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CONTENTS

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An Uncertainty Principle like Hardy's Theorem for Nilpotent Lie Group G_n

CHET RAJ BHATTA
Central Department of Mathematics
Tribhuvan University, Kirtipur
Email: crbhatta@yahoo.com

Abstract

Let f: $\Re \to \mathbb{C}$ be measurable and for all x, y $\in \Re$ and if

(i) $|f(x)| \leq C \exp(-a\pi x^2)$

(ii) $|\hat{f}(y)| \leq C \exp(-b\pi y^2)$

where C, a, b > 0. If ab > 1 then f = 0 a.e. If ab = 1 then f(x) = C exp ($-a\pi x^2$). Iϕ $\alpha\beta < 1$ then there exist infinitely many linearly independent function satisfying (i) and (ii).

In this paper we extend an uncertainty principle due to Cowling and Price to threadlike nilpotent Lie groups G_n.

Keywords and Phrases: Uncertainty principle, nilpotent Lie group, Fourier transform, Hilbert Schmidt norm.

2000 mathematics subject classification: Primary 22E25; secondary 43A30.

Introduction

A classical theorem of Hardy [6] on Fourier transform pairs says that a non-zero function f on the real line $\Re$ and its Fourier transform $\hat{f}$ can not both be very rapidly decreasing. More precisely, let the Fourier transform be defined by,

$$\hat{f}(y) = \int_{\Re} f(x) \exp(-2\pi i x y)\, dx,\ y \in \Re$$

The following is a generalization of this theorem due to Cowling and Price [3].

Theorem (Cowling and Price): Let f:$\Re \to \mathbb{C}$ be measurable and

(i) $\|e_a\, f\|_{L^p(\Re)} < \infty$

(ii) $\|e_b\, \hat{f}\|_{L^q(\Re)} < \infty$

Where a, b > 0, $e_k(x) = \exp(k\pi x^2)$ and $1 \le \min(p, q) < \infty$. If $ab \ge 1$, then f = 0 a.e. If ab < 1, then there exist infinitely many linearly independent functions satisfying (i) and (ii)

An analogue of the Cowling-Price theorem has been proved in [3] for Eucledian space the Heisenberg group H_n and the Eucledian motion group of the plane.

Main Results

Theorem: let $f{:}G_n \to \mathbb{C}$ be a measurable function such that $|f(x)| \le C \exp(-a\pi \|x\|^2)$for some c, a > 0 and all $x \in G_n$ then the function $h(\xi) = \|\pi_\xi\,(f)\|_{HS}$ is bounded.

Proof:

$$|\xi_1|\, \|\pi_\xi(f)\|^2{}_{HS} = \int_{\Re^2} |\mathcal{F}_1 \ldots (n-1)\, f(\xi_1, t, q_3(\xi, t), q_{n-1}(\xi, t), s)|^2\, dt\, ds$$

$$\le \int_{\Re^{n-1}} |f(x_1, \ldots x_{n-1}, s)|\, |e^{2\pi i \xi_1 x_1}|\, |e^{2\pi i t x}2|\, dx_1 dx_2 \ldots dx_{n-1}$$

$$\le C \int_{\Re^{n-1}} \exp(-a\pi(x_1{}^2 + \ldots + x_n{}^2 + s^2)\, |\exp(2\pi i\, \xi_1 x_1)\, |\exp(2\pi i x_2)|\, dx_1 \ldots dx_{n-1}$$

$$\le \text{constant} \exp(-a\pi s^2) \int_{\Re^2} |\exp(-a\pi(x_1{}^2 - \frac{2}{a} i\, \xi_1 x_1)|\, |\exp\left(-a\pi\left(x_2{}^2 - \frac{2}{a}\, i t x_2\right)\right| dx_1\, dx_2$$

$$\int_{-\infty}^{\infty} |\exp(-a\pi\left(x^2 - \frac{2}{a} i\, \xi_1 x\right)|dx = \int_{-\infty}^{\infty} |\exp(-a\pi(x + iy)^2 - \frac{2}{a}\, i\, \xi_1(x + iy)|\, dx$$

By the change of contour $x \to x + iy$

$$= \int_{-\infty}^{\infty} \exp \operatorname{Re}\,(-a\pi((x + iy)^2 - \frac{2}{a} i\, \xi_1(x + iy))\, dx$$

$$= e^{-2\pi\xi_1 y} \int_{-\infty}^{\infty} \exp(-a\pi(x^2 - y^2))\, dx$$

$$= e^{-2\pi}(\xi y - \frac{a}{2} y^2) \int_{-\infty}^{\infty} \exp(-a\pi x^2) dx$$

$$\le \text{const. } e^{-2\pi}(\xi, y - \frac{a}{2} y^2)$$

Taking infimum over y, we have

$$\int_{-\infty}^{\infty} |\exp(-a\pi(x^2 - \frac{2}{a} i\xi_1 x)|\, dx \le \text{const } e^{-2\pi}\sup\left(\xi_1\, y - \frac{a}{2}\, y^2\right)$$

$$= \text{const } e^{-2\pi\xi^2{}_1/2a}$$

$$= \text{const } e^{-\pi\xi^2{}_1/a}$$

$\therefore\quad ||\pi_\xi(f)||_{HS} \le \text{const } (|\xi_1|^{-1} \exp(-\pi\xi_1{}^2/a))^{1/2}$

Thus h is bounded.

The sharpness of the constant

Let G_4 be the low dimensional nilpotent Lie groups for $ab < 1$, let $\alpha \in (a, 1/b)$, but it works for G_n using lemms 2.1, 2.2 in [7]

For $x \in G_4$, let $f(x) = \exp(-\alpha\pi\, ||x||^2)$

We know that,

$$||\pi\xi_1, \xi_2(f)||^2{}_{HS} = |\xi_1)^{-1} \int_{\Re} |\mathcal{F}_{123} f(\xi_1, u, \frac{1}{2}\frac{u^2}{\xi_1} + \xi_3, v)|^2 \, du\, dv$$

For $u, v \in \Re$,

$$\mathcal{F}_{123} f(\xi_1, u, \frac{1}{2}\frac{u^2}{\xi_1} + \xi_3, v) = \int_{\Re^3} e^{-\propto\pi(x_1 2 + x_2 2 + x_3 2 + v 2)}\, e^{-2\pi i x}{}_1{}^{\xi}{}_1\, e^{-2\pi i x}{}_2{}^{u} e^{-2\pi i x}{}_3{}^{\{\xi_3 + (1/2)(u^2/\xi_1)\}}\, dx_1 dx_2 dx_3$$

$$= e^{-\propto\pi v^2} \mathcal{F}(e^{-\propto\pi x}{}_1{}^2)(\xi_1) \mathcal{F}(e^{-\propto\pi x}{}_2{}^2)\, (u)\, \mathcal{F}(e^{-\propto\pi x}{}_3{}^2)\, {}^{\{\xi_3 + (1/2)\,(u^2/\xi_1)\}} \quad \ldots (A)$$

$$= e^{-\propto\pi v^2} \mathcal{F}(e^{-\propto\pi x_1 2})(\xi_1) \mathcal{F}(e^{-\propto\pi x_2 2})\, (u)\, \mathcal{F}(e^{-\propto\pi x_3 2 - 2\pi i_\xi 3 x_3})\, {}^{\{(1/2)\,(u^2/\xi_1)\}}$$

where $\mathcal{F}(e^{-\propto\pi x}{}_1{}^2)\, (\xi_1)$ is the Fourier transform of the function $g(x) = e^{-\propto\pi x^2}$ at ξ_1

Now, $\mathcal{F}(e^{-\propto\pi x}{}_1{}^2)\, (\xi_1) = \text{const } e^{-\propto\pi x}{}_1{}^{2/\propto}$

And $\mathcal{F}(e^{-\propto\pi x}{}_2{}^2)(u) = \text{const}^{-\pi u}{}_2{}^{2/\propto}$

So, $|\mathcal{F}_{123} f(\xi_1, u, \frac{1}{2}\frac{u^2}{\xi_1} + \xi_3, v)|$

$$= \text{const. } e^{(-\pi\xi_1{}^2)/\propto}\, e^{(-\pi u_2{}^2)/\propto} e^{-\propto\pi v^2} |\mathcal{F}(e^{-2\pi i\xi_3 x_3 - \propto\pi x_3{}^2}) \left(\frac{1}{2}\frac{u^2}{\xi_1}\right)|$$

$$\le \text{const. } e^{(-\pi\xi_1{}^2)/\propto}\, e^{(-\pi u^2)/\propto}\, e^{-\pi u v^2} \int_{\Re} |\, e^{-2 i\pi i x_3 x_3 - \propto\pi x_3{}^2}|\, dx_3$$

$$= \text{const. } e^{(-\pi\xi_1^2)/\alpha} e^{(-\pi u^2)/\alpha} e^{-\pi uv^2} e^{(-\pi\xi_3^2)/\alpha} \int_{\Re} |e^{-\alpha\pi\{x_3 + (i\xi_3/\alpha)^2\}^2}| dx_3$$

$$= \text{const. } e^{(-\pi\xi_1^2)/\alpha} e^{(-\pi u^2)/\alpha} e^{-\pi uv^2} e^{(-\pi\xi_3^2)/\alpha}$$

$$\therefore \quad \|\pi\xi_1\xi_3(f)\|^2_{HS} \le \text{const } |\xi_1|^{-1} e^{-2\pi(\xi_1^2+\xi_3^2)/\alpha} \int_{\Re} e^{-2\pi u^2/\alpha} e^{-2\pi\alpha v^2} \, du \, dv$$

$$= \text{const. } |\xi_1|^2 e^{-2\pi(\xi_1^2+\xi_3^2)/\alpha} \qquad \ldots \text{(B)}$$

$$\int_{\Re^2} e^{2\pi b(\xi_1^2+\xi_2^2)} |\xi_1| \, \|\pi\xi_1, \xi_3(f)\|^2_{HS} \, d\xi_1 \, d\xi_3$$

$$\le \text{const } \int_{\Re^2} e^{2\pi(b-1/\alpha)(\xi_1^2+\xi_3^2)} \, d\xi_1 \, d\xi_3$$

$$< \infty \text{ since } b - \frac{1}{\alpha} < 0$$

Thus there is a non zero function satisfying the condition (i) and (ii) of theorem 2.4 [7] for $ab < 1$.

Since $|f(x)| = \exp(-\alpha\pi x^2) < \exp(-a\pi x^2)$ for $q = 2$.

In case the condition (ii) is replaced by

$$\int_{\Re^2} e^{2\pi b\xi_1^2} |\xi_1| \, \|\pi_{\xi_1,\xi_3}(f)\|^2_{HS} \, d\xi_1 \, d\xi_3 < \infty$$

which is really the condition used in the proof of $q = 2$, the things are more simpler.

Write the equation (A) as

$$\mathcal{F}_{123} f\left(\xi_1, u, \frac{1}{2}\frac{u^2}{\xi_1} + \xi_{3,v}\right)$$

$$= \text{const } e^{-\alpha\pi v^2} e^{(-\pi\xi_1^2)/\alpha} e^{(-\pi u_1^2)/\alpha} e^{-(\pi/\alpha)\{\xi_3 + (1/2)(u^2/\xi_1)\}^2}$$

$$\therefore \|\pi_{\xi_1,\xi_2}(f)\|^2_{HS} = \text{const. } |\xi_1|^{-1} e^{(-2\pi\xi_1^2)/\alpha} \int_{\Re^2} e^{-2\alpha\pi v^2} e^{-2\pi/\alpha[u^2 + \{\xi_3 + (1/2)(u^2/\xi_1)\}^2]} \, du \, dv$$

$$= \text{const. } |\xi_1|^{-1} e^{(-2\pi\xi_1^2)/\alpha} \int_{\Re} e^{-2\pi/\alpha[u^2 + (\xi_3 + 1/2\, u^2/\xi_1)^2]} \, du$$

$$\int_{\Re^2} e^{-2\pi b\xi_1^2} |\xi_1| \, \|\pi_{\xi_1,\xi_3}(f)\|^2_{HS} \, d\xi_1 \, d\xi_3$$

$$= \text{const. } \int_{\Re^3} e^{2\pi(b-1/\alpha)\xi_1^2} e^{(-2\pi u^2)/\alpha} e^{-2\pi/\alpha\{\xi_3 + (1/2)(u^2/\xi_1)\}^2} \, du \, d\xi_1 \, d\xi_3$$

Applying $\xi_3 \to \xi_3 - \frac{1}{2}\frac{u^2}{\xi_1}$

$$= \text{const.} \int_{\Re^3} e^{2\pi\{b - 1/\alpha\}\, \xi_1^{\,2}}\, e^{(-2\pi u^2)/\alpha}\, e^{-2\pi/\alpha\, \xi_3^{\,2}}\, du\, d\xi_1\, d\xi_3$$

$$< \infty$$

For $q \geq 2$ or $1 \leq q < 2$ and $ab < 1$.

$$\int_{\Re^2} e^{qb\pi\{\xi_1^{\,2} + \xi_3^{\,2}\}}\, |\xi_1|\, \|\pi_{\xi_1,\xi_3}(f)\|_{HS}^{q}\, d\xi_1\, d\xi_3 \qquad \ldots(1)$$

$$\int_{\Re^2} e^{qb\pi\{\xi_1^{\,2} + \xi_3^{\,2}\}}\, |\xi_1|^{\{1-(q/2)\}}\, e^{-(ap/\infty)(x_1^{\,2} + x_3^{\,2})}\, d\xi_1\, d\xi_3$$

$$= \int_{\Re^2} e^{\pi q(b - 1/\alpha)(\xi_1^{\,2} + \xi_3^{\,2})}\, |\xi_1|^{1-q/2}\, dx_1\, d\xi_3 < \infty \text{ for } 1 \leq q < 2$$

and for $q \geq 2$ considering the case $|\xi_1| > 1$ we obtain integral in (i) is finite.

Theorem: Let a and b be positive real numbers and $1 \leq \min(p, q) < \infty$. Suppose that

$f \in L^1(Gn) \cap L^2(Gn)$ satisfies the following conditions:

(i) $\displaystyle\int_{Gn} e^{pa\pi\|\xi\|^2}\, |f(x)|^p\, dx < \infty$

(ii) $\displaystyle\int_{\Re^{n-2}} |\xi_1|\, e^{b\pi q\|\xi\|^2}\, \|\pi_\xi(f)\|_{HS}^{q}\, d\xi < \infty$

If $1 \leq q < 2$ and $ab \geq \dfrac{2}{q}$ then $f = 0$ a.e.

Proof: Let p be such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Clearly $p \geq 2$ and $q < p$ by Lemma 2.2 [7] we have

$$\|e_{b\{1+q/p\}}\, \hat{g}\|_1 = \int_{\Re} e_{b\{1+(q/p)\}}(\xi_1)\, |\hat{g}(\xi_1)|\, d\xi_1$$

$$= \int_{\Re^{n-2}} |\xi_1|\, e_{b\{1+(q/p)\}}(\xi_1)\, \|\pi_\xi(f)\|^2\, d\xi_1\, d\xi_3 \ldots d\xi_{n-1}$$

Define the function u and v in $\Re^{n-2}$ by

$u(\xi) = e_b(\xi_1)\, |\xi_1|^{1/q}\, \|\pi_\xi(f)\|_{HS}$ and

$v(\xi) = e_{bq/p}(\xi_1)\, |\xi_1|^{1/p}\, \|\pi_\xi(f)\|_{HS}$

Then,

$$\int_{\Re^{n-2}} |u(\xi)|^q\, d\xi = \int_{\Re^{n-2}} e_{bq}(\xi_1)\, |\xi_1|\, \|\pi_\xi(f)\|_{HS}^{q}\, d\xi < \infty$$

and

$$\int_{\Re^{n-2}} |v(\xi)|^p\, d\xi = \int_{\Re^{n-2}} e_{bq}(\xi_1)\, |\xi_1|\, \|\pi_\xi(f)\|_{HS}^{p}\, d\xi$$

$$= \int_{\Re^{n-2}} e_{bq}(\xi_1)\,|\xi_1|\,\|\pi_\xi(f)\|_{HS}^{q}\,\|\pi_\xi(f)\|_{HS}^{p-q}\,d\xi \quad (p > q)$$

$$\leq K \int_{\Re^{n-2}} e_{bq}(\xi_1)\,|\xi_1|\,\|\pi_\xi(f)\|_{HS}^{q} > \infty$$

So using Holders inequality we have

$$\|e_{b\{1+(q,p)\}}\,\hat{g}\|_1 \leq \|u\|_q\,\|v\|_p < \infty$$

Since $\frac{a}{2} b\left(1 + \frac{q}{p}\right) \geq \frac{1}{2} \cdot \frac{2}{q}\left(1 + \frac{q}{p}\right) = \frac{1}{2} \cdot \frac{2}{q} \cdot q = 1$

The Cowling price theorem shows that $g = 0$ and hence $f = 0$ a.e.

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A Topological Criterion for Starlikeness, Piecewise Convex and Piecewise α - Convex Functions

CHINTA MANI POKHREL
Nepal Engineering College

Abstract: In [3] C.N. Genther, St Ruscheweyh and L. C. Salinas introduced the concept of quasi - simple curve and have given a criterion for it. In this article we shall use the concept of quasi - simple curves to establish a topological criterion for starlike, piecewise convex and piecewise α - convex functions.

1. Introduction

Definition 1.1 Let C denote the entire complx plan. A positively oriented close curve γ is siad to be quasi - simple if it is the positively oriented boundary of a simply connected domain. An arc is said be a quasi - simple if it is a subarc of a quasi- simple curve. For any closed curve or arc $\gamma : [a\ b] \longrightarrow C$, let $S_\gamma := \{\gamma(t) : a \leq t \leq b\}$, be its support in C.

Definition 1.2 Let γ be a positively oriented closed curve and ω_0 be any point in the complex plane C. We say that ω_0 is attainable with respect to γ from ∞ if there exist simply connected domain G such that $0 \in \partial G$, the function $f(z) = z^2 + \omega_0$ is univalent in G, and there exists a closed curve γ^* such that $S_{\gamma^*} \subseteq \overline{G}$ and $f(\gamma^*(t)) = \gamma(t)$; $t \in [0\ 2\pi]$.

In a More descriptive language we can say that ω_0 is attainnable with respect to γ from ∞ if there exists a curve connecting ω_0 with ∞ which does not intersect the curve γ (it may, however, touch γ).

Let A_γ denote the set of all attainable points with respect to γ from ∞. Clearly A_γ is the union of the closure of some of the connected components of S_γ^c, the complement of S_γ, including the unbounded component plus possibly certain segment of S_γ. Let D_γ is the simply connected domain bounded by the quasi - simple curve γ. Then $A_\gamma = D_\gamma^c$ and in particular

$$\gamma \text{ is quasi - simple} \Longrightarrow S_\gamma \subseteq A_\gamma \qquad (1)$$

Definition 1.3 An oriented closed curve $\gamma : [0\ 2\pi] \longrightarrow C$ is said to be in the class **C** if it has the following properties :

C$_1$: γ is piecewise smooth.

C$_2$: γ is locally quasi - simple i.e. for each $t \in [\ 0\ 2\pi]$ there exists $\varepsilon(t) > 0$ such that the arc $\gamma_t := \gamma/[t - \in\ \ t + \in]$ is quasi - simple.

C$_3$: for every $t \in [\ 0\ 2\pi)$ let G_t be a simply connected domain which has γ_t in its (positively oriented) boundary. Then there exists an open neighbourhood U of $z_t = \gamma(t)$ for which $(U \cup G_t) \cup A_\gamma = \Phi$.

C$_4$: the function $\beta_t = \lim\limits_{\tau \to t^+} \arg \dot{\gamma}(\tau), t \in R$ satisfies $\beta(t + 2\pi) - \beta(t) = 2\pi, t \in R$

2. Statement and Proof of the Main Result

Theorem 2.1[1]

Let f be a function holomorphic in the closed unit disk $\bar{\mathrm{D}}$, except possibly at a finite number of points in $\partial\mathrm{D}$, and continuous throughout $\bar{\mathrm{D}}$, normalized by

1 this theorem has been proved in [1], but here we have given a completely different and very short proof, as compared to the proof given in [1], by using the concept of quasi – simple curves introduced and defined in [3]

$f(0) = 0, f'(0) = 1$. Let f be locally univalent on D, $f(z) \neq 0$ on ∂D, the curve $\gamma(\theta) = f(e^{i\theta}) \in$ and

$$Re\left(1 + \frac{zf''}{f'}\right) \geq 0 \text{ on } \partial D \tag{2}$$

except on the set $\mathcal{M} = \{z \in \partial D : f'(z) = 0 \text{ or } f \text{ is not holomorphic at } z\}$.

Let the values of θ with $e^{i\theta} \in$ M be $\theta_0, \theta_1, \theta_2 \ldots, \theta_{n-1}, \theta_n$, where $\theta_0 < \theta_1 < \theta_2 < \ldots < \theta_{n-1} < \theta_n = \theta_0 + 2\pi$. Furthermore, let for any such point $e^{i\theta j}$, if we measure the argument of the tangent to the curve $\{f(e^{i\theta}) : 0 \leq \theta \leq 2\pi\}$ from a point $f(e^{i\tilde{\theta}})$ where $< \theta_j$ and $\tilde{\theta}$ close to θ_j, there exist $k_j \in N$, $\alpha_j, \beta_j \in [0\ \pi)$ such that

$$\lim_{\theta \to \theta_j^+} \arg\left[ie^{i\theta} f'(e^{i\theta})\right] = \arg f(e^{i\theta}) + 2\pi k_j + \alpha_j \tag{3}$$

$$\lim_{\theta \to \theta_j^-} \arg\left[ie^{i\theta} f'(e^{i\theta})\right] = \arg f(e^{i\theta}) + (2\pi k_j + 1)\pi - \beta_j \tag{4}$$

then f is starlike.

Proof: Since the curve $\gamma(\theta) = f(e^{i\theta}) \in \mathbf{C}$ we first show that each arc $\gamma_j = f(e^{i\theta})$, $\theta_j \leq \theta \leq \theta_{j+1}$ is quasi -simple and the points $f(e^{i\theta})$ are attainable with respect to γ from ∞ which guarantees the univalence of the function f on D [3]

If $\mathcal{M}$ has no element then we are back in the classical case and the function f is in fact convex and hence starlike so nothing has to be proved. Now suppose that $\mathcal{M}$ has at least one element. It is easily seen that the quasi - simple property of the arc γ_j can not be destroyed by a negative loop, since this would mean that arg of the tangent at γ_j decreases over a certain interval but this has been ruled out by (2). The other way to destroy the quasi - simple property, namely a positive loop, can not occur either as one can readily verifies by using the construction in [3] that maps the situation on to D preserving the loops (and their orientation). So we see that any positive loop on one of γ_j would increase the total tangent rotation by 2π, but there are no negative loops available to compensate for that. Since $\gamma \in \mathbf{C}$ and hence condition C_4 limits this total rotation to the minimal value of 2π, so there is no room for positive loops, and the γ_j must be quasi - simple.

To show that the points $f(e^{i\theta})$ are attainable with respect to $\gamma = f(e^{i\theta})$ from the ∞ we first note that (3) and (4) implies that there exists $\delta > 0$ such that arg $f(e^{i\theta})$ increases on the intervals $[\theta_{j+1} - \delta, \theta_j)$ and $[\theta_j, \theta_j + \delta)$ for $j = 0, 1, 2, \ldots, n$. See [1] for detail. And we already have shown that there is no rooms for loops so it is clear that the straight line emanating from the point $f(e^{i\theta_j})$ goes to ∞ without intersecting the curve γ. Hence the point $f(e^{i\theta_j})$ are attainable with respect to γ from the ∞. Therefore the curve $\gamma = f(e^{i\theta})$ is quasi – simple and hence [3] the function f is univalent.

To show that f is starlike it remains to show that age $f(e^{i\theta})$ is increasing and the total change in argument of $f(e^{i\theta})$ as θ varies form 0 to 2π is 2π. The last assertion is clear because $f(0) = 0$, f is univalent on $\mathbb{D}$ and f has no zeros on $\partial\mathbb{D}$, so by argument principle $\Delta_{C(=e^{i\theta})} f(e^{i\theta}) = 2\pi$.

To show that arg $f(e^{i\theta})$ is increasing it suffices to show that it is increasing on $[\theta_0, \theta_1]$. We know already that arg $f(e^{i\theta})$ is increasing on $[\theta_0, \theta_0 + \delta]$ and on $[\theta_1 - \delta, \theta_1]$ for some $\delta > 0$. Hence it remains to show that arg $f(e^{i\theta})$ is increasing on $[\theta_0 + \delta, \theta_1 - \delta]$. But if are $f(e^{i\theta})$ is strictly decreasing on certain interval of $[\theta_0 + \delta, \theta_1 - \delta]$ then either there should be at least one loop (positive) on γ or arguments of the tangent should decrease over certain subinterval of $\gamma_1 = f(e^{i\theta})$, $\theta_0 \le \theta \le \theta_1$. But both possibilities are ruled out so are $f(e^{i\theta})$ is increasing for $\theta_0 \le \theta \le \theta_1$ and hence on whole γ. This completes the proof.

3. Topological Criterion For Piecewise Convex And Piecewise α-Convex Functions

Definition 3.1 A quasi – simple curve γ in C is said to be n – piecewise convex curve if there are n points $z_k = \gamma(t_k)$ on S_γ with $t_1 < t_2 < \ldots < t_n < t_{n+1} = t_1 + 2\pi$ such that the function

$$\beta(t) = \lim_{\tau \to t^+} \arg \dot{\gamma}(\tau)$$

is increasing on $S_{\gamma k} = \gamma([t_k, t_{k+1}])$. In other words the quasi – simple curve γ is n – piecewise convex if

$$Re\left(1 + \frac{zf''}{f'}\right) \ge 0$$

is increasing on $S_{\gamma k} = \gamma([t_k, t_{k+1}])$ and the corresponding normalized univalent function f which maps the unit disk D conformally on to the simply connected domain D_γ is called the n-piecewise convex function.

Definition 3.2 Let γ be a quasi – simple curve in C and $\omega \notin S_\gamma$. Then the curve γ is said to be n piecewise α - convex $(0 \leq \alpha \leq 1)$ curve if there are n-points $z_k = \gamma(t_k)$ on S_γ with $t_1 < t_2 < \ldots t_n < t_{n+1} := t_1 + 2\pi$ such that

$$(1 - \alpha)\arg(\gamma(t - \omega_0) + \alpha\beta(t)$$

is increasing on $S_{\gamma k} = \gamma([t_k, t_{k+1}])$, where

$$\beta(t) = \lim_{\tau \to t^+} \arg \gamma(t)$$

and the corresponding normalized univalent function f which maps the unit disk D conformally on to the simply connected domain D_γ is called the n - piecewise α - convex function.

Theorem 3.1 Let f be a function holomorphic in $\bar{D}$, except possibly at a finite number of points on ∂D, and continuous throughout $\bar{D}$, with the normalization $f(0) = 0, f'(0) = 1$. Suppose f is locally univalent on D, $f(z) \neq 0$ on ∂D, and the curve $\gamma(\theta) = f(e^{i\theta}) \in \mathbf{C}$. Let

$$\mathcal{M} = \{z \in \partial D : f'(z) = 0 \text{ or f is not holomorphic at } z\}.$$

and the values of θ such that $e^{i\theta} \in \mathcal{M}$ be $\theta_0, \theta_1, \ldots, \theta_{n-1}, \theta_n$, where $\theta_0 < \theta_1 < \ldots < \theta_{n-1} < \theta_n := \theta_0 + 2\pi$. Then the curve $\gamma = f(e^{i\theta})$ is n-piecewise convex curve and the function f is n-piecewise convex function if

$$Re\left(1 + \frac{zf''}{f'}\right) \geq 0 \text{ on } \partial D \setminus \boldsymbol{M}. \qquad (5)$$

Proof: Since the curve $\gamma(\theta) = f(e^{i\theta}) \in \mathbf{C}$. Therefore from [3], to prove the theorem it suffices to show that each arc $\gamma_j = f(e^{i\theta})$, $\theta_j \leq \theta \leq \theta_{j+1}$ is quasi-simple and the points $z_j = f(e^{i\theta_j})$ are attainable with respect to γ from ∞.

Suppose $\mathcal{M}$ has n elements. If $n = 1$ then the result is obvious. Now suppose $n > 1$. From (5) it is clear that neither any single arc γ_k nor any arc of the form $(\gamma_k \cup \gamma_j)$, the union of any two arcs γ_k and γ_j, with $j \neq k$ can offer a negative loop otherwise the function $\beta(t)$ is strictly decreasing on certain subinterval of either γ_j or γ_k; which contradicts the condition (5). Hence there is no room for negative loop on the whole curve γ at all. The other possibility to destroy the quasi-simple property of the arc γ_k is by forming a positive loop on it. But any positive loop on γ_k would increase the total tangent rotation by 2π, but there are no negative loop available to compensate for that. As $\gamma \in C$, so the condition C_4 limits this total rotation to the minimal value of 2π. Therefore there is no room for a positive loop on the arc γ_k. Hence each arc γ_k, $k = 1, 2, \ldots, n$ are quasi - simple.

It remains to show that the points $f(e^{i\theta})$ are attainable with respect to γ from the ∞. But by using the more or less same arguments we can prove it. We already know that there is no room for negative loop on whole γ so the only possibility to destroy the attainable property of the points $f(e^{i\theta_k})$ is by a positive loop formed on an arc of the form $(\gamma_k \cup \gamma_j)$, the union of any two arcs γ_k and γ_j, with $j \neq k$. But again any positive loop on $(\gamma_k \cup \gamma_j)$ would also increase the total tangent rotation by 2π, but there are no negative loop available to compensate for that. And since $\gamma \in \mathbf{C}$ so the condition C_4 again limits this total rotation to the minimal value of 2π. Therefore there is no room for a positive loop on the arc $(\gamma_k \cup \gamma_j)$, with $j \neq k$ too. Therefore the points $f(e^{i\theta_k})$, for $k = 1, 2, \ldots, n$ are all attainable with respect the curve γ from ∞. Hence by definition γ is a n-piecewise convex curve and the function f is n-piecewise convex function.

By using more or less same techniques and arguments one can easily proved the following theorem on n-piecewise α.

Theorem 3.2 Let f be a function holomorphic in $\overline{\mathrm{D}}$, except possibly at a finite number of points on $\partial\mathrm{D}$, and continuous throughout $\overline{\mathrm{D}}$, with the normalization $f(0) = 0, f'(0) = 1$. Suppose f is locally univalent on D, $f(z) \neq 0$ on $\partial\mathrm{D}$, the curve $\gamma(\theta) = f(e^{i\theta}) \in \mathbf{C}$. Let

$$\mathcal{M} = \{z \in \partial\mathrm{D} : f'(z) = 0 \text{ or } f \text{ is not holomorphic at } z\}.$$

and the values of θ such that $e^{i\theta} \in \mathcal{M}$ be $\theta_0, \theta_1, \ldots, \theta_{n-1}, \theta_n$, where $\theta_0 < \theta_1 < \ldots \theta_{n-1} < \theta_n := \theta_0 + 2\pi$. Then the curve $\gamma = f(e^{i\theta})$ is n-piecewise α-convex curve and the function f is n-piecewise α-convex function if

$$(1 - \alpha)\, Re\left(1 + \frac{zf'}{f(z)}\right) + \alpha Re\left(1 + \frac{zf''}{f'}\right) \geq 0 \text{ on } \partial\mathrm{D} \setminus \mathcal{M} \qquad (6)$$

Proof: Since the curve $\gamma(\theta) = f(e^{i\theta}) \in C$. Therefore from [3], to prove the theorem it suffices to show that each arc $\gamma_j = f(e^{i\theta})$, $\theta_j \leq \theta \leq \theta_{j+1}$ is quasi – simple and the points $z_j = f(e^{i\theta_j})$ are attainable with respect to γ from ∞. Suppose $\mathcal{M}$ has n element. If $n = 1$ then the result is obvious. Now suppose $n > 1$. From (6) it is clear that neither any single arc γ_k nor any arc of the form $(\gamma_k \cup \gamma_j)$, the union of any two arcs γ_k and γ_j, with $j \neq k$ can offer a negative loop otherwise the function $k(t)$ is strictly decreasing on certain subinterval of either γ_j or γ_k; which contradicts the condition (6). Hence there is no room for negative loop on the whole curve γ at all. The other possibility to destroy the q-s property of the arc γ_k is by forming a positive loop on it. But any positive loop on γ_k would increase the total tangent rotation by 2π, but there are no negative loop available to compensate for that. And condition C_4 limits this total rotation to the minimal value of 2π. Therefore there is no room for a positive loop on the arc γ_k. Hence each arc γ_k, $k = 1, 2, \ldots, n$ are

q-s.

It remains to show that the points $f(e^{i\theta_k})$ are attainable with respect to γ from the ∞. But by using the more or less same arguments we can prove it. We already know that there is no room for negative loop on whole γ so the only possibility to destroy the attainable property of the points $f(e^{i\theta_k})$ is by a positive loop formed on an arc of the form $(\gamma_k \cup \gamma_j)$, the union of any two arcs γ_k and γ_j, with $j \neq k$. But again any positive loop on $(\gamma_k \cup \gamma_j)$ would also increase the total tangent rotation by 2π, but there are no negative loop available to compensate for that . And

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Five Layered Temperature Distribution in Human Dermal Part

D. B. GURUNG
Department of Natural Sciences (Mathematics)
School of Science, Kathmandu University
and
SARASWATI ACHARYA
Department of Natural Sciences (Mathematics)
School of Science, Kathmandu University

Abstract: The paper discusses finite element method approach to study one-dimensional steady state temperature distribution in the five layers of dermal part - Stratum Corneum, Stratum Germinativum, Papillary Region, Reticular Region and Subcutaneous Tissue. The physical and physiological parameters in each layer that affect the heat regulation in human body is taken as a function of position dependent. The loss of heat from the outer surface of body to the environment is taken due to convection, radiation and sweat evaporation. The CAS software MATLAB has been used to compute the numerical results. The numerical results so obtained have been exhibited graphically.

Key words: Human dermal part, Bio heat equation, FEM

2000 AMS subject classification: 92C35

1. INTRODUCTION

The skin is a complicated structure with many functions. If any one of the structures in the skin is not working properly, a rash or abnormal sensation is the result. Skin form the largest organ of the body accounting for about 14-16 percentage of an adult person's weight. The transport of heat in living tissue is a complex process involving multiple phenomenological mechanisms including conduction, convection, radiation, metabolism, evaporation and phase change

The temperature regulation in human body occurs due to several mechanisms. These are blood flow, temperature of incoming arterial blood and heat exchange with environment [6]. The various physiological and physical functions performed by the body require energy, which must be supplied by ingested food materials. The body temperature is determined by the balanced between heat produced and heat lost by our body. The skin plays the vital role in thermal balance of our body in addition to its protective activity. The body loses heat mainly through skin to its surrounding by radiation, conduction and evaporation. Thus, the body temperature maintains constant level under normal physiological and atmospheric conditions. Any abnormality arising in its condition disturbs the thermal balance. Body temperature is the result of equilibrium between heat production and heat loss. Blood circulation is the major sources of heat transfer inside the body. The body's ability to regulate temperature is critical to sustaining life. Death is the ultimate result if the body temperature strays to far from the normal range. Human's low critical temperature is about 27^0C and in fact if our temperature raises much above 42^0C death occurs. So the temperature of structures below the skin and subcutaneous tissue should be maintaining 37^0C. The maintenance of body temperature is a dynamic system. If heat loss is greater than heat production then the core temperature drops. Likewise if heat loss is less than heat production then the core temperature rises. A drop or rise in core temperature is equally dangerous, so body temperatures are kept constant. If heat gain does not equal heat loss, the extra heat is stored or lost from the body which affects the temperature distribution in the body. The core temperature in humans is kept relatively constant in an environment with temperatures ranging from values below the lower critical temperatures. This implies that despite large variations in ambient temperature heat production balances heat loss, resulting in a stable core temperature. The situation is deal when heat loss and heat production occur at the same rate.

The model consist five elements as five layers of dermal part with six nodes as nodal points of interface temperatures (Figure 1). The shape function for temperatures in the layers has been considered as a linear function of depth. The thickness of layers has been measured perpendicularly from the outer skin surface towards body core. The outer surface of the skin is assumed to expose to the environment and the loss of heat from the skin surface is considered due to

convection, radiation and sweat evaporation. The nude human subject is assumed to expose to an environment temperature less than 37^0C resulting loss of heat from human body.

2. Mathematical Model

The heat transfer through dermal part is due to conduction in tissue, blood perfusion and metabolic heat generation. The rate of change of tissue temperature T is given by the partial differential equation [10]

$$\rho c \frac{\partial T}{\partial t} = \operatorname{div}\left(K \operatorname{grad} T \right) + M\left(T_a - T\right) + S \tag{1}$$

The differential equation (1) in one dimensional steady state case reduces to

$$K \frac{d^2 T}{dx^2} + M\left(T_a - T\right) + S = 0 \tag{2}$$

where T (^{0}C) denotes the temperature of tissue element at any time t at a distance of x measured perpendicularly into the tissue element from the skin surface, and

ρ	=	Tissue density (g/cm^3)
c	=	Tissue specific heat (cal/g°C)
K	=	Tissue thermal conductivity (cal/cm – min °C)
M	=	$m_b c_b$ (cal/cm^3 – min°C)
m_b	=	Blood mass flow rate (g/cm^3 -min)
c_b	=	Blood specific heat (cal/g°C)
T_a	=	Arterial blood temperature (°C)
S	=	Metabolic heat generation rate (cal/cm^3 – min)

The loss of heat from the skin surface due to convection, radiation and evaporation is considered. So the mixed boundary condition is

$$\left. \frac{\partial T}{\partial x} \right|_{\text{Skin surface}} = h\left(T - T_\infty\right) + LE \tag{3}$$

where, h = Combined heat transfer coefficient due to convection and radiation

T_∞ = Surrounding temperature

L = Latent heat of evaporation

E = Rate of sweat evaporation

The inner body core temperature T_b is assumed to be 37^0C. So, the initial boundary condition is

(4) $$T(L) = T_b = 37^0C$$

where L is the total thickness of skin considered.

The thickness of stratum corneum, stratum germinativum, papillary region, reticular region and subcutaneous tissue have been considered as l_1, $l_2 - l_1$, $l_3 - l_2$, $l_4 - l_3$, $l_5 - l_4$ respectively and T_0, T_1, T_2 T_3, T_4 and $T_5 = T_b$ are the nodal temperatures at a distances $x = 0$, $x = l_1$, $x = l_2$, $x = l_3$, $x = l_4$ and $x = l_5$. $T^{(i)}$, $i = 1, 2, 3, 4, 5$ be the temperature function in the layers stratum corneum, stratum germinativum, papillary region, reticular region and subcutaneous tissue respectively (Figure 1).

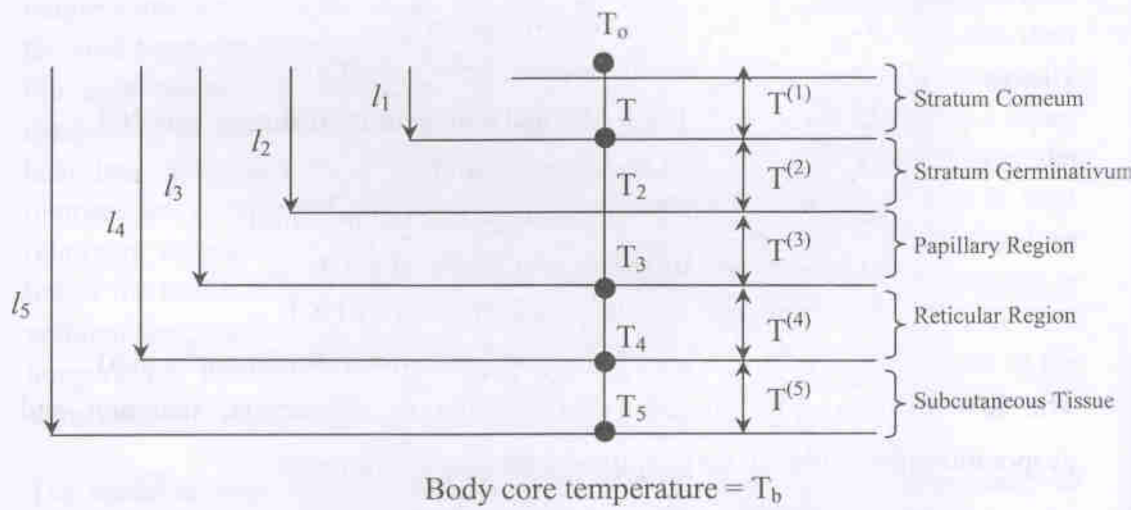

Figure 1: Schematic diagram of five layers of dermal part with nodal points

The anatomical structure of human dermal part makes it reasonable to consider M and S zero in stratum corneum. In the model, the thermal conductivity in dermal

part is considered as constant varying in each layer. All the assumptions for parameters in the layers of dermal part can be summed up as:

Stratum Corneum: $(0 \leq x \leq l_1)$

$$T^{(1)} = T_0 + \frac{T_1 - T_0}{l_1} x$$

$$T_a^{(1)} = 0; \quad K = K^{(1)}; \; M = M^{(1)} = 0; \quad S^{(1)} = 0$$

Stratum Germinativum: $(l_1 \leq x \leq l_2)$

$$T^{(2)} = \frac{l_2 T_1 - l_1 T_2}{l_2 - l_1} + \frac{T_2 - T_1}{l_2 - l_1} x$$

$$T_a^{(2)} = 0; \quad K = K^{(2)}; \quad M = M^{(2)} = 0; \quad S^{(2)} = \left(\frac{x - l_1}{l_4 - l_1}\right) s$$

Papillary Region: $(l_2 \leq x \leq l_3)$

$$T^{(3)} = \frac{l_3 T_2 - l_2 T_3}{l_3 - l_2} + \frac{T_3 - T_2}{l_3 - l_2} x$$

$$T_a^{(3)} = T_b; \quad K = K^{(3)}; \quad M = M^{(3)} = \left(\frac{x - l_2}{l_4 - l_2}\right) m; \quad S^{(3)} = \left(\frac{x - l_1}{l_4 - l_1}\right) s$$

Reticular Region: $(l_3 \leq x \leq l_4)$

$$T^{(4)} = \frac{l_4 T_3 - l_3 T_4}{l_4 - l_3} + \frac{T_4 - T_3}{l_4 - l_3} x$$

$$T_a^{(4)} = T_b; \quad K = K^{(4)}; \quad M = M^{(4)} = \left(\frac{x - l_2}{l_4 - l_2}\right) m; \quad S^{(4)} = \left(\frac{x - l_1}{l_4 - l_1}\right) s$$

Subcutaneous Tissue: $(l_4 \leq x \leq l_5)$

$$T^{(5)} = \frac{l_5 T_4 - l_4 T_5}{l_5 - l_4} + \frac{T_5 - T_4}{l_5 - l_4} x$$

$$T_a^{(5)} = T_b; \quad K = K^{(5)}; \; M = M^{(5)} = m; \quad S = S^{(5)} = s$$

Equation (1) has been used by Perl [11] and Cooper and Trezek [2] to study the solution of some simple problems assuming all the parameters as constant throughout the dermal region. Saxena [14] considered the equation (1) to find

analytic solution for temperature in the layers of dermal part by taking various values of parameters.

The equation (1) has been used by Saxena et al. [15] to study the general formulation for temperature distribution in the natural three layers of human dermal part. Patterson [9] made the experimental study for the measurement of temperature profiles in human skin. Agrawal et al. [1] used FEM technique using equation (1) to construct thermal distribution models in dermal layers of elliptic shaped of human limbs involving metastasis of tumor. Gurung et al. [4] used FEM approach to study the temperature distribution pattern in the three layers of dermal part considering the quadratic shape function. Khanday and Saxena [7] used equation (1) for the estimation of cold effect in dermal regions. Gurung and Saxena [5] used equation (1) to construct an unsteady state mathematical model for temperature distribution in human dermal part under cooling.

3. Solution of the Problem

In the figure 1 the total thickness of skin considered is $L = l_5$ and skin thickness is assumed to measure perpendicularly from skin surface towards body core with interface thickness of the layers as l_1, l_2, l_3, l_4 and l_5.

The boundary condition (4) is automatically incorporated during the computational process of variational integral of the equation (2). So we need only to incorporate the variational integral form of boundary condition (3) in the variational integral form of equation (2). Therefore the variational integral form of equation (2) together with the boundary condition (3) is [3, 8, 12].

$$I = \frac{1}{2}\int_L \left[K\left(\frac{dT}{dx}\right)^2 + M\,(T_a - T)^2 - 2ST \right] dx + \frac{1}{2}h\,(T_o - T_\infty)^2 + LET_o \tag{5}$$

The term without integral sign is employed only for Stratum Corneum.

We write I separately for the five layers – I_1 for stratum corneum, I_2 for stratum germinativum, I_3 for papillary region, I_4 for reticular region and I_5 for subcutaneous tissue. Then

$$(6) \qquad I = \sum_{i=1}^{5} I_i$$

where,

$$(7) \qquad I_1 = \frac{1}{2}\int_{l_0}^{l_1}\left[K^{(1)}\left(\frac{dT^{(1)}}{dx}\right)^2 + M^{(1)}(T_a^{(1)} - T^{(1)})^2 - 2S^{(1)}T^{(1)}\right]dx + \frac{1}{2}h(T_o - T_\infty)^2 + LET_o$$

$$(8) \qquad I_2 = \frac{1}{2}\int_{l_1}^{l_2}\left[K^{(2)}\left(\frac{dT^{(2)}}{dx}\right)^2 + M^{(2)}(T_a^{(2)} - T^{(2)})^2 - 2S^{(2)}T^{(2)}\right]dx$$

$$(9) \qquad I_3 = \frac{1}{2}\int_{l_2}^{l_3}\left[K^{(3)}\left(\frac{dT^{(3)}}{dx}\right)^2 + M^{(3)}(T_a^{(3)} - T^{(3)})^2 - 2S^{(3)}T^{(3)}\right]dx$$

$$(10) \qquad I_4 = \frac{1}{2}\int_{l_3}^{l_4}\left[K^{(4)}\left(\frac{dT^{(4)}}{dx}\right)^2 + M^{(4)}(T_A^{(4)} - T^{(4)})^2 - 2S^{(4)}T^{(4)}\right]dx$$

$$(11) \qquad I_5 = \frac{1}{2}\int_{l_4}^{l_5}[K^{(5)}\left(\frac{dT^{(5)}}{dx}\right)^2 + M^{(5)}(T_a^{(5)} - T^{(5)})^2 - 2S^{(5)}T^{(5)})]dx$$

Evaluating the integrals (7) to (11) with the help of the assumptions for parameters considered in the layers, we get the following system of equations

$$I_1 = A_1 + B_1T_o + D_1T_o^2 + E_1T_1^2 + F_1T_oT_1$$

$$I_2 = A_2 + B_2T_1 + C_2T_2 + D_2T_1^2 + E_2T_2^2 + F_2T_1T_2$$

$$I_3 = A_3 + B_3T_2 + C_3T_3 + D_3T_2^2 + E_3T_3^2 + F_3T_2T_3$$

$$I_4 = A_4 + B_4T_3 + C_4T_4 + D_4T_3^2 + E_4T_4^2 + F_4T_3T_4$$

$$I_5 = A_5 + B_5T_4 + C_5T_5 + D_5T_4^2 + E_5T_5^2 + F_5T_4T_5$$

where A_i, B_i, D_i, E_i, F_i , $1 \le i \le 5$ and C_j, $2 \le j \le 5$ are all constants depending upon the values of physical and physiological parameters of dermal part and are defined in Appendix.

Now as a next step of finite element method, we differentiate I with regard to the nodal temperatures T_o, T_1, T_2, T_3, and T_4. Since $T_5 = T_b$ (the body core temperature), we get

$$2D_1T_o + F_1T_1 = -B_1$$

$$F_1T_o + 2(E_1 + D_2)\,T_1 + F_2T_2 = -B_2$$

$$F_2T_1 + 2(E_2 + D_3)\,T_2 = -C_2 - B_3 - F_3T_3$$

$$F_3T_2 + 2(E_3 + D_4)\,T_3 + F_4T_4 = -C_3 - B_4$$

$$F_4T_3 + 2(E_4 + D_5)\,T_4 = -C_4 - B_5 - F_5T_5$$

The above system of equations in matrix form can be expressed as

(12) $$PT = W$$

where,

$$P = \begin{bmatrix} 2D_1 & F_1 & 0 & 0 & 0 \\ F_1 & 2(E_1+D_2) & F_2 & 0 & 0 \\ 0 & F_2 & 2(E_2+D_3) & F_3 & 0 \\ 0 & 0 & F_3 & 2(E_3+D_4) & F_4 \\ 0 & 0 & 0 & F_4 & 2(E_4+D_5) \end{bmatrix};$$

$$T = \begin{bmatrix} T_o \\ T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} \text{ and } W = \begin{bmatrix} -B_1 \\ -B_2 \\ -C_2 - B_3 \\ -C_3 - B_4 \\ -C_4 - B_5 - F_5T_5 \end{bmatrix}$$

4. Numerical Results and Discussion

In the system of equations (12), the unknown variables are the nodal values T_0, T_1, T_2, T_3, and T_4. The temperature distribution profiles in the layers of dermal part rely on these nodal values. To determine these nodal values, we consider the thicknesses of interfaces of layers as given in Table - 1. The values of M, S, and E at different atmospheric temperatures are considered as shown in Table - 2. The other physical and physiological parameters considered as [3, 4]

$$K_1 = 0.030\text{cal/cm} - \text{min}\ ^\circ\text{C}$$

$$K_2 = 0.030\text{cal/cm} - \text{min}\ ^\circ\text{C}$$

$$K_3 = 0.045\text{cal/cm} - \text{min}\ ^\circ\text{C}$$

$$K_4 = 0.045\text{cal/cm} - \text{min}\ ^\circ\text{C}$$

$$K_5 = 0.06\text{cal/cm} - \text{min}\ ^\circ\text{C}$$

$$h = 0.009\text{cal/cm}^2 - \text{min}\ ^\circ\text{C}$$

$$L = 579\text{cal/g}$$

Table -1: Thicknesses of interface of layers considered [15]

Sets	l_1(cm)	l_2(cm)	l_3(cm)	l_4(cm)	l_5(cm)
I	0.05	0.10	0.20	0.35	0.5
II	0.05	0.10	0.25	0.40	0.90

Table - 2: Parameter values at different atmospheric temperatures [15]

Atm. Temp. (T_a^0C)	S (cal/cm^3 – min)	M = $m_b c_b$ (cal/cm^3 – min^{0}c)	E (× 10^{-3}) gm/cm^2 – min
15	0.0357	0.003	0
23	0.018	0.018	0, 0.48
33	0.018	0.0315	0.48, 0.96

Using the above numerical values in equations (12) the profiles for temperature distribution in the layers of dermal part obtained is as shown in figure 2 and figure 3. The numeral put on the curves in each graph indicate the followings:

(i) $T_\infty = 23^0C, E = 0$

(ii) $T_\infty = 15^0C, E = 0$

(iii) $T_\infty = 33^0C, E = 0.48 \times 10^{-3}$

(iv) $T_\infty = 23^0C, E = 0.48 \times 10^{-3}$

(v) $T_\infty = 33^0C, E = 0.96 \times 10^{-3}$

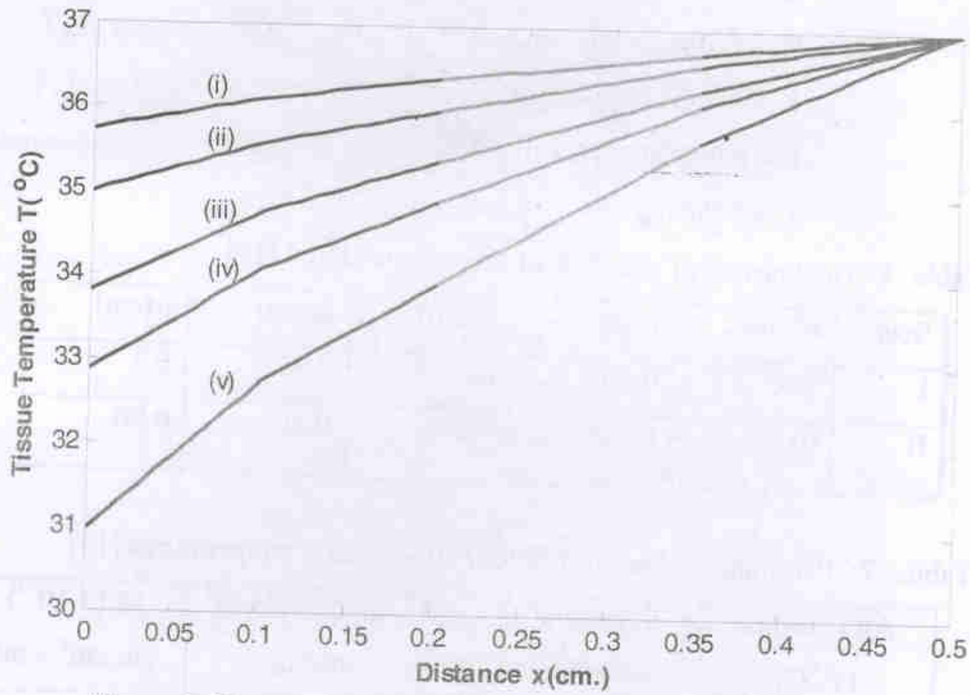

Figure 2: Temperature profiles for Set – I of dermal layers

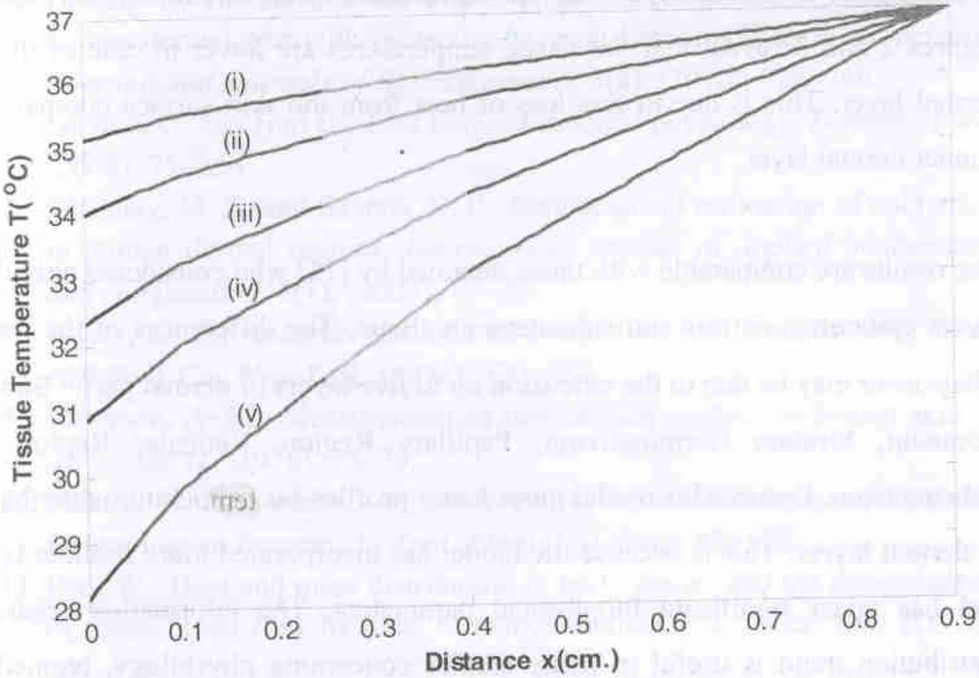

Figure 3: Temperature profiles for Set – II of dermal layers

From, the figure 2 and figure 3, we find that the tissue temperature at $T_\infty = 15^0C$ is less than that at $T_\infty = 23^0C$ at same evaporation rate $E = 0$. This is due to the high rate of blood mass flow at $T_\infty = 23^0C$ than $T_\infty = 15^0C$. Same reason we can discuss for the tissue temperature at $T_\infty = 33^0C$ and $T_\infty = 23^0C$ having the same evaporation rate $E = 0.48\times 10^{-3}$. Also, we can observe that at $T_\infty = 23^0C$ and $E = 0.48 \times 10^{-3}$ the tissue temperature is higher than at $T_\infty = 33^0C$ and $E = 0.96\times 10^{-3}$ even though blood mass flow rate is higher in the case of $T_\infty = 33^0C$ than $T_\infty = 23^0C$. So at this temperature evaporation rate has more effect in tissue temperature. Thus sweat evaporation rate E has major effect in temperature distribution. If E increases under same atmospheric temperature, the tissue temperature decreases and vice versa.

The thickness of dermal layers has the significance effect in temperature profiles. Figures 2 and 3 reveal that the tissue temperatures are lower in case of thicker dermal layer. This is due to less loss of heat from the skin surface compared to thinner dermal layer.

The results are comparable with those obtained by [15] who considered only three layers epidermis, dermis and subcutaneous tissue. The differences of the results whatsoever may be due to the extension up to five layers of dermal part – Stratum Corneum, Stratum Germinativum, Papillary Region, Reticular Region and Subcutaneous Tissue. This model gives better profiles for temperature distribution in dermal layers. This is because the model has incorporated more feasible layers and has taken significant biophysical parameters. The information regarding distribution trend is useful in many studies concerning physiology, biomedical and allied sciences.

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Appendix

$$A_1 = \frac{1}{2}hT_\infty^{\,2};\quad B_1 = LE - hT_\infty;\quad D_1 = \frac{1}{2}\left(\frac{K^{(1)}}{l_1} + h\right);\quad E_1 = \frac{K^{(1)}}{2l_1};\quad F_1 = -\frac{K^{(1)}}{l_1}$$

$$A_2 = 0;\ \ B_2 = -P_1;\ C_2 = -2P_1;\ D_2 = \frac{1}{2R_1};\ E_2 = \frac{1}{2R_1};\ \ F_2 = -R_1;$$

$$A_3 = \frac{T_b^2(l_3-l_2)^2}{4(l_4-l_2)};\ \ B_3 = -P_2(l_3-l_2)^2 - N_1(l_3^2+l_2l_3-2l_2^2-3l_1l_3+3l_1l_2)$$

$$C_3 = P_2(4l_2l_3 - 2l_2^2 - 2l_3^2) + N_1(l_2l_3 + l_2^2 - 2l_3^2 - 3l_1l_2 + 3l_1l_3)$$

$$D_3 = \frac{1}{2}R_2 + \frac{1}{24}Q_1(l_3^3 - 3l_2l_3^2 + 3l_2^2l_3 - l_2^3)$$

$$E_3 = \frac{1}{2}R_2 + \frac{1}{8}Q_1(l_3-l_2)^3$$

$$F_3 = -R_2 + \frac{1}{12}Q_1(l_3-l_2)^3$$

$$A_4 = \frac{T_b^2 m}{4(l_4-l_2)}(l_4^2 - l_3^2 - 2l_2l_4 + 2l_2l_3)$$

$$B_4 = -P_2(l_4^2+l_3l_4-2l_3^2-3l_2l_4+3l_2l_3) - N_1(l_4^2+l_3l_4-2l_3^2-3l_1l_4+3l_1l_2)$$

$$C_4 = P_2(l_3l_4+l_3^2-2l_4^2-3l_2l_3+3l_2l_4) + N_1(l_3l_4+l_3^2-2l_4^2-3l_1l_3+3l_1l_4)$$

$$D_4 = \frac{1}{2}R_3 + \frac{1}{24}Q_2(l_4^3+l_3l_4^2-5l_3^2l_4+3l_3^3-4l_2l_4^2-4l_2l_3^2+8l_2l_3l_4)$$

$$E_4 = \frac{1}{2}R_3 + \frac{1}{24}Q_2(l_3^2l_4+l_3^3-5l_3l_4^2+3l_4^3-4l_2l_4^2-l_2l_3^2+8l_2l_3l_4)$$

$$F_4 = -R_3 - \frac{1}{12}Q_2(l_3l_4^2+l_3^2l_4-l_4^3-l_3^3-4l_2l_3l_4+2l_2l_4^2+2l_2l_3^2)$$

$$A_5 = \frac{1}{2}mT_b^2(l_5-l_4);\ \ B_5 = -P_3(l_5-l_4);\ \ C_5 = P_3(l_4-l_5)$$

$$D_5 = \frac{1}{2}R_4 + Q_3;\ E_5 = \frac{1}{2}R_4 + Q_3;\ F_5 = -R_4 + Q_3;$$

where,

$$P_1 = \frac{s(l_2-l_1)^2}{6(l_4-l_1)};\ \ P_2 = \frac{T_b m}{6(l_4-l_2)};\ P_3 = \frac{1}{2}(T_b m + s);$$

$$R_1 = \frac{K^{(2)}}{(l_2-l_1)};\ \ R_2 = \frac{K^{(3)}}{(l_3-l_2)};\ R_3 = \frac{K^{(4)}}{(l_4-l_3)};\ R_4 = \frac{K^{(5)}}{(l_5-l_4)};$$

$$Q_1 = \frac{m}{(l_3-l_2)(l_4-l_2)};\ \ Q_2 = \frac{m}{(l_4-l_3)(l_4-l_2)};\ \mathbf{Q_3} = \frac{\mathbf{m(l_5-l_4)}}{\mathbf{6}};\ \ \mathbf{N_1} = \frac{\mathbf{s}}{\mathbf{6(l_4-l_1)}}$$

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On Locally Convex Space Valued Function Spaces $c_0(X,E,M,\lambda,p), c(X,E,M,\lambda,p)$ and $l_\infty(X,E,M,\lambda,p)$ defined by Orlicz Function

J. K. SRIVASTAV[a] and N. P. PAHARI[b]
[a] Department of Mathematics and Statistics,
D.D.U. Gorakhpur University, India
Email : jks_ddugu@yahoo.com
[b] Department of Mathematics,
Tribhuvan University, Trichandra Campus,
Kathmandu, Nepal
Email : nppahari @gmail.com

Abstract: *The aim of this paper is to display some of the new properties of locally convex space valued function spaces* $c_0(X,E,M,\lambda,p), c(X,E,M,\lambda,p)$ and $l_\infty(X,E,M,\lambda,p)$ *as a generalization of well known sequence spaces and function spaces.*

Keywords and Phrases: Orlicz function, seminorm, locally convex topological vector space.

1. Prerequisites:

We begin with recalling some of the basic definitions that are used in this paper. A subset A *of a linear space over the field* **C** *of complex numbers* is *called convex if* $x_1, x_2 \in A$, $\alpha+\beta=1$, $\alpha \geq 0$, $\beta \geq 0$ *implies* $\alpha x_1 + \beta x_2 \in A$ *and balanced if* $x \in A, |\alpha| \leq 1$ *implies* $\alpha x \in A$.

A real valued function f: $A \to R$ *defined on a convex set* A *in a linear space is said to be convex if for any two points* $x_1, x_2 \in A$, *the inequality* $f(\alpha x_1 + \beta x_2) \leq \alpha f(x_1) + \beta f(x_2)$, $\alpha+\beta=1, 0 \leq \alpha, \beta \leq 1$ *is satisfied. A function* $M:[0,\infty) \to [0,\infty)$ *is called an Orlicz if it is continuous, non decreasing and convex with* $M(0)=0, M(x)>0$ *for* $x>0$ *and* $M(x) \to \infty$ *as* $x \to \infty$. *If* M *takes only finite values, then we write* M $< \infty$, *(see[1,2]). A seminorm* p *on a linear space* X *is a function* p: $X \to R^+$ *such that*

$p(\alpha x)=|\alpha|p(x)$ *and* $p(x_1+x_2)\le p(x_1)+p(x_2)$ *for all* $x_1,x_2\in X$ *and scalar* α.

A topological vector space (TVS) is a linear space X which has a topology $\mathcal{T}$ such that the algebraic operations of addition and scalar multiplication in X are continuous. It is called locally convex if there exists a local base $\mathcal{B}$ whose members are convex sets and Hausdorff if the underlying topology $\mathcal{T}$ is separated .The gauge or Minkowski functional of a set U in a vector space X is a $x\to p_U(x)$ from X into the extended set $R^+\cup\{\infty\}$ of non negative real numbers defined as :

$$p_U(x)=\inf\{\rho>0:x\in\rho U\}.$$

In a vector space X, the gauge of an absorbing and convex set is a seminorm (see [3,5,6]).

2. Notations and Terminology:

Let M be the Orlicz function, X be an arbitrary set (not necessarily countable) and $\mathcal{F}(X)$ be the collection of all finite subsets of X. Let (E, $\mathcal{T}$) be a Hausdorff locally convex topological vector space (lc TVS) over the field of complex number **C**. Let u(E) denote the fundamental system of balanced, convex and absorbing neighbourhoods of zero vector θ.

p_U will denote gauge or Minkowski functional of $U\in u(E)$. Thus, $D=\{P_U:U\in u(E)\}$ is the collection of all continuous seminorms generating the topology $\mathcal{T}$ of E (see [4,5,6,7]).

We shall write p,q for the functions on $X\to R^+$ and

$$l_\infty(X,R^+)=\{p:X\to R^+ \text{ such that} \sup_x p(x)<\infty\}.$$

Further we write λ , μ for the functions on $X\to C-\{0\}$ and the collection of all such functions will be denoted by $s(X,C-\{0\})$.We shall also frequently use the notations:

$$t(x)=\left|\frac{\lambda(x)}{\mu(x)}\right|^{p(x)},\ H(x)=\sup_x p(x),\ L=\max\{1,H\} \textit{ and for scalar } \alpha,$$

$$A[\alpha]=\max(1,|\alpha|).$$

But when the function p(x) and q(x) occur, then to distinguish H, we use the notations H(p) and H(q) respectively.

3. *The Classes* $c_0(X,E,M,\lambda,p), c(X,E,M,\lambda,p), l_\infty(X,E,M,\lambda,p)$

We introduce the following classes of lc TVS E valued function spaces:

(*i*) $c_0(X,E,M,\lambda,p) = \{\phi : X \to E :$ *there exists* $\rho > 0$ *such that for given* $\varepsilon > 0$ *and* $p_\upsilon \in D$, there exists $J \in \mathcal{F}(X)$ satisfying

$$M\left(\frac{[p_\upsilon(\lambda(x)\phi(x))]^{p(x)}}{\rho}\right) < \varepsilon, \textit{ for each } x \in X \setminus J\},$$

(*ii*) $c(X,E,M,\lambda,p) = \{\phi : X \to E :$ *there exists* $\rho > 0$ *and* $l \in E$, *such that for given* $\varepsilon > 0$ *and* $p_\upsilon \in D$, there exists $J \in \mathcal{F}(X)$ satisfying

$$M\left(\frac{[p_\upsilon(\lambda(x)\phi(x)) - l]^{p(x)}}{\rho}\right) < \varepsilon, \textit{ for each } x \in X \setminus J\}, \textit{and}$$

(*iii*) $l_\infty(X,E,M,\lambda,p) = \{\phi : X \to E :$ *there exists* $\rho > 0$ *such that for* given $p_\upsilon \in D$, there exists $J \in \mathcal{F}(X)$ satisfying

$$\sup_x M\left(\frac{[p_\upsilon(\lambda(x)\phi(x))]^{p(x)}}{\rho}\right) < \infty, \textit{ for each } x \in X \setminus J\}.$$

Further, when $\lambda : X \to C - \{0\}$ is a function such that $\lambda(x) = 1$ for all $x \in X$, then $c_0(X,E,M,\lambda,p)$ will be denoted by $c_0(X,E,M,p)$ and when $p : X \to R^+$ is a function such that $p(x)=1$ for all $x \in X$, then $c_0(X,E,M,\lambda,p)$ will be denoted by $c_0(X,E,M,\lambda)$. Similarly, we define *c(X,E, M ,p)*, $c(X,E,M,\lambda), l_\infty(X,E,M,p)$ and $l_\infty(X,E,M,\lambda)$.

4. Main Results:

In the present paper, we study the containment of the classes $c_0(X,E,M,\lambda,p), c(X,E,M,\lambda,p)$ *and* $l_\infty(X,E,M,\lambda,p)$ in terms of different p and λ so that such a class is contained in or equal to another class of same kind. Throughout the paper, proofs of the results of $c(X,E,M,\lambda,p)$ and $l_\infty(X,E,M,\lambda,p)$ are analogous to those of $c_0(X,E,M,\lambda,p)$ and hence are omitted.

Theorem 4.1 : *If* $p \in l_\infty(X, R^+)$*, then for any* $\lambda, \mu \in s(X, C-\{0\})$,

a) $c_0(X,E,M,\lambda,p) \subset c_0(X,E,M,\mu,p)$ *if and only if* $\liminf_x t(x) > 0$

b) $l_\infty(X,E,M,\lambda,p) \subset l_\infty(X,E,M,\mu,p)$ *if and only if* $\liminf_x t(x) > 0$.

Proof: We prove the result for $c_0(X,E,M,\lambda,p)$ only. For sufficiency, assume that $\lim\ \inf_x t(x) > 0$. *Then* $\exists$ *a* constant $m > 0$ such that for all but finitely many $x \in X$, we have

$$m|\mu(x)|^{p(x)} < |\lambda(x)|^{p(x)}$$

Let $\phi \in c_0(X,E,M,\lambda,p), \varepsilon > 0$ *and* $p_\upsilon \in D$. Then for all but finitely many $x \in X$ and for some $\rho_1 > 0$, we have

$$M\left(\frac{[p_\upsilon(\lambda(x)\,\phi(x))]^{p(x)}}{\rho_1}\right) < \varepsilon.$$

Let us choose ρ such that $\rho_1 < m\rho$. Since M is non decreasing, we have

$$M\left(\frac{[p_\upsilon(\mu(x)\,\phi(x))]^{p(x)}}{\rho}\right) = M\left(\frac{[|(\mu(x)|p_\upsilon(\phi(x))]^{p(x)}}{\rho}\right)$$

$$\le M\left(\frac{[|(\lambda(x)|p_\upsilon(\phi(x))]^{p(x)}}{m\rho}\right)$$

$$\le M\left(\frac{[p_\upsilon(\lambda(x)\,\phi(x))]^{p(x)}}{\rho_1}\right) < \varepsilon,$$

for all but finitely many $x \in X$. Since $p_U \in D$ is an arbitrary. So it clearly shows that $\phi \in c_0(X,E,M,\lambda,P)$. This proves that $c_0(X,E,M,\lambda,p) \subset c_0(X,E,M,\mu,p)$.

For necessity, suppose that $c_0(X,E,M,\lambda,p) \subset c_0(X,E,M,\mu,p)$ but $\lim \inf_x t(x) = 0$.

Then we can find a sequence (x_k) in X such that

$$\forall k \geq 1,\ |t(x_k)|^{p(x_k)} < \frac{1}{k} \qquad \text{i.e;} \qquad k\ |\lambda(x_k)|^{p(x_k)} < |\mu(x_k)|^{p(x_k)}.$$

Now, choose $u \in E$ and $p_V \in D$ such that $p_V(u)=1$ and define $\phi : X \to E$ by

$$\phi(x) = (\lambda(x))^{-1} k^{-1/p(x)} u\ ,\ for\ x = x_k,\ k = 1,2,3,..$$
$$= \theta\ ,\ \text{otherwise.}$$

Then for each $p_v \in D$ *and each* $k \geq 1$, we have

$$M\left(\frac{[p_U(\lambda(x_k)\phi(x_k))]^{p(x_k)}}{\rho}\right) = M\left(\frac{[p_v(k^{-1/p(x_k)}u)]^{p(x_k)}}{\rho}\right) \qquad \leq M\left(\frac{\frac{1}{k}[p_v(u)]^{p(x_k)}}{\rho}\right)$$

$$\leq \frac{1}{k} M\left(\frac{A[p_v(u)]^{H(P)}}{\rho}\right) \text{ for some finite subset of } X.$$

This shows that $\phi \in c_0(X, E, M, \lambda, p)$. But for each $k \geq 1$, we have

$$M\left(\frac{[p_v(\mu(x_k)\phi(x_k))]^{p(x_k)}}{\rho}\right) = M\left(\frac{[p_v(\mu(x_k).(\lambda(x_k))^{-1} k^{-1/p(x_k)} u]^{p(x_k)}}{\rho}\right)$$

$$= M\left(\frac{1}{k\rho}\left|\frac{\mu(x_k)}{\lambda(x_k)}\right|^{p(x_k)} .[p_v(u)]^{p(x_k)}\right) > M\left(\frac{1}{\rho}\right)$$

which is independent of k. This shows that $\phi \notin c_0(X, E, M, \mu, p)$, a contradiction.

This completes the proof.

Theorem 4.2 : *Let* $p \in l_\infty(X, R^+)$. *Then for any* $\lambda, \mu \in s(X, C - \{0\})$,

a) $c_0(X, E, M, \mu, p) \subset c_0(X, E, M, \lambda, p)$ *if and only if lim* $sup_x t(x) < \infty$,

b) $l_\infty(X, E, M, \mu, p) \subset l_\infty(X, E, M, \lambda, p)$ *if and only if lim* $sup_x t(x) < \infty$.

Proof : a) For sufficiency, assume that *lim sup*$_x t(x) < \infty$. Then $\exists$ a constant $L > 0$ such that

$$|t(x)|^{p(x)} < L \quad \text{i.e.,} \qquad |\lambda(x)|^{p(x)} < L|\mu(x)|^{p(x)}$$

for all but finitely many $x \in X$.

If $\phi \in c_0(X, E, M, \mu, p)$, then analogous to the proof of the Theorem 4.1, we can show that $\phi \in c_0(X, E, M, \lambda, p)$.

For necessity, suppose that $c_0(X, E, M, \mu, p) \subset c_0(X, E, M, \lambda, p)$ but *lim sup*$_x\, t(x) = \infty$. Then $\exists$ a sequence (x_k) in X such that for each $k \geq 1$,

$$|t(x_k)|^{p(x_k)} \;>\; k \quad i.e; \quad |\lambda(x_k)|^{p(x_k)} < k|\mu(x_k)|^{p(x_k)}.$$

Choose $u \in E$ *and* $p_r \in D$ *with* $p_r(u) = 1$, *define* $\phi : x \to E$ *by*

$$\phi(x) = (\mu(x))^{-1} k^{-1/p(x)} u \;\; for \;\; x = x_k,\; k = 1,2,3..... \; and$$

$$= \theta, otherwise.$$

Then as in Theorem 4.1 a), we can show that $\phi \in c_0(X, E, M, \mu, p)$, but $\phi \notin c_0(X, E, M, \lambda, p)$, a contradiction. This completes the proof.

On combining Theorems 4.1 and 4.2, we have

Theorem 4.3: *Let* $p \in l_\infty(X, R^+)$.*Then for any* $\lambda, \mu \in s(X, C - \{0\})$,

a) $c_0(X, E, M, \lambda, p) = c_0(X, E, M, \mu, p)$

if and only if $0 < \liminf_x t(x) \leq \limsup_x t(x) < \infty$.

b) $l_\infty(X, E, M, \lambda, p) = l_\infty(X, E, M, \mu, p)$

if and only if $0 < \liminf_x t(x) \leq \limsup_x t(x) < \infty$.

Corollary 4.4 : *If* $p \in l_\infty(X, R^+)$ *and* $\lambda \in s(X, C - \{0\})$. *Then*

i) $c_0(X, E, M, \lambda, p) \subset c_0(X, E, M, p)$ *if and only if lim inf*$_x$ $|\lambda(x)|^{p(x)} > 0$.

ii) $c_0(X,E,M,p) \subset c_0(X,E,M,\lambda,p)$ *if and only if* $\lim\sup_x |\lambda(x)|^{p(x)} < \infty$.

iii) $c_0(X,E,M,\lambda,p) = c_0(X,E,M,p)$ *if and only if*

$$0 < \lim\inf_x |\lambda(x)|^{p(x)} \leq \lim\sup_x |\lambda(x)|^{p(x)} < \infty.$$

Proof: If we consider $\lambda : X \to C-\{0\}$ *such that* $\lambda(x)=1$, *for each* x, in Theorems 4.1, 4.2 and 4.3, we obtain the results i), ii) and iii) respectively. Theorem 4.3 and Corollary 4.4 equally hold for $l_\infty(X,E,M,\lambda,p)$.

Theorem 4.5 : *If* $p \in l_\infty(X,R^+), q : X \to R^+$ *and* $\lambda \in s(X,C-\{0\})$, *then*

a) $c_0(X,E,M,\lambda,p) \subset c_0(X,E,M,\lambda,q)$ *if and only if* $\lim\inf_x \dfrac{q(x)}{p(x)} > 0$.

b) $c(X,E,M,\lambda,p) \subset c(X,E,M,\lambda,q)$ *if and only if* $\lim\inf_x \dfrac{q(x)}{p(x)} > 0$.

Proof :a) For sufficiency, assume that $\lim\inf_x \dfrac{q(x)}{p(x)} > 0$. Then $\exists$ a positive constant m such that $\dfrac{q(x)}{p(x)} > m$ *i.e* $q(x) > m\,p(x)$ for all but finitely many $x \in X$.

Let $\phi \in c_0(X,E,M,\lambda,p)$ *and* $p_U \in D$. Then for some $\rho > 0$, we have

$$M\left(\frac{[p_U(\lambda(x)\phi(x))]^{p(x)}}{\rho}\right) < 1.$$

But M is non decreasing and

$$M\left(\frac{[p_U(\lambda(x)\phi(x))]^{q(x)}}{\rho}\right) \leq M\left(\frac{[p_U(\lambda(x)\phi(x))]^{p(x)}}{\rho}\right)^m < 1$$

holds for all but finitely many $x \in X$. This implies that $\phi \in c_0(X,E,M,\lambda,q)$.

Analogous to the Theorem 4.1, necessity of the theorem follows.

Corresponding theorem for the function space $l_\infty(X,E,M,\lambda,p)$,we have:

Theorem 4.6 :

If $q \in l_\infty(X,R^+), p : X \to R^+$ *and* $\lambda \in s(X, C-\{0\})$, *then*

$$l_\infty(X,E,M,\lambda,q) \subset l_\infty(X,E,M,\lambda,p) \text{ if and only if } \liminf_x \frac{q(x)}{p(x)} > 0.$$

Next two Theorems 4.7 and 4.8 follow on the lines of theorem 4.2 a).

Theorem 4.7 : *Let* $p : X \to R^+, q \in l_\infty(X,R^+)$ *and* $\lambda \in s(X, C-\{0\})$,*then*

a) $c_0(X,E,M,\lambda,q) \subset c_0(X,E,M,\lambda,p)$ *if and only if* $\limsup_x \dfrac{q(x)}{p(x)} < \infty.$

b) $c(X,E,M,\lambda,q) \subset c(X,E,M,\lambda,p)$ *if and only if* $\limsup_x \dfrac{q(x)}{p(x)} < \infty.$

Theorem 4.8 : *Let* $q : X \to R^+, p \in l_\infty(X,R^+)$ *and* $\lambda \in s(X, C-\{0\})$,*then*

$$l_\infty(X,E,M,\lambda,p) \subset l_\infty(X,E,M,\lambda,q) \text{ if and only if } \limsup_x \frac{q(x)}{p(x)} < \infty.$$

On combining Theorems 4.7 and 4.8, one can easily obtain

Theorem.4.9: *If* $p,q \in l_\infty(X,R^+)$ *and* $\lambda \in s(X, C-\{0\})$, *then the relations*

a) $c_0(X,E,M,\lambda,p) = c_0(X,E,M,\lambda,q)$

b) $c(X,E,M,\lambda,p) = c(X,E,M,\lambda,q)$

c) $l_\infty(X,E,M,\lambda,p) = l_\infty(X,E,M,\lambda,q)$

hold if and only if

$$0 < \liminf_x \frac{q(x)}{p(x)} \le \limsup_x \frac{q(x)}{p(x)} < \infty.$$

Corollary 4.10: *Assume that* $p \in l_\infty(X,R^+)$ *and* $\lambda \in s(X, C-\{0\})$, *then*

i) $c_0(X,E,M,\lambda) \subset c_0(X,E,M,\lambda,p)$ *if and only if* $\liminf_x p(x) > 0;$

ii) $c_0(X,E,M,\lambda,p) \subset c_0(X,E,M,\lambda)$ *if and only if* $\limsup_x p(x) < \infty$;

iii) $c_0(X,E,M,\lambda,p) = c_0(X,E,M,\lambda)$ *if and only if*

$$0 < \liminf_x p(x) \le \limsup_x p(x) < \infty.$$

Proof follows easily if we take p(x) =*1 for all* x *and the function* q *is replaced by the function* p *in the Theorems 4.7 , 4.8 and 4.9.*

Corollary 4.10 equally holds for $c(X,E,M,\lambda,p)$ *and* $l_\infty(X,E,M,\lambda,p)$.

Theorem 4.11 **:** If $\lambda,\mu \in s(X, C-\{0\})$, $p \in l_\infty(X,R^+)$ *and* $q : X \to R^+$, *then*

$c_0(X,E,M,\lambda,p) \subset c_0(X,E,M,\mu,q)$ *if and only if*

$$\text{i) } \liminf_x t(x) > 0 \text{ and ii) } \liminf_x \frac{q(x)}{p(x)} > 0.$$

Proof of the theorem follows immediately from the Theorems 4.1 a) and 4.5.

Theorem 4.12 **:** If $\lambda,\mu \in s(X, C-\{0\})$, $p \in l_\infty(X,R^+)$ *and* $q : X \to R^+$, *then*

$l_\infty(X,E,M,\lambda,p) \subset l_\infty(X,E,M,\mu,q)$ *if and only if*

$$\text{i) } \liminf_x t(x) > 0 \text{ and ii) } \limsup_x \frac{q(x)}{p(x)} < \infty.$$

Proof follows from the Theorems 4.1 b) and 4.8.

In the following example, we show that $c_0(X,E,M,\lambda,p)$ may strictly be contained in $c_0(X,E,M,\mu,q)$ although the conditions *(i)* and *(ii)* of Theorem 4.11 hold.

ii) $c_0(X,E,M,\lambda,p) \subset c_0(X,E,M,\lambda)$ *if and only if lim sup*$_x$ $p(x) < \infty$;

iii) $c_0(X,E,M,\lambda,p) = c_0(X,E,M,\lambda)$ *if and only if*

$$0 < \liminf_x p(x) \le \limsup_x p(x) < \infty.$$

Proof follows easily if we take p(x) *=1 for all* x *and the function* q *is replaced by the function* p *in the Theorems 4.7 , 4.8 and 4.9.*

Corollary 4.10 equally holds for $c(X,E,M,\lambda,p)$ *and* $l_\infty(X,E,M,\lambda,p)$.

Theorem 4.11 : If $\lambda,\mu \in s(X, C-\{0\})$, $p \in l_\infty(X,R^+)$ *and* $q : X \to R^+$, *then*

$c_0(X,E,M,\lambda,p) \subset c_0(X,E,M,\mu,q)$ *if and only if*

$$\textit{i)}\ \liminf_x t(x) > 0 \ \ \textit{and ii)}\ \liminf_x \frac{q(x)}{p(x)} > 0.$$

Proof of the theorem follows immediately from the Theorems 4.1 a) and 4.5.

Theorem 4.12 : If $\lambda,\mu \in s(X, C-\{0\})$, $p \in l_\infty(X,R^+)$ *and* $q : X \to R^+$, *then*

$l_\infty(X,E,M,\lambda,p) \subset l_\infty(X,E,M,\mu,q)$ *if and only if*

$$\textit{i)}\ \liminf_x t(x) > 0 \ \ \textit{and ii)}\ \limsup_x \frac{q(x)}{p(x)} < \infty.$$

Proof follows from the Theorems 4.1 b) and 4.8.

In the following example, we show that $c_0(X,E,M,\lambda,p)$ may strictly be contained in $c_0(X,E,M,\mu,q)$ although the conditions *(i)* and *(ii)* of Theorem 4.11 hold.

Example:4.13 Let X be any set and (x_k) be a sequence of distinct points of X. Consider $u \in E$ *and* $p_\upsilon \in D$ such that $p_V(u)=1$ and define $\phi : X \to E$ by

$$\phi(x) = k^{-k}u, \text{ if } x = x_k, k = 1,2,3,.. \quad and$$
$$= \theta, \text{ otherwise}.$$

Further, if $x = x_k$, we define $p(x_k)=k^{-1}$, if k is odd integer, $p(x_k)=k^{-2}$, if k is even integer,

$q(x_k) = k^{-1}$ for all values of k. $\lambda(x_k) = 3^k$, $\mu(x_k) = 2^k$ for all values of k; and

$p(x) = \frac{1}{2}, q(x) = 1, \lambda(x) = 3, \mu(x) = 2$ otherwise. Then for $x = x_k$ *and* $k \geq 1$, we have

$$t(x_k) = \left|\frac{\lambda(x_k)}{\mu(x_k)}\right|^{p(x_k)} = \left|\frac{3^k}{2^k}\right|^{k-1} = \frac{3}{2}, \text{ if } k \text{ is odd intger;}$$

$$t(x_k) = \left|\frac{3^k}{2^k}\right|^{k-2} = \left|\frac{3}{2}\right|^{1/k}, \text{ if } k \text{ is even integer and } t(x) = |3/2|^{1/2} \text{ otherwise.}$$

Thus, $\liminf_x t(x) > 1$, i.e; condition (i) of theorem 4.11 is satisfied. Further,

$\frac{q(x_k)}{p(x_k)} = 1$, if k is an odd integer, $\frac{q(x_k)}{p(x_k)} = k$, if k is an even integer and

$\frac{q(x)}{p(x)} = 2$ otherwise.

Therefore $\liminf_x \frac{q(x)}{p(x)} = 1 > 0$, i.e; condition (ii) of theorem 4.11 is also satisfied.

Now, for *any* $p_\upsilon \in D$,

$$M\left(\frac{[p_\upsilon(\mu(x_k)\phi(x_k))]^{q(x_k)}}{\rho}\right) = M\left(\frac{[p_\upsilon(2^k k^{-k}u)]^{1/k}}{\rho}\right) = M\left(\frac{2}{k\rho}[p_\upsilon(u)]^{1/k}\right) \leq \frac{1}{k} M\left(\frac{2A[p_U(u)]}{\rho}\right), k \geq 1,$$

and $M\left(\dfrac{[p_v(\mu(x)\phi(x))]}{\rho}\right) = 0,\ if\ x \neq x_k$ for any $k \geq 1$, shows that $\phi \in c_0(X, E, M, \mu, q)$. But for k an even integer,

$$M\left(\frac{[p_v(\lambda(x_k)\phi(x_k))]^{p(x_k)}}{\rho}\right) = M\left(\frac{[p_v(3^k k^{-k} u)]^{1/k2}}{\rho}\right) = M\left(\frac{(3/k)^{1/k}}{\rho}\right) > M(1/2\rho)$$

implies that $\phi \notin c_0(X, E, M, \lambda, p)$. Thus, the containment of $c_0(X, E, M, \lambda, p)$ in $c_0(X, E, M, \mu, q)$ is strict inspite of the fact that (i) and (ii) of the theorem 4.11 are satisfied.

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Favorable and Unfavorable Steady States of the Flow in a Natural Circulation Steam Generator with Many Pipes

KEDAR NATH UPRETY
Central Department of Mathematics
Kirtipur Campus
Tribhuvan University, Kathmandu, Nepal.

Abstract: *A natural circulation steam generator consisting of a network of heated pipes is considered and the steady state behavior of the flow is studied. The model is focused on a geometry consisting of three pipes connecting a drum at the top and a collector at the bottom. The flow in the pipes is modeled by the one dimensional Euler equations with source terms describing the impact of heating, gravity, and friction. The equations describing the drum and the collector provide necessary boundary conditions for the pipe flow. The equation of state is represented by a surface in pressure, density, and temperature space depending upon the complex properties of water. This model is analyzed with three phases: the liquid, the wet steam and the steam phase for the favorable and unfavorable steady state solutions. Dimensional analysis and asymptotic methods are used as the important tools in the computation of stationary solutions.*

Key words: *Asymptotic methods, dimensional analysis, Euler equations, scaling and phases.*

A natural circulation steam generator is a complicated network of pipes, as in figure. The generator consists of a drum at the top and a collector at the bottom, connected by a number of pipes. Water flows downwards in one of the pipes from the drum to the collector and water flows upwards from the collector to the drum in many pipes. The size of the drum is very big in comparison to the size of the collector. The drum is divided into an upper part filled with steam and a lower part filled with liquid water. The drum is connected to a feed water inlet and also there is a steam outlet connected to the drum. When the pipes leading upwards are heated externally, circulation starts and due to the circulation the whole system is heated. Then the fluid in the heated part of the network becomes a mixture of water and steam, called wet steam. Thus we have liquid, wet steam, and steam phases. Steam collected in the drum is taken out from the steam outlet in a controlled way such that the pressure in the drum is kept constant. The

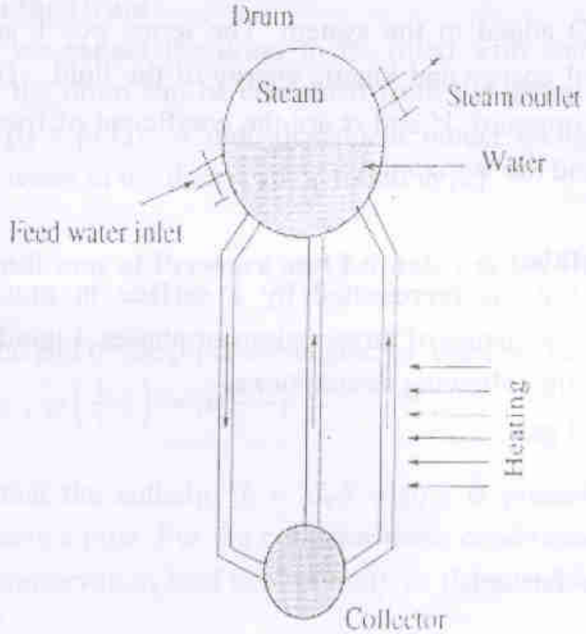
volume of the steam taken out is replaced by feed water such that the water level in the drum is kept constant. Thus, the volume of water in the drum is always constant. There exist phase change in the system due to the decrease or increase of temperature.

In this mathematical model the flow in the pipe is governed by:

1. The continuity equation or law of conservation of mass.
2. The law of conservation of momentum.
3. The law of conservation of energy.
4. The equation of state of the fluid giving the dependence of the fluid density on pressure and temperature.

The pressure in the drum is given and assumed independent of position and time. The temperature in the drum is dependent of time and independent of position. There is conservation of mass and energy in the drum. While the specific enthalpy of water in the drum is smaller than the specific enthalpy of saturated water the steam chamber is empty. Due to the loss of pressure there is no momentum conservation in the drum and collector, only mass and energy are conserved. Pressure as well as enthalpy in the drum and collector provide boundary conditions for the flow in the pipes.

Constitutive relations are represented by a surface in pressure, temperature, and density space, depending on the complex properties of liquid, wet steam, and steam. The constitutive relation also holds in the collector. After presenting and scaling the model, we approximate the steady state solution explicitly for three pipes considering the simple systems. We consider three straight vertical pipes joining the drum and the collector as described by the given figure. In the first pipe (without heating) water flows downwards. In the second and third pipe having a constant heating per length, water flows upwards. We assume the first pipe and the collector are in the liquid phase, and the drum in the wet steam phase. The second and third pipe has a phase change from liquid to wet steam.. We show the multiple steady state solution exists for the system described by three pipes as long as the heating is small enough.



1. Derivation of the Model in the Pipe

The pipe flow is modeled by the stationary one-dimensional Euler equations:

Conservation of mass

(1)

Conservation of momentum

(2)

Conservation of energy

— (3)

In the above equations ρ(x), u(x), T(x), and p(x) are the density, velocity, temperature, and pressure of the fluid at the point x. Here x ∈ (0, 1/2) represents the unheated and x ∈ (1/2, 1) the heated pipe. The terms on the right hand sides

of (2) and (3) represent the force and work done due to gravity and friction, as well as the heat Q added to the system. The terms $\rho c_v T$ and $\rho u^2/2$ are the densities of internal energy and kinetic energy of the fluid. The term u p is the work done due to pressure k' and α are the coefficient of friction and the angle between the pipe and the horizontal.

1.1. Equation of State

The equation of state is represented by a surface in pressure density, and temperature space, consisting of three regions or phases: Liquid water, wet steam, and steam. We use the following assumptions:

- Steam is an ideal gas:

$$p = \rho RT \tag{4}$$

where R is the gas constant.

- Liquid water has constant density:

$$\rho = \rho_w \tag{5}$$

The approximation of the pressure-temperature relation for wet steam i

$$p = b(T - T_o)^{\theta} = \tilde{p}\,(T) \tag{6}$$

where T_o is the freezing temperature of water.

Now the above mentioned surface consists of the following three parts representing the three different phases.

Liquid water phase

$$\rho = \rho_w \,;\, p \geq max \geq \{\, \tilde{p}\,(T), \rho_w RT \,\} \tag{7}$$

- **Wet steam phase**

$$p = \tilde{p}\,(T)\,;\, \frac{\tilde{p}\,(T)}{RT} \leq \rho \leq \rho_w \tag{8}$$

- **Steam phase**

$$p = \rho RT \,; \leq \min \{\, \rho_w, \frac{\tilde{p}\,(T)}{RT} \,\} \tag{9}$$

1.2. Conditions in the Drum

In a steady state we expect the drum to be filled with wet steam. Thus, the temperature T_d in the drum can be computed from the given pressure p_d by the equation of state $p_d = p(T_d)$. A time dependent model taking into account the possibility of cold water in the drum can be found in [2].

1.3 Boundary Conditions of Pressure and Enthalpy at the Drum and the Collector

Pressure loses at the end of the pipes are neglected. thus, we have

$p(0) = p(1) = p_d\,,\ p\left(\frac{1}{2}+\right) = p(\frac{1}{2}-).$

Also we assume that the enthalpy $h = C_V T + p/\rho$ is prescribed at these ends where the flow enters a pipe. For the collector these conditions together with the mass and energy conservation lead to continuity of the variables ρ, u, T and p at the point $x = 1/2$.

Since we will study for the solutions with $u > 0$ (downward flow in the unheated pipe and upward flow in the heated pipe), we have the boundary conditions

$C_V T + \frac{p}{\rho|_{x=0}} = C_v T_d + \frac{p_d}{\rho_w}$ at the drum.

2. Approximations of Multiple Steady States for a System with three Pipes:

The modeled equation and approximation of steady state for a system with two pipes is explained in detail in [3]. Here a system with three pipes, all of them straight and vertical, is considered. Pipe 1 is unheated and oriented downward and identified it with the interval (0, 1). The subscripts will denote the number of pipe for the quantities held in the system. Thus

$\alpha_1 = -\frac{\pi}{2}\,,\ Q_1 = 0\ \text{ in } (1, \frac{1}{2})$

Pipes 2 and 3 are oriented upward, have the same amount of heating and are both identified with the interval $(\frac{1}{2}, 1)$. Therefore

$\alpha_{2,\ 3} = \frac{\pi}{2}\,,\ Q_{2,\ 3} = 1\quad \text{in } (1/2,\ 1)$ holds.

1.2. Conditions in the Drum

In a steady state we expect the drum to be filled with wet steam. Thus, the temperature T_d in the drum can be computed from the given pressure p_d by the equation of state $p_d = p(T_d)$. A time dependent model taking into account the possibility of cold water in the drum can be found in [2].

1.3 Boundary Conditions of Pressure and Enthalpy at the Drum and the Collector

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Also we assume that the enthalpy $h = C_V T + p/\rho$ is prescribed at these ends where the flow enters a pipe. For the collector these conditions together with the mass and energy conservation lead to continuity of the variables ρ, u, T and p at the point $x = 1/2$.

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Pipes 2 and 3 are oriented upward, have the same amount of heating and are both identified with the interval $(\frac{1}{2}, 1)$. Therefore

$\alpha_{2,\ 3} = \frac{\pi}{2}\,,\ Q_{2,\ 3} = 1 \quad \text{in } (1/2,\ 1\,)$ holds.

The steady states equations described in [3] are

$$\frac{dm}{dx} = 0$$

$$\frac{dp}{dx} = -\rho\, sin\alpha - k\epsilon\, \frac{m^2}{\rho}$$

$$\frac{dT}{dx} + \epsilon p \frac{d}{dx}\left(\frac{1}{p}\right) = \frac{Q}{m} \tag{10}$$

hold for m_1, p_1, ρ_1 and T_1 in $(0, \frac{1}{2})$ and for m_i, p_i, ρ_i and T_i, $i = 2, 3$ in $(1/2, 1)$.

We assume that the cross section areas of the three pipes to be equal. Then mass conservation, energy conservation and the pressure condition at the collector become, as explained in [2]

$$m_1 = m_2 + m_3$$

$$m_1\left(T_1 + \epsilon\, \frac{p_1}{\rho_1}\right) = m_2\left(T_2 + \epsilon\, \frac{p_2}{\rho_2}\right) + m_3\left(T_3 + \epsilon\, \frac{p_3}{\rho_3}\right)$$

$$p_1 = p_2 = p_3 \quad ,$$

at $x = 1/2$.

At the drum we have the pressure conditions

$$p_1(0) = p_2(1) = p_3(1) = p_d$$

Note that the assumptions, pipes 2 and pipes 3 are indistinguishable. This symmetry property will be used in the following :

2.1 A Favorable Steady State

We look for a solution with downward flow in the unheated pipe and identical upward flows in the heated pipes. With the flow being in the liquid phase close to the collector and mass conservation, energy conservation at the collector becomes in the limit $\varepsilon \to 0$

$$(m_2 + m_3)\, T_1 = m_2\, T_2 + m_3\, T_3 \qquad \text{at } x = 1/2.$$

Also, by $m_2, m_3 > 0$, we have from the enthalpy boundary conditions $T_2(1/2) = T_3(1/2)$, and therefore

$$T_1 = T_2 = T_3 \qquad \text{at } x = 1/2.$$

The solutions in pipes 2 and 3 are the same as that in pipe 2 and also the solution in pipe 1 is the same except the fact that the mass flux should be multiplied by 2, because pipe 1 now feeds the upwards flow of two identical pipes, is explained in [3].

As a measure of the quality of the computed flow we compute the amount m_s of steam produced per time. Mass conservation and energy conservation in the drum are described by

$$m_2 + m_3 - m_1 + m_f - m_s = 0$$

$$m_2\left(T_2 + \epsilon\,\frac{p_2}{\rho_2}\right)_{x=1} + m_3\left(T_3 + \epsilon\,\frac{p_3}{\rho_3}\right)_{x=1} - m_1\left(T_1 + \epsilon\,\frac{p_1}{\rho_1}\right)_{x=0} + m_f h_f - m_s\gamma T_d = 0$$

Where m_f is the amount of feed water per time, and h_f and γT_d are the enthalpies of feed water and steam leaving the drum, respectively. By the mass conservation in the collector, the first equation gives $m_f = m_s$. Then from the second equation m_s is computed, assuming $h_f < \gamma T_d$ which is perfectly reasonable, because as the enthalpy of feed water in the drum increases the drum contains steam. Using the results described in [3],

$m = 4(p_d + 1 - x_0)/k$, $\rho = k\epsilon m^2/2$, the approximated density,
$p_1(0) = p_2(1) = p_3(1) = p_d$ and $m_2 = m_3 = m$ we get

$$m_s(\gamma T_d - h_f) = \frac{p_d}{p_d + 1 - x_0} \qquad (11)$$

2.2 Two Unfavorable Steady States

Here will be shown that under certain assumptions solutions with downward flow in one of the heated pipes exist. We shall construct a solution with downward flow in pipes 1 and 3, i.e., $m_1 > 0$ and $m_3 < 0$, and upward flow in pipe 2, i.e., $m_2 > 0$. Then of course an additional solution can be constructed by interchanging pipes 2 and 3 using the symmetry mentioned above.

We look for a solution with pipes 1 and 3 and the collector in the liquid phase and with a phase change from liquid to we steam in pipe 2.

Pipe 1: Since pipe 1 is in the liquid phase we have

$\rho_1(x) = 1 \quad \text{for } x \in [0, 1/2]$

with $\in \to 0$ the momentum and energy equations reduce to

$$\frac{dp}{dx} = 1, \quad \frac{dT}{dx} = 0$$

and the boundary conditions imply

$p_1(x) = p_d + x,\ T_1(x) = T_d$ for $x \in [1, 1/2]$

Pipe 3: Since the pipe 3 is also in the liquid phase we have

$p_3(x) = 1\ ,\ x \in \left[\frac{1}{2}, 1\right]$

and the momentum and energy equations

$$\frac{dp}{dx} = -1,\ \frac{dT}{dx} = \frac{1}{m}$$

Continuity at $x = \frac{1}{2}$ implies

$p_3(x) = p_d + 1 - x,\ T_3(x) = T_d + (x - 1)/m_3$ for $x \in [1/2, 1]$.

However it is necessary to check that solution remains in the liquid phase, i.e.,

$p_3 \geq \widetilde{p}\,(T_3)$

holds. It is easily verified that

$p_d + 1 - x \geq \widetilde{p}\left(T_d + \frac{x-1}{m_3}\right)$ for $x \in \left[\frac{1}{2},\ 1\right]$

holds if m_3 is large enough. This will be seen later.

Pipe 2, liquid phase: With the above results, mass and energy conservation in the collector are

$m_1 = m_2 + m_3$

$m_1 T_d = m_2 T_2\left(\frac{1}{2}\right) + m_3\left(T_d - \frac{1}{2m_3}\right)$

Implying $T_2(1/2) = T_d + 1/2m_3$.

This leads to

$\rho_2(x) = 1,\ p_2(x) = p_d + 1 - x,\ T_2(x) = T_d + x/m_3$, for $x \in \left[\frac{1}{2}, x_0\right]$

Pipe 2, Wet steam phase: Following the procedure of the preceding section,
$m_2 = 4\,(p_d + 1 - x_0)/k\ , \rho_2 = \epsilon m_2^2/2$ (12)

and x_0 must satisfy the equation
$h(x_0) = \tilde{p}\ \left(T_d + \frac{kx_0}{2(p_d+1-x_0)}\right) + x_0 - 1 - p_d = 0$

Clearly,
$h'(x_0) > 0$, for $x_0 \in \left(\frac{1}{2}, 1\right)$
$h(1) = \tilde{p}\ \left(T_d + \frac{k}{2p_d}\right) - p_d > 0$

hold. However for the second condition for existence of a solution
$$h\left(\frac{1}{2}\right) = \tilde{p}\ \left(T_d + \frac{k}{2(2p_d+1)}\right) - p_d - \frac{1}{2} < 0$$

we need the assumption that k is small enough. Since the dimensional parameter k is proportional to the square of the strength of the heating , this can be interpreted as a small heating assumption. This k and approximation of steady state is explained in section [3].

For small enough k we have completely determined a solution up to the fact that we only know the value of $m_1 - m_3 = m_2$, but not the separate values of m_1 and m_3 .

An additional equation can be derived by considering certain higher order terms in the asymptotic treatment for small . If the $O(\epsilon)$ friction terms are included in the computations for pipes 1 and 3, we obtain

$p_1(x) = p_d + x\,(1 - k\,\epsilon m_1^2)$ for $x \in (0, 1/2)$
$p_3(x) = p_d + (1 - x)\,(1 - k\,\epsilon m_3^2)$ for $x \in (1/2, 1)$

The pressure condition $p_1(1/2) = p_3(1/2)$ at the collector (which is trivially satisfied if only the leading order terms are included) leads to $m_1^2 = m_3^2$ and , thus,
$$m_1 = -\,m_3 = m_2/2$$

But by equation (12) we have

$$\frac{2p_d}{k} \leq m_2 \leq (2p_d + 1)/k$$

This shows that the largeness of $-m_3$, which was required for the validity of the solution in pipe 3, can again be guaranteed if k is small enough.

Finally , we again compute the steam production. As above, we again obtain the equation for m_3. However, now the phase point is the zero of function g explained equation in [3]. By the obvious relation $g < h$ in (1/2, 1), the phase change point x_0 for the present solution is smaller than in the preceding subsection. Therefore, by equation (11) , also the steam production is smaller. This result explains why we called the solution from the preceding subsection section 2.1 favorable compared to the two solutions of this subsection.

These computed approximations are qualitatively as the numerical solution of [1] and the multiple stationary solutions for more complex geometries have been found in [2].

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Markov Chain Model to Describe the Distribution of Intergenerational Occupational Mobility

KNS YADAVA[1] AND TR ARYAL[2]

[1]Department of Statistics
Banaras Hindu University, Varanasi, India
e-mail: knsy@bhu.ernet.in

[2]Central Department of Statistics
Tribhuvan University, Kirtipur Kathmandu, Nepal
e-mail: traryal@gmail.com & traryal@rediffmail.com

Abstract: This paper attempts to apply a Markov chain model to study intergenerational occupational mobility distribution among the residents of Palpa and Rupandehi districts. Markov chain model was found to be the appropriate model to describe the distribution of intergenerational occupational mobility. The attraction of sons was more toward non-manual and non-agricultural occupations as compared to their fathers. Indeed, intergenerational occupational mobility was found moderately high for both the generations. Findings may help planners and policy-makers in designing proper policies especially in integrated rural development program.

1. Introduction

The dynamic structure of social phenomena is linked with the movement of the people across social, economic or occupational categories. In this line several studies on occupational mobility have been carried out by scholars [1,2,3,4,5,6,7]. Models for mobility involved probabilistic terms which are useful for the future prediction and also for assessing the likely error of these predictions [8]. The need of the measurement of social mobility was first realized in connection with the empirical research [3]. Prais [6] was probably the first researcher to apply Markov

chain theory to social mobility followed by the measurement of occupational mobility based on the semi-Markov process [9].

Most of the mobility measures are developed based on the elements of transition matrix, which may be recognized by the transition probability matrix. A number of researchers have been studied the intergenerational occupational mobility in different societies and communities [1,2,5,7,10,11]. Markov chains or mixtures of Markov chains, like mover-stayer model, have been commonly used in social sciences to study various forms of dynamic behavior of human being including occupational mobility [1,2,4,5,12].

This paper attempts to apply a Markov chain model to study the distribution of intergenerational occupational mobility among the residents of Palpa and Rupandehi districts. In brief, Markov chain model is given below.

2. Markov Chain Model

Markov chain model developed by Sampson [12] is applied to study the inter-generational occupational mobility pattern. Let $Q_u = (p_{ij})$ be a k x k unrestricted transition matrix, where p_{ij} is the probability that a process is in state j (j=1,2,3 . . .k), the existing period, given that it was in state i(i=1,2,3 . . .k) in the initial period, and k is the occupational group, hence forth known as state of a Markov chain. The class of restricted transition matrices is given as:

$$Q_r = \theta + (I - \theta) v p^* \ ;$$

$$\text{or} \quad p_{ij} = \theta_i \delta_{ij} + (1-\theta_i) p_j^* \tag{1}$$

where, I is the identity matrix, and δ_{ij} is the kronecker delta, v be the k x 1 vector of ones, and $p^*=(p_j^*)$ be a 1x k vector of probabilities, such that $p^*v=1$, and $\theta=\text{diag}(\theta_i)$ be an k x k diagonal matrix with $0 \le \theta \le 1$. $(1-\theta_i)$ represents to the probability that a son would look for a different occupation that of his father and θ_i is the probability that a son would look for the same occupation that of his father.

Now, pre-multiplying equation (1) by p, we have,

$$p^* = p(I - \theta) / p(I - \theta) v \ ,$$

or $p_j^* = p_j(1-\theta_j) / \sum p_k (1-\theta_k)$ (2)

On substituting equation (2) into equation (1), we get

$$Q = \theta + \frac{(1 - \theta)vp(I - \theta)}{p(I - \theta)v},$$

$$\text{or} \quad p_{ij} = \theta_i \partial_{ij} + \frac{(1-\theta_i)(1-\theta_j)p_j}{\sum_k p_k(1-\theta_k)} \tag{3}$$

Let $p=(p_j)$ be the equilibrium distribution of Q, so that, $pQ = p$. In general, $p^* \neq p$, except for the condition that all the values of θ_i's are identical.

On solving equation (1) by using $pQ=p$, we get the equilibrium distribution as given below,

$$p = p*(I-\theta)^{-1}/p*(I-\theta)^{-1}v,$$

$$\text{or} \quad p_j = p_j*(1-\theta_j)^{-1}/\sum_k p_k*(1-\theta_k)^{-1} \tag{4}$$

The occupation of sons will have the probability p_j^* of being in occupational category j, which is independent of occupational category i. Furthermore, p and v be the two eigen vectors, which are associated with the eigen values 1 to the left and v to the right respectively. It was also shown that the remaining eigen values are the root of the function $f(\lambda)= p^*(\theta - \lambda I)^{-1}v$. where λ is the scalar quantity having only one eigen value lying between θ_i and θ_{i+1} when $\theta_1 \leq \theta_2 \leq \ldots \leq \theta_s$. The right and left eigen vectors corresponding to the eigen values λ are treated as $p^*(\lambda I-\theta)^{-1}$ and $(\lambda I-\theta)^{-1}(I-\theta)v$ respectively.

Two parameters θ and p^* are to be estimated by maximum likelihood (ML) method. In a Markov chain model, let N be the independent realization of length T+1 period for t= 0, 1, 2, ..., T, and log-likelihood function is given by

$$l=\sum n_{ij}\ln(p_{ij}) \tag{5}$$

where n_{ij} is the number of transitions in the sample from state i to a state j (i.e. father's occupational state i to the son's occupational state j). The unrestricted ML estimates are given as,

$$P_{ij} = \frac{n_{ij}}{n_i} \tag{6}$$

It is observed that,

$$n_i = \sum_j n_{ij},\ \ddot{n}_i = \sum_j n_{ji} \text{ and } n = \sum \sum_j n_{ij} = \sum_i n_i = \sum_i \ddot{n}_i$$

where n_i is the number of the occurrence of i for the initial period of state and $\ddot{n}_i$ is the number of the occurrences of i for the existing period of state.

So the 1xk probability vector is $\bar{p} = (n_i / n)$ and $\ddot{p} = (\ddot{n}_i/n)$ satisfies the following equation,

$$\ddot{p} = \bar{p}\, Q_u \tag{7}$$

where $Q_u = (p_{ij})$ is the unrestricted ML transition matrix.

Similarly, restricted ML estimates are obtained by substituting equation (1) in equation (5), and we obtain the restricted log-likelihood function as given below,

$$l = \sum\{n_{ii} \ln[\theta_i + (1-\theta_i)p^*] + (\ddot{n}_{.i} - n_{ii}) \ln(p_i^*) + (n_{i.} - n_{ii})\ln(1-\theta_i)\} \tag{8}$$

Maximizing equation (8) with respect to θ_i and p^* and some simplification was made by Aryal [1,2], we get,

$$\hat{\theta}_{it} = \hat{\theta}_{it-1} - G_i a_i + \frac{a_i \sum_k \bar{p}_k a_k b_k G_k}{1 + \sum_k \bar{p}_k a_k b_k} \tag{9}$$

$$a_i(\theta) = [1 + (2\theta_i \bar{p}_i - \ddot{p}_i - \bar{p}_i)/\{\bar{p}(I-\theta)v\}]^{-1} \text{ and}$$

$$b_i(\theta\theta = (1-\theta_i)(\ddot{p}_i - \bar{p}_i\theta_i)/[\bar{p}(I-\theta)v]^2.$$

Once we obtain $\hat{\theta}_i$, $\hat{p}_i^*$ is easily estimated by the equation

$$\hat{p}_i * = 1 - \left(1 - \frac{n_{ii}}{n_i}\right) / \left(1 - \hat{\theta}_i\right) \tag{10}$$

3. Application

The model is tested with the real sets of data from a sample survey of Palpa and Rupandehi districts. A total of 811 households were surveyed. The information on occupation was collected from each household. This paper deals with a sample of 777 fathers and sons of the first two generations and 303 fathers (sons) and sons (grandsons) of last two generations. The occupational categories are noted as: 1=Agricultural Laborer (Land less), 2=Non- Agricultural Laborer, 3=Agriculturist (landowner), 4=Contractual workers and Workers in Abroad, 5=Business/Trade and household industry, and 6=Professional, Administrative and Govt./Pvt. Service. The details of data and measurements of variables are found in Aryal [13,14,15]

Table 1: Estimated transition probability matrix (first two generations)

Occupational categories		Son					
		1	2	3	4	5	6
Father	1	.3482	.1696	.1071	.1339	.0893	.1518
	2	.2299	.4138	.0690	.0690	.0920	.1264
	3	.0862	.0366	.3190	.1638	.1228	.2716
	4	.0364	.0182	.1636	.4000	.1273	.1818
	5	.0385	.0000	.1538	.0000	.4231	.3846
	6	.0000	.0303	.3333	.0606	.2121	.3636

Table 1 shows the occupational distribution of sons by occupation of their fathers in the first two generations (from father to son). The estimated transition probability matrices are also shown in the respective table for the first two generations. The occupational categories 1 to 6 represent the different states of a Markov chain model under consideration and the mobility either forward or backward is of one step (father and son).

Table 2: Estimated transition probability matrix (last two generation)

		Son					
		1	2	3	4	5	6
	1	.2826	.1087	.1522	.2174	.0217	.2174
	2	.1905	.3810	.0476	.2619	.0000	.1190
Father	3	.0400	.0400	.2800	.2000	.1200	.3200
	4	.0750	.0250	.1750	.4750	.0750	.1750
	5	.0000	.0000	.1333	.0333	.5333	.3000
	6	.0000	.0889	.1111	.0889	.0889	.6222

Table 2 presents the occupational distribution of the last two generations (from sons to grandsons). The estimated transitional probability matrices are also displayed in this table for the last two generations.

Table 3: Estimated parameters (first two generations)

Occupational categories	θ	p*	p
1	.2601	.1158	.1250
2	.3801	.0527	.0680
3	.1850	.1599	.1567
4	.2634	.1803	.1955
5	.3159	.1523	.1778
6	.0231	.3390	.2770
-2logL(Null)			1227.086
-2logL(Model)			1225.783
Model(LR) chi-square			2.61
Degrees of freedom			16
Critical values(5% level)			26.30

The estimated values of parameters θ and p* are given in Tables 3 and 4 for both the successive generations. An estimate of θ explains that the son follow the same occupation under taken by his father whereas p* shows the chance of getting an

occupation different from his father's occupation. However, when θ_j's are identical, the estimated value of parameter p* gives a state of equilibrium. This is considered as the persistence class of Markov chains [1,2,5] whereas when θ_js are equal to zero, the process exhibits an inter-temporal independence [1,5].

Thus, θ_j determines the extent to which state i (the initial structure) influences the next period of state i.e. the existing structure. Similarly, when θ_j=0, then the initial state has no influence on the existing period of state. Moreover, when θ_j=1, and obviously, p_{ii}=1, then i becomes an absorbing state of Markov chain, the process in that state remains there forever, i.e. immobile [1,2].

Table 4: Estimated parameters (last two generations)

Occupational categories	θ	p*	p
1	.2358	.0661	.0560
2	.3510	.0502	.0501
3	.1211	.1930	.1432
4	.3032	.2642	.2474
5	.4991	.0731	.0950
6	.4330	.3553	.4089
-2logL(Null)	459.10		
-2logL(Model)	423.65		
Model chi-square	70.80		
Degrees of freedom	15		
Critical values (5% level)	25.00		

Thus, a person in the occupational category of agricultural laborer, for example, has had an estimate (1-0.2601) =0.7399 probability of getting a different occupation of son from his father whereas the son has a chance of 0.0527 of joining an occupation in non- agricultural laborer (Table 3). Similarly, a person in the occupational category non- agricultural laborer, has had an estimate (1-.3510)=. 6490 probability of getting a different occupation than that of his father whereas the son has a chance of .3553 of joining an occupation in professional,

administrative and government and private service for the last two generations (Table 4) and so on. For the other elements of the estimates, a similar interpretation can be given.

The observed log-likelihood (L_0), is -1227.086 and expected log-likelihood (L_a), is -1225.783 (Table 3). Thus, the likelihood ratio (LR) test statistic for H_0: $Q=\theta+(I-\theta)vp^*$ is $-2(L_0 - L_a)=2.61$ at the 16 degrees of freedom which is accepted as the tabulated $\chi^2(16) = 26.30$ at the 5% level of significance for the first type of data set i.e. first two generations (fathers to sons). An insignificant value of chi-square suggests that the Markov chain model fitted the data very well. But from Table 7.4, the likelihood ratio (LR) test statistic for $H_0 : Q=\theta+(I-\theta)vp^*$ is =70.80 at the 15 degrees of freedom and comparing with $\chi^2(15) = 25$ (at the 5% level), the null hypothesis is rejected for the second set of the data i.e. the last two generations (sons to grandsons).

The main diagonal elements of the transition probability matrix show the inheritance (son's occupation is same as that of their father's) occupational situations among the six occupational categories for both the generations (Tables 1 and 2). Occupational inheritance was considerably similar (.3482, .4138, .3190, .4000, .4231 and .3636) but low in nature among all the occupational categories (Table 1) whereas occupational inheritance was fluctuating in behavior (.2826, .3810, .2800, .4750, .5333 and .6222) among the six occupational categories (Table 2). Occupational inheritance was low for the first three occupational categories i.e. agricultural laborer, non-agricultural laborer and agriculturist (landowner) whereas it was high for the last three occupational categories i.e. professional, administrative & government and private services; contractual workers and workers abroad for the last two generations.

It is found that the attraction of sons was more toward the non-manual and non-agricultural occupations and they were highly mobile towards these occupational categories. It may be due to changed occupational status of the rural people because of many influencing factors such as spread of education, industrial development, constant land resources for cultivation, rise of the population,

climatic conditions, rural-urban migration, dissolution of the households and government policies. Moderate amount of mobility also occurred towards the manual and agricultural occupations. This may be due to some specific characteristics of individual who need for joining non-manual and non-agricultural occupations.

4. Conclusions

Markov chain model has been applied to describe the distribution of intergenerational occupational mobility among the residents of Palpa and Rupandehi districts. It was found that the attraction of sons was more toward the non-manual and non-agricultural occupations. The intergenerational occupational mobility was found moderately high for both the generations. Findings may have a number of policy implications particularly for the developing countries, like Nepal. This may also help the planners and policy makers in designing policies such as integrated rural development program, etc. to be adopted for specific region.

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Fixed Point Theorems in Dislocated Quasi D-Metric Spaces

P. RANGA SWAMY
Department of Mathematics
St.Ann's College of Engineering & Technology
Chirala, Prakasam (Dt)523 187
Andhra Pradesh, India
Email: puligeti@yahoo.co.in

K.P.R. RAO
Department of Applied Mathematics
A.N.U.P.G. Center
Nuzvid, Krishna (Dt) 521 201
Andhra Pradesh, India.

Abstract: In this paper we have proved fixed point theorem for continuous contraction mapping in dislocated quasi D-metric space. Also we obtain a common fixed point theorem for pairs of mappings and four mappings in dislocated D-metric space.

Keywords: dislocated quasi D-metric, fixed point.

1 Introduction:

B.E. Rhoades [1] established various definitions of contractive mappings. In 1992 Banach proved fixed point theorem for contraction mapping in complete metric space. In 1992 , Dhage [2] introduced a generalized metric space or D-metric space, and proved the existence of unique fixed point of a self map satisfying a contractive condition. Rhoades [3] generalized Dhage's contractive condition and obtained some fixed point theorems.

In 2005 F.M. Zeyada, G.H. Hassan, M.A. Ahmed [6] established various definitions of dislocated quasi metric space. A Isufati [11] proved some fixed point theorems for a single and a pairs of mappings in dislocated metric space.

In this paper we proved fixed point theorem for continuous contraction mapping in dislocated quasi D-metric space and also proved common fixed point theorems for pairs of mappings and four mappings in dislocated quasi D-metrics space.

2 Preliminaries:

Definition 2.1 Let X be a non-empty set and let $D: X\times X\times X \to [0,\infty)$ be a function satisfying following conditions

(i) $D(x,y,z) = 0 \Leftrightarrow x = y = z$.

(ii) $D(x,y,z) \leq D(x,y,a) + D(x,a,z) + D(a,y,z)\ \forall\, x,y,z,a \in X$.

Then D is called a dislocated quasi D-metric on X. If D satisfies $D(x,y,z) = D(y,z,x) = D(z,x,y)$ then it is called dislocated D-metric.

Definition 2.2 A sequence $\{x_n\}$ in dislocated quasi D-metric space (X.D) is called Cauchy sequences if for given $\varepsilon > 0$, $\exists n_0 \in N$ such that $\forall\ m,n \geq n_0$ $\Rightarrow D(x_m,x_n,x) < \varepsilon$ (or) $D(x_n,x_m,x) < \varepsilon$ i.e $\min\{D(x_m,x_n,x), D(x_n,x_m,x)\} < \varepsilon$.

Definition 2.3 A sequence $\{x_n\}$ in dislocated quasi D-converges to x

$$\lim_{m,n\to\infty} D(x_m, x_n, x) = 0 = \lim_{m,n\to\infty} D(x_n, x_m, x) = 0$$

In this case x is called a dislocated quasi-limit of $\{x_n\}$ and we write $x_n \to x, x_m \to x$.

Lemma 2.4 Dislocated quasi-limit in a dislocated quasi D-metric space are unique.

Definition 2.5 A dislocated quasi D-metric space (X,D) is called complete if every Cauchy's sequence In it is a dislocated quasi D-convergent.

Example 2.6 [5]
Let $X = \{1/2^n : n \in N\}$ Define $D: X x X x X \to [0,\infty)$ as follows

$$D(x,y,z) = \begin{cases} 0 \; if \; x = y = z \\ \min\{\max\{x,y\}, \max\{y,x\}, \max\{z,x\}\} \; otherwise \end{cases}$$

Define $T:X \to X$ as $Tx = x/2$ for all $x \in X$.
Then X is a complete bounded dislocated quasi D-metric space

Definition: Let (X, D_1) and (Y, D_2) be dislocated quasi D-metric spaces and let $f: X \to Y$ be a function. Then f is continuous to $x_0 \in X$, if for each sequence $\{x_n\}$ which is d_1-q D-convergent to x_0 the sequence $\{f(x_n)\}$ d_2 -q D-convergent to $f(x_0)$ in Y.

Definition 2.8 Let (X,D) be a dislocated quasi D-metric space. A mapping $T:X \to X$ is called contraction if $\exists\; 0 \le \lambda < 1$ such that $D(Tx,Ty,Tz) \le \lambda D(x,y,z)$ $\forall x, y.z \in X$

Proposition 2.9 Every convergent sequence in a dislocated quasi D-metric space is 'bi' Cauchy. converse of proposition may not be true.

Lemma 2.10 Let (X,D) be a dislocated quasi D-metric space. If $F:X \to X$ is a contraction function, then $\{f^n(x_0)\}$ is a D-Cauchy's sequence for each $x_0 \in X$

3 Main Results:

Theorem 3.1 Let (X,D) be a complete dislocated quasi D-metric space and let $T: X \to X$ be a continuous mapping satisfying the follows condition

$$D(Tx,Ty,Tz) \le \alpha \left[\frac{1+D(x,Tx,z)}{1+D(x,y,z)}\right] D(y,Ty,Tz) + \beta D(x,y,z)$$

$for\; all\; x,y \in X, \alpha > 0, \beta > 0, \alpha + \beta < 1. Then\; T\; has\; unique\; fixed\; point.$

Proof: Let $\{x_n\}$ be a sequence in X, defined as follows.

$Let\; x_0 \in X$

$T(x_0) = x_1, (Tx_1) = x_2, \ldots\ldots\ldots\ldots, T(x_n) = x_{n+1}, \ldots\ldots$

Consider

$$D(x_n, x_{n+1}, x_{n+2}) = D(Tx_{n-1}, Tx_n, Tx_{n+1})$$

$$\leq \alpha \left[\frac{1 + D(x_{n-1}, Tx_{n-1}, x_{n+1})}{1 + D(x_{n=1}, x_n, x_{n+1})} \right] D(x_n, Tx_n, Tx_{n+2}) + \beta D(x_{n=1}, x_n, x_{n+1})$$

$$\leq \alpha \left[\frac{1 + D(x_{n-1}, x_n, x_{n+1})}{1 + D(x_{n=1}, x_n, x_{n+1})} \right] D(x_n, x_{n+1}, x_{n+2}) + \beta D(x_{n=1}, x_n, x_{n+1})$$

Therefore,

$$D(x_n, x_{n+1}, x_{n+2}) = \frac{\beta}{1-\alpha} D(x_{n-1}, x_n, x_{n+1})$$

$$D(x_n, x_{n+1}, x_{n+2}) = \lambda D(x_{n-1}, x_n, x_{n+1})$$

$where\, \lambda = \dfrac{\beta}{1-\alpha}\, with 0 \leq \lambda < 1. Similarly, we\, will\, show that$

$$D(x_{n-1}, x_n, x_{n+1}) \leq \lambda D(x_{n-2}, x_{n-1}, x_n)$$

and so $D(x_n, x_{n+1}, x_{n+2}) \leq \lambda^2 D(x_{n-2}, x_{n-1}, x_n)$

In this way we have

$$D(x_n, x_{n+1}, x_{n+2}) \leq \lambda^n D(x_2, x_1, x_0)$$

Since o $\leq \lambda < 1$, as $\lambda^n \to 0$ as $n \to \infty$. Thus $\{x_n\}$ is dislocated quasi D-sequence in the complete dislocated quasi D-metric space X. Thus $\{x_n\}$ is dislocated quasi D-converges to some t_0. Since T is continuous,

$we\, have\, T(t_0) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = t_0.$

Thus $T(t_0) = t_0$. Thus T has a fixed point.

Uniqueness: Let x be a fixed point of T. Then by given condition, we have

D(x,x,x) = D(Tx,Tx,Tx)

$$\leq \alpha \left[\frac{1 + D(x, Tx, x)}{1 + D(x, x, x)} \right] D(x, Tx, Tx) + \beta D(x, x, x)$$

D(x,x,x) = D(Tx,Tx,Tx,) $\leq (\alpha+\beta)$D(x,x,x)

Which gives D(x,x,x) = 0, since $0 \leq (\alpha+\beta) < 1$ and D(x,x,x) ≥ 0.

Thus D(x,x,x) = 0, if x is fixed point of T.

$Let\, x, y \in X\, be\, fixed\, po\text{int}\, of\, T. That\, is, Tx = x, Ty = y.$

Then by condition $D(x,y,z) = D(Tx,Ty,Tz) \leq \beta D(x,y,z)$
Which gives $D(x,y,z) = 0$, since $0 \leq \beta < 1$ and $D(x,y,z) \geq 0$. Similarly $D(y,x,z) = 0$, and hence $x = y = z$.
Thus T has unique fixed point.

Theorem 3.2 Let (X,d) be a complete dislocated D-metric space.
Let $S,T : X \to X$ be D-continuous mapping satisfying:

$$D(Sx,Ty,z) \leq h \max\{D(x,y,z), D(x,Sx,z), D(y,Ty,z)\}$$

for all $x,y \in X$ and $0 < h < 1$. Then S and T have common fixed point.

Proof: Let $x_0 \in X$. Define the sequence x_n by
$X_1 = S(x_0)$, $x_2 = T(x_1)$,...............$x_n = S(x_{n-1})$, $x_{n+1} = T(x_n)$,.........
Consider
$D(x_n, x_{n+1}, x_{n+1}) = D(Sx_{n-1}, Tx_n, x_{n+1})$
$\leq h \max\{D(x_{n-1}, x_n, x_{n+1}), D(x_{n-1}, x_n, x_{n+1}), D(x_n, x_{n+1}, x_{n+1})\}$
$\leq h\{D(x_{n-1}, x_n, x_{n+1})\}$

Similarly
$D(x_{n-1}, x_n, x_{n+1}) \leq h\{D(x_{n-2}, x_{n-1}, x_n)\}$
and so $D(x_n, x_{n+1}, x_{n+1}) \leq h^2\{D(x_{n-2}, x_{n-1}, x_n)\}$
In this way we have
$D(x_n, x_{n+1}, x_{n+1}) \leq h^n\{D(x_0, x_1, x_2)\}$
Since $o < h < 1$, as $h^n \to 0$ as $n \to \infty$. Thus $\{x_n\}$ is a cauchy sequence in a Complete dislocated D-metric space X. There exists a point $u \in X$ such that $x_n \to u$.
Therefore the subsequences $\{Sx_{n-1}\} \to u$ and $\{Tx_n\} \to u$. Since S and T are continuous functions. So we have $Su = u$ and $Tu = u$.
Uniqueness of common fixed point: Let u,v be a common fixed point of S and G. Then

$D(u,v,v) = D(Su, Tv, v)$

$\leq h \ \max\{D(u,v,v), D(u,u,v), D(v,v,v)$

Replacing v by u, we get $D(u,u,u) \leq h\, D(u,u,u)$. Since $0 < h < 1$, we have $D(u,u,u) = 0$. Similarly we have $D(v,v,v) = 0$ and so $u = v$.
Hence the proof is completed.

Theorem 3.3 Let (X, D) be a complete dislocated D-metric space. Let A, B, S, T: $X \to X$ be D-continuous mappings satisfying:
$D(Sx, Ty, z) \leq h \max\{D(Sx, Ty, z), D(Sx, Ax, z), D(Ty, By, z)\}$
for all $x,y \in X$ and $0 < h < 1$. Then A, B, S, T have common fixed point.

Proof: Suppose x_0 is an arbitrary point of X. Define the sequence $\{y_n\}$ by
$Y_{2n} = Ax_{2n} = Tx_{2n+1}$
$Y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$
Consider
$D(y_{2n}, y_{2n+1}, y_{2n+1}) = D(Ax_{2n}, Bx_{2n+1}, y_{2n+1})$
$\leq h \ \max\{ D(Sx_{2n}, Tx_{2n+1}, y_{2n+1}),\ D(Sx_{2n}, Ax_{2n}, y_{2n+1}),\ D(Tx_{2n+1}, Bx_{2n+1}, y_{2n+1}) \}$
$= h \max\{ D(y_{2n-1}, y_{2n}, y_{2n+1}), D(y_{2n-1}, y_{2n}, y_{2n+1}), D(y_{2n}, y_{2n+1}, y_{2n+1}) \}$
$\leq h\, D(y_{2n-1}, y_{2n}, y_{2n+1})$

Similarly
$D(y_{2n-1}, y_{2n}, y_{2n+1}) \leq h\, D(y_{2n-2}, y_{2n-1}, y_{2n})$
and so $D(y_{2n}, y_{2n+1}, y_{2n+1}) \leq h^2\, D(y_{2n-2}, y_{2n-1}, y_{2n})$
In this way we have
$D(y_{2n}, y_{2n+1}, y_{2n+1}) \leq h^n\, (y_2, y_1, y_0)$
Since $0 < h < 1$, as $h^n \to \infty$. Thus $\{y_n\}$ is Cauchy sequence in a complete dislocated D-metric space X.
There exists a point $u \in X$ such that $\{y_n\} \to u$.
Therefore the subsequences $\{Ax_{2n}\} \to u$, $\{Bx_{2n+1}\} \to u$, $\{Sx_{2n+2}\} \to u$ and $\{Tx_{2n+1}\} \to u$.

So we have $Au = u$, $Bu = u$, $Su = u$, and $Tu = u$.

Uniqueness of common fixed point :

Let u and v be a common fixed point of A, B, S, T. Then

$$D(u,u,v) \leq h \max \{D(Su,Tu,v), D(Su,Au,v), D(Tu,Bu,v)\}$$
$$\leq h\, D(u,u,v)$$

Replacing v by u, we get $D(u,u,u) \leq hD(u,u,u)$.

Since $0 < h < 1$, we have $D(u,u,u) = 0$. Similarly we have $D(v,v,v) = 0$ and so $u = v$.

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Finite Capacity Queueing System with Vacations and Server Breakdowns

R.P.GHIMIRE* and RITU BASNET
Department of Mathematical Sciences,
Kathmandu University, Dhulikhel (Nepal)
*Corresponding author: r12ghimire@yahoo.com

Abstract: This investigation deals with the finite queuing system with vacation and server breakdowns in which customers arrive in the system in Poisson fashion at rate λ_0 during vacation, faster rate λ_f during active service and slower rate $\lambda_s \geq 0$ during the breakdown. Customers are served exponentially with the rate μ. Server breakdowns at rate b and it immediately repaired exponentially with the rate r. We derive the explicit formulas for queue length distribution, average queue length, average number of customers in the system, average waiting times for a customer in queue and in the system. Numerical illustrations have been cited to show the model proposed is practically sound.

Keywords: *Queue, Vacation, Breakdown, Poisson, Exponential.*

1. Introduction

The queuing system with vacations and server breakdowns in infinite capacity system can be found in several research papers [1, 2, 3, 4, 5] but from the practical view point such infinite system may not always be the case .In many real life situations the finite capacity system plays the vital role such as in optimized manufacturing systems, maintenance activities and telecommunication network centers where the multi-task employees are to be deployed so it is worthwhile to mention some significant work done on the line. Ibe et al. [6] made the petri net analysis of finite-population vacation queuing systems. Lee [7] developed M/G/1/N queue with the provision of vacation time and limited service discipline.

Zhang et al. [8] analyzed M/M/1/N queue with balking, reneging and server vacations and obtained the various measures of effectiveness. From the practical view point the arrival of customers in heterogeneous rates is of paramount importance and very few journals can be found in the case. Some of the authors who contributed their efforts are [9, 10].Recently Lam et al. [11] employed geometric process model for M/M/1 queueing system with a repairable service station. Wang et al.[12] made the maximum entropy analysis of the $M^x/M/1$ queueing system under the consideration of multiple vacations and server breakdowns. Jain and Agrawal [13] proposed optimal policy for bulk queue with multiple types of server breakdown and made the comparisons with model without vacations. Chen et al.[14] studied a retrial queue with general retrial times, a modified vacation policy and server breakdowns and obtained some analytical results for the system size distribution as well as some other performance measures of the system.Khalaf and Alenezy [15] studied the batch arrival queueing system $M^{[X]}$ / G / 1 in which the server has the option to take a vacation after any service .completion, when the server finished any period of vacation it does not start serving in the system and there is a period of delay time before starting the service and obtained steady state results in explicit and closed form in terms of the probability generating functions for the number of customers in the queue, the average number of customers, and the average waiting time in the queue.

In this paper under the vacation policy, server completely stops serving customers during a vacation and starts serving the customer whenever there is at least one customer in the queue. Once service starts, there can be interruption due to server breakdown and it is sent to repair facility. As soon as repair process completes, the server starts to serve to the same earlier customer to whom the service was interrupted. Vacation starts at rate υ if the system is empty. Also the server takes another vacation, if upon his arrival to the system, finds the system empty. Customers arrive in the system in Poisson fashion at rate λ_0 during vacation, faster rate λ_f during active service and slower rate $\lambda_s \geq 0$ during the breakdown. Customers are served exponentially with the rate μ. Server breakdowns at rate b and it immediately repaired exponentially with the rate r. We derive the explicit formulas for queue length distribution, average queue length, average number of

customers in the system, average waiting times for a customer in queue and in the system. Numerical illustrations have been cited to show the model proposed is practically sound.

2. Mathematical Model and Analysis

The states for the model are as follows:

$(0, i)$ is the state in which there are i customers in the queue and the server is on vacation, $0 \le i \le N$. Its probability is $p(0, i)$.

(a) $(1, i)$ is the state in which there are i customers in the system during active service, $1 \le i \le N$. Its probability is $p(1, i)$.

(b) $(2, i)$ is the state in which there are i customers in the system during repair process, $1 \le i \le N$. Its probability is $p(2, i)$.

The generating function for the queue length distribution is

$$F(z) = F_0(z) + F_f(z) + F_s(z), \tag{1}$$

Where the partial generating functions are:

$$F_0(z) = \sum_{i=0}^{N} p(0, i) z^i,$$

$$F_f(z) = \sum_{i=1}^{N} p(1, i) z^i,$$

$$F_s(z) = \sum_{i=1}^{N} p(2, i) z^i.$$

The balance equations for the queue length distribution are:

$$\lambda_0\, p(0, 0) = \mu\, p(1, 1), \tag{2}$$

$$(\lambda_0 + \upsilon)\, p(0, i) = \lambda_0\, p(0, i-1), \qquad 1 \le i < N, \tag{3}$$

$$\upsilon\, p(0, N) = \lambda_0\, p(0, N-1), \tag{4}$$

$$(\lambda_f + \mu + b)\, p(1, 1) = \upsilon p(0, 1) + \mu\, p(1, 2) + r\, p(2, 1), \tag{5}$$

$$(\lambda_f + \mu + b)\, p(1, i) = \lambda_f\, p(1, i-1) + \upsilon p(0, i) + \mu p(1, i+1) + r\, p(2, i), \quad 2 \le i < N, \tag{6}$$

$$(\mu + b)\, p(1, N) = \lambda_f\, p(1, N-1) + \upsilon\, p(0, N) + r\, p(2, N), \tag{7}$$

$$(\lambda_s + r)\, p(2,1) = b\, p(1,1), \tag{8}$$

$$(\lambda_s + r)\, p(2, i) = b\, p(1, i) + \lambda_s\, p(2, i-1), \quad 2 \le i < N, \tag{9}$$

$$r\, p(2, N) = b\, p(1, N) + \lambda_s\, p(2, N-1). \tag{10}$$

Equation (2) gives

$$p(1,1) = \frac{\lambda_0}{\mu}\, p(0,0). \tag{11}$$

From equation (4), we have

$$p(0, N) = \frac{\lambda_0}{\upsilon}\, p(0, N-1). \tag{12}$$

Substituting (11) into (8), we obtain

$$p(2,1) = \frac{b}{\lambda_s + r}\, p(1,1) = \frac{b\,\lambda_0}{\mu(\lambda_s + r)}\, p(0,0). \tag{13}$$

From equation (3), we get

$$p(0, i) = \rho_0\, p(0, i-1), \qquad \text{Where} \qquad \rho_0 = \frac{\lambda_0}{\lambda_0 + \upsilon}.$$

For $1 \le i \le N$, recursively

$$p(0, i) = \rho_0^i\, p(0,0) \tag{14}$$

Now

$$F_0(z) = \sum_{i=0}^{N} p(0, i)\, z^i$$

$$= p(0,0) + p(0,1)\, z + p(0,2)\, z^2 + \ldots + p(0, N-1)\, z^{N-1} + p(0, N)\, z^N$$

Using (12) and (14), we have

$$F_0(z) = \left\{ \frac{1 - (\rho_0 z)^N}{1 - (\rho_0 z)} + \frac{\lambda_0}{\upsilon}\, \rho_0^{N-1} z^N \right\} p(0,0). \tag{15}$$

Multiplying equation (6) by z^i and sum for $i = 2, 3, \ldots, N-1$

$$\left(\lambda_f + \mu + b\right)\sum_{i=2}^{N-1} p(1,i) z^i =$$

$$\lambda_f \sum_{i=2}^{N-1} p(1, i-1)\, z^i + \upsilon \sum_{i=2}^{N-1} p(0, i)\, z^i + \mu \sum_{i=2}^{N-1} p(1, i+1)\, z^i + r \sum_{i=2}^{N-1} p(2, i)\, z^i$$

$$\lambda_f\, z \sum_{i=1}^{N-1} p(1, i)\, z^i - \lambda_f\, p(1, N-1)\, z^N + \upsilon \sum_{i=0}^{N} p(0, i) z^i - \upsilon\, p(0, N)\, z^N$$

$$-\upsilon\left\{p(0,0) + p(0,1)\, z\right\} + \frac{\mu}{z}\sum_{i=1}^{N} p(1, i)\, z^i - \frac{\mu}{z}\left\{p(1,1)\, z + p(1,2)\, z^2\right\}$$

$$+ r \sum_{i=1}^{N} p(2, i)\, z^i - r\, p(2, N)\, z^N - r\, p(2,1)\, z$$

or, $$\lambda_f \sum_{i=1}^{N-1} p(1, i)\, z^i + (\mu + b)\, F_f(z) - (\mu + b)\, p(1, N)\, z^N - (\lambda_f + \mu + b)\, p(1,1)\, z =$$

$$\lambda_f\, z \sum_{i=1}^{N-1} p(1, i)\, z^i - \lambda_f\, p(1, N-1)\, z^N + \upsilon F_0(z) - \upsilon\, p(0, N)\, z^N - \upsilon\left\{p(0,0) + p(0,1) z\right\} +$$

$$\frac{\mu}{z} F_f(z) - \frac{\mu}{z}\left\{p(1,1) z + p(1,2) z^2\right\} + r\, F_s(z) - r\, p(2, N)\, z^N - r\, p(2,1)\, z$$

Using equation (7), we have

$$\left\{\mu + b - \frac{\mu}{z} - \lambda_f (z-1)\right\} F_f(z) = (\lambda_f + \mu + b)\, p(1,1)\, z +$$

$$\upsilon\left\{\frac{1 - (\rho_0\, z)^N}{1 - (\rho_0\, z)} + \frac{\lambda_0}{\upsilon}\rho_0^{N-1} z^N - 1 - \rho_0\, z\right\} p(0,0) -$$

$$\frac{\mu}{z}\left\{p(1,1) z + p(1,2)\, z^2\right\} + r\, F_s(z) - r\, p(2,1)\, z. \qquad (16)$$

Now, we find $F_s(z)$ in terms of $F_f(z)$, then using equation (16) we find $F_f(z)$. Multiply equation (9) by z^i and sum for $i = 2, 3, \ldots, N-1$, we obtain

$$\lambda_s \sum_{i=1}^{N-1} p(2,i)\, z^i + r\, F_s(z) - r\, p(2,N)\, z^N - (\lambda_s + r)\, p(2,1)\, z = b\, F_f(z) - b\, p(1,N)\, z^N -$$

$$b\, p(1,1)\, z + \lambda_s\, z \sum_{i=1}^{N-1} p(2,i)\, z^i - \lambda_s\, p(1, N-1)\, z^N$$

Using equations (8) and (10), we have

$$F_s(z) = \frac{b}{(\lambda_s + r - z\lambda_s)} F_f(z). \tag{17}$$

Substituting the equation (17) into (16) and simplification gives

$$\frac{(z-1)\,Q(z)}{z\,(\lambda_s + r - \lambda_s z)} F_f(z) = (\lambda_f + \mu + b)\, p(1,1)\, z + \upsilon \left\{ \frac{1-(\rho_0 z)^N}{1-(\rho_0 z)} + \frac{\lambda_0}{\upsilon} \rho_0^{N-1} z^N - 1 - \rho_0 z \right\} p(0,0) -$$

$$\frac{\mu}{z} \left\{ p(1,1)\, z + p(1,2)\, z^2 \right\} - r\, p(2,1)\, z, \tag{18}$$

Where $Q(z) = \lambda_f\, \lambda_s\, z^2 - (\lambda_f\, \lambda_s + \lambda_f\, r + b\, \lambda_s + \mu\, \lambda_s)\, z + \mu\,(r + \lambda_s)$. (19)

In order for the queue length distribution to exist, the R.H.S. of equation (18) must vanish when $z = 1$.

$$F_f(z) = \frac{(\lambda_s + r - \lambda_s z)\, \lambda_0}{Q(z)(1-\rho_0 z)} \left[(1-\rho_0^N)\, z^2 - (z-1)\left\{ z + \rho_0^N\, \Phi(z) \right\} \right] p(0,0). \tag{20}$$

Where $\Phi(z) = z^N + z^{N-1} + \ldots + z^2$. (21)

For $\lambda_s > 0$, discriminant Δ of the quadratic expression (19) satisfies

$$\Delta = (\lambda_f\, \lambda_s + \lambda_f\, r + b\, \lambda_s + \mu\, \lambda_s)^2 - 4\, \lambda_f\, \lambda_s\, \mu\,(r + \lambda_s)$$

or, $\quad \Delta \geq \lambda_s^2\, b^2 + \lambda_s^2\, \mu^2 + \lambda_f^2\, \lambda_s^2 + \lambda_f^2\, r^2 - 2\,\lambda_f\, \lambda_s^2 \mu - 2\lambda_f\, \lambda_s \mu r + 2\,\lambda_f^2\, \lambda_s\, r$

or, $\quad \Delta \geq \lambda_s^2\, b^2 + (\lambda_f\, \lambda_s + \lambda_f\, r - \lambda_s\, \mu)^2 > 0.$

So the equation $Q(z)=0$ has two distinct real roots Z_1 and Z_2.

In order for the steady-state queue length distribution to exist, both roots of the equation $Q(z)=0$ must be greater than 1. Since in $Q(z)$, the coefficient of z^2 is positive, the two roots of $Q(z)=0$ will be greater than 1 iff $Q(1)>0$ and $Q'(1)<0$. Since $Q(1)=\mu r-\lambda_s b-\lambda_f r$, we assume that

$$\mu r>\lambda_s b+\lambda_f r \tag{22}$$

Now (22) implies that $\mu>\lambda_f$, so if (22) holds, then

$$Q'(1)=\lambda_s(\lambda_f-\mu)-\lambda_s b-\lambda_f r<0.$$

Thus, if we assume that (22) holds, then the roots Z_1 and Z_2 of $Q(z)=0$ will be greater than 1.

Substituting equations (15), (17) and (20) into (1), we get

$$F(z)=F_0(z)+F_f(z)+F_s(z)$$

$$F(z)=\frac{\left\{\frac{1-(\rho_0 z)^V}{1-(\rho_0 z)}+\frac{\rho_0^N z^N}{1-\rho_0}\right\}Q(z)(1-\rho_0 z)+(\lambda_s+r-z\lambda_s+b)\lambda_0\begin{bmatrix}(1-\rho_0^N)z^2- \\ (z-1)\left\{z+\rho_0^V\,\Phi(z)\right\}\end{bmatrix}}{Q(z)(1-\rho_0 z)}\,p(0,0). \tag{23}$$

From equation (23) and the normalizing condition $F(1)=1$, we obtain

$$p(0,0)=\frac{(\mu r-\lambda_s b-\lambda_f r)(1-\rho_0)}{(\mu r-\lambda_s b-\lambda_f r)+(b+r)\lambda_0(1-\rho_0^N)}. \tag{24}$$

Now assuming $\lambda_s>0$ the equation $Q(z)=0$ becomes

$$\mu(r+\lambda_s)\left(\frac{1}{z}-\frac{1}{Z_1}\right)\left(\frac{1}{z}-\frac{1}{Z_2}\right)=0, \tag{25}$$

Substituting $\alpha=1/Z_1$ and $\beta=1/Z_2$, the roots of equation (25), we obtain

$$\mu(r+\lambda_s)(1-\alpha z)(1-\beta z)=0. \tag{26}$$

For $z=1$,

$$\mu(r+\lambda_s)(1-\alpha)(1-\beta)=0. \tag{27}$$

Then from (24)

$$p(0,0)=\frac{\mu(r+\lambda_s)(1-\alpha)(1-\beta)(1-\rho_0)}{\mu r-\lambda_s b-\lambda_f r+(b+r)\lambda_0(1-\rho_0^N)}. \tag{28}$$

Using it in equation (23), we obtain

$$F(z)=R(z)\frac{(1-\alpha)(1-\beta)(1-\rho_0)}{(1-\alpha z)(1-\beta z)(1-\rho_0 z)}, \tag{29}$$

Where

$$R(z)=\frac{\left\{\frac{1-(\rho_0 z)^N}{1-(\rho_0 z)}+\frac{\rho_0^N z^N}{1-\rho_0}\right\}Q(z)(1-\rho_0 z)+(\lambda_s+r-z\lambda_s+b)\lambda_0\begin{bmatrix}(1-\rho_0^N)z^2-\\(z-1)\left\{z+\rho_0^N\Phi(z)\right\}\end{bmatrix}}{\mu r-\lambda_s b-\lambda_f r+(b+r)\lambda_0(1-\rho_0^N)}, \tag{30}$$

This reduces to $R(1)=1$.

In case when there is no customer admitted in the queue during a repair process, $\lambda_s=0$. Then the equation (19) takes the form

$$Q(z)=\mu r(1-\rho_f z), \tag{31}$$

Where $\rho_f=\dfrac{\lambda_f}{\mu}<1$.

Substituting equation (31) into equation (23), we get

$$F(z)=\frac{\left\{\frac{1-(\rho_0 z)^N}{1-(\rho_0 z)}+\frac{\rho_0^N z^N}{1-\rho_0}\right\}\mu r(1-\rho_f z)(1-\rho_0 z)+(b+r)\lambda_0\begin{bmatrix}(1-\rho_0^N)z^2-\\(z-1)\left\{z+\rho_0^N\Phi(z)\right\}\end{bmatrix}}{\mu r(1-\rho_f z)(1-\rho_0 z)}p(0,0) \tag{32}$$

$$p(0,0)=\frac{\mu r(1-\rho_f)(1-\rho_0)}{\mu r(1-\rho_f)+\lambda_0(b+r)(1-\rho_0^N)} \tag{33}$$

And $$F(z) = R(z)\frac{(1-\rho_f)(1-\rho_0)}{(1-\rho_f z)(1-\rho_0 z)}, \tag{34}$$

Where

$$R(z) = \frac{\left\{\frac{1-(\rho_0 z)^N}{1-(\rho_0 z)} + \frac{\rho_0^N z^N}{1-\rho_0}\right\}\mu r(1-\rho_f z)(1-\rho_0 z) + (r+b)\lambda_0\begin{bmatrix}(1-\rho_0^N)z^2 - \\ (z-1)\left\{z+\rho_0^N\Phi(z)\right\}\end{bmatrix}}{\mu r - \lambda_s b - \lambda_f r + (b+r)\lambda_0(1-\rho_0^N)} \tag{35}$$

Equations (32) and (35) are the queue length distribution for $\lambda_s > 0$ and $\lambda_s = 0$ respectively.

If $\lambda_s = 0$ the expression (23) becomes the necessary and sufficient condition for the queue length distribution to exist and it gives the utilization factor for the M/M/1 queue which is independent of service rates and breakdown.
For $\lambda_s > 0$, using (33) into (32), we obtain

$$F(z) = \frac{\left\{\frac{1-(\rho_0 z)^N}{1-(\rho_0 z)} + \frac{\rho_0^N z^N}{1-\rho_0}\right\}Q(z)(1-\rho_0 z) + (\lambda_s + r - z\lambda_s + b)\lambda_0\begin{bmatrix}(1-\rho_0^N)z^2 - \\ (z-1)\left\{z+\rho_0^N\Phi(z)\right\}\end{bmatrix}}{(1-\alpha z)(1-\beta z)(1-\rho_0 z)} \times$$

$$\frac{(1-\alpha)(1-\beta)(1-\rho_0)}{\mu r - \lambda_s b - \lambda_f r + (r+b)\lambda_0(1-\rho_0^N)},$$

The average queue length L_q can be obtained by computing $F'(1)$

$$L_q = \frac{\alpha}{1-\alpha} + \frac{\beta}{1-\beta} + \frac{\rho_0}{1-\rho_0} + \frac{\lambda_s(\lambda_f - \mu - b) + \lambda_0\left\{b + r - \lambda_s(1-\rho_0^N)\right\} - \lambda_f r - \left\{\frac{\rho_0^{N+1}}{1-\rho_0}(\mu r - \lambda_s b - \lambda_f r)\right\}}{\mu r - \lambda_s b - \lambda_f r + \lambda_0(b+r)(1-\rho_0^N)}. \tag{36}$$

The average number of customer in the system L_s can also be obtained as

$$L_s = L_q + \frac{\lambda_0}{\mu} + \frac{\lambda_f}{\mu} + \frac{\lambda_s}{\mu}.$$

Average waiting times per customer in the queue and the system are respectively:

$$W_q = \frac{L_q}{\lambda_0} + \frac{L_q}{\lambda_f} + \frac{L_q}{\lambda_s}$$

And $W_s = L_s + \dfrac{1}{\mu}$.

For $\lambda_s = 0$, using (35) into (34), we have

$$F(z) = \frac{\left\{\dfrac{1-(\rho_0 z)^N}{1-(\rho_0 z)} + \dfrac{\rho_0^N z^N}{1-\rho_0}\right\} \mu r (1-\rho_f z)(1-\rho_0 z) + (b+r)\lambda_0 \begin{bmatrix} (1-\rho_0^N) z^2 + \\ (z-1)\left\{z + \rho_0^N \Phi(z)\right\} \end{bmatrix}}{(1-\rho_f z)(1-\rho_0 z)} \times$$

$$\frac{(1-\rho_f)(1-\rho_0)}{\mu r - \lambda_s b - \lambda_f r + (b+r)\lambda_0 (1-\rho_0^N)},$$

Then the average queue length is

$$L_q = \frac{\rho_f}{1-\rho_f} + \frac{\rho_0}{1-\rho_0} + \frac{\lambda_0 (b+r) - \mu r \rho_f - \dfrac{\rho_0^{N+1}}{1-\rho_0}\left\{\mu r\left(1-\rho_f\right)\right\}}{\mu r\left(1-\rho_f\right) + \lambda_0 (b+r)\left(1-\rho_0^N\right)}, \tag{37}$$

and the average number of customers in the system L_s is

$$L_s = L_q + \frac{\lambda_0}{\mu} + \frac{\lambda_f}{\mu},$$

Average waiting times in the queue and in the system are respectively:

$W_q = \dfrac{L_q}{\lambda_0} + \dfrac{L_q}{\lambda_f}$ and $W_s = L_s + \dfrac{1}{\mu}$

3. Special Cases

If $N \to \infty$, that is, when the system capacity is infinite then the results worked out here coincide with the results obtained by Gray *et al.* [2].

4. Numerical results and concluding remarks

We provide the numerical results for various performance indices using equations (37) to the equations involving L_s, W_q, W_s MATLAB software has been used to develop the computer program. For the computation purpose, we fix different system parameters as follows:

Table 1: Average queue length

N=25, $\lambda_0=1$, $\lambda_s=1$, $\upsilon=1$, $r=3$, $b=2$

	$\mu=7$	$\mu=7.5$	$\mu=8$
λ_f	L_q	L_q	L_q
2	1.5000	1.4439	1.3988
3	1.7667	1.6509	1.5641
4	2.3452	2.0588	1.8667
5	4.0833	3.0693	2.5238
6	19.1667	7.2667	4.5000

Table 2: Average queue length

N=25, $\lambda_0=1$, $\lambda_f=2$, $\lambda_s=1$, $\upsilon=1$, $r=3$

	$b=1$	$b=1.5$	$b=2$
μ	L_q	L_q	L_q
3	4.1667	5.8333	9.1667
4	2.1333	2.4444	2.8333
5	1.6667	1.8167	1.9881
6	1.4667	1.5619	1.6667
7	1.3571	1.4259	1.5000

Table 3: Average queue length

N=25, $\lambda_0=1$, $\lambda_f=2$, $\lambda_s=1$, $\upsilon=1$, $b=2$

	$r=3$	$r=3.5$	$r=4$
μ	L_q	L_q	L_q
3	9.1667	6.8810	5.7500
4	2.8333	2.5810	2.4167
5	1.9881	1.8761	1.8000
6	1.6667	1.5976	1.5500
7	1.5000	1.4508	1.4167

For λ_s=0,

Table 4: Average queue length
N=25, $\lambda_0=1$, $\upsilon=1$, $r=3$, $b=2$

	$\mu=7$	$\mu=7.5$	$\mu=8$
λ_f	L_q	L_q	L_q
2	1.3500	1.3171	1.2899
3	1.5147	1.4505	1.4000
4	1.8333	1.6912	1.5882
5	2.5909	2.2000	1.9524
6	5.3750	3.6316	2.8182

Table 5: Average queue length
N=25, $\lambda_0=1$, $\lambda_f=2$, $\upsilon=1$, $r=3$

	$b=1$	$b=1.5$	$b=2$
μ	L_q	L_q	L_q
3	2.7143	2.8000	2.8750
4	1.8000	1.8571	1.9091
5	1.5128	1.5556	1.5952
6	1.3750	1.4091	1.4412
7	1.2947	1.3231	1.3500

Table 6: Average queue length
N=25, $\lambda_0=1$, $\lambda_f=2$, $\upsilon=1$, $b=2$

	$r=3$	$r=3.5$	$r=4$
μ	L_q	L_q	L_q
3	2.8750	2.8333	2.8000
4	1.9091	1.8800	1.8571
5	1.5952	1.5729	1.5556
6	1.4412	1.4231	1.4091
7	1.3500	1.3348	1.3231

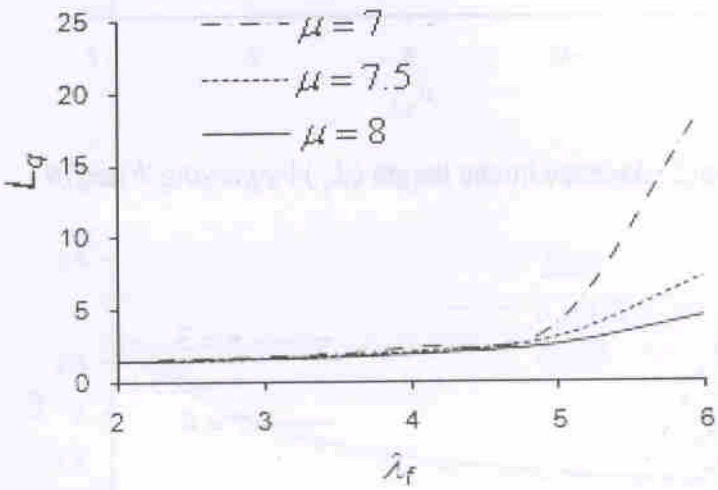

Figure 1: Average queue length (L_q) by varying μ and λ_f

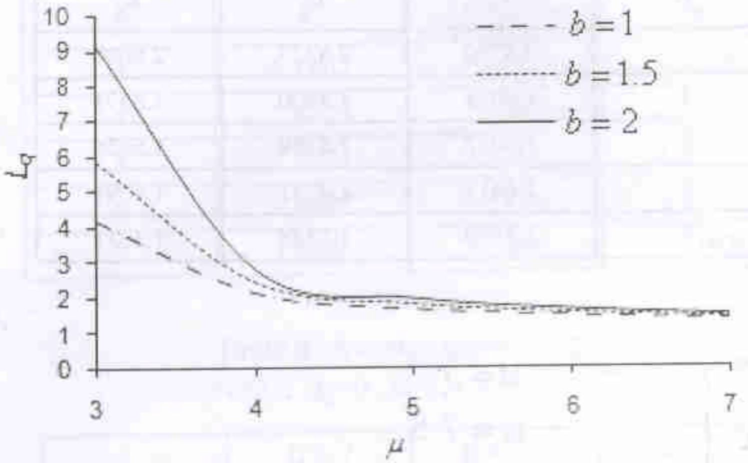

Figure 2: Average queue length (L_q) by varying b and μ

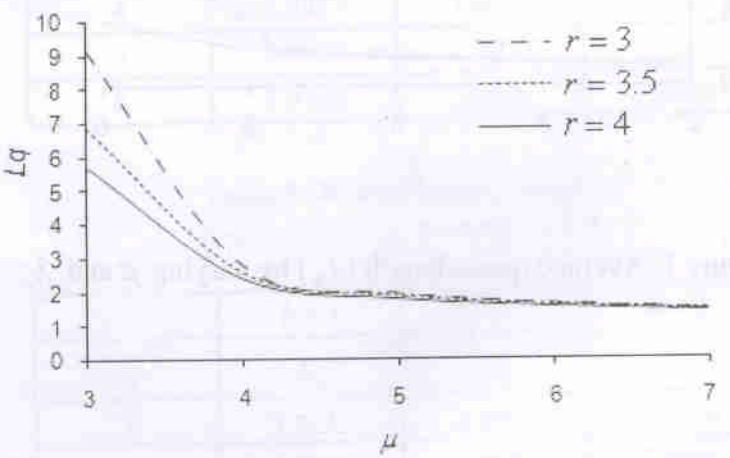

Figure 3: Average queue length (L_q) by varying r and μ

For λ_s=0,

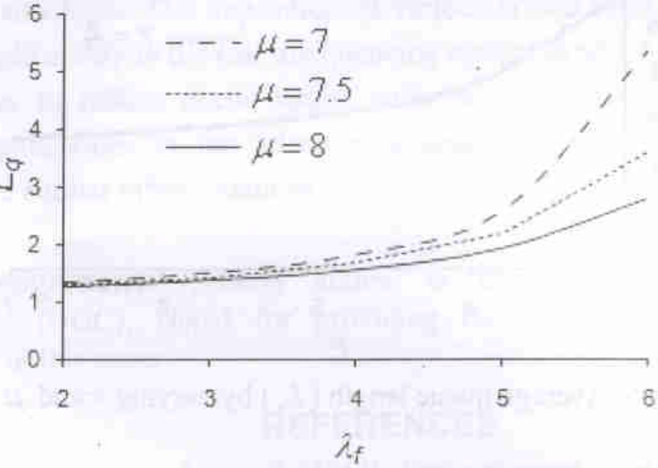

Figure 4: Average queue length (L_q) by varying μ and λ_f

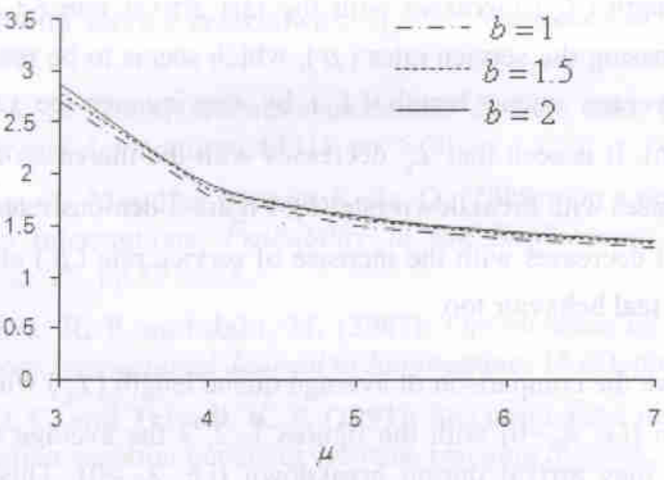

Figure 5: Average queue length (L_q) by varying b and μ

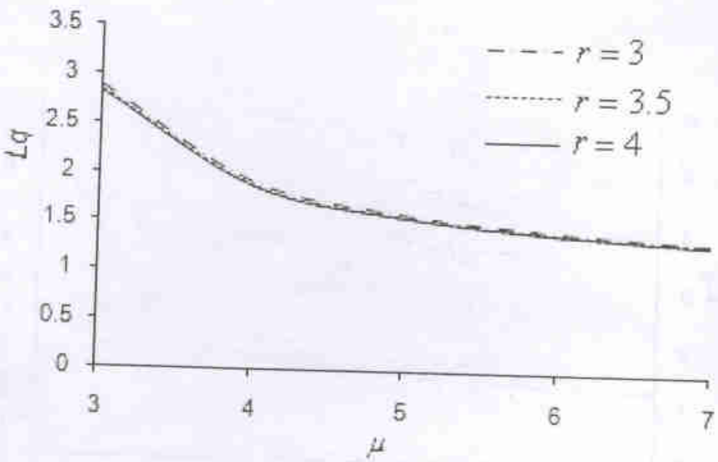

Figure 6: Average queue length (L_q) by varying r and μ

Figure 1 displays the correlation between average queue length (L_q) vs. fast arrival rate (λ_f) by varying the service rates (μ). We also observe that the average queue length (L_q) increases with the fast arrival rate (λ_f) whereas it decreases by increasing the service rates (μ), which seems to be realistic. Figure 2 exhibits the average queue length (L_q) by varying service rate (μ) and breakdown rate (b). It is seen that L_q decreases with the increase of service rate (μ) and L_q increases with breakdown rate (b). Figure 3 demonstrates the average queue length (L_q) decreases with the increase of service rate (μ) and repair rate (r) that shows the real behavior too.

Figures 4, 5, 6 give the comparison of average queue length (L_q) when no arrival during breakdown (i.e. λ_s =0) with the figures 1, 2, 3 the average queue length (L_q) when there may arrival during breakdown (i.e. λ_s >0). This comparison shows the average queue length (L_q) in latter figures increases or decreases gradually than in former cases.

We obtained various performance measures of a queue such as average queue length, average number of customer in the system, average waiting time for a customer in queue and a single server finite capacity queuing system with the

provision of vacations and server breakdowns. The correlation between average queue length and the various parameters gives the applications of proposed model in real life situations. The imposition of various arrival rates to the system may have the applicability in the real life queueing system in which arrival rates can be varied so as to reduce queue length sufficiently. Proposed model may have potential applications in the telecommunication systems, systems, machining systems, and similar other situations.

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Optimality of the Cyclic Sequence on Bottleneck Product Rate Variation Problem with a General Objective

SHREE RAM KHADKA* and TANKA NATH DHAMALA
Central Department Mathematics,
Tribhuvan University, Kathmandu, Nepal
P.O.Box, 13143
*Corresponding author: shreeramkhadka@gmail.com

Abstract: The bottleneck product rate variation problem minimizes the maximum variation in the rate at which different models are produced. The problem has mathematically interesting base model with theoretical value and real world applications. In this paper, cyclic sequence to the problem has been shown to be optimal. Cyclic sequence reduces the time complexity.

Keywords: Product rate variation problem; sequencing problem; nonlinear integer programming

1. Introduction

The product rate variation problem (PRVP) minimizes the variation in the rate at which different models of a common base product are produced on the assembly lines [6]. The problem minimizes both the earliness and the tardiness penalties that respond to the customer demands for a variety of models without holding large inventories or incurring large shortages. This is a problem of finding a sequence of different models distributed as evenly as possible on the assembly lines with the assumption of negligible switch-over cost and unit processing time for each copy of each model.

The problem has been formulated as a non-linear integer programming with the objective of minimizing the deviation between the actual and the ideal production under the assumption that the system has sufficient capacity with negligible changeover costs from one model to another and each model is produced in a unit

time [9, 10]. The problem has mathematically interesting base model with theoretical value and real world applications, see [2].

The problem has been extensively studied and solved in pseudo-polynomial time. The bottleneck PRVP i.e. the problem with the objective of minimizing the maximum deviation between the actual and the ideal production has been solved in O(DlogD) time [11, 3, 4] . Solution with pseudo-polynomial time may be expensive since this time depends on the size of the demands. Existence of cyclic sequences substantially reduces the time.

In this paper, cyclic sequence of the bottleneck product rate variation problem with a general objective is shown to be optimal.

The plan of the paper is as follows. Section 2 reviews the mathematical model. Section 3 describes the solution procedure. Section 4 shows that the cyclic sequence is optimal. The last section concludes the paper.

2. Mathematical Models

Given $d_i \in N$ demand for a model i, $i = 1,...,n$, N being the set of positive integers, with total demand $D = \sum_{i=1}^{n} d_i$ and demand ratio $r_i = \frac{d_i}{D}$, let the time horizon be partitioned into D equal units and each product is produced in a unit time. There will be k complete units of various products during the first k, $k = 1,...,D$ time units. Let x_{ik} be the quantity of product i produced during the time units 1 through k. Consider f_i, $i = 1,...,n$, unimodal symmetric convex function with minimum 0 at 0.

The mathematical model of the bottleneck PRVP [7, 8] is

(1) $\min \max f_i(x_{ik} - kr_i)$

subject to

(1.1) $\sum_{i=1}^{n} x_{ik} = k \qquad k = 1,...,D$

(1.2) $\quad x_{i(k-1)} \le x_{ik} \qquad i = 1,...,n;\ k = 2,...,D$

(1.3) $\quad x_{iD} = d_i;\ x_{i0} = 0 \qquad i = 1,...,n$

(1.4) $\quad x_{ik} \ge 0$, integer

Constraint (1.1). shows the cumulative production during the time units 1 through k. Constraint (1.2) ensures that the total production of every product over k time units is a non-decreasing function of k. Constraint (1.3) guarantees that the demands for each product are met exactly. Constraint (1.2) and (1.4) ensure that exactly one unit of a product is scheduled during one time unit. In this paper, we consider a general objective function $f_i(x_{ik} - kr_i) = |x_{ik} - kr_i|^m$, m being a positive integer.

2. Solution procedure

The perfect matching with a bisection search, appeared in [11] for the bottleneck product rate variation problem with absolute-deviation objective, can also be applied for the Problem with necessary modifications.

The method relies on the level curves $f_{ij}(k) = |j - kr_i|^m$, $j = 0,1,...,d_i$, ; $i = 0,1,...,n; k = 1,...,D$ and the bottleneck (bound) $B > 0$. The time horizon is assumed to be continuous though is partitioned into D equal time-buckets i.e. $T = [1, D]$. A model (i, j) is sequenced in a time-bucket $k \in [1, D]$ such that the level curves do not exceed B. This introduces the earliest sequencing time $E_m(i, j)$ and the latest sequencing time $L_m(i, j)$ for (i, j), for all i, j.

For a given B, $E_m(i, j)$ and $L_m(i, j)$, $i = 0,1,...,n; j = 1,...,d_i$ are the unique integers $E_m(i,j) = \left\lceil \frac{j - \sqrt[m]{B}}{r_i} \right\rceil$ and $L_m(i,j) = \left\lfloor \frac{j-1+\sqrt[m]{B}}{r_i} + 1 \right\rfloor$, [5].

The earliest sequencing time $E_m(i, j)$ and the latest sequencing time $L_m(i, j)$ form a time window $T_m = [E_m(i,j), L_m(i,j)]$ within which $(i,j), i = 0,1,...,n; j = 1,...,d_i$, can be sequenced with the level curves not exceeding the bottleneck.

A V_1-convex bipartite graph $G = (V_1, V_2, E)$ is constructed sequencing (i, j) within T_m, where $V_1 = \{1,...,D\}$ stands for the set of sequencing models, $V_2 = \{(i,j) \mid i = 0,1,...,n; j = 1,...,d_i\}$ the set of (i, j) and $E = \{(k,(i,j)) \mid k \in T_m\}$.

The earliest due date (EDD) algorithm that matches each $k \in V_1$ to the unmatched (i, j) with the smallest $L_m(i, j)$ and $(k,(i,j)) \in E$ finds a perfect matching. The algorithm stops if no such (i, j) exists [11]. The perfect matching is order-preserving. An order-preserving perfect matching in G is analogous to a feasible solution to the Problem [11].

A perfect matching in G exists if and only if $|N(K)| \geq |K|$,where $N(K) = \{(i, j) : (i, j) \in V_2, \exists k \in K s.t. (k,(i, j)) \in E\}$ and K is either an interval in V_1 or the neighborhood of an interval in V_1, [1]. This is the Hall's theorem for the existence of a perfect matching that yields a feasible solution to the problem. Existence of a perfect matching depends on B. A perfect matching exists if B satisfies the inequalities in the following theorem. This is a certificate for the existence of a feasible solution.

Theorem 1 [5] *Problem F_m has a feasible solution if and only if, for all $k_1, k_2 \in V_1$ with $k_1 \leq k_2$ and $[E_m(i, j), L_m(i, j)] \cap [k_1, k_2] \neq \phi$, B satisfies the inequalities* $\sum_{i=1}^{n} (\lfloor k_2 r_i + \sqrt[m]{B} \rfloor - \lceil (k_1 - 1) r_i - \sqrt[m]{B} \rceil) \geq k_2 - k_1 + 1$ *and* $\sum_{i=1}^{n} (\lceil (k_2 r_i - \sqrt[m]{B} \rceil - \lfloor k_1 - 1) r_i + \sqrt[m]{B} \rfloor) \leq k_2 - k_1 + 1$.

A feasible solution with a minimum B is optimal. The minimum B can be obtained using a bisection search that runs between the lower and upper bottlenecks. The lower and upper bottlenecks for the problem are $(1 - r_{\max})^m$ and $(1 - \frac{1}{D})^m$, respectively.

Theorem 2 [5] *A bisection search in the interval $[(1 - r_{\max})^m, (1 - \frac{1}{D})^m]$ determines the minimum B in $O(\log D)$ time.*

The time complexity to yield an optimal sequence using the bisection search is O(DlogD) since $E_m(i,j)$ and $L_m(i,j)$ can be calculated in O(D).

3. Optimality of Cyclic Sequence

The time complexity can substantially be reduced when cyclic sequence exists. When $u = \gcd(d_1, \ldots, d_n) > 1$, cyclic sequence consisting of u subsequences with the same length exists. Furthermore, cyclic sequence is optimal.

Lemma 1 If jth copy of a model i, i = 1,…, n, is not sequenced within $[E_m(i,j), L_m(i,j)]$, the level curves exceed B.

Proof:

Suppose that j^{th} copy of a model i, $i = 1, \ldots, n$, be sequenced such that $k < E_m(i,j)$.

$\Rightarrow k < \left| \frac{j-\sqrt[m]{B}}{r_i} \right|$

$\Rightarrow k < \frac{j-\sqrt[m]{B}}{r_i} \Rightarrow$ if $\frac{j-\sqrt[m]{B}}{r_i}$ is not an integer.

If $\frac{j-\sqrt[m]{B}}{r_i}$ is an integer, $k = E_m(i,j)$

$\Rightarrow \sqrt[m]{B} < j-kr_i$

$\Rightarrow B < \left| j-kr_i \right|^m$

Further, suppose that $L_m(i,j) < k$.

$\Rightarrow \left\lfloor \frac{j-1+\sqrt[m]{B}}{r_i} + 1 \right\rfloor < k$

$\Rightarrow \frac{j-1+\sqrt[m]{B}}{r_i} + 1 < k$ if $\frac{j-1+\sqrt[m]{B}}{r_i} + 1$ is not an integer.

If $\frac{j-1+\sqrt[m]{B}}{r_i} + 1$ is an integer, $k = L_m(i,j)$.

$\Rightarrow \sqrt[m]{B} < (k-1)r_i(j-1)$

$\Rightarrow B < \left| (j-1)-(k-1)r_i \right|^m$.

Theorem 3 If $u = \gcd(d_1, \ldots, d_n) > 1$, a cyclic sequence to the problem, consisting
of u repetition of optimal subsequence, exists and is optimal.

Proof:

For a feasible solution, j^{th} copy of of model I, $I = 1, \ldots, n$, must be sequenced within $[E_m(i.j), L_m(i,j)]$, $I = 1, \ldots, n$; $j = 1, \ldots, d_i$.

Otherwise, the level curves exceed B.

Let $u = \gcd(d_1, \ldots, d_n) > 1$ be a factor of d_i and D.

We write, $d_i = uv_i$, $D = uv$, $v = \sum_{i=1}^{n} v_i$ and $r_i = \frac{v_i}{v}$, $I = 1, \ldots, n$.

We have, $E_m(i,(e-1)v_i+1)$

$= \left| \frac{(e-1)v_i+1-\sqrt[m]{B}}{r_i} \right|$, $e = 1, \ldots, u$

$= \left| (e-1)v + \frac{1-\sqrt[m]{B}}{r_i} \right|$

$= (e-1)v + \left| \frac{1-\sqrt[m]{B}}{r_i} \right|$

$> (e-1)v$ since $B < 1$.

Further, $L_m(i, ev_i)$
$= \left\lfloor \frac{ev_i - 1 + \sqrt[m]{B}}{r_i} + 1 \right\rfloor$
$= \left\lfloor ev + \frac{\sqrt[m]{B} - 1}{r_i} + 1 \right\rfloor$
$= ev + \left\lfloor \frac{\sqrt[m]{B} - 1}{r_i} + 1 \right\rfloor$
$\leq ev$.

Thus, $(e-1)v$
$< E_m(i, (e-1)v_i + 1)$
$\leq L_m(i, (e-1)v_i + 1)$
$\leq E_m(i, ev_i)$
$\leq L_m(i, ev_i)$
$\leq ev$.

This implies that v_i copies $(e-1)v_i + 1, \ldots, ev_i$ of model i, $i = 1, \ldots, n$ occupy positions in
$[(e-1)v+1, ev]$.

This shows that the sequence consists of u periods.
So, cyclic sequence is periodic. Each period is a subsequence of the sequence.
Now, we show that each period consists of the same models in the same order.
The e^{th} period of copies of model i, $i = 1, \ldots, n$ is labeled as $(e-1)v_i + f$, $e = 1, \ldots, u$; $f = 1, \ldots, v_i$.

For $E_m(i, j)$, $E_m(i, ev_i + f)$
$= \left\lceil \frac{ev_i + f - \sqrt[m]{B}}{r_i} \right\rceil$
$= \left\lceil \frac{(e-1)v_i + f - \sqrt[m]{B}}{r_i} + v \right\rceil$
$= \left\lceil \frac{(e-1)v_i + f - \sqrt[m]{B}}{r_i} \right\rceil + v$
$= E_m(i, (e-1)v_i + f) + v$
$= E_m(i, f) + ev$, for $f = 1, \ldots, v_i$; $I = 1, \ldots, n$.

For $L_m(i, j)$, $L_m(i, ev_i + f)$
$= \left\lfloor \frac{ev_i + f - 1 + \sqrt[m]{B}}{r_i} + 1 \right\rfloor$
$= \left\lfloor \frac{(e-1)v_i + f - 1 + \sqrt[m]{B}}{r_i} + 1 + v \right\rfloor$
$= \left\lfloor \frac{(e-1)v_i + f - 1 + \sqrt[m]{B}}{r_i} + 1 \right\rfloor + v$

$= L_m(i,(e-1)v_i + f) + v$
$= L_m(i,f) + ev$, for $f = 1, \ldots, v_i$; $i = 1, \ldots, n$.

The linear relations $E_m(i,ev_i + f) = E_m(i,f) + ev$ and $L_m(i,ev_i + f) = L_m(i,f) + ev$ imply that each period consists of v units of models and all units in thev isame order after sequencing the v units in the eth period.

An optimal subsequence can be determined for the first period. Then an optimal sequence consisting of u repetitions of this subsequence exists.

Corollary 1 The optimal bottlenecks of a subsequence and of its sequence are optimal.

Proof: Assume that a subsequence consists of D' copies with demands d_i' for model i, $i = 1, \ldots, n$
such that $\sum_{i=1}^{n} d_i'$ and $uD' = D$

We can write, $(x_{ik} - kr_i)^m$
$= (x_{i,(\theta D' + k')} - (\theta D' + k)r_i)^m$ for $0 \le \theta < u$ and $1 \le k' < D'$
$= (\theta d_i' + x_{ik'} - \theta d_i' - k' r_i)^m$
$= (x_{ik'} - k' r_i)^m$

4. Conclusion

The bottleneck product rate variation problem can be solved in pseudo-polynomial time. The complexity would be expensive for large size instances of the problem. So, it is natural to seek the cyclic sequence that is optimal. The cyclic sequence to the problem if exists is optimal.

The existence of optimal cyclic sequence for the bottleneck product rate variation problem with significant setup time and arbitrary processing time would be an area for future research.

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A Brief Review on the Reducibility of Shop Sequences Minimizing Some Regular Objectives

TANKA NATH DHAMALA
Central Department of Mathematics
Institute of Science and Technology
Tribhuvan University, Kathmandu, Nepal
Email: **dhamala@yahoo.com**

Abstract: The minimal spaces of irreducible sequences in shop scheduling problems have been investigated since last 20 years. However, the question if there exist a polynomial time algorithm for the decision whether a given sequence is irreducible is unsolved. Neither the computational complexity status of this decision problem (DP) is known.

Recently, one polynomial heuristic and one enumeration algorithm have been presented to solve the problem.with makespan objective. Under some restrictions the proposed exact algorithm reduces to the polynomial time algorithm. However, one of the crucial issues that remains to be solved is to justify the validity of the proposed conditions. We review the solution approaches of this problem and highlight the recent results. Various alternatives of its modeling in terms of H-comparability graphs are presented.

Keywords: scheduling/sequencing theory, H-comparability graph, mathematical modeling, reducibility/irreducibility, complexity analysis.

1. Introduction

We consider the nonpreemptive scheduling problems, open shop (*OSP*) and job shop (*JSP*) with regular objective γ and no additional constraints,

respectively, denoted by $O \| \gamma$ and $J \| \gamma$, in the usual modeling scheme $\alpha | \beta | \gamma$. A job i with $i \in I = \{1,....,n\}$ has to be processed on a machine j with $j \in J = \{1,..,m\}$ for the positive processing time. Each machine can process at most one job and each job can be processed on at most one machine at a time. All processing orders are arbitrarily in the *OSP*, whereas the machine orders are given in advance in a *JSP*. Flow shop is a special case of the *JSP* where all jobs have the identical machine orders. A sequence is an acyclic combination of all machine orders and all job orders. A schedule is the corresponding time table. A sequence is optimal if it yields a schedule with minimum objective value among all others.

Let $OIJ = I \times J$ and $PIJ \subseteq OIJ$. The numerical data release time $r = [r_i]$, due date $d = [d_i]$, weight $w = [w_i]$ and processing time $p = [p_{ij}]$, with $i \in I$ and $j \in J$, are contained in $N_D \in \boldsymbol{N_D}$. Let $C = [c_{ij}]$ be the matrix of completion times c_{ij}, a schedule. Given a sequence and the numerical date, we obtain the associated semiactive schedule (each operation is started as early as possible for given processing orders) in polynomial time. For a regular objective γ, that is, $\gamma(C_1, \dots, C_n) \leq \gamma(D_1, \dots, D_n)$ whenever $C_i \leq D_i$ for all $i \in I$, one can restrict the investigation to semiactive schedules for the optimality.

A *DP* is in the class *P* if there exists a deterministic polynomial time algorithm solving it. A *DP* is in *NP* if there exists a nondeterministic polynomial time algorithm solving it. The co-*NP* class contains the DPs whose complements are in *NP*. A DP is *NP* - complete if it belongs to *P*, then $NP = P$ holds. A DP in *NP* which is neither polynomial solvable nor *NP*-complete is *NP*-incomplete. An optimization problem whose DP is *NP* - complete is *NP*-hard. We refer to [7] for a systematic analysis.

The problems $O\|C_{max}$ for $m \geq 3$, $F3 \| C_{max}$ and $J2\|C_{max}$ are *NP*-hard , [8, 10].

A set of sequences is a solution space if it contains an optimal element for arbitrary numerical inputs. A solution space is potentially (universally) optimal if it contains an optimal sequence (solution) for arbitrary numerical data, for example, the set of all semiactive schedules. An important issue is the question of determination whether there exists such a (unique) minimal set, whose existence seems unlikely, in general, [6].

The idea of such spaces is very applicable when the processing times are erroneous, difficult to find out in advance or simply unknown, for instance, in manufacturing and service industries, satellite communications, examination scheduling and teacher class assignments, [2].

A sequence A is reducible to B, we write $B \leq A$, if $C_{max}(B) \leq C_{max}(B)$ for all P. It is strongly reducible, denoted by $B < A$, if $B \leq A$ but not $A \leq B$. They are similar, denoted by $A \sim B$ if $B \leq A$ and $A \leq B$. A sequence is irreducible if there exists no other non-similar sequence to which it can be reduced.

A sequence A is general-reducible to B, written as $B \leq_g A$, if $C_i(B) \leq C_i(A)$ holds for all jobs i and all possible numerical data N_D. A sequence A is r-reducible to B, denoted by $B \leq_r A$, if $C_{max}(B) \leq C_{max}(B)$ for all $P = [p_{ij}]$ and all $r = [r_{ij}]$. Generalizations of similarity and irreducibility relations are done likewise.

We review the open questions: Does there exist a polynomial time algorithm for the decision whether a given sequence is irreducible? What is the complexity status of this DP? Some interesting open problems are stated in at the end.

A complete conclusion of these problems are awaited. We shortly sketch the mathematical modeling, the solution strategies and the structures of the sets of all solutions, and then the set of irreducible sequences. The complexity status is reviewed very shortly.

2. Mathematical Modeling

2.1 Basic Terminology

A comparability graph (CG) is an undirected graph $G = (V, E)$ that has a transitive orientation $G^{tr} = (V, E^{tr})$. It is prime if uniquely orientable. A Hamming graph $K_n \times K_m$, denoted by G_{IJ}, restricted on $PIJ \subseteq OIJ$ is H-graph. A CG = (PIJ, E) which contains an H-graph is H-CG. For any (MO, JO), define the shop graph $G_{MO,JO} = (PIJ, E_{MO,JO})$. An acyclic shop graph is a sequence graph. For each sequence graph $G_{MO,JO}$ we can describe the sequence (MO, JO) by a special matrix $A = [a_{ij}]$, where $a_{ij} = \mathrm{rank}(o_{ij})$, such that for each integer $a_{ij} > 1$ there exists $a_{ij} - 1$ in row i or in column j or in both. An arc from (o_{ij}, o_{kl}) exists if and only if $i = k$ or $j = l$ and $a_{ij} <$

a_{kl} hold. There is a one-to-one correspondence between the sets of all sequences and all sequence graphs. A sequence graph $G_A = (PIJ, E)$, is an acyclic orientation of the H-graph $G_{I,J}$.

For the sequence A, we denote the transitive orientation of the G_A and its symmetric closure by G_A^{tr} and $[G_A^{tr}]$, respectively. Let $E_{r(A)}$ and $E_{d(A)}$ represent the sets of all regular edges (edges in $G_{I,J}$) and diagonal edges, respectively. Here, for the given sequence the symmetric closure $[G_A^{tr}] = (OIJ, G_A^{tr} + (G_A^{tr})^{-1}) = (OIJ, E_{r(A)} \cup E_{d(A)})$ is the underline undirected graph where G^{-1}denotes the reversed graph of G with all arcs in reversed direction.

For two edges ab, cd in $G = (V, E)$ a Γ-relation $ab\Gamma cd$ is defined if and only if either $a = c$, $bd \notin E$ or $b = d$, $ac \notin E$ or $ab = cd$. The transitive relation Γ^{tr} decompose the set of all edges into equivalent ICs in the CG. The set of all these classes of the sequence A is denoted by $I([G_A^{tr}]) = \{I_1, ..., I_l, I_1^{-1}, ..., I_l^{-1}\}$. A graph is a CG if and only if there is no implication class (IC) containing both an arc and its reverse.

Two edges ab, cd in G_A are in Γ_A-relation denoted by $ab\Gamma_A cd$, if and only if $ab\Gamma cd$ in G_A^{tr}. Two edges e, e' in $E_{r(A)}$ are connected by a Γ_A path if there exist $e = e_0, e_1, ..., e_m, e_{m+1} = e'$ from $E_{r(A)}$ such that $e\, \Gamma_A\, e_1\, \Gamma_A\, e_2 ... e_m\, \Gamma_A\, e'$ which defines the transitive closure Γ^{tr} on $E_{r(A)}$. The extended sequence ICs (SICs) are the minimal sets containing all transitive edges of the corresponding classes. The relation Γ^{tr} partitions $E_{r(A)}$ into equivalent SICs. The set of all SICs of the sequence A is denoted by $P([G_A^{tr}]) = \{P_1, ..., P_k, P_1^{-1}, ..., P_k^{-1}\}$.

Given $G = (V, E)$, we define the Γ-graph $G_\Gamma = (E, \Gamma)$ with an edge $e_1 e_2$ in Γ if and only if $e_1 \Gamma e_2$ in G. For a given sequence A, from the Γ-graph $G_\Gamma = (E_{r(A)} + E_{d(A)}, \Gamma)$ with contraction of edges in Γ-relation, we define the factor graph $G_{F(A}$. The vertex set $G_{F(V)}$ contains an arc v in E_d or the extended SICs in P_A and $P_{A^{-1}}$. An undirected edge $e_1 e_2$ belongs to $G_{F(E)}$ if and only if there exists a Γ-relation between nodes or set of nodes e_1 and e_2.

Let $(OIJ, E_{r(A)} + E_{d(A)})$ with $|E_d| = d$ be the H-CG to the given sequence $A \in SIJ$, $P_A = \{P_1, \ldots, P_k\}$ and $E_{r(A)} = P_1 + \ldots + P_k + P_1^{-1} + \ldots + P_k^{-1}$. The consequence graph $G_{K(V)} = (V_K, E_K)$ is defined as follows. The set of nodes is $V_K \subseteq E_{d(A)} + P_A + P_{A^{-1}}$. Two edges e' and e'' from the vertex set V_K are connected by an un directed edge of color $I \in \{1, \ldots, d\}$ when the removal of e_i in E_d forms a Γ-relation between e' and e'' or between the SICs they represent, respectively, i.e., $E_K = \{e'e''\, with\, color\, i : e'\Gamma e''\, in\, [G_A''] - e_i, e_i \in E_{d(A)}\}$.

The set G_{Ki} represents the subgraph of G_K with i^{th} color such that $G_K = G_{K1} + \ldots + G_{Kd}$.

For a given set $M \subseteq E_{d(A)}$, the reduction graph $G_{RM}(A) = (V_{RM}, E_{RM})$ is defined by inserting into G_F all edges from G_F which are colored from M and deleting the nodes which represent edges in M as $G_{RM}(A) = G_F + \cup_{e \in M} G_{Ke} - M$.

The removal of an edge result new Γ-relations. The consequence graph informs which SICs are merged by the removal of $\hat{e}_i \in E_d(S)$. The reduction graph informs about the deletion of nodes from G_F and addition of edges between the remaining nodes in G_F which induce a new Γ-relation between SICs.

2.2 Alternate Formulations

Given a A in SIJ for the $O||C_{max}$ we reformulate different versions of the DP: Does there exist a polynomial time algorithm for the decision whether a given sequence is irreducible? The irreducibility and reducibility are complement DPs. Reducing is the constructive optimization problem to the reducibility.

Irreducibility 1 Is the sequence A irreducible?

Reducibility 1 Does there exist a sequence $B \in SIJ$ such that $B < A$?

Reducing: Find a sequence B, if it exists, such that $B < A$.

A path w_A with vertex set $V(w_A)$ in G_A (equivalently, in A) is maximal if there is no path w_A^* in A with $V(w_A) \subset V(w_A^*)$. The set $\mathbf{W_A}$ of all maximal

paths contains the longest path. A sequence A is reducible to B if and only if for all maximal paths w_B in B, there exists a maximal path w_A in A such that $V(w_B) \subseteq V(w_A)$ holds. If $B < A$, then there exists w_B in $\boldsymbol{W_B}$ with $V(w_B) \subseteq V(w_A)$ for some w_A in w_A . However, this test is exponential.

Theorem 1 [3] Let $A,\ B \in SIJ$ be on the same *OIJ* for $O||C_{max}$. Then A is reducible, strongly reducible or similar to B if and only if $G_B^{tr} \subseteq G_{A,}^{tr} G_B^{tr} \subset G_{A,}^{tr} G_B^{tr} = G_{A,}^{tr}$ respectively.

Since Theorem 1 can be tested in time $O(n^2 m^2)$, it yields an answer in polynomial time to the question of irreducibility, reducibility or similarity between two given sequences. For a reducible sequence, the reducibility can be proved with nondeterministic polynomial time. As this is constructive, such a procedure answers not only to the reducibility but also to the problem reducing. The problem reducibility is in *NP* and the problem irreducibility is in co-*NP*. Furthermore, if there exists a *NP* -test for irreducibility, then this problem is either polynomially solvable or *NP* -incomplete, as far as $P = NP$ holds, [1].

The irreducible sequences are the minimal elements of the half-order $\prec$ on H-CG $[G_A^{tr}]$ containing H-graph G_{IJ}, for given A in *SIJ*. The DPs for the $O||C_{max}$ can be reformulated as the question of the existence of an H-CG G as follows.

Irreducibility 2 Is there no H-CG G with $G_{IJ} \subseteq G \subset [G_A^{tr}]$?

Reducibility 2 Does there exist an H-CG G with $G_{IJ} \subseteq G \subset [G_A^{tr}]$?

The reducibility concerns the reduction of a sequence through the reversion of a IC in its transitive closure. One of the most fundamental properties states that a sequence A whose H-CG $[G_A^{tr}]$ is not prime is either reducible or is similar to an irreducible sequence B with $B \neq A$ and $B \neq A^{-1}$, [11].

A sequence can be obtained from every transitive orientation of an H-CG. If the H-CG G has a sequence orientation G_A^{tr}, then G = G_A^{tr}, is the H-CG to a sequence A in *SIJ*. A $T \in T_{G_A^{tr}}$ is a sequence orientation if every

diagonal edges in T is transitive. If G_B^{tr} of $[G_A^{tr}]$ is not a sequence orientation, then some diagonal edges of $[G_A^{tr}]$ are not in the orientation G_B^{tr}, and $B < A$.

One may reduce the given sequence by reversing an IC. One method to reverse the ICs is the deletion of a single diagonal edge. Deletion of an edge from a transitive reduction can be done easily. However, if $[G_A^{tr}]$ can be transitively oriented such that neither e nor e^{-1} are transitive edges, then the edge $\hat{e}$ can be deleted and $[G_A^{tr}]$ - $\hat{e}$ is a CG whose sequence orientation reduces A strongly.

As transitive orientation of an H-CG can be found in polynomial time and the number of diagonal edges for an $n \times m$ OSP is of order $O(n^2 m^2)$, it can be tested in polynomial time whether a give sequence can be strongly reduced by deleting a diagonal edge.

Thus, A in SIJ can be strongly reduced to B in SIJ, in polynomial time, which cannot be further reduced by reversing an arbitrary IC. The H-CG $[B^{tr}]$ is then either prime or there exist similar sequences to B other than B^{-1}. The set of all such reducible sequences cannot be obtained in polynomial time as the recombination of all ICs is of size $O(2^k)$ for k ICs and every edge may represent an IC in the worst case. The reversion of only ICs and their recombination does not generate the sequence space.

Not every recombination of the SICs of a sequence A is acyclic, and it yields a sequence B if it is acyclic. The set of all recombination of the SICs is sufficient. Therefore, taking SICs as basis for the space of sequences, we reformulate

Irreducibility 3 Is every produced feasible recombination of the SICs is similar to A?

Reducibility 3 Does there exist a feasible recombination of the SICs of A missing at least one diagonal edge?

A removable set with respect to a given sequence A is a set of undirected diagonal edges $M \subseteq E_{d(A)}$. The set M is feasible if $[G_A^{tr}]$ - M is an H-CG, and it is feasibly extendable if there exists a feasible removable set M^* of

diagonal edges of $[G_A'']$ such that $M \subset M^*$. The set M is infeasible if it is not feasibly extendable.

A removable set which is not feasible may not be necessarily infeasible. A removable set can be feasible and, in addition, feasibly extendable, too. Removal of only one edge may not yield a strongly reduced sequence. But removal of more than two edges simultaneously may reduce the sequence.

Irreducibility 4 Is every removable set $M \subseteq E_{d(A)}$ in $[G_A'']$ infeasible?

Reducibility 4 Does there exist a feasibly extendable removable set $M \subseteq E_{d(A)}$ in $[G_A'']$?

In any $[G_A'']$, the ICs which consist exclusively of diagonal edges can be deleted and the reduction through the reversion of a group of ICs can be done in polynomial time. A sequence is normal if it cannot be reduced in either of these ways. We restrict the space of sequences into the class of normal sequences [1].

For a reduction of a normal sequence by the reversion of a SIC $P1$ against the SIC $P2$ from the same IC, all Γ-paths between them which contains at least one diagonal edge have to be cut keeping the comparability property. For feasibility of the set M each such path has to be broken, for disconnection in $[G_A''] - M$.

3. Results and Discussions

We review the classical results with respect to the minimization of the maximum completion times and some general regular objectives. Moreover, the recent results based on the amazing roles of diagonal stable edges are also presented.

3.1 Classical Results

A criterion in terms of free machines to eliminate a large number of non-optimal sequences is given for $J|n = 2|C_{max}$. Different sequence decompositions for the JSP are given. For flow shop, it is sufficient to examine those sequences in which the job orders for the last two and the first two machines, respectively, are the same. The set of all irreducible sequences for $O2||C_{max}$ is presented, where each element contains only one SIC. See, [6] for the references.

The dominance relation is formulated as a mixed integer programming and unavoidable set of sequences is computed for small formats, see [1]. Among seven classes of all 3 × 3 irreducible sequences only three are unavoidable such that these together with their reverses form unique minimal set containing at least one optimal sequence. For $O3|n = 2|C_{max}$, the minimal cardinality of two distinct potentially optimal sets is 3. Reducibility can be tested without computing the transitive closures under some sufficient conditions, [3, 9].

The irreducible sequences for the *OSP* on an operation set with spanning tree structure are studied, [5]. A necessary and sufficient condition can be tested in polynomial time on tree-like operation sets, see [1]. To test whether a given sequence can be strongly reduced to another sequence by deleting an operation and reinserting it as a sink or a source, it has to be ensured that at least one path is destroyed in the former and no new path is created in the latter. This can be done in $O(n^2 m^2)$ time and space, given an n × m sequence, [3].

An enumeration algorithm computes all irreducible sequences constructing inclusion minimal CGs by successively inserting diagonal arcs into $[G_A]$, [4]. Each sequence in such a set is similar to exactly one sequence in this class, namely its reverse one. This algorithm constructs graphs G such that $G = [G''_A]$ for some sequence A.

An enumeration algorithm is presented in [3]. There a set of nonisomorphic sequences is computed and, thereafter tested for irreducibility. One sequence per isomorphic class is sufficient. The ratio between the number of irreducible sequences and all sequences decreases with growing problem size. Note that the concept of sequences isomorphisms plays a central role in determining the class of irreducible sequences. A test whether two given sequences are isomorphic can be performed in polynomial time [6].

A generalized decomposition on irreducibility is introduced in [6]. This investigates the properties of irreducibility on the sequences of larger sizes based on smaller sizes.

3.2 Conflict Resolution

The factor, consequence and reduction graphs help to recognize a feasible removable set. They decide a possibility of a transitive orientation of $[G_A^{\prime\prime}]$ - M, if M is going be feasible. Otherwise, the question is how a none feasible but feasibly extendable set M can be expanded or prove that the set M is not only none feasible but is infeasible. The diagonal edges in the irreducible sequences must not be removed. The problem of merging two ICs occurs when an edge play a role of conflict.

If there exists a path $W \subseteq V_{\mathrm{RM}}$ from a P_{i} in V_{RM} to its reversion P_i^{-1} in V_{RM} we call it a conflict in $G_{\mathrm{RM}}(\mathrm{A})$. The number $l \geq 0$ of the diagonal edges in the inclusion-minimal path W is its order. A conflict is direct if $l = 0$. Every conflict in G_{RM} reflects a Γ-path in $[G_A^{\prime\prime}]$ - M from P_{i} to its reversion P_i^{-1}. For the feasible extension of M all these conflicts must be dissolved and every one of these Γ-paths must be broken. A Γ-path between two edges will only be destroyed when at least one edge from this path is removed.

A diagonal edge is stable if it is in every irreducible sequence of A. A diagonal edge is trivial-stable if it is in an extended SIC. A stable diagonal edge which is not in an extended SIC is non-trivial-stable. If all edges in $[G_A^{\prime\prime}]$ are trivial-stable, then the irreducibility of a sequence is decidable in polynomial time. A diagonal edge $e \in E_d^{\prime\prime}(A)$ is magic-stable with respect to M if it does not lead to a direct conflict in G_{RM^*}, with $M + e \subseteq M^*$, through a series of conflicts of order 1.

It has not been found any sequence which contains a magic-stable edge. If one could prove that there exists no magic-stable edges, then the problem of irreducibility is polynomially solvable. Therefore, one of the main issues in irreducibility is to decide the existence or non-existence of magic-stable edges.

Two algorithms are proposed in [1]. They base on the characteristics of the diagonal edges of the associated H-CG. A number of open problems are posed. A key role lies on the diagonal edges while resolving the conflicts.

3.3 Regular Objectives

The concept of irreducibility for arbitrary numerical input data and some additional regular objective functions γ can be found in [11].

Theorem 2 [11] Let A, $B \in SIJ$, and let the set $v_i(A)$ of operations contains the predecessors of an operation of the job i. Then it holds $B \leq_g A \Leftrightarrow ([G''_B] \subseteq [G''_A]) \wedge (v_i(B) \subseteq v_i(A), \forall i \in I)$.

Along with a number of results in terms of the CGs and precedence relations between operations, the relations between the general-reducibility and r-reducibility are established.

Theorem 3 [11] For any sequences A, B in SIJ, it holds $B \lneq A$ if and only if $B^{-1} \lneq_g A^{-1}$ holds.

4. Conclusions

In this paper, we consider the open problem: does there exist a polynomial time algorithm for the decision whether a given sequence is irreducible? Then we review all mathematical formulations and all solution approaches of the *OSP* and *JSP*.

Following questions are of main interest subject of our due course.

How can be the concept of conflict resolution on makespan for other regular objectives or other shop environments, like the job shop, generalized? The extensions are likely though not considered yet.

It is still necessary to improve the achieved complexity results of the irreducibility. The enumeration algorithm recently presented is expected to be polynomial if the set of diagonal magic stable edges is empty. However, this result seems to be quite challenging.

Likewise, a development of neighborhood structure of irreducible sequences is an emerging problem. Because of the small class of irreducible sequences in comparison to the space of all sequences, a research on this smaller class has been motivated.

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Weibull Distribution to Describe the Patterns of the Duration of Post-partum Amenorrhea

TIKA RAM ARYAL
Central Department of Statistics
Tribhuvan University, Kirtipur Kathmandu, Nepal
E-Mail: traryal@gmail.com & traryal@rediffmail.com

Abstract: This paper tries to fit Weibull distribution to describe the distribution of the duration of Post-partum amenorrhea of rural Nepal. Weibull distribution was found well fitted to the data of Nepal for describing the distribution of PPA. The duration of PPA period estimated using the retrospective and current status reporting data gave the modal value to be about 8 months. The amount of over or under-reporting of PPA period was found to be about 6 days. Average expected waiting time for returning menstruation at delivery was found to be 8.47 and 8.59 months for current and retrospective reporting data respectively.

1 Introduction

Post-partum amenorrhea (PPA) refers to the period after a pregnancy during which conception does not occur i.e. the period of infecundable (sterility) immediately following the termination of pregnancy in a live birth or a stillbirth or an abortion [1, 2]. It is considered as the temporary infecundable period where attainment of first menstruation after delivery is treated as the termination of PPA. Davis and Blake [3] proposed a list of eleven key variables as intermediate variables that directly affect natural fertility. These intermediate variables were divided into three categories as intercourse, conception, and gestation. In a society where contraception use rate is very low, PPA plays a vital role in controlling fertility by lengthening the inter-birth interval [4,5,6]. It can be considered as the most powerful fertility inhibiting effect among other variables [1,7,8].

PPA distribution is complex in nature [9,10,11]. Talwar [12] suggested a triangular distribution as an approximation of the duration of PPA. Barret [13] used modified Pascal distribution, which was generalized by Potter and Kobrin

[10, 14]. They also proposed a mixture of geometric and negative binomial distribution in order to describe the bimodality of PPA duration. Lesthaeghe and Page [15] applied logit model for describing the duration of PPA. Ford and Kim [11] reviewed a number of models and finally used a mixture of two type I extreme value distribution to describe the pattern of amenorrhea period. Biswas [16] proposed a mixture of two gamma distributions and following Singh and Bahaduri [17], suggested a mixture of type III distribution for this purpose.

Several researchers have discussed the major problems for applying various distributions, and most of them may be found in Ford and Kim [11] and Biswas [16]. Rahman [18] fitted a modified exponential curve to describe the pattern of PPA considering several marriage cohorts of Bangladeshi mothers. Sinha [19] proposed an inflated type I extreme value distribution to describe the pattern of PPA for Indian mothers. Islam [20] applied type I extreme value, inflated type I extreme value distributions and modified exponential curve for describing the duration of PPA by using national data of Bangladeshi mothers. Type I extreme value distribution and modified exponential curve has been found to be good approximation for the PPA duration of Bangladeshi mothers. But these studies do not take into account problems regarding estimation of the parameters and test of the goodness of fit of the used models. Modified exponential curve is deterministic in nature and was discussed by Islam [20]. To check suitability of these distributions, researchers have taken cumulative proportions of PPA distribution. Moreover, testing of goodness of fit and suitability of the distribution by such methods leads to close approximation of observed and expected distributions in most cases and was discussed by Aryal [2]. Due to these limitations, this paper tries to fit Weibull distribution to describe the patterns of the duration of PPA for Nepalese mothers. In brief, the distribution is given below.

2 Weibull Distribution

Weibull distribution has been proposed to describe the distribution of PPA for Nepalese data. In brief, the model is given below.

Let X denote the length of amenorrhea period. We assume that X has a Weibull distribution whose density function is given by

$$f(x)=\beta\alpha^{-1}\{(x-\xi_0)/\alpha\}^{\beta-1}\exp[-\{(x-\xi_0)/\alpha\}^{\beta}] \qquad (1)$$

and cumulative probability distribution function by

$$F(x)=1-\exp\left[-\{(x-\xi_0)/\alpha\}^{\beta}\right] \qquad (2)$$

where α and β are the scale and location parameters of the Weibull distribution respectively.

The parameters are estimated by using the method of least squares. On simplification of equation (2) and by taking double log on both of its sides, we get

$$\log\left[\log\left\{\frac{1}{1-F(x)}\right\}\right]=\beta\log(x-\xi_0)-\beta\log\alpha \qquad (3)$$

This is a linear function and can be re-written in the form of

$$\mathrm{Y}=\mathrm{A}+\beta\mathrm{X} \text{ for } \beta>0, \qquad (4)$$

where, $Y=\log\left[\log\left\{\frac{1}{1-F(x)}\right\}\right]$, $A=-\beta\log\alpha$ and $X=\beta\log(x-\xi_0)$.

A and β can easily be estimated from equation (4). If we assume ξ_0=0, then other parameters can be easily obtained by using the least square principle. For ξ_0=0 and $\alpha=1$, the validity of the standard Weibull distribution may be found in literature [21]. We have assumed that ξ_0=0 for the approximation of the duration of PPA for Nepalese mothers.

3 Construction of Amenorrheic Life-table

Aryal [2] constructed amenorrheic life-table through model specification for the Nepalese data. The components of the life table are given below.

Let M_t be the proportion of mothers who terminated their amenorrheic periods at t months or less. Let S_t be the proportion of surviving (not terminating amenorrheic period) at t months (S_t= 1-M_t). If l_o is the cohort of the life table, then l_t=l_o*S_t and the probability of terminating amenorrheic period between t and t+1 months is

given as:

$$q_t = \frac{(M_{t+1} - M_t)}{(1 - M_t)} \qquad (5)$$

Other remaining components of the amenorrheic life-table such as L_t (person-months returned amenorrheic period by the cohort at exact t month), T_t (total person-months terminating amenorrheic period by the cohort after exact t months) and e_t (average number of months expected to return menstruation at exact t months) are defined and computed by the usual process followed in construction of a life-table [2, 22, 23].

4 Application

Data are taken from a sample survey entitled 'Demographic Survey on Fertility and Mobility (DSFM) A Study of Palpa and Rupandehi Districts of Western rural Nepal'. A total of 811 households were surveyed. A sample of 1019 ever-married women of reproductive age were interviewed. A total of 642 mothers provided information on the duration of PPA for their last child birth (current status reporting) who had given at least one birth in the last 7 years preceding the survey date. About 85 per cent (544 mothers) reported that their menstruation had resumed and the rest (98 mothers) were still amenorrheic at the date of interview. A total of 481 mothers provided information on the duration of PPA for their last but one child birth (retrospective reporting). Finally, 544 mothers provided the current status reporting data of PPA period while 481 mothers provided the retrospective reporting data of PPA period.

Figure 1 Observed and expected distribution of PPA for current status data

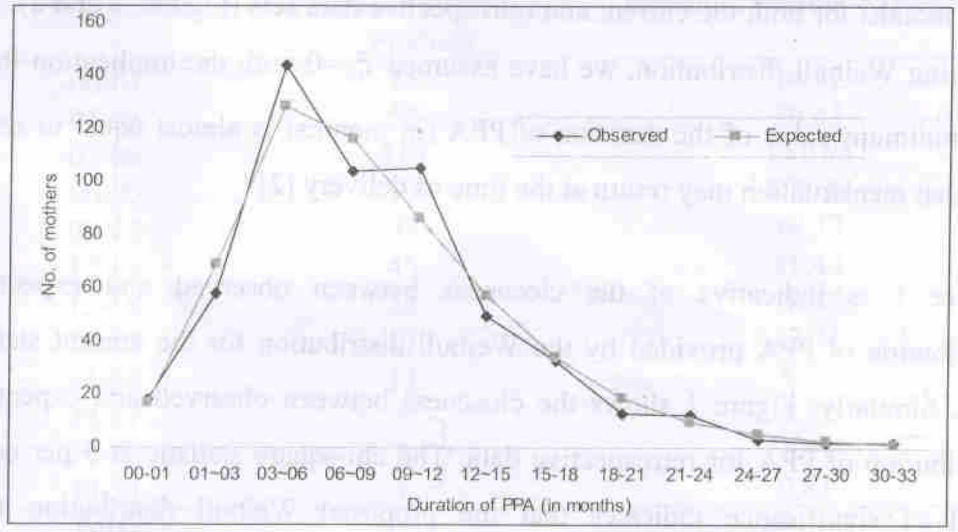

Table 1: Observed and expected distribution of PPA for current status reporting data

PPA (in months)	Observed number of mothers	Expected number of mothers
00-01	18	17.03
01-03	58	69.18
03-06	144	128.15
06-09	104	115.98
09-12	105	86.71
12-15	50	57.71
15-18	33	34.08
18-21	13	18.72
21-24	12	9.54
24-27	3	5.03
27-30	2	2.33
30-33	2	1.03
Total	544	544.00
	χ^2	12.41
	d.f.	6
	Parameters	α=9.4150
		β=1.5369
		ξ_0=0

$\chi^2_{(0.05,6)}$=12.59

The observed distribution of PPA duration showed bimodality at 3-6 months and 9-12 months for both the current and retrospective data sets (Figures 1 and 2). In applying Weibull distribution, we have assumed $\xi_0=0$ with the implication that the minimum value of the duration of PPA (in months) is almost equal to zero and that menstruation may return at the time of delivery [2].

Figure 1 is indicative of the closeness between observed and expected distribution of PPA provided by the Weibull distribution for the current status data. Similarly, Figure 1 shows the closeness between observed and expected distribution of PPA for retrospective data. The chi-square statistic at 5 per cent level of significance indicates that the proposed Weibull distribution fits reasonably well to the data of Nepal.

Figure 2 Observed and expected distribution of PPA for retrospective status data

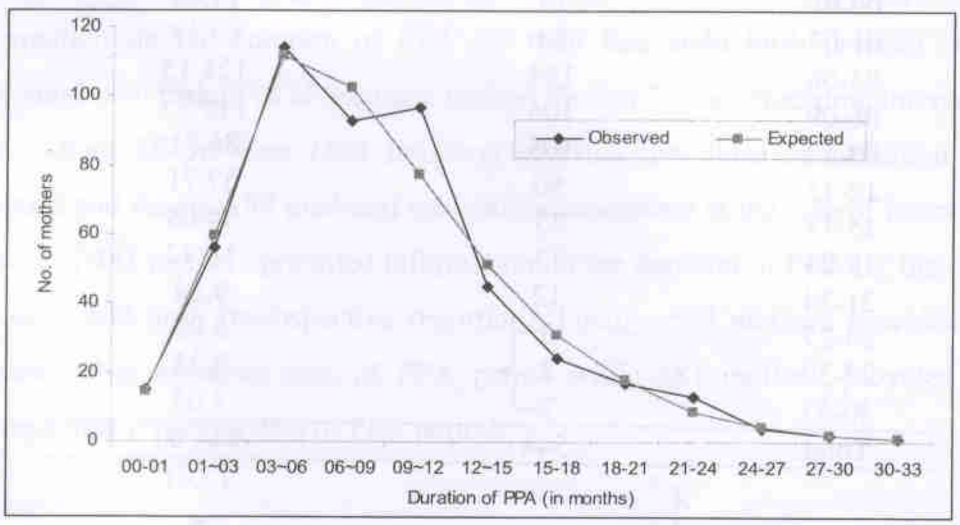

Table 2: Observed and expected distribution of PPA for retrospective status data

PPA (in month)	Observed number of mothers	Expected number of mothers
00-01	15	14.42
01-03	56	59.43
03-06	114	111.68
06-09	93	102.39
09-12	97	77.37
12-15	45	51.43
15-18	24	30.94
18-21	17	17.71
21-24	13	8.79
24-27	4	4.22
27-30	2	1.91
30-33	1	0.81
Total	481	481.00
	χ^2	10.38
	d.f.	6
	Parameters	$\alpha=9.5476$ $\beta=1.5478$ $\xi_0=0$
	$\chi^2_{(0.05,6)}=12.59$	

The estimated values of parameters and chi-square value along with degrees of freedom provided by Weibull distribution are presented in Tables 1 and 2. The graph shows a peak at 6-9 months, which may be considered as the modal class of the distribution of PPA. Mode values of the PPA were 8.05 and 8.24 months for current and retrospective status data respectively. The difference of mode values obtained from the data of current and retrospective status provides the amount of time due to memory lapse in reporting the PPA period. The amount of time due to memory lapse in reporting the data was about 6 days. The over-reporting tendency of the PPA duration has been observed in the retrospective reporting data whereas under-reporting tendency has been observed in the current status reporting data [1], though, the extent of over or under-reporting of PPA period was very small and insignificant.

Table 3 Amenorrheic life-table (current status reporting data)

PPA in months	M_t	S_t	l_t	q_t	L_t	T_t	e_t
0	0.0000	1.0000	10000.0000	0.0314	9843.18	84749.57	8.47
1	0.0314	0.9686	9686.3580	0.0588	9401.57	74906.39	7.73
2	0.0883	0.9117	9116.7895	0.0769	8766.46	65504.81	7.19
3	0.1584	0.8416	8416.1373	0.0914	8031.41	56738.35	6.74
4	0.2353	0.7647	7646.6848	0.1040	7249.22	48706.94	6.37
5	0.3148	0.6852	6851.7638	0.1151	6457.47	41457.71	6.05
6	0.3937	0.6063	6063.1681	0.1252	5683.61	35000.25	5.77
7	0.4696	0.5304	5304.0570	0.1345	4947.35	29316.64	5.53
8	0.5409	0.4591	4590.6457	0.1432	4262.06	24369.28	5.31
9	0.6067	0.3933	3933.4714	0.1513	3635.97	20107.23	5.11
10	0.6662	0.3338	3338.4681	0.1589	3073.19	16471.26	4.93
11	0.7192	0.2808	2807.9201	0.1662	2574.61	13398.06	4.77
12	0.7659	0.2341	2341.3090	0.1731	2138.68	10823.45	4.62
13	0.8064	0.1936	1936.0570	0.1797	1762.11	8684.76	4.49
14	0.8412	0.1588	1588.1615	0.1860	1440.44	6922.65	4.36
15	0.8707	0.1293	1292.7245	0.1921	1168.55	5482.21	4.24
16	0.8956	0.1044	1044.3787	0.1980	941.00	4313.66	4.13
17	0.9162	0.0838	837.6197	0.2036	752.34	3372.66	4.03
18	0.9333	0.0667	667.0527	0.2091	597.31	2620.32	3.93
19	0.9472	0.0528	527.5678	0.2144	471.01	2023.01	3.83
20	0.9586	0.0414	414.4531	0.2196	368.96	1552.00	3.74
21	0.9677	0.0323	323.4597	0.2245	287.14	1183.05	3.66
22	0.9749	0.0251	250.8281	0.2294	222.06	895.90	3.57
23	0.9807	0.0193	193.2872	0.2341	170.66	673.85	3.49
24	0.9852	0.0148	148.0323	0.2387	130.36	503.19	3.40
25	0.9887	0.0113	112.6908	0.2432	98.99	372.82	3.31
26	0.9915	0.0085	85.2802	0.2476	74.72	273.84	3.21
27	0.9936	0.0064	64.1626	0.2519	56.08	199.12	3.10
28	0.9952	0.0048	47.9991	0.2561	41.85	143.04	2.98
29	0.9964	0.0036	35.7063	0.2602	31.06	101.18	2.83
30	0.9974	0.0026	26.4152	0.2642	22.93	70.12	2.65
31	0.9981	0.0019	19.4357	0.2682	16.83	47.20	2.43
32	0.9986	0.0014	14.2238	0.2720	12.29	30.37	2.14
33	0.9990	0.0010	10.3548	0.2758	8.93	18.08	1.75
34	0.9993	0.0007	7.4990	0.2795	6.45	9.15	1.22
35	0.9995	0.0005	5.4030	0.2831	2.70	2.70	0.50

Tables 3 and 4 show the amenorrheic life-table constructed for the Nepalese mothers for both the current as well as retrospective data sets respectively. These life-tables give the average expected waiting time to return to menstruation after some specified months of amenorrheic state t. Figures in column 2 (M_t) are based on the proportion of mothers who returned to amenorrheic period after ending the pregnancy or delivery, as estimated by using the Weibull distribution, at a particular months t. The expected time to return to menstruation among Nepalese mothers on ending the pregnancy or delivery was 8.47 months for the current status data and 8.59 months for the retrospective data. For mothers who did not return to menstruation after delivery until 3, 6, 12, 24 and 30

months, the waiting times were 6.74, 5.77, 4.60, 3.40 and 2.60 months respectively. About 40, 77 and 93 per cent of the mothers returned to menstruation before 6, 12 and 18 months respectively.

Amenorrheic life-table provides additional information on the duration of PPA and also provides trends on expected waiting time to return to menstruation after delivery or some specified duration of amenorrheic state at any t months. Amenorrheic life-table may be used to identify the duration of PPA period for any population if PPA distribution of any other population is known.

Table 4 Amenorrheic life-table (retrospective status reporting data)

PPA in months	M_t	S_t	l_t	q_t	L_t	T_t	e_t
0	0.0000	1.0000	10000.0000	0.0300	9850.13	85876.03	8.59
1	0.0300	0.9700	9700.2698	0.0569	9424.49	76025.90	7.84
2	0.0851	0.9149	9148.7068	0.0747	8806.82	66601.41	7.28
3	0.1535	0.8465	8464.9427	0.0892	8087.23	57794.58	6.83
4	0.2290	0.7710	7709.5142	0.1018	7317.28	49707.35	6.45
5	0.3075	0.6925	6925.0418	0.1129	6534.11	42390.08	6.12
6	0.3857	0.6143	6143.1860	0.1230	5765.26	35855.96	5.84
7	0.4613	0.5387	5387.3286	0.1324	5030.72	30090.70	5.59
8	0.5326	0.4674	4674.1194	0.1411	4344.38	25059.98	5.36
9	0.5985	0.4015	4014.6389	0.1493	3715.02	20715.60	5.16
10	0.6585	0.3415	3415.3990	0.1570	3147.32	17000.58	4.98
11	0.7121	0.2879	2879.2432	0.1643	2642.70	13853.26	4.81
12	0.7594	0.2406	2406.1602	0.1713	2200.08	11210.56	4.66
13	0.8006	0.1994	1994.0063	0.1780	1816.57	9010.48	4.52
14	0.8361	0.1639	1639.1324	0.1844	1488.02	7193.91	4.39
15	0.8663	0.1337	1336.9128	0.1905	1209.54	5705.88	4.27
16	0.8918	0.1082	1082.1766	0.1965	975.86	4496.34	4.15
17	0.9130	0.0870	869.5490	0.2022	781.63	3520.48	4.05
18	0.9306	0.0694	693.7092	0.2078	621.64	2738.85	3.95
19	0.9450	0.0550	549.5760	0.2132	491.00	2117.21	3.85
20	0.9568	0.0432	432.4325	0.2184	385.22	1626.20	3.76
21	0.9662	0.0338	338.0002	0.2234	300.24	1240.98	3.67
22	0.9738	0.0262	262.4744	0.2284	232.50	940.75	3.58
23	0.9797	0.0203	202.5290	0.2332	178.91	708.25	3.50
24	0.9845	0.0155	155.3004	0.2379	136.83	529.33	3.41
25	0.9882	0.0118	118.3574	0.2425	104.01	392.50	3.32
26	0.9910	0.0090	89.6610	0.2469	78.59	288.49	3.22
27	0.9932	0.0068	67.5218	0.2513	59.04	209.90	3.11
28	0.9949	0.0051	50.5545	0.2556	44.09	150.86	2.98
29	0.9962	0.0038	37.6350	0.2597	32.75	106.77	2.84
30	0.9972	0.0028	27.8599	0.2638	24.18	74.02	2.66
31	0.9979	0.0021	20.5097	0.2678	17.76	49.84	2.43
32	0.9985	0.0015	15.0165	0.2718	12.98	32.07	2.14
33	0.9989	0.0011	10.9355	0.2756	9.43	19.10	1.75
34	0.9992	0.0008	7.9214	0.2794	6.81	9.67	1.22
35	0.9994	0.0006	5.7081	0.2831	2.85	2.85	0.50

5 Conclusions

Weibull distribution was found well fitted to the data of Nepal for describing the distribution of PPA. The duration of PPA period estimated using the retrospective and current status reporting data gave the modal value to be about 8 months. The amount of over or under-reporting of PPA period was found to be about 6 days. Average expected waiting time for returning menstruation at delivery was found to be 8.47 and 8.59 months for current and retrospective reporting data respectively.

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