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Summability of Legendre Series by Uniform Lower Triangular Matrix Method

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Abstract: In this paper a new theorem on the uniform matrix summability of Legendre series has been established. This theorem is generalization of previously known all theorems of this direction.

Key words: Matrix Summability, Legendre series, orthogonal polynomial, monotonic function.

Subject classification: 40C05, 42C10.

1. Definitions

The Legendre series associated with the Lebesgue- integrable function of f (x) in the interval (-1, 1) is given by

$$f(x)=\sum_{n=0}^{\infty} a_n P_n(x) \tag{1}$$

where $a_n=(n+\frac{1}{2})\int_{-1}^{1} f(t)\, P_n(t)\, dt$ (2)

and the Legendre polynomials $P_n(x)$, which are orthogonal in the interval (–1 ,1) are defined by the generating function

$$\frac{1}{\sqrt{1-2xz+z^2}}=\sum_{n=0}^{\infty} P_n(x)\, z^n . \tag{3}$$

Let $T=(a_{n,k})$ be an infinite triangular matrix satisfying the Silverman- Töeplitz [4] conditions of regularity i.e.

$$\sum_{k=0}^{n} a_{n,k} \to 1 \; as \; n \to \infty,$$

$$a_{n,k}=0 \;, \text{ for } k > n$$

and

$$\sum_{k=0}^{n} \left| a_{n,k} \right| \leq M, \text{ a finite constant.}$$

Let $\sum_{m=0}^{\infty} u_m(x)$ be infinite series defined in $a \leq x \leq b$.

Write

$$S_n(x) = \sum_{\nu=0}^{n} u_\nu(x).$$

If there exist a function $S(x)$ such that

$$\sum_{k=0}^{n} a_{n,k} \left(S_k(x) - S(x)\right) = o(1) \text{ as } n \to \infty \tag{4}$$

Uniformly in set E in which $S(x)$ is bounded, then the series $\sum_{m=0}^{\infty} u_m(x)$ is summable by matrix means (T) uniformly in set E to sum $S(x)$.

We use the following notations.

$$N_n(t) = \sum_{k=0}^{n} a_{n,k} \frac{\sin(k+1)t}{\sin\frac{t}{2}}$$

$$\psi(t) = \psi(\theta, t) = f\{\cos(\theta - t)\} - f(\cos\theta)$$

$$\Psi(t) = \int_0^t |\psi(u)| \, du$$

2. Introduction

Tripathi [5], Prasad & Tripathi [2] and Prasad [3] have studied Legendre series by ordinary Nörlund summability methods as well as uniform Nörlund summability methods. The objective of this paper is to obtain a more general result than those

of Tripathi [5], Prasad & Tripathi [2] and Prasad [3] so that their results come out as particular cases.

Tripathi [5] proved the following theorem on Nörlund summability of Legendre series:

Theorem 1: If

$$\int_0^t |f(x \pm u) - f(x)|\, du = o\left(\frac{p_\tau}{P_\tau}\right) \text{ as } n \to \infty, \tag{5}$$

then Legendre series (1) is summable (N, p_n) to $f(x)$ at an internal point x of the interval $(-1+\varepsilon, 1-\varepsilon)$, $\varepsilon > 0$, where $\{p_n\}$ is a real non negative and monotonic non increasing sequence such that $P_n \to \infty$ as n tends to infinite.

Prasad [3] generalized above theorem for uniform Nörlund summability in the following form:

Theorem 2: If $\alpha(t)$ denotes a function of t, $\alpha(t)$ and $\frac{t}{\alpha(t)}$ ultimately increase steadily with t,

$$\int_0^t |f(x \pm u) - f(x)|\, du = o\left(\frac{t}{\alpha(P_\tau)}\right) \tag{6}$$

uniformly in a set E defined in the interval $(-1, 1)$, in which $f(x)$ is bounded as $t \to +0$ then the series (1) is summable (N, p_n) uniformly in E to the sum $f(x)$, where $\{p_n\}$ is real non negative and monotonic non increasing sequence such that $P_n \to \infty$ as $n \to \infty$, provided that

$$\log n = O(\alpha(P_n)) \text{as } n \to \infty \text{ holds.}$$

Main theorem:

If $$\int_t^\eta \frac{|\psi(u)|\, A_{n,\left(\frac{1}{u}\right)}}{u}\, du = o\left(\frac{A_{n,\left(\frac{1}{t}\right)}}{\gamma\left(\frac{1}{t}\right)}\right) \text{ as } t \to +0, \tag{7}$$

uniformly in set E defined in the interval (-1, 1), $0 < \eta < 1$, then the Legendre series (1) is summable by triangular matrix method (T) uniformly in E to the sum $f(x)$ which is bounded in E, where $\gamma(t)$ is positive monotonic increasing function of t provided $\gamma(n) \to \infty$ as $n \to \infty$ and $T = (a_{n,k})$ be an infinite lower triangular

matrix such that the element $(a_{n,k})$ be positive, monotonic non decreasing with $k \leq n$, $A_{n,u} = \int_0^u (a_{n,n-u})\, du$, for $0 \leq u \leq n$ and $A_{n,n} = 1\ \forall n$.

Lemmas:

Following lemmas are required for the proof of the theorem.

Lemma 1: The condition (7) implies that

$$\Psi(t) = \int_0^t |\psi(u)|\, du = o\left(\frac{t}{\gamma(\frac{1}{t})}\right) \text{ as } t \to +0. \tag{8}$$

Prof. We have

$$\sigma(t) = \int_t^\eta \frac{|\psi(u)| A_{n,(\frac{1}{u})}}{u}\, du = o\left(\frac{A_{n,(\frac{1}{t})}}{\gamma(\frac{1}{t})}\right)$$

therefore

$$\int_0^t |\psi(u)| A_{n,(\frac{1}{u})}\, du = \int_0^t \frac{|\psi(u)| A_{n,(\frac{1}{u})}}{u}\, u\, du$$

$$= [u\,\sigma(u)]_0^t + \int_0^t \sigma(u)\, du$$

$$= o\left(\frac{t\, A_{n,(\frac{1}{t})}}{\gamma(\frac{1}{t})}\right) + o\left(\frac{A_{n,(\frac{1}{t})}}{\gamma(\frac{1}{t})}\right) \int_0^t du$$

$$= o\left(\frac{t\, A_{n,(\frac{1}{t})}}{\gamma(\frac{1}{t})}\right).$$

Since

$$\int_0^t |\psi(u)| A_{n,(\frac{1}{u})}\, du \geq A_{n,(\frac{1}{t})} \int_0^t |\psi(u)|\, du.$$

We get

$$\int_0^t |\psi(u)|\, du = o\left(\frac{t}{\gamma(\frac{1}{t})}\right).$$

Lemma 2: Let $N_n(t) = \sum_{k=0}^{n} a_{n,k} \frac{\sin(k+1)t}{\sin\frac{t}{2}}$, than

$$|N_n(t)| = O(n) \text{ uniformly in } 0 < t < \tfrac{1}{n}. \tag{9}$$

Proof:

$$
\begin{aligned}
|N_n(t)| &= \left|\sum_{k=0}^{n} a_{n,k} \frac{\sin(k+1)t}{\sin\frac{t}{2}}\right| \\
&\le \sum_{k=0}^{n} a_{n,k} \frac{(k+1)|\sin t|}{|\sin\frac{t}{2}|} \\
&\le (n+1) \sum_{k=0}^{n} a_{n,k} \\
&= (n+1) A_{n,n} \\
&= O(n).
\end{aligned}
$$

Lemma 3: If $(a_{n,k})$ be non negative and non decreasing with $k \le n$, then for $0 \le a < b \le \infty$, $0 \le t \le \pi$ and for any n,

$$\left|\sum_{k=a}^{b} a_{n,k}\, e^{ikt}\right| = O(A_{n,\tau})$$

where $\tau =$ Integral part of $\frac{1}{t} = [\frac{1}{t}]$.

Lemma 3 may be proved by the following technique of Lemma 4.1 in Lal [1].

Lemma 4: Let $N_n(t)$ be given as in Lemma 2 and using lemma 3, we have

$$|N_n(t)| = O\left(\frac{A_{n,\tau}}{t}\right) \text{ uniformly in } \tfrac{1}{n} < t < \eta. \tag{11}$$

Proof. Since, $\sin\frac{t}{2} > \frac{t}{\pi}, 0 < t < \eta < \pi$ therefore, $\tau \le n$, we have

$$
\begin{aligned}
|N_n(t)| &= \left|\sum_{k=0}^{n} a_{n,k} \frac{\sin(k+1)t}{\sin\frac{t}{2}}\right| \\
&= O\left(\frac{1}{t}\right)\left|\operatorname{Im}.\sum_{k=0}^{n} a_{n,k}\, e^{i(k+1)t}\right| \\
&= O\left(\frac{1}{t}\right)\left|\sum_{k=0}^{n} a_{n,k}\, e^{ikt}\right| \left|e^{it}\right| \\
&= O\left(\frac{A_{n,\tau}}{t}\right).
\end{aligned}
$$

3. Proof of the Theorem

Following Prasad & Tripathi [2], the k th partial sum of Legendre series (1) is given by

$$S_k(x)-f(x) = \frac{1}{\pi\sqrt{\sin\theta}} \int_0^{\eta} \frac{f\{\cos(\theta-t)\}-f(\cos\theta)}{\sin\frac{1}{2}} \sin(k+1)t\,\sqrt{\sin(\theta-t)}\,dt + o(1)$$

uniformly in E.

where

$$0 \le \eta \le \delta < 1,\ \ x=\cos\theta,\ \ y=\sin\theta,\ \ 0<\theta<\pi,\ \ \theta-\pi=t,\ \ 0<\phi<\pi \text{ etc.}$$

Now,

$$\sum_{k=0}^{n} a_{n,k}\,(S_k(x)-f(x))$$

$$= \frac{1}{\pi\sqrt{\sin\theta}} \int_0^{\eta} f\{\cos(\theta-t)\}-f(\cos\theta)\sin(k+1)t\,\sqrt{\sin(\theta-t)} \sum_{k=0}^{n} a_{n,k}\frac{\sin(k+1)t}{\sin\frac{1}{2}}\,dt + o(1)$$

$$= O\left[\int_0^{\eta} |\psi(t)|\,|N_n(t)|\,dt\right] + o(1) \text{ uniformly in set E.}$$

$$= O\left[\int_0^{\frac{1}{n}} |\psi(t)|\,|N_n(t)|\,dt\right] + O\left[\int_{\frac{1}{n}}^{\eta} |\psi(t)|\,|N_n(t)|\,dt\right] + o(1) \text{ uniformly in set E.}$$

$$= I_1 + I_2 + o(1), \text{ say.} \tag{12}$$

In order to prove the theorem, we have to show that under our assumption

$I_1 = o(1)$ and $I_2 = o(1)$ as $n \to \infty$, uniformly in set E.

Now considering I_1, we have

$$I_1 = O\left[\int_0^{\frac{1}{n}} |\psi(t)|\,|N_n(t)|\,dt\right]$$

$$= O(n)\int_0^{\frac{1}{n}} |\psi(t)|\,dt$$

$$= O(n)\left(\Psi\left(\tfrac{1}{n}\right)\right)$$

$$= o\left(\frac{1}{\gamma(n)}\right)$$

$$= o(1) \text{ as } n \to \infty, \text{ uniformly in set E.} \tag{13}$$

Again, for I_2, we have

$$I_2 = O\left[\int_{\frac{1}{n}}^{\eta} |\psi(t)|\,|N_n(t)|\,dt\right]$$

$$= O\left[\int_{\frac{1}{n}}^{\eta} \frac{|\psi(t)|\,A_{n,\tau}}{t}\,dt\right]$$

$$= o\left(\frac{A_{n,n}}{\gamma(n)}\right)$$

$$= o\left(\frac{1}{\gamma(n)}\right)$$

$$= o\,(1) \quad \text{as } n \to \infty, \text{ uniformly in set E.} \tag{14}$$

Collecting (12) – (14), we get

$$I = o\,(1) \quad \text{as } n \to \infty, \text{ uniformly in set E.}$$

This completes the proof of the theorem.

4. Applications

I. Particular cases are

1. If $a_{n,k} = \frac{p_{n-k}}{P_n}$ and $\gamma(t) = \alpha(P_{[t]})\ \forall t$, result of Prasad [3] becomes the particular case of main theorem.

2. If $a_{n,k} = \frac{q_{n-k}}{Q_n}$ and $\gamma(t) = \frac{\beta(Q_{[t]})}{t\,\lambda(t)\,q_{[t]}}\ \forall t$, result of Prasad & Tripathi [2] becomes the particular case of main theorem.

3. If $a_{n,k} = \frac{p_{n-k}}{P_n}$ and $\gamma(t) = \frac{P_{[t]}}{t\,p_{[t]}}\ \forall t$, result of Tripathi [5] becomes the particular case of main theorem.

II. Following corollary can be derived easily from our theorem.

Let a sequence $\{p_n\}$ be defined as $p(u)$, monotonic decreasing and strictly positive for $u \geq 0$;

$$P(u) = \int_0^u p(x)\,dx\,;\; p_n = p(n)\,.$$

If

$$\int_{\frac{1}{n}}^{\eta} \frac{|\psi(u)|P(\frac{1}{u})}{u}\,du = o\left(\frac{P_n}{\gamma(n)}\right),\;\; 0<\eta<1$$

uniformly in set E defined in the interval (-1, 1), then the Legendre series (1) is summable by Nörlund method uniformly in E to the sum f (x), where $\gamma(t)$ is positive monotonic increasing function of t provided $\gamma(n) \to \infty$ as $n \to \infty$.

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Wiener Tauberian Theorem for Locally Compact Abelian Group

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Abstract:

If $X = \Re^+$, then X is locally compact abelian group under multiplication. Our aim is to prove uniform version of the wiener Tuberian Theorem for $\Re^+$ under new product.

Key words: Winer-Tuberian Theorem, Locally compact abelian group, Translation invariant subspace.

1. Introduction

Wiener in his attempt to characterize rigorously the spectral analysis of a white light signal $\phi \in L^\infty(\Re)$ had to introduce Tuberian arguments, an interesting description of which can be found in Wieners classical work ([4], [5], [6])

The illustration is

Theorem 1.1:

Let $f \in L^1(\Re)$ have a non vanishing Fourier transform $\hat{f}$ and $\phi \in L^\infty(\Re)$. Assume that $\int_\Re |xf(x)|\, dx < \infty$ and $\lim\limits_{x\to\infty} f*\phi(x) = r\int_\Re f(y)dy$ where $*$ denotes the convolution then for each $g \in L^1(\Re)$ $\lim\limits_{x\to\infty} g*\phi(x) = r\int_\Re g(y)\, dy$.

The general Tauberian theorem proved by N. Weiner [6] says that if $g \in L^1(\Re)$is a uniqueness function in the sense that its Fourier transform $\hat{g}$ vanishes nowhere on $\Re$ (and thus the closed translation invariant subspace generated by g is $L^1(\Re)$ and $\phi \in L^\infty(\Re)$ is such that $g * \phi(\underline{x}) \to A\, \hat{g}(o)$ (A is a complex number) as $x \to \infty$ then for every $f \in L^1(\Re)$, $f * \phi(x) \to A\, \hat{f}(o)$ as $x \to \infty$.

We may take the uniqueness function g as

(i) $g(x) = \exp.(-\alpha\, |x|)$ where α is +ve constant then

$$\hat{g}(y) = \left(\frac{2}{\pi}\right)^{1/2} \frac{\alpha}{\alpha^2 + y^2}$$

(ii) For $\alpha > 0$, let g_a be the function on $\Re$ defined by $g_a(x) = \left(1 - \frac{|x|}{a}\right) \xi_{[-a,\, a]}(x)$

Then,

$$\hat{g}_a(y) = (2\pi)^{-1/2}\, \alpha \left[\frac{\sin \frac{1}{2}\alpha y}{\frac{1}{2}\alpha y}\right]^2 \text{ for } y \neq 0, \text{ and } \hat{g}_a(o) = (2\pi)^{-1/2}\, \alpha$$

2. Uniform Version

Let G be a locally compact group with left Haar measure μ and Δ be the modular function on G. For $x \in G$ and $f \in L^p(G)$, $1 \leq p \leq \infty$, let ${}_xf$ and f_x be the left and right translates of f. Let $\phi_f : G \to L^\infty(G)$ be defined by $\phi_f(x) = f_x$, $x \in G$, $f \in L^\infty(G)$. We denote by S_1 and S_∞, the unit balls of $L_1(G)$ and $L^\infty(G)$ respectively. Let Y be the set of all bounded continuous function on G.

Define $U = \{g : G \to \mathbb{C}\ |g$ is measurable function such that $\left(\frac{1}{\Delta}\right)g \in S_1$ and for $a \in Y$, $a * g = 0 \Rightarrow a = 0\}$.

In this paper we are going to prove uniform version of Weiner Tayberian theorem for $X = \Re^+$ under multiplication and new convolution product as used in the technique of following theorem.

Theorem 2.1:

Let G be a separable locally compact group. For $\mathcal{H} \subset L^1(G)$, suppose that there exists $h_0 \in S_1$ such that $|h(t)| \le |h_0(t)|$ for all $h \in \mathcal{H}$ and $t \in G$. Let $\boldsymbol{u} \subset S_\infty$ be such that the family $\{\phi_a : a \in \boldsymbol{u}\}$ is right uniformly equicontinuous. If $g \in U$ and $a * g(x) \to 0$ as $x \to \infty$ uniformly for $a \in \boldsymbol{u}$ then $h * a(x) \to 0$ as $x \to \infty$ uniformly for $a \in \boldsymbol{u}$ and $h \in \mathcal{H}$.

Remark 2.2:

If $X = \Re^+$, then X is locally compact abelian group under multiplication, let μ be the Lebesgue measure on X. For $g \in L^1(X)$ and $\psi \in L^\infty(X)$, the product given by $g \odot \psi(x) = \frac{1}{x}\int_X g(t/x)\, \psi(t)\, d\mu(t)$ is not a convolution product but the uniform version of the Wiener Tauberian theorem can be proved for this product also using the techniques of theorem 2.1. However we need to prove the followings:

(i) $\| g_x \|_1 = \frac{1}{x}\|g\|_1$ for $x \in X$ and $g \in L^1(X)$

(ii) ${}_y(g \odot \psi) = g \odot {}_y\psi$

(iii) ${}_yg \odot \psi = \left(\frac{1}{y}\right) g \odot {}_{1/y}\psi$

(iv) $\|g \odot \psi\|_\infty \le \|g\|_1\, \|\psi\|_\infty$

Proof:

(i) $$\| g_x \|_1 = \int_X |g_x(y)| d\mu(y) = \int_X |g(yx)|\, d\mu(y)$$

$$= \int_X |g(t)| \frac{d\mu(t)}{x}$$

$$= \frac{1}{x}\|g\|_1$$

(ii) $${}_y(g \odot \psi)(x) = g \odot \psi(yx) = \frac{1}{yx}\int_X g\left(\frac{t}{yx}\right) \psi(t)\, d\mu(t)$$

$$= \frac{1}{yx}\int_X g\left(\frac{t'}{x}\right) \psi(t'y)\, y\, d\mu(t')$$

$$= \frac{1}{x}\int_X g\left(\frac{t'}{x}\right)_y \psi(t')\, d\mu(t')$$

$$= (g \odot {}_y\psi)(x)$$

(iii) $({}_y g \odot \psi)(x) = \frac{1}{x}\int_X {}_y g\left(\frac{t}{x}\right) \psi(t)\, d\mu(t)$

$$= \frac{1}{x}\int_X g\left(\frac{yt}{x}\right) \psi(t)\, d\mu(t)$$

$$= \frac{1}{x}\int_X g\left(\frac{t'}{x}\right) \psi\left(\frac{t'}{y}\right) \frac{1}{y}\, d\mu(t')$$

$$= \frac{1}{yx}\int_X g\left(\frac{t'}{x}\right) ({}_{1/y}\psi)(t')\, d\mu(t')$$

$$= \frac{1}{y}\, g \odot {}_{1/y}\psi(x)$$

(iv) $\|g \odot \psi\|_\infty = \sup_{x\in X} |g \odot \psi(x)| = \sup_{x\in X} \left|\frac{1}{x}\int_x g\left(\frac{t}{x}\right) \psi(t)\, d\mu(t)\right|$

$$\leq \sup_{x\in X}\int_X g\left(\frac{t}{x}\right) |\psi(t)|\, \frac{d\mu(t)}{x} \leq \|g\|_1 \|\psi\|_\infty.$$

Let $d\mu(x)$ be the usual Lebesgue measure on X.

Main Result:

Theorem 2.3:

Let X be locally compact abelian group under multiplication. For $\mathcal{H} \subset L^1(X)$, suppose that there exist $h_o \in S_1$ such that $|h(t)| \leq |h_o(t)|$ for all $h \in \mathcal{H}$ and $t \in G$. Let $\mathcal{u} \subset S_\infty$ be such that the family $\{\phi_\psi : \psi \in \mathcal{u}\}$ is uniformly equicontinuous from X to $L^\infty(X)$: If $g \in U$ and $\frac{1}{x}\int_X g\left(\frac{t}{x}\right) \psi(t)\, d\mu(t) \to 0$ as $x \to \infty$ uniformly for ψ in $\mathcal{u}$ then $\frac{1}{x}\int_X h\left(\frac{t}{x}\right) \psi(t)\, d\mu(t) \to 0$ as $x \to \infty$ uniformly for h in $\mathcal{H}$ and ψ in $\mathcal{u}$.

Proof:

Assuming to the contrary. Then there must exists $\delta' > 0$ such that for each $n > 0$ there exist $x_n \in X$ with $x_n > n$, $h_n \in \mathcal{H}$ and $\psi_n \in \boldsymbol{u}$ satisfying $|h_n \odot \psi_n)(x_n) > \delta'$.

Let us define the sequence of functions on X by

$$S_n(x) = (h_n \odot \psi_n)_{x_n}(x) = (h_n \odot \psi_n)(xx_n),\ n \in \mathcal{N}.$$

We shall show that it is bounded and equicontinuous on X.

$$||S_n||_\infty = ||(h_n \odot \psi_n)x_n||_\infty = ||h_n \odot \psi_n||_\infty \leq ||h_n||_1||\psi_n||_\infty \leq ||h_o||_1\psi_n||_\infty \leq 1.$$

Therefore $\{S_n\}_n \in \mathcal{N}$ is bounded.

Let $x, y \in X$, then

$$|S_n(x) - S_n(y)| = |(h_n \odot \psi_n)_{x_n}(x) - (h_n \odot \psi_n)_{x_n}(y)|$$
$$= |(h_n \odot \psi_n)_x(x_n) - (h_n \odot \psi_n)_y(x_n)|$$
$$= |(h_n \odot (\psi_n)_x(x_n) - (h_n \odot \psi_n)_y)\ (x_n)|$$
$$\leq ||h_n \odot (\psi_n)_x - (\psi_n)_y||_\infty$$
$$\leq ||h_n||_1||\psi_n)_x - (\psi_n)_y||_\infty \leq ||h_o||_1||(\psi_n)x - (\psi_n)_y||_\infty.$$

Since $x \to \psi_x$ is uniformly equicontinuous on X to $L^\infty(X)$, $\psi \in \boldsymbol{u}$ so given $\mathcal{E} > 0\ \exists\ \delta > 0$ such that for $|x - y| < \delta$ and $\psi \in \boldsymbol{u}$ we have, $||\psi_x - \psi_y||_\infty < \mathcal{E}$.

Therefore, $S_n(x) - S_n(y)| < \in$ for $|x - y| < \delta$.

Hence, S_n is equicontinuous.

By Ascol's Theorem, there exists a subsequence (S_{n_k}) converging to a continuous function s pointwise.

For each $x \in X$ and $t \in X$.

$$S_{n_k}\left(\frac{t}{x}\right) \to s\left(\frac{t}{x}\right)'\ k \to \infty$$

and therefore $S_{n_k}\left(\frac{t}{x}\right) g(t) = s\left(\frac{t}{x}\right) g(t)$ as $k \to \infty$.

Since $|S_{n_k}\left(\frac{t}{x}\right) g(t)| = \left|S_{n_k}\left(\frac{t}{x}\right)\right| |g(t)| \leq g(t)|$.

Thus by Lebesgue dominated convergence theorem,

$$\int_X S_{n_k}\left(\frac{t}{x}\right) g(t)d\mu(t) \to \int_X s\left(\frac{t}{x}\right) g(t)d\mu(t),\ k \to \infty$$

$$\equiv x(s \odot g)(x)$$

Now,

$$\begin{aligned}
\int_x S_{n_k}\left(\frac{t}{x}\right) g(t)d\mu(t) &= \int_X h_{n_k} \odot \psi_{n_k})\left(\frac{tx_{n_k}}{x}\right)g(t)d\mu t \\
&= \int_X \frac{x}{tx_{n_k}} \int_X h_{n_k}\left(\frac{xu}{tx_{n_k}}\right)\psi_{n_k}(u)g(t)\, d\mu(u)d\mu(t) \\
&= \int_X \frac{x}{tx_{n_k}} \int_X h_{n_k}(u)\psi_{n_k}\left(\frac{tx_{n_k}u}{x}\right)g(t)\left(\frac{tx_{n_k}}{x}\right)d\mu(u)\, d\mu(t) \\
&= \int_X \int_X h_{n_k}(u)\psi_{n_k}\left(\frac{tx_{n_k}u}{x}\right)g(t)d\mu(u)\, d\mu(t), \\
&= \int_x \frac{x}{x_{n_k}u} h_{n_k}(u)\left[\int_x g\left(\frac{tx}{x_{n_k}\, u}\right)\psi_{n_k}(t)d\mu(t)\right] d\mu(u) \\
&= \int_x h_{n_k}(u)(g \odot \psi_{n_k})\left(\frac{x_{n_k}u}{x}\right) d\mu(u) \\
&= \int_x F_{k,x}(u)d\mu(u)
\end{aligned}$$

Where $F_{k,x}(u) = (g \odot \psi_{n_k})\left(\frac{x_{n_k}u}{x}\right) h_{n_k}(u)$.

Since we know, $(g \odot \psi)(t) \to 0$ as $t \to \infty$ uniformly for ψ in $\boldsymbol{u}$ we have for given $\mathcal{E} > 0$ there exists $\Delta : |(g \odot \psi)(t)| < \mathcal{E}$ for every $t \geq \Delta$ and ψ in $\boldsymbol{u}$.

Therefore,

$$\left|(g \odot \psi_{n_k})\left(\frac{x_{n_k}u}{x}\right)\right| \leq \mathcal{E} \text{ for } \left(\frac{x_{n_k}u}{x}\right) \geq \Delta$$

Thus for fixed x and u, $(g \odot \psi_{n_k})\left(\frac{x_{n_k}u}{x}\right) \to 0$ and $k \to \infty$, and therefore $F_{k,x}(u) \to 0$ as $k \to \infty$.

But $\|(g \odot \psi_{n_k})\|_\infty \leq \|g\|_1 \|\psi_{n_k}\|_\infty \leq 1$ and therefore

$$|F_{k,x}(u)| = \left|(g \odot \psi_{n_k})\left(\frac{x_{n_k}u}{x}\right) h_{n_k}(u)\right| \leq \left|(g \odot \psi_{n_k})\left(\frac{x_{n_k}u}{x}\right)\right| |h_{n_k}(u)| \leq |h_{n_k}(u)| \leq |h_o(u)|.$$

Thus again applying Lebesgue dominated convergence theorem $s \odot g(x) = 0$, since $g \in U$, $s = 0$.

But $0 = |s(1)| = \lim_{k \to \infty} s_{n_k}(1)| = \lim_{k \to \infty} | h_{n_k} \odot \psi_{n_k})(x_{n_k})|$.

Thus, we obtain a contradiction. Therefore

$$\frac{1}{x}\int_X h\left(\frac{t}{x}\right)\psi(t)d\mu(t) \to 0 \text{ as } x \to \infty$$

Uniformly for h in $\mathcal{H}$ and ψ in $\boldsymbol{u}$.

This completes the proof.

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On Weakly Symmetric and Weakly Ricci-Symmetric Lorentzian Para-Sasakian Manifolds

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Abstract: The aim of the present paper is to study weakly symmetric and weakly Ricci symmetric an LP-Sasakian manifolds and proved that if weakly symmetric LP-Sasakian manifold satisfies η-parallel Ricci condition then the scalar curvature of the manifolds is equal to rank of ϕ. If weakly symmetric an LP-Sasakian manifolds satisfies Ricci tensor of Coddazi type then the manifolds is R-Harmonic, further we have obtained the necessary condition for weakly symmetric an LP-Sasakian manifolds has Cyclic Ricci is that vanishing the 1-forms $\alpha + \beta + \delta$ for all the vector field X, Y and Z.

Mathematical Subject Classification (2000): 53C50, 53C15, 53C25, 53B30.

Key words: Coddazi type Ricci tensor, R-harmonic manifold, CyclicRicci tensor, η-parallel Ricci tensor, and Quasi Einstein manifold.

1. Introduction

K. Matsumoto 1989, Introduced the notion of Lorentzian Para-Sasakian manifold $(M^n g)$ $(n \geq 2)$ Mihai and Rosca [2] define same notion independently. This type of manifolds is also discussed in [3]. A non-flat Riemannian manifolds if called weakly symmetric if there exist 1-form α, β, δ and σ such that the relation.

(1.1) $(\nabla_X R)(Y, Z, U, V) = \alpha(X)R(Y, Z, U, V) + \beta(Y)R(X, Z, U, V) + \gamma(Z)R(Y, X, U, V) \ldots \delta(U)R(Y, Z, X, V) + \sigma(V)R(Y, Z, U, V)$

Holds for all vector fields X, Y, Z… $\in (M^n, g)$ ([4][5])

Where R is the curvature tensor of the manifold (M^n, g) of type (0, 4)

A weakly symmetric manifolds is said to be proper if $\alpha = \beta = \gamma = \delta = \sigma = 0$ is not the case.

A Riemannian manifolds is called weakly Ricci symmetric if there exist 1-form ρ, μ and ν such that the relation

(1.2) $(\nabla_X S)(Y, Z) = \rho(X)S(Y, Z) + \mu(Y)S(X, Z) + \nu(Z)S(X, Y)$

Hold for vector fields A, Y and Z where S is the Ricci tensor of type (0, 2) of the manifold.

A weakly Ricci symmetric manifolds is said to be proper if $\rho = \mu = \nu = 0$ is not the cases.

Let $[e_I]$, I = 1, 2, 3, 4…, n be an orthonarmal basis of the tangent space at a point of the manifolds.

Putting $Y = V = e_I$ in (1.1) and taking summation over i, $1 \le i \le n$, we get

(1.3) $(\nabla_X S)(Z, U) = \alpha(X)S(Z, U) + \beta(Z)S(X, U) + \delta(U)S(Z, X) + \beta(R(X, Z)U) + \delta(R(X, U)Z)$

L. Tamassy and T. Q. Binh ([4], [5]) introduced the notion of weakly symmetric and weakly Ricci symmetric, Sasakian manifolds. M. Kon [6], U. C. De, T. Q. Binh and A.A Shaikh [7] obtained the necessary condition for the compatibility of several k-Contact structure with weak symmetric and weak Ricci symmetric and provided they do not reduce to conman local symmetric and Pseudo Ricci symmetric respectively. In this paper we study LP–Sasakian manifolds with weakly symmetric and weakly Ricci symmetric tensor, Quasi Einstein, Cyclic Ricci tensor and Ricci tensor of Codaggi type.

2. Preliminaries

Let (M^n, g) be n-dimensional. Differentiable manifolds with a tensor filed f of type (1, 1), a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g of type (0, 2) which satisfying

(2. 1) $\phi^2 = 1 + \eta \otimes \xi$

(2. 2) $\eta\xi = -1$

(2.3) $g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$

(2.4) $g(X, \xi) = \eta(X)$

It is called Lorentzian Para-Sasakian manifolds briefly (LP-Sasakian manifold) and the structure (ϕ, η, ξ, g) is called a Lorentzian Para-contact structure. An LP-Contact manifolds is called LP-Sasakian manifold if it satisfying.

(2.5) $(\nabla_X\phi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi$

Where ∇ denoted the operator of covariant differention with respect to Lorentzian g metric of type (0, 2).

In a LP-Sasakian manifolds the following relations hold ([1][2]).

(2.6) (a) $\phi\xi = 0$. (b) $\eta(\phi X) = 0$ (c) rank $(\phi) = \eta - 1$

(2.7) $(\nabla_X\xi = \phi X$

(2.8) $\eta(R(\xi, X)Y = g(X, Y) - \eta(Y)\eta(X)$

(2.9) $R(\xi, X)\xi = X + \eta(X)\xi$

(2.10) $S(X, \xi) = (n - 1)\eta(X)$

(2.11) $R(X, Y).\xi = \eta(Y)X - \eta(X)Y$

(2.12) $S(\phi X, \phi Y) = S(X, Y) + (n - 1)\eta(X)\eta(Y)$

For any vector field X, Y and Z where $R^*(X, Y)Z$ is the Riemannian curvature tensor.

Lemma 2.1:

In a Lorentzian Para-Sasakian manifolds if ξ be a killing vector fields then we get.

(a) $L_\xi S = 0$

(b) $L_\xi r = 0$, where L denoted Li-derivation

3. Weakly symmetric and weakly Ricci symmetric LP-Sasakian manifolds with $\nabla_\xi S)(Z, U) = 0$

Theorem 3.1:

If weakly symmetric and weakly Ricci symmetric LP-Sasakian manifolds (M^n, g) $n \geq 3$ satisfies the condition then the sum of the 1-form $\alpha + \beta + \delta$ are equal to the sum of the 1-form $\rho + \mu + \nu$ over the killing vector field ξ.

Proof:

Consider the manifolds (M^n, g) is weakly symmetric. Then form (1.3) putting $X = \xi$ and using (2.10), we get

(3.1) $(\nabla_\xi S)(Z, U) = \alpha(\xi)S(Z, U) + (n-1)[\beta(Z)\eta(U) + \delta(U)\eta(Z)] + \beta(P(\xi, Z)U) + \delta(R(\xi, U)Z)$

From (2.1-a), we get

(3.2) $(\nabla_\xi S)(Z, U) = -S(\nabla_Z\xi, U) - S(Z, \nabla_U\xi)$

Using (2.7) in (3.2), we get

(3.3) $(\nabla_\xi S)(Z, U) = -S(\phi Z, U) + S(Z, \phi U)$

Also from (2.11) we have

$S(\phi Z, U) = ng(\phi Z, U) - g(\phi Z, U)$ and

$S(Z, \phi U) = ng(Z, \phi U) - g(Z, \phi U)$

Using this result in (3.3), we get

(3.4) $(\nabla_\xi S)(Z, U) = 0$

From (1.2) We get

(3.5) $(\nabla_\xi S)(Z, U) = \rho(\xi)S(Z, U) + \beta(Z)S(\xi, U) + \nu(U)S(\xi, Z)$

Also from (3.1) we get

(3.6) $(\nabla_\xi S)(Z, U) = \alpha(\xi)S(Z, U) + \beta(Z)S(\xi, U) + \delta(U)S(\xi, Z) + \beta(R(\xi, U)Z)U) + \delta(R(\xi, U)Z)$

From (3.4), (3.5) and (3.6), we get

(3.7) $\alpha(\xi)S(Z, U) + \beta(Z)S(\xi, U) + \delta(U)S(\xi, Z) + \beta(R(\xi, U)Z)U) + \delta(P(\xi, U)Z)$
$= \rho(\xi)S(Z, U) + \mu(Z)S(\xi, U) + \nu(U)S(\xi, Z)$

Putting $Z = U = \xi$ in (3.7), we get

$\alpha(\xi) + \beta(\xi) + \delta(\xi) = \rho(\xi) + \mu(\xi) + \nu(\xi)$

We get the result as required.

4. Weakly Symmetric Quasi-Einstein an LP-Sasakian Manifolds

Definition 4.1 A non-flat LP-Sasakian manifold is called Quasi-Einstein manifold if its Ricci tensor S of type (0, 2) satisfies the condition

(4.1) $S(X, Y) = ag(X, Y) + bA(X)A(Y)$

Where a, b are scalar and $b \neq 0$ and A is non-zero 1-form such that $A(X) = g(X, U)$ for every X and U is unit vector fields.

Theorem 4.10:

There exist no weakly symmetric Quasi Einstein LP-Sasakian manifolds if

$r\alpha + 2\alpha\beta + 2\beta\delta + 2bA(L)U + 2bA(M)U - b\|U\|^2$ is not everywhere zero.

Proof:

We consider a weakly symmetric manifold is a Quasi Einstein LP-Sasakian manifold (M^n, g)

Then from (1.3), we get

(4.2) $(\nabla_X S)(Z, U) = \alpha(X)S(X, Y) + \beta(Y)S(X, Y) + \delta(Z)S(X, Y) + \beta(R(X, Y)Z) + \delta(R(X, Z)Y)$

Where $g(X, L) = \beta(X)$ and $g(X, M) = \delta(X)$

From (4.1) we get

(4.3) $(\nabla_X S)(Y, Z) = b[\nabla_X A)(Y)A(Z) + (\nabla_X A)(Z)A(Y)]$

Then from (4.2) and (4.3), we get

(4.4) $\alpha(X)S(X, Y) + \beta(Y)S(X, Y) + \delta(Z)S(X, Y) + \beta(R(X, Y)Z) + \delta(R(X, Z)Y)$
$= b[(\nabla_X A)(Y)A(Z) + (\nabla_X A)(Z)A(Y)]$

Putting $Y = Z = e_i$ in (4.4) and taking summation $1 \leq i \leq n$, we get

(4.5) $r(X) + 2S(X, L) + 2S(X, M) = 2b \sum_{i=1}^{n} (\nabla_X A)(e_i)A(e_i)$

Using (4.1) in (4.5), we get

(4.6) $r\alpha(X) + 2[ag(X, L) + bA(X)A(L)] + 2[Ag(X, M) + bA(X)A(M)] = 2b \sum_{i=1}^{n} (\nabla_X A)(e_1)A(e_1)$

The right hand side of equation (4.6) can be written as follows

$$\sum_{i=1}^{n} X(A(e_1) - A(\nabla_X e_1)]A(e_1)$$

Where $\{e_1\}$ be the orthonormal bases at $T_u(M)$, $u \in M$ let us translate these parallel from u in any direction then $(\nabla_X e_1) = 0$ using this fact in above relation, we get

(4.7) $2b\sum_{i=1}^{n} X(A(e_1))A(e_1) = 2b\sum_{i=1}^{n} g(\nabla_X U, e_1)g(e_1, U) = 2bg(\nabla_X U, U) = bXg(U, U) = bX\|U\|^2$

Then from (4.6) and (4.7) we get

$$R\alpha + 2\alpha\beta + 2\beta\delta + 2bA(L)U + 2bA(M)U - b\|U\|^2$$

We get the result as required.

5. Weakly Ricci-Symmetric and LP-Sasakian Manifolds with η-Parallel Ricci tensor.

Definition 5.1: The Ricci tensor of weakly Ricci symmetric LP-Sasakian manifolds is called η-parallel if it satisfies.

(5.1) $(\nabla_X S)(\phi X, \phi Y) = 0$, for all vector field X, Y and Z.

Theorem 5.1: If weakly Ricci symmetric LP-Sasakian manifolds has η-parallel Ricci tensor then the scalar curvature r of the manifolds is equal to rank (ϕ).

Proof:

We consider weakly Ricci symmetric LP-Sasakian manifolds with η-parallel Ricci tensor. Then from (1.2) we get

(5.2) $(\nabla_X S)(X, Y) = M(X)S(\phi X, \phi Z) + \mu(\phi Y)S(X, \phi Z) + \nu(\phi Y)S(X, \phi Y)$

Using (2.12) in (5.2) we get

(5.3) $(\nabla_X S)(\phi Y, \phi Z) = M(X)S(Y, Z) + (n-1)\eta(Y)\eta(Z) + M(\phi Y)S(X, \phi Y)$

From (5.1) and (5.3), we get.

(5.4) $M(X)[S(Y, Z) + (n-1)\eta(Y)\eta(Z)] + \mu(\phi Y)S(X, \phi Z) + \nu(\phi Z)S(X, \phi Y)$

Putting $Z = \xi$ in (5.4) and using (2.6-b) and (2.10), we get.

$r = n - 1 = \text{rank}(f)$

We get the result as required.

6. Weakly Symmetric an LP-Sasakian Manifolds with Cyclic Ricci Tensor

Theorem 6.1: The necessary condition for LP-Sasakian manifolds has Cyclic Ricci tensor if vanishing the sum of 1-form $\alpha + \beta + \delta$ for all vector fields S, Y and Z.

Proof:

From (1.3), we get,

(6.1) $(\nabla_X S)(Y, Z) = \alpha(X)S(Y, Z) + \beta(Y)S(X, Z) + \delta(Z)S(Y, X) + \beta(R(X, Y)Z) + \delta(R(X, Y)Z)$

Now taking the cyclic permutation of equation (6.1) in X, Y, Z and adding them, we get.

(6.2) $(\nabla_X S)(Y, Z) + (\nabla_Y S)(X, Z) + (\nabla_Z S)(X, Y) = S(X, Y)[\alpha(X) + \beta(X) + \delta(X)] + S(X, Z)[\alpha(Y) + \beta(Y) + \delta(Y)] + S(Y, X)[\alpha(Z) + \beta(Z) + \delta(Z)] + \beta(R(X, Y)Z) + \beta(R(Y, Z)X) + \beta(R(Z, X)Y) + \delta(R(X, Z)Y) + \delta(R(Y, X)Z) + \delta(R(Z, Y)X)$

Using (2.8) in equation (6.2), we get

(6.3) $(\nabla_X S)(Y, Z) + (\nabla_Y S)(X, Z) + (\nabla_Z S)(X, Y) = S(X, Y)[\alpha(X) + \beta(X) + \delta(X)] + S(X, Z)[\alpha(Y) + \beta(Y) + \delta(Y)] + S(Y, X)[\alpha(Z) + \beta(Z) + \delta(Z)]$

From (6.3) it follows that

(6.4) $(\nabla_X S)(Y, Z) + (\nabla_Y S)(X, Z) + (\nabla_Z S)(X, Y) = 0$

If vanishing the sum of 1-form $\alpha + \beta + \delta$ for all vector fields X, Y and Z.

Corollary 6.1: If weakly symmetric KLP-Sasakian manifold (M^n, g), $n \geq 3$ satisfies the Cyclic Ricci tensor condition then the vanishing the sum of 1-form $\alpha + \beta + \delta$ over the killing vector fieldsξ.

Theorem 6.2: If weakly symmetric LP-Sasakian manifolds has Ricci tensor of Coddazzi type then the manifolds is R-Harmonic. (that is symmetric)

Proof:

We suppose weakly Ricci symmetric LP-Sasakian manifold has Ricci tensor of Coddazzi type that is LP Sasakian manifolds with the condition (6.7)

(6.7) $(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z) + (\nabla_X S)(X, Y)$

From equation (6.7) and equation (6.4), we get $(\nabla_X S)(Y, Z) = 0$

We get the result as required.

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Mathematical Model of Cross-docking Operation in Supply Chain Logistics under Multi-level Just-in-time Production Environment

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ABSTRACT: We give brief literature review and mathematical models of multi-level just-in-time sequencing problem with a view of cross-docking approach for supply chain logistics. Describing the cross-docking operations, we have proposed a new mathematical model for the cross-docking supply chain logistics to minimize the operation time of inbound and outbound trucks as truck sequencing problem. We state a proposition as the synthesis of the production and logistics problems.

Keywords: just-in-time, supply chain, cross-dock, logistics, operation time.

1. Introduction

Integration of overall production processes within and among companies is the most recent fundamental trend in the domain of production management. A synchronized view of these processes includes all traditional areas of supplier-buyer relationships coping with the production of goods and their distribution as well, in particular production and logistics. Just-in-time *(JIT)* production is one of the most commonly used technologies due to its noticeable characteristics in operating with very low work-in-progress inventory

and often with low finished goods inventory. *JIT* production system *(JITPS)* is a pull production system where products are assembled just before they are sold; subassemblies are made just before the products are assembled and the components are fabricated just before the subassemblies are made. The *JITPS* is a management philosophy of continuous improvement including three sequential components: people involvement → total quality control → *JIT* flow; jointly called productivity triad [18]. Sequenced delivery of the materials and products throughout overall supply chain of manufacturing companies is the ultimate realization of *JIT* principles- zero inventories, zero defects, zero waste. In an ideal *JIT* production environment, parts should be delivered to the workstations at the exact time they are needed and in the exact quantity required. Reduced inventory, improved quality, shorter lead times, lower production costs and increased productivity are some of the benefits of *JITPS*. The two types of *JIT* sequencing problems are studied in the literature: single-level [7] and multi-level [9, 16]. The single-level problem is to minimize the variations in the product rates at which different products are produced on the production line, called product rate variation *(PRV)* problem. The multi-level problem is to minimize the variations in demand rates for outputs of supplying processes, called output rate variation *(ORV)* problem.

Supply chain logistics is the task of integrating organizational units along a supply chain and coordinating materials, information and financial flows to fulfill the final customer demands with the aim of improving competitiveness of the manufacturing company as a whole. This is best visualized by the house of supply chain management and logistics [19]. To realize the best quality production and timely distribution for the customer in a rapidly changing technical environment, it is essential to create a cross-docking environment throughout the whole supply chain system that is capable to address the diversified demands. Cross-docking is the movement of products directly from the receiving dock to the shipping dock with minimum dwell time in between. By arranging for immediate cross-docking of incoming products, retailers are able to reduce to a minimum in-transit time for their incoming products. Moreover, cross-docking is a logistics technique used in the retail and trucking industries with operations seeking to move materials from inbound locations to outbound locations as quickly as possible. However, cross-docking operations require good information systems and close synchronization of all inbound and outbound shipments. The mutual coordination among all the independent firms (*viz.*, raw-material suppliers, manufacturers, distributors and retailers) is the crux to attain the flexibility required to enable them in the progressive improvement of logistics processes in response of rapidly changing market conditions.

The rest of the paper is planned as follows: Section 2 gives brief literature survey of *JITPS* as the *ORV* and the *PRV* problems together with the supply chain logistics, which also describes the cross-docking operations in multi-level production and distribution. Section 3 presents the *ORV* problem formulation. Sections 4 and 5 provide the model

description and the model formulation of cross-docking supply chain logistics respectively. Finally, Section 6 concludes the paper.

2. Brief Literature Reviews

2.1 Just-in-Time Production Systems

The single-level *JIT* sequencing problem aims to minimize the maximum deviation and the total deviation between the actual and ideal productions; *i.e.*, it deals only with the production level. The *PRV* problem is studied providing a polynomial time solution in terms of assignment formulation [7, 8]. A recent survey of *PRV* problem with the approach of discrete apportionment can be found in [18, 20]. Another review on sequencing approaches for mixed-model *JITPS* can be found in [4]. An efficient frontier is established for the sum deviation *JIT* sequencing problem via apportionment in [3]. The single-level problem is extended into the multi-level [12, 14], which deals with several levels such as raw materials →components →subassembly →final product →distribution centers →retailers →customers [19]. Most of the discrete manufacturing systems are multi-level in nature, characterized by the condition where several parts are used to produce a particular part at a higher level, terminating at the last level yielding the final product with direct consumer utility. The sequence of products on the final assembly line impacts greatly on inventory levels of parts used directly for assembly and other parts in the system. Recently, the problem of determining an appropriate product sequence has attracted a lot of attentions. The *ORV* problem is proved *NP*-hard, even in special cases [7, 9]. However, the dynamic programming solution is proposed in [9, 13]. The *ORV* problem with pegging assumption is effectively solved in [16], which reduced the problem into the weighted *PRV* problem. Modifying the solution techniques used for the unweighted single-level problem, the pegged multi-level problem may be solved to optimum in time which is polynomial in the total product demand and the weighted factors.

The *JIT* production logistics forms a specific part of the supply chain, which deals with the planning and control of materials and information flows throughout the production and distribution supply chains of manufacturing companies with the mission to get the right materials to the right place at the right time in perfect quality at the lowest possible costs [5, 18]. The *JIT* logistics is performed to optimize some sort of given performance measures, such as to minimize total operating costs, transportation costs, operating time and to satisfy a given set of constraints. It is the true realization of *JIT* production and delivery systems: a management philosophy which uses a set of integrated activities to achieve manufacturing flexibility with minimum shortages and inventories. As its pull nature, the *JITPS* starts a supplying process only when a consuming process demands the supplying process. The role of information logistics in supply chain production process is briefly studied in [17]. The extended scenario of the supply chain network in multi-level *JITPS* is shown in terms of inbound logistics and outbound logistics [19].

2.2 Cross-Docking Operations

As a dynamic *JIT* distribution centre, cross-docking has been widely applied in both manufacturing systems and logistics, since cross-docking operation *(CDO)* favors the timely distribution of freight and a better synchronization with the demand to provide the improved customer service. For instance, perishable products reach the marketplace faster preserving quality and freshness. *CDO* is considered as the best method to reduce inventory and to improve responsiveness of various customer demands. It is a process where products are received in a distribution center occasionally merged with other products going to the same destination, then shipped at the earliest opportunity. A cross-docking supply chain logistics *(CDSCL)* system is a material handling and distribution concept in which the products move directly from the receiving dock to the shipping dock without being stored in a warehouse or distribution center [19] (also see Fig. 2.1 below). Buffa [2] showed that logistics cost could be reduced by integrating the inbound and outbound vehicles in the distribution system. A framework for understanding and designing cross-docking systems is provided in [21] discussing the techniques that can improve the overall efficiency of the logistics and distribution operations. Only few research papers deal with the short-term scheduling problems arising from the daily operations of cross-docking terminals. Material handling inside the cross-dock terminal for a given truck schedule is considered in [11]. The advantages of *CDO* include minimal inventory, low handling costs, low space requirement, centralized processing and low transportation costs.

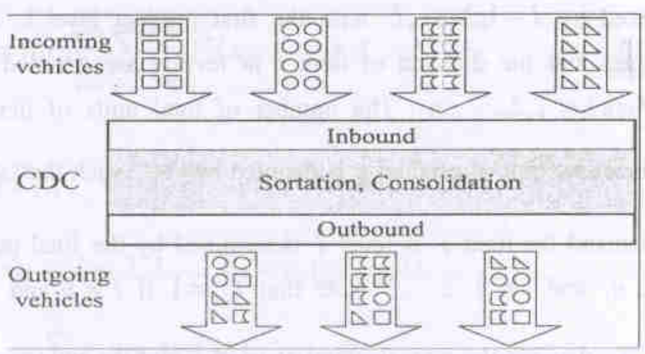

Fig. 2.1 Operational Scenario at Cross-Docking Center *(CDC)*

Once a set of inbound and outbound trucks is docked, jobs consisting of products to be handled have to be assigned to resources, i.e., workers and means of conveyance like fork-lifts, in such a way that efficient unloading, sorting and loading operations render possible. It is modeled as a machine scheduling problem and proposed a meta-heuristic suited for its solution [10]. A specific truck scheduling problem is covered at a parcel hub

and solved by a simulation-based optimization approach [11]. A special kind of cross-dock terminal with a conveyor belt system is treated in [22] where the transportation of goods within the dock is modeled as a detailed scheduling problem providing a priority rule based start heuristic. Boysen et al. [1] treated a stylized one inbound door serves one outbound door setting in order to generate fundamental insights to the underlying real-world problem structure. Exact handling times for inbound trailers depend on the exact packing of goods and the sequence in which they can be obtained, whereas those for outbound trailers have to account for load stability and the sequence in which customers are served. Furthermore, the determination of transportation times between doors results to a complex optimization problem in itself. Thus, handling times used in a detailed truck scheduling model are merely estimated average times and often bound to heavy inaccuracies. Under such prerequisites, detailed models may lead to more misleading or even infeasible plans when compared to aggregate models. So individual handling times for products are merged to service slots to which inbound and outbound trucks are assigned. A slot comprises the time required for completely unloading an inbound truck and completely loading an outbound truck respectively. Handling times in between dock doors are considered by a delay which covers the time span until incoming products are available at an outbound door. By a simultaneous scheduling of inbound and outbound trucks, incoming flows of products are harmonized with outbound flows, so that a *JIT* supply of products, and thus, a reduced turnover time is enabled.

3. The *ORV* Problem Formulation

Assume that the mixed-model multi-level *JITPS* consist of L levels of manufacturing operations, indexed by $l = 1, 2, ,..., L$ with the first product level 1. The number of different part types and the demand of item i in level l are denoted by n_l and d_{il} respectively, where $i = 1, 2, \ldots, n_l$. The number of total units of item i at level l required to produce one unit of product p is denoted by t_{ilp} such that $d_{il} = \sum_{p=1}^{n_1} t_{ilp} d_{p1}$ is the dependent demand for item i at level l determined by the final product demands d_{p1}, $p = 1, 2, ..., n_l$ and $l = 1, 2, \ldots, L$. Note that $t_{ilp} = 1$ if $i = p$ and *0* if otherwise. Finally, $D_l = \sum_{i=1}^{n_l} d_{il}$ denotes the total demand at level l, and the ratio $r_{il} = \frac{d_{il}}{D_l}$ gives the demand rate for item i of level l such that $\sum_{i=1}^{n_l} r_{il} = 1$ at each level $l = 1, 2, \ldots, L$. It is noteworthy that the model of the *ORV* problem is assumed to be non-preemptive- once commenced production of a product at level 1 must be completed prior to switch into another. This creates the concept of various stages or cycles in the production system.

The production schedule at level 1 consists of D_1 stages in total and at each stage a single unit of an end-product can be processed. An item is said to be in stage $k,(k = 1, 2, \ldots, D_1)$, if k units of product have been produced at level 1 and there will be k complete units of various products p at level 1 during the first k time units.

Let x_{ilk} be the necessary quantity of item i produced at level l during the time units 1 through k and $y_{lk} = \sum_{i=1}^{n_l} x_{ilk}$ be the cumulative quantity of item i produced at level l during the same time units such that $y_{1k} = \sum_{i=1}^{n_l} x_{i1k} = k$. Due to the pull nature of the *JITPS*, the particular combination of the highest level products produced during the k time units (the x_{p1k} values) determines the necessary cumulative production at every other level. Thus, the required cumulative production for item i at level l with $l \geq 2$ through k time units is given by $x_{ilk} = \sum_{p=1}^{n_l} t_{ilp} x_{p1k}$. For a unimodal convex penalty function $F_i, i = 1,2,\ldots,n_l$ with minimum 0 at 0, the maximum and the sum deviation multi-level *JIT* sequencing *(i.e.,ORV)* problems in mixed-model systems are formulated to minimize the objectives Z_{max} and Z_{sum} as the followings [6, 12]:

$$Z_{\max} = \min \max_{i,l,k} F_i(x_{ilk} - y_{lk} r_{il}) \tag{1}$$

$$Z_{sum} = \min \sum_{k=1}^{D_1} \sum_{l=1}^{L} \sum_{i=1}^{n_l} F_i(x_{ilk} - y_{lk} r_{il}) \tag{2}$$

subject to

$$x_{ilk} = \sum_{p=1}^{n_l} t_{ilp}\, x_{p1k}, \quad i = 1,2,\ldots,n_l;\ l = 1,2,\ldots, L;\ k = 1,2,\ldots D_1 \tag{3}$$

$$y_{lk} = \sum_{i=1}^{n_l} x_{ilk}, \quad l = 2,3,\ldots, L;\ k = 1,2,\ldots,D_1 \tag{4}$$

$$y_{1k} = \sum_{p=1}^{n_1} x_{p1k} = k, \quad k = 1, 2, \ldots, D_1 \tag{5}$$

$$x_{p1k} \geq x_{p1,k-1}, \quad p = 1,2,\ldots,n_1;\ k = 1,2,\ldots,D_1 \tag{6}$$

$$x_{p1D_1} = d_{p1};\ x_{p10} = 0, \quad p = 1, 2, \ldots, n_1 \tag{7}$$

$$x_{ilk} \geq 0 \in Z^+, i = 1,2,\ldots,n_l; l = 1,2,\ldots L;\ k = 1,2,\ldots,\ D_1 \tag{8}$$

The objective functions (1) and (2) minimize the maximum and sum deviation measures respectively. The constraint (3) ensures that the necessary cumulative production of part i

at level l by the end of time unit k is determined explicitly by the quantity of products produced at level 1. Constraints (4) and (5) show the total cumulative production of level l and level 1 respectively during the time slots 1 through k. Constraint (6) ensures that the total production of every product over k time units is a non-decreasing function of k. Constraint (7) guarantees that the demands for each product are met exactly. Constraints (5), (6), (8) jointly ensure that exactly one unit of a product is scheduled during one time unit in the product level. The particular cases of the *ORV* objectives (1) and (2) are studied in the literature as the absolute and the squared deviation objectives in both maximum and sum deviation cases as follows:

$$Z^{a}_{\max} = \min \max_{i,l,k} |x_{ilk} - y_{lk} r_{il}| \tag{9}$$

$$Z^{s}_{\max} = \min \max_{i,l,k} (x_{ilk} - y_{lk} r_{il})^2 \tag{10}$$

$$Z^{a}_{\min} = \min \sum_{k=1}^{D_1} \sum_{l=1}^{L} \sum_{i=1}^{n_l} |x_{ilk} - y_{lk} r_{il}| \tag{11}$$

$$Z^{s}_{\min} = \min \sum_{k=1}^{D_1} \sum_{l=1}^{L} \sum_{i=1}^{n_l} (x_{ilk} - y_{lk} r_{il})^2 \tag{12}$$

Being *NP*-hard in nature, there is no efficient algorithm for the above general *ORV* objectives with the assigned constraints therewith. The only published algorithm for scheduling mixed-model multi-level *JIT* production systems is the goal-chasing method *(GCM)*, developed and used by Toyota to schedule automobile final assembly lines [15]. This algorithm considers only two levels- final assembly and sub-assemblies. We refer [12, 15], for the extensive study of *GCM* and extended *GCM*. We consider the cross-docks in between each level so that the operation time is minimized and inventory level is reduced.

4. The *CDSCL* Model Description

The *CDSCL* problem is considered as truck sequencing problem *(TRSP)* over here. The notational convention is described as follows: Let I and O be the sets of inbound and outbound trucks at the single receiving door and the single shipping door respectively of the cross-docking terminal. Each inbound truck is loaded with units of different products $p \in P$. Suppose $a_{\alpha p}$ be the number of units of product type p arriving in an inbound truck α and b_{β_z} be the number of product type p to be loaded onto outbound truck β. All product units are completely unloaded within a service slot t to which the respective inbound truck is assigned, so that all handling operations (e.g., docking,

unloading and undocking) required to process the truck are executed within this time span, e.g., an hour or two. Moreover, all inbound trucks are assumed to be available for processing at the beginning of the planning horizon, so that a static problem with identical arrival dates of inbound trucks is considered. The assumption of equally long service times can be seen as a reasonable approximation of reality, whenever vehicle capacities and the number of products per vehicle do not strongly differ. As trailers are typically of a standardized size and cross-docking aims at moving only full truck loads, this premise is fulfilled whenever all processed products are of comparable size (e.g. mail distribution centers) or all truck loads resemble a representative average truck load (e.g., rotational deliveries of special promotional offers to all stores of a retail chain).

Once unloaded, the delivered products have to undergo several subsequent operations before they are available for being loaded onto the outbound trucks at the shipping door. These operations include recording of any product unit in the information system, examining the product correctness and quality, collecting, sorting, rearranging and packing to recombine products from different inbound trucks to form the load of a certain outbound truck. Finally, the products have to be transported to the shipping door, where they wait in an intermediate buffer of sufficient size until they are needed. This variety of tasks from recording to transporting is assumed to last a fixed movement time m irrespective of the truck load actually processed. Then, all products arriving in a slot t are available for loading at the shipping dock not before slot $t + m$ if the movement process can be started for any unloaded unit immediately, e.g. when applying a conveyor belt system. If the movement starts not before the complete inbound truck has been unloaded completely (e.g., a worker stacks all units behind the receiving door before moving them), the units are first available at slot $t + m + 1$. However, the displacement m or $m + 1$, respectively, can be ignored (set to zero) when modeling and solving the problem, because after having determined a solution, an appropriate re-indexing of slots outbound trucks are assigned to allow the exact determination of the outbound schedule. Similarly to constant unloading times, it is assumed that the movement time m is independent of the inbound truck and the loaded products, because handling full truck loads, which may always consist of almost the same number of product units, should take very similar times. This assumption is realistic especially within an aggregated medium-term scheduling approach.

The set O of outbound trucks is to be loaded at the shipping door for each $\beta \in O$ with a predetermined number of units $b_{\beta p}$ of the different products $p \in P$. Also, it is assumed that all handling operations per truck are completed within a single slot. An outbound truck can be assigned to a slot t not before enough stock has accumulated in the intermediate buffer to serve all demanded product units of the truck. As only temporary stock is allowed (or desired) within a cross dock, it is assumed that temporary stock is empty before the first inbound truck arrives and is emptied out again after the last

outbound truck was served. Thus, within our model the following equality holds: $\sum_{\alpha\in I} a_{\alpha p} = \sum_{\beta\in O} b_{\beta p}\ \forall p \in P$. The simplifying assumptions applied to our base model are described in [1].

5. The *CDSCL* Model Formulation

We set the following notations to formulate the *CDSCL* problem:

I = set of inbound trucks (indexed by α); O = set of outbound trucks (indexed by β)

P = set of products (indexed by p); T = total number of time slots (indexed by t)

$a_{\alpha p}$ = quantity of p arriving in truck α; $b_{\beta p}$ = quantity of p to be loaded in truck β

$x_{\alpha t}$ = 1, if inbound truck α is assigned to slot t; 0, if otherwise

$y_{\beta t}$ = 1, if outbound truck β is assigned to slot t; 0, if otherwise

$tx_{\alpha t}$ = operation time of α trucks; $ty_{\beta t}$ = operation time of β trucks

As a direct result of the simplifying assumptions in [1], the inbound and outbound schedule can be readily derived by the sequence of inbound and outbound trucks, so that the problem reduces to *TRSP*. The objective is to sequence the trucks in such a way that the operation time is minimized which comprises the time span starting from the first slot to which an inbound truck is assigned and lasts until the final slot in which an outbound truck is processed. In our model, the time slots *(t)* of operations of inbound and outbound trucks are considered equal. With the above notations, we formulate the *TRSP* problem to minimize the operation time as follows:

$$\text{Minimize} \qquad M = Max\left|t\,x_{\alpha t} - t\,y_{\beta t}\right| \quad \forall \alpha \in I \ \ and \ \ \beta \in O \tag{13}$$

subject to

$$\sum_{t=1}^{T} x_{\alpha t} = 1, \quad \forall \alpha \in I \tag{14}$$

$$\sum_{\alpha\in I} x_{\alpha t} \le 1, \quad \forall t = 1, 2, ..., T \tag{15}$$

$$\sum_{t=1}^{T} y_{\beta t} = 1, \quad \forall \beta \in O \tag{16}$$

$$\sum_{\beta\in O} y_{\beta t} \le 1, \quad \forall t = 1, 2, ..., T \tag{17}$$

$$\sum_{\tau=1}^{t} \sum_{\alpha\in I} x_{\alpha\tau}\cdot a_{\alpha p} \ge \sum_{\tau=1}^{t} \sum_{\beta\in O} y_{\beta\tau}\cdot b_{\beta p}, \quad \forall t = 1, ..., T;\ p \in P \tag{18}$$

$$x_{\alpha t} \in \{0, 1\} \quad \forall \alpha \in I;\ t = 1, ..., T \tag{19}$$

$$y_{\beta t} \in \{0, 1\}, \quad \beta \in O;\ t = 1, ..., T \tag{20}$$

The objective (13) minimizes the maximum of the absolute differences of operation times of outbound trucks β and inbound trucks α. The constraint (14) ensures that each inbound truck is processed in exactly one slot, whereas (15) enforces that in each slot at most one inbound truck can be assigned. In analogy, these two conditions hold true for outbound trucks by constraints (16) and (17). Constraints (18) ensure that an outbound truck can only be assigned to a slot t whenever all required products are available (delivered by preceding inbound trucks yet not consumed by preceding outbound trucks) to satisfy the demand for product units of each type p. So the available stock accumulated by all inbound trucks assigned to slots $\tau = 1, 2, \ldots, t$ has to exceed the total demand for product units of outbound trucks scheduled up to the actual slot t (recall that it will actually be slot $t + m$ or even $t + m + 1$ when realizing the schedule). The constraints (19) and (20) represent the binary variables for inbound and outbound trucks respectively.

On providing the *left shifting property*, Boysen et al. [1] established the following proposition:

Proposition 1: The *TRSP* is *NP*-hard in the strong sense.

The overall *TRSP* problem is decomposed into two sub-problems [1], namely inbound and outbound *TRSPs*, written as *IBD-TRSP* and *OBD-TRSP* respectively. It is divided into sub-problems by fixing a particular inbound (outbound) sequence and then finding the optimal outbound (inbound) sequence respectively. A comparison of *IBD-TRSP* and *OBD-TRSP* reveals that their mathematical structures are identical. As a consequence, any algorithm for *OBD-TRSP* can be used to solve *IBD-TRSP* and vice versa. In fact, *IBD-TRSP* can be seen as a reverted *OBD-TRSP*, in the sense that the solution of an instance of *IBD-TRSP* with an algorithm designed for *OBD-TRSP* requires the following steps:

1. Revert the given outbound sequence and set it as the modified inbound sequence. Consider the set of inbound trucks I to be scheduled as the modified set of outbound trucks O.
2. Solve *OBD-TRSP* with the modified input data.
3. The reverted optimal outbound sequence constitutes the optimal inbound sequence for the original *IBD-TRSP* instance.

An exact dynamic programming approach is introduced and a heuristic starting procedure is proposed to solve the identified sub-problems [1]. The algorithmic descriptions are limited to *OBD-TRSP*, as they are directly transferable to *IBD-TRSP*. We conjecture that the multi-level *JIT* production problem and the cross-docking supply chain logistics problem are counterparts of each others. In this regard, we propose the following proposition to integrate *ORV* and *CDSCL* problems:

Proposition 2: The solution of the *ORV* problem is balanced if and only if the solution of the *CDSCL* problem is balanced.

6. Conclusion

We have proposed a mathematical model for minimization of operation time in terms of absolute differences of operation times of inbound and outbound trucks. Our model is slightly different comparing with the model of Boysen et al. [1] that considers the operation time of outbound trucks only. The *CDSCL* problem is the multi-level distribution problem. In this paper, we have considered the *ORV* model and the logistics model only. The proper coordination between multi-level production lines and distribution lines plays an important role to balance the overall supply chain management of the manufacturing companies. The simultaneous study of multi-level *JIT* production and *CDSCL* problem will be our due course.

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Neighborhood Classes of H-Comparability Graphs

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Abstract: Every *H*-comparability graph contains a Hamming graph as spanning subgraph. Any acyclic orientation of an *H*-comparability graph contains an acyclic orientation of the spanning Hamming graph, which has an interpretation in scheduling theory. We investigate the neighborhood of *H*-comparability graphs in the family of graph classes. We find a class of H-graphs, which can also be transitively oriented. In the most cases the neighborhood classes are identified as proper superclasses or proper subclasses of the class of H-comparability graphs.

Keywords: Hamming graphs, comparability graphs, graph inclusions.

1. Introduction

An undirected graph G = (V, E) is called a *comparability graph* if there exists a transitive orientation of its edges, i.e., if the arcs (uv) and (vw) are contained in the orientation then the arc (uw) must be contained in it. The transitive orientation is denoted by $D = (V, A)$. Comparability graphs are perfect graphs. A graph $G = (V, E)$ is called *perfect* if for each of its induced subgraphs G^*, the chromatic number of G^* is equal to the clique number of G^*. The *chromatic number* $\chi(G)$ of a graph $G = (V, E)$ is the smallest number of colors that can be assigned to the vertices in V in such a way that any pair of adjacent vertices receive two distinct colors. The *clique number* $\omega(G)$ of G is defined as the largest number of pairwise adjacent vertices in V.

By assignment of a positive integral weight $w(v)$ to each vertex v of the graph G, this can be extended: for each induced subgraph G^* of a vertex weighted comparability graph G, the weighted chromatic number $\chi_{weight}(G^*)$ is equal to the weighted clique number $\chi_{weight}(G^*)$. The weighted chromatic number $\chi_{weight}(G)$ is the smallest number of colors for a weight coloring of the given graph, where to each vertex v a set of colors $F(v)$ of cardinality $w(v)$ is assigned with $F(v) \cap F(w) = \Phi$ for

all adjacent vertices v and w. The weighted clique number ω_{weight} is equal to the weight of a maximum weighted clique in the considered graph.

Vertex weighted comparability graphs are *super-perfect* graphs, i.e. the interval chromatic number $\chi_{int}(G)$ is equal to the weighted clique number $\omega_{weight}(G)$. An Interval Coloring of G is an assignment of each vertex v to an open interval I_v of length $w(v)$ such that the intervals corresponding to adjacent vertices are disjoint. The number of colors needed for an interval coloring is the length of $|\cup_v I_v|$. The *Interval Chromatic Number* $\chi_{int}(G)$ is the minimum number of colors needed for an interval coloring of G. The calculation of all introduced chromatic numbers and clique numbers belongs to *NP*-hard. However, there exist polynomial algorithms for the comparability graphs.

In this paper, properties of vertex weighted Hamming graphs and the *H*-comparability graphs with a Hamming graph as a spanning subgraph are of interest. They have an interpretation in scheduling theory. Especially, we investigate the embedding of H-comparability graphs in the family of graph classes.

The paper is organized as follows. Section 2 describes some basic properties of comparability graphs. Section 3 is devoted to the study of graph class inclusions with respect to the H-comparability graphs. The paper closes with an overview on the obtained results and concluding remarks.

2. Comparability Graphs

A basic introduction into graph classes, especially, into comparability graphs is given by in [5]. If there exists a transitive orientation of a given graph G, then the *reverse* orientation is transitive, too. We call a comparability graph *unique orientable* if only both these orientations of G are transitive. Note that each induced subgraph of a comparability graph is also transitive orientable. There are two distinct approaches for the orientation of a comparability graph. Both can be used to decide if a given graph is transitive orientable.

The first approach is based upon color classes or implication classes. In this context we understand each edge {ab} of G as two arcs (ab) and (ba). Then the transitive closures Γ^* and Γ^*_d of the following relations Γ and Γ_d are equivalence relations on the set of undirected or directed edges of the comparability graph $G = (V, E)$, respectively.

$$\forall \{ab\}, \{bc\} \in E\colon \{ab\}\ \Gamma\ \{bc\} \leftrightarrow \begin{Bmatrix} \{ab\} = \{bc\} \text{ or} \\ \{ab\} \neq \{bc\} \wedge \{ac\} \notin E \end{Bmatrix}$$

We say, the edges $\{ab\}$, $\{bc\}$ form a *V*-shape, if $\{ab\}\Gamma\{bc\}$ and $\{ab\} \neq \{bc\}$ is valid. The orientation of one edge forces the orientation of the second one. The generated equivalence classes are denoted as color classes.

If we interpret each edge $\{ab\}$ as the set of the two arcs (ab) and (ba), the transitive closure of the following relation gives the equivalence classes, called implication classes:

$$\forall (ab), (cd) \in E\colon (ab)\ \Gamma_d\ (cd) \ \leftrightarrow \ \begin{Bmatrix} (\mathrm{ab}) = (\mathrm{cd}) \text{ or} \\ \mathrm{a} = \mathrm{c} \wedge \{\mathrm{bd}\} \notin \mathrm{E} \text{ or} \\ \mathrm{b} = \mathrm{d} \wedge \{\mathrm{ac}\} \notin \mathrm{E} \end{Bmatrix}$$

If IC is an implication class of a graph generated by the arc (ab), then the implication class IC^{-1} is generated by the reverse arc (ba). The set of edges is split into the implication classes $IC_1, \ldots, IC_r, IC_1^{-1}, \ldots, IC_r^{-1}$ and any transitive orientation has to contain exactly one of each pair ICk, ICk^{-1}, $k = 1, \ldots, r$. There exists an $O(n^2)$ time algorithm for the orientation of a comparability graph by means of these classes [10]. Clearly, if $\{ab\}$ and $\{bc\}$ form a V-shape, (ab) and (cb) belong to the same implication class.

The second approach is a dual one and uses the modular decomposition of a comparability graph. The modular decomposition generates an acyclic orientation of G, which is also transitive, if the graph G is a comparability graph, see [2, 7] for details.

An orientation $D = (V, A)$ of a graph $G = (V, E)$ is called *pseudo-transitive* if (ab) and (bc) in D implies that either (ac) or (ca) is contained in it, i.e. it is allowed that D contains triangles, which are not transitively oriented.

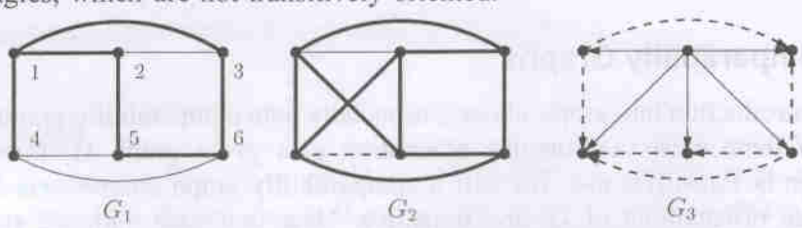


Figure 1: The non-comparability graph G_1 and the comparability graphs G_2 and G_3

In Figure 1, G_1 has an odd closed walk (41256314), where no pair of vertices at distance 2 apart on the walk are adjacent. Therefore, G_1 is not a comparability graph. The graphs G_2 and G_3 are comparability graphs. The color classes are demonstrated for graph G_2 and for graph G_3 a transitive orientation together with two implication classes are given.

The transitive closure of an acyclic oriented graph G is the smallest transitive oriented graph which contains G. The transitive reduction of a graph G is the smallest subgraph of G whose transitive closure is equal to the transitive closure of G. The symmetric closure of a directed graph G is generated from G by adding all arcs (ab) where $(ba) \in E(G)$, i.e. this graph is undirected. From any given undirected graph G a comparability graph can be easily constructed: calculate an acyclic orientation of G

and determine the transitive closure of this orientation. Obviously, the symmetric closure of the obtained graph is a comparability graph.

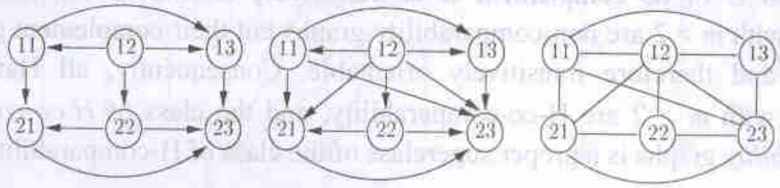


Figure 2: From an acyclic orientation of $K_2 \times K_3$ to the H-comparability graph

We define the rank matrix $RK = [rk_{ij}]$ of the corresponding sequence graph (acyclic orientation of the Hamming graph $K_n \times K_m$), where the entry $rk_{ij} = l$ means that a path from a source to operation (ij) with maximal number of operations has l operations.

We denote a simple graph as H-graph, if it contains a $K_n \times K_m$ as spanning subgraph. An H -graph HG is usually drawn into the plane as n row-cliques of size m connected to m column-cliques of size n together with diagonal edges. Therefore, $E(HG) = E(K_n \times K_m) \cup E_D$ holds, where E_D is the set of all diagonal edges. An *H-comparability graph* is an *H*-graph which can be transitively oriented.

The comparability graphs are widely applied to solve a decision problem in open shop problem. A number of properties have been invented and a number of algorithms have been proposed from past, see [1, 3] for recent results.

3. Graph Class Inclusions

Inclusions of graph classes are investigated in the following. We recommend to wwwteo.informatik.uni-rostock.de/isgci for Information System on Graph Class Inclusions (ISGCI) for an overview. Here, the H-graphs and H-comparability graphs are mainly focused. The family of graph classes directly connected with comparability graphs is specified to the family of H-X-graph classes, where X denotes a special property. We classify each considered H-X-graph class as a proper superclass or a proper subclass of the class of H-comparability graphs or we show that the both classes are identical [6].

3.1 Superclasses of the *H*-Comparability Graphs

To show that an H-X-graph class is a proper superclass of the class of H-comparabilty graphs it is sufficient to construct a graph which is an H-X-graph but not a comparability.

a. A graph G belongs to the class of *co-comparability* $\cup$ *comparability* graphs if one of the graphs G or its complement $\overline{G}$ is transitively orientable. All Hamming graphs $K_2 \times K_m$ with $m > 2$ are non-comparability graphs but their complement graphs $K_{2,m}$ are bipartite and therefore transitively orientable. Consequently, all Hamming graphs $K_2 \times K_m$ with $m > 2$ are H-co-comparability, and the class of *H-co-comparability* $\cup$ *comparability* graphs is a proper superclass of the class of H-comparability graphs.

b. A graph G is called $\overline{BW_3}$*-free* if it does not contain any graph $\overline{BW_3}$ as its induced subgraph, Figure 3. The Hamming graph $K_2 \times K_3$ must be $\overline{BW_3}$-free as the graph $\overline{BW_3}$ contains more than the number of vertices of the former. Since the graph $K_2 \times K_3$ is not a comparability graph, the class of H-$\overline{BW_3}$-free graphs is a proper superclass of the class of H-comparability graphs.

c. A vertex weighted graph G is called super-perfect if for every non-negative integral weights the interval chromatic number is equal to the weight of a maximal weighted clique. All Hamming graphs $K_n \times K_2$ and $K_2 \times K_m$ are super-perfect. Because they cannot be transitively oriented except for $K_2 \times K_2$, the class of H-super-perfect graphs is a proper superclass of the class of H-comparability graphs.

d. A graph G is called alternately orientable if there exists an orientation of G in which no chordal free cycle of length $k \geq 4$ contains a directed path P_3. By definition, a deletion of one or more edges in any minimal H-comparability graphs yields a non-comparability graph.

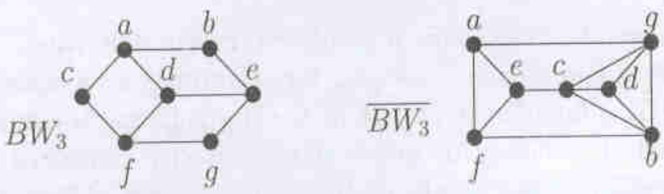


Figure 3: The graphs BW_3 and $\overline{BW_3}$

In Figure 4, the graph HG is a minimal H-comparability graph because the Hamming graph can be uniquely oriented by one sequence implication class, described by the rank matrix RK. Therefore, the graph $HG - \{b, g\}$ is not transitively orientable, but there exists an alternating orientation (see Figure 4). Thus, the class of H-alternately orientable graphs is a proper superclass of the class of H-comparability graphs.

$$RK = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 4 \\ 5 & 4 & 2 \end{bmatrix}$$

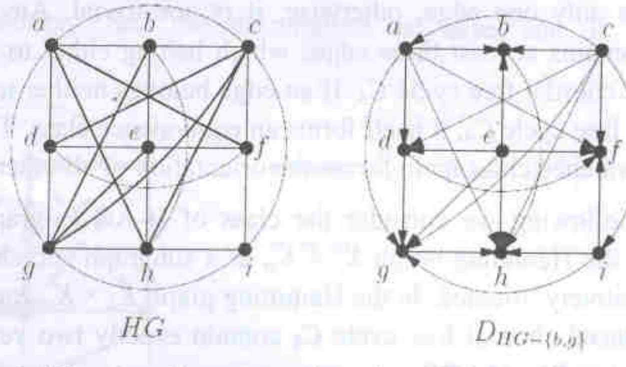


Figure 4: The graph HG and an alternating orientation of the $HG - \{b, g\}$.

e. A graph G is a P_4-comparability graph if there exists an orientation of G in which no induced path P_4 contains a directed path P_3. We did not find a direct relationship between H-P_4-comparability graphs and H-comparability graphs. Therefore, we define a new class of graphs which are both H-alternately orientable graphs and H-P_4-comparability graphs and show that this class is a superclass of H -comparability graphs.

f. A graph G is called AO-P_4-graph if there exists an acyclic orientation D with the following properties.

1. There is no induced chordal free cycle of length $k \geq 4$ which contains a directed path P_3.
2. There is no induced path P4 which contains a directed path P_3.

The graph is denoted by H-AO-P_4-graph, if it contains a Hamming graph as spanning subgraph. Note that each induced subgraph of H-AO-P_4-graphs is an AO-P_4-graph, too. If a graph G contains an induced chordal free cycle of odd length $k \geq 5$, then it cannot be an H-AO-P_4-graph since each orientation of such a cycle contains a directed path P_3. On the other side, each chordal free cycle of even length $k \geq 6$ is made up of induced adjacent paths P_4. Therefore, it is sufficient to consider only induced chordal free cycles of length 4 and paths P_4.

We define the relation $\mathcal{R}$ on the set of edges E of a given graph as follows.

$$\forall\, e_1, e_2 \in E : e_1 \mathcal{R} e_2 \Leftrightarrow \left\{ \begin{array}{ll} \text{either} & e_1 = e_2, \\ \text{or} & e_1, e_2 \in P_4, \\ \text{or} & e_1, e_2 \in C_4, \\ \text{or} & e_1, e_2 \in P_4 \cup C_4 \text{ with } P_4 \cap C_4 \neq \Phi \end{array} \right\}$$

The transitive closure $\mathcal{R}^*$ of this relation is an equivalence relation partitioning the set of edges into disjoint equivalence clas.es. An $\mathcal{R}^*$-equivalence class is trivial if it

contains only one edge, otherwise, it is non-trivial. Any non-trivial $\mathcal{R}^*$-equivalence class contains at least three edges which belong either to an induced path P_4 or to an induced chordal free cycle C_4. If an edge belongs neither to an induced path P_4 nor to a chordal free cycle C_4, it itself forms an equivalence class. The orientation of one edge in an equivalence class again forces the orientation of all other edges in this class.

In the following we consider the class of H-AO-P_4-graphs with $2m$ vertices which contain the Hamming graph $K_2 \times K_m$ as a subgraph and show that all these graphs can be transitively oriented. In the Hamming graph $K_2 \times K_m$ each induced path P_4 as well as any induced chordal free cycle C_4 contain exactly two vertices from each of the both row cliques RC_1 and RC_2.

Lemma 1 Let $HG = (V, E) = K_2 \times K_m \cup E_D$ be an H-AO-P_4-graph and $D = (V, A)$ be an AO-P_4-orientation of HG. If both arcs (a, b) and (b, c) belong to A and one of the edges $\{a, b\}$ or $\{b, c\}$ belongs to a non-trivial $\mathcal{R}^*$-equivalence class of HG, then $\{a, c\} \in E(HG)$.

Proof. Let $HG = (V, E) = K_2 \times K_m \cup E_D$ be an H-AO-P_4-graph and $D = (V, A)$ be an AO-P_4 -orientation of HG. Let $e_1 = \{a, b\}$ and $e_2 = \{b, c\}$ be edges in HG with (a, b), $(b, c) \in$ A(D), and let e_1 be in a non-trivial $\mathcal{R}^*$-equivalence class. Then, there exists either an induced path P_4 that contains the edge e_1 but not the edge e_2, or a chordal free cycle C_4 which contains e_1 but not e_2. The proof requires the following distinction of cases.

Case 1: The vertices a, b and c are contained in the same row clique. Then $\{a, c\} \in E(HG)$.

Case 2: Not all vertices a, b, c are contained in the same row clique. Suppose that $a, b \in RC_1$ and $c \in RC_2$. Assume that $\{a, c\}$ is not an edge in HG.

Case 2.1: The edge e_1 belongs to an induced path P_4.

Case 2.1.1: The vertices a, b, u and v with $u, v \in RC_2$ induce the path $P_4 = (u, v, a, b)$ in HG. Then the edges $\{u, a\}$, $\{u, b\}$ and $\{v, b\}$ are not edges of HG, see Figure 5. Since $\{v, b\} \notin E(HG)$ holds and by assumption $\{a, c\} \notin E(HG)$ is satisfied, the vertices a, b, c and v induce a chordal free cycle C_4 in HG. This cycle contains a directed path P_3 in the orientation D which contradicts that D is an alternating orientation. Thus, $\{a, c\} \in E(HG)$ is valid.

Case 2.1.2: The vertices a, b, u, v with $u, v \notin RC_2$ induce the path $P_4 = (a, b, u, v)$ in HG. This implies that $\{a, u\}$, $\{a, v\}$, $\{b, v\} \notin E(HG)$. Assume that $\{a, c\} \notin E(HG)$. Then the vertices a, b, c, v induce a path P_4 in HG, Figure 6. This implies that in the orientation D the path $P_4 = (a, b, c, v)$ contains the directed path $P_3 = (a, b, c)$

contradicting to the precondition. So, the assumption is false and it holds $\{a, c\} \in E(HG)$.

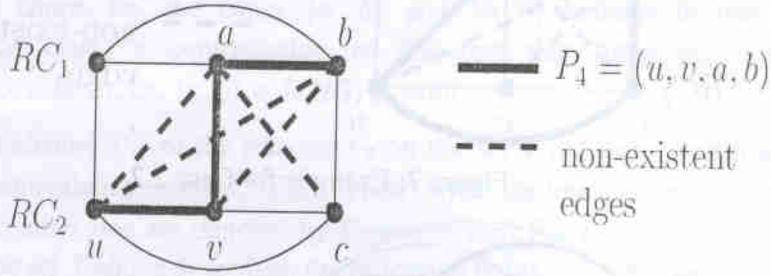


Figure 5: Example for Case 2.1.1

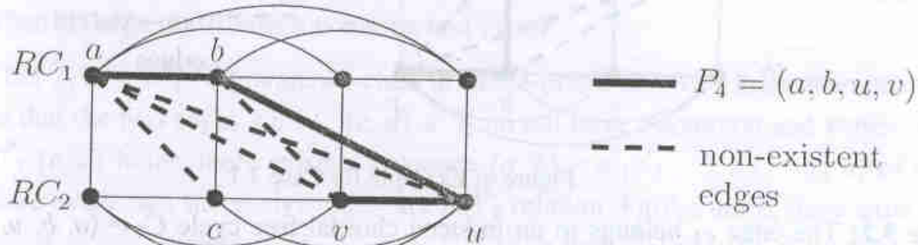


Figure 6: Example for case 2.1.2

Case 2.2: The edge e_1 belongs to an induced chordal free cycle $C_4 = (a, b, u, v)$. Then $\{a, u\}$, $\{b, v\} \notin E(HG)$ holds, see Figure 7. With $\{a, c\} \notin E(HG)$, the vertices a, b, c, v induce a chordal free cycle $C_4 = (a, b, c, v)$ which contains a directed path P_3 in the orientation D. Then D is not an AO-P_4-orientation, in contradiction to the precondition. Therefore, it also holds $\{a, c\} \in E(HG)$ in this case.

Case 3: Not all vertices a, b, and c are contained in the same row clique. Suppose that $a \in RC_1$ and $b, c \in RC_2$. Assume again that $\{a, c\}$ is not an edge in HG.

Case 3.1: The edge e_1 belongs to an induced path $P_4 = (u, a, b, v)$ with $u \in RC_1$ and $v \in RC_2$. Since $\{u, b\}$ and $\{a, c\}$ do not belong to the set of edges of HG, as illustrated in Figure 8, the vertices a, b, c and u induce a chordal free cycle C_4 ($\{u, c\} \in E(HG)$) or path P_4 ($\{u, c\} \notin E(HG)$). But then the cycle $C_4 \notin (a, b, c, u)$ or the path $P_4 = (a, b, c, u)$ contains the directed path $P_3 = (a, b, c)$ in the orientation D, a contradiction to the precondition. Therefore, $\{a, c\} \in E(HG)$ holds.

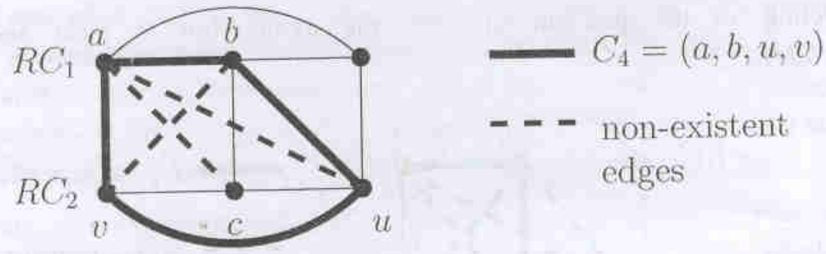


Figure 7: Example for Case 2.2

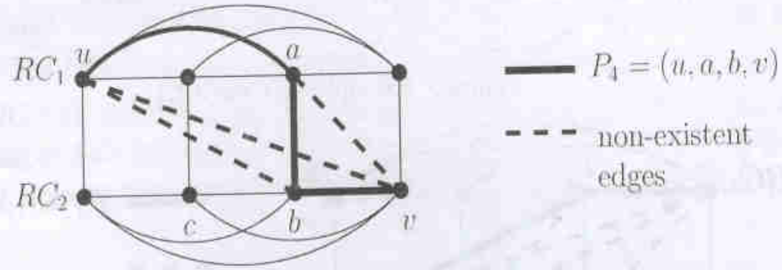


Figure 8: Example for Case 3.1

Case 3.2: The edge e_1 belongs to an induced chordal free cycle $C_4 = (a, b, u, v)$ with $v \in RC_1$ and $u \in RC_2$. By assumption $\{a, u\}$, $\{b, v\}$ and $\{a, c\}$ are not in $E(HG)$, see Figure 9. If the edge $\{v, c\}$ is contained in $E(HG)$, the vertices a, b, c, v induce a chordal free cycle C_4. If $\{v, c\}$ does not belong to $E(HG)$, the vertices v, a, b, c induce a path P_4. Both cases contradict the condition that D is an AO-P_4-orientation of HG. Thus the assumption is false and $\{a, c\} \in E$ holds.

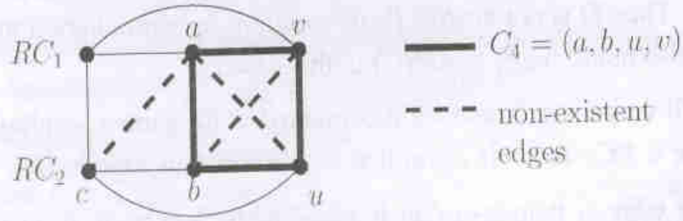


Figure 9: Example for case 3.2

The proof is completed, as in all cases $\{a, c\} \in E(G)$ is valid.

Let $E_1, \ldots, E_k$ be the $\mathcal{R}^*$-equivalence classes of H-graph $HG = K_2 \times K_m \cup E_D$, and let T be the union of all trivial $\mathcal{R}^*$-equivalence classes. Note that T exactly contains those edges which belong neither to an induced path P_4 nor to a chordal free cycle C_4. Now we investigate all edges in T by the relation Γ. To be sure that we are only in T we denote this relation by Γ_T. If $\{a, b\} \Gamma_T \{b, c\}$ is valid, then not all of the vertices a, b and c can be in the same row clique of HG because then $\{a, c\} \in E(HG)$ holds.

Furthermore, there exists a vertex x in $V(HG)$ such that the vertices a, b, c, x induce a cycle of length four in HG. We assume that $\{x, b\} \notin E(HG)$ holds. Then the cycle (a, b, c, x) has no chord. So, the edges $\{a, b\}$ and $\{b, c\}$ belongs to one non-trivial $\mathcal{R}^*$-equivalence class, a contradiction to the fact that these edges are trivial $\mathcal{R}^*$-equivalence classes, i.e. $\{x, b\} \in E(HG)$ is valid.

The transitive closure Γ^*_T of the relation Γ_T on the set $\boldsymbol{T}$ induces a partition of $\boldsymbol{T}$ into disjoint Γ^*_T-equivalence classes. Each class with cardinality one is denoted by Γ_T*-trivial*. The other one are denoted by Γ_T*-non-trivial*. For any H-graph HG with $2m$ vertices and the set $\boldsymbol{T}$ above described, the following holds.

Lemma 2 Let T_i be a Γ^*_T-equivalence class in the H-graph HG with more than one edge. Then all edges in T_i have a common end vertex.

Proof. Let T_i be a Γ^*_T- equivalence class in the H-graph HG with more than one edge. Assume that the two edges $\{a, b\}, \{c, d\} \in T_i$ do not have a common end vertex. Since $\{a, b\}\Gamma^*_T\{c, d\}$ holds, there exists a sequence $\{a, b\} = e^*_1 e^*_2 \ldots, e^*_k = \{c, d\}$ of edges in T where every two successive edges are in Γ_T relation. Furthermore, there exist three edges e^*_i, e^*_{i+1}, e^*_{i+2} with $i \in \{1, 2, \ldots, k-2\}$ such that e^*_i and e^*_{i+2} do not meet at a common end vertex. W.l.o.g., let it be $e^*_i = \{u, v\}$, $e^*_{i+1} = \{v, w\}$ and $e^*_{i+2} = \{w, x\}$. Now we have to consider two cases.

Case 1: It holds $u, x \in RC_1$ and $v, w \in RC_2$, then in particular, $\{u, x\} \in E(HG)$. Because by assumption $\{u, w\}$ and $\{v, x\} \notin E(HG)$ the vertices u, v, w, x form a chordal free cycle C_4. So, the edges $\{u, v\}$, $\{v, w\}$ and $\{w, x\}$ belong to a non-trivial $\mathcal{R}^*$-equivalence class which contradictions to $\{u, v\}, \{v, w\}, \{w, x\} \in T$.

Case 2: It holds $u, v \in RC_1$ and $w, x \in RC_2$ as shown in Figure 10. If $\{u, x\} \notin E(HG)$, the vertices u, v, w, x induce a path P_4 in HG. If $\{u, x\} \in E(HG)$, the vertices u, v, w, x induce a chordal free cycle C_4. In both cases the edges $\{u, v\}$, $\{v, w\}$ and $\{w, x\}$ belong to a non-trivial $\mathcal{R}^*$-equivalence class which is a contradiction to the choice of $\{u, v\}$, $\{v, w\}$ and $\{w, x\}$.

Thus the assumption is false and all edges from T_i have a common vertex. □

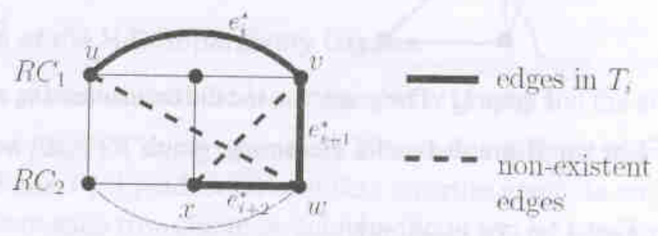


Figure 10: An example for Case 2 in the proof of Lemma 2

If we apply the Γ_T relation to the edges of T_i with the common vertex a, all edges $\{x, a\}$ of T_i has to be oriented either from x to a or from a to x. Now we are able to formulate and to prove the following theorem.

Theorem 1 If $HG = (V, E) = K_2 \times K_m \cup E_D$ is an H-AO-P_4-graph, then HG is a *comparability graph*.

Proof. Let $HG = (V, E) = K_2 \times K_m \cup E_D$ be an H -AO-P_4-graph and let $E_1, \ldots, E_k$ be $\mathcal{R}^*$-equivalence classes from HG. We construct a pseudo-transitive orientation $D = (V, A)$ of HG: the non-trivial $\mathcal{R}^*$-equivalence classes are oriented as in an AO-P_4-orientation through fixing of a direction of an edge in each class. The set $\boldsymbol{T}$ of trivial $\mathcal{R}^*$-equivalence classes of HG are partitioned in Γ^*_T-equivalence classes. The edges of each non-trivial Γ^*_T-equivalence classes form a star and are oriented according to the Γ_T-relation by fixing an arbitrary arc. All remaining edges $\{u, v\}$ build trivial Γ_T-equivalence classes, they get either the orientation (u, v) or (v, u). The conditions $\{a, b\}, \{b, c\} \in E$ and $\{a, c\} \notin E$ implies that either both edges belong to the same non-trivial $\mathcal{R}^*$-equivalence class or they belong to a non-trivial Γ_T-equivalence class. In both cases either $(a, b), (c, b) \in A$ or $(b, a), (b, c) \in A$ is satisfied by Lemma 1, Lemma 2 and the construction of D. Since the condition $(a, b), (b, c) \in A$ implies either (a, c) or (c, a) in $A(D)$, the orientation D is pseudo-transitive. Therefore, also a transitive orientation of HG exists, and hence, HG is an H-comparability graph. □

A generalization of Lemma 1 does not remain true if we consider the H-AO-$P4$-graphs which contain the Hamming graph $K_n \times K_m$ as their subgraphs. Figure 11 illustrates an AO-P_4-graph $G_{P4} = (V_{P4}, E_{P4})$ which is not a comparability graph. An AO-P_4-orientation is represented by $D_{P4} = (V_{P4}, A_{P4})$.

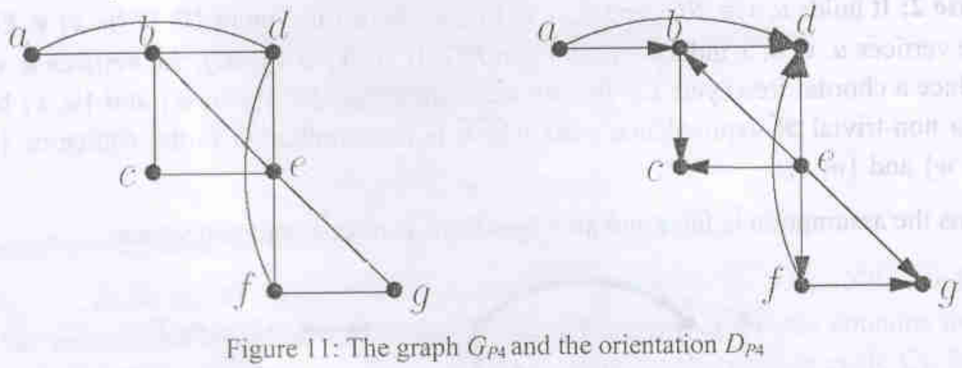


Figure 11: The graph G_{P4} and the orientation D_{P4}

Let $HG = (V, E)$ be an H-graph on the Hamming graph $K_3 \times K_4$ with the following properties.

1. $V(HG) = V_{P4} \cup \{v_1, v_2, v_3, v_4, v_5\}$,

2. The vertices in V_{P4} induced the graph G_{P4} illustrated in Figure 11,
3. For all $x \in V \setminus \{v_i\}, i = 1, \ldots, 5, \{v_i, x\} \in E(HG)$.

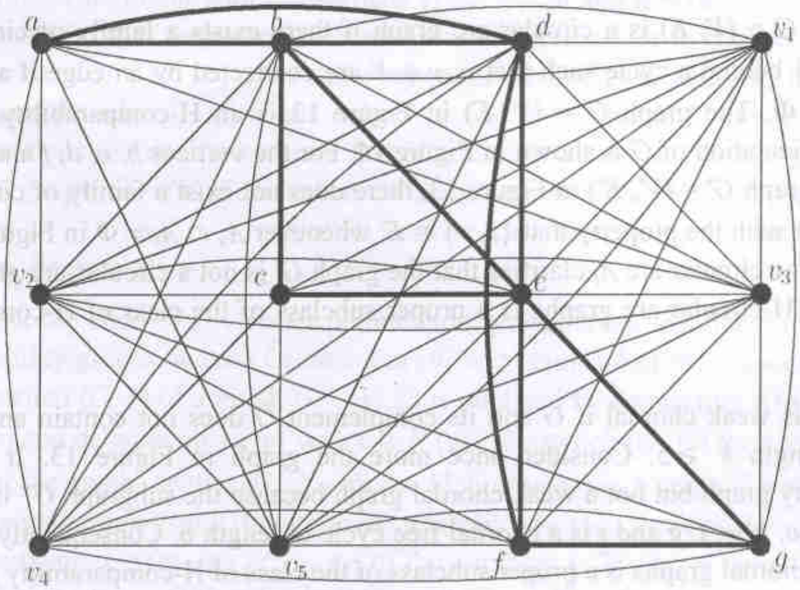


Figure 12: An H-AO-P$_4$-comparability graph which is not an H-comparability graph.

Then the graph HG (see Figure 12) is an H-AO-P_4-graph as there exists an AO-P_4-orientation of it. The edges of the vertex set V_{P4} which build an induced subgraph form the only non-trivial $\mathcal{R}^*$-equivalence class of HG. The orientation $D = (V, A)$ have the set of arcs $A = \{(v_i, x) \mid x \in N(v_i) \text{ and } x \neq v_j, j \leq i\} \cup A_{P4}$.

where A_{P4} is the arc set of the orientation of D_{P4} as illustrated in Figure 11. Clearly, D is an AO-P_4-orientation as it satisfies the required properties. But the graph HG is not an H-comparability graph since the induced subgraph with respect to the vertices in V_{P4} is not transitive orientable.

So, the class of H -AO-$P4$-graphs is a proper superclass of the class of H-comparability graphs It is shown that the class of H-$P4$-comparability graphs and the class of H-comparability graphs are not identical.

3.2 Subclasses of the H-Comparability Graphs

We investigate the relationship of the H-comparability graphs and the subclasses.

a. A graph is called *bisplit* if its set of vertices V can be partitioned into V_1, V_2, V_3 where V_1 is a stable set S and V_2, V_3 induces a complete bipartite graph. In any H-bisplit graph only one vertex from each row-clique or column-clique can be contained in V_1, V_2, V_3.

Therefore, all H-graphs which contain the Hamming graphs $K_n \times K_m$ with $n > 3$ or $m > 3$ are not bisplit. Thus, the class of H-bisplit graphs is a proper subclass of the H-comparability graphs.

b. A graph $G = (V, E)$ is a circular-arc graph if there exists a family of circular arcs $\{A_v \mid v \in V\}$ behind a cycle such that $u, v \in V$ are connected by an edge if and only if $A_u \cap A_v \neq \Phi$. The graph $G = (V, E)$ in Figure 13 is an H-comparability graph. A transitive orientation of G is shown in Figure 14. For the vertices b, c, d, f and g, which induce the graph $G' = (V', E')$ in Figure 13, there does not exist a family of circular arcs $\{A_v \mid v \in V'\}$ with the property that $\{u, v\} \in E'$ whenever $A_u \cap A_v \neq \Phi$ in Figure 14. The location of the circular arc A_f clarifies that the graph G' is not a circular-arc graph. Thus the class of H-circular-arc graphs is a proper subclass of the class of H-comparability graphs.

c. A graph is weak chordal if G and its complement $\overline{G}$ does not contain any induced cycle of length $k \geq 5$. Consider once more the graph in Figure 13. It is an H-comparability graph but not a weak chordal graph because the subgraph G'' induced by the vertices a, b, e, f, g and i is a chordal free cycle of length 6. Consequently, the class of H-weak chordal graphs is a proper subclass of the class of H-comparability graphs.

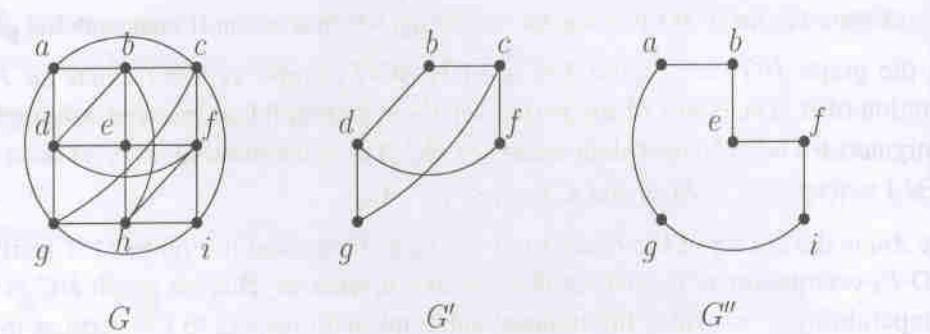


Figure 13: An H-comparability graph G and the induced subgraphs G' and G"

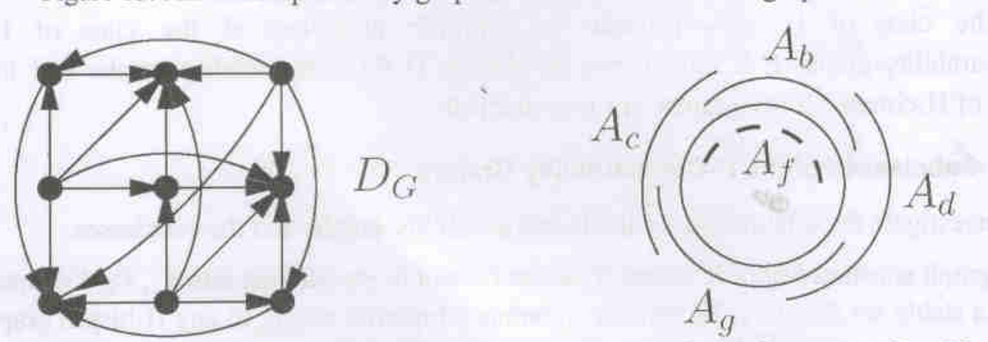


Figure 14: A transitive orientation of G and a family of circular arcs to G – {f}

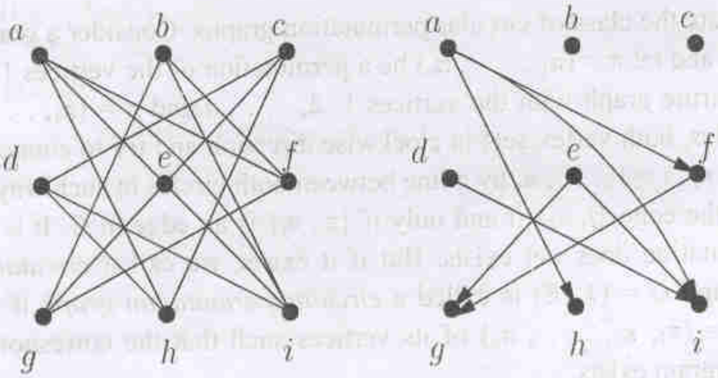
d. We investigate the class of circular permutation graphs. Consider a graph $G = (V, E)$ with n vertices and let $\pi = (\pi_1, \ldots, \pi_n)$ be a permutation of the vertices $1, \ldots, n$. Now we draw a bipartite graph with the vertices $1, 2, \ldots, n$ and $\pi = (\pi_1, \ldots, \pi_n)$ on two concentric circles, both vertex sets in clockwise direction and try to connect the vertex i with the vertex π_i, $i = 1, \ldots, n$, by a line between both circles in such way that the edge $\{i, \pi_i\}$ crosses the edge $\{j, \pi_j\}$ if and only if $\{\pi_i, \pi_j\}$ is an edge in G. It is possible, that such a representation does not exists. But if it exists, we call it *circular permutation diagram*. A graph $G = (V, E)$ is called a *circular permutation graph* if there exists a permutation $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ of its vertices such that the corresponding circular permutation diagram exists.

To show that the class of H-circular permutation graphs is not identical to the class of H-comparability graphs, we use Lemma 3 of [9] and Lemma 4 of [8], respectively. Here the graph switch (G, v) of a graph $G = (V, E)$ is obtained by connecting v to all vertices $x \in V \setminus N(v)$ and deleting all $\{v, y\}$ with $y \in N(v)$. The switch (G, V') for a set $V' \subseteq V$ is the union of the graphs switch (G, z) with $z \in V'$. Furthermore, a graph G with n vertices is called a *permutation graph* if there exists a labeling $v_1, v_2, \ldots, v_n$ of the vertices of G and a permutation $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ of the numbers $1, 2, \ldots, n$ such that v_i and v_j are joined by an edge in G if and only if $(i - j)(\pi_i^{-1} - \pi_j^{-1}) < 0$, where $\pi^{-1}{}_k$ is the integer which π maps π onto k.

Lemma 3 Let $v \in E$ in $G = (V, E)$. The graph G is a circular permutation graph if and only if G is a comparability graph and the switch $(G, N(v))$ is a permutation graph.

Lemma 4 A graph G is a permutation graph if and only if both G and the complement $\overline{G}$ are comparability graphs.

Consider the H-comparability graph $G = (V, E)$ in Figure 13. Assume that G is a circular permutation graph. Then by Theorem 3 the graph switch$(G, N(a))$ is a permutation graph. This is false, because the graph switch $(G, N(a)) = G$ and the graph $\overline{G}$ in Figure 15 are not transitively orientable. If the vertices in $\overline{G}$ would be oriented by means of the relation Γ, the edges $\{a, h\}$, $\{a, f\}$, $\{a, i\}$, $\{d, i\}$, $\{e, i\}$, $\{e, g\}$, $\{f, g\}$ belongs to one Γ-equivalence class. Then a result is the path $P_3 = (a, f, g)$ with $\{a, g\} \notin E(\overline{G})$, Figure 15. So, the class of H-circular permutation graphs is a proper subclass of the class of H-comparability graphs.



$\overline{G}$ partial orientation

Figure 15: The complement $\overline{G}$ of G from Figure 13 and a partial orientation

e. We consider the partial orders and their comparability graphs. Let $P = (X, \trianglelefteq)$ be a poset of dimension k. The graph $G = (X, E)$ with $\{u, v\} \in E \Leftrightarrow$ either $u < v$ or $u > v$ in P is called a comparability graph of k-dimensional poset P. The orientation of G with $A = \{(u, v) \mid u, v \in V$ with $u < v$ in $P\}$ is transitive. A graph $G = (V, E)$ is called a comparability graph of circular order if there exists a family $C = \{C_{vi} \mid v_i \in V\}$ of the cycles in the plane such that the vertices v_i, $v_j \in V$ are adjacent if and only if C_{vi} is contained in C_{vj} or C_{vj} is contained in C_{vi}.

It is an open question whether the class of H -comparability graphs of circular orders and the H-comparability graphs of 4–dimensional, respectively, 3–dimensional posets are proper lower classes of the class of H-comparability graphs. Recall [11] that the determination of the dimension of a given partial order is a NP-complete problem for $\dim(P) \geq 3$. Therefore, the connection between partial ordered sets and the comparability graphs, especially, the H-comparability graphs is interesting but seems difficult.

In the following the lower classes of the comparability graphs of circular orders, of 4-dimensional, respectively, 3-dimensional posets on the basis of H-comparability graphs are considered to establish the border between the H-comparability graphs and their relatives. The classes of the chordal comparability graphs, the co-interval graphs and the permutation graphs belong to this class.

f. A graph G is chordal if it does not contain a chordal free cycle of length $k \geq 4$. If a graph is comparability as well as chordal, it is called chordal comparability graph. The Hamming graph $K_2 \times K_2$ is a comparability graph but not a chordal. Following that the class of H-chordal comparability graphs is a proper subclass of the class of H-comparability graphs.

g. A graph is called *co-interval graph* if its complement is an interval graph, where a graph is an interval graph if there exists a set of real intervals $\{I_v \mid v \in V\}$ such that the vertices $u, v \in V$ are adjacent if and only if $I_v \cap I_u \neq \Phi$. see [4]for a proof.

Lemma 5 A graph is an interval graph if and only if it does not contain any chordal free cycle of length k $\geq$4 and its complement $\overline{G}$ is transitively orientable.

The H-comparability graph G in Figure 16 is not a co-interval graph, because the complement $\overline{G}$ contains a chordal free cycle C4 = (a, g, b, h) and, therefore, is not an interval graph by Lemma 5. Therefore, the class of H-co-interval graphs is a proper subclass of the class of H-comparability graphs.

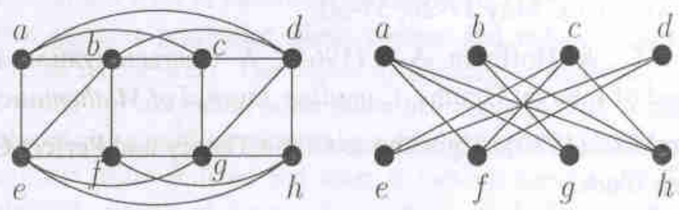


Figure16: An H-comparability graph that is not co-interval graph

h. Recall, a graph G with n vertices is called a permutation graph if there exists a labeling $v_1, v_2, \ldots, v_n$ of the vertices of G and a permutation $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ of the numbers 1, 2, . . . , n such that v_i and v_j are joined by an edge in G if and only if $(i-j)(\pi_i^{-1} - \pi_j^{-1}) < 0$, where π_k^{-1} is the integer which π maps onto k. Because the H-comparability graph in Figure 11 is not a permutation graph, the class of H–permutation graphs is a proper subclass of the H-comparability graphs.

4. Conclusions

All contained superclasses of the class of H-comparability graphs are proper superclasses. Moreover, it is shown, that all H-AO-P_4 graphs which contain the Hamming graph $K_2 \times K_m$ as spanning subgraph, are H-comparability graphs. Most of the considered subclasses are proper subclasses. But the relationship between H-comparability graphs of circular order and 4-dimensional, respectively, 3-dimensional posets and H-comparability graphs could not be clarified. There are some open questions in this connection of graph classes for further research.

These investigations are important not only from graph theoretical viewpoints but also from practical viewpoints since the comparability graphs have quite important roles in solving the classical shop scheduling problems.

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Mathematical Model of Slider Bearing

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Abstract: To reduce tear and wear of machinery lubrication is essential. Lubricants form a layer between two surfaces preventing direct contact and reduce friction between moving parts and hence reduce wear. The choice of lubricant is important for a given application. In this model the lubrication of the slider bearing is studied. A simple slider bearing has two plates of given profile separated by a gap between the plates is filled with the lubricant. One of the plates is fixed and other is moving horizontally. Due to the viscosity of the lubricant, motion of the plate's results in work done on the lubricant increasing the temperature. This study will be helpful in finding the condition under which the safe operation of the bearing is ensured. That is, in finding the condition under which the temperature of the lubricant is lower than the ignition temperature. When the variable viscosity is considered the case becomes complicated. Further investigations are necessary.

Key words: Navier-Stokes Equations, dimensional analysis, scaling, load.

1. Slider Bearing with Parallel Plates

The incompressible Navier- Stokes equation is

$$\frac{\partial \mathrm{u}}{\partial t} + \mathrm{u}\nabla \mathrm{u} - f - \nu\Delta \mathrm{u} + \frac{\nabla p}{\rho} = 0 \tag{1}$$

$$\nabla . \mathrm{u} = 0 \tag{2}$$

where (1) represents the conservation of momentum. $\mathrm{u} = (u, w)$ be a vector, p is pressure and ρ is the density, ν is kinematic viscosity. The body force f is usually absent and other parameters remain constant.

Let $u(x, z, t)$ and $w(x, z, t)$ be the components of velocity of the fluids in horizontal and vertical directions, respectively then by (1) and (2)

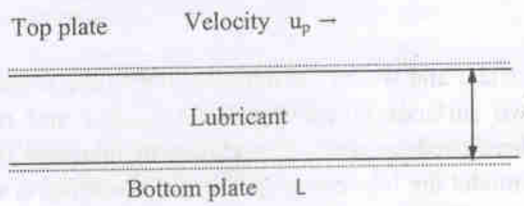
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{3}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right) \quad (4)$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad (5)$$

By imposing the boundary conditions

$$(u, w) = (0, 0)\, at\, z = 0;$$
$$(u, w) = (u_p, 0)\, at\, z = l$$



Bearing with parallel plates

Here, u_p is the typical horizontal velocity of the plate and l is the typical separation between the plates which is shown as in figure. Let the following parameters for bearing used are :

$L = 5cm$, $l = 5\mu m$, $u_p = 1m/sec$, $\rho = 1 \times 10^3 Kgm^{-3}$, $\mu = 1 \times 10^{-4} m^2/sec$

To non- dimensionalize the equations parameters are scaled as

$$x = \bar{x}L\ , z = \bar{z}l\ , u = \bar{u}u_p\ , w = \bar{w} \in u_p\ ,\ t = \bar{t}\, L/u_p\ , p = \bar{p}P$$

From (3) and (4), we get

$$\frac{u_p^2}{L}\frac{\partial \bar{u}}{\partial \bar{t}} + \frac{\bar{u}u_p^2}{L}\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\bar{w}\epsilon u_p^2}{l}\frac{\partial \bar{u}}{\partial \bar{z}} + \frac{1}{\rho}\frac{P}{L}\frac{\partial \bar{p}}{\partial \bar{x}} = \nu\left(\frac{u_p}{L^2}\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{u_p}{l^2}\frac{\partial^2 \bar{u}}{\partial \bar{z}^2}\right) \quad (6)$$

$$\frac{\epsilon u_p^2}{L}.\frac{\partial \bar{w}}{\partial \bar{t}} + \frac{\bar{u}\epsilon u_p^2}{L}.\frac{\partial \bar{w}}{\partial \bar{x}} + \bar{w}\frac{\epsilon^2 u_p^2}{l}\frac{\partial \bar{w}}{\partial \bar{z}} + \frac{1}{\rho}\frac{P}{L}\frac{\partial \bar{p}}{\partial \bar{z}}$$
$$= \nu\left\{\frac{\epsilon u_p}{L^2}\frac{\partial^2 \bar{w}}{\partial \bar{x}^2} + \frac{\epsilon u_p}{l^2}\frac{\partial^2 \bar{w}}{\partial \bar{z}^2}\right\} \quad (7)$$

Multiplying (6) by $\frac{L}{u_p^2}$ and (7) by $\frac{L}{\epsilon u_p^2}$ and setting, $\epsilon = \frac{l}{L}$ is typically 1×10^{-4} . P is undecided scaling factor for the pressure. Eliminating the terms which have small coefficients as compared to $\frac{1}{\epsilon}$, from (6)

$$\frac{P}{\rho u_p^2}\frac{\partial \bar{p}}{\partial \bar{x}} = \frac{\nu L}{l^2 u_p}\frac{\partial^2 \bar{u}}{\partial \bar{z}^2}$$

Here, P is chosen such that

$$\frac{P}{\rho u_p^2} = \frac{\nu L}{l^2 u_p}$$

$$P = \frac{\rho v u_p L}{l^2}$$

$$P = \frac{\mu u_p L}{l^2} \tag{8}$$

where $\mu = \rho v$

With the choice of P, we get from (6)

$$\frac{\partial \bar{p}}{\partial \bar{x}} = \frac{\partial^2 \bar{u}}{\partial \bar{z}^2} \tag{9}$$

and from (7)

$$\frac{1}{\rho} \cdot \frac{PL}{\epsilon u_p^2} \frac{\partial \bar{p}}{\partial \bar{z}} = 0$$

$$\therefore \frac{\partial \bar{p}}{\partial \bar{z}} = 0 \tag{10}$$

From (5), we obtain

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{w}}{\partial \bar{z}} = 0 \tag{11}$$

The boundary condition for $\bar{u}$ and for $\bar{w}$

$$\bar{u} = 0 \text{ at } \bar{z} = 0 \text{ and } \bar{u} = 1 \text{ at } \bar{z} = 1$$

$$\bar{w} = 0 \text{ at } \bar{z} = 0 \text{ and } \bar{w} = 0 \text{ at } \bar{z} = 1,$$

Since the pressure $\bar{p}$ is independent of $\bar{z}$ then from equation (10)

$$\frac{\partial \bar{p}}{\partial \bar{z}} = 0$$

Integrating, $\bar{p} = \emptyset(\bar{x})$ with the boundary conditions;

$$\bar{p} = 0 \text{ at } \bar{x} = 0$$

$$\bar{p} = 0 \text{ at } \bar{x} = 1$$

Integrating first we obtain,

$$\frac{\partial \bar{u}}{\partial \bar{z}} = \frac{\partial \emptyset(\bar{x})}{\partial \bar{x}} \bar{z} + c_1$$

Again integrating,

$$\bar{u} = \frac{\partial \emptyset(\bar{x})}{\partial \bar{x}} \cdot \frac{\bar{z}^2}{2} + c_1 \bar{z} + c_2 \tag{12}$$

where c_1 and c_2 are constants.

With the boundary condition $\bar{u} = 0$ at $\bar{z} = 0$ and $\bar{u} = 1$ at $\bar{z} = 1$

We get, $c_2 = 0$ and $c_1 = 1 - \frac{1}{2}\frac{d\emptyset}{d\bar{x}}$

$$\therefore \quad \bar{u} = \left(\frac{\bar{z}^2 - \bar{z}}{2}\right)\frac{d\emptyset}{d\bar{x}} + \bar{z}$$

Again from equation (11), $\frac{\partial \bar{w}}{\partial \bar{z}} = -\frac{\partial \bar{u}}{\partial \bar{x}}$

$$\frac{\partial \bar{w}}{\partial \bar{z}} = -\frac{\partial}{\partial \bar{x}}\left[\left(\frac{\bar{z}^2-\bar{z}}{2}\right)\frac{d\emptyset}{d\bar{x}} + \bar{z}\right]$$

$$\frac{\partial \bar{w}}{\partial \bar{z}} = -\frac{\partial}{\partial \bar{x}}\left[\left(\frac{\bar{z}^2-\bar{z}}{2}\right)\frac{d\emptyset}{d\bar{x}} + \bar{z}\right]$$

$$\therefore \bar{w} = -\left[\frac{\bar{z}^3}{6} - \frac{\bar{z}^2}{4}\right]\frac{d^2\emptyset}{d\bar{x}^2} + c_3$$

Where c_3 is constant of integration with the boundary conditions;

$\bar{w} = 0$ at $\bar{z} = 0$ and $\bar{z} = 1$, we have

$$c_3 = 0 \text{ and } c_3 = -\frac{1}{12}\frac{d^2\emptyset}{d\bar{x}^2}$$

$$\therefore \quad \frac{d^2\emptyset}{d\bar{x}^2} = 0$$

Integrating we get,

$$\frac{d\emptyset}{d\bar{x}} = c_4$$

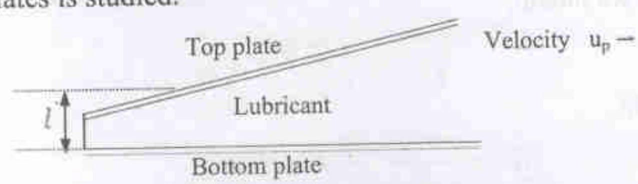
$$\therefore \quad \emptyset = c_4\,\bar{x} + c_5$$

With the boundary conditions on $\emptyset$ begin zero, we must have that $\bar{p} = 0$.

Here, it is shown that the pressure is zero, the bearing with parallel plates can support no load. Therefore, it is not helpful in physical problem. But this problem acts as a clue for the further work.

2. Slider Bearing with Non-parallel Plates

When a bearing with parallel plates cannot support the load, Slider bearing with non-parallel plates is studied.



Bearing with non-parallel plates

We assume that the bottom plate is flat and the top plate is given by $z = h(x)$. Other parameters remain the same as bearing with parallel plates.

We scale the above equations (3), (4) and (5) with

$$x = \bar{x}L\,,\ z = \bar{z}l\,,\ u = u\bar{u}_p,\ w = \bar{w} \in u_p, h(x) = l\bar{h}\,(\bar{x})$$

$$t = \bar{t}\frac{l}{u_p}, p = \bar{p}P$$

We obtain the non-dimensional equations, (9), (10) and (11)

Imposing the boundary conditions

$$\bar{u} = 0 \; at \; \bar{z} = 0$$
$$\bar{u} = 1 \; at \; \bar{z} = \bar{h}(\bar{x})$$

From (12) we get,

$$0 = \frac{d\emptyset}{d\bar{x}} . 0 + c_1 0 + c_2 \quad \therefore \; c_2 = 0$$

and $\quad 1 = \frac{d\emptyset}{d\bar{x}} \frac{\bar{h}^2(\bar{x})}{2} + c_1 \bar{h}(\bar{x})$

$$\therefore \quad c_1 = \frac{1}{\bar{h}(\bar{x})} - \frac{1}{2}\frac{d\emptyset}{d\bar{x}} \bar{h}(\bar{x})$$

Thus.

$$\bar{u} = \frac{d\emptyset}{d\bar{x}}\left[\frac{\bar{z}^2}{2} - \frac{\bar{z}\bar{h}}{2}\right] + \frac{\bar{z}}{\bar{h}} \tag{13}$$

Denoting $\frac{d\emptyset}{d\bar{x}}$ by $\emptyset'(\bar{x})$. Now from equation

$$\frac{\partial \bar{w}}{\partial \bar{z}} = -\frac{\partial \bar{u}}{\partial \bar{x}} = -\frac{\partial}{\partial \bar{x}}\left[\left(\frac{\bar{z}^2 - \bar{z}\bar{h}}{2}\right)\frac{d\emptyset}{d\bar{x}} + \frac{\bar{z}}{\bar{h}}\right]$$

$$= -\left[\frac{\bar{z}^2 - \bar{z}\bar{h}}{2}\emptyset''(\bar{x}) - \frac{\bar{z}}{2}\frac{d\bar{h}}{d\bar{x}}\emptyset'(x) + \frac{\bar{h}.0 - \bar{z}\frac{d\bar{h}}{d\bar{x}}}{\bar{h}^2}\right]$$

$$\frac{\partial \bar{w}}{\partial \bar{z}} = -\left[\frac{\bar{z}^2 - \bar{z}\bar{h}}{2}\emptyset''(\bar{x}) - \frac{\bar{z}}{2}\frac{d\bar{h}}{d\bar{x}}\emptyset'(x) - \frac{\bar{z}\frac{d\bar{h}}{d\bar{x}}}{\bar{h}^2}\right]$$

Integrating we get,

$$\bar{w} = -\left[\left[\frac{\bar{z}^3}{6} - \frac{\bar{z}^2\bar{h}}{4}\right]\emptyset''(\bar{x}) - \frac{\bar{z}^2}{4}\frac{d\bar{h}}{d\bar{x}}\emptyset'(\bar{x}) - \frac{\bar{z}^2}{2\bar{h}^2}\frac{d\bar{h}}{d\bar{x}}\right] + c_3 \tag{14}$$

By imposing the boundary condition,

$$\bar{w} = 0 \; at \; \bar{z} = 0$$
$$\bar{w} = 0 \; at \; \bar{z} = \bar{h}(\bar{x})$$

we get $c_3 = 0$

Again,

$$0 = -\left[\left(\frac{\bar{h}^3(\bar{x})}{6} - \frac{\bar{h}^2(\bar{x})}{4}\bar{h}\right)\emptyset''(\bar{x}) - \frac{\bar{h}^2(\bar{x})}{4}\frac{d\bar{h}}{d\bar{x}}\emptyset'(\bar{x}) - \frac{\bar{h}^2(\bar{x})}{2\bar{h}^2}\frac{d\bar{h}}{d\bar{x}}\right] + 0$$

$$0 = -\left[-\frac{\bar{h}^3(\bar{x})}{12}\emptyset''(\bar{x}) - \frac{\bar{h}^2\bar{x}}{4}\frac{d\bar{h}}{d\bar{x}}\emptyset'(\bar{x}) - \frac{1}{2}\frac{d\bar{h}}{d\bar{x}}\right]$$

or, $\quad \frac{\bar{h}^3(\bar{x})}{12}\emptyset''(\bar{x}) + \frac{\bar{h}^2(\bar{x})}{4}\frac{d\bar{h}}{d\bar{x}}\emptyset'(\bar{x}) + \frac{1}{2}\frac{d\bar{h}}{d\bar{x}} = 0$

This can be simplified as

$$\frac{d}{d\bar{x}}\left[\frac{\bar{h}^3(\bar{x})}{12}\,\emptyset'(\bar{x})\right]+\frac{1}{2}\frac{d\bar{h}}{d\bar{x}}=0$$

Assume a linear profile with $\bar{h}(\bar{x})=k_1\bar{x}+k_2$ and further assume that $k_1=k_2=1$. With this assumption, we get

$$\frac{d}{d\bar{x}}\left[\frac{(k_1\bar{x}+k_2)^3}{12}\,\emptyset'(\bar{x})\right]+\frac{1}{2}\frac{d}{d\bar{x}}(k_1\bar{x}+k_2)=0$$

Integrating first,

$$\frac{(k_1\bar{x}+k_2)^3}{12}\,\emptyset'(\bar{x})+\frac{1}{2}(k_1\bar{x}+k_2)=r_1$$

$$\emptyset'(\bar{x})=\frac{-\frac{1}{2}(k_1\bar{x}+k_2)+r_1}{\frac{(k_1\bar{x}+k_2)^3}{12}}$$

$$\emptyset'(\bar{x})=\frac{-6}{(k_1\bar{x}+k_2)^2}+\frac{r_1}{(k_1\bar{x}+k_2)^3}$$

Integrating we get,

$$\emptyset(x)=\frac{6k_1}{k_1\bar{x}+k_2}+\frac{k_1r_1}{(k_1\bar{x}+k_2)^2}+r_2$$

The constants of integration r_1 and r_2 are determined by boundary conditions on $\emptyset$ which are at $\emptyset=0$ at $\bar{x}=0$ and $\bar{x}=1$.

$$r_1+r_2=-6$$
$$r_1+4r_2=-12$$
$$\therefore\; r_1=-4,\quad r_2=-2$$

$$\emptyset(\bar{x})=\frac{6}{1+\bar{x}}+\frac{(-4)}{(1+\bar{x})^2}+(-2)$$
$$=\frac{6}{1+\bar{x}}-\frac{4}{(1+\bar{x})^2}-2$$
$$=\frac{2\bar{x}-2\bar{x}^2}{(1+\bar{x})^2}=\frac{2\bar{x}(1-\bar{x})}{(1+\bar{x})^2}$$
$$\therefore\;\bar{p}=\frac{2x(1-x)}{(1+x)^2}$$

which is positive for $\bar{x}\in(0,1)$

Hence, pressure is developed inside the fluid and the bearing supports a load given by the integral of pressure between the limits $\bar{x}=0$ and $\bar{x}=1$

Hence,

$$\text{load }\emptyset=\int_0^1\frac{2\bar{x}(1-\bar{x})}{(1+\bar{x})^2}\,d\bar{x}$$
$$\text{load}=6\,ln\,(2)-4$$

Substituting the expression for pressure in equation (13) and (14), we get the expression of u and w respectively.

Having determined the velocities of the fluid, we now consider the viscous dissipation caused by motion. Assuming constant viscosity with respect to temperature, we have energy balance equation

$$\rho C_p \frac{DT}{Dt} = \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x_j}\left(k \frac{\partial T}{\partial x_j}\right) + \emptyset$$

$$\text{where,} \qquad \emptyset = \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\frac{\partial v_i}{\partial x_j}$$

Where ρ and k are density and thermal conductivity. The external energy q_i is neglected because it is not present.

Now, from energy Balance equation, we have

$$\rho c_p \left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + w\frac{\partial T}{\partial z}\right) = 2\mu\left[\frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

where c_p is the specific heat of the fluid and k is its thermal conductivity. Suppose that the plates are kept at a constant temperature T_1 and lubricant catches fire at critical temperature T_c. We proceed to non-dimensionalize this equation by scaling as before with

$$T = T_1 + \bar{\theta}(T_c - T_1)$$

Here $\bar{\theta}$ is the non-dimensional variable which varies between 0 and 1. The boundary conditions on $\bar{\theta}$ are given by,

$$\bar{\theta} = 0 \ at\ \bar{z} = 0, \bar{z} = \bar{h}(\bar{x})$$

Scaling and neglecting the terms that are small as compared to $\frac{1}{\epsilon}$, we get the simplified energy equation given by

$$\mu \frac{u_p^2}{l^2}\left(\frac{\partial \bar{u}}{\partial \bar{z}}\right)^2 = -\frac{k}{l^2}\frac{\partial^2 \bar{\theta}}{\partial \bar{z}^2}(T_c - T_1)$$

$$\left(\frac{\partial \bar{u}}{\partial \bar{z}}\right)^2 = -\frac{k}{\mu u_p^2}\frac{\partial^2 \bar{\theta}}{\partial \bar{z}^2}(T_c - T_1)$$

$$\left(\frac{\partial \bar{u}}{\partial \bar{z}}\right)^2 = -B\frac{\partial^2 \bar{\theta}}{\partial \bar{z}^2}$$

Where,

$$B = \frac{k}{\mu u_p^2}(T_c - T_1)$$

Substituting the value of $\bar{u}$ from equation we get

$$\frac{\partial}{\partial \bar{z}}\left[\frac{(\bar{z}^2 - \bar{z}\bar{h})}{2}\frac{d\emptyset}{d\bar{x}} + \frac{\bar{z}}{\bar{h}}\right]^2 = -B\frac{\partial^2 \bar{\theta}}{\partial \bar{z}^2}$$

Integrating twice the equation, we get,

$$\frac{\bar{z}^4}{12}\emptyset'(\bar{x})^2 + \left[\frac{1}{\bar{h}} - \frac{\bar{h}}{2}\emptyset'(\bar{x})\right]^2\frac{\bar{z}^2}{2} + \frac{\bar{z}^3}{3}\emptyset'(\bar{x})\left[\frac{1}{\bar{h}} - \frac{\bar{h}}{2}\emptyset'(\bar{x})\right] + c_1\bar{z} + c_2 = -B\bar{\theta}$$

Using the boundary condition on $\bar{\theta}$, we get the expression for $\bar{\theta}$ as

$$-B\bar{\theta} = \emptyset'(\bar{x})^2\frac{(\bar{z}^4 - \bar{h}^3\bar{z})}{12} + \left[\frac{1}{\bar{h}} - \frac{\bar{h}}{2}\emptyset'(\bar{x})\right]^2\left(\frac{\bar{z}^2 - \bar{z}\bar{h}}{2}\right)$$
$$+ \emptyset'(\bar{x})\left[\frac{1}{\bar{h}} - \frac{\bar{h}}{2}\emptyset'(\bar{x})\right]\left(\frac{\bar{z}^3 - \bar{h}^2\bar{z}}{3}\right)$$

We note that $\bar{\theta}$ satisfies the boundary conditions at $\bar{z} = 0$ and $\bar{z} = \bar{h}(\bar{x})$ and for the case of linear profile, $\emptyset'(\bar{x})$ is

$$\emptyset'(\bar{x}) = -\frac{6k_1^2}{(k_1\bar{x} + k_2)^2} + \frac{8k_1^2}{(k_1\bar{x} + k_2)^3}$$

So $\bar{\theta}$ can be determined from equation

Here pressure is not zero and the bearing with non-parallel plates can support load easily.

Thus, the expression for temperature of a lubricant in slider bearing is derived. For the case when viscosity is constant, the expression is

$$B = \frac{k}{\mu u_p^2}(T_c - T_1)$$

This gives conditions on possible values of various parameters of the bearing. However, when viscosity of the lubricant changes with temperature, due to nature of PDE involved, an explicit solution could not be derived.

3. Variable Viscosity

In the previous works, viscosity was considered constant. Here, variable viscosity is considered. For the liquid, the viscosity decreases with the temperature. From (3) and (4) we have

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\left(v\,\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial z}\left(v\,\frac{\partial u}{\partial x}\right) \tag{15}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial z}\left(v\,\frac{\partial w}{\partial x}\right) \tag{16}$$

As in the previous work, the dimensional analysis give, where, $v = v_0 \bar{v}$

$$\frac{\partial \bar{p}}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}\left(v \frac{\partial \bar{u}}{\partial \bar{z}}\right) \tag{17}$$

$$\frac{\partial \bar{p}}{\partial \bar{z}} = 0 \tag{18}$$

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{w}}{\partial \bar{z}} = 0$$

From (17), using $v = v_0 \bar{v}$

$$\frac{\partial \bar{p}}{\partial \bar{z}} = v_0 \frac{\partial}{\partial \bar{z}}\left(\bar{v} \frac{\partial \bar{u}}{\partial \bar{z}}\right) \text{z}$$

Similarly, the energy equation reduces to

$$\bar{v} \left(\frac{\partial \bar{u}}{\partial \bar{z}}\right)^2 = -B \frac{\partial^2 \bar{\theta}}{\partial \bar{z}^2} \tag{19}$$

where,

$$B = \frac{k}{v_0 u_p^2} (T_c - T_1)$$

Let $\bar{v} = \frac{\alpha}{\bar{\theta}}$, as the viscosity changes with temperature, for the liquid.

From (18),

$$\bar{p} = \emptyset(\bar{x})$$

From (17)

$$\frac{\partial \bar{u}}{\partial \bar{x}} = \frac{(\emptyset'(x)\bar{z} + c_1(\bar{x}))}{v}$$

From (19),

$$\frac{\partial^2 \bar{\theta}}{\partial \bar{z}^2} = -\frac{\bar{\theta}}{\alpha B} \left(\emptyset'(x)\bar{z} + c_1(\bar{x})\right)^2$$

This equation is not easy to solve. If we can show that viscosity of the liquid changes with the increase in the temperature. Then the solution will have stability. This is the case of further investigation.

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A Generalized Common Fixed Point in Fuzzy Metric Space

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Abstract: The aim of the present paper is to prove a common fixed point theorem for weakly compatible self mappings in a fuzzy metric space which generalizes and improves various well-known comparable results.

Key Words: Common fixed point, Fuzzy metric space, Weakly compatible maps.

1. Introduction

The study of common fixed points of mappings in a fuzzy metric spacesatisfying certain contractive conditions has been at the center of vigorous research activity. The concept of fuzzy sets was initiated by Zadeh [19] in 1965. With the concept of fuzzy sets, the fuzzy metric space was introduced by Kramosil and Michalek [7]. Grabiec [5] proved the contraction principle in the setting of the fuzzy metric space which was further generalization of results by Subrahmanyam [17] for a pair of commuting mappings. Also, George and Veeramani [4] modified the notion of fuzzy metric spaces with the help of continuous t-norm, by generalizing the concept of probabilistic metric space to fuzzy situation. In 1999, Vasuki [18] introduced the concept of R-weak commutativity of mappings in fuzzy metric space and Pant [10] introduced the notion of reciprocal continuity of mappings in

metric spaces. Also, Jungck and Rhoades [6] defined a pair of self mappings to be weakly compatible if they commute at their coincidence points. Balasubramaniam *et.al* [1] proved a fixed point theorem, which generalizes a result of Pant for fuzzy mappings in fuzzy metric space.

Pant and Jha [11] proved a fixed point theorem that gives an analogue of the results by Balasubramaniam *et. al* [1] by obtaining a connection between the continuity and reciprocal continuity for four mappings in fuzzy metric space. Recently, Kutukcu *et. al* [8] has established a common fixed point theorem in a fuzzy metric space by studying the relationship between the continuity and reciprocal continuity which is a generalization of the results of Mishra [9] and also gives an answer to the open problem of Rhoades [13] in fuzzy metric space.

The purpose of this paper is to prove a common fixed point theorem for four self mappings in fuzzy metric space under the weak contractive conditions, by relaxing the continuity and reciprocal continuity conditions of mappings and even the completeness. Our result generalizes and improves various other similar results of fixed points. We also give an example to illustrate our main theorem.

We have used the following notions:

Definition 1.1([19]) Let X be any set. A fuzzy set A in X is a function with domain X and values in $[0, 1]$.

Definition 1.2([4]) A binary operation $*:[0,1] \times [0, 1] \rightarrow [0, 1]$ is called a continuous *t*-norm if, $([0, 1], *)$ is an abelian topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for all a, b, c, d in $[0, 1]$.

Example: $a * b = ab$, $a * b = \min\{a, b\}$.

Definition 1.3([4]) The triplet $(X, M, *)$ is called a fuzzy metric space (shortly, a FM-space) if, X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions: for all x, y, z in X, $s, t > 0$,

(i) $M(x, y, 0) = 0$, $M(x, y, t) > 0$;

(ii) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,

(iii) $M(x, y, t) = M(y, x, t)$,

(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(v) $M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$ is left continuous and $s, t > 0$,

In this case, M is called a fuzzy metric on X and the function $M(x, y, t)$ denotes the degree of nearness between x and y with respect to t. Also, we consider the following condition in the fuzzy metric space $(X, M,*)$:

(vi) $\lim_{t\to\infty} M(x, y, t) = 1$, for all $x, y \in X$.

It is important to note that every metric space (X, d) induces a fuzzy metric space $(X, M, *)$ where $a * b = \min\{a, b\}$ and for all $a, b \in X$, we have $M(x, y, t) = \frac{t}{t + d(x, y)}$, for all $t > 0$, and $M(x, y, 0) = 0$, so-called the fuzzy metric space induced by the metric d.

Definition 1.4([4]) A sequence $\{x_n\}$ in a fuzzy metric space $(X, M,*)$ is called Cauchy sequence if, $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ for every $t > 0$ and for each $p > 0$.

A fuzzy metric space $(X, M, *)$ is complete if, every Cauchy sequence in X converges in X.

Definition 1.5([4]) A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be convergent to x in X if, $\lim_{n\to\infty} M(x_n, x, t) = 1$ for each $t > 0$.

It is noted that since $*$ is continuous, it follows from the condition (iv) of Definition (1.3.) that the limit of a sequence in a fuzzy metric space is unique.

Definition 1.6([1]) Two self mappings A and S of a fuzzy metric space $(X, M, *)$ are said to be compatible if, $\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = p$ for some p in X.

Definition 1.7([6]) Two self mappings A and S of a fuzzy metric space $(X, M, *)$ are said to be weakly compatible if, they commute at coincidence points. That is, $Ax = Sx$ implies that $ASx = SAx$ for all x in X.

It is important to note that a compatible mappings in a metric space are weakly compatible but weakly compatible mappings need not be compatible [16].

Lemma 1.8([14]) Let $(X, M,*)$ be a fuzzy metric space. If there exists $k \in (0, 1)$ such that $M(x, y, kt) \geq M(x, y, t)$ then $x = y$.

Lemma 1.9([2]) Let $\{y_n\}$ be a sequence in a fuzzy metric space (X, M, *) with the condition (vi) of Definition (1.3). If there exists $k \in (0, 1)$ such that $M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t)$ for all $t > 0$ and $n \in \aleph$, the set of natural numbers, then the sequence $\{y_n\}$ is a Cauchy sequence in X.

If A, B, S and T are self mappings of fuzzy metric space $(X, M, *)$ in the sequel, we shall denote

$N(x, y, t) = M(Ax, Sx, t) * M(By, Ty, t) * M(Sx, Ty, t) * M(Ax, Ty, \alpha t) * M(Sx, By, (2-\alpha), t)$,

for all $x, y \in X$, $\alpha \in (0, 2)$ and $t > 0$.

2. Main Result

Theorem 2.1. Let $(X, M, *)$ be a fuzzy metric space with additional condition (vi) and with $a * a \geq a$ for all $a \in [0, 1]$. Let A, B, S and T be mappings from X into itself such that

(i) $AX \subseteq TX$, $BX \subseteq SX$,

(ii) there exists $k \in (0, 1)$ such that $M(Ax, By, kt) \geq N(x, y, t)$,

for all $x, y \in X$, $\alpha \in (0, 2)$ and $t > 0$.

If one of AX, BX, SX and TX is complete subspace of X and if the pair (A, S) and (B, T) are weakly compatible then A, B, S and T have a unique common fixed point in X.

Proof: Let $x_0 \in X$ be an arbitrary point. Then, since $AX \subseteq TX$, $BX \subseteq SX$, there exists $x_1, x_2 \in X$ such that $Ax_0 = Tx_1$ and $Bx_1 = Sx_2$. Inductively, we construct the sequences $\{y_n\}$ and $\{x_n\}$ in X such that

$y_{2n} = Ax_{2n} = Tx_{2n+1}$ and $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$, for $n = 0, 1, 2, \ldots$.

Now, putting $x = x_{2n}$, $y = x_{2n+1}$ for all $t > 0$ and $\alpha = 1 - q$ with $q \in (0, 1)$ in (ii), we have

$M(Ax_{2n}, Bx_{2n+1}, kt) \geq M(Ax_{2n}, Sx_{2n}, t) * M(Bx_{2n+1}, Tx_{2n+1}, t) * M(Sx_{2n}, Tx_{2n+1}, t) * M(Ax_{2n}, Tx_{2n+1}, (1-q)\,t) * M(Sx_{2n}, Bx_{2n+1}, (1+q)\,t)$.

That is,

$M(y_{2n}, y_{2n+1}, kt) \geq M(y_{2n-1}, y_{2n}, t) * M(y_{2n}, y_{2n+1}, t) * M(y_{2n-1}, y_{2n}, t) * M(y_{2n-1}, y_{2n+1}, (1+q)t) \geq M(y_{2n-1}, y_{2n}, t) * M(y_{2n}, y_{2n+1}, t) * M(y_{2n-1}, y_{2n}, t) * M(y_{2n-1}, y_{2n+1}, qt) \geq M(y_{2n-1}, y_{2n}, t) * M(y_{2n}, y_{2n+1}, t) * M(y_{2n}, y_{2n+1}, qt)$.

Since t – norm $*$ is continuous, letting $q \to 1$, we have

$M(y_{2n}, y_{2n+1}, kt) \geq M(y_{2n-1}, y_{2n}, t) * M(y_{2n}, y_{2n+1}, t) * M(y_{2n}, y_{2n+1}, t) \geq M(y_{2n-1}, y_{2n}, t) * M(y_{2n}, y_{2n+1}, t)$.

It follows that, $M(y_{2n}, y_{2n+1}, kt) \geq M(y_{2n-1}\}, y_{2n}, t) * M(y_{2n}, y_{2n+1}, t)$.

Similarly, $M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t)$. Therefore, for all n even or odd, we have

$$M(y_n, y_{n+1}, kt) \geq M(y_{n-1}\}, y_n, t) * M(y_n, y_{n+1}, t).$$

Consequently, $M(y_n, y_{n+1}, t) \geq M(y_{n-1}\}, y_n, k^{-1}t) * M(y_n, y_{n+1}, k^{-1}t)$.

Using a simple induction, we have

$$M(y_n, y_{n+1}, t) \geq M(y_{n-1}, y_n, k^{-1}t) * M(y_n, y_{n+1}, k^{-m}t).$$

Since $M(y_n, y_{n+1}, k^{-m}t) \to 1$ as $m \to \infty$, it follows that

$M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t)$, for all $n \in$ N and $t > 0$. Therefore, by Lemma (1.8.), the sequence $\{y_n\}$ is a Cauchy sequence.

Now, suppose TX is a complete subspace of metric space X. Then, the subsequence $y_{2n} = Tx_{2n+1}$ is also a Cauchy sequence in TX and hence has a limit u. Let $v \in T^{-1}u$, then $Tv = u$. Since y_{2n} is convergent, so y_n is convergent to u and hence y_{2n+1} also converges to u.

Now, setting $x = x_{2n}$ and $y = v$ in (ii), with $\alpha = 1$, we have

$M(Ax_{2n}, Bv, kt) \geq M(Ax2n, Sx_{2n}, t) * M(Bv, Tv, t) * M(Sx_{2n}, Tv, t) * M(Ax_{2n}, Tv, t) * M(Sx_{2n}, Bv, t)$,

letting $n \to \infty$, we have $M(u, Bv, kt) \geq M(u, Bv, t)$. This implies that $Bv = u$. Also, since $BX \subseteq SX$, so

$u = Bv$ implies that $u \in SX$. Let $w \in S^{-1}$u, then $Sw = u$. Again, setting $x = w$ and $y = x_{2n+1}$ in (ii) with

$\alpha = 1$, we get

$M(Aw, Bx_{2n+1}, kt) \geq M(Aw, Sw, t) * M(Bx_{2n+1}, Tx_{2n+1}, t) * M(Sw, Tx_{2n+1}, t) * M(Aw, Tx_{2n+1}, t) * M(Sw, Bx_{2n+1}, t)$.

Letting $n \to \infty$, we get $M(Aw, u, kt) \geq M(Aw, u, t)$, which implies that $u = Aw$.

Thus, we have $u = Tv = Bv = Aw = Sw$ ... (1)

Now, since $u = Tv = Bv$, so by the weak compatibility of (B, T), it follows that $BTv = TBv$ and so we get $Bu = BTv = TBv = Tu$. Also, since $u = Sw = Aw$, so by the weak compatibility of (A, S), it follows that $ASw = SAw$ and so we have $Au = ASw = SAw = Su$. Thus, from (ii) with $\alpha = 1$, we have

$M(Aw, Bu, kt) \geq M(Aw, Sw, t) * M(Bu, Tu, t) * M(Sw, Tu, t) * M(Aw, Tu, t) * M(Sw, Bu, t)$,

That is, $M(u, Bu, kt) \geq M(u, Bu, t)$, which is a contradiction. This implies that $u = Bu$. Similarly, using (ii) with $\alpha = 1$, one can show that $Au = u$. Therefore, we have $u = Bu = Tu = Au = Su$. Hence, the point u is a common fixed point of A, B, S and T. If we assume SX is complete, then the argument analogue to the previous completeness argument proves the theorem. If AX is complete, the $u \in AX \subseteq TX$. Similarly, if BX is complete, then $u \in BX \subseteq SX$.

The uniqueness of a common fixed point of the mappings A, B, S and T be easily verified by using (ii). In fact, if u' be another fixed point for mappings A, B, S and T. Then, for $\alpha = 1$, we have

$M(u, u', kt) = M(Au, Bu', kt) \geq M(Au, Su, t) * M(Bu', Tu', t) * M(Su, Tu', t) * M(Au, Tu', t) * M(Su, Bu', t), \geq M(u, u', t)$, and hence, we get $u = u'$.

This completely establishes the theorem.

We now give an example to illustrate the above theorem.

Example: Let $X = [2, 20]$ and M be the usual fuzzy metric space on $(X, M, *)$. Define A, B, S and $T : X \to X$ as follows:

$A2 = 2$, $\quad Ax = 3$ if, $x > 2$;

$Bx = 2$ if, $x = 2$ or > 5, $Bx = 6$ if, $2 < x \leq 5$;

$S2 = 2$, $\quad Sx = 6$ if, $x > 2$;

$T2 = 2$, $\quad Tx = 12$ if, $2 < x \leq 5$, $\quad Tx = x - 5$ if, $x > 5$.

Also, we define $M(Ax, By, t) = \dfrac{t}{[t + d(x, y)]}$, for all x, y in X and for all $t > 0$.

Then, for $\alpha = 1$, the

pair (A, S) and (B, T) are weakly compatible mappings. Also, these mappings satisfy all the conditions of the above theorem and have a unique common fixed point $x = 2$.

Remarks: As the earlier fixed point theorems have been established using stronger contractive conditions, so our results generalizes the results of Kutukcu *et.al*[8] and that of Sharma [14], Mishra [9]. Consequently, it improves and unifies the results of Balasubramaniam *et.al* [1], Chugh and Kumar

[3], Pant and Jha [11], Pant [12], Sharma *et.al* [15] and other similar results for fixed points.

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Common Fixed Point Theorem for Four Mappings in Dislocated Quasi–Metric Spaces

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Abstract: We have proved a common fixed point theorem for four mappings in dislocated metric spaces.

Keywords: dislocated quasi-metric,dq-limit, dq-convergent, dq-cauchy's sequence, fixed point, weakly compatible.

1. Introduction

B.E. Rhoades [2] established various definitions of contractive mappings. In 1992 Banach proved fixed point theorem for contraction mapping in complete metric space. In 1996 Jungck [5] introduced the concept of weakly compatible.

In 2005 F.M. Zeyada, G.H. Hassan, M.A. Ahmed [4] established various definitions of dislocated quasi-metric space. A Isufati [1] proved some fixed point theorems for a single and a pairs of mappings in dislocated metric space. In this paper we proved common fixed point theorem for four maps in dislocated quasi-metric space.

2. Preliminaries

Definition 2.1 [4] Let X be a nonempty set and let $d:X\times X \to [0, \infty) \to [0, \infty)$ be a function satisfying following conditions:

(i) $d(x,y) = d(y,x) = 0$, implies $x = y$,

(ii) $d(x,y) \leq d(x,z)+ d(z,y)$, for all $x,y,z \in X$.

Then d is called a dislocated quasi-metric on X.

If d satisfies d(x,y) = d(y,x), then it is called dislocated metric.

Definition 2.2 [4] A sequence $\{x_n\}$ in dq-metric space (dislocated quasi-metric space) (X,d) is called Cauchy sequences if for given $\varepsilon > 0$, $\exists$ $n_0 \in N$, such that $\forall$ m,n $\geq n_0$, implies $d(x_m,x_n) < \in$ or $d(x_n,x_m) < \in$ i.e. $\min\{d(x_m, x_n), d(x_n, x_m)\} < \varepsilon$.

Definition 2.3 [4] A sequence $\{x_n\}$ dislocated quasi-converges to x if

$$Lim_{n\to\infty}\, d(x_n, x) = lim_{n\to\infty}\, d(x, x_n) = 0$$

In this case x is called a dq-limit of $\{x_n\}$ and we write $x_n \to x$.

Lemma: dq-limit in a dq-metric space are unique.

Definition 2.4 [4] A dq-metric space (X,d) is called complete if every Cauchy sequence in it is a dq-convergent.

Definition 2.5 [4] Let (X,d_1) and (Y,d_2) be dq-metric spaces and let $f{:}X \to Y$ be a function. Then f is continuous to $x_0 \in X$, if for each sequence $\{x_n\}$ which is d_1 -q convergent to x_0, the sequence $\{f(x_n)\}$ is d_2-q convergent to $f(x_0)$ in Y.

Definition 2.6 [4] Let (X,d) be a dq-metric space. A map T: X$\to$X is called contraction if there exists $0 \leq \lambda < 1$ such that

$$d(Tx,Ty) \leq \lambda d(x,y) \text{ for all } x,y \in X.$$

Definition 2.7[9] Let A and S be mapping from a metric space (X,d) into itself. A and S are said to be weakly compatible if they commute at their coincidence points, that is, Ax = Sx for some x$\in$ X implies that ASx = Sax.

Theorem 2.8 [4] Let (X,d) be a dq-metric space and let T: X$\to$X be a continuous contraction mapping. Then T has unique fixed point.

3. Main Results

Theorem 3.1 Let (X,d) be a complete dislocated metric space. Let F,G, S,T : X$\to$X be continuous mapping satisfying:

(i) $S(X) \subseteq G(X), T(X) \subseteq F(X)$ and T(X) or S(X) is a closed subset of X

(ii) The pairs (S,F) and (T,G) are weakly compatible

$$d(Sx,Ty) \le h\max\left\{d(Fx,Gy), d(Fx,Sx), d(Gy,Ty), \frac{d(Fx,Ty)+d(Gy,Sx)}{2}\right\}$$

for all x,y$\in X$ and 0<h<1. Then f , g, s and t have common fixed point.

Proof: Suppose x_0 is an arbitrary point of X and define the sequence $\{y_n\}$ in X such that

$$Y_{2n} = Sx_{2n} = Gx_{2n+1}$$

$$Y_{2n+1} = Tx_{2n+1} = Fx_{2n+2}$$

Consider

$$d(y_{2n}, y_{2n+1}) = d(Sx_{2n}, Tx_{2n+1})$$

$$\le h\max\left\{d(Fx_{2n}, Gx_{2n+1}), d(Fx_{2n}, Sx_{2n}), d(Gx_{2n+1}, Tx_{2n+1}), \frac{d(Fx_{2n}, Tx_{2n+1})+d(Gx_{2n+1}, Sx_{2n})}{2}\right\}$$

$$= h\max\left\{d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \frac{d(y_{2n-1}, y_{2n})+d(y_{2n}, y_{2n})}{2}\right\}$$

$$d(y_{2n}, y_{2n+1}) \le hd(y_{2n-1}, y_{2n})$$

Similarly

$$d(y_{2n-1}, y_{2n}) \le hd(y_{2n-2}, y_{2n-1})$$

and so $d(y_{2n}, y_{2n+1}) \le h^2 d(y_{2n-2}, y_{2n-1})$

In this way we have

$$d(y_{2n}, y_{2n+1}) \le h^n d(y_1, y_0)$$

Since o<h<1, as $h^n \to 0$ as $n \to \infty$. Thus $\{y_n\}$ is Cauchy sequence in a complete dislocated metric space X. There exists a point u$\in X$ such that $\{y_n\} \to$u.

Therefore the subsequences $\{Sx_{2n}\} \to u$, $\{Tx_{2n+1}\} \to u$, $\{Gx_{2n+1}\} \to u$ and $\{Fx_{2n+2}\} \to u$.

Assume that G(X) is a closed subset of X. Then $\exists$ v$\in X$ such that

Gv = u.

If Tv $\neq$u then, we obtain

$$d(Sx_{2n}, Tv)$$

$$\leq h\max\left\{d(Fx_{2n}, Gv), d(Fx_{2n}, Sx_{2n}), d(Gv, Tv), \frac{d(Fx_{2n}, Tv) + d(Gv, Sx_{2n})}{2}\right\}$$

$$= h\max\left\{d(y_{2n-1}, Gv), d(y_{2n-1}, y_{2n}), d(Gv, Tv), \frac{d(y_{2n-1}, Tv) + d(Gv, y_{2n})}{2}\right\}$$

As $n \to \infty$, we get

$$d(u, Tv) \leq h\max\left\{d(u, Gv), d(u,u), d(Gv, Tv), \frac{d(u, Tv) + d(Gv, u)}{2}\right\}$$

$$< d(u,Tv)$$

It follows that Tv = u = Gv. Since B and T are weakly compatible,

we have TGv = GTv and so Tu = Gu.

If u $\neq$Bu we get

$$d(Sx_{2n}, Tu)$$

$$\leq h\max\left\{d(Fx_{2n}, Gu), d(Fx_{2n}, Sx_{2n}), d(Gu, Tu), \frac{d(Fx_{2n}, Tu) + d(Gu, Sx_{2n})}{2}\right\}$$

$$= h\max\left\{d(y_{2n-1}, Gu), d(y_{2n-1}, y_{2n}), d(Gu, Tu), \frac{d(y_{2n-1}, Tu) + d(Gu, y_{2n})}{2}\right\}$$

As $n \to \infty$, we get

$$d(u, Tu) \leq h\max\left\{d(u, Gu), d(u,u), d(Gv, Tu), \frac{d(u, Tu) + d(Gu, u)}{2}\right\}$$

$$< d(u,Tu)$$

and so Tu = u

since $T(X) \subseteq F(X)$, $\exists$ w$\in$ X such that Fw = u.

If Sw $\neq$u, we have

$$d(Sw, Tu) \leq h\max\left\{d(Fw, Gu), d(Fw, Sw), d(Gu, Tu), \frac{d(Fw, Tu) + d(Gu, Sw)}{2}\right\}$$

And it follows that

$$d(Sw, u) \leq h\max\left\{d(Fw, u), d(Fw, Sw), d(u, Tu), \frac{d(Fw, Tu) + d(u, Sw)}{2}\right\}$$

$$< d(Sw,u)$$

This implies that Sw = u and hence Sw = Fw = u.

Since S and F are weakly compatible

SFw = FSw and so

Su = Fu.

If Su ≠ u then, we get

d(Su,u) = d(Su,Tu)

$$\leq h\max\left\{d(Fu,u), d(Fu,Su), d(u,Tu), \frac{d(Fu,u)+d(u,Su)}{2}\right\}$$

$$= d(Su,u)$$

And so Su = u. Thus Su = Tu = Fu = Gu = u, that is u is a common fixed point for S,T,F,G.

Uniqueness of common fixed point:

Let u and v be a common fixed point of F , G, S and T. Then

$$d(u,v) \leq d(Su,Tv) \leq h\max\left\{d(Fu,Gv), d(Fu,Su), d(Gv,Tv), \frac{d(Fu,Tv)+d(Gv,Su)}{2}\right\}$$

$$\leq h\max\left\{d(u,v), d(u,u), d(v,v), \frac{d(u,v)+d(v,u)}{2}\right\}$$

$$\leq h\max\{d(u,v), d(u,u), d(v,v),\}$$

Replacing v by u, we get d(u,v) ≤ h d(u,u). Since 0<h<1, we have d(u,u) = 0. Similarly we have d(v,v) = 0.

In this way we have d(u,v) ≤ h d(u,v). Since 0<h<1,

we have d(u,v) = 0.

Similarly, we have d(v,u) = 0 and so u = v.

Hence u is unique common fixed point of F, G, S and T.

Hence the proof is completed.

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Some Properties and Applications of Conformal Metrics

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Abstract: Some basic and invariant properties as well as some applications of conformal metrics will be discussed in this paper. Applications discussed here are based on Poincar´e metric.

Keywords: Conformal metric, Poincar´e metric, isometric property, distance decreasing property, holomorphic functions, rotation.

1. Introduction

Conformal metrics have been instrumental in connecting complex analysis, partial differential equations, and differential geometry. Schwartz, Poincare, Picard, Ahlfors and other Mathematician used them in different context. Conformal metrics have been used to gain deep insights into complex analysis on the disc and aren useful to find beautiful proofs of some of the important theorems in complex analysis. They are commonly used in hyperbolic geometry or Riemann surfaces. The study of conformal metrics leads to the study of the curvature and its applications. Some of the metrics, for example: Caratheodory and Kobayashi, are invariant under conformal mappings and are motivated by the external problem that arises from the Riemann Mapping Theorem.

In section two of this paper, I will define conformal metric on a connected open set $\Omega \subset C$, the length of continuously differentiable curve and distance between two points in the metric. Section three discusses several examples of the conformal metrics. In section four, some basic properties of the conformal metrics will be discussed in the form of two propositions. Invariant properties of conformal metric will be discussed in section five. The last section is devoted in explaining some applications of conformal metrics, specially the Poincare metric.

This paper does not contain new results but some of the proofs have been worked out in detail in a systematic manner. Sections are designed to present the material as simple as possible. No referencing in the main text have been included but they are provided at the end of the paper.

2. Definitions Relevant to Conformal Metrics

In this section, starting with the classical definition of metric, we define the conformal metric, the length of a curve in the conformal metric and the distance between two points in the conformal metric.

Definition 2.0.1 (Metric) Let X be a set. A metric γ for X is a function $\gamma : X \times X \to R$ satisfying for all x, y, z in X

a) $\gamma(x, y) = \gamma(y, x)$

b) $\gamma(x, y) \geq 0$

c) $\gamma(x, y) = 0$ iff $x = y$

d) $\gamma(x, y) \leq \gamma(x, z) + (z, y)$.

The function γ is also called a distance function and the space (X, γ) is called a metric space.

Examples: a) The set of real numbers, $\mathbb{R}$, is a metric space with metric $\gamma(x, y) = |x - y|$. More generally, Euclidean n-space with the Euclidean distance is a metric space.

b) Any normed vector space is a metric space with metric defined by $\gamma(x, y) = \|y - x\|$.

Definition 2.0.2 (Conformal Metric) If $\Omega \subseteq \mathbb{C}$ is a domain(i.e. a connected open set), then a conformal metric on is a continuous function $\rho(z) \geq 0$ in Ω. If $z \in \Omega$ and $\xi \in \mathbb{C}$, then we define the length of ξ at z to be $\|\xi\|_{\rho, z} = \rho(z) \cdot |\xi|$ where $|\xi|$ denotes Euclidean length.

Definition 2.0.3 (The length of curve) Let $\Omega \subseteq \mathbb{C}$ be a domain and ρ a metric on Ω. If γ :

$[a, b] \to \Omega$ is a continuously differentiable curve then we define its length in the metric ρ to be,

$$l\rho(\gamma) = \int_a^b \| \gamma(t)\|\rho,\gamma(t)dt$$

It is important to note that the length of a piecewise continuously differentiable curve is defined to be the sum of the lengths of its continuously differentiable pieces and is independent of its parametrization.

Definition 2.0.4 (The distance between two points in the metric) Let ρ be a metric given on a planar domain Ω. Suppose $C_\Omega(P,Q)$ is the collection of all piecewise continuously differentiable curves $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = P$ and $\gamma(1) = Q$. Then the ρ -metric distance from P to Q is defined by

$$d_\rho(P,Q) = \inf\{l_\rho(\gamma) : \gamma \in C_\Omega(P,Q)\}$$

We will see that this notion of distance between two points satisfies the metric axioms listed above with the possible exception of (c) because $d_\rho(P,Q)$ may equal to 0 even if $P \neq Q$.

3. Examples of Conformal Metrics

Example 3.0.1 (Euclidean Metric) Let Ω be any domain in $\mathbb{C}$. Define $\rho(z) = 1$ $\forall z \in \Omega$. Then ρ is a conformal metric.

This metric yields that if $z \in \Omega$ and $\xi \in \mathbb{C}$ then,

$$\| \xi \|\rho, z = \rho(z) \,.\, |\xi| = |\xi|$$

This choice of metric gives the standard Euclidean notion of vector length and is independent of z and is called the Euclidean metric. If Ω is a convex subset of $\mathbb{C}$ and ρ is the Euclidean metric, then $d\rho(P,Q)$ is the ordinary Euclidean distance between P and Q. If Ω is a non-convex subset of $\mathbb{C}$ and ρ is the Euclidean metric, then there may be no shortest curve connecting P and Q.

Example 3.0.2 (Poincar´e Metric) Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, the unit disc. Let $\rho(z) = \frac{1}{1 - |z|^2}$.

Then ρ is a conformal metric. This metric is called the Poincare metric.

We calculate the length of the curve $\gamma(t) = t$, $0 \leq t \leq 1 - \mathcal{E}$ for fixed $\mathcal{E} > 0$ in the Poincare metric

$\rho(z) = \frac{1}{1 - |z|^2}$. Since $\gamma(t) = t$, we have, we have that $\gamma(t) = 1$ and hence $|\gamma(t)| = 1$.

But,

$$l\rho(\gamma) = \int_0^{1-\varepsilon} ||\gamma(t)||_{\rho,\gamma(t)}dt$$

$$= \int_0^{1-\varepsilon} \frac{|\gamma(t)|}{1-|\gamma(t)|^2}dt$$

$$= \int_0^{1-\varepsilon} \frac{1}{1-t^2}dt = \frac{1}{2}\ln\left(\frac{2-\mathcal{E}}{\mathcal{E}}\right)$$

If we consider $\mathcal{E} \to 0^+$, then we get $l_\rho(\gamma) \to \infty$. This suggests that the boundary $\partial\mathbb{D}$ is infinitely far from the origin in the Poncare metric.

Example 3.0.3 (Spherical Metric) The metric defined by $\sigma(z) = \frac{2}{1+|z|^2}$ on $\mathbb{C}$ is called the spherical metric. This metric induces the distance between two points in the sphere under the stereographic projection.

4. Some Basic Properties of Conformal Metrics

We now state and prove two propositions that summerize some basic properties of onformal metrics.

Proposition 4.1 The distance between two points P and Q in the conformal metric ρ i.e. $d_\rho(P,Q)$ is a metric when $\rho > 0$.

Proof:

The properties $d_\rho(P,Q) = d_\rho(Q, P)$, $d_\rho(P,Q) \geq 0$ and $d_\rho(P, P) = 0$ are obvious. To show $d_\rho(P,Q) = 0$ implies $P = Q$, we use contrapositive. Fix P and Q in Ω with $P \neq Q$. Let $C_\Omega(P,Q) = \{\gamma : [0, 1] \to$ s.t. γ is piecewise continuously differentiable, $\gamma(0) = P$ and $\gamma(1) = Q\}$ then,

$$d_\rho(P,Q) = \inf\{(l_\rho(\gamma) : \gamma \in C_\Omega(P,Q)\}$$

$$= \inf\{\int_0^1 \rho(\gamma(t))|\gamma'(t)|dt : \gamma \in C_\Omega(P,Q)\} \tag{4.1}$$

Since Ω is open, there exists $r > 0$ s.t. $\overline{D(P, r)} \subset \Omega$ and $|P - Q| > r$. As ρ is continuous and

$D(P, r)$ is compact, there exists $c > 0$ such that $c = \min\{\rho(z) : z \in \overline{D(P, r)}\}$. Fix $\gamma \in C_\Omega(P,Q)$.

Let $t_0 = \inf\{t : \gamma(t) \notin D(P, r)\}$. Then,

$$\int_0^1 \rho(\gamma((t))|\gamma(t)|dt \geq \int_o^{to} \rho(\gamma((t))|\gamma(t)|dt \geq c \int_o^{to} |\gamma(t)|dt \geq cr \quad (4.2)$$

Since this is true for any $\gamma \in C_\Omega(P,Q)$, we have $d_\rho(P,Q) \geq cr$. Hence $d_\rho(P,Q) > 0$.
For the triangle inequality, let P,Q and R be points in Ω. Then by definition,

$$d_\rho(P,R) = \inf\{l_\rho(\gamma) : \gamma \in C_\Omega(P,R)\}.$$

Choose $\gamma_1 \in C_\Omega(P,Q)$ and $\gamma_2 \in C_\Omega(Q,R)$ and join them to form a curve $\gamma_3 \in C_\Omega(P,R)$. Then

$l_\rho(\gamma_3) = l_\rho(\rho_1) + l_\rho(\gamma_2)$.

Now,

$d_\rho(P,R) \leq l_\rho(\gamma_3) = l_\rho(\gamma_1) + l_\rho(\gamma_2)$.

Taking infimum over γ_1 and γ_2, we get,

$$\begin{aligned} d_\rho(P,R) &\leq \inf\{l_\rho(\gamma_1) : \gamma_1 \in C_\Omega(P,Q)\} + \inf\{l_\rho(\gamma_2) : \gamma_2 \in C_\Omega(Q,R)\} \\ &= d_\rho(P,Q) + d_\rho(Q,R). \end{aligned}$$

Thus, we conclude that the distance between two points in a conformal metric ρ is metric when $\rho > 0$.

Proposition 4.2 The metric $d_\rho(P,Q)$ gives the same topology as the Euclidean metric when $\rho > 0$.

Proof:

From the previous proposition, we know that $d_\rho(P,Q)$ is a metric. We wish to show this metric gives the same topology as Euclidean metric.

Let $\{pn\} \subset$ and $p \in \Omega$. It is enough to show that $|pn - p| \to 0$ iff $d_\rho(pn, p) \to 0$.

($\Rightarrow$) Suppose that $d_\rho(pn, p) \to 0$. We need to show $|pn - p| \to 0$. Suppose on the contrary that $|pn - p| \nrightarrow 0$, so there exists $r > 0$ such that we can find infinitely many pn outside D(p, r). But, by the same construction as proposition 4.1 and from equation 4.2, since there exists $c > 0$ such that $\rho \geq c$ in D(p, r), for infinitely many n, $d(pn, p) \geq cr > 0$, a contradiction.

($\Leftarrow$) Suppose $|pn - p| \to 0$, we need to show that $d_\rho(pn, p) \to 0$. Choose $r > 0$ s.t. $\overline{D(p, r)} \subset \Omega$.

Since $|pn - p| \to 0$, there exist N s.t. $pn \in D(p, r)$ if $n \geq N$. Since ρ is continuous and $\overline{D(p, r)}$ is compact, there exists M such that $M = \max\{\rho(z) : z \in \overline{D(p, r)}\}$. Fix n. If γ_0 is defined on [0,1] by

$\gamma_o(t) = (1 - t)p + tpn$, then

$d_\rho(pn, p) \leq l_\rho(\gamma_0)$

$$= \int_0^1 ||\gamma_0(t)||_\rho, \gamma_0(t)dt$$

$$= \int_0^1 \rho(\gamma_0(t))|\gamma 0(t)| \leq M|pn - p|$$

Hence, $d_\rho(pn, p) \to 0$ as $n \to \infty$. This completes the proof that the metric $d_\rho(P,Q)$ gives the same topology as the Euclidean metric when $\rho > 0$.

5. Invariance Properties of Conformal Metrics

On Euclidean spaces, we have the Euclidean metric which is invariant under translations and which scales properly under dilations. The conformal metric also enjoys similar features. Before we explain those properties, we give the definitions of some important terms and state the Schwarz lemma and Schwarz-Pick theorem without proof.

Definition 5.0.5 (Pullback of a metric) Let Ω_1 and $\Omega 2$ be two planar domains and let $f : \Omega_1 \to \Omega_2$ be a continuously differentiable function. Assume that Ω_2 is equipped with a conformal metric ρ. Then the pullback of ρ under the map f is the conformal metric on Ω_1 defined by

$$f^* \rho(z) = \rho(f(z)) \cdot \left|\frac{\partial f}{\partial z}\right|.$$

Remark: The pullback of any conformal metriç under a conjugate holomorphic function f is a zero metric: If $f = \bar{g}$ and g is holomorphic then

$$f^* \rho(z) = \rho(f(_z)) \cdot \left|\frac{\partial f}{\partial z}\right|$$

But

$$\left|\frac{\partial f}{\partial z}\right| = \left|\frac{\partial \bar{g}}{\partial z}\right| = \left|\frac{\partial \bar{g}}{\partial z}\right| = |\bar{o}| = 0$$

Definition 5.0.6 (Isometry) Let Ω_1 and Ω_2 be two planar domains equipped with conformal metrics Ω_1 and Ω_2 respectively. Let $f : \Omega_1 \to \Omega_2$ be a one-to-one, onto continuosly differentiable map. If $f^* \rho_2(z) = \rho_1(z) \forall_z \Omega_1$ then f is called an isometry of the pair (Ω_1, ρ_1) with the pair (Ω_2, ρ_2).

We now state without proof two standard results from complex analysis.

Theorem 5.1 (The Schwarz Lemma) If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and $f(0) = 0$ then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. If either $|f(z)| = |z|$ for some $z \neq 0$ or $|f'(0)| = 1$ then f is rotation i.e.

$$f(z) = e^{iT}.z \text{ for some } T \in \mathbb{R}.$$

Theorem 5.2 (Schwarz-Pick) If $f : \mathbb{D} ! \mathbb{D}$ is holomorphic, $f(z_1) = w_1$ and $f(z_2) = w_2$ then

$$\frac{w_1 - w_2}{1 - w_1 - w_2} \leq \frac{z_1 - z_2}{1 - z_1 - z_2} \text{ and } |f'(z_1)| \leq \frac{1 - |w_1|^2}{1 - |z_1|^2}$$

Proposition 5.1 (Isometric Property) Let Ω_1 and Ω_2 be two planar domains equipped with metrics ρ_1 and ρ_2 respectively. If $f : \Omega_1 \to \Omega_2$ is holomorphic and is an isometry of (Ω_1, ρ_1) to (Ω_2, ρ_2) then the following properties hold.

a) If $\gamma: [a, b] \to \Omega_1$ is a continuously differentiabe curve then so is $f_*\gamma = f \circ \gamma$ and
$l_{\rho 1}(\gamma) = l_{\rho 2}(f_*\gamma)$.

b) If $P,Q \in \Omega_1$ then, $d\rho_1(P,Q) = d\rho_2(f(P), f(Q))$.

c) f^{-1} is also an isometry.

Proof:

a) By definition,

$$l\rho_2(f_*\gamma) = \int_a^b \| (f_*\gamma)'(t) \|_{\rho 2, f^*(t)}\, dt$$

$$= \int_a^b \| f\gamma((t)).\gamma(t) \|_{\rho 2, f^*(t)}\, dt \tag{5.1}$$

But,

$$\begin{aligned} \| f'(\gamma(t)) \cdot \gamma(t) \|_{\rho 2, f^*(t)} &= \rho_2(f(\gamma(t))) \cdot |f'(\gamma(t)) \cdot \gamma(t)| \\ &= |f'(\gamma(t))|\, \rho_2(f(\gamma(t))|\gamma(t)| \\ &= f^* \rho_2(\gamma(t)) \cdot |\gamma(t)| \\ &= \| \gamma(t) \|f_{*\rho 2,\gamma(t)} = k\ \gamma(t) \|_{\rho 1,\gamma(t)} \end{aligned} \tag{5.2}$$

So from 5.1, we get

$$l\rho_2\,(f_*\gamma) = \int_a^b \| \gamma(t) \|_{\rho 1,\gamma(t)}\, dt = l\rho_1\,(\gamma)$$

This completes the proof for a)

b) This part is immediate from part a)

c) To show f^{-1} an isometry, we have to show that $f^{-1*}\rho_1(w) = \rho_2(w) \forall w\ \Omega_2$. But, since f is an isometry and if $f(z) = w$, we have
$f^{-1*}\rho_1(w) = \rho_1(f^{-1}(w)) \,.\, |(f^{-1})'(w)|$

$$= \rho_1 \cdot \frac{1}{|f'(f^{-1}(w))|}$$

$$= \rho_2(f(z)) = \rho_2(w)$$

Before we see another important property of the Poincare metric, we define the distance decreasing in the metric.

Definition 5.2.1 (distance decreasing property) Let Ω_1 and Ω_2 be domains in $\mathbb{C}$ equipped with metrics Ω_1 and Ω_2 respectively, and $f : \Omega_1 \rightarrow \Omega_2$ be a holomorphic function. Then f is said to be a distance decreasing in the metric if

$$f^{*}\rho_2(z) \leq \rho_1(z) \text{ for every } z \in \Omega_1.$$

Now we prove that the Poincare metric satisfies the distance decreasing property.

Proposition 5.2 (Distance Decreasing Property of the Poincare metric) Let $f : \mathbb{D} \rightarrow \mathbb{D}$ be a holomorphic function and let ρ denote the Poincare metric on $\mathbb{D}$. Then

a) f is distance decreasing in the Poincare metric.

b) If $\gamma: [0, 1] \rightarrow D$ is a continuously differentiable curve then $l_{\rho 2}(f_{*}\gamma) \leq l\rho_1(\gamma)$.

c) If $P,Q \in \mathbb{D}$ then, $d_{\rho 2}(f(P), f(Q)) \leq d_{\rho 1}(P,Q)$.

Proof:

a) We know that

$$f^{*}\rho(z) = |f'(z)|\rho(f(z)) = |f'(z)|\cdot\frac{1}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2} = \rho(z),$$

where the inequality is true by Schwarz-Pick Theorem.

b) Following similar steps as 5.1 and 5.2, we get with the help of part a) that

$$l\rho(f_{*}\gamma) = \|\gamma(t)\|_{f^{*}\rho,\gamma(t)} \leq \|\gamma(t)\|_{\rho,\gamma(t)} = l_{\rho}(\gamma)$$

c) This part is an immediate consequence of part b).

6. Application to Poincar`e Metric

Recall that the Poincar`e metric is the conformal metric on the unit disc, $\mathbb{D}$ and is given by

$$\rho(z) = \frac{1}{1-|z|^2}.$$

The unit disc, $\mathbb{D}$, equipped with the Poincare metric is called the Poincare disc.

We use the notation Aut($\mathbb{D}$) to denote the group of invertible holomorphic functions from $\mathbb{D}$ to itself. So, $f \in$ Aut($\mathbb{D}$) means "f is conformal self map of the unit disc" or "f is the holomorphic map from $\mathbb{D}$ to $\mathbb{D}$ which is one to one and onto". These are two examples of automorphisms.

1. The rotation $\rho_T : \mathbb{D} ! \mathbb{D}$ defined by $\rho_T(z) = e^{iT}.z$, $_T \in \mathbb{R}$.

2. Consider a Mobius transformation $fa(z) = \dfrac{z - a}{1 - \bar{a}z}$ for fixed $a \in \mathbb{D}$. Then f_a maps $\mathbb{D}$ to $\mathbb{D}$, $fa(0) = a$, $fa(a) = 0$. It is an element of Aut($\mathbb{D}$) taking an arbitrary point of $\mathbb{D}$ to the origin.

 Here are two important facts. The proofs of these facts are not supplied.

Fact 6.0.1 If $f \in$ Aut($\mathbb{D}$) and $f(0) = 0$, then f is a rotation.

Fact 6.0.2 The rotation and the Mobius transformation completely characterize all $f \in$ Aut($\mathbb{D}$), i.e., if $f \in$ Aut($\mathbb{D}$) then there exist $a \in \mathbb{D}$ and $_T \in \mathbb{R}$ s.t. $f(z) = f_a \circ {}_{\rho T}(z)$. Similarly, every $f \in$ Aut($\mathbb{D}$) can be written in the form $\rho_T \circ f_a$.

We now state and prove some important applications of the Poincare metric in the form of theorems.

Theorem 6.1 Let ρ denotes the Poincare metric on $\mathbb{D}$. If $f \in$ Aut($\mathbb{D}$) then f is an isometry of (D, ρ) with ($\mathbb{D}$, ρ).

Proof:

By the above fact, any $f \in$ Aut($\mathbb{D}$) is a composition of a rotation and a Mobius transformation, so the proof is complete if we can show the result for the following two cases.

Case 1: f is rotation then $f(z) = \mu z$ for some $\subset \mu \in \mathbb{C}$ with $|\mu| = 1$. But, $f'(z) = \mu$ and hence

$|f'(z))| = 1$. Now,

$$f^*\rho(z) = \rho(f(z))\,|f'(z)|$$
$$= \frac{|f'(z)|}{1 - |f(z)|^2}$$
$$= \frac{1}{1 - |\mu z|^2}$$
$$= \frac{1}{1 - |z|^2} = \rho(z)$$

This shows that f is an isometry if f is a rotation.

Case 2: If f is a Mobius transformation then $f(z) = \dfrac{z - a}{1 - \bar{a}z}$, $a \in \mathbb{D}$. Hence

$$|f'(z)| = \frac{1 - |a|^2}{|1 - \bar{a}z|^2}.$$

But,

$$f^*\rho(z) = \rho(f(z))|f'(z)|$$

$$= \frac{1}{1 - \left|\frac{z-a}{1-\bar{a}z}\right|^2} \cdot \frac{1-|a|^2}{|1-\bar{a}z|^2}$$

$$= \frac{1-|a|^2}{|1-\bar{a}z|^2 - |z-a|^2} = \rho(z)$$

The last equality follows because

$|1-\bar{a}z|^2 - |z-a|^2 = (1-\bar{a}z)(1-a\bar{z}) - (z-a)(\bar{z}-\bar{a}) = 1+|a|^2|z|^2 - |z|^2 - |a|^2 = (1-|a|^2)(1-|z|^2)$

This implies that f is an isometry when f is a Mobius transformation.

This theorem and Prop 1.14 imply that any $f \in \mathrm{Aut}(\mathbb{D})$ preserves the Poincare distance. Note that f_a defined as above preserves the Poincare distance but it does not preserve Euclidean distance.

Theorem 6.2 Let ρ be the Poincare metric on $\mathbb{D}$. Fix $r > 0$ with $r < 1$. Among all continuously differentiable curves in $\mathbb{D}$ of the form

$\mu(t) = t + iw(t), 0 \le t \le r$,

which satisfy $\mu(0) = 0$ and $\mu(r) = r + i \,.\, 0$, the one of the least length is $\gamma(t) = t$.

Proof:

For any candidate μ defined above, the length of the curve is given by

$$l\rho(\mu) = \int_0^r \|\mu'(t)\|_{\rho,\mu(t)}dt$$

$$= \int_0^r \frac{|\mu'(t)|}{|1-|\mu(t)|^2} dt$$

$$= \int_0^r \frac{1}{1 - t2 - (w(t))^2} \,.\, 1 + (w'(t))^2)^{1/2}\, dt. \tag{6.1}$$

But,

$$\frac{1}{1-t^2-(w(t))^2} \ge \frac{1}{1-t^2} \text{ and } (1+(w'(t))^2)^{1/2} \ge 1.$$

So from equation 6.1, we get

$$l\rho(\mu) \ge \int_0^r \frac{1}{1-t^2} dt = l\rho(\gamma)$$

Note: This theorem together with our calculation in example 3.0.2 implies that the shortest length of the curve joining two points 0 and $r + i.0$ in $\mathbb{D}$ is

$$d\rho(0, r) = \frac{1}{2}\ln\left(\frac{1+r}{1-r}\right). \qquad (6.2)$$

Theorem 6.3 If P and Q are two points in the unit disc $\mathbb{D}$ then the Poncare distance from P to Q is given by

$$d\rho(P,Q) = \frac{1}{2}\log\left(\frac{1+\dfrac{P-Q}{1-\bar{P}.Q}}{1-\dfrac{P-Q}{1-\bar{P}.Q}}\right)$$

Proof:

Define $f_P : \mathbb{D} \to \mathbb{D}$ by $f_P(z) = \frac{z-P}{1-\bar{P}z}$ then by 6.1, f_P is an isometry. Hence

$$\begin{aligned} d\rho(P,Q) &= d\rho(f_P(P), f_P(Q)) \\ &= d\rho(0, f_P(Q)) = d\rho(0, |f_P(Q)|) \end{aligned}$$

The last equality holds because a rotation is an isometry. But, $|f_P(Q)| = \left|\frac{P-Q}{1-\bar{P}Q}\right|$ and by applying equation 6.2, we get that the Poncare distance from P to Q is

$$\begin{aligned} d\rho(P,Q) &= d\rho(0, |f_P(Q)|) \\ &= \frac{1}{2}\log\left(\frac{1+|f_P(Q)|}{1-|f_P(Q)|}\right) \\ &= \frac{1}{2}\log\left(\frac{1+\dfrac{P-Q}{1-\bar{P}.Q}}{1-\dfrac{P-Q}{1-\bar{P}.Q}}\right) \qquad (6.3) \end{aligned}$$

Now we show that, upto a constant multiple, the Poincare metric ρ is the only metric so that every $f \in \mathrm{Aut}(\mathbb{D})$ is an isometry.

Theorem 6.4 If $\tilde{\rho}(z)$ is a conformal metric on $\mathbb{D}$ and every $f \in \mathrm{Aut}(\mathbb{D})$ is an isometry of (D, ρ) with $(D, \tilde{\rho})$ then $\tilde{\rho}$ is a constant multiple of the Poncare metric ρ.

Proof:

We fix $a \in \mathbb{D}$ and define $f_a(z) = \frac{z-a}{1-\bar{a}z}$. Then clearly, $f_a \in \mathrm{Aut}(\mathbb{D})$. By hypothesis, f_a is an isometry. Hence,

$f^{*}\tilde{\rho}(z)(z) = \tilde{\rho}(z)$, in particular, $f^{*}\tilde{\rho}(a) = \tilde{\rho}(a)$

But,

$$F_a^{*}\tilde{\rho}(a) = \tilde{\rho}(f_a(a)).|f'a(a)| \Rightarrow \tilde{\rho}(a) = \tilde{\rho}(0).\frac{1}{1-|a|^2}$$

since $f_a(a) = 0$ and $f'_a(z) = \dfrac{1-|a|^2}{(1-\bar{a}z)^2}$. As a is arbitrary, we obtain that for all $z \in \mathbb{D}$

$\tilde{\rho}(z) = \tilde{\rho}(0).\ \rho(z)$

This proves that $\tilde{\rho}$ is constant multiple of Poincare metric.

Theorem 6.5 Let ρ be the Poincare metric on the unit disc, $\mathbb{D}$. Then $\mathbb{D}$ equipped with ρ, is a complete metric space.

Proof:

If $\{p_j\} \subset \mathbb{D}$ is a Cauchy sequence in ρ, then the sequence is bounded in ρ. So, there exists $R > 0$ such that $d_\rho(0, p_j) \le R\ \forall j$. But, by equation 1.6, we get

$$d\rho(0, p_j) = \frac{1}{2}\log\left(\frac{1+|p_j|}{1-|p_j|}\right) \le R$$

and with some simple calculations, we get

$$|pj| \le \frac{e^{2R}-1}{e^{2R}+1} < 1$$

This shows that for all j, $\{p_j\}$ is contained in a relatively compact subset of $\mathbb{D}$ in the Euclidean topology. We have proved earlier in prop 1.9 that ρ generates the same topology as Euclidean metric, so the sequence is also Cauchy in the Euclidean metric. Hence, it converges to a limit point in $\mathbb{D}$ in the Euclidean metric, so in the metric induced by ρ. Hence the metric space induced by ρ is complete. Note: Of course, $\mathbb{D}$ with the Euclidean metric is not complete.

We now state a fact without proof

Fact: Let $f : \mathbb{D} \to \mathbb{D}$ be a continuously differentiable function and ρ the Poincare metric. If f is an isometry of $(\mathbb{D}, \rho)$ with (D, ρ) then f is holomorphic.

Theorem 6.6 Let $f : \mathbb{D} \to \mathbb{D}$ be a continuously differentiable function and ρ the Poincare metric. If f is an isometry of (D, ρ) with (D, ρ), $f(0) = 0$ and $f'(0) = 1$ then f is the identity.

Proof:

By above fact, f is holomorphic. Since f fixes the origin and is an isometry, by Schwarz Lemma,

There exist $\tau \notin \mathbb{R}$ such that $f(z) = e^{i\tau}z$. But, $f'(0) = 1$ implies that $f(z) = z \;\forall z \in \mathbb{D}$. This
completes the proof.

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Some Results on LP-Sasakian Manifolds

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Abstract: The object of the present paper is to study LP-Sasakian manifolds satisfying certain conditions.

Key words: LP-Sasakian manifolds, quasi-conformal curvature tensor, pseudo projective curvature tensor.

1. Introduction

In 1989 K. Matsumoto [2] introduced the notion of LP-sasakian manifold. Then I. Mihai & R. Rosca [1] obtained several results in this manifold. Other geometers also studied it.

In 1968 Yano & Sawaki [4] defined & studied quasi-conformal curvature tensor which includes both the conformal and concircular curvature tensor. The present paper deals with a study of LP-Sasakian manifolds of dimension $(2n+1)$ satisfying certain conditions. After preliminaries, in section 3 we study an LP-Sasakian manifolds satisfying $R(X,Y).W=0$ & it is shown that such a manifold in an η- Einstein manifold. Also in such a manifold we obtain the value of scalar curvature & show that manifold is quasi-confromally flat. Section 4 deals with a ϕ-pseudo projectively flat LP-Sasakian manifold & it is proved that such a manifold is η-Einstein.

2. Preliminaries

A $(2n+1)$ dimensional differentiable manifold M is said to be an LP-Sasakian manifold [2], if admits a (1,1) tensor field ϕ, a contravariant vector field ξ, a 1-form η and a Lorentzian metric g which satisfy

$$\eta(\xi)=-1,\phi^2(X)=X+\eta(X)\xi,\phi\xi=0,\eta(\phi X)=0 \tag{2.1}$$

$$g(\phi X,\phi Y)=g(X,Y)+\eta(X)\eta(Y), \quad g(X,\xi)=\eta(X) \tag{2.2}$$

$$\nabla_X\xi=\phi X,(\nabla_X\phi)(Y)=g(X,Y)\xi+\eta(Y)X+2\eta(X)\eta(Y)\xi \tag{2.3}$$

Where ∇ denotes the operator of covariant differentiation with respect to g.
The fundamental 2-form Ω is defined as

$$\Omega(X,Y)=g(X,\phi Y)=g(\phi X,Y) \tag{2.4}$$

for any vector fields X and Y, $\Omega(X,Y)$ is the symmetric (0,2) tensor field [2].

Also the vector field η is closed in an LP-Sasakian manifold we have [2], [3]

$$(\nabla_X\eta)(Y)=\Omega(X,Y)=g(X,\phi Y),\Omega(X,\xi)=0 \tag{2.5}$$

for any vector fields X and Y.

A $(2n+1)$-dimensional LP-Sasakian manifold is said to be an Einstein & an η- Einstein manifold if its Ricci tensor S is of the form

$$S(X,Y)=\alpha g(X,Y)\,\&\,S(X,Y)=\beta g(X,Y)+\gamma\eta(X)\eta(Y) \tag{2.6}$$

respectively where $\alpha,\beta\,\&\,\gamma$ are smooth functions on M.

Also we have the following important basic results [2]

$$\eta(R(X,Y)Z)=g(Y,Z)\eta(X)-g(X,Z)\eta(Y) \tag{2.7}$$

$$R(\xi,X)Y=g(X,Y)\xi-\eta(Y)X \tag{2.8}$$

$$R(X,Y)\xi=\eta(Y)X-\eta(X)Y \tag{2.9}$$

$$R(\xi,X)\xi=X+\eta(X)\xi=\phi^2X \tag{2.10}$$

$$S(X,\xi) = 2n\eta(X) \tag{2.11}$$

$$S(\phi X,\phi Y) = S(X,Y) + 2n\eta(X)\eta(Y) \tag{2.12}$$

for any vector fields X,Y,Z where R(X,Y)Z is the Riemannian curvature tensor. The quasi-conformal curvature tensor W on a manifold M of dimension $(2n+1)$ is defined by [4]

$$W(X,Y)Z = aR(X,Y)Z + b\left[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\right]$$
$$-\frac{r}{2n+1}\left\{\frac{a}{2n} + 2b\right\}\left[g(Y,Z)X - g(X,Z)Y\right] \tag{2.13}$$

Where a,b are arbitrary constants, $a,b \neq 0$ and other symbols have their usual meanings.

3. LP-Sasakian manifolds satisfying *R(X, Y)W* = 0

Let us consider an LP-Sasakian manifold $M^{2n+1}(\phi,\xi,\eta,g)$ satisfying the condition [6]

$$R(X,Y).W = 0 \tag{3.1}$$

Now,

$$\begin{aligned}(R(X,Y).W)(U,V)Z &= R(X,Y)W(U,V)Z - W(R(X,Y)U,V)Z \\ &\quad - W(U,R(X,Y)V)Z - W(U,V)R(X,Y)Z\end{aligned} \tag{3.2}$$

From relations (3.1) and (3.2) we have

$$R(X,Y)W(U,V)Z - W(R(X,Y)U,V)Z - W(U,R(X,Y)V)Z - W(U,V)R(X,Y)Z = 0 \tag{3.3}$$

Taking $X=\xi$ in (3.3) we obtain by virtue of (2.8)

$$\begin{aligned}&g(W(U,V)Z,Y)\xi - \eta(W(U,V)Z)Y - g(Y,U)W(\xi,V)Z + \eta(U)W(Y,V)Z \\ &-g(Y,V)W(U,\xi)Z + \eta(V)W(U,Y)Z - g(Y,Z)W(U,V)\xi + \eta(Z)W(U,V)Y = 0\end{aligned} \tag{3.4}$$

Taking innerproduct on both sides by ξ in (3.4) & using $\eta(W(U,V)\xi) = 0$, we get

$$-g(W(U,V)Z,Y)-\eta(Y)\eta(W(U,V)Z)-g(Y,U)\eta(W(\xi,V)Z)+\eta(U)\eta(W(Y,V)Z)$$
$$-g(Y,V)\eta(W(U,\xi)Z)+\eta(V)\eta(W(U,Y)Z)+\eta(Z)\eta(W(U,V)Y)=0 \tag{3.5}$$

Let $\{e_i : i=1,2,......,2n+1\}$ be an orthonormal basis of the tangent space at any point of the manifold. Then setting $U=Y=e_i$ and taking summation over $i, 1\le i\le 2n+1$ in (3.5) we get

$$-\sum_{i=1}^{2n+1} g(W(e_i,V)Z,e_i)-(2n+1)\eta(W(\xi,V)Z)-\sum_{i=1}^{2n+1} g(e_i,V)\eta(W(e_i,\xi)Z)+$$
$$\eta(Z)\sum_{i=1}^{2n+1}\eta(W(e_i,V)e_i)=0 \tag{3.6}$$

Since, $\sum_{i=1}^{2n+1}\eta(V)\eta(W(e_i,e_i)Z)=0$

After straight forward calculation we get

$$S(V,Z)=\frac{1}{a-b}\left[2n\{a+2nb\}-br\right]g(V,Z)+\frac{b}{a-b}\left[2n(2n+1)-r\right]\eta(V)\eta(Z) \tag{3.7}$$

Provided that $a-b\neq 0$

Hence we can state

Theorem 3.1 An LP- Sasakian manifold $M^{2n+1}(\phi,\xi,\eta,g)$ satisfying the condition $R(X,Y).W=0$ is an η-Einstein manifold, providing $a-b\neq 0$.

Let $\{e_i : i=1,2,......,2n+1\}$ be an orthonormal basis of the tangent space at any point of the manifold. Putting V=Z=e_i in relation (3.7) & then summing over $i, 1\le i\le 2n+1$, we get $r=2n(2n+1)$, if $a-b\neq 0$, (3.8)

This leads to the corollary:

Corollary 3.1: An LP- Sasakian manifold $M^{2n+1}(\phi,\xi,\eta,g)$ satisfying the condition $R(X,Y).W=0$ has scalar curvature given by (3.8).

Using (3.8) in the relation (3.7) we get

$$S(V,Z) = 2ng(V,Z), \quad a-b \neq 0, \tag{3.9}$$

Thus we can state the following theorem:

Theorem3.2: An LP- Sasakian manifold $M^{2n+1}(\phi,\xi,\eta,g)$ satisfying the condition $R(X,Y).W = 0$ is an Einstein manifold providing $a-b \neq 0$.

Putting $Z = \xi$ in relation (3.9) we get

$$S(V,\xi) = 2n\eta(V) \tag{3.10}$$

Now, taking inner product on both sides of (2.13) by ξ & using the relations (3.8), (3.9), (3.10) and (2.7) we obtain

$$\eta(W(U,V)Z) = 0 \qquad \text{for all } U,V,Z \tag{3.11}$$

Again, using (3.11) in the relation (3.5) we get

$$g(W(U,V)Z,Y) = 0 \qquad \text{for all } U,V,Z$$

This implies

$$W(U,V)Z = 0 \qquad \text{for all } U,V,Z \tag{3.12}$$

Hence we can state

Theorem 3.3: An LP- Sasakian manifold $M^{2n+1}(\phi,\xi,\eta,g)$ satisfying the condition $R(X,Y).W = 0$ is quasi-conformally flat.

4. ϕ-pseudo Projectively flat LP- Sasakian Manifold

Definition: An LP- Sasakian manifold $M^{2n+1}(\phi,\xi,\eta,g)$ is said to be ϕ-pseudo projectively flat if it satisfies

$$\phi^2(\bar{P}(\phi X,\phi Y)\phi Z) = 0 \tag{4.1}$$

for arbitrary vector fields $X,Y,Z \in T_pM$.

The pseudo projective curvature tensor is defined by [5]

$$\bar{P}(X,Y)Z = aR(X,Y)Z + b\left[S(Y,Z)X - S(X,Z)Y\right] - \frac{r}{2n+1}\left\{\frac{a}{2n} + b\right\}$$

$$\left[g(Y,Z)X - g(X,Z)Y\right] \tag{4.2}$$

Where a,b are constants such that $a, b \neq 0$, R,S and r are the curvature tensor, Ricci-tensor & scalar curvature respectively.

Consider an LP- Sasakian manifold $M^{2n+1}(\phi,\xi,\eta,g)$ which is ϕ-pseudo projectively flat then (4.1) holds. Now from relations (4.1) & (4.2) we have

$$g\left(\bar{P}(\phi X,\phi Y)\phi Z,\phi W\right) = ag\left(R(\phi X,\phi Y)\phi Z,\phi W\right) + b\left[S(\phi Y,\phi Z)g(\phi X,\phi W) - S(\phi X,\phi Z)g(\phi Y,\phi W)\right]$$
$$-\frac{r}{(2n+1)}\left\{\frac{a}{2n} + b\right\}\left[g(\phi Y,\phi Z)g(\phi X,\phi W) - g(\phi X,\phi Z)g(\phi Y,\phi W)\right]$$

or

$$\tilde{R}(\phi X,\phi Y,\phi Z,\phi W) = -\frac{b}{a}\left[S(\phi Y,\phi Z)g(\phi X,\phi W) - S(\phi X,\phi Z)g(\phi Y,\phi W)\right] \tag{4.3}$$
$$+\frac{r}{(2n+1)a}\left\{\frac{a}{2n} + b\right\}\left[g(\phi Y,\phi Z)g(\phi X,\phi W) - g(\phi X,\phi Z)g(\phi Y,\phi W)\right]$$

Where $\tilde{R}(\phi X,\phi Y,\phi Z,\phi W) = g\left(R(\phi X,\phi Y)\phi Z,\phi W\right)$.

Using, (2.2) and (2.12) in (4.3) we get

$$g(R(X,Y)Z,W) + g(Y,Z)\eta(X)\eta(W) + g(X,W)\eta(Y)\eta(Z) - g(X,Z)\eta(Y)\eta(W) - g(Y,W)\eta(X)\eta(Z)$$
$$= \frac{b}{a}\left[-\{S(Y,Z) + 2n\eta(Y)\eta(Z)\}\{g(X,W) + \eta(X)\eta(W)\} + \{S(X,Z) + 2n\eta(X)\eta(Z)\}\{g(Y,W) + \eta(Y)\eta(W)\}\right]$$
$$+\frac{r}{(2n+1)a}\left\{\frac{a}{2n} + b\right\}\times$$
$$\left[\{g(Y,Z) + \eta(Y)\eta(Z)\}\{g(X,W) + \eta(X)\eta(W)\} - \{g(X,Z) + \eta(X)\eta(Y)\}\{g(Y,W) + \eta(Y)\eta(W)\}\right] \tag{4.4}$$

Where $g(R(X,Y)Z,W) = g(Y,Z)g(X,W) - g(X,Z)g(Y,W)$.

Let $\{e_i : i = 1,2,\ldots\ldots,2n+1\}$ be an orthonormal basis of the tangent space at any point of the manifold. Putting $X = W = e_i$ in (4.4) & taking sum over $i, 1 \leq i \leq 2n+1$, we get

$$S(Y,Z)=\left[\frac{1}{a+(2n+1)b}\left\{\frac{(a+2nb)r}{2n}-a\right\}\right]g(Y,Z)+\left[\frac{(a+2nb)(2n-1)}{a+(2n+1)b}\left\{\frac{r}{2n(2n+1)}-1\right\}\right]\eta(Y)\eta(Z) \tag{4.5}$$

Provided that $a,b\neq 0$.

This leads to the following:

Theorem 4.1: A ϕ-pseudo projectively flat LP- Sasakian manifold $M^{2n+1}(\phi,\xi,\eta,g)$ is an η-Einstein manifold, providing $a,b\neq 0$.

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Amazing Roles of Diagonal Edges on the Reducibility Problem of Open Shop Sequences Minimizing the Makespan

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Abstract: In this survey, we investigate the solution spaces of open shop irreducible sequences, which contain an optimal element for arbitrary processing times. The objective function is the minimization of makespan. The computational status of deciding whether a given sequence is irreducible remained unsolved for last 20 years. Here, we explore the roles of diagonal edges in the given sequence, which play vital roles for the conclusion of this decision problem.

Keywords: Scheduling, sequencing, open shop problem, reducibility, complexity.

1. Introduction

We consider the strongly *NP*-hard nonpreemptive open shop problem $O||C_{\max}$, [13]. Assume that, at a time, each job $i \in I = \{1, \ldots, n\}$ has to be processed on each machine $j \in J = \{1, \ldots, m\}$ exactly once for the positive time such that each machine can process at most one job and each job can be processed on at most one machine. The 2-jobs *m*-machines problem $O|n = 2|C_{max}$ is solvable in time $O(m)$, [13, 5]. Let $OIJ = I \times J$, $P = [p_{ij}]_{n\times m}$ and $C = [c_{ij}]_{n\times m}$ be the sets of all operations o_{ij}, matrix of processing times p_{ij} and matrix of completion times c_{ij}, respectively. The objective function is $C_{max} = \max_{i\in I} C_i$, where C_i is the completion time of job i. All machine orders (jobs processed on machines) and job orders (machines processing jobs) are arbitrarily. The $n \times m$ matrices of all job orders and machine orders are denoted by *JO* and *MO*, respectively. An interest is to find an optimal schedule (solution) $C = (A, P)$ which minimizes C_{max} (A) for given P. A schedule C is the time table corresponding to a sequence A (a feasible combination of all processing orders).

Informally, a decision problem is said to be in the class *P* if there exists a deterministic algorithm which solves the problem in polynomial time. A decision problem is in *NP* if there exists a nondeterministic polynomial time algorithm solving it. The co-*NP* class

contains the decision problems whose complements are in *NP*. A decision problem is called *NP*-complete if the problem belongs to *P*, then *NP* = *P* holds. A decision problem in *NP* which is neither polynomial solvable nor *NP*-complete is called *NP*-incomplete. We refer to [10] for a systematic analysis of the complexity classes.

The set of all semiactive schedules in which each operation is started as early as possible with respect to the given processing orders is sufficient for an optimal solution. An infinite set of schedules can be assigned to each sequence. We can define an equivalence relation on the set of all schedules decomposing the set into finite number of equivalence classes. Two schedules belong to the same class if and only if they base on the same sequence. The semiactive schedules under unit processing times, i.e., a finite set of all sequences, are used to find a set of distinct representatives. Obtaining the associated semiactive schedule (a scheduling problem) for a given sequence is an easy problem. Therefore, investigations on the theory of irreducibility are focused in obtaining an optimal sequence (a sequencing problem). For such difficult problems, the study concentrated either on the determination of polynomial solvable subproblems or on the development of an algorithm for an approximate solution.

A set of sequences is called a solution space if it contains an optimal element for arbitrary processing times. Obviously, the set of all sequences, i.e. of all semiactive schedules, is the largest solution space. A study focused on searching such potentially (universally) optimal solution spaces of smaller cardinalities. But, the obtained results show that the existence of unique minimal one is unlikely in general, [8]. The concept of potentially optimal solution spaces is very applicable when the processing times are erroneous, difficult to find out in advance or simply unknown, for instance, in manufacturing and service industries, satellite communications, examination scheduling and teacher class assignments, [2, 17].

A sequence A is called reducible to a sequence *B*, we write $B \preceq A$, if $Cmax\ (B) \leq C_{max}\ (A)$ for all $P \in \mathbf{P}nm$. It is called strongly reducible, denoted by $B \prec A$, if $B \preceq A$ but not $A \preceq B$. They are called similar, denoted by $A \simeq B$, if $B \preceq A$ and $A \preceq B$. A sequence is irreducible if there exists no other non-similar sequence to which it can be reduced. The dominance relation $\preceq$ on the set of all sequences with fixed format $n \times m$, introduced in [15], determines the minimal sequences with respect to the partial order $\prec$ independent of the given processing times. The solution set of all these locally optimal sequences is potentially optimal of smaller cardinality, however, still not the minimal. Two algorithms (one polynomial time and the other exponential) are proposed in [1]. They base on the characteristics of the diagonal edges of the associated H- comparability graph [7]. A number of open problems are raised, for instance strong conditions under

which both algorithms coincide. A key role lies on the diagonal edges while resolving the conflicts.

In this paper, we consider the following open questions: Does there exist a polynomial time algorithm for the decision whether a given sequence is irreducible? What is the computational complexity status of this decision problem? We characterize the H-comparability graphs in Section 2. We give different mathematical formulations of the problem of reducibility in Section 3. This section also presents the main characteristics of the diagonal edges in connection. Section 4 briefly overviews the status of the theory of irreducibility. The final section concludes the paper.

2. Comparability Graphs

A comparability graph is an undirected graph $G = (V, E)$ that has a transitive orientation $G^{tr} = (V, E^{tr})$. It is prime if it is uniquely orientable, [11]. We denote the Hamming graph $K_n \times K_m$ by G_{IJ}. A Hamming graph restricted on a partial operation set $PIJ \subseteq OIJ$ is called H-graph. A comparability graph $G = (PIJ, E)$ which contains an H-graph is called H-comparability graph. There exist Hamming graphs which are not comparability graphs, for instance $K_2 \times K_3$.

For any pair (MO, JO) in the open shop, define the shop graph $G_{MO,JO} = (PIJ, E_{MO,JO})$ where the arc set reflects the union of all machine orders and all job orders, [9]. An acyclic shop graph is called a sequence graph that is an acyclic orientation of the disjunctive graph. For each sequence graph $G_{MO,JO}$ we can describe the sequence (MO, JO) by a special latin rectangle $A = [a_{ij}]$, where $a_{ij} = \text{rank}\,(o_{ij})$ (the number of vertices on a longest path from a source to o_{ij}), such that for each integer $a_{ij} > 1$ there exists $a_{ij} - 1$ in row i or in column j or in both. An arc from o_{ij} to o_{kl} exists if and only if $i = k$ or $j = l$ and $a_{ij} < a_{kl}$ hold. There is a one-to-one correspondence between the sets of all sequences and all sequence graphs which can be done in linear time on the number of operations, see [9].

A sequence graph, denoted by $G_A = (PIJ, E)$, is an acyclic orientation of the H-graph G_{IJ}. For the sequence A, we denote the transitive orientation of a sequence graph and its symmetric closure by G_A^{tr} and $[G_A^{tr}]$, respectively. Given a sequence, both graphs can be determined in polynomial time $O(n^2m^2)$, [16]. Let $E_{r(A)}$ and $E_{d(A)}$ represent the sets of all regular edges (edges in H-graph GIJ) and diagonal edges, respectively. Here, $[G_A^{tr}] = (OIJ, G_A^{tr} + (G_A^{tr})^{-1}) = (OIJ, E_{r(A)} \cup E_{d(A)})$ is undirected graph, where G^{-1} denotes the reversed graph of a graph G with all arcs in the reversed direction.

Consider a 3-jobs 4-machines open shop with machine order $J_1 : M_2 \to M_4 \to M_1$, J_2: $M_3 \to M_2 \to M_4 \to M_1$, $J_3 : M_1 \to M_3 \to M_4$ and job orders $M_1 : J_3 \to J_1 \to J_2$,

$M_2 : J_1 \to J_2$, $M_3 : J_2 \to J_3$, $M_4 : J_1 \to J_3 \to J_2$. The corresponding and rank matrices and the graphs are given.

$$MO = \begin{bmatrix} 3 & 1 & & 2 \\ 4 & 2 & 1 & 3 \\ 1 & & 2 & 3 \end{bmatrix} \quad JO = \begin{bmatrix} 2 & 1 & & 1 \\ 3 & 2 & 1 & 3 \\ 1 & & 2 & 2 \end{bmatrix} \quad A = \begin{bmatrix} 3 & 1 & & 2 \\ 5 & 2 & 1 & 4 \\ 1 & & 2 & 3 \end{bmatrix}$$

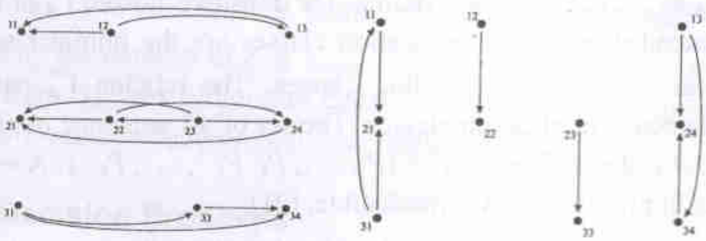

Figure 1: Machine order graph G_{MO} and job order graph G_{JO}.

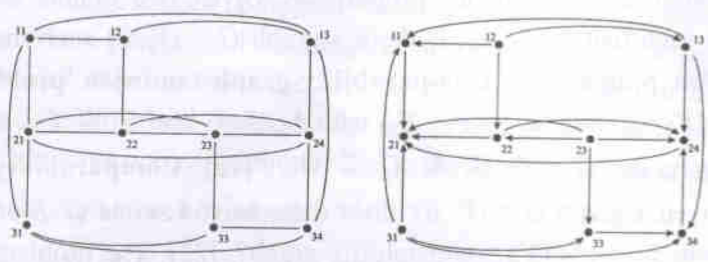

Figure 2: The H-graph $G_{IJ}(A)$ and the sequence graph G_A.

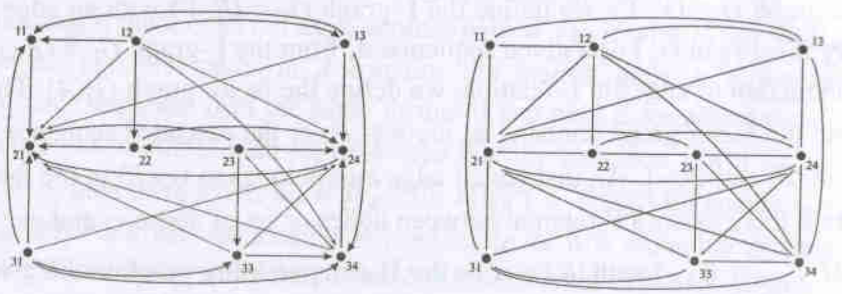

Figure 3: The transitive closure G_A^{tr} and the symmetric closure $[G_A^{tr}]$.

For two edges ab, cd in a graph $G = (V, E)$ a Γ-relation, denoted by $ab\Gamma cd$, is defined if and only if either $a = c$, $bd \notin E$ or $b = d$, $ac \notin E$ or $ab = cd$. The transitive relation Γ^{tr} decompose the set of all edges into equivalent implication classes in the

comparability graph. The set of all implication classes of the sequence A is denoted by $\mathcal{I}_{[G_A^{tr}]} = \{I_1, \ldots, I_l, I_1^{-1}, \ldots, I_l^{-1}\}$. Note that a graph is a comparability graph if and only if there is no implication class containing both an arc and its reverse. A sequence with only one implication class is irreducible, [3]. Two edges ab, $cd \in G_A$ are said to be in Γ_A-relation, denoted by $ab\Gamma_A\, cd$, if and only if $ab\Gamma cd$ in $[G_A^{tr}]$. Two edges e, $e' \in E_{r(A)}$ are connected by a Γ_A-path if there exist $e = e_0, e_1, \ldots, e_m, e_{m+1} = e'$ from $E_{r(A)}$ such that $e\Gamma_A e_1\, \Gamma_A\, e_2 \ldots e_m \Gamma_A\, e'$ which defines the transitive closure Γ_A^{tr} of the Γ_A-relation on $E_{r(A)}$. The extended sequence implication classes are the minimal sets containing all transitive edges of the corresponding classes. The relation Γ_A^{tr} partitions $E_{r(A)}$ into equivalent sequence implication classes. The set of all sequence implication classes of the sequence A is denoted by $\mathcal{P}_{[G_A^{tr}]} = \{P_1, \ldots, P_k, P_1^{-1}, \ldots, P_k^{-1}\}$. A sequence with only one sequence implication class is irreducible, [21].

The problem of irreducibility is closely related to the following problems (see Section 3).

The graph sandwich problem for property P: Given two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ such that $E_1 \subseteq E_2$, is there a graph $G = (V, E)$ such that $E_1 \subseteq E \subseteq E_2$ which satisfies property P? **Comparability-graph-sandwich problem:** Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$, does there exist a comparability graph G with $G_1 \subseteq G \subseteq G_2$?, [12]. **Comparability-graph-deletion problem:** Given a graph $G = (V, E)$, does there exist a set $M \subseteq E$ of at most k edges deletion of which yields G a comparability-graph?, [22]. The problems comparability-graph-sandwich and comparability-graph-deletion are NP-complete. The problem of irreducibility is a special case with $G_1 = K_n \times K_m$ and $G_2 = [G_A^{tr}]$.

Given a graph $G = (V, E)$, we define the Γ-graph $G_\Gamma = (E, \Gamma)$ with an edge $e_1e_2 \in \Gamma$ if and only if $e_1\Gamma e_2$ in G. For a given sequence A, from the Γ-graph $G_\Gamma = (E_{r(A)} + E_{d(A)}, \Gamma)$ with contraction of edges in Γ-relation, we define the factor graph $G_{\mathfrak{F}}(A)$. The vertex set $G_{\mathfrak{F}}(V)$ in the factor graph contains an arc $v \in E_d$ or the extended sequence implication classes in $\mathcal{P}_A$ and $\mathcal{P}_A$–1. An undirected edge e_1e_2 belongs to $G_{\mathfrak{F}}(E)$ in the factor graph if and only if there exists a Γ-relation between nodes or set of nodes e_1 and e_2.

Let $(OIJ, E_{r(A)} + E_{d(A)})$ with $|E_d| = d$ be the H-comparability graph to the given $A \in SIJ$, $\mathcal{P}_A = \{P_1, \ldots, P_k\}$ and $E_{r(A)} = P_1 + \ldots + P_k + P_1^{-1} + \ldots P_k^{-1}$. The consequence graph $G_k(A) = (V_K, E_K)$ is defined as follows. The set of nodes is $V_k \subseteq E_{d(A)} + \mathcal{P}_A + \mathcal{P}_A$–1. Two edges e' and e'' from V_K are connected by an undirected edge of color $i \in \{1, \ldots, d\}$ when the removal of $e_1 \in E_d$ forms a Γ-relation between e' and e'' or between the sequence implication classes they represent, respectively, i.e., $E_K = \{e'{}'e''$ with color i

$|e'\Gamma e''$ in $[G_A^{tr}] - e_i, e_i \in E_{d(A)}\}$. The set G_{Ki} represents the subgraph of G_K with i^{th} color such that $G_K = G_{k1} + \ldots + G_{Kd}$.

For a given set $M \subseteq E_{d(A)}$, the reduction graph $G_{R_M}(A) = (V_{R_M}, E_{R_M})$ is defined by inserting into $G_{\mathfrak{F}}$ all edges from G_K which are colored from M and deleting the nodes which represent edges in M as $G_{R_M}(A) = [G_{\mathfrak{F}} + \cup_{e \in M} G_{K_e}] - M$. The removal of an edge result new Γ-relations. The consequence graph informs which sequence implication classes are merged by the removal of $e_i \in E_{d(S)}$. The reduction graph informs about the deletion of nodes from $G_{\mathfrak{F}}$ and addition of edges between the remaining nodes in $G_{\mathfrak{F}}$ which induce a new Γ-relation between sequence implication classes.

3. Modeling Decision Problems

Given a sequence A ∈ SIJ on the same operation set OIJ for the open shop sequencing problem $O \parallel C_{max}$, we reformulate different versions of the following recognition problem. This question has not been completely answered since 1990's raised implicitly in [15]. Does there exist a polynomial time algorithm for the decision whether a given sequence is irreducible? What is the computational complexity status of this decision problem? The irreducibility and reducibility are complement decision problems. Reducing is the constructive optimization problem to the decision problem reducibility.

Irreducibility 1 *Is the sequence A irreducible?*

Reducibility 1 *Does there exist a sequence $B \in SIJ$ such that $B \prec A$?*

Reducing: Find a sequence B, if it exists, such that $B \prec A$.

A path w_A with vertex set $V(w_A)$ in the sequence graph G_A (equivalently, in the sequence A) is called maximal if there does not exist another path w^*_A in it with $V(w_A) \subset V(w^*_A)$. The set W_A of all maximal paths in A contains the longest path. A sequence A is reducible to another sequence B of the same format if and only if for all maximal paths w_B in B, there exists a maximal path w_A in A such that the inclusion $V(w_B) \subseteq V(w_A)$ is satisfied. If $B \prec A$, then there exists $w_B \in W_B$ with $V(w_B) \subset V(w_A)$ for some $w_A \in W_A$. The decision whether a given sequence A is irreducible, or it is reducible or similar to another sequence B simply by using the related definitions takes exponential time.

Lemma 3.1 [18] Let be the operation sets such that $OIJ' \subseteq OIJ$. Then there exists a path w_A in the closure A^{tr} with $V(w_A) = OIJ'$ if and only if OIJ' is a clique in $[G_A^{tr}]$. Moreover, such a path is unique for the clique OIJ' in $[G_A^{tr}]$.

Since the subgraphs can be tested and the transitive closures can be constructed with the same time complexity $O(n^2m^2)$ for given sequences of the same size $n \times m$, Theorem 3.1

yields an answer in polynomial time to the question of irreducibility, reducibility or similarity between two given sequences.

Theorem 3.1 [3] Let A, $B \in SIJ$ be on the same operation set OIJ for $O||C_{max}$. The sequence A is reducible, strongly reducible or similar to the sequence B if and only if $[G_B^{tr}] \subseteq [G_A^{tr}]$, $[G_B^{tr}] \subset [G_A^{tr}]$ or $[G_B^{tr}] = [G_A^{tr}]$ for the corresponding H-comparability *graphs, respectively.*

For a reducible sequence, the reducibility can be proved with nondeterministic polynomial time. As this proof is constructive, such a procedure answers not only to the reducibility but also to the problem reducing. The problem reducibility is in N P and the problem irreducibility is in co-*NP*. Furthermore, if there exists a *NP*-test for irreducibility, then this problem is either polynomially solvable or *NP*-incomplete, as far as $P \neq NP$ holds.

The relation $\prec$ induces a partial order on the set of all sequences of the same operation set SIJ. The irreducible sequences are the minimal elements of this half-order on H-comparability graph $[G_A^{tr}]$ containing H-graph G_{IJ}, for given $A \in SIJ$. Let $A \in SIJ$ on the operation set OIJ for the open shop sequencing problem $O||C_{max}$. Let $[G_A^{tr}]$ be the corresponding H-comparability graph containing the H-graph G_{IJ} on the same OIJ. With this the problems of irreducibility and reducibility can be reformulated as the question of the existence of an H-comparability graph G as follows.

Irreducibility 2 Is there no H-comparability graph G with $G_{IJ} \subseteq G \subset [G_A^{tr}]$?

Reducibility 2 Does there exist an H-comparability graph G with $G_{IJ} \subseteq G \subset [G_A^{tr}]$?

The theory of reducibility concerns the reduction of a sequence through the reversion of an implication class in its transitive closure. One of the most important fundamental properties states that a sequence A whose H-comparability graph $[G_A^{tr}]$ is not prime is either reducible or is similar to an irreducible sequence B with $B \neq A$ and $B \neq A^{-1}$, [21].

A sequence can be obtained from every transitive orientation of an H-comparability graph. If the H-comparability graph G has a sequence orientation G_A^{tr}, then $G = [G_A^{tr}]$ is the H-comparability graph to a sequence $A \in SIJ$. A transitive orientation $T \in T_{[G_A^{tr}]}$ is called a sequence orientation if every diagonal edges in T is transitive. If a transitive orientation G_B^{tr} of $[G_A^{tr}]$ is not a sequence orientation, then some diagonal edges of $[G_A^{tr}]$ are not in the orientation G_B^{tr}, and $B \prec A$.

One may reduce the given sequence by reversing an implication class. One method to reverse the implication classes is the deletion of a single diagonal edge. Deletion of an edge from a transitive reduction can be done easily. However, if $[G_A^{tr}]$ can be transitively

oriented such that neither e nor e^{-1} are transitive edges, then the edge $\hat{e}$ can be deleted and the graph $[G_A^{tr}] - \hat{e}$ is a comparability graph whose sequence orientation reduces A strongly. As transitive orientation of an H-comparability graph can be found in polynomial time and the number of diagonal edges for an n × m open shop sequencing problem is of order $O(n^2m^2)$, it can be tested in polynomial time whether a give sequence can be strongly reduced by deleting a diagonal edge.

Theorem 3.2 [21] If there exists $\hat{e} \in E_{d(A)}$ in $[G_A^{tr}]$ such that $[G_A^{tr}] - \hat{e}$ is a comparability graph, then ever transitive orientation of $[G_A^{tr}] - \hat{e}$ induces a sequence which strongly reduces the sequence A.

Thus a sequence $A \in SIJ$ can be strongly reduced to a sequence $B \in SIJ$ which cannot be further reduced by reversing an arbitrary implication class. This can be done in polynomial time. The H-comparability graph $[B^{tr}]$ is then either prime or there exist similar sequences to B other than B^{-1}. The set of all such reducible sequences cannot be obtained in polynomial time as the recombination of all implication classes is of size $O(2^k)$ for k implication classes and every edge may represent an implication class in the worst case. The reversion of only implication classes and their recombination does not generate the sequence space.

Not every recombination of the sequence implication classes of a sequence A is acyclic, and it yields a sequence B if it is acyclic. The set of all recombination of the sequence implication classes is sufficient. Therefore, taking sequence implication classes as basis for the space of sequences, we reformulate

Irreducibility 3 *Does every feasible recombination of the sequence implication classes of A produce a sequence B similar to A?*

Reducibility 3 *Does there exist a feasible recombination of the sequence implication classes of A where at least one diagonal edges of A is missing?*

Removal of one edge may not yield a strongly reduced sequence but with more than two edges removed may yield. A removable set with respect to a given sequence A is a set of undirected diagonal edges $M \subseteq E_{d(A)}$. The set M is called feasible if $[G_A^{tr}] - M$ is an H-comparability graph, and it is called feasibly extendable if there exists a feasible removable set M^* of diagonal edges of $[G_A^{tr}]$ such that $M \subset M^*$. The set M is called infeasible if it is not feasibly extendable. A removable set which is not feasible may not be necessarily infeasible. A removable set can be feasible and, in addition, feasibly extendable, too. We reformulate the problem of reducibility as follows.

Irreducibility 4 Is every removable set $M \subseteq E_{d(A)}$ in $[G_A^{tr}]$ infeasible?

Reducibility 4 Does there exist a feasibly extendable removable set $M \subseteq E_{d(A)}$ in $[G_A^{ir}]$?

In any $[G_A^{ir}]$, the implication classes which consist exclusively of diagonal edges can be deleted and the reduction through the reversion of group of implication classes can be done in polynomial time. A sequence is called normal if it cannot be reduced in either of these ways. Since any sequence can thus be reduced to a normal sequence in polynomial time, we restrict the space of sequences into the class of normal sequences [1].

For a reduction of a normal sequence by the reversion of a sequence implication class P_1 against the sequence implication class P_2 from the same implication class, all Γ-paths between them which contains at least one diagonal edge have to be cut keeping the comparability property. For feasibility of M each such path has to be broken, in order to avoid a connection in $[G_A^{ir}] - M$. Not all Γ-paths are destroyed if P_1 and P_2 belong to one connected component of $G_F - M$.

With the help of factor, consequence and reduction graphs, one can recognize a feasible removable set. These graphs inform existence of a transitive orientation of $[G_A^{ir}] - M$, if M terns out to be feasible. However, if this is not the case, the question remains how a none feasible but feasibly extendable set M can be expanded or to prove that the set M is not only none feasible but is infeasible.

The way to decide which additional diagonal edges should be added to a none feasible but feasibly extendable removable set in order to get a feasible removable set is another problem. One of the main issues while doing this process is that the diagonal edges, which belong to the irreducible sequences, must not be removed and the problem of merging two implication classes occurs when an edge play a role of conflict.

If there exists a path $W \subseteq V_{RM}$ from a sequence implication class $P_i \in V_{RM}$ to its reversion $P_i^{-1} \in V_{RM}$, we call it a conflict in $G_{RM}(A)$. The number $l \geq 0$ of the diagonal edges contained in the inclusion-minimal path W is called the order of the conflict. A direct conflict is a conflict with $l = 0$. Clearly, every conflict in G_{R_M} reflects a Γ-path in $[G_A^{ir}] - M$ from P_i to P_i^{-1}. For M, in order to be a feasibly extendable, all these conflicts must be dissolved and every one of these Γ-paths must be broken. A Γ-path between two edges in a graph will only be destroyed removing edges from this graph when at least one edge from the Γ-path is removed.

A diagonal edge is called stable if it is contained in every irreducible sequence of A. A diagonal edge is called trivial-stable if it in an extended sequence implication class. A stable diagonal edge which is not in an extended sequence implication class is called non-trivial-stable. If all edges in $[G_A^{ir}]$ are trivial-stable, then the irreducibility of a sequence is decidable in polynomial time.

We construct a sequence with two sequence implication classes such that a diagonal edge is contained between the classes which is transitive in each combination of these classes.

$$B = \begin{bmatrix} 2 & & & \\ 1 & 2 & 3 & \\ & 1 & 4 & 2 \\ & & 1 & \end{bmatrix} \quad C = \begin{bmatrix} 2 & & & \\ 1 & 3 & 4 & \\ & 2 & 1 & 3 \\ & & 2 & \end{bmatrix}$$

The H-comparability graph of the sequence B is uniquely orientable and this transitive orientation contains three diagonal edges $\{(o_{32}, o_{23}), (o_{22}, o_{33}), (o_{21}, o_{33})\}$. The set of regular edges form two sequence implication classes, say P_1 and P_2. The four inner nodes $\{o_{22}, o_{23}, o_{32}, o_{33}\}$ form the complete graph K_4 which contains trivial-stable edge $\{o_{32}, o_{23}\}$ and nontrivial-stable edge $\{o_{22}, o_{33}\}$. If we consider the combination $P_1 + P_2^{-1}$, then the resulting graph of the irreducible sequence C removes the third diagonal edge $\{o_{21}, o_{33}\}$ and converts the nontrivial-stable edge $\{o_{22}, o_{33}\}$ into trivial-stable edge.

A diagonal edge $e \in [G_d^{tr}]\,(A)$ is called magic-stable with respect to M if it does not lead to a direct conflict in $G_{R_{M^*}}$, with $M + e \subseteq M^*$, through a series of conflicts of order 1. It has not been found any sequence yet which contains a magic-stable edge. If one could prove that there exists no magic-stable edges, then the problem of irreducibility is polynomially solvable. Therefore, the difficulty of the problem of irreducibility depends upon the existence or non-existence of magic-stable edges in a given sequence graph. For recent algorithms and some conjectures on the complexity status of this problem based on the category of diagonal stable edges, we refer to [1].

4. Results on Irreducibility

We briefly review the status of irreducibility in the open shop $O||C_{max}$, see [8, 1] for details. A sequence B is optimal if $B \preceq A$ for all sequences A. It holds $C_{max}(A) = C_{max}(A^{-1})$. Moreover, $B \preceq A$ implies $B \preceq A^{-1}$, $B^{-1} \preceq A$ and $B^{-1} \preceq A^{-1}$.

An unavoidable set of sequences is computed for small formats, [19]. This approach formulates the dominance relation as a mixed integer programming. Among seven classes of all 3×3 irreducible sequences only three of them are unavoidable in the sense that these together with their reverses form unique minimal optimal set ensuring of at least one optimal sequence. A sequence of biggest rank five among all irreducible sequences cannot be missed. For the problem $O3|n = 2|C_{max}$, the minimal cardinality of two distinct potentially optimal solution sets is 3. Thus, the minimal set is not unique. The properties of sequence isomorphisms play important roles for the enumeration of

open shop sequences, [14]. The sequence A is irreducible if and only if the sequence B is also irreducible in the same isomorphism class.

The set of all irreducible sequences for $O2||C_{max}$ is presented in [5]. For every irreducible sequence A, there exists a $k \in \{2, \ldots, m\}$ such that A can be obtained by a permutation of the columns of the sequence A_k . Any sequence in this class is irreducible and any sequence not belonging to this class reduces to a sequence belonging to this class. This can be done in polynomial time. Note that the latter sequence A_k is irreducible having only one implication class. This result also holds even if the sets of operations is partial.

$$A_k = \begin{pmatrix} 1 & \ldots & k-1 & k & k+1 & \ldots m \\ m-k+2 & \ldots & m & 1 & 2 & \ldots m-k+1 \end{pmatrix}$$

The asymptotic ratio of all irreducible sequences to all sequences for $O|n = 2|C_{max}$ is $\lim_{m\to\infty} \frac{m!(m-1)}{m!(m!+\sum_{k=1}^{m}\binom{m}{k}\frac{m!}{k!})} = 0$, motivating a research in the narrow search space.

There are several sufficient conditions for the reducibility which can be tested without computing the transitive closures. An $n \times m$ sequence $[a_{ij}]$, where $\min\{n, m\} \geq 3$, having an operation o_{ij} with $a_{ij} \geq nm - 2$ is strongly reducible. Any sequence with o_{ij} such that o_{ij} has at least one successor but none of its successors in row i or column j has a direct predecessor outside row i and column j, respectively, is strongly reducible. If A be a sequence such that each job i is first processed on the same machine j, then $B \prec A$ for some sequence B. We refer to [3, 14] for more polynomial time test conditions, enumeration algorithms and computations of irreducible sequences for small formats. A motivating result is that only a very small fraction of all sequences has been found irreducible.

The irreducible sequences for the open shop on an operation set with spanning tree structure are studied in [5]. They describe in detail the set of all locally minimal elements for $O2||C_{max}$. A necessary and sufficient condition for the irreducibility can be tested in polynomial time on tree-like operation sets, [20]. Given $OIJ \subseteq I \times J$, the bipartite graph is defined as follows. For each node $i \in I$ and $j \in J$, there exists an edge between them if and only if the operation $o_{ij} \in OIJ$. If this graph is a tree then the operation set is called tree-like. The problem of irreducibility in this structure is polynomially solvable as all diagonal edges in a sequence of this structure are trivial-stable [1].

A test whether a given sequence can be strongly reduced to another sequence by deleting an operation and reinserting it as a sink or a source, it has to be ensured that at least one path is destroyed in the former and no new path is created in the latter. This

can be done in $O(n^2m^2)$ time and space, given an $n \times m$ sequence, [3].

An enumeration algorithm computes all irreducible sequences constructing inclusion minimal comparability graphs by successively inserting diagonal arcs into $[G_A]$, [4]. Each sequence in such a set is similar to exactly one sequence in this class, namely its reverse one. This algorithm constructs graphs G such that $G = [G_A^{tr}]$ for some sequence A. For $\min\{n, m\} \geq 2$, a lower bound on the number of diagonal arcs of an $n \times m$ sequence on the complete operation set is $\frac{1}{4}n(n-2)\binom{m}{2}$.

An insertion method for the enumeration of all sequences can be found in [6]. This method is modified and a new enumeration algorithm is presented in [3]. In their algorithm, a set of nonisomorphic sequences is computed and, thereafter tested for irreducibility. One sequence per isomorphic class is sufficient. They compute that the ratio between the number of irreducible sequences and all sequences decreases with growing n and m. There exist irreducible sequences that are not rank minimal, [3].

A generalized decomposition on irreducibility is introduced in [9]. For this, he considers an underlying 2×2 open shop problem by the assignment of an operation to each part. This approach invents the properties of irreducibility on the sequences of larger sizes based on similar properties on the sequences of smaller sizes.

The concept of irreducibility for arbitrary numerical input data, release time $r_i \geq 0$, due date $d_i \geq 0$, weight $w_i \geq 0$ and the processing times $p_{ij} \geq 0$ with respect to some additional regular objective functions γ can be found in [21].

A sequence A is called general-reducible to another sequence B, written as $B \preceq_g A$, if $C_i(B) \leq C_i(A)$ holds for all jobs i and all possible instances of numerical data N_D; release times, due dates, weights and processing times. Obviously, if $C_i(B) \leq C_i(A)$ for all jobs and all numerical data, then $\gamma(B) \leq \gamma(A)$ holds for all regular γ.

Theorem 4.1 Let $A, B \in SIJ$, and let the set $v_i(A)$ of operations contains the predecessors of an operation of the job i. Then it holds

$$B \preceq_g A \;\Leftrightarrow\; ([G_B^{tr}] \subseteq [G_A^{tr}]) \wedge (v_i(B) \subseteq v_i(A) \;\; for\ all\ \ i \in I).$$

Proof. First we show that the condition $([G_B^{tr}] \subseteq [G_A^{tr}]) \wedge (v_i(B) \subseteq v_i(A))$ for all jobs implies that $B \preceq_g A$. On the contrary assume that there exists a matrix P of processing times and a job k such that $C_k(B) > C_k(A)$. Let $w_B = (o_{ij}, \ldots, o_{kl})$ be a path in sequence B which ends at an operation o_{kl} of the job k with $\Sigma o_{uv \in w_B}\, p_{uv} = C_k(B)$. All nodes in the path w_B are contained in $v_k(B)$ and hence are contained in $v_k(A)$ because $v_k(B) \subseteq v_k(A)$ holds.

Because $B \preceq A$, the nodes $o_{ij}, \ldots, o_{kl}$ form a clique in A, and therefore, there exists a path w_A in A which contains all nodes of w_B and ends at an operation of job k. Then the inequality $\Sigma_{o_{uv} \in w_A} p_{uv} \leq C_k(A)$ contradicts to the assumption $C_k(B) > C_k(A)$.

On the other hand, we assume that $B \preceq_g A$ and show the validity of the condition $([G_B^{tr}] \subseteq [G_A^{tr}]) \wedge (v_i(B) \subseteq v_i(A))$ for all jobs. If $B \npreceq A$, then there exists an arc $e = (o_{uv}, o_{kl}) \in [G_B^{tr}]$ but not in $[G_A^{tr}]$. We define the matrix such that

$$p_{ij} = \begin{cases} 1 & \text{if } o_{ij} \in \{o_{uv}, o_{kl}\} \\ 0 & \text{otherwise} \end{cases}$$

This follows that $C_k(B) = 2$ and $C_k(A) = 1$, which is a contradiction.

If $v_k(B) \nsubseteq v_k(A)$ for a job k, then there exists an operation o_{uv} which is a predecessor of an operation of job k in B but not in A. If we define a matrix $P = [p_{ij}]$ as follows:

$$p_{ij} = \begin{cases} 1 & \text{if } o_{ij} = o_{uv} \\ 0 & \text{otherwise,} \end{cases}$$

the inequality $C_k(B) > C_k(A)$ follows immediately. □

A sequence A is called r-reducible to another sequence B, denoted by $B \preceq_r A$, if $C_{max}(B) \leq C_{max}(A)$ for all $P = [p_{ij}]$ and all $r = [r_i]$.

Along with a number of results in terms of the comparability graphs and precedence relations between operations, the interesting relations between the general-reducibility and r-reducibility are established.

Theorem 4.2 For any $A, B \in SIJ$, it holds $B \preceq_r A$ if and only if $B^{-1} \preceq_g A^{-1}$ holds.

Proof. If $B^{-1} \preceq_g A^{-1}$, then $v_i(B^{-1}) \subseteq v_i(A^{-1})$ for all jobs i and $[G_B^{tr}] \subseteq [G_A^{tr}]$. Furthermore, each induced subgraph from $v_i(B^{-1})$ of $[G_B^{tr}]$ is also an induced subgraph from $v_i(A^{-1})$ of $[G_A^{tr}]$. Because the maximal completion time of jobs in A corresponds to the maximal sum of one of the release times r_i and the maximal clique value of the induced subgraph from $v_i(A^{-1})$ of $[G_A^{tr}]$, the completion time of jobs in B must not be greater than that of in A.

On the other hand, if $B^{-1} \npreceq A^{-1}$ then it is easy to verify that there exists a matrix of processing times such that $C_{max}(B) > C_{max}(A)$.

Similarly, if $v_k(B^{-1}) \nsubseteq v_k(A^{-1})$ for some job k, then there exists an operation o_{uv} which is a successor of the job k in B but not in A. Then the following settings

$$p_{ij} = \begin{cases} 1 & \text{if } o_{ij} = o_{uv} \\ 0 & \text{otherwise} \end{cases} \text{ and } r_i = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

implies that $C_{max}(B) = 2$ and $C_{max}(A) = 1$. □

The definitions of strong reducibility, similarity and irreducibility have been extended similarly for general case and the r-reducibility.

5. Conclusions

In this paper we consider the problem: does there exist a polynomial time algorithm for the decision whether a given sequence is irreducible? Then we review all mathematical formulations and all solution approaches of the problem.

We mainly deal with the makespan objective function in the open shop. The polynomial solvability of this problem depends on the diagonal edge as all regular edges are stable.

One of the main issues raised from this research is whether the concept of stability of the diagonal edges in the corresponding H-comparability graph of the sequencing problem with makespan objective can be extended for other regular objectives. The results are likely to be similar, however, some extensions have to be made.

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A New Technique to Construct the Female Marriage Life-Table

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Abstract: This article tries to construct female marriage life-table by using models. Models developed by Aryal [1] and Mishra [2] have been applied to construct the female marriage life-table by using a sample survey data of Palpa and Rupandehi districts and Nepal Demographic and Health Survey (NDHS). The constructed marriage life-table provided the distribution to describe the waiting time for getting married among Nepalese girls. The expected waiting time for getting married at birth was found to be 16.4 years among Nepalese girls whereas it was 17.5 years among hilly girls, 16.9 years among Tarai girls and 17.2 years among rural girls. Similarly, expected waiting time for getting married among girls at age 12 years was found to wait another 5 years and at age 24 years was found to wait next 2 years and at age 30 years was found to wait another one and half years. Thus expected waiting time for getting married at birth was around 17 years, which is consistent value to the average age at marriage of 17 years [1].

1. Introduction

Marriage is one of the social events, which is an important demographic parameter because it direct influences the human fertility [1]. Naturally it is considered to be the foundation of family formation and it is an important determinant of fertility associated with the duration of exposure for the risk of childbearing [1,3,4]. Indeed marriage is universal in Nepal. The timing of getting married is generally early for both males and females in Nepal.

A number of probability/mathematical models were used in order to describe the distribution of age at marriage in different societies. They are: lognormal distribution[5], convolution of a normal and exponential distribution [6], linear function of the logarithm

of a standard gamma distribution [7], two parameter log-logistic model, Gompertz curve, simple polynomials and logistic curve [8,9,10] convolution of two exponential distributions[11] type I extreme value distribution [12,13,14]. Of course, so used models are being conceptually difficult to apply and to understand as well as the computation of parameters. Since all these models provided a big discrepancy between observed and expected parameters. Likewise a number of researchers were also applied several mathematical models for describing the distribution demographic phenomena [15,16,17,18,19]. Indeed Aryal [1] and Mishra [2] developed simple mathematical models for describing the distribution of female age at marriage. In brief, the description of the developed models is given in the following section.

2. The data

The data are used from a sample survey of Demographic Survey on Fertility and Mobility in Rural Nepal (DSFM): A Study of Palpa and Rupandehi districts. A total of 811 households were surveyed. The details of the data are available in Aryal [1, 3]. Another set of data are also used by extracting from the Nepal Family Health Survey.

3. Models

The basic assumptions of the models developed by Aryal [1] and Mishra [2] are given below. X is a random variable where X=0, if the female getting married before the age of 12 years; X=1, if female getting married at the age of 12 to 15 years; X=2, if female getting married at the age of 15 to 18 years; X=3, if female getting married at the age of 18 to 21 years; and so on. The random variable X can also be regarded as a number of failures preceding the first success. However researchers have not assumed constant probability of success and independence of trials by discussing the distribution of marriage and first birth [20, 21]. Probability p_i of getting a success in the $(i+1)^{th}$ trial when it is known that first i trials resulted in failure, increases as i (i = 0, 1, 2, 3,. . . k) moves from zero to a certain value s and decreases monotonically as i moves from s to a value t ($\geq$ s) and thereafter remains constant. That is, the probability that the female is married in i^{th} age group (i.e. x = i, for i= 0, 1, 2, 3 . . . k) then,

$$P(x=i) = P(\geq i)\cdot P(x=i/x\geq i)$$

In other words, the probability can be expressed as the product of the probabilities that the same did not marry in the preceding i age groups i.e. failure (1- p_i) and the same marries in the i^{th} age-group given that she did not marry in the preceding i^{th} age-groups i.e. success (p_i). i.e.

$$P(x=i/x\geq i) = p_i \text{ and } P(x=i) = 1-p_i = q_i$$

So that $P(x=0)=p_0$ *and* $P(x \geq 0)=1$

$$P(x=1)=q_0 \cdot p_1$$
$$P(x=2)=q_0 \cdot q_1 \cdot p_2$$
$$P(x=3)=q_0 \cdot q_1 \cdot q_2 \cdot p_3$$
$$\ldots$$
$$\ldots$$
$$P(x=k)=q_0 \cdot q_1 \cdot q_2 \cdot p_3 \ldots\ q_{k-1} p_k$$
$$P(x \geq k)=q_0 \cdot q_1 \cdot q_2 \cdot p_3 \ldots\ q_{k-1} p_k .$$

Of this, we have the different probabilities $p_0, p_1, p_2, p_3,\ p_4, \ldots$ and are defined as:

$$p_0=\alpha,\ p_1=\alpha+\beta \text{ and } p_2=\alpha+\gamma\beta$$

Finally the following expression developed by Mishra [1] is:

(M-1) $\quad p_i=\alpha+(\gamma+\delta)\beta$ for all i = 3, 4, 5, ...k.

Where α, β, γ *and* δ are the parameters of the model.

Aryal [1] highlighted that in some of the societies, the probability p_i (i= 3, 4, 5 . . . k) was not found constant always due to the occurrence of some marriages at the late ages and it decreases as i increases. So that a number of decreasing factors have been applied for getting the decreasing probability p_i after i= 4, 5, . . k. Aryal [1] found the appropriate decreasing factor, $\delta=\dfrac{\delta}{i-3}$ where δ is replacing by $\dfrac{\delta}{i-3}$ in model (M-1). Then his expression is given as:

(M-2) $\quad p_i=\alpha+(\gamma+\dfrac{\delta}{i-3})\beta$

i.e. start decreasing from i=5, where i = 4, 5, … k and i ≠3

and the respective different probabilities are:

$$p_4=\alpha+(\gamma+\delta)\beta,\ p_5=\alpha+(\gamma+\frac{\delta}{2})\beta,\ p_6=\alpha+(\gamma+\frac{\delta}{3})\beta,\ p_7=\alpha+(\gamma+\frac{\delta}{4})\beta$$

and so on.

In addition to this, Aryal [1] also highlighted that the model (M-2) has four parameters and the expected frequencies would be exactly equal to the observed frequencies for x=0, 1, 2. The difference between the observed and expected frequencies would start only from x=3 onwards. It is therefore, for a small value of k, the model (M-2) may not be appropriate so that Aryal [1] further developed a model and it is given below.

(M-3) $\quad p_i=\alpha+(\alpha+\beta)\gamma$ for all i = 3, 4, 5 . . . k.

The above said models have three and/or four parameters, which were estimated by using either iteration or maximum likelihood methods. The fitted models and estimated parameters were adopted from Aryal [1, 3] and presented in Tables 1 and 4.

4. Construction of female marriage life-table

Female marriage life-table may be constructed through the estimated parameters of the models discussed above. The procedure for constructing marriage life-table is similar to that of the construction of mortality life-table. Since marriage is a universal event, and each and every female would eventually have experienced it. Under this assumption, the expected waiting time for getting married in a certain age or age group is obtained by using life-table technique with the help of the models. Number of females who got married in a certain age or age group was considered as the number of death cases. The proportions of surviving i.e. not having married in a specified age or age interval are obtained by fitting the above discussed models. Once we get the proportions of surviving, i.e. proportions of not having get married in a specified age or age group; the following procedure can be followed for the construction of marriage life-table.

Let M_x be the proportion of females who got married at age x years or less. This value can be computed by fitting the above discussed models to estimate the value of proportion of surviving i.e not getting married. S_x be the proportion of surviving or not getting married at age x ($S_x = 1-M_x$). If l_o is the cohort of the life table, then $l_x = l_o * S_x$ and the probability of getting married between x and x+1 years of age is thus obtained by using following expression.

$$q_x = \frac{(M_{x+1} - M_x)}{(1 - M_x)} .$$

The other functions of the marriage life-table, L_x (person-year getting married by the cohort at age x), T_x (total person-year getting married by the cohort after age x) and e_x (average number of years expected to get married at age x) are computed by the usual process followed in the construction of a mortality life-table. The respective formulae of life-table functions are given below.

$$L_x = \frac{n(l_x + l_{x+n})}{2}, \; T_x = \sum_x^n L_x + L_{x+1} + \ldots + L_{x+n} \; ; \; T_{x+1} = T_x - L_x \text{ and } \; e_x = \frac{T_x}{l_x} .$$

5. Applications

Construction of female marriage life-table through mathematical models is given in the Tables 2, 3, 5 and 6 for the data consists of hilly, Tarai, rural and whole Nepal respectively. The female marriage life-table of Nepal was constructed using estimated

parameters of the fitted models. Model parameters were adopted by Aryal [1, 3] and portrayed in Tables 1 and 4. Table 1 presents the fitted models to the data of hilly and Tarai regions of Nepal. The corresponding constructed female marriage life-table is given in Tables 2 and 3 for hilly and Tarai regions of Nepal respectively.

Table 1 Observed and expected distribution of female age at marriage by different models

Re-scale age	Hill, DSMF, 2000				Tarai, DSMF, 2000			
	Obs	Exp.			Obs.	Exp.		
		M-1	M-2	M-3		M-1	M-2	M-3
0	10	10.00	10.00	10.00	26	26.00	26.00	26.00
1	82	82.00	82.00	82.00	104	104.00	104.00	104.00
2	197	197.00	197.00	205.86	206	206.00	206.00	205.60
3	103	114.58	115.11	106.45	166	166.88	167.31	163.92
4	50	42.56	42.76	41.19	35	36.20	36.28	38.09
5	19	15.81	14.53	15.94	8	7.85	6.41	8.85
6	6	5.87	5.99	6.17	2	1.70	2.15	2.06
7	3	2.18	2.61	2.39	2	0.37	0.85	0.48
Total	470	470.00	470.00	470.00	549	549.00	549.00	549.00
$\chi 2$.		3.22	3.87	2.29		0.484	0.765	0.131
d.f.		2	2	3		1	1	2
Parameter								
a		0.021277		0.021277		0.047359		0.047359
b		0.156984		0.156984		0.151494		0.151494
r		3.184320		3.293200		2.932711		2.925300
c		0.683560		-		1.923783		-

This table adopted from Aryal [1,3]

Table 2 Female marriage life-table for hilly region of Nepal using different models

Model: M-1							
Age(x)	Mx	Sx	lx	qx	Lx	Tx	ex
0-12	0.021277	1.000000	100000.00	0.178261	1187234.00	1755006.00	17.55
12-15	0.195745	0.978723	97872.34	0.521164	267446.82	567772.31	5.80
15-18	0.614894	0.804254	80425.53	0.633039	178404.34	300325.52	3.73
18-21	0.858681	0.385106	38510.64	0.640771	78963.83	121921.32	3.16
21-24	0.949234	0.141319	14131.91	0.662615	28812.77	42957.45	3.03
24-27	0.982872	0.050765	5076.60	0.729193	10184.04	14144.68	2.78
27-30	0.995362	0.017127	1712.77	0.869821	3264.89	3960.64	2.31
30+	0.999999	0.004636	463.83	0.987872	695.75	695.75	1.50

Model: M-2							
0-12	0.021277	1.000000	100000.00	0.178261	1187234.00	1754828.00	17.54
12-15	0.195745	0.978723	97872.34	0.521164	267446.82	567593.61	5.79
15-18	0.614894	0.804255	80425.53	0.635967	178404.31	300146.82	3.73
18-21	0.859809	0.385106	38510.64	0.64896	78794.68	121742.61	3.16
21-24	0.950787	0.140191	14019.15	0.628188	28410.64	42947.87	3.06
24-27	0.981702	0.049212	4921.28	0.696512	10126.62	14537.23	2.95
27-30	0.994447	0.018297	1829.79	0.879821	3577.66	4410.63	2.41
30+	0.999991	0.005553	555.32	0.988872	832.97	832.97	1.50
Model: M-3							
0-12	0.021277	1.000000	100000.00	0.178261	1187234.00	1749753.00	17.49
12-15	0.195745	0.978723	97872.34	0.544603	267446.82	562519.10	5.74
15-18	0.633745	0.804255	80425.53	0.618392	175576.61	295072.30	3.66
18-21	0.860234	0.366255	36625.53	0.627036	75903.19	119495.70	3.26
21-24	0.947872	0.139765	13976.60	0.650612	28784.04	43592.55	3.11
24-27	0.981787	0.052124	5212.77	0.720794	10551.06	14808.51	2.84
27-30	0.994915	0.018212	1821.28	0.879821	3494.68	4257.44	2.33
30+	0.999991	0.005085	508.51	0.988872	762.76	762.76	1.50

The values presented in the last column of each table (Tables 2,3,5 and 6), e_x, show the expected waiting time for getting married at the specific age x. The expected waiting time for getting married at birth was found to be 17.5 years for hilly girls (Table 2) whereas it was 16.9 years for Tarai girls (Table 3). The different models were provided the consistent expected waiting time for getting married among Nepalese girls both at hilly as well as Tarai regions. The waiting time for getting married at age of 12 years was found to be 5.8 years and at the age of 24 years was found to be 2.8 years among hilly girls.

The values presented in the last column, e_x, show the expected waiting time for getting married at the specific age x (Table 3). Among Tarai girls, the expected waiting time for getting married at birth was found to be 16.9 years. The waiting time for getting married at age of 12 years was found to be 5.5 years and at age of 24 years was found to be 2.2 years among the Tarai girls (Table 3). Slightly different waiting time for getting married among hilly and Tarai girls of Nepal where girls belong to Tarai got married early than hilly girls, which might be the climatic as well as nutritional effects.

Table 3 Female marriage life-table for Tarai region of Nepal using different models

Age(x)	Mx	Sx	lx	qx	Lx	Tx	ex
Model: M-1							
0-12	0.047359	1.000000	100000.00	0.198853	1171585.00	1691792.00	16.91
12-15	0.236794	0.952641	95264.12	0.491647	257377.00	520207.70	5.46
15-18	0.612022	0.763205	76320.58	0.783474	172677.60	262830.60	3.44
18-21	0.915993	0.387978	38797.81	0.784909	70797.81	90153.01	2.32
21-24	0.981931	0.084007	8400.73	0.791331	15311.48	19355.19	2.30
24-27	0.996230	0.018069	1806.92	0.821256	3275.95	4043.716	2.23
27-30	0.999326	0.003770	377.05	0.896777	666.66	767.759	2.03
30+	0.999999	0.000673	67.40	0.999998	101.09	101.09	1.50
Model: M-2							
0-12	0.047359	1.000000	100000.00	0.198853	1171585.00	1692049.00	16.92
12-15	0.236794	0.952641	95264.12	0.491647	257377.00	520464.50	5.46
15-18	0.612022	0.763205	76320.58	0.785493	172677.60	263087.40	3.44
18-21	0.916776	0.387978	38797.81	0.794047	70680.33	90409.84	2.33
21-24	0.98286	0.083224	8322.40	0.68119	15054.64	19729.51	2.37
24-27	0.994536	0.017140	1714.03	0.716667	3390.71	4674.86	2.72
27-30	0.998452	0.005464	546.45	0.896777	1051.91	1284.15	2.35
30+	0.999999	0.001548	154.83	0.999998	232.24	232.24	1.50
Model: M-3							
0-12	0.047359	1.000000	100000.00	0.198853	1171585.00	1694967.00	16.94
12-15	0.236794	0.952641	95264.12	0.490692	257377.00	523382.50	5.49
15-18	0.611293	0.763205	76320.58	0.768135	172786.90	266005.50	3.48
18-21	0.909872	0.388706	38870.67	0.769806	71825.14	93218.58	2.39
21-24	0.979253	0.090127	9012.75	0.776997	16631.15	21393.44	2.37
24-27	0.995373	0.020746	2074.68	0.811024	3806.01	4762.29	2.29
27-30	0.999126	0.004626	462.66	0.896777	825.13	956.28	2.06
30+	0.999999	0.000874	87.43	0.999998	131.14	131.14	1.50

Table 4 Observed and expected distribution of female age at marriage different models

Rescale age	Rural Nepal, DSMF, 2000				NEPAL, NFHS, 1996			
	Obs	Exp			Obs	Exp		
		M-1	M-2	M-3		M-1	M-2	M-3
0	36	36.00	36.00	36.00	309	309.00	309.00	309.00
1	186	186.00	186.00	186.00	2189	2189.00	2189.00	2189.00
2	403	403.00	403.00	422.90	3741	3741.00	3741.00	3701.74
3	269	277.84	279.05	241.78	1549	1568.09	1568.09	1598.01
4	85	82.36	82.72	86.44	450	445.89	446.15	453.12
5	27	24.41	21.07	30.89	139	126.80	119.34	128.44
6	8	7.24	7.88	11.04	39	36.05	38.18	36.43
7	5	2.15	3.28	3.95	6	10.25	12.88	10.33
8	-	-	-	-	7	2.92	4.45	2.93
Total	1019	1019.0	1019.0	1019.00	8429	8429.00	84296.00	8429.00
χ^2.		4.52	2.98	5.63		9.22	8.87	9.89
d.f.		3	3	4		4	4	5
Parameter								
a			0.035329	0.035329			0.036666	0.03666
b			0.153888	0.153888			0.232922	0.23292
r			3.056233	3.209100			2.550610	2.52165
c			1.286165	-			0.36450	-

This table adopted from Aryal [1,3]

Table 5 Female marriage life-table for Rural Nepal using different models

Age(x)	Mx	Sx	lx	qx	Lx	Tx	ex
Model: M-1							
0-12	0.035329	1.000000	100000.00	0.189217	1178803.00	1721688.0(	17.21
12-15	0.217861	0.964671	96467.12	0.505646	262021.60	542885.20	5.62
15-18	0.613346	0.782139	78213.94	0.705178	175318.90	280863.60	3.59
18-21	0.886006	0.386653	38665.36	0.709022	75097.15	105544.70	2.72
21-24	0.966830	0.113994	11399.41	0.722189	22074.58	30447.50	2.67
24-27	0.990785	0.033169	3316.98	0.771033	6357.70	8372.91	2.52
27-30	0.997890	0.009214	921.49	0.892662	1698.72	2015.21	2.18
30+	0.999999	0.002109	210.99	0.987647	316.48	316.48	1.50
Model: M-2							
0-12	0.035329	1.000000	100000.00	0.189217	1178803.00	1721723.0(	17.21
12-15	0.217861	0.964671	96467.12	0.505646	262021.60	542920.50	5.62

15-18	0.613346	0.782139	78213.94	0.708249	175318.90	280898.90	3.59
18-21	0.887193	0.386653	38665.36	0.719617	74919.04	105580.00	2.73
21-24	0.968371	0.112806	11280.67	0.653739	21665.36	30660.94	2.71
24-27	0.989048	0.031629	3162.90	0.706093	6387.14	8995.58	2.84
27-30	0.996781	0.010951	1095.19	0.892662	2125.61	2608.44	2.38
30+	0.999999	0.003218	321.88	0.987647	482.82	482.82	1.50
Model: M-3							
0-12	0.035329	1.000000	100000.00	0.189217	1178803.00	1726322.00	17.26
12-15	0.217861	0.964671	96467.12	0.530615	262021.60	547519.10	5.67
15-18	0.632875	0.782139	78213.94	0.646298	172389.60	285497.50	3.65
18-21	0.870147	0.367124	36712.46	0.653265	74546.61	113107.90	3.08
21-24	0.954975	0.129852	12985.28	0.673278	26231.60	38561.33	2.96
24-27	0.985289	0.045024	4502.45	0.736491	8960.25	12329.74	2.73
27-30	0.996124	0.014710	1471.05	0.892662	2788.02	3369.48	2.29
30+	0.999999	0.003876	387.63	0.987647	581.45	581.45	1.50

The waiting time for getting married was found to be 17.2 years among girls of rural Nepal. The waiting time for getting married at age of 12 years was found to be 5.6 years and at the age of 24 years was found to be 2.5 years for rural girls. Likewise the waiting time for getting married at 30 years was found to wait another one and half years (Table 5).

Table 6 Female marriage life-table for Nepal using by different models

Model: M-1							
Age(x)	Mx	Sx	lx	qx	Lx	Tx	ex
0-12	0.036659	1.000000	100000.00	0.269581	1178005.00	1642267.00	16.42
12-15	0.296358	0.963340	96334.08	0.630754	250047.50	464262.90	4.81
15-18	0.740183	0.703642	70364.22	0.716023	144518.90	214215.40	3.04
18-21	0.926218	0.259817	25981.73	0.716969	50039.92	69696.52	2.68
21-24	0.979117	0.073782	7378.22	0.720373	14199.73	19656.60	2.66
24-27	0.994161	0.020882	2088.27	0.732426	4008.30	5456.87	2.61
27-30	0.998438	0.005839	583.94	0.778284	1110.27	1448.57	2.48
30-33	0.999654	0.001562	156.25	0.892662	286.33	338.29	2.16
33+	0.999999	0.000346	34.64	0.987647	51.96	51.96	1.50

Model: M-2							
0-12	0.036659	1.000000	100000.00	0.269581	1178005.00	1642798.00	16.42
12-15	0.296358	0.963340	96334.08	0.630754	250047.50	464793.20	4.82
15-18	0.740183	0.703642	70364.22	0.715982	144518.90	214745.80	3.05
18-21	0.926207	0.259817	25981.73	0.717283	50041.52	70226.84	2.70
21-24	0.979138	0.073792	7379.29	0.678647	14198.30	20185.31	2.73
24-27	0.993296	0.020862	2086.25	0.675633	4135.01	5987.01	2.86
27-30	0.997825	0.006704	670.42	0.702673	1331.83	1851.99	2.76
30-33	0.999353	0.002174	217.46	0.816514	423.18	520.16	2.39
33+	0.999881	0.000646	64.66	0.987647	96.98	96.98	1.50
Model: M-3							
0-12	0.036659	1.000000	100000.00	0.269581	1178005.00	1644093.00	16.44
12-15	0.296358	0.963340	96334.08	0.624134	250047.50	466088.00	4.83
15-18	0.735525	0.703642	70364.22	0.716834	145217.60	216040.60	3.07
18-21	0.92511	0.264475	26447.50	0.717814	50904.79	70822.99	2.67
21-24	0.978867	0.074890	7489.03	0.721046	14403.49	19918.20	2.65
24-27	0.994105	0.021132	2113.30	0.733146	4054.21	5514.71	2.60
27-30	0.998427	0.005895	589.51	0.779035	1120.24	1460.49	2.47
30-33	0.999652	0.001573	157.31	0.892662	288.13	340.25	2.16
33+	0.999881	0.000347	34.76	0.987647	52.14	52.14	1.50

The waiting time for getting married was found to be 16.42 years for Nepalese girls (Table 6). The waiting time for getting married at age of 12 years was found to be 4.8 years and at the age of 24 years was found to be 2.6 years among Nepalese girls. Likewise the expected waiting time for getting married at 30 years was found to wait another one and half years.

6. Conclusions

This paper developed a technique to construct female marriage life-table of Nepal by using model estimated parameters. Female marriage life-table was constructed for girls belong to hills, Tarai, and rural Nepal. The expected waiting time for Nepalese girls at birth was found to be 16.4 years whereas it was 17.5 years among hilly girls, 16.9 years

among Tarai girls and 17.2 years among rural girls. Expected waiting time for getting married at the age of 12 years was found to wait more than 5 years and at the age of 24 years was found to wait more than 2 years and at the age of 30 years was found to wait another one and half years. The expected waiting time at birth was around 17 years, which is consistent value to the average age at marriage of 17 years [1]. The finding of this paper may help planners and policy-makers to design proper policy of a country.

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α- Conformal change of a Finsler Space with (α, β)-Metric of Douglas Type

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Abstract: A change of Finsler metric $L(\alpha, \beta) \rightarrow L(e^{\sigma}\alpha, \beta)$ is called an α- conformal change, where σ is a function of position x^i only, α is Riemannian fundamental function and β a differentiable one-form [6]. M. Matsumoto found the several conditions under which a Finsle space with (α, β)-metric is of Douglas type ([1], [7]). The purpose of the present paper is to find the condition that α- conformal change of Finsler space with (α, β)-metric of Douglas type remains to be Douglas type.

Key words: (α, β)-metric, Douglas space, conformal change.

Mathematics subject classification: 2000: 53B 20, 53B 28, 53B 40, 53B 18.

1. Introduction

First time M. Matsumoto introduce (α, β)-metric notation in 1972 on the study of C-reducible Finsler space [4] and in 1991 he studied about its Berwald connection [5]. Y. Ichijyo and M. Hashiguchi [2] studied about conformal change of (α, β)-metric. Some properties of α- conformal change was studied by M. Matsumoto in 1992 [6]. The notion of Douglas space and the condition that the Finsler space with (α, β)-metric of Douglas type has been investgated by M. Matsumoto and S. Bacso ([1], [7]). The present paper is devoted to study the α- conformal change of

Finsler space with (α, β)-metric of Douglas type and to found the conditions on conformal factor, so that a Douglas space is α- conformally changed to a Douglas space.

2. Preliminaries

Let $\alpha(x, y) = \sqrt{a_{ij}(x)y^i y^j}$ be Riemannian metric and $\beta(x, y) = b_i(x)y^i$ be a differentiable one-form in an n-dimensional differentiable manifold M^n. If a Finsler fundamental metric function $L(\alpha, \beta)$ is positively homogeneous of degree one in α and β in M^n, then $F^n =$ (M^n, $L(\alpha, \beta)$ is called a Finsler space with (α, β)-metric [5]. The space $R^n = (M^n, \alpha)$ is called a Riemannian space associated with F^n [5] and Christoffel symbols of R^n are indicated by $\gamma_j{}^i{}_k$, and covariant differentiation with respect to $\gamma_j{}^i{}_k(x)$ by ∇.

We shall use the symbols as follows:

$$\text{(2.1)} \quad r_{ij} = \frac{1}{2}(\nabla_j b_i + \nabla_i b_j) \; s_{ij} = \frac{1}{2}(\nabla_j b_i - \nabla_i b_j), \qquad s^i{}_j = a^{ir} s_{rj},$$
$$s_j = b_r s^r{}_j .$$

It is to be noted that $s_{ij} = \frac{1}{2}(\partial_j b_i - \partial_i b_j)$. Throughout the paper the symbols ∂_j and $\dot\partial_j$

stand for $\frac{\partial}{\partial_{x^j}}$ and $\frac{\partial}{\partial_{y^j}}$ respectively. We are concerned with the Berwald connection $B\Gamma = (G^i_{jk}, G^i_j)$ which is given by

$$2G^i(x,y) = g^{ij}(y^r \dot\partial_j \partial_r F - \partial_j F), \text{where } F = L^2/2, \; G^i_j = \dot\partial_j G^i \text{ and } G^i_{jk} = \dot\partial_k G^i_j .$$

The Finsler space F^n is said to be of Douglas type (Douglas space) [1] if $D^{ij} = G^i(x, y)\, y^j - G^j(x, y)\, y^i$ are homogeneous polynomial in y^i of degree three. We shall denote the "homogeneous polynomials in y^i of degree r" by hp(r).

For a Finsler space F^n with (α, β)-metric ([3], [5]), we have

$$\text{(2.2)} \qquad 2G^i = \gamma^i_{00} + 2B^i ,$$

where

(2.3) $$B^i = \frac{E}{\alpha}y^i + \frac{\alpha L_\beta}{L_\alpha}s_0^i - \frac{\alpha L_{\alpha\alpha}}{L_\alpha}C^*\left(\frac{y^i}{\alpha} - \frac{\alpha}{\beta}b^i\right), \qquad E = \frac{\beta L_\beta}{L}C^*,$$

$$C^* = \frac{\alpha\beta(r_{00}L_\alpha - 2\alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha r^2 L_{\alpha\alpha})}, \quad b^i = a^{ij}\, b_j, \quad r^2 = b^2\,\alpha^2 - \beta^2, \qquad b^2 = a^{ij}\, b_i\, b_j$$

and the subscript α and β in L denote the partial differentiation with respect to α and β respectively. Since $\gamma^i_{00} = \gamma^i_{jk}(x)y^j y^k$ is homogenous polynomial in (y^i) of degree two, we have

Proposition (2.1) [7]. A Finsler space with (α, β)-metric is a Douglas space if and only if $B^{ij} = B^i y^j - B^j y^i$ are hp(3).

Equation (2.3) gives

(2.3) $$B^{ij} = \frac{\alpha L_\beta}{L_\alpha}(s_0^i y^j - s_0^j y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}C^*(b^i y^j - b^j y^i).$$

3. α- conformal change of Douglas space with (α, β)-metric

Let $F^n = M^n, L)$ and $\overline{F}^n = (M^n, \overline{L})$ be two Finsler spaces on the same underlying manifold M^n. If we have a function $\sigma(x)$ in each coordinate neighbourhoods of M^n such that $\overline{L}(\alpha, \beta) = L(e^\sigma\alpha, \beta)$ then F^n is called α-conformal to $\overline{F}^n$, and change $L \to \overline{L}$ of metric is called α-conformal. Therefore we have

(3.1) $$\overline{\alpha} = e^\sigma\alpha, \quad \overline{\beta} = \beta, \quad \overline{a_{ij}} = e^{2\sigma}\, a_{ij}, \quad \overline{b_i} = b_i, \quad \overline{a^{ij}} = e^{-2\sigma}a^{ij}, \quad \overline{b}^{\,i} = e^{-2\sigma}\, b^i.$$

From (3.1), it follows that the α-conformal change of Christoffel symbols is given by

(3.2) $$\overline{\gamma}^i_{jk} = \gamma^i_{jk} + \delta^i_j\sigma_k + \delta^i_k\sigma_j - \sigma^i a_{jk}$$

where $\sigma_j = \partial_j\sigma$ and $\sigma^i = a^{ij}\sigma_j$. From (2.1), (3.1) and (3.2) we have the following α-conformal changes (3.3) (a) $\overline{\nabla}_j\overline{b}_i = \nabla_j b_i - (b_j\sigma_i + b_i\sigma_j) + \rho a_{ij}$,

where $\rho = b_r\sigma^r$,

(b) $\overline{r}_{ij} = r_{ij} - (b_i\sigma_j + b_j\sigma_i) + \rho a_{ij}$ (c) $\overline{s}_{ij} = s_{ij}$.

(d) $\bar{s}^i_j = e^{-2\sigma} s^i_j$, (e) $\bar{s}_j = e^{-2\sigma} s_j$.

From (3.2) and (3.3), we can easily obtain the following:

(3.4) (a) $\bar{\gamma}^i_{0\,0} = \gamma^i_{0\,0} + 2\sigma_0\, y^i - \alpha^2\, \sigma^i$, (b) $\bar{r}_{00} = r_{00} + \rho\, \alpha^2 - \sigma_0\, \beta$

(c) $\bar{s}^i_0 = e^{-2\sigma} s^i_0$ (d) $\bar{s}_0 = e^{-2\sigma} s_0$.

To find the α-conformal change of B^{ij} given in (2.4), we first find the α-conformal change of C* given in (2.3). Since $\bar{L}(\bar{\alpha}, \bar{\beta}) = L(e^{\sigma}\alpha, \beta)$, we have

(3.5) $\bar{L}_{\bar{\alpha}} = e^{-\sigma} L_{\alpha}$, $\bar{L}_{\bar{\alpha}\bar{\alpha}} = e^{-2\sigma} L_{\alpha\alpha}$, $\bar{L}_{\bar{\beta}} = L_{\beta}$, $\bar{b}^2 = e^{-2\sigma} b^2$, $\bar{r}^2 = r^2$.

Thus we have the following:

Proposition (3.1). In a Finsler space with (α, β) –metric r is invariant under α-conformal change.

From (2.3), (3.4) and (3.5), we have

$$\bar{C}^* = e^{\sigma}(C^* + D^*), \tag{3.6}$$

where

$$D^* = \frac{\alpha\beta L_{\alpha}(\rho\alpha^2 - 2\sigma_0\beta)}{2(\beta^2 L_{\alpha} + r^2 \alpha L_{\alpha\alpha})} \tag{3.7}$$

Hence from equation (2.4), the α-conformal change of B^{ij} is written in the form

$$\bar{B}^{ij} = B^{ij} + C^{ij}, \tag{3.8}$$

where

$$C^{ij} = \frac{D^* \alpha^2 L_{\alpha\alpha}(b^i y^j - b^j y^i)}{\beta L_{\alpha}}. \tag{3.9}$$

Theorem (3.1). The α-conformal change of a Douglas space with (α, β) –metric is a Douglas space with (α, β) –metric if and only if C^{ij} is hp(3).

4. α-conformal change of Douglas space with Randers metric.

For a Randers metric we have $L = \alpha + \beta$, so that $L_\alpha = L_\beta = 1$ and $L_{\alpha\alpha} = 0$. Hence from (3.9), we get $C^{ij} = 0$. Thus we have the following:

Theorem (4.1). The α-conformal change of Douglas space with Randers metric is always a Douglas space with Randers metric.

Remark 1. We know that a Finsler space with Randers metric is Douglas space if and only if $s_{ij} = 0$ [7]. From equation (3.3)(c) it follows that the α-conformal change of Douglas space with Randers metric is always a Douglas space with Randers metric.

5. α-conformal change of Douglas space with Kropina metric.

For a Kropina metric, we have $L = \frac{\alpha^2}{\beta}$, so that $L_\alpha = \frac{2\alpha}{\beta}$, $L_{\alpha\alpha} = \frac{2}{\beta}$, $L_\beta = -\frac{\alpha^2}{\beta^2}$.

Hence the value of D* given by (3.7) reduces to

$$D^* = \frac{\beta(\rho\alpha^2 - \sigma_0\beta)}{2b^2\alpha}. \tag{5.1}$$

Therefore, the value of C^{ij} given by (3.9) reduces to

$$C^{ij} = \frac{1}{2b^2}(\rho a_{hk} - \sigma_k b_h)y^h y^k (b^i y^j - b^j y^i),$$

which shows that C^{ij} is hp(3).

Theorem (5.1). The α-conformal change of Douglas space with Kropina metric with $b^2 \neq 0$ is a Douglas space with Kropina metric.

Remark 2. We know that a Finsler space with Kropina metric is Douglas space if and only if $A_{ij} = 0$ [7], where $A_{ij} = s_{ij} - \frac{1}{b^2}(s_j b_i - s_i b_j)$. From equations (3.1) and (3.3) it follows that $\overline{A}_{ij} = A_{ij}$. This proves the theorem (5.1) in another way.

6. α-conformal change of Douglas space with generalized Kropina metric.

We know that the Finsler space with genetalized Kropina metric $L = \frac{\alpha^{m+1}}{\beta^m}$ $(m \neq 0.\ 1)$ is a Douglas space [7] if and only if there exists a scalar k(x) such that

$$s_{ij} = \frac{1}{b^2}(s_j b_i - s_i b_j). \qquad (b^2 \neq 0)$$

and $$r_{ij} = \frac{k}{m(m+1)}\{(1-m)b_i b_j - mb^2 a_{ij}\} + \frac{1-m}{(1+m)b^2}(s_i b_j + s_j b_i).$$

We put $A_{ij} = s_{ij} - \frac{1}{b^2}(s_j b_i - s_i b_j)$,

and $$K_{ij} = r_{ij} - \frac{k}{m(m+1)}\{(1-m)b_i b_j - mb^2 a_{ij}\} - \frac{1-m}{(1+m)b^2}(s_i b_j + s_j b_i).$$

From (3.1) and (3.3), it follows that $\overline{A}_{ij} = A_{ij}$ and $\overline{K}_{ij} = K_{ij} - (b_i \sigma_j + b_j \sigma_i) + \rho a_{ij}$, with the assumption that the scalar k of F^n and scalar $\overline{k}$ of $\overline{F}^n$ are related by $\overline{k} = k$. Hence α-conformal change of Douglas space with generalized Kropina metric is a Douglas space with generalized Kropina metric if and only if

(6.1) $\qquad \rho a_{ij} = b_i \sigma_j + b_j \sigma_i.$

Contracting (6.1) by g^{ij} we get $\rho(n-2) = 0$, which gives $\rho = 0$ for $n > 2$. Putting $\rho = 0$ in (6.1) and contracting by b^j we get $\sigma_i = 0$ provided $b^2 \neq 0$. Thus α-conformal change is α- homothetic. Conversely, if α-conformal change is α- homothetic, then $\sigma_i = 0$ and hence $\rho = 0$, so that $\overline{A}_{ij} = A_{ij}$ and $\overline{K}_{ij} = K_{ij}$. Therefore, we have the following:

Theorem (6.2). The α-conformal change of Douglas space with generalized Kropina metric ($b^2 \neq 0$, $n > 2$) is a Douglas space with generalized Kropina metric if and only if α-conformal change is α- homothetic.

Remark 3. If α-conformal change is α- homothetic then $\sigma_i = 0$ implies that $\rho = 0$. Then from equations (3.7) and (3.9), we get $D^* = 0$ and $C^{ij} = 0$. So from equation (3.8) we get $\overline{B}_{ij} = B_{ij}$. This shows that Douglas space with (α, β)-metric is α-homothetic to a Douglas space with (α, β)-metric.

7. α-conformal change of Douglas space with Matsumoto metric metric.

It is known that [8] Matsumoto space $F^n = (M^n, L = \frac{\alpha^2}{\alpha - \beta})$ is a Douglas space if and only if $\nabla_j b_i = 0$. Then from equation (3.3)(a), the Finsler space $\overline{F}^n$ $(n > 2)$ which is obtained by α- conformal change of Matsumoto space F^n of Douglas

type is also of Douglas type if and only if $\rho\, a_{ij} = b_i\, \sigma_j + b_j\, \sigma_i$ As in section 6 this equation gives $\sigma_i = 0$ provided $b^2 \neq 0$, $n > 2$. Also, it has been shown [8] that Matsumoto space is a Berwald space if and only if $\nabla_j b_i = 0$. Thus a Matsumoto space is a Douglas space if and only if it is a Berwald space. Hence we have the following:

Theorem (7.1). The α-conformal change of Douglas space with Matsumpto metric ($b^2 \neq 0$, $n > 2$) is a Douglas space with Matsumoto metric if and only if α-conformal change is α- homothetic.

8. α-conformal change of Douglas space with metric $L = \alpha + (\beta^2/\alpha)$.

A Finsler space with metric $L = \alpha + (\beta^2/\alpha)$ ($b^2 \neq 0,1$) is a Douglas space if and only if there exist a scalar function $k(x)$ such that $H_{ij} = 0$ [5], where

(8.1) $$H_{ij} = \nabla_j b_i - k\{(1+2b^2)a_{ij} - 3b_i b_j\}.$$

From (8.1), (3.1) and (3.2), we get

$$\overline{H}_{ij} = [\nabla_j b_i - (b_i\, \sigma_j + b_j\, \sigma_i) + \rho\, a_{ij}] - \overline{k}[(1 + 2e^{-2\sigma}\, b^2)e^{2\sigma}\, a_{ij} - 3\, b_i\, b_j]$$

which may be written as

(8.2) $$\overline{H}_{ij} = H_{ij} + a_{ij}\,(k - k\, e^{2\sigma} + \rho) - (b_i\, \sigma_j + b_j\, \sigma_i)$$

with $\overline{k} = k$ Hence the α-conformal change of Douglas space with metric $L = \alpha + (\beta^2/\alpha)$ ($b^2 \neq 0,1$) remains to be Douglas space if and only if

(8.3) $$a_{ij}\,(k - k\, e^{2\sigma} + \rho) = (b_i\, \sigma_j + b_j\, \sigma_i).$$

Contracting (8.2) by g^{ij} we get $n\,(k - k\, e^{2\sigma} + \rho) = 2\,\rho$. Hence (8.2) reduces to

(8.4) $$2\rho\, a_{ij} = n\,(b_i\, \sigma_j + b_j\, \sigma_i).$$

Contracting (8.4) by b^j, we get

(8.5) $$\rho(2 - n)b_i = b^2\, \sigma_i\,.$$

Eliminating σ_i from equations (8.4) and (8.5)

(8.6) $$\rho\, a_{ij} = \frac{n\rho}{(2-n)b^2}\, b_i\, b_j.$$

provided $n > 2$. Since rank of matrix $||a_{ij}||$ is n where as that of $||b_i\ b_j||$ is one equation (8.6) is only possible when $\rho = 0$. Hence (8.5) gives $\sigma_i = 0$. Thus the transformation is α- homothetic.

Theorem (8.1). The α-conformal change of Douglas space with metric $L = \alpha + (\beta^2/\alpha)$ ($b^2 \neq 0,1$, $n > 2$) is a Douglas space if and only if α-conformal change is α-homothetic.

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Additive Derivation on Involution *- Rings

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Abstract: In this paper author gives some results on additive derivation on involution * -rings.

Key words: Additive mapping, Jordan subring, Jordan ideal, torsion free.

AMS classification: 16W10, 17C50, 17B60

1. Introduction

A ring R is prime if $xRy = 0$, $x, y \in R$ implies $x = 0$ or $y = 0$. We say that R is 2 torsion free if in R, $2x = 0$ forces $x = 0$. An additive mapping $x \mapsto x^*$ satisfying $(x\,y)^* = y^* x^*$ and $(x^*)^* = x$ is an involution. A ring equipped with an involution is a *- ring. We say that a subset A of a ring R is a Jordan subring of R if A is an additive subgroup such that for x, y in A, $xoy = xy + yx$ must also in A. A subset A of a ring is called a Lie subring of R if A is an additive subgroup such that for x, y in A, [x,y] = xy-yx must also in A.

Let A be a Jordan subring of R. The additive subgroup $U \subset A$ is said to be a Jordan ideal of A if whenever $u \in U$ and $x \in A$ then $uox = ux + xu$ is in U. Let A be a Lie subring of R. The additive subgroup $U \subset A$ is said to be a Lie ideal of A if whenever $u \in U$ and $x \in A$ then $[u,x] = ux - xu$ is in U.

Combining (1) and (2), we have

$$2D(x)\,y^*x^*+2xD(y)\,x^*+2xyD(x) = 2D(xyx)$$

$$D\,(xyx) = D(x)\,y^*x^*+xD(y)x^*+xyD(x)$$

Proof of the theorem 2.3: Since $D\,(xyx) = D(x)y^*x^*+xD(y)x^*+xyD(x)$ for all $x,y \in U$.

Linearizing on x

$$D\,((x+z)y(x+z)) = D(x+z)y^*(x+z)^*+(x+z)D(y)(x+z)^*+(x+z)yD(x+z)$$

$$D(xyx+xyz+zyx+zyz) = (D(x)+D(z))y^*(x^*+z^*)+(x+z)D(y)(x^*+z^*)+(x+z)y(D(x)+D(z))$$

$$D(xyx)+D(zyz)+D(xyz+zyx) = D(x)y^*x^* +D(x)y^*z^*+ D(z)y^*x^*+D(z)y^*z^*+xD(y)x^*+ xD(y)z^* +zD(y)x^*+zD(y)z^*+xyD(x)+xyD(z)+zyD(x)+zyD(z)$$

$$D\,(xyz+zyx) = D(x)\,y^*z^*+ xD(y)z^*+xyD(z)+D(z)y^*x^*+zD(y)x^*+zyD(x)$$

for all $x,y,z \in U$.

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