forthcoming Theorems, we shall deal with the class $S\,(X,(Y,\|.\|),\Phi,\gamma,u)$ to investigate the conditions in terms of different u and γ so that it is contained in or equal to another class of similar nature.

Theorem 3.5: If $v: X\to R^{+}$, $u\in \ell_\infty(X,R^{+})$ and $\gamma\in s\,(X,C\backslash\{0\})$, then

$$S(X,(Y,\|.\|),\Phi,\gamma,u)\subset S(X,(Y,\|.\|),\Phi,\gamma,v)$$

if $\lim\sup_x\frac{v(x)}{u(x)}<\infty$.

Proof: Assume that $\lim\sup_x\frac{v(x)}{u(x)}<\infty$. Then there exists a constant $d>0$ such that

$$v(x)<d\,u(x)$$

for all but finitely many $x \in X$.

Now, if $\phi \in S(X, (Y, \|.\|), \Phi, \gamma, u)$, $r > 0$ is associated with ϕ ,then we have

$$\sup_{x \in X} \Phi\left(\frac{\| \gamma(x)\, \phi(x) \|^{u(x)}}{r} \right) < \infty .$$

This shows that there exists some positive real number η satisfying

$$\Phi\left(\frac{\| \gamma(x)\, \phi(x) \|^{u(x)}}{r} \right) \leq \Phi\left(\frac{\eta}{r} \right),$$

for all but finitely many $x \in X$. Since Φ is non decreasing, therefore

$$\| \gamma(x)\, \phi(x) \|^{u(x)} < \eta$$

Since $v(x) < d\, u(x)$ and so if $\| \gamma(x)\, \phi(x) \| \leq 1$, then obviously

$$\| \gamma(x)\, \phi(x) \|^{v(x)} \leq 1;$$

and on the other hand if $\| \gamma(x)\, \phi(x) \| > 1$, then

$$\| \gamma(x)\, \phi(x) \|^{v(x)} < \| \gamma(x)\, \phi(x) \|^{d\, u(x)} < \eta^{d}.$$

Therefore

$$\| \gamma(x)\, \phi(x) \|^{v(x)} \leq \max(1, \eta^{d}),$$

for all but finitely many $x \in X$. This shows that for all but finitely many $x \in X$,

$$\Phi\left(\frac{\| \gamma(x)\, \phi(x) \|^{v(x)}}{r} \right) \leq \Phi\left(\frac{\max(1, \eta^{d})}{r} \right),$$

and therefore

$$\sup_{x \in X} \Phi\left(\frac{\| \gamma(x)\, \phi(x) \|^{v(x)}}{r} \right) < \infty .$$

This completes the proof.

Theorem 3.6: If $v : X \to \boldsymbol{R}^{+}$, $u \in \ell_{\infty}(X, \boldsymbol{R}^{+})$, $\gamma \in s(X, \boldsymbol{C} \setminus \{0\})$ and

$$S(X, (Y, \|.\|), \Phi, \gamma, u) \subset S(X, (Y, \|.\|), \Phi, \gamma, v),$$

then $\limsup_{x} \frac{v(x)}{u(x)} < \infty$.

Proof: Assume that $S(X, (Y, \|.\|), \Phi, \gamma, u) \subset S(X, (Y, \|.\|), \Phi, \gamma, v)$ but $\limsup_{x} \frac{v(x)}{u(x)} = \infty$. Then there exists a sequence (x_k) of distinct points in X such that for each $k \geq 1$,

$$v(x_k) > k u(x_k). \tag{3.3}$$

Now, taking $y \in Y$ such that $\| y \| = 1$, we define $\phi : X \to Y$ by

$$x_k = \begin{cases} (\gamma(x_k))^{-1}\, 2^{1/u(x_k)} y, & \text{for } x = x_k,\ k \geq 1, \text{ and} \\ \theta, & \text{otherwise.} \end{cases} \tag{3.4}$$

Let $r > 0$ .Then we have

$$\sup_{x\in X}\Phi\left(\frac{\|\gamma(x)\,\phi(x)\|^{u(x)}}{r}\right) = \sup_{k\geq 1}\Phi\left(\frac{\|\gamma(x_k)\,\phi(x_k)\|^{u(xk)}}{r}\right)$$
$$= \sup_{k\geq 1}\Phi\left(\frac{\|2^{1/u(x_k)}y\|^{u(x_k)}}{r}\right)$$
$$= \sup_{k\geq 1}\Phi\left(\frac{2\|y\|^{u(x_k)}}{r}\right)$$
$$\leq\Phi\left(\frac{2A\,[\|y\|^{M(u)}]}{r}\right).$$

This shows that $\phi\in S\,(X,(Y,\|.\|),\Phi,\gamma,u)$ but in view of (3.3) and (3.4) ,we have

$$\sup_{x\in X}\Phi\left(\frac{\|\gamma(x)\,\phi(x)\|^{v(x)}}{r}\right) = \sup_{k\geq 1}\Phi\left(\frac{2^{v(x_k)/u(x_k)}\,.\,\|y\|^{v(x_k)}}{r}\right)$$
$$\geq \sup_{k\geq 1}\Phi\left(\frac{2^k}{r}\right) = \infty$$

and hence $\phi\notin S\,(X,(Y,\|.\|),\Phi,\gamma,v)$, a contradiction. This completes the proof: After combining the Theorems 3.5 and 3.6, we get:

Theorem 3.7: If $v: X\to \boldsymbol{R}^+$, $u\in \ell_\infty(X,\boldsymbol{R}^+)$ and $\gamma\in s\,(X,\boldsymbol{C}\backslash\{0\})$, then

$$S(X,(Y,\|.\|),\Phi,\gamma,u)\subset S(X,(Y,\|.\|),\Phi,\gamma,v)$$

if and only if $\limsup_x \frac{v(x)}{u(x)}<\infty$.

Theorem 3.8: If $u: X\to \boldsymbol{R}^+$, $v\in \ell_\infty(X,\boldsymbol{R}^+)$ and $\gamma\in s(X,\ \boldsymbol{C}\backslash\{0\})$, then

$$S(X,(Y,\|.\|),\Phi,\gamma,v)\subset S(X,(Y,\|.\|),\Phi,\gamma,u)$$

if $\liminf_x \frac{v(x)}{u(x)}> 0$.

Proof: Assume that $\liminf_x \frac{v(x)}{u(x)}> 0$ .Then there exists a constant $m> 0$ such that $v(x) > m\,u(x)$ for all but finitely many $x\in X$.

Let $\phi\in S\,(X,(Y,\|.\|),\Phi,\gamma,v)$, $r>0$ is associated with ϕ . Then we have

$$\sup_{x\in X}\Phi\left(\frac{\|\gamma(x)\,\phi(x)\|^{v(x)}}{r}\right)<\infty .$$

Hence we can find some positive real number η satisfying

$$\Phi\left(\frac{\|\gamma(x)\,\phi(x)\|^{v(x)}}{r}\right)<\Phi\left(\frac{\eta}{r}\right),$$

for all but finitely many $x\in X$. Since Φ is non decreasing, we have

$$\|\gamma(x)\,\phi(x)\|^{v(x)}\leq\eta.$$

Since $v(x) > m\,u(x)$ and so if $\| \gamma(x)\, \phi(x) \| \geq 1$, then

$$\| \gamma(x)\, \phi(x) \|^{u(x)} \leq \| \gamma(x)\, \phi(x) \|^{v(x)/m} \leq \eta^{1/m}.$$

But on the other hand if $\| \gamma(x)\, \phi(x) \| < 1$, then obviously $\| \gamma(x)\, \phi(x) \|^{u(x)} < 1$. Therefore

$$\| \gamma(x)\, \phi(x) \|^{u(x)} \leq \max(1, \eta^{1/m}),$$

for all but finitely many $x \in X$. This shows that for all but finitely many $x \in X$,

$$\Phi\left(\frac{\| \gamma(x)\, \phi(x) \|^{u(x)}}{r} \right) \leq \Phi\left(\frac{\max(1, \eta^{1/m})}{r} \right).$$

and therefore

$$\sup_{x \in X} \Phi\left(\frac{\| \gamma(x)\, \phi(x) \|^{u(x)}}{r} \right) < \infty.$$

This follows that $\phi \in S(X, (Y, \|.\|), \Phi, \gamma, u)$ and hence

$$S(X, (Y, \|.\|), \Phi, \gamma, v) \subset S(X, (Y, \|.\|), \Phi, \gamma, u).$$

This completes the proof.

Theorem 3.9: If $u : X \to R^+$, $v \in \ell_\infty(X, R^+)$, $\gamma \in s(X, C \setminus \{0\})$ and

$$S(X, (Y, \|.\|), \Phi, \gamma, v) \subset S(X, (Y, \|.\|), \Phi, \gamma, u),$$

then $\liminf_x \dfrac{v(x)}{u(x)} > 0$.

Proof: Assume that $S(X, (Y, \|.\|), \Phi, \gamma, v) \subset S(X, (Y, \|.\|), \Phi, \gamma, u)$ holds but $\liminf_x \dfrac{v(x)}{u(x)} = 0$. Then there exists a sequence (x_k) of distinct points in X such that for $k \geq 1$,

$$k v(x_k) < u(x_k). \tag{3.5}$$

Now, taking $y \in Y$ with $\| y \| = 1$, define $\phi : X \to Y$ by

$$x_k = \begin{cases} (\gamma(x_k))^{-1}\, 2^{1/v(x_k)}\, y, & \text{for } x = x_k,\ k \geq 1, \text{ and} \\ \theta, & \text{otherwise.} \end{cases} \tag{3.6}$$

Let $r > 0$. Then we have

$$\sup_{x \in X} \Phi\left(\frac{\| \gamma(x)\, \phi(x) \|^{v(x)}}{r} \right) = \sup_{k \geq 1} \Phi\left(\frac{\| \gamma(x_k)\, \phi(x_k) \|^{v(x_k)}}{r} \right)$$

$$= \sup \Phi\left(2 \| y \|^{v(x_k)} \right)$$

$$\leq \Phi\left(\frac{2A\, [\, \| y \|^{M(v)}]}{r} \right).$$

This implies that $\phi \in S(X, (Y, \|.\|), \Phi, \gamma, v)$.

But on the other hand, in view of (3.5) and (3.6) ,we get

$$\sup_{x\in X} \Phi\left(\frac{\|\gamma(x)\,\phi(x)\|^{u(x)}}{r}\right) = \sup_{k\geq 1} \Phi\left(\frac{\|\gamma(x_k)\,\phi(x_k)\|^{u(x_k)}}{r}\right)$$
$$= \sup_{k\geq 1} \Phi\left(\frac{2^{u(x_k)/v(x_k)}\,\|y\|^{u(x_k)}}{r}\right)$$
$$\geq \sup_{k\geq 1} \Phi\left(\frac{2^k}{r}\right) = \infty,$$

shows that $\phi \notin S(X, (Y, \|.\|), \Phi, \gamma, u)$, a contradiction. This completes the proof.

After combining the Theorems 3.8 and 3.9, we get:

Theorem 3.10: If $u : X \to R^+$, $v \in \ell_\infty(X, R^+)$ and $\gamma \in s(X, C\setminus\{0\})$, then

$$S(X, (Y, \|.\|), \Phi, \gamma, v) \subset S(X, (Y, \|.\|), \Phi, \gamma, u)$$

if and only if $\liminf_x \frac{v(x)}{u(x)} > 0$.

After combining the Theorems 3.7 and 3.10, it follows that:

Theorem 3.11: If $u, v \in \ell_\infty(X, R^+)$ and $\gamma \in s(X, C\setminus\{0\})$, then

$$S(X, (Y, \|.\|), \Phi, \gamma, u) = S(X, (Y, \|.\|), \Phi, \gamma, v)$$

if and only if $0 < \liminf_x \frac{v(x)}{u(x)} \leq \limsup_x \frac{v(x)}{u(x)} < \infty$.

Theorem 3.12: If $u \in \ell_\infty(X, R^+)$ and $\gamma \in s(X, C\setminus\{0\})$, then

(i) $S(X, (Y, \|.\|), \Phi, \gamma) \subset S(X, (Y, \|.\|), \Phi, \gamma, u)$
if and only if $\limsup_x u(x) < \infty$;

(ii) $S(X, (Y, \|.\|), \Phi, \gamma, u) \subset S(X, (Y, \|.\|), \Phi, \gamma)$
if and only if $\liminf_x u(x) > 0$; and

(iii) $S(X, (Y, \|.\|), \Phi, \gamma, u) = S(X, (Y, \|.\|), \Phi, \gamma)$
if and only if $0 < \liminf_x u(x) \leq \limsup_x u(x) < \infty$.

Proof:If we consider $u : X \to R^+$ such that $u(x) = 1$ for all $x \in X$ and v is replaced by u in Theorems 3.7,3.9 and 3.10, the assertions (i), (ii) and (iii) follows.

Theorem 3.13: If $u \in \ell_\infty(X, R^+)$, then for any $\gamma, \mu \in s(X, C\setminus\{0\})$,

$$S(X, (Y, \|.\|), \Phi, \gamma, u) \subset S(X, (Y, \|.\|), \Phi, \mu, u)$$

if $\liminf_x \left|\frac{\gamma(x)}{\mu(x)}\right|^{u(x)} > 0$.

Proof: Assume that $\liminf_x \left|\frac{\gamma(x)}{\mu(x)}\right|^{u(x)} > 0$. Then there exists $m > 0$ such that

$$m\,|\mu(x)|^{u(x)} < |\gamma(x)|^{u(x)}$$

for all but finitely many $x \in X$. Let $\phi \in S(X, (Y, \|.\|), \Phi, \gamma, u)$, $r_1 > 0$ is associated with ϕ ,so that

$$\sup_{x\in X} \Phi\left(\frac{\|\gamma(x)\,\phi(x)\|^{u(x)}}{r_1}\right) < \infty.$$

Let us choose r such that $r_1 < m\,r$. Then for such r ,using non decreasing property of Φ, we have

$$\Phi\left(\frac{\|\mu(x)\,\phi(x)\|^{u(x)}}{r}\right) = \Phi\left(\frac{[|\mu(x)|\,\|\phi(x)\|^{u(x)}}{r}\right)$$

$$\leq \Phi\left(\frac{[|\gamma(x)|\,\|\phi(x)\|^{u(x)}}{m\,r}\right) \leq \Phi\left(\frac{\|\gamma(x)\,\phi(x)\|^{u(x)}}{r_1}\right),$$

and therefore $\sup_{x\in X} \Phi\left(\frac{\|\mu(x)\,\phi(x)\|^{u(x)}}{r}\right) < \infty.$

This shows that $\phi \in S(X, (Y, \|.\|), \Phi, \mu, u)$ and hence

$$S(X, (Y, \|.\|), \Phi, \gamma, u) \subseteq S(X, (Y, \|.\|), \Phi, \mu, u).$$

This completes the proof.

Theorem 3.14: If $u \in \ell_\infty(X, R^+)$, then for any $\gamma, \mu \in s(X, C\setminus\{0\})$ and

$$S(X, (Y, \|.\|), \Phi, \gamma, u) \subset S(X, (Y, \|.\|), \Phi, \mu, u),$$

then $\liminf_x \left|\frac{\gamma(x)}{\mu(x)}\right|^{u(x)} > 0.$

Proof:Assume that $S(X, (Y, \|.\|), \Phi, \gamma, u) \subset S(X, (Y, \|.\|), \Phi, \mu, u)$ but $\liminf_x \left|\frac{\gamma(x)}{\mu(x)}\right|^{u(x)} = 0$. Then there exists a sequence (x_k) in X of distinct points such that for each $k \geq 1$, we have

$$k|\gamma(x_k)|^{u(x_k)} \leq |\mu(x_k)|^{u(x_k)} \tag{3.7}$$

We now choose $y \in Y$ such that $\|y\| = 1$ and define $\phi : X \to Y$ by

$$\phi(x) = \begin{cases} (\gamma(x_k))^{-1} y, \text{ for } x = x_k,\ k \geq 1, \text{ and} \\ \theta, \text{ otherwise.} \end{cases} \tag{3.8}$$

Let $r > 0$. Then we have

$$\sup_{x\in X} \Phi\left(\frac{\|\gamma(x)\,\phi(x)\|^{u(x)}}{r}\right) = \sup_{k\geq 1} \Phi\left(\frac{\|\gamma(x_k)\,\phi(x_k)\|^{u(x_k)}}{r}\right)$$

$$= \sup_{k\geq 1} \Phi\left(\frac{\|y\|^{u(x_k)}}{r}\right)$$
$$\leq \Phi\left(\frac{A\,[\| y \|^{M(u)}]}{r}\right).$$

This clearly shows that $\phi \in S\,(X, (\,Y, \|\,.\,\|\,), \Phi, \gamma, u)$.

But on the other hand ,in view of (3.7) and (3.8) ,we have

$$\sup_{x\in X} \Phi\left(\frac{\|\,\mu\,(x)\;\phi(x)\,\|^{u(x)}}{r}\right) = \sup_{k\geq 1} \Phi\left(\frac{\|\,\mu\,(x_k)\;\phi(x_k)\,\|^{u(x_k)}}{r}\right)$$
$$= \sup_{k\geq 1} \Phi\left(\frac{1}{r}\left|\frac{\mu(x_k)}{\gamma\,(x_k)}\right|^{u(x_k)} \|\,y\,\|^{u(x_k)}\right)$$
$$\geq \sup_{k\geq 1} \Phi\left(\frac{k}{r}\right) = \infty.$$

This shows that $\phi \notin S\,(X, (\,Y, \|\,.\,\|\,), \Phi, \mu, u)$, a contradiction. This completes the proof.

After combining the Theorems 3.13 and 3.14, we get:

Theorem 3.15: If $u\in \ell_\infty\,(X, R^+)$, then for any $\gamma, \mu \in s\,(X, C\setminus\{0\})$,

$$S(X, (\,Y, \|\,.\,\|\,), \Phi, \gamma, u) \subset S(X, (\,Y, \|\,.\,\|\,), \Phi, \mu\,, u)$$

if and only if $\liminf_x \left|\dfrac{\gamma\,(x)}{\mu(x)}\right|^{u(x)} > 0.$

Theorem 3.16: If $\gamma,\mu \in s\,(X, C\setminus\{0\})$, $u, v\in \ell_\infty\,(X, R^+)$, then

$$S(X,(\,Y, \|\,.\,\|\,),\Phi,\gamma\,, u) \subset S(X, (\,Y, \|\,.\,\|\,), \Phi, \mu, v)$$

if and only if

(i) $\limsup_x \dfrac{v(x)}{u(x)} < \infty$; and (ii) $\liminf_x \left|\dfrac{\gamma\,(x)}{\mu(x)}\right|^{u(x)} > 0.$

Proof: Proof easily follows from Theorems 3.7 and 3.15.

Theorem 3.17: Let $u\in \ell_\infty\,(X, R^+)$. Then for any $\gamma, \mu \in s(X, C\setminus\{0\})$,

$$S(X, (\,Y, \|\,.\,\|\,), \Phi, \mu, u) \subset S(X, (\,Y, \|\,.\,\|\,), \Phi, \gamma, u)$$

if $\limsup_x \left|\dfrac{\gamma\,(x)}{\mu(x)}\right|^{u(x)} < \infty.$

Proof:Assume that $\limsup_x \left|\dfrac{\gamma\,(x)}{\mu(x)}\right|^{u(x)} < \infty$. Then there exists a positive constant d such that $|\gamma\,(x)|^{u(x)} < d\,|\mu\,(x)|^{u(x)}$ for all but finitely many $x\in X$.

Let $\phi\in S\,(X, (\,Y, \|\,.\,\|\,), \Phi, \mu, u)$, $r_1 > 0$ is associated with ϕ . Then

$$\sup_{x\in X} \Phi\left(\frac{\|\,\mu\,(x)\,\phi\,(x)\,\|^{u(x)}}{r_1}\right)<\infty.$$

Let us choose $r> 0$ such that $d\,r_1\leq r$, then for such r ,using non decreasing property of Φ we have

$$\Phi\left(\frac{\|\,\gamma\,(x)\,\phi\,(x)\,\|^{u(x)}}{r}\right)\leq\Phi\left(\frac{[\,|\gamma(x)|\,\|\,\phi\,(x)\,\|^{u(x)}}{r}\right)$$

$$\leq\Phi\left(\frac{d\,|\mu\,(x)|^{\,u(x)}\,\|\,\phi\,(x)\,\|^{u(x)}}{r}\right)$$

$$\leq\Phi\left(\frac{\|\,\mu\,(x)\,\phi\,(x)\,\|^{u(x)}}{r_1}\right)$$

and therefore $\sup_{x\in X} \Phi\left(\frac{\|\,\gamma\,(x)\,\phi\,(x)\,\|^{u(x)}}{r}\right)<\infty.$

This shows that $\phi\in S\,(X, (\,Y, \|\,.\,\|\,), \Phi, \gamma, u)$ and hence

$$S\,(X, (\,Y, \|\,.\,\|\,), \Phi, \mu, u) \subset S\,(X, (\,Y, \|\,.\,\|\,), \Phi, \gamma, u).$$

This completes the proof.

Theorem 3.18: Let $u\in \ell_\infty\,(X, \boldsymbol{R}^+)$. Then for any γ, $\mu\in s(X, \boldsymbol{C}\setminus \{0\})$,

$$S(X, (\,Y, \|\,.\,\|\,), \Phi, \mu, u) \subset S(X, (\,Y, \|\,.\,\|\,), \Phi, \gamma, u),$$

then $\limsup_x \left|\frac{\gamma\,(x)}{\mu(x)}\right|^{u(x)}<\infty.$

Proof:Assume that

$$S\,(X, (\,Y, \|\,.\,\|\,), \Phi, \mu, u) \subset S\,(X, (\,Y, \|\,.\,\|\,), \Phi, \gamma, u)$$

but $\limsup_x \left|\frac{\gamma\,(x)}{\mu(x)}\right|^{u(x)} = \infty$.Then we can find a sequence (x_k) of distinct points in X such that for each $k\geq 1$,

$$|\gamma\,(x_k)|^{u(x_k)}>k\,|\mu\,(x_k)|^{u(x_k)} \qquad (3.9)$$

We now choose $y\in Y$ such that $\|\,y\,\| = 1$ and define $\phi : X\rightarrow Y$ by

$$\phi\,(x) \;=\; \begin{cases}(\mu(x_k))^{-1}y, \text{ for } x = x_k,\ k\geq 1, \text{ and} \\ \theta, \text{ otherwise}\end{cases} \qquad (3.10)$$

Let $r> 0$. Then we have

$$\sup_{x\in X} \Phi\left(\frac{\|\,\mu(x)\,\phi(x)\,\|^{u(x)}}{r}\right) \;=\; \sup_{k\geq 1}\Phi\left(\frac{\|\,\mu(x_k)\,\phi(x_k)\,\|^{u(x_k)}}{r}\right)$$

$$=\; \sup_{k\geq 1}\Phi\left(\frac{\|\,y\,\|^{u(x_k)}}{r}\right) \leq \Phi\left(\frac{A\,[\|\,y\,\|^{M(u)}]}{r}\right).$$

This clearly shows that $\phi \in S(X, (Y, \|\cdot\|), \Phi, \mu, u)$. But in view of (3.9) and (3.10), we have

$$\sup_{x\in X} \Phi\left(\frac{\|\gamma(x)\,\phi(x)\,\|^{u(x)}}{r}\right) = \sup_{k\geq 1} \Phi\left(\frac{\|\,\gamma(x_k)\,(\mu(x_k))^{-1} y\,\|^{u(x_k)}}{r}\right)$$

$$= \sup_{k\geq 1} \Phi\left(\frac{\left|\dfrac{\gamma(x_k)}{\mu(x_k)}\right|^{u(x_k)} \cdot \|\,y\,\|^{u(x_k)}}{r}\right)$$

$$\geq \sup_{k\geq 1} \Phi\left(\frac{k}{r}\right) = \infty,$$

implies that $\phi \notin S(X, (Y, \|\cdot\|), \Phi, \gamma, u)$. This leads to a contradiction and completes the proof.

When the Theorems 3.17 and 3.18 are combined, we get

Theorem 3.19: Let $u \in \ell_\infty(X, R^+)$. Then for any $\gamma, \mu \in s(X, C\setminus\{0\})$,

$$S(X, (Y, \|\cdot\|), \Phi, \mu, u) \subset S(X, (Y, \|\cdot\|), \Phi, \gamma, u)$$

if and only if $\limsup_x \left|\dfrac{\gamma(x)}{\mu(x)}\right|^{u(x)} < \infty$.

After combining the Theorems 3.16 and 3.18, we get

Theorem 3.20: If $u \in \ell_\infty(X, R^+)$ and $\gamma, \mu \in s(X, C\setminus\{0\})$, then

$$S(X, (Y, \|\cdot\|), \Phi, \gamma, u) = S(X, (Y, \|\cdot\|), \Phi, \mu, u)$$

if and only if $0 < \liminf_x \left|\dfrac{\gamma(x)}{\mu(x)}\right|^{u(x)} \leq \limsup_x \left|\dfrac{\gamma(x)}{\mu(x)}\right|^{u(x)} < \infty$.

Corollary 3.21: Let $u \in \ell_\infty(X, R^+)$ and $\gamma \in s(X, C\setminus\{0\})$. Then

(i) $S(X, (Y, \|\cdot\|), \Phi, \gamma, u) \subset S(X, (Y, \|\cdot\|), \Phi, u)$
if and only if $\liminf_x |\gamma(x)|^{u(x)} > 0$;

(ii) $S(X, (Y, \|\cdot\|), \Phi, u) \subset S(X, (Y, \|\cdot\|), \Phi, \gamma, u)$
if and only if $\limsup_x |\gamma(x)|^{u(x)} < \infty$; and

(iii) $S(X, (Y, \|\cdot\|), \Phi, u) = S(X, (Y, \|\cdot\|), \Phi, \gamma, u)$
if and only if $0 < \liminf_x |\gamma(x)|^{u(x)} \leq \limsup_x |\gamma(x)|^{u(x)} < \infty$.

Proof: By considering the function μ on X such that $\mu(x) = 1$ for all $x \in X$ in Theorems 3.15 and 3.19 and 3.20, one can easily obtain the assertions (i), (ii) and (iii) respectively.

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Two Different Ways to Show a Function is an A_1 Weight Function

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Abstract
In this paper, we briefly discuss the theory of weights and then define A_1 and A_p weight functions. Finally we explore two different ways to show a function is an A_1 weight function.

Introduction

The theory of weights play an important role in various fields such as extrapolation theory, vector-valued inequalities and estimates for certain class of non linear differential equation. Moreover, they are very useful in the study of boundary value problems for Laplace's equation in Lipschitz domains. In 1970, Muckenhoupt characterized positive functions w for which the Hardy-Littlewood maximal operator M maps $L^p(R^n, w(x)dx)$ to itself. Muckenhoupt's characterization actually gave the better understanding of theory of weighted inequalities which then led to the introduction of A_p class and consequently the development of weighted inequalities. Before we explore two different ways to show a function is and A_1 weight, some definitions are in order.

Definition: A locally integrable function on R^n that takes values in the interval $(0,\infty)$ almost everywhere is called a weight. So by definition a weight function can be zero or infinity only on a set whose Lebesgue measure is zero.

We use the notation $w(E) = \int_E w(x)dx$ to denote the w-measure of the set E and we reserve the notation $L^p(R^n,w)$ or $L^p(w)$ for the weighted L^p spaces. We note that $w(E) < \infty$ for all sets E contained in some ball since the weights are locally integrable functions.

Definition: The uncentered Hardy-Littlewood maximal operators on R^n over balls B is defined as

$$M(f)(x) = \sup_{x\in B} \operatorname{Avg}_{B}|f| = \sup_{x\in B} \frac{1}{|B|}\int_{B} |f(y)|dy.$$

Similarly the uncentered Hardy-Littlewood maximal operators on R^n over cubes Q is defined as

$$M_c(f)(x) = \sup_{x\in Q} \operatorname{Avg}_{Q}|f| = \sup_{x\in Q} \frac{1}{|Q|}\int_{Q} |f(y)|dy.$$

In each of the definition above, the suprema are taken over all balls B and cubes Q containing the point x. H-L maximal functions are widely used in Harmonic Analysis. For the details about the H-L maximal operators, see [2].

Definition: A positive measure dμ is called doubling measure if for some positive constant C< ∞,

$$\mu(2B) \leq C\mu(B)$$

for all balls B. This means that the size of a ball with certain radius can be controlled by the ball of half of the given radius.

Definition: A function $w(x)\geq 0$ is called an A_1 weight if there is a constant $C_1>0$ such that

$$M(w)(x)\leq C_1\, w(x)$$

where M(w) is uncentered Hardy-Littlewood Maximal function given by

$$M(w)(x) = \sup_{x\in B} \frac{1}{|B|}\int_{B} w(t)dt.$$

If w is an A_1 weight, then the quantity (which is finite) given by

$$[w]_{A_1} = \sup_{Q\ cubes\ in R^n} \left(\frac{1}{|Q|}\int_{Q} |w(t)|dt\right) ||w^{-1}||_{L^\infty(Q)}$$

is called the A_1 characteristic constant of w.

Definition: Let $1 < p < \infty$. A weight w is said to be of class A_p if $[w]_{A_p}$ is finite where $[w]_{A_p}$ is defined as

$$[w]_{A_p} = \sup_{Q\ cubes\ in\ R^n} \left(\frac{1}{|Q|}\int_{Q} |w(x)|dx\right)\left(\frac{1}{|Q|}\int_{Q} |w(x)|^{\frac{-1}{p-1}}dx\right)^{p-1}.$$

We remark that in the above definition of A_1 and A_p one can also use set of all balls in R^n instead of all cubes in R^n. Readers are suggested to read [1] for motivation, properties of A_p weights and much more about the A_p weights.
Consider the following function:

$$u(x) = \begin{cases} log\frac{1}{|x|}, & |x| < \frac{1}{e} \\ 1, & otherwise. \end{cases}$$

We introduce two different ways to show that the above function is an A_1 weight function.

First approach: Set $v_n = \frac{|s(0,1)|}{n}$, where $|s(0,1)|$ is the surface area of the unit ball. Let $B_\epsilon = \{x \in R^n: |x| \leq \epsilon\}$. Then

$$\int_{B(0,1/e)} log\left(\frac{1}{|x|}\right) dx = \lim_{\epsilon\to 0}\int_{B(0,1/\epsilon)\setminus B_\epsilon} \log\left(\frac{1}{|x|}\right)\, dx$$

$$= \lim_{\epsilon\to 0}\int_\epsilon^{1/e} \log\left(\frac{1}{r}\right) r^{n-1}\, dr.|B(0,1/e)\setminus B_\epsilon|$$

$$= \frac{1}{ne^n}\,(\log(e)+1)|B(0,1/e)|$$

$$= \frac{1}{ne^{2n}}\,(\log(e)+1)v_n$$

$$\leq 2v_n.$$

Note that $\sup_{B_{1/e}}\left(\log\left(\frac{1}{|x|}\right)\right)^{-1} = ||u^{-1}||_{L^\infty(B_{1/e})} \leq 1$. Let $T_1 := \{B(x_0,R): |x_0| \geq 3R\}$ and $T_2 := \{B(x_0,R): |x_0| < 3R\}$. *For all the balls in* T_1 *we have* $2R \leq |x_0| - R < |x| < |x_0| + R$. In the case $\frac{1}{e} \leq 2R$, we have u=1 and so $[u]_{A_1} \leq 1$. Note that

$$\left(\frac{1}{|B|}\int_B u(x)dx\right)||u^{-1}||_{L^\infty(B)} = \left(\frac{1}{|B|}\int_{B\cap B_{1/e}} u(x)dx\right)||u^{-1}||_{L^\infty(B\cap B_{1/e})}$$

$$+\left(\frac{1}{|B|}\int_{B\setminus B_{1/e}} u(x)dx\right)||u^{-1}||_{L^\infty(B\setminus B_{1/e})}.$$

For $2R < \frac{1}{e} < |x_0| + R$, we have $B\cap B_{1/e} \neq \phi$ and $B\setminus B_{1/e} \neq \phi$. In this case we have,

$$\left(\frac{1}{|B|}\int_{B\cap B_{1/e}} u(x)dx\right)||u^{-1}||_{L^\infty(B\cap B_{1/e})} \leq \left(\frac{1}{|B|}\int_{B_{1/e}} u(x)dx\right)||u^{-1}||_{L^\infty(B_{1/e})} \leq 2v_n$$

and

$$\left(\frac{1}{|B|}\int_{B\setminus B_{1/e}} u(x)dx\right)||u^{-1}||_{L^\infty(B\setminus B_{1/e})} \leq 1.$$

Thus we have, $[u]_{A_1} \leq 1 + 2\nu_n$.
In the case, $|x_0| + R < \frac{1}{e}$ one has $B \cap B_{1/e} \neq \phi$ and $B \setminus B_{1/e} = \phi$ and so

$$\left(\frac{1}{|B|}\int_B u(x)dx\right) ||u^{-1}||_{L^\infty(B)} = \left(\frac{1}{|B|}\int_{B_{1/e}} u(x)dx\right) ||u^{-1}||_{L^\infty(B_{1/e})} \leq 2\nu_n.$$

Putting all together, we get

$$\left(\frac{1}{|B|}\int_B u(x)dx\right) ||u^{-1}||_{L^\infty(B)} \leq 1 + 2\nu_n$$

for all balls in T_1.
For the balls in T_2 we have, $0 \leq |x| \leq |x_0| + R < 4R$. Using the same calculation as above, one gets for $\frac{1}{e} < |x_0| + R$,

$$\left(\frac{1}{|B|}\int_B u(x)dx\right) ||u^{-1}||_{L^\infty(B)} \leq 1 + 2\nu_n$$

and

$$\left(\frac{1}{|B|}\int_B u(x)dx\right) ||u^{-1}||_{L^\infty(B)} \leq 1 + 2\nu_n$$

for $|x_0| + R < \frac{1}{e}$. In the case, $\frac{1}{e} \leq |x_0| - R$ we get $[u]_{A_1} \leq 1$. Thus,

$$\left(\frac{1}{|B|}\int_B u(x)dx\right) ||u^{-1}||_{L^\infty(B)} \leq 1 + 2\nu_n$$

for all balls in T_2. Thus, $u \in A_1$.

Second approach: We need to show that there exists M>0 such that

$$\frac{1}{|B|}\int_B u(x)dx \leq M \text{ ess.}\inf_{x\in B} u(x), \qquad \forall B \subset R^n \qquad (1).$$

First, we assume that $B(x_0, R)$ is such that $|x_0| > 3R$. Let's say this is of type I.
Case 1: $|x_0| \leq \frac{1}{16}$. Then $\leq \frac{1}{16}$. We have,

$$\frac{2}{3}|x_0| \leq |x_0| - R \leq |x| \leq |x_0| + R \leq \frac{4}{3}|x_0| \leq \frac{1}{e}, \qquad \forall x \in B.$$

Thus,

$$1 \leq \ln\frac{1}{|x_0|+R} \leq u(x) \leq \ln\frac{1}{|x_0|-R}, \ \forall x \in B.$$

Hence,

$$\frac{1}{|B|}\int_B u(x)dx \leq \ln\frac{1}{|x_0|-R} \leq \ln\frac{3}{2|x_0|}.$$

Moreover we have,

$$\ln\frac{3}{2|x_0|} = \ln\frac{3}{4|x_0|} + \ln 2 \le 2\ln\frac{3}{4|x_0|} \le 2\ln\frac{1}{|x_0|+R} \le 2u(x), \quad \forall x \in B.$$

Therefore, $\frac{1}{|B|}\int_B u(x)dx \le M \text{ ess.}\inf_{x\in B} 2u(x).$

Case 2: $|x_0| > \frac{3}{16}$. Then $|x| \ge |x_0| - R \ge \frac{2}{3}|x_0|$. Thus,

$$1 \le u(x) \le \max\left(1, \ln\frac{16}{3}\right) = \ln\frac{16}{3}.$$

$$\frac{1}{|B|}\int_B u(x)dx \le \ln\frac{16}{3} \le M \text{ ess.}\inf_{x\in B} u(x), M = \ln\frac{16}{3}.$$

Therefore (1) holds when B is of type I.

Secondly consider the case B is such that $B(x_0, R)$ is such that $|x_0| < 3R$. Let's call this is of type II. In this case one has, $B(x_0, R) \subset B(0, 5R)$. Note that,

$$0 < a := \int_{\mathbb{R}^n}[u(x) - 1]dx = \int_{B(0,1/e)}[u(x) - 1]dx < \infty.$$

Case 1: 5R > $\frac{1}{5}$. We have

$$\frac{1}{|B|}\int_B u(x)dx = 1 + \frac{1}{|B|}\int_B [u(x) - 1]dx$$
$$\le 1 + \frac{1}{v_n R^n} a \le 1 + \frac{(5e)^n}{v_n} a.$$

Thus, (1) is satisfied with $M = 1 + \frac{(5e)^n}{v_n} a$.

Case 2: 5R $\le \frac{1}{5}$. We have

$$\frac{1}{|B|}\int_B u(x)dx \le \frac{1}{v_n R^n}\int_{B(0,5R)} u(x)dx =: J.$$

Let $x = 5Ry$. We have,

$$J = \frac{1}{v_n R^n}\int_{B(0,1)} \ln\frac{1}{5R|y|}(5R)^n dy$$
$$= \frac{5^n}{v_n}\int_{B(0,1)}\left(\ln\frac{1}{5R} + \ln\frac{1}{|y|}\right)dy$$
$$= 5^n\left(\ln\frac{1}{5R} + b\right)$$

where $b := \int_{B(0,1)} \ln\frac{1}{|y|}\, dy$. Since $\ln\frac{1}{5R} \nearrow \infty$ as R$\to$ 0, there exists c>0 such that

$$b \leq c \ln \frac{1}{5R}, \quad \forall R < \frac{1}{5e}.$$

Here we have used the fact that $1 \leq \ln \frac{1}{5R}, \ \forall R < \frac{1}{5e}$. Therefore,

$$\frac{1}{|B|} \int_B u(x)dx \leq J \leq 5^n (1 + \mathrm{c}) \ln \frac{1}{5R} \leq 5^n (1 + \mathrm{c}) \ln \frac{1}{|x|}, \qquad \forall x \in B(x_0, R).$$

Thus (1) holds for $M = 5^n (1 + \mathrm{c})$. So in all possible cases, we have shown that the given function satisfies the criteria to be an A_1 weight function.

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A Relationship between the Assignment and the Perfect Matching Approaches for the Product Rate Variation Problem with a General Objective

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Abstract

In this paper, the product rate variation problem is considered which arises as a sequencing problem in mixed-model just-in-time production systems. In particular, we establish a relationship between the assignment and the perfect matching approaches. During the past decades, this problem has been extensively investigated. Several heuristics and an exact pseudo-polynomial assignment method by Kubiak (1993) with a complexity of $O(D^3)$ have been given for the total product rate variation case, where D is the total demand for all models of the product and an exact pseudo-polynomial perfect matching by Steiner and Yeomans (1993) with a complexity of $O(DlogD)$ for its bottleneck case. In this paper, we propose a relationship between the two approaches.

Key words: Product rate variation problem, Sequencing, Non-linear integer programming problem.

Introduction

One of the most important problems for the effective utilization of the mixed-model just-in-time production systems consists in sequencing different models with keeping the rate of usage of all parts used by the assembly lines as constant as possible. This problem is known as the mixed-model just-in-time sequencing problem (abbreviated as MMJITSP). The problem of minimizing the variations in the rate at which different models are produced on the line is called the product rate variation problem (abbreviated as PRVP). The latter problem is the single-level case of MMJITSP. The

problem of minimizing the total deviations between the actual cumulative productions from the ideal one is called the total PRVP (abbreviated as TPRVP), and the problem of minimizing the maximum deviation is called the bottleneck PRVP (abbreviated as BPRVP), see Kubiak [7]. PRVP has been widely investigated in the literature since it has a model with a strong mathematical base and wide real-world applications [2, 3, 4, 5, 7, 12]. Both TPRVP and BPRVP have been solved with pseudo-polynomial complexity solution procedures. The solution procedure for the TPRVP is the assignment approach with $O(D^3)$ time, see Kubiak [7] and that for the BPRVP is the perfect matching with a bisection search approach with $O(DlogD)$, see Steiner and Yeomans [10].

In this paper, we prpose a relationship between the assignment and the perfect matching with a bisection search approaches which shows that any feasible sequence obtained by the assignment method is feasible if and only if all the copies are assigned at time units within their sequencing time windows. However, this is not true for optimality.

The remainder of the paper is as follows. In Section 2 we present the mathematical modeling of the product rate variation problem. In Section 3, we describe the level curve, the bounds and the ideal position in which both the assignment and the perfect matching approaches are based on. In section 4, we describe the proposed relation between the two approaches. The last section concludes the paper.

Mathematical Modeling

Let D be the total demand of n different models with d_i copies of model $i, i = 1, 2, \dots, n,$ where $n \geq 2$ and $D = \sum_{i=1}^{n} d_i$. The time horizon is partitioned into D equal time units under the assumption that each copy of a model $i, i = 1, \dots, n,$ has equal processing time. A copy of a model is produced in a time unit $k, k = 1, \dots, D,$ means that the copy of the model is produced during the time period from $k - 1$ to k. Let $r_i = \frac{d_i}{D}$ be the demand rate. Let x_{ik} and kr_i be the actual and the ideal cumulative productions, respectively, of model i produced during the time units 1 through k. An inventory holds if $x_{ik} - kr_i > 0$, and a shortage incurs if $kr_i - x_{ik} > 0$. We assign the same cost for both inventory and shortage. Miltenburg (1989) [10] and Kubiak and Sethi (1991, 1994) [8, 9] gave an integer programming formulation for PRVP as follows with m being a positive integer:

$$\text{minimize } \max \left[F_m = |x_{ik} - kr_i|^m\right]$$

$$\text{minimize } [G_m = \sum_{k=1}^{D}\sum_{i=1}^{n}|x_{ik} - kr_i|^m]$$

subject to

$$\sum_{i=1}^{n} x_{ik} = k, \qquad k = 1,2,\ldots,D$$

$$x_{i(k-1)} \leq x_{ik}, \quad i = 1,2,\ldots,n;\ k = 2,3,\ldots,D$$

$$x_{iD} = d_i, x_{io} = 0, \qquad i = 1,2,\ldots,n$$

$$x_{ik} \geq 0, \text{ integer } i = 1,2,\ldots,n;\ k = 1,2,\ldots,D.$$

Level Curve, Bounds and Ideal Position

There exist nD deviations between the actual and the ideal cumulative productions of D copies of n models. The value of the actual cumulative production x_{ik}, $i = 1,2,\ldots,n$; $k = 1,2,\ldots,D$, is sequence-dependent integer from $\{0,1,\ldots,d_i\}$. However, the value of the ideal cumulative production kr_i, $i = 1,2,\ldots,n$; $k = 1,2,\ldots,D$, is sequence-independent rational number. Let j be the number of copies of a model and (i,j) be the j^{th} copy of model $i, i = 1,2,\ldots,n$. The actual cumulative productionx_{ik}, $i = 1,2,\ldots,n$; $k = 1,2,\ldots,D$, has nD values with $x_{ik} \in \{j|j = 0,1,2,\ldots,d_i;\ i = 1,2,\ldots,n\}$. There exist at most $n + D$ different values of x_{ik} for PRVP. Hence, one can replace x_{ik} by j with $j = 0,1,\ldots,d_i$; $i = 1,2,\ldots,n$, in the level curve of the objective value of the function of PRVP. The level curve for copy (i,j) of the objective function of PRVP is defined as

$$f_{ij}{}^m = |j - kr_i|^m, i = 1,2,\ldots,n;\ j = 0,1,\ldots,d_i.$$

A perfect matching of copies and time units relies on the level curves and the bound $B > 0$ of the function F_m of BPRVP that are drawn over the planning horizon. The points at which the bound intersects the level curves are useful to find the sequencing times. A copy (i,j) is sequenced at a time unit $k \in \{1,\ldots,D\}$ such that the level curve does not exceed the bound B. An upper bound on the absolute deviation objective function $|x_{ik} - kr_i|$ for BPRVP is $UB^*_1 = 1$, see Steiner and Yeomans (1993) [10], and a better one has been given as $UB_1 = 1 - \frac{1}{D}$, see Brauner and Crama (2004) [1]. Since the points, where the bound $UB_1 = 1$ intersects the level curves of BPRVP with the objective function $|x_{ik} - kr_i|^m$ for different values of m, are the same, the bound $UB_1 = 1$ is also an upper bound for BPRVP with the objective function $|x_{ik} - kr_i|^m$ for all values of m. However, the upper bounds corresponding to $UB_1 = 1 - \frac{1}{D}$ are different for BPRVP with the objective functions $|x_{ik} - kr_i|^m$ for different values of

m. An upper bound on the largest value of the objective function $|x_{ik} - kr_i|^m$ of BPRVP has been established as

$$UB_m = \left(1 - \frac{1}{D}\right)^m,$$

Dhamala et al. (2010) [3] and Khadka (2012) [6].

The lower bound $LB_1 = 1 - r_{max}$ on the absolute deviation objective function $|x_{ik} - kr_i|$ for BPRVP has been established by Steiner and Yeomans (1993) [10], and it has been modified as

$$LB_m = (1 - r_{max})^m,$$

for this problem with the objective function $|x_{ik} - kr_i|^m$ (Dhamala et al. (2010) [3] and Khadka (2012) [6]. The earliest and the latest sequencing times are determined by the level curve and a suitably chosen bound. The selection of an upper bound always yields the sequencing times that give rise to a feasible solution. A feasible solution corresponding to the lower bound is optimal.

The assignment approach is based on the assignment cost on the level curve $f^m{}_{ij}$, $i = 1, \dots, n\,; j = 0, 1, \dots, d_i$ and the position at which (i, j), denoting the copy j of model i, is sequenced. The level curves are drawn on the interval $[0, D]$. The ideal position of (i, j) denoted Z_{ij} is obtained from the unique intersection point satisfying

$$f^m{}_{ij} = f^m{}_{i(j-1)}, i = 1, \dots, n\,; j = 1, \dots, d_i.$$

The ideal position is the unique integer near to the unique intersection point defined to be $Z_{ij} = \left\lceil \frac{2j-1}{2r_i} \right\rceil$. When copy (i; j) is sequenced at the time unit Z_{ij}, copy (i, j) contributes only the unavoidable cost

$$inf\{|j - kr_i|^m, i = 1, \dots, n\,; j = 1, \dots, d_i\}$$

to the total cost.

Relationship Between the Assignment and the Perfect Matching Approches

A copy (i,j), $i = 1; \dots, n;\ j = 1, \dots, d_i$, is sequenced at the ideal position Z_{ij} unless another copy competes for the same position in the assignment method. In the competition, one copy is sequenced at that time unit whereas other competing copies have to be sequenced in the neighboring unassigned time units. The ideal position is the

unique integer obtained from the intersection point of the level curves ${f^m}_{ij}$ and ${f^m}_{i(j-1)}$, $i = 1, \dots, n;\ j = 1, \dots, d_i$.
Similarly, the earliest sequencing time $E_m(i,j)$ and the latest sequencing time $L_m(i,j)$ of copy (i,j) are obtained from the intersection points of ithe level curves ${f^m}_{ij}$ and ${f^m}_{i(j-1)}$ with the bound B, respectively. Both the feasibility and the optimality occur only if each copy (i,j) is sequenced within the time window $T_{(i,j)m}, i = 1, \dots, n;\ j = 1, \dots, d_i$.

With this situation, one may be interested in finding a relationship between the ideal position and the sequencing time window. We can prove that the ideal position of each copy (i,j), $i = 1, \dots, n;\ j = 1, \dots, d_i$, lies within the corresponding sequencing time window.

Lemma 1 Let Z_{ij} be the ideal position and

$$T_{(i,j)m}, i = 1, \dots, n;\ j = 1, \dots, d_i$$

be the sequencing time window for copy (i,j). Then the ideal position lies in the sequencing time window, i.e., we have

$$Z_{ij} \in T_{(i,j)m}.$$

Proof:
For any copy (i,j), we have

$$E_{(i,j)m} \le L_{(i,j)m},$$

i.e., inequality

$$\left\lceil \frac{j - \sqrt[m]{B}}{r_i} \right\rceil \le \left\lfloor 1 + \frac{j-1+\sqrt[m]{B}}{r_i} \right\rfloor$$

holds.
Since one can take $B = UB_m$, i.e.,

$$B = (1 - \tfrac{1}{D})^m,$$

we obtain

$$\left\lceil \frac{j-(1-\frac{1}{D})}{r_i} \right\rceil \le \left\lfloor 1 + \frac{j-1+(1-\frac{1}{D})}{r_i} \right\rfloor$$

which yields the inequality

$$\left\lceil \frac{2j-1}{2r_i} + \frac{1-2(1-\frac{1}{D})}{2r_i} \right\rceil \le \left\lfloor \frac{2j-1}{2r_i} + \frac{2(1-\frac{1}{D})-1}{2r_i} + 1 \right\rfloor$$

i.e., we have

$$\left\lceil \frac{2j-1}{2r_i} + \frac{1-2(1-\frac{1}{D})}{2r_i} \right\rceil \le \left\lceil \frac{2j-1}{2r_i} \right\rceil \le \left\lfloor \frac{2j-1}{2r_i} + \frac{2(1-\frac{1}{D})-1}{2r_i} + 1 \right\rfloor$$

Hence,

$$Z_{ij} \in T_{(i,j)m}$$

holds for all i, j with $i = 1, \dots, n;\ j = 1, \dots, d_i$.

If two or more copies compete for a time unit k, it is clear that the time unit k is in the sequencing time window of all the copies.

Lemma 2 If two copies (i,j) and (i^*,j^*) with $(i,j) \neq (i^*,j^*)$ compete for a time unit k, then

$$T_{(i,j)m} \cap T_{(i^*,j^*)m} \neq \emptyset .$$

Proof: If two copies (i,j) and (i^*,j^*) with $(i,j) \neq (i^*,j^*)$ compete for a time unit k, we have

$$\left\lceil \frac{2j-1}{2r_i} \right\rceil = \left\lceil \frac{2j^*-1}{2r^*_i} \right\rceil$$

Since

$$\left\lceil \frac{2j-1}{2r_i} \right\rceil \in T_{(i,j)m} \text{ and } \left\lceil \frac{2j^*-1}{2r^*_i} \right\rceil \in T_{(i^*,j^*)m}$$

we obviously have

$$T_{(i,j)m} \cap T_{(i^*,j^*)m} \neq \emptyset .$$

Assume that $T^p{}_{(i,j)m}$ with p being a positive integer, is the sequencing time window for copy $(i,j), i = 1, \dots, n;\ j = 1, \dots, d_i$, such that p copies including (i,j) compete for the same time unit.

Lemma 3 If $l \geq 2$ (l integer) copies compete for a time unit $k \in \{1, \dots, D\}$, there exist at least $l-1$ unassigned time units within their sequencing time windows

$$\bigcup_{p=1}^{l} T^p{}_{(i,j)m} .$$

Proof: Suppose that l copies, $l \geq 2$ (l integer), compete for a time unit $k, k = 1, \dots, D$.

Then

$$k \in \bigcap_{p=1}^{l} T^p{}_{(i,j)m} \subseteq [1, D] .$$

Since there exist exactly D time units for D copies, exactly one copy can be sequenced at one time unit. Assume that copy (i^*,j^*) among l copies competing for time unit k can be sequenced at this time unit. There remain $l-1$ copies competing for time unit k that have to be assigned to other $l-1$ time units. Suppose that there exist only $l-2$

unassigned time units within the interval $\bigcup_{p=1}^{l} T^{p}{}_{(i,j)m}$. Then there exists a copy (i^{**}, j^{**}) with $j^{*} \neq j^{**}$ competing for time unit k to be sequenced at time unit $k^{*} \notin \bigcup_{p=1}^{l} T^{p}{}_{(i,j)m}$. The level curve of the objective function for copy (i^{**}, j^{**}) exceeds the bound which necessarily leads to infeasibility. Hence, there exist at least $l-1$ unassigned time units within the interval $\bigcup_{p=1}^{l} T^{p}{}_{(i,j)m}$.

Theorem A sequence s obtained by the assignment method is feasible if and only if all the copies are assigned at time units within their sequencing time windows.

Proof: Let s be a feasible sequence obtained by the assignment method. Feasibility is assured only if the level curve $f^{m}{}_{ij}$ of the objective function of PRVP does not exceed the bound for each copy $(i,j), i = 1, \dots, n;\ j = 1, \dots, d_i$. This is only possible if each copy $(i,j), i = 1, \dots, n;\ j = 1, \dots, d_i$, is assigned at a time unit k with $k \in T_{(i,j)m}$.

Conversely, consider an assignment that each copy $(i,j), i = 1, \dots, n;\ j = 1, \dots, d_i$, assigns to a time unit k such that $k \in T_{(i,j)m}$. The level curve $f^{m}{}_{ij}$ of the objective function of PRVP does not exceed the bound B for all copies $(i,j),\ i = 1, \dots, n;\ j = 1, \dots, d_i$. Hence, the corresponding sequence s is feasible.

However, the perfect matching method does not yield optimal solution for the total PRVP though it gives exact solution to the bottleneck PRVP. An instance $d_1 = 1, d_2 = 1, d_3 = 4, d_4 = 4, n = 4$ is a counter example. A sequence $3-4-1-3-4-3-4-2-3-4$ obtained by the perfect matching method is optimal for the bottleneck PRVP with maximum value 0.7 and total value 11.8. However, $3-4-3-4-2-1-4-3-4-3$ obtained the assignment method is optimal for the total PRVP with maximum value 0.8 and the total value 11.4.

Concluding Remarks

Product rate variation problem has been extensively studied with heuristics and an exact assignment method for its total PRVP case and an exact perfect matching method for its bottleneck PRVP case. The perfect matching approach does not optimally solve the total PRVP case. However, we have established through a relationship between the two procedures that feasibility holds by the both approaches.

A study on the relationship between the two approaches for the batching case of the PRVP would be an interesting area to work for further research.

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