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Approximation of the Lip ($\xi(t)$, p) Class Functions by Matrix-cesáro Summability Method

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Abstract. The degree of approximation of functions belonging to Lipα, Lip(α,p) and Lip($\xi(t)$, p) class by Cesàro, Nörlund, Euler and matrix summability method has been determined by number of researchers of Modern Analysis. Most of the summability methods are derived from matrix method. In this paper, I have taken product of two summability methods, matrix and Cesàro; and established a new theorem on the degree of approximation of the function f belonging to Lip ($\xi(t)$, p) class by matrix- Cesàro method.

Subject classification: 40C05, 40G05, 42A10, 42B08.

Key words and phrases: Degree of approximation, the Lip ($\xi(t)$, p) class functions, matrix-Cesàro summability method, Fourier series.

1. INTRODUCTION

Bernstin [3], used (C,1) means to obtain the degree of approximation function f by lip 1 class. Jackson [6] determined the degree of approximation by using (C,δ) method in Lip α class, for $0<\alpha<1$. Results of Alexits [2], Chandra [4], Sahney & Goel [14], Sahney & Rao [15], Alexits & Leindler [1] for the degree of

approximation of functions $f \in Lip\alpha$ are not satisfied for n= 0, 1 or $\alpha = 1$. Above mentioned results have been generalized by number of researchers like Khan [7], Qureshi [10, 11, 12 & 13], Lal & Nigam [8], Lal & Singh [9] and Dhakal [5]; and determined the degree of approximation of a function f belonging to Lip α, Lip (α,p) and Lip $(\xi\ (t),\ p)$ by using Cesàro, Nörlund, generalized Nörlund, Riesz, matrix and (C,1)(E,1) summability method. But till now no work seems to have been done to obtain the degree of approximation of functions by product summability of matrix means and Cesàro means of order one i.e. $T(C_1)$. In an attempt to make an advanced study in this direction, in this paper, a new theorems on the approximation of function $f \in$ Lip $(\xi(t),p)$ class has been established.

2. DEFINITIONS AND NOTATIONS

Let f be 2π-periodic, integrable over $(-\pi,\pi)$ in the sense of Lebesgue, then its Fourier series is given by $f(t) \sim \frac{1}{2}a_o + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$

(1)

with partial sum $S_n(x)$.

The L^p norm is defined by $\|f\|_p = \left(\int_0^{2\pi} |f(x)|^p dx\right)^{\frac{1}{p}}$, $p \geq 1$

and the degree of approximation $E_n(f)$ under norm $\| \ \|_p$ is given by(Zygmund [17])

$$E_n(f) = \min \|T_n - f\|_p,$$

where $T_n(x)$ is a trigonometric polynomial of degree n.

A function $f \in Lip\alpha$ if $|f(x+t) - f(x)| = O\left(|t|^{\alpha}\right)$, for $0 < \alpha \leq 1$.

$f \in \mathrm{Lip}\,(\alpha, p)$, for $0 \le x \le 2\pi$, if $\left(\int_0^{2\pi} |f(x+t)-f(x)|^p dx\right)^{\frac{1}{p}} = O\left(|t|^{\alpha}\right)$, $0 < \alpha \le 1$,

$p \ge 1$.

Given a positive increasing function $\xi(t)$, $p \ge 1$, $f \in \mathrm{Lip}\,(\xi(t), p)$ if

$$\left(\int_0^{2\pi} |(f(x+t)-f(x))|^p dx\right)^{\frac{1}{p}} = O\,(\xi(t)).$$

It is observed that $\mathrm{Lip}\,(\xi(t), p) \xrightarrow{\xi(t)=t^{\alpha}} \mathrm{Lip}(\alpha, p) \xrightarrow{p\to\infty} \mathrm{Lip}\alpha$.

Let $\sum_{n=0}^{\infty} u_n$ be the infinite series whose nth partial sum is given by $S_n = \sum_{ki=0}^{n} u_k$.

Cesàro means (C, 1) of sequence $\{S_n\}$is given by $\sigma_n = \frac{1}{n+1} \sum_{k=0}^{n} S_k$.

If $\sigma_n \to S$, as $n \to \infty$ then sequence $\{S_n\}$ or the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable by Cesàro means (C,1) to S.

Let $T = (a_{n,k})$ be an infinite lower triangular matrix satisfying the Silverman-Töeplitz [16] conditions of regularity i.e.

$\sum_{k=0}^{n} a_{n,k} \to 1$ as $n \to \infty$, $a_{n,k} = 0$, for $k > n$ and $\sum_{k=0}^{n} |a_{n,k}| \le M$, a finite constant.

Matrix- Cesàro means T (C_1) of the sequence $\{S_n\}$ is given by

$$t_n = \sum_{k=0}^{n} a_{n,n-k}\, \sigma_{n-k} = \sum_{k=0}^{n} a_{n,n-k} \frac{1}{n-k+1} \sum_{r=0}^{n-k} S_r .$$

If $t_n \to S$ as $n \to \infty$, then sequence $\{S_n\}$ or the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable by matrix- Cesàro means T (C_1) method to S.

Important particular cases of matrix- Cesàro means are:

(i) $(N,p_n)C_1$ means, when $a_{n,n-k} = \frac{p_k}{P_n}$, where $P_n = \sum_{k=0}^{n} p_k \neq 0$.

(ii) $(\tilde{N},p_n)C_1$ means, when $a_{n,n-k} = \frac{p_{n-k}}{P_n}$

(iii) (N, p, q) C_1 means, $a_{n,n-k} = \frac{p_k\, q_{n-k}}{R_n}$, where $R_n = \sum_{k=0}^{n} p_k q_{n-k} \neq 0$.

We write

$$\phi(t) = f(x+t) + f(x-t) - f(x) \tag{2}$$

$$K(n,t) = \frac{1}{2\pi} \sum_{k=0}^{n} \frac{a_{n,n-k}}{(n-k+1)} \frac{\sin^2(n-k+1)\frac{t}{2}}{\sin^2 \frac{t}{2}} \tag{3}$$

3. THEOREM

Quite a good amount of works are known for the degree of approximation of the function $f \in Lip\alpha$, Lip (α, p) and Lip $(\xi(t), p)$ class by various summability methods. The purpose of the present paper is to obtain the degree of approximation of a function $f \in$Lip $(\xi(t), p)$ class by matrix- Cesàro T (C_1) summability method. We prove the following theorem:

Theorem. Let f: $R \to R$ is 2π-periodic, Lebesgue integrable on $[-\pi, \pi]$ and belonging to Lip$(\xi(t), p)$ class then the degree of approximation of f matrix-Cesàro means of Fourier series (1) is given

$$\| t_n - f \|_p = O\left((n+1)^{\frac{1}{p}} \xi\left(\frac{1}{n+1}\right)\right),$$

(3)

provided $T = (a_{n,k})$ be an infinite lower triangular matrix whose elements $(a_{n,k})$ positive and monotonic increasing in k with $0 \le k \le n$ such that

$$\sum_{k=0}^{n} a_{n,n-k} = 1 \text{ and}$$

(4)

$$\sum_{k=0}^{n} \frac{a_{n,n-k}}{n-k+1} = O\left(\frac{1}{n+1}\right). \tag{5}$$

And $\xi(t)$ satisfies the following conditions:

$$\left\{\frac{\xi(t)}{t}\right\} \text{ is monotonic decreasing} \tag{6}$$

$$\left[\int_0^{\frac{1}{n+1}} \left(\frac{t\,|\phi(t)|}{\xi(t)}\right)^p dt\right]^{\frac{1}{p}} = O\left(\frac{1}{n+1}\right), \tag{7}$$

$$\left[\int_{\frac{1}{n+1}}^{\pi} \left(\frac{t^{-\delta}\,|\phi(t)|}{\xi(t)}\right)^p dt\right]^{\frac{1}{p}} = O\left((n+1)^{\delta}\right) \tag{8}$$

where δ is an arbitrary number such that $q(1-\delta)-1 > 0$, q the conjugate index of p and conditions (7) & (8) hold uniformly in x.

4. LEMMAS

Lemma 1: For $0 < t < \frac{1}{n+1}$ and fact that $\frac{1}{\sin t} \le \frac{\pi}{2t}$ for $0 < t \le \frac{\pi}{2}$,

$$K(n,t) = O(n+1). \tag{9}$$

Proof:

$$K(n,t) = \frac{1}{2\pi} \sum_{k=0}^{n} \frac{a_{n,n-k}}{n-k+1} \frac{\sin^2 (n-k+1)\frac{t}{2}}{\sin^2 \frac{t}{2}}$$

$$= \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,n-k}\,(n-k+1)$$

$$\left(\because \sin n\theta \le n \sin\theta \le n\theta \text{ for } 0 < \theta < \frac{1}{n}\right)$$

$$\le \frac{n+1}{2\pi} \sum_{k=0}^{n} a_{n,n-k}$$

$$= \frac{n+1}{2\pi}$$

$$= O(n+1).$$

Lemma 2: For $\frac{1}{n+1} < t < \pi$

$$K(n,t) = O\left(\frac{1}{(n+1)t^2}\right). \tag{10}$$

Proof:
$$K(n,t) = \frac{1}{2\pi} \sum_{k=0}^{n} \frac{a_{n,n-k}}{n-k+1} \frac{\sin^2(n-k+1)\frac{t}{2}}{\sin^2\frac{t}{2}}$$

$$\le \frac{1}{2\pi} \sum_{k=0}^{n} \frac{a_{n,n-k}}{n-k+1} \frac{\pi^2}{t^2}, \text{ by Jordan's Lemma}$$

$$= \frac{\pi}{2t^2} \sum_{k=0}^{n} \frac{a_{n,n-k}}{n-k+1}$$

$$= \frac{\pi}{2t^2} O\left(\frac{1}{n+1}\right), \text{ from condition (5).}$$

$$= O\left(\frac{1}{(n+1)t^2}\right).$$

5. PROOF OF THE THEOREM

n^{th} partial sum $S_n(x)$ of the Fourier series (1) is given by

$$S_n(x) - f(x) = \frac{1}{2\pi}\int_0^{\pi} \phi(t)\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\,dt$$

The (C,1) transform i.e. σ_n of S_n is given by

$$\frac{1}{n+1}\sum_{k=0}^{n}\left(S_k(x) - f(x)\right) = \frac{1}{2(n+1)\pi}\int_0^{\pi}\frac{\phi(t)}{\sin\frac{t}{2}}\sum_{k=0}^{n}\sin\left(k+\tfrac{1}{2}\right)t\,dt$$

$$\sigma_n(x) - f(x) = \frac{1}{2(n+1)\pi}\int_0^{\pi}\phi(t)\,\frac{\sin^2(n+1)\frac{t}{2}}{\sin^2\frac{t}{2}}\,dt$$

The matrix means of the sequence $\{\sigma_n\}$ is given by

$$\sum_{k=0}^{n} a_{n,k}\left(\sigma_k(x) - f(x)\right) = \int_0^{\pi}\phi(t)\frac{1}{2\pi}\sum_{k=0}^{n}\frac{1}{(k+1)}\frac{\sin^2(k+1)\frac{t}{2}}{\sin^2\frac{t}{2}}\,dt$$

or $$\sum_{k=0}^{n} a_{n,n-k}\left(\sigma_{n-k}(x) - f(x)\right) = \int_0^{\pi}\phi(t)\frac{1}{2\pi}\sum_{k=0}^{n}\frac{1}{(n-k+1)}\frac{\sin^2(n-k+1)\frac{t}{2}}{\sin^2\frac{t}{2}}\,dt$$

$$t_n(x) - f(x) = \int_0^{\pi}\phi(t)\,K(n,t)\,dt$$

$$= \int_0^{\frac{1}{n+1}}\phi(t)\,K(n,t)\,dt + \int_{\frac{1}{n+1}}^{\pi}\phi(t)\,K(n,t)\,dt$$

$$= J_1 + J_2,\ \text{say.} \tag{11}$$

Applying Hölder inequality, Lemma 1 and fact that $\phi(t) \in Lip(\xi(t), p)$, we have,

$$|J_1| \le \left\{\int_0^{\frac{1}{n+1}}\left(\frac{t|\phi(t)|}{\xi(t)}\right)^p dt\right\}^{\frac{1}{p}}\left\{\int_0^{\frac{1}{n+1}}\left(\frac{\xi(t)|K(n,t)|}{t}\right)^q dt\right\}^{\frac{1}{q}}$$

$$= O\left(\xi\left(\frac{1}{n+1}\right)\right)\left\{\int_{\epsilon}^{\frac{1}{n+1}} t^{-q}dt\right\}^{\frac{1}{q}}$$, for some $0 < \epsilon < \frac{1}{n+1}$, by second mean value theorem for integrals.

$$= O\left(\xi\left(\frac{1}{n+1}\right)\right)\left[\left(\frac{t^{-q+1}}{-q+1}\right)_{\epsilon}^{\frac{1}{n+1}}\right]^{\frac{1}{q}}$$

$$= O\left((n+1)^{1-\frac{1}{q}}\,\xi\left(\frac{1}{n+1}\right)\right)$$

$$= O\left((n+1)^{\frac{1}{p}}\,\xi\left(\frac{1}{n+1}\right)\right). \qquad (12)$$

Similarly for the J_2, we have

$$|J_2| \le \left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\right)^p dt\right\}^{\frac{1}{p}}\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi(t)|K(n,t)|}{t^{-\delta}}\right)^q dt\right\}^{\frac{1}{q}}$$

$$= O\left((n+1)^{\delta}\right)\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi(t)}{t^{-\delta+2}(n+1)}\right)^q dt\right\}^{\frac{1}{q}}$$

$$= \left((n+1)^{\delta}\,\xi(\tfrac{1}{n+1})\right)\left\{\int_{\frac{1}{n+1}}^{\pi} t^{-q(1-\delta)}\,dt\right\}^{\frac{1}{q}} \qquad \text{by condition (6)}$$

$$= O\left((n+1)^{\delta}\,\xi(\tfrac{1}{n+1})\right)\left\{\left(\frac{t^{-q(1-\delta)+1}}{-q(1-\delta)+1}\right)_{\frac{1}{n+1}}^{\pi}\right\}^{\frac{1}{q}}$$

$$= O\left((n+1)^{\delta}\,\xi(\tfrac{1}{n+1})\right)\ (n+1)^{1-\delta-\frac{1}{q}}$$

$$= O\left((n+1)^{1-\frac{1}{q}}\,\xi(\tfrac{1}{n+1})\right)$$

$$= O\left((n+1)^{\frac{1}{p}} \xi(\tfrac{1}{n+1}) \right).$$

(13)

By (11), (12) and (13), we have

$$|t_n - f| = \left((n+1)^{\frac{1}{p}} \xi(\tfrac{1}{n+1}) \right)$$

or

$$\|t_n - f\|_p = O\left\{ \int_0^{2\pi} \left((n+1)^{\frac{1}{p}} \xi(\tfrac{1}{n+1}) \right)^p dx \right\}^{\frac{1}{p}}$$

$$= O\left((n+1)^{\frac{1}{p}} \xi(\tfrac{1}{n+1}) \right) \left\{ \int_0^{2\pi} dx \right\}^{\frac{1}{p}}$$

$$= O\left((n+1)^{\frac{1}{p}} \xi(\tfrac{1}{n+1}) \right). \tag{14}$$

6. APPLICATIONS

Following corollaries may be derided from the theorem.

Corollary 1. If $\xi(t) = t^{\alpha}$, $0 < \alpha \le 1$, then the degree of approximation of a function

$f \in \mathrm{Lip}\,(\alpha, p)$, $\frac{1}{p} < \alpha \le 1$, is given by

$$\|t_n - f\|_p = O\left(\frac{1}{(n+1)^{\alpha - \frac{1}{p}}} \right).$$

Corollary 2. If $p \to \infty$ in corollary 1, then the degree of approximation of a function $f \in \mathrm{Lip}\,\alpha$, for $0 < \alpha \le 1$, is

$$\| t_n - f \|_\infty = \begin{cases} O\left(\dfrac{1}{(n+1)^\alpha}\right), & \text{for } 0<\alpha<1 \\ O\left(\dfrac{\log(n+1)\pi e}{(n+1)}\right), & \text{for } \alpha=1. \end{cases}$$

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Parseval's Identity for Low-Dimensional Nilpotent Lie Groups $G_{5,6}$ and $G_{6,15}$

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Abstract. We prove the Parseval's identity for low-dimensional Nilpotent Lie groups such as $G_{5,6}$ and $G_{6,15}$ which are important for proving Hardy uncertainty principles type results.

Key Words: Fourier transform, Hilbert Schmidt norm, kernel function.

1. INTRODUCTION

Let g be an n-dimensional real Nilpotent Lie algebra and G = exp g be the associated connected and simply connected Nilpotent Lie group. Let $\{x_1, ..., x_n\}$ be a strong Malcev basis of g through the ascending central series of g. In particular, RX_1 is contained in the centre of g. We introduce a norm function on G by setting for

$$x = \exp(x_1X_1 + ... + x_nX_n) \in G,\ x_j \in \mathfrak{R}$$
$$\|x\| = (x_1^2 + x_2^2 + ... + x_n^2)^{1/2}$$

The composed map

$$\mathfrak{R}^n \to g \to G,\ (x_1, ..., x_n) \to \sum_{j=1}^{n} x_j X_j \to \exp\left(\sum_{j=1}^{n} x_i X_j\right)$$

is a diffeomosphism and maps Lebesgue measure on $\Re^n$ to Haar measure on G. In this manner, we shall always identify g and sometimes G_1 as sets with $\Re^n$. The measurable (integrable) functions on G can be viewed as such functions on $\Re^n$. The measurable (integrable) functions on G can be viewed as such functional on $\Re^n$.

Let g denote the vector space dual of g and $\{X_1^*, ..., X_n^*\}$ the basis of g^* which is dual to $\{X_1, ..., X_n\}$. Then $\{X_1^*, ..., X_n^*\}$ is Jordon Holder basis for the coadjoint action of G on g^*. We shall identify g^* with $\Re^n$ via the map $\xi = (\xi_1, ..., \xi_n) \to \sum_{j=1}^{n} \xi_j X_j^*$ and on g^*. We shall identify g^* with $\Re^n$ via the map $\xi = \{\xi_1, ..., \xi_n) \to \sum_{j=1}^{n} \xi_j X_j^*$ and on g^* we introduce the Eucledian norm relative to the basis $\{X_1^*, ..., X_n^*\}$, that is

$$\left\|\sum_{j=1}^{n} \xi_j X_j^*\right\| = (\xi_1^2 + \xi_2^2 + ... + \xi_n^2)^{1/2} = ||\xi||.$$

For an operator T in a Hilbert space such that T*T is a trace class. $||T||_{HS}$ will denote the Hillbert Schmidt norm of T.

2. THREAD LIKE NILPOTENT LIE GROUPS

For $n \geq 3$, let g_n be the n-dimensional real Nilpotent Lie algebra with basis $X_1, ..., X_n$ and non trivial lie brackets $[X_1, X_{n-1}] = X_{n-2}, ..., [X_1, X_2] = X_1$.

g_n is a $(n - 1)$ step Nilpotent and is a product of RX_n and the abelian ideal $\sum_{j=1}^{n-1} RX_j$. Note that g_3 is the Heisenberg Lie algebra. Let $G_n = \exp(g_n)$.

For $\xi = \sum_{j=1}^{n-1} \xi_j X_j^* \in g_n^*$, the coadjoint action of G_n is given by

$$Ad^* (\exp (tX_n)\, \xi = \sum_{j=1}^{n-1} P_j (\xi, t)\, X_j^*,$$

where for $i \leq j \leq n - 1$, $P_j (\xi, t)$ is the polynomial in t defined by

$$P_j(\xi, t) = \sum_{k=1}^{j-1} (1/k!)\, (-1)^k\, t^k\, \xi_{j-k}$$

The orbit of ξ is generic with respect to the basis $\{X_1^*, \ldots, X_n^*\}$ if and only if $\xi_1 \neq 0$, and the jumping indices are 2 to n. The cross section X_{ξ_1} for the set of generic orbit is given by,

$$X_{\xi_1} = \{\xi = (\xi_1, 0, \xi_3, \ldots, \xi_{n-1}, 0) : \xi_1 \in \Re, \xi_1 \neq 0\}$$

For $\xi \in g_n^*$, let π_ξ denote the irreducible representation of G_n, associated with ξ. Then the mapping $\xi \to \pi_\xi$ is bijection of X_ξ and the set of all generic irredicible representation. Plancherel measure on $\hat{G}_n$ is supported by these π_ξ. Denoting by F the fourier transform on R^{n-1}, it follows that the Hilbert Schmidt norm of the operator. π_ξ (f) , $f \in L^1 \cap L^2 (G_n)$ is given by

$$\|\pi_\xi (f)\|^2_{HS} = \int_{R^2} F\, f\{p_1 (\xi, t), \ldots, P_{n-1} (\xi, t), t - s\}^2 \, ds\, dt$$

The following group of lower dimensions such as $G_{5,6}$ and $G_{6,15}$ are found in [8].

3. PARSEVAL IDENTITY FOR $G_{5,6}$

Let $G = G_{5,6} = R^5$

$$(x_1, \ldots, x_5)\,(y_1, \ldots, y_5)$$

$$= \quad (x_1 + y_1 + x_4y_3 + x_5y_2 + x_4x_5y_4 + \frac{1}{2}x_5\, y_4^2 + \frac{1}{2}x_5^2\, y_3 + \frac{1}{6}x_2 + y_2 + x_5y_3 + \frac{1}{2}x_5^2\, y_4,$$

$$x_3 + y_3 + x_5y_4,\ x_4 + y_4,\ x_5)$$

$$(x_1, \ldots, x_5)^{-1} = (-x_1 + x_2x_5 + x_3x_4 - \frac{1}{2}x_3x_5^2 - \frac{1}{2}x^2_4\, x_5 + \frac{1}{6}x_4x^3_5, -x_2 + x_3x_5$$

$$-\frac{1}{2}x_4\, x^2_5, -x_3 + x_4x_5, -x_4, -x_5)$$

For $y_1, y_2 \in R^2$

$$\pi_{\xi_1} (f)\, \phi\, (y_1, y_2) = \int_{\Re^5} f(x)\ \pi_{\xi_1}\ (-x_1 + x_2x_5 + x_3x_4 - \frac{1}{2}\, x_3x^2_5 - \frac{1}{2}\, x^2_4x_5 + \frac{1}{6}\, x_4x^3_5,$$

$$-x_2 + x_3x_5 - \frac{1}{2}\, x_4x^2_5, -x_3 + x_4x_5, -x_4, -x_5)\ \phi\ (y_1, y_2)\ dx$$

$$= \int_{\Re^5} f(x) \exp 2\pi i\, [-x_1 + x_2x_5 + x_3x_4 - \frac{1}{2}\, x_3x^2_5 - \frac{1}{2}\, x^2_4\, x_5 + \frac{1}{6}\, x_4x^3_5$$

$$+\frac{1}{2}\, x^2_4x_5 - \frac{1}{6}\, x_4x^3_5 - (-x_3 + x_4x_5)\, y_1 + x_4x_5y_1 - \frac{1}{6}\, x^3_5y_1 +$$

$$\frac{1}{2}x_5y^2_1 - (-x_2 + x_3x_5 - \frac{1}{2}x_4x^2_5)\,y_2 - \frac{1}{2}x_4x^2_5\,y_2 - \frac{1}{2}x_4x_5\,y^2_2 -$$

$$\frac{1}{2}x^3_5\,y^2_2 - \frac{1}{2}x^2_5y_1y_2 - \frac{1}{2}x_5y_1y^2_2]\ \phi\,(y_1 + x_4,\, y_2 + x_5)\,dx$$

$$x_4 \to x_4 - y_1,\ x_5 \to x_5 - y_2$$

$$= \int_{\Re^5} f(x_1, x_2, x_3, x_4 - y_1, x_5 - y_2)\exp 2\pi i\,[-x_1 + x_2(x_5 - y_2)$$

$$+ x_3(x_4 - y_1) - \frac{1}{2}x_3\,(x_5 - y_2)^2 + x_3y_1 - \frac{1}{6}(x_5 - y_2)^3\,y_1 +$$

$$\frac{1}{2}(x_5 - y_2)\,y_1{}^2 + x_2y_2 - x_3(x_5 - y_2)\,y_2 - \frac{1}{2}(x_4 - y_1)\,(x_5 - y_2)\,y_2{}^2 -$$

$$\frac{1}{2}(x_5 - y_2)^3\,y_2{}^2 - \frac{1}{2}(x_5 - y_2)^2\,y_1y_2 - \frac{1}{2}(x_5 - y_2)\,y_1y_2{}^2)\xi_1]\ \phi(x_4, x_5)\,dx$$

$$= \int_{\Re^5} f(x_1, x_2, x_3, x_4 - y_1, x_5 - y_2)\exp[2\pi i\,((-x_1 + x_2x_5 - x_3)\,[x_4 - \frac{1}{2}$$

$$(x_5 - y_2)^2 - (x_5 - y_2)\,y_2)\xi_1 + U(y_1, y_2, x_4, x_5)\,\xi_1)\ \phi(x_4, x_5)\,dx$$

$$K^f_{\xi_1}(y_1, y_2, x_4, x_5) = \int_{\Re^3} f(x_1, x_2, x_3, x_4 - y_1, x_5 - y_2)\exp - 2\pi i\,(x_1\xi_1 -$$

$$x_2x_5\xi_1 - x_3\,[x_4 - \frac{1}{2}(x_5 - y_2)^2 - (x_5 - y_2)\,y_1]\,\xi_1 -$$

$$U(y_1, y_2, x_4, x_5)\xi_1)\,dx_1\,dx_2\,dx_3$$

$$= F_{123}\,f(\xi_1, -x_5\,\xi_1, -[x_4 - \frac{1}{2}(x_5 - y_2)^2 - (x_5 - y_2)\,y_2]$$

$$\xi_1, x_4 - y_1, x_5 - y_2)\exp 2\pi\,i\,U(y_1, y_2, x_4, x_5)\,\xi_1$$

$$= \int_{\Re^4} |\,K^f_{\xi_1}(y_1, y_2, x_4, x_5)|^2\,dy_1\,dy_2\,dx_4\,dx_5$$

$$\|\pi_{\xi_1}(f)\|^2_{HS} = \int_{\Re^4} |F_{123}\,f(\xi_1, -x_5\xi_1, -[x_4 - \frac{1}{2}(x_5 - y_2)^2 - (x_5 - y_2)\,y_2]$$

$$\xi_1, x_4 - y_1, x_5 - y_2]\,dy_1\,dy_2\,dx_4\,dx_5$$

$$x_5 \to \frac{-1}{\xi_1}x_5,\ x_4 \to \frac{-1}{\xi_1}x_4$$

$$= \frac{1}{\xi_1{}^2} \int_{\Re^4} |F_{123} f(\xi_1, x_5, x_4 + \left(\frac{1}{2}\left(-\frac{x_5}{\xi_1} - y_2\right)^2 + \left(\frac{-x_5}{\xi_1} - y_2\right) y_2\right) \xi_1$$

$$-\left(\frac{1}{\xi_1} x_4 - y_1 - \frac{x_5}{\xi_1} - y_2\right) dy_1\, dy_2\, dx_4\, dx_5$$

$$y_1 \to -y_1 - \frac{1}{\xi_1} x_4,\ y_2 \to -y_2 - \frac{1}{\xi_1} x_5$$

$$= \frac{1}{\xi_1{}^2} \int_{\Re^4} |F_{123} f(\xi_1, x_5, x_4 + \left(\frac{1}{2} y_2{}^2 - y_2\left(y_2 + \frac{1}{\xi_1} x_5\right)\right)$$

$$\xi_1, y_1, y_2)|^2\, dy_1\, dy_2\, dx_4\, dx_5$$

$$= \frac{1}{\xi_1{}^2} \int_{\Re^4} |F_{13} f(\xi_1, u_1\, x_4 + \left(\frac{-1}{2} y_2{}^2\, \xi_1 - y_2\, u\right) y_1, y_2)|^2$$

$$dy_1 dy_2\, dx_4\, du$$

$$x_4 \to x_4 + \frac{1}{2} y_2{}^2\, \xi_1 + y_2\, u$$

$$= \frac{1}{\xi_1{}^2} \int_{\Re^4} |F_{13} f(\xi_1, u_1\, x_4, y_1, y_2)|^2\, dy_1\, dy_2\, dx_4\, du$$

$$= \frac{1}{\xi_1{}^2} \int_{\Re^4} |F_1 f(\xi_1\, u, w, y_1, y_2)|^2\, dy_1\, dy_2\, dw\, du$$

4. PARSEVAL IDENTITY FOR $G_{6,\,15}$

$G = G_{6,\,15} = \Re^6$

$(x_1, ..., x_6)\,(y_1, ..., y_6) = (x_1 + y_1 + x_6 y_4,\ x_2 + y_2 + x_5\, y_4,\ x_3 + y_3 + x_6 y_5,\ x_4 + y_4,\ x_5 + y_5,\ x_6 + y_6)$

$(x_1, x_2, ..., x_6)^{-1} = (-x_1 + x_4 x_6, -x_2 + x_4 x_5, -x_3 + x_5 x_6, -x_4, -x_5, -x_6)$

For $\phi \in L^2(\Re)$

$\hat{f}\,(\pi_{\xi_1, \xi_2, \xi_3, \xi_6})\, \phi\,(y),\ \xi_2 \neq 0$

$$= \int_{\Re^6} f(x)\, \pi_{\xi_1, \xi_2, \xi_3, \xi_6}\ (-x_1 + x_4 x_6, -x_2 + x_4 x_5, -x_3 + x_5 x_6, -x_4, -x_5, -x_6)\, \phi\,(y)\, dx$$

$$= \int_{\Re^6} f(x) \exp 2\pi i\, [(-x_1 + x_4 x_6)\, \xi_1 + (-x_2 + x_4 x_5)\, \xi_2 + (-x_3 + x_5 x_6 - x_5 x_6)\, \xi_3 + \frac{\xi_3}{\xi_2}$$

$$(-x_6y - \frac{1}{2}x^2{}_6\,\xi_1) - x_6\xi_1 + x_4y]\,\phi(y + \xi_2x_5 + \xi_1x_6)\,dx$$

Applying $x_5 \to \frac{1}{\xi_2}(x_5 - y - \xi_1x_6)$

$$= \frac{1}{|\xi_2|}\int_{\Re^6} f(x_1, x_2, x_3, x_4, \frac{1}{\xi_2}(x_5, -y - \xi, x_6), x_6)\exp[(-x_1 + x_4x_6)\,\xi_1 + (-x_2 + x_4$$

$$(\frac{1}{\xi_2}(x_5 - y - \xi_1x_6))\,\xi_2 + (-x_3\,\xi_3) + \frac{\xi_3}{\xi_2}(-x_6y - \frac{1}{2}x^2{}_2\,\xi_1) - x_6\xi_6 + x_4y]\,\phi\,(x_5)\,dx$$

$$= \frac{1}{|\xi_2|}\int_{\Re^6} f(x_1, x_2, x_3, x_4, \frac{1}{\xi_2}(x_5 - y - \xi_1, x_6), x_6)\exp.\,2\pi i\,[-x_1\xi_1 - x_2\xi_2 - x_3\xi_3 +$$

$$x_4x_5 + \frac{\xi_3}{\xi_2}(-x_6y - \frac{1}{2}x_6{}^2\,\xi_1) - x_6\xi_6]\,\phi.(x_5)\,dx$$

$\hat{f}\,(\pi_{\xi_1,\xi_2,\xi_3,\xi_6})$ is the integral operator on $L^2\,(\Re)$ where kernel is

$$K^f{}_{(\xi_1,\xi_2,\xi_3,\xi_6)}\,(y, x_5) = \frac{1}{|\xi_2|}\int_{\Re^5} f(x_1, x_2, x_3, x_4, \frac{1}{\xi_2}(x_5 - y - \xi_1\,x_6), x_6)\exp.(2\pi i)$$

$$[\sum_{i=1}^{3} x_i\xi_i - x_4x_5 + \frac{\xi_3}{\xi_2}[x_6y + \frac{1}{2}\int_{\Re^5} x^2{}_6\,\xi_1 + x_6\xi_6]\,dx_1\,dx_2\,dx_3\,dx_4\,dx_6$$

$$= \frac{1}{|\xi_2|}\int_{\Re} F_1F_2F_3F_4\,(\xi_1, \xi_2, \xi_3, x_5\,\frac{1}{\xi_2}(x_5 - y - \xi_1x_6), x_6)\exp(-2\pi i)$$

$$[\frac{\xi_3}{\xi_2}(x_6y + \frac{1}{2}x^2{}_6\,\xi_1) + x_6x_6]\,dx_6$$

$$\|\hat{f}\,(\pi_{\xi_1,\xi_2,\xi_3,\xi_6})\|^2 = \int_{\Re^2} \|k^f{}_{(\xi_1,\xi_2,\xi_3,\xi_6)}\,(y, x_5)|^2\,dy\,dx_5$$

$$= \frac{1}{|\xi_2|}\int_{\Re^2}\left|\int_{\Re} F_1\,F_2\,F_3\,F_4\,(\xi_1, \xi_2, \xi_3, -x_5, \frac{1}{\xi_2}(x_5 - y - \xi_1\,x_6, x_6)\exp.(2\pi i)\right.$$

$$(\frac{\xi_3}{\xi_2}(x_6y + \frac{1}{2}x^2{}_6\;\;\xi_1) + x_3\xi_6)\,dx_6|^2\,dy\,dx_5$$

$$y \to y - \frac{1}{2}x_6\,\xi_1$$

$$= \frac{1}{|\xi_2|} \int_{\Re^2} \left| \int_{\Re} F_1F_2F_3F_4 \left(\xi_1, \xi_2, \xi_3, -x_5, \frac{1}{\xi_2}(x_5 - y - \frac{1}{2}\xi_1 x_6), x_6\right) \exp(-2\pi i) \right.$$

$$\left. \left(\frac{\xi_3}{\xi_2}(x_6 y + x_6 \xi_6)\right) dx_6 \right|^2 dy\, dx_5$$

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DCP Property of a Certain Combinations of de la Vallée Poussin Kernels

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Abstract: We shall established the DCP Property of certain Combinations of de la Vallée Poussin Kernels for some particular cases.

1. INTRODUCTION

Let $\mathcal{A}$ denote the set of analytic functions in D, $f * g$ the Hadamard product or convolution between two members of $\mathcal{A}$. A domain $\Omega \subseteq C$ is said to be *convex in the direction* $e^{i\phi}$, $\phi \in \Re$, if and only if for every $a \in C$ the set.

$$\Omega \cap \{a + te^{i\phi} : t \in \Re\}$$

is either connected or empty. Accordingly we define the class $\mathcal{K}(\phi) \subset \mathcal{A}$, $\phi \in \Re$, of the functions *convex in the direction* $e^{i\phi}$ as

$$\mathcal{K}(\phi) := \{f \in \mathcal{A} : f \text{ univalent and } f(D) \text{ convex in the direction } e^{i\phi}\}.$$

Finally, a function $g \in \mathcal{A}$ is called *Direction-Convexity-Preserving* ($g \in$ DCP) if and only if

$$g * f \in \mathcal{K}\text{ (f) for all } f \in \mathcal{K}(\phi) \text{ and all } \phi \in \Re.$$

Functions in DCP have many other intriguing convolution-type properties, for instance the preservation of convex harmonic functions in D, and of Jordan curves

in the plane with convex interior domain; we refer to [7], [8] for more details. There one also finds a complete description of the members of DCP, namely

$$g \in DCP \Longleftrightarrow g(z) + itzg'(z) \in \not\mathcal{K}\left(\frac{\pi}{2}\right) \text{ for all } t \in \Re.$$

Further it is known, that DCP functions are convex univalent.

The following criterion for membership in DCP is a slight variant of [7, Theorem 4].

Lemma 1 Let g be analytic in $\bar{D}$, convex univalent and let $u(t) := \operatorname{Re} g(e^{it})$, $t \in \Re$. Then

$g \in DCP$ if and only if

$$\sigma_u := (u''(t))^2 - u'(t)\, u'''(t) \geq 0,\ t \in \Re$$

The classical definition of the de la Vallée Poussin Kernel of order $n \in N$ is

$$w_n(t) := \frac{2^n (n!)^2}{(2n)!} (1 + \cos(t))^n \tag{1}$$

$$= \frac{1}{\binom{2n}{n}} \sum_{k=-n}^{n} \binom{2n}{n+k} e^{ikt}.$$

But here we are interested in the analytic version of the de la Vallée Poussin Kernel

$$V_n(z) = \frac{1}{\binom{2n}{n}} \sum_{k=1}^{n} \binom{2n}{n+k} z^k,\ z \in C. \tag{2}$$

Note that

$$2\operatorname{Re} V_n(e^{it}) = w_n(t) - 1,\ n \in N. \tag{3}$$

2. MAIN RESULTS

In this section, we again come back to the analytic version of the classical de la Vallee Poussin Kernels. Let us recall that the function

$$V_n(z) = \frac{1}{\binom{2n}{n}} \sum_{k=1}^{n} \binom{2n}{n+k} z^k,\ z \in D,$$

is the de la Vallee Poussin kernel of order n.

In [9], St. Ruscheweyh and J. K. Wirths proved that for $0 < x < \infty$ and for $n \in N$, the function

$$f_n(z) = \sum_{k=1}^{n} \binom{n}{k} x^k \binom{2k}{k} V_k(z), z \in D, \tag{4}$$

is convex. When they proved this, the class DCP was not even defined. Later in 1989 [7], St. Ruscheweyh and L. Salinas introduced the class DCP, which is a sublcass of the class of convex functions. Now one can ask a natural question whether the functions $f_n(z)$ belong to the class DCP instead of just to the class of convex functions. We shall prove that in general the function $f_n(z)$ does not belong to the class DCP. Already for the special case $x = 1$ in (4), we get the following result.

Theorem 1 For $n \in N$, let

$$f_n(z) = \sum_{k=1}^{n} \binom{n}{k} \binom{2k}{k} V_k(z), \ z \in D$$

Then $f_n \in DCP$ for $n \leq 6$ and $\notin DCP$ for $n = 7$.

3. PROOF OF THE MAIN RESULT

Proof: Just like in the previous sections, put

$$w_k(t) := \text{Re}\, V_k(e^{it}) = -\frac{1}{2} + \frac{2^{k-1}(k!)^2}{(2k)!}(1 + \cos(t))^k$$

and let

$$u_n(t) := \text{Re}\, f_n(e^{it}) = \sum_{k=1}^{n} \binom{n}{k} \binom{2k}{k} w_k(t)$$

$$= \sum_{k=1}^{n} \binom{n}{k} \binom{2k}{k} \left(-\frac{1}{2} + \frac{2^{k-1}(k!)^2}{(2k)!}(1 + \cos(t))^k\right)$$

$$= \sum_{k=1}^{n} \left(-\frac{n!\,(2k)!}{2(n-k)!\,(k!)^3} + \frac{2^{k-1}\, n!}{k!\,(n-k)!}(1 + \cos(t))^k\right)$$

Then, from lemma 1, $f_n \in DCP$ if and only if

$$v_n(t) := u''_n(t)\, u''_n(t) - u_n'''(t)\, u_n'(t) \geq 0 \text{ for } 0 \leq t \leq 2n.$$

After simplification (using Mathematica 3.0), we get:

$$\begin{aligned}
v_1(t) &= 1,\\
v_2(t) &= 52 + 54\cos(t) - 6\cos(3t)\\
v_3(t) &= 9\,(3+2\cos(t))^2\,(15 + 15\cos(t) - 2\cos(2t) - 3\cos(3t)),\\
v_4(t) &= 8\,(3+2\cos(t))^4\,(34 + 33\cos(t) - 8\cos(2t) - 9\cos(3t)),\\
v_5(t) &= 25\,(3+2\cos(t))^6\,(19 + 18\cos(t) - 6\cos(2t) - 6\cos(3t)),\\
v_6(t) &= 18\,(3+2\cos(t))^8\,(42 + 39\cos(t) - 16\cos(2t) - 15\cos(3t)),\\
v_7(t) &= 49\,(3+2\cos(t))^{10}\,(23 + 21\cos(t) - 10\cos(2t) - 9\cos(3t)).
\end{aligned}$$

We shall show one by one that

$$u_n(t) \geq 0,\ 0 \leq t \leq 2\pi, \tag{5}$$

for $n \leq 6$, while $v_7(t)$ does not satisfy this condition.

The case $n = 1$ is obvious. For the case $n = 2$,

$$v_2(t) = 52 + 54\cos(t) - 6\cos(3t) = 52 + 72\,x - 24\,x^3 =: p_2(x)$$

where $x = \cos(t)$. Therefore $v_2(t) \geq 0$ on $0 \leq t \leq 2\pi$ if and only if the polynomial $p_2(x) \geq 0$ on $-1 \leq x \leq 1$. Now for $-1 \leq x \leq 1$,

$$\begin{aligned}
p_2(x) &= 52 + 24x\,(3 - x^2)\\
&\geq 52 + 48x\\
&\geq 4.
\end{aligned}$$

Therefore (5) holds for the case $n = 2$.

Now consider the case $n = 3$. After a simple calculation, we can write

$$v_3(t) = 9(3 + 2\cos(t))^2\, p_3(x), \tag{6}$$

where

$$p_3(x) = 17 + 24\,x - 4x^2 - 12x^3 \text{ and } x = \cos(t).$$

From (6), we see that $v_3(t) \geq 0$ for $0 \leq t \leq 2\pi$ if and only if $p_3(x) \geq 0$ for $-1 \leq x \leq 1$. Now

$$p_3(-1) = 1,\ p_3(1) = 25,$$

while

$$p_3(x_1) \approx 0.871904,\quad p_3(x_2) \approx 27.7289$$

at the critical points

$$x_1 = \frac{1}{9}(-1 - \sqrt{55}) \approx -0.935133,\ x_2 = \frac{1}{9}(-1 + \sqrt{55}) \approx 0.712911,$$

both of which lie inside the interval $[-1, 1]$. This shows that

$$p_3(x) \geq p_3(x_1) > 0 \text{ for } -1 \leq x \leq 1,$$

and hence $v_3(t) \geq 0$ for $0 \leq t \leq 2\pi$.

Consider the case n = 4. As in the case of $v_3(t)$, we can write

$$v_4(t) = 8\,(3 + 2\cos(t))^4\, p_4(x), \tag{7}$$

where

$$p_4(x) = 42 + 60x - 16x^2 - 36x^3 \text{ and } x = \cos(t)).$$

It is clear from (7) that $v_4(t) \geq 0$ on $0 \leq t \leq 2\pi$ if and only if $p_4(x) \geq 0$ on $-1 \leq x \leq 1$.

If we study the behaviour of the polynomial $p_4(x)$ on $[-1, 1]$, we see that

$$p_4(-1) = 2,\; p_4(1) = 50,$$

and for $x \in (-1, 1)$, $p_4(x)$ has critical points at $x_1 = \frac{1}{27}(-4 - \sqrt{421}) \approx -0.908085 > -1$ and $x_2 = \frac{1}{27}(-4 + \sqrt{421}) \approx 0.611788 < 1$, and $p_4(x)$ takes positive values at both of these points. In fact, $p_4(x_1) \approx 1.27865$ and $p_4(x_2) \approx 64.4753$. From this we conclude that $p_4(x) \geq 0$ on $-1 \leq x \leq 1$, and hence $v_4(t) \geq 0$ on $0 \leq t \leq 2\pi$.

For the case n = 5, we can write

$$v_5(t) = 25\,(3 + 2\cos(t))^6\, p_5(x), \tag{8}$$

where

$$p_5(x) = 25 + 36x - 12x^2 - 24x^3 \text{ and } x = \cos(t).$$

We see here also that $v_5(t) \geq 0$ on $0 \leq t \leq 2\pi$ if and only if $p_5(x) \geq 0$ on $-1 \leq x \leq 1$.

Now

$$p_5(-1) = 1,\quad p_5(1) = 25,$$

and for $x \in (-1, 1)$, $p_5(x)$ has critical points: one at $x_1 = \frac{1}{6}(-4 - \sqrt{19}) \approx -0.89315 > -1$ and the other at $x_2 = \frac{1}{6}(-1 + \sqrt{19}) \approx 0.559816 < 1$. $p_5(x)$ takes positive values at both of these points; in fact, $p_5(x_1) \approx 0.373538$ and $p_5(x_2) \approx 37.182$. From this we conclude that $p_5(x) \geq 0$ on $-1 \leq x \leq 1$, and hence $v_5(t) \geq 0$ on $0 \leq t \leq 2\pi$.

Case n = 6. As in the previous case, let us write

$$v_6(t) = 18\,(3 + 2\cos(t))^8\, p_6(x), \tag{9}$$

where

$$p_6(x) = 58 + 84x - 32x^2 - 60x^3 \text{ and } x = \cos(t).$$

If we study the behaviour of $p_6(x)$ on $[-1, 1]$, we see that $p_6(-1) = 2$, $p_6(1) = 50$. And for $x \in (-1, 1)$, $p_6(x)$ has critical points at $x_1 = \frac{1}{45}(-8 - \sqrt{1009}) \approx -0.883661$ and $x_2 = \frac{1}{45}(-8 + \sqrt{1009}) \approx 0.528106$, and $p_6(x)$ takes positive values at both of

these points. In fact, $p_6(x_1) \approx 0.18582$ and $p_6(x_2) \approx 84.599$. We thus see that $p_6(x) \geq 0$ on $-1 \leq x \leq 1$, and hence, from (9), we conclude that $v_6(t) \geq 0$ on $0 \leq t \leq 2\pi$. In this way we have shown that the functions $v_n(t) \geq 0$ on $0 \leq t \leq 2\pi$ for $n = 1, ..., 6$. and hence the functions

$$f_n(z) = \sum_{k=1}^{n} \binom{n}{k} \binom{2k}{k} V_k(z)$$

are in the class DCP for $n \in N, n \leq 6$.

Moving on to the case $n = 7$, we now show that the condition $v_7(t) \geq 0$ for $0 \leq t \leq 2\pi$ does not hold. After a simple calculation, we can write

$$v_7(t) = 49\,(3 + 2\cos(t))^{10}\, p_7(x) \tag{10}$$

where

$$p_7(x) = 33 + 48x - 20x^2 - 36x^3 \text{ and } x = \cos(t).$$

Here also, we see that $v_7(t) \geq 0$ on $0 \leq t \leq 2\pi$ if and only if $p_7(x) \geq 0$ on $-1 \leq x \leq 1$. But for this case, we have

$$p_7(-1) = 1, \quad p_7(1) = 25$$

and

$$p_7(x_1) \approx -0.19564, \; p_7(x_2) \approx 47.5034$$

at the critical points $x_1 = \frac{1}{27}(-5 - \sqrt{349}) \approx -0.877094 > -1$ and $x_2 = \frac{1}{27}(-5 + \sqrt{349}) \approx 0.506724$, both of which lie inside the closed interval $[-1, 1]$. This shows that $p_7(x)$ takes also negative values in $-1 \leq x \leq 1$ and consequently $v_7(t)$ takes also negative values in $0 \leq t \leq 2\pi$. Hence f_n can not belong to the class DCP for $n = 7$. This completes the proof of this case and of the theorem as well.

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Just-in-time sequencing in mixed-model production systems relating with fair representation in apportionment theory

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Abstract: The just-in-time sequencing problem in mixed-model production systems is dealt relating with the well known apportionment problem. In addition of brief historical background of both the problems, the mathematical models of them have been reviewed. The sequencing approaches to solve the just-in-time sequencing problem are explained in brief, and some apportionment methods are discussed to resolve the apportionment problem. We have proposed two mean-based divisors, indicating the objective function for better solution, which work for just-in-time sequencing too. The linkage of both the problems is characterized in terms of similar type of objective functions. The problems are shown equivalent via suitable notational transformations and similar properties. Some sequencing algorithms are analyzed for joint approaches to tackle the problems proposing equitably efficient solutions.

Keywords: just-in-time sequencing, heuristics, perfect matching, assignment, apportionment, divisor methods, mean-based divisor methods, global and local indices.

1. INTRODUCTION

Both of the problems, just-in-time *(JIT)* sequencing of products in mixed-model production systems and apportionment of representatives in legislature, are the promising research problems over the years from the domain of discrete optimization. The *JIT* sequencing of different products is a well-known socio-industrial problem which aims to minimize the maximum and the total deviations between the actual and ideal productions; whereas the apportionment problem is a socio-political problem which aims to allocate representatives to states/parties as close as to their exact quota. *JIT* sequencing is highly applied in many industries and all kinds of organizations to optimize several production and other operational activities. Any industry has to interplay mainly among the three major intersecting components: *finance*, *operations* and *marketing*; of which our focus lies on the operational part. The discrete apportionment problem has been in existence since more than two hundred years with the target to allocate integral seats to states or parties and to achieve so-called equity between states' representation.

JIT production system *(JITPS)* is the most commonly used technology due to its noticeable characteristics in operating with very low work-in-progress *(WIP)* inventory and often with low finished goods inventory. It is a *pull system:* products are assembled just before they are sold, subassemblies are made just before the products are assembled and the components are fabricated just before the subassemblies are made. As a result, *WIP* inventory is low and production lead times are short. So *JITPS* is a broad philosophy of continuous improvement including three mutually supportive components: people involvement→ total quality control→ *JIT* flow; jointly called productivity triad. Any unnecessary delays and inventories are considered as waste and so *WIP* inventory is kept as minimum as possible. The goal is to achieve a smooth and synchronized flow of small lots of materials at a uniform rate. Instead of producing one product for a long period and then shifting to another product, *JITPS* uses a technique called mixed-model assembly. A mix of the models is produced each day in short repetitive sequences, so that each model is frequently repeated in proportion to its relative demand.

Section 2 gives the state of art of the two problems with brief historical background. Section 3 formulates mathematical models. The sequencing approaches for *JIT*PS are explained in Section 4. Some apportionment methods including mean-based divisor methods are presented in Section 5. The linkage and

algorithmic characterizations are posed in Section 6. Finally, Section 7 concludes the paper.

2. POSING THE STATUS OF THE PROBLEMS

2.1 Just-in-Time Sequencing

Since the time of Henry Ford, product requirements and hence the requirements of production systems have been changed in rapid speed. Assembly lines were originally developed for a cost efficient mass production of a single standardized product. Nowadays, varieties of options are available to the customers, so that manufacturers need to handle product varieties which exceed several billions of models. So the very first cornerstone of *JITPS* can be traced out from Ford production system (Ford Motor Company in 1903). However, the present idea of the system is developed and perfected by T. Ohno, who is credited as father of *JITPS*, while working as an assembly manager in Toyota motor company around 1970s. After the World War II, Toyota realized that Japanese automotive manufactures were far behind the American motor companies and the president of Toyota made a comment about the gap: "Catch up with America in three years; otherwise the automobile industry of Japan will not survive". *JITPS* within Toyota is a result of low demands, limited space and resources in Japan compared to America. Toyota production system *(TPS)*, comprised of two pillars: *JIT* and autonomation, provides the highest quality, lowest cost and shortest lead time in order to achieve stable production systems. Standardized work, smoothing production schedule via mixed-model sequencing and change for better are the main bases of *TPS*. The *JITPS* is a management philosophy based on the planned elimination of all wastages, continuous improvement of productivity and reduction of inventories in all level; performed by producing only the necessary amount of necessary products in perfect quality at right place and time. To achieve this goal, *JITPS* penalizes the early-tardy jobs by using the limited resources in optimal way. The main target is to satisfy customers for various demands of different products without holding large inventories and incurring large shortages of products.

In *JIT* sequencing (scheduling) environment, products (jobs) that complete early must be held in finished goods inventory till their due dates, while products that complete after their due dates may cause customers to shut down operations. So an ideal schedule is one in which all products are finished exactly on their

assigned due dates. The concept of penalizing both earliness and tardiness has spawned a new and rapidly developing line of research in scheduling theory. The mixed-model *JIT* sequencing is the problem of determining production sequence of different models of the same product produced on the assembly line, assuming that products require an approximately equal number and mix of parts. This is formulated as a non-linear integer programming problem in [48], which seeks to minimize the sum of squared deviations between average and cumulative quantity of products, and thereby nearest integer point algorithm for optimality is presented. Kubiak [37] well-studied this level schedule problem, referring as product rate variation (*PRV*) problem. Kubiak and Sethi [38] proposed an assignment formulation for sum deviation problem to determine an optimum solution at a smaller computational cost and further gave some properties that relate to the problems involving each of the products considered separately. Several open questions and conjectures for balanced *JIT* optimization problem are formulated and dealt by means of extensive computational testing [36].

The intuitive similarity between the *JIT PRV* problem and the single machine scheduling problem is used to propose a heuristic for the *PRV* problem [31]. This heuristic shows that an earliest due date (*EDD*) sequence minimizes the sum of squared deviation between the ideal time and actual time in which given unit is actually produced. Balinski and Shahidi [5] proposed another type of deviation for two products, which aims to minimize the variation of production rates from product to products. The mixed model assembly line (*MMAL*) is one where a variety of different items are assembled or processed at different stations in small batch sizes. Such a line serves in flexible manufacturing systems *(FMS)* to meet diverse demands of the customers. Most *FMS* adopt the *JIT* philosophy in their effort to minimize inventory. Thus, *MMAL* finds good applications in *JITPS* [66]. In this assembly environment, workers are expected to be more versatile and have better skills than those working in traditional systems [18]. The *TPS* used the *JIT* sequencing to distribute production volumes and mix of models as evenly as possible over the production sequences [28, 52]. The *JIT* sequencing process has become a universal and robust concept to balance the two sequencing goals of the manufacturing companies: *usage goal* and *loading goal* [48, 49, 52]. The former maintains a constant rate of usage of all items in the production sequence whereas the latter smoothes the workload on final assembly process to reduce the chance of production delays and stoppages. *JIT* sequencing is used to balance workloads throughout *JIT* supply chains intended for low-volume high-mix family of

products [41]. The purpose of optimal sequence is to keep the actual production level and the desired production level as close to each other as possible all the time. See [14, 21, 22, 42] for more details.

The single-level problem is extended into the multi-level [50, 51], which is referred as the output rate variation (*ORV*) problem [37]. Most discrete manufacturing systems are multi-level in nature, characterized by the condition where several parts are used to produce a particular part at a higher level, terminating at the last level yielding the final product with direct consumer utility. The sequence of products on the final assembly line impacts greatly on inventory levels of parts used directly for assembly and other parts in the system. Recently, the problem of determining an appropriate product sequence has been attracting a lot of attention. The *ORV* problem is proved *NP*-hard, even in special cases [37, 39]. However, the dynamic programming solution is proposed in [39, 49]. The *ORV* problem with pegging assumption is effectively solved in [61], which reduced the problem into the weighted *PRV* problem. Modifying the solution techniques used for the unweighted single level problem, the pegged multi-level problem may be solved to optimum in time which is polynomial in the total product demand and the weighted factors.

2.2 Apportionment Problem

There is a very large class of real life problems related to fair division of resources among competing interests in many areas of applications in the real world, which plays a significant role in decision sciences. Several types of equity problems arise in allocating available resources in integral parts to different subdivisions. Some of the problems are efficiently solved (e.g., assignment) whereas others are not solved well yet (e.g., timetabling). One particular problem having wide applications in governmental decision-making is the apportionment problem, which may be of continuous-type (e.g., taxes) and of discrete-type (e.g., seats). The discrete apportionment problem (*DAP*) has its origin in the proportional election system developed for House of Representatives in *USA*, where each state receives seats in the house in proportion to its population [8]. *DAP* occurs in all kinds of electoral systems such as in: *(a) Federal system:* regional representation based on population, e.g., in *USA*. *(b) Proportional system:* political representation based on votes, e.g., in Israel. *(c) Mixed system:* mixture of federal and proportional systems, e.g., in Nepal.

The *DAP* is the problem of determining how to divide a given integer number of representatives or delegates proportionally among the given constituencies according to their respective sizes. It is a quite complex kind of discrete fair-division problem in electoral systems, because all possible apportionment methods contradict the principle of fairness criteria [16]. In fact, no method equalizes states under fixed house size allocating minimum requirement of one seat and states not crossing the house size. Mainly two fairness ideas have been studied: the *first* is each state should get either its lower quota or upper quota and the *second* is to look at pairwise equity between states. Practically, there always exists a certain inequality between two states yielding one of the states a slight advantage over the other. A transfer of one seat from the more favored state to the less favored state will ordinarily reverse the sign of inequality, so that the more favored state now becomes the less favored, and vice-versa. Whether such a transfer should be made or not, depends on whether the amount of inequality between the two states, after the transfer, is less or greater than it was before. If the amount of inequality is reduced by the transfer, then it is obvious that the transfer should be made. The fundamental problem of quite unexpected complexity is, therefore, how to measure the *amount of inequality* between two states and how to minimize it as far as possible, since it cannot be eliminated perfectly. The philosophy of apportionment must obey political legitimacy and the solutions must be acceptable to nation [8].

Though there is not a single method meeting all the requirements imposed by political needs, a so-called perfect apportionment method is supposed to satisfy some basic properties [57]: *(a) Quota condition:* each state should have its seats within the lower quota and upper quota *(b) House monotonicity:* when total number of representatives (house size) increases, then any state's number of representatives should not decrease *(c) Population monotonicity:* the number of representatives of any state should not decrease as its population increases. Also, any method should not artificially favor large states at the expense of smaller ones and vice-versa *(d) Quota monotonicity:* the actual apportionment of any state should not decrease as its quota increases *(e) Minimum requirement:* every state must have at least one representative. As a result of using one method or another, some surprising apportionment paradoxes are found: *(a) Alabama paradox:* an increase in the size of the house can cause a state to lose a seat. Hamilton method in 1880 assigned Alabama 8 seats from house size 299, whereas it gave only 7 seats from increased house size 300, *(b) Population Paradox:* an increase in a

state's population can cause it to lose a seat faced around 1900 in Hamilton method. If the population of one state is increased while holding the other state's population and house size fixed, the former state may lose a seat [58], *(c) New States paradox:* adding new state and increasing house size can cause another state to lose seats, found in 1907 when Oklahoma became a new state. As a new state, it got 5 new seats increasing the old house size from 386 to 391. As a result, Maine's apportionment went up from 3 to 4 and New York's went down from 38 to 37. But the intent was to leave unchanged for other states, *(d) Quota paradox:* a state may receive number of seats less than its lower quota or more than its upper quota; faced while applying Jefferson's method *(e) Migration paradox:* under the fixed total population, house size, number of states and fixed population of a state, it is possible that the state can lose seats if there is a population shift between other two states. This paradox affects both Hamilton and divisor methods. These paradoxes are visualized via geometric representation in [17].

3. MATHEMATICAL MODELS OF THE PROBLEMS

3.1 *JIT* Sequencing Problem

Suppose there are n products to be produced within the specified time horizon with the integer demands $d_1, d_2, \ldots, d_n$ such that $\sum_{i=1}^{n} d_i = D$. The time needed to produce one unit is assumed to be independent on the product and time needed to switch from one product to another is assumed to be negligible. Without loss of generality, it can be supposed that it takes one unit of time to produce one unit of product and thus the time horizon is equal to D time units. If $r_i = \frac{d_i}{D}$ is the ideal production rate for the parts of type i such that $\sum_{i=1}^{n} r_i = 1$, then the scheduling goal for the assembly line is to maintain the total cumulative production of product i to the total production as close to r_i as possible. This means exactly kr_i units of product i should be produced in the first k time periods $(k = 1, 2, \ldots, D)$, which is the ideal production.

Let x_{ik}, $i = 1, 2, \ldots, n$; $k = 1, 2, \ldots, D$, be the actual cumulative production of product i in the time period 1 through k. For a convex symmetric penalty function F_i, $i = 1, 2, \ldots, n$ with minimum $F_i(0) = 0$; the maximum deviation and the sum deviation just-in-time sequencing problems are formulated as follows:

$$Z_{\max} = \min \max_{i,k} F_i\left(x_{ik} - kr_i\right) \tag{3.1}$$

$$Z_{\min} = \min \sum_{i=1}^{n} \sum_{k=1}^{D} F_i\left(x_{ik} - kr_i\right) \tag{3.2}$$

subject to

$$\sum_{i=1}^{n} x_{ik} = k, \; k = 1, 2, \ldots, D \tag{3.3}$$

$$x_{i(k-1)} \le x_{ik}, \; i = 1, 2, \ldots, n; \; k = 1, 2, \ldots, D \tag{3.4}$$

$$x_{iD} = d_i, \; i = 1, 2, \ldots, n \tag{3.5}$$

$$x_{ik} \text{ is a non-negative integer} \tag{3.6}$$

The constraint (3.3) ensures that exactly k units are scheduled in periods 1 through k, whereas the constraint (3.4) represents the monotone condition (guarantees that the total production of every product over k is non-decreasing function). The constraint (3.5) ensures that production requirements are met for each product (obviously satisfied by any optimal solution); whereas (3.6) is integrality constraint. These four constraints jointly indicate that exactly one product is produced during each stage. Moreover, the above formulation (3.1) to (3.6) is an integer programming problem with cardinality, monotonicity and integrality constraints. The optimization problem is to find the sequence $z = s_1, s_2, \ldots, s_D$ that minimizes one of the objectives (3.1) or (3.2) under the constraints (3.3) to (3.6). Hereafter, we use *MDJIT* and *SDJIT* for maximum deviation and sum deviation *JIT* sequencing objectives respectively. Several scientists have studied above problems via different angles with little-varied objective functions, which are discussed in Section 4. The mathematical formulation of multi-level is beyond of this paper.

3.2 Discrete Apportionment Problem

Assume that there are s states (or parties) indexed $i = 1, 2, \ldots, s$, which are to receive representatives (or seats) from the house of size h. Suppose that the state i has a population p_i and the total population is $\sum_{i=1}^{s} p_i = p$. The fundamental problem is to apportion a_{ih} integer seats to state i, under the constraints $\sum_{i=1}^{s} a_{ih} = h$ and $a_{ih} \in Z^+$. An ideal apportionment is assumed to satisfy the equation $\frac{p_i}{p} = \frac{a_{ih}}{h}$ for all states, which gives $a_{ih} = \frac{p_i h}{p}$, called ideal quota or fair share for state i denoted

by q_{ih}, not necessarily integer. Since only the integral a_{ih} can be assigned to any state, the crucial point is how to handle this problem fairly. One immediate idea is *rounding*: for each state, ideal apportionment should either be rounded down to the next lower integer or rounded up to the next higher integer; but should never exceed these bounds.

To this point, Balinski and Young [6] used the following concept (see [58]): The ideal apportionment $\frac{p_i h}{p}$ is called exact quota denoted by q_{ih}. The largest integer less than or equal to q_{ih} is called the lower quota l_i; the smallest integer greater than or equal to q_{ih} is called the upper quota u_i. An apportionment is said to satisfy *lower quota* if it never gives a state less than its lower quota of seats, i.e., if $l_i \le a_{ih}$ for all i; to satisfy *upper quota* if it never gives a state more than its upper quota of seats, i.e., if $a_{ih} \le u_i$ for all i; and to satisfy *quota* if it does both, i.e., if $l_i \le a_{ih} \le u_i$ for all i. This is known as quota condition.

4. SOME *JIT* SEQUENCING APPROACHES

4.1 Heuristic Frontiers

The heuristic approaches provide comparatively good solutions, not necessarily optimal. Some of the mixed-model sequencing heuristics are developed by academic research scientists, while others are emerging as a result of practical applications. A complex heuristic for selecting the production sequence when the objective is to minimize the chance of stopping the line due to overloading individual stations is proposed in [54]. Monden [52] developed the two greedy heuristics at Toyota, which he referred as goal chasing methods: *GCM I* and *GCM II* (see [35]). The heuristics *GCM I* and *GCM II*, designed with product level and sub-assembly level, constructed a sequence filling one position at a time from first slot to the last one, considering the variability at the sub-assembly level. In comparison of *GCM I*, the *GCM II* represented a decrease in computational time, since the sum is formed only on the components of a given product [62]. However, the comparative research in [62] and in [63] showed that *GCM I* performed better than *GCM II* when compared on the basis of maintaining a constant usage of component parts. These heuristics has been found to yield very good results in the Toyota [34]. Hyundai's heuristic (*HH*) used an alternative way, which was developed to approximate the result given by *GCM I* while reducing the steps of computation. Duplaga and Bragg [26] concluded that the reduction in

computational effort related to *HH* may be significant in situations similar to automobile assembly where many options and choices are available for final product configurations. *GCM I* is advanced to the extended *GCM* to consider all levels in a multi-level production system [51] and introduced another polynomial heuristic to reduce the myopic nature of the previous heuristic. Moreover, the myopic nature of the *GCM I* has been reduced and an exact procedure based on the bounded dynamic programming is developed in [9].

Miltenburg [48] suggested the squared and absolute *SDJIT* sequencing objectives to be minimized as:

$$\text{square deviation:} \quad f_s(x) = \sum_{i=1}^{n} \sum_{k=1}^{D} (x_{ik} - kr_i)^2 \tag{4.1}$$

$$\text{absolute deviation:} \quad f_a(x) = \sum_{i=1}^{n} \sum_{k=1}^{D} |x_{ik} - kr_i| \tag{4.2}$$

He proposed three algorithms and two heuristics with their mutual assimilation to solve the problem. The first algorithm finds the nearest integer point to x_{ik}, the second algorithm solves squared deviation problem (using first algorithm) testing the feasibility of the schedule, and the third algorithm determines whether the schedule is feasible. The first heuristic is used with third algorithm to calculate an entire schedule for the mixed-model *JIT* production system considering the product rates, not the parts usage rates. It is one-stage myopic heuristic doing one calculation for each product and then making a selection with complexity $O(nD)$ for each stage. But it does not consider the effect of its current decision on the variation in future stages. Due to the myopic nature of the first heuristic, Miltenburg further developed the two-stage second heuristic with the complexity $O(n^2D)$ for each stage, which together with the third algorithm approximates the variability over the two stages and schedules in such a way that this variability is as small as possible.

A good heuristic is a simple heuristic with good average performance and reasonable time complexity. The analysis of the heuristics in [25, 62] showed that Miltenburg's second heuristic (with third algorithm) is of the highest quality heuristic. Due to the large size of the products n and their units D, the impression of Miltenburg's heuristics is not so effective. The various heuristics utilizing large-size problems and representing realistic situations are compared in [26] and examined relative performance of those mixed-model sequencing heuristics based on their ability to develop a sequence for final assembly which smoothes out the

rate of use of each component part feeding the assembly line. Inman and Bulfin [31] proposed the min-sum squared sequencing objective

$$f(y)=\sum_{i=1}^{n}\sum_{k=1}^{D}(y_{ik}-t_{ik})^2 \qquad (4.3)$$

to be minimized and developed a pseudo polynomial heuristic with complexity $O(nD)$ by defining ideal time t_{ik} and needed time y_{ik} of production of each product with an efficient *EDD* rule. They reduced the problem into single-machine scheduling with due date t_{ik}. The heuristic yielded better solutions and considered computationally faster than Miltenburg's heuristics. Ding and Cheng [24, 25] gave a simple two-stage algorithm with complexity $O(nD)$ which minimizes the variation of the two stages and produces a good solution. Miltenburg's third algorithm with second heuristic and Ding and Cheng's heuristic work in sequential manner and make use of special structure of the *PRV* problem. Sumichrast et al. [63] constructed the time spread (*TS*) heuristic employing similar procedure as *GCM I* with function in which time required to assemble products are applied. They compared different methods through simulation analysis and found that *TS* and Miltenburg's third algorithm with second heuristic seem to be effective. Groeflin et al. [28] developed a local search heuristic attempting to swap the order of assembly of a pair of products providing near-optimal sequence for realistic-size problems in a reasonable time. The problem with bi-criterion objective functions of part usage and setup time is studied in [19, 43, 46, 47], whose values are inversely correlated; *viz.*, maximization of feasibility and minimization of setup time is simultaneously desired. This is achieved by using local search heuristics, such as tabu search, simulated annealing, genetic algorithm, ant colony optimization, beam search, artificial neural network etc. Similarly, the extensive study of objective for parts usage and workload using heuristics can be found in [32, 45, 50, 63].

4.2 Perfect Matching Approach for *MDJIT*

MDJIT sequencing problem was introduced and studied in the context of *JIT* car production systems [52], where the processor represents a mixed-model assembly line, and the d_i's are the quantity of each type of car to be produced. Steiner and Yeomans [59] proposed *MDJIT* problem in absolute form

$$g(x)=\min\max_{i,k}\left|x_{ik}-kr_i\right| \qquad (4.4)$$

to be minimized subject to the constraints (3.3) to (3.6) and solved this problem by reducing to an order preserving perfect matching problem via single machine scheduling release date/due date decision problem in a bipartite graph. In a graph, a matching is a subset of edges such that no two edges are incident to the same node and a perfect matching is incident to every vertex. A graph is bipartite if and only if it has no circuit of odd length [56]. For a given bound B, the earliest starting time $E(i,j)$ and the latest starting times $L(i,j)$ are respectively given by unique integers such that $\left(\frac{1}{r_i}\right)(j-B)-1 \le E(i,j) < \left(\frac{1}{r_i}\right)(j-B)$ and $\left(\frac{1}{r_i}\right)[(j-1)+B]-1 < L(i,j) \le \left(\frac{1}{r_i}\right)[(j-1)+B]$ which can be computed for each copy of each product in a one pass procedure with time complexity $O(D)$, where $i=1,2,\dots,n;\ j=1,2,\dots,d_i$. Hereafter, the j^{th} copy of product i is denoted by the pair (i,j).

The release date /due date decision problem may be represented as a perfect matching problem which is constructed in a V_1-convex bipartite graph $G=(V_1 \cup V_2, E')$ where $V_1=\{0,1,\dots,D-1\}$ is the starting time, $V_2=\{(i,j): i=1,2,\dots,n;\ j=1,2,\dots,d_i\}$, i.e., V_2 corresponds to the copies of each product, and $E'=\{(k,(i,j)): k \in [E(i,j), L(i,j)] \subseteq V_1\}$. The graph G is said to be V_1-convex graph if $(i,j), (k,j) \in E$ with $i<k \in V_1$ implies that $(l,j) \in E$ for $i \le l \le k$. To find a feasible sequence in the release date/due date decision problem is similar as to find a perfect matching in bipartite graph G, with additional property that lower numbered copies of product are always matched to earlier starting times than higher numbered copies. This type of matching is called as order preserving matching [59].

Theorem 4.1 The objective (4.4) has a feasible solution if and only if the graph G has a perfect matching.

Among various forms of *EDD* algorithm (e.g., Hodgson and Moore 1968, Frederickson 1983, Derigs et al. 1984), a modified version of Glover's (1967) $O(|E|)$ *EDD* algorithm in V_1-convex bipartite graph $G=(V_1 \cup V_2, E')$ found an order preserving perfect matching for $B \le 1$. Using a certain bound obtained via bisection search in the interval $[1-r_{\max}, 1]$, this rule obtained an optimal sequence of D units of n products in $O(D \log D)$ time [21]. In fact, Steiner and Yeomans [59] explained the algorithm giving two deficiency cases of stoppage of the algorithm

in case of too few and too many products for the available time. That is, they concluded that the *EDD* algorithm can stop at time $k < D-1$ for one of the two cases. Defining the lower bound $1-r_{\max}$ for target value B, they proved the following theorem to find a perfect matching:

Theorem 4.2 For the release date / due date decision problem with the objective (4.4), an upper bound is the target value of $B=1$.

Taking $V_1 = \{1,2,\ldots,D\}$, Brauner and Crama [15] redefined the earliest and the latest starting times respectively as $E(i,j) = \left\lceil \frac{j-B}{r_i} \right\rceil$ and $L(i,j) = \left\lfloor \frac{j-1+B}{r_i} + 1 \right\rfloor$. Using Hall's condition to the convex bipartite graph associated with the objective (4.4), they proved the following theorem:

Theorem 4.3 For all $k_1, k_2 \in \{1,2,\ldots,D\}$ with $k_1 \le k_2$, the objective (4.4) with B < 1 has a feasible solution if and only if $\sum_{i=1}^{n} \max(0, \lfloor k_2 r_i + B \rfloor - \lceil (k_1-1)r_i - B \rceil) \ge k_2 - k_1 + 1$

and $\qquad \sum_{i=1}^{n} \max(0, \lceil k_2 r_i - B \rceil - \lfloor (k_1-1)r_i + B \rfloor) \le k_2 - k_1 + 1$.

They further characterized the several algebraic properties of the problem with result oriented lemmas and conjectures. They proved that the *MDJIT* problem (4.4) is in *Co-NP*. For fixed *n*, they concluded that the optimization version of the problem can be reduced to integer linear programming with O(n) variables, in particular, the minimum value of B for the feasibility of the problem can be computed in time polynomial in O(log D). From the point of quota apportionment method, which can be described as a version of the *EDD* algorithm applied with the bound B = 1, Brauner and Crama [15] obtained slightly stronger bounds:

Theorem 4.4 If $\Delta_i = \frac{D}{\gcd(d_i, D)}$, $i = 1, \ldots, n$, then optimal value B^* of *MDJIT* problem (4.4) satisfies the inequality $\frac{1}{\Delta_i}\left\lfloor \frac{\Delta_i}{2} \right\rfloor \le B^* \le 1 - \frac{1}{D}$.

The *MDJIT* problem (4.4) with weighted objective function $g_w(x) = \min \max_{i,k} w_i |x_{ik} - kr_i|$ can also be reduced to the perfect matching problem [39, 61]. The $E(i,j)$ and the $L(i,j)$ for this problem are computed as the unique integers given by $E(i,j) = \left\lceil \frac{jw_i - B}{r_i w_i} \right\rceil$ and $L(i,j) = \left\lfloor \frac{(j-1)w_i + B}{r_i w_i} + 1 \right\rfloor$. The heavy weightage for particular copies of a product restricts the time window $[E(i,j), L(i,j)]$ and increases the separation of consecutive copies of that product in the production

sequence. For the optimal value of the problem $g_n(x)$, the upper and lower bounds are computed as $\max_i\{w_i\}$ and $\min_i w_i(1-r_i)$ respectively [21].

4.2.1 Problem with Small Deviations

Some conjectures are observed in [15] for the instances of *MDJIT* with $B^* \leq \frac{1}{2}$ and all instances are identified with small deviation, i.e., with $B^* < \frac{1}{2}$ for $n \leq 6$. An instance of the problem (4.4) with demands $d_1, d_2, \dots, d_n$ is called *standard* if $d_1 \leq d_2 \leq \dots \leq d_n$ and $\gcd(d_1, \dots, d_n, D) = 1$.

Small deviation conjecture (*SDC*): [15] For $n \geq 3$, a *standard* instance $(d_1, d_2, \dots, d_n)$ of the *MDJIT* problem (4.4) has optimal value $B^* = \frac{2^{n-1}-1}{2^n-1} < \frac{1}{2}$ if and only if $d_i = 2^{i-1}$, $i = 1, 2, \dots, n$.

The small deviation approach is further characterized via the concept of balanced words [15]. Induction on n is used to show that a periodic, symmetric and balanced word with $r_1 < r_2 < \dots < r_n$, $n \geq 3$, exists if and only if $r_i = \frac{2^{i-1}}{2^n-1}$, which is known as Fraenkel's conjecture for symmetric case [65]. For any sequence z with maximum deviation B^*, Jost [32] proved that any infinite periodic word with period z is *1*- balanced, 2-balanced, and 3-balanced on each product i, if $B^* < \frac{1}{2}$, $B^* < \frac{3}{4}$ and $B^* < 1$, respectively. It is observed in [22] that the sequences with bounds $\frac{1}{2}, \frac{3}{4}, 1$ are properly contained in the sets of *1*-, *2*- and *3*- balanced words respectively. Furthermore, [21] observed that there is no instance $(d_1, d_2, \dots, d_n)$ with $n > 3$ of the *MDJIT* problem that has a feasible solution with $B^* < \frac{1}{3}$.

According to [15], the validity of *SDC* is $3 \leq n \leq 6$, whereas Kubiak [40] proved *SDC* for any $n \geq 3$ by showing that if $B^* < \frac{1}{2}$, then all copies of all products must be sequenced in their ideal positions $\left\lceil \frac{2j-1}{2r_i} \right\rceil$ for copy j of product i ($j = 1, 2, \dots, d_i$). His geometric approach relied on the ideal positions and on symmetry of regular product polygons inscribed in a circle of circumference D such that each polygon corresponds to a different product having d_i corners (the number $\frac{2j-1}{2r_i}$ is referred as the ideal corner of copy j) for product i at $\left\lceil \frac{2j-1}{2r_i} \right\rceil$ points on the perimeter of the circle. He proved for $n = 2$ that there are infinitely many instances with optimal value less than $\frac{1}{2}$ in the following theorem.

Theorem 4.5 For $n = 2$, the optimal value of the *MDJIT* objective (4.4) is less than $\frac{1}{2}$ if and only if one of the demands d_1 or d_2 is even and the other is odd.

Kubiak further presented a complete characterization of instances with small deviations (i.e., less than $\frac{1}{2}$) for two products, and consequently characterized the instances with small deviations for any number of products. Finally, he exploited these results to prove special cases of the well known Fraenkal's conjecture.

4.2.2 Problem with two Products

Brauner and Crama [15] studied the *MDJIT* problem with two products $(n = 2)$ with ideal production rates r_1 and $r_2 = 1 - r_1$, formulating the problem to find a $2 \times D$ matrix $X = (x_{ik})$ that minimizes

$$\max_{1 \le k \le D} \left(|x_{1k} - kr_1|, |x_{2k} - kr_2| \right) \tag{4.5}$$

subject to

$$x_{1k} + x_{2k} = k, \quad k = 1, 2, ..., D$$
$$x_{i(k-1)} \le x_{ik} \quad i = 1, 2;\ k = 2, 3, ..., D$$
$$x_{io} = 0,\ x_{iD} = d_i,\ i = 1, 2$$
$$x_{ik} \in Z^+, \quad i = 1, 2;\ k = 1, 2, ..., D$$

The optimal solution of this problem is obtained and computed in polynomial time as follows: A matrix X defined by $x_{1k} = [kr_1]$ and $x_{2k} = k - [kr_1]$, $k = 1, 2, ..., D$ is an optimal solution of the two-product-typed *MDJIT* sequencing problem (4.5). Consequently, at every instant k, this fact permits to determine efficiently which product should be produced at time k. And the optimal value B^* is computed by the formula $B^* = \frac{1}{\Delta}\left\lfloor \frac{\Delta}{2} \right\rfloor$, where $\Delta = \frac{D}{\gcd(d_1, D)} = \frac{D}{\gcd(d_2, D)}$. It is noteworthy that the matrix X defined as above actually minimizes the deviation $x_{ik} - kr_i$ for all k and i. So it is optimal for the *MDJIT* problem and the *SDJIT* problem as well, with any convex and symmetric penalty function F_i.

4.3 Assignment Approach for *SDJIT*

Kubiak [37] and, Kubiąk and Sethi [38] reduced the *SDJIT* objective (3.2) under the given constraints therewith into an equivalent solvable assignment problem, taking F_i as a unimodal convex symmetric penalty function satisfying $F_i(0) = 0$ and

$F_i(y) > 0,\ y \neq 0,\ i = 1,2,\dots,n.$ The core idea of the assignment algorithm is calculation of ideal position and the assignment costs: The ideal position for each product i is computed by the formula $Z_j^{i^*} = \left\lceil \frac{2j-1}{2r_i} \right\rceil$, $n = 1, 2, \dots\dots, n$ and $j = 1,2,\dots\dots, d_i$, which is the ceiling of the unique crossing point of (i, j) satisfying $F_i(j - kr_i) = F_i(j-1-kr_i)$. Let $C_{jk}^i \geq 0$ be the cost of assigning (i, j) to the period k. This sequencing cost is computed by the formula:

$$C_{jk}^i = \begin{cases} \sum_{l=k}^{z_j^{i^*}} \psi^i{}_{jl} & ; \quad k < Z_j^{i^*} \\ 0 & ; \quad k = Z_j^{i^*} \\ \sum_{l=Z_j^{i^*}}^{k-1} \psi^i{}_{jl} & ; \quad k > Z_j^{i^*} \end{cases}$$

where $\psi_{jl}^i = \left\| j - lr_i \right| - \left| j - 1 - lr_i \right\|$, and $(i, j) \in I = \{(i, j) : i = 1,2,\dots,n;\ j = 1,2,\dots,d_i\}$, $l = 1,2,\dots,D$

Defining the production variable $x^i{}_{jk} = 1$, if (i, j) is assigned to period k and *0*, if otherwise, Kubiak and Sethi [38] obtained the following result:

Theorem 4.6 An optimal solution to minimize *SDJIT* objective (3.2) subject to the constraints (3.3) to (3.6), can be obtained from any optimal solution of the assignment problem:

$$\text{minimize} \qquad \sum_{k=1}^{D} \sum_{(i,j)}^{(n,d_i)} C^i{}_{jk} x^i{}_{jk} \tag{4.6}$$

subject to

$$\sum_{(i,j)}^{(n,d_i)} x^i{}_{jk} = 1,\ k = 1, 2, \dots, D \tag{4.7}$$

$$\sum_{k=1}^{D} x^i{}_{jk} = 1, \quad (i, j) \in I \tag{4.8}$$

$$x^i{}_{jk} = 0 \ \text{ or } 1,\ k = 1,2,\dots\dots, D,\ (i, j) \in I. \tag{4.9}$$

The feasibility and the optimality of this problem are subsequently proved in [38]. The assignment problem is efficiently solvable; viz. the problem with $2D$ nodes can be solved in $O(D^3)$ time [56]. Moreover, there are D^2 values ψ^i_{jl} to calculate and then D^2 values $C^i{}_{jk}$ to calculate, each taking $O(D)$ steps, which shows that the computation of the assignment costs takes $O(D^3)$ steps. Balas et al. [2] developed a parallel algorithm which can efficiently solve assignment problems with 900 million variables.

It is proved in [22] that the assignment problem cannot be solved at optimality only under the constraints (4.7) to (4.9), however needs another constraint, not of

assignment type: If (i, j, k) and (i, j', k') are feasible schedules with $k < k'$, then $j < j'$, i.e., lower indices copies are produced earlier, which imposes an order on copies of a product. This constraint is essential as it ties up the copy j of a product with the j^{th} ideal position for the product. Looking on the efficiency, the order of the copies can be reordered for the optimality and the latter copies can be produced earlier in the production sequence. The approach proposed in [37, 38] for *SDJIT* is applicable to any l_p-norm; and in particular to l_∞- norm minimizing *MDJIT* objective [22]. Some structural properties of the min-sum *PRV* problem are described in [1].

4.3.1 Problem with Cyclic Sequences

The optimal *JIT* sequences for min-sum problem are cyclic [40] which is a factoring idea having potential to reduce the computational effort in constructing optimal sequences. It is proved that if β is an optimal sequence for the instance $d_1, d_2, \ldots, d_n$, then concatenation β^m of m copies of β is an optimal sequence for instance $md_1, md_2, \ldots, md_n$. The even instances of the form $2d_1, 2d_2, \ldots, 2d_n$ for some positive integers $d_1, d_2, \ldots, d_n$ are considered with feasible sequences of length $2D$ where $D = \sum_{i=1}^{n} d_i$. The three operations folding, shuffling and unfolding are used to prove the existence of optimal cyclic sequences rather than to determine optimal sequences. The optimal sequence to the original problem can be obtained in the three steps (i) by calculating the greatest common divisor m of $d_1, d_2, \ldots, d_n$, (ii) by using the algorithm in Kubiak and Sethi [38] to obtain an optimal sequence for $\frac{d_1}{m}, \frac{d_2}{m}, \ldots, \frac{d_n}{m}$ and (iii) by concatenating the sequence m times to construct an optimal sequence for the original demands $d_1, d_2, \ldots, d_n$.

Theorem 4.7 (Existence of cyclic sequence) If $z = s_1, s_2, \ldots, s_D, s_{D+1}, \ldots, s_{2D}$ is a feasible sequence for $2d_1, 2d_2, \ldots, 2d_n$, then a sequence $z' = z_1, z_2, \ldots, z_D, z_{D+1}, \ldots, z_{2D}$, where i occurs exactly d_i times in each of the two halves $z_1, z_2, \ldots, z_D$ and $z_{D+1}, \ldots, z_{2D}$, can be constructed s. t. $Z_{\min}(z') \le Z_{\min}(z)$.

Theorem 4.8 (Optimality of cyclic sequence) If β is an optimal sequence for (3.2) with the integer demands $d_1, d_2, \ldots, d_n$, then β^m $(m \ge 1)$ is optimal sequence with the demands $md_1, md_2, \ldots, md_n$.

For further knowledge of cyclical property, we refer to [11, 40, 61].

4.4 Dynamic Programming Approach

A dynamic programming (*DP*) procedure to determine the optimal *JIT* production schedule for single-level min-sum problem is presented in [49], considering usage and loading goals simultaneously. The usage goal maintains a constant rate of usages of all items in the system whereas the second goal smooth the work load on the final assembly process to reduce the chance of the production delays and stoppages by balancing the products having long production times with the products having relatively short production times.

The *DP* formulation: Taking the notations as in Section 3.1, we let t_i be the time required to produce one unit of product i and define usage and loading variabilities at stage k by $U_k = \sum_{i=1}^{n}(x_{ik}-kr_i)^2$ and $L_k = \sum_{i=1}^{n} t_i^2(x_{ik}-kr_i)^2$. In this case, the joint problem is to seek integer x_{ik} minimizing the objective $\sum_{k=1}^{D}(\alpha_U U_k + \alpha_L L_k)$ under constraints (3.3) to (3.6), where α_U and α_L are relative weights for usage and loading goals. If $\alpha_U = 1,\ \alpha_L = 0$, then the problem converts into the usage problem. If $\alpha_U = 0,\ \alpha_L = 1$, then the problem converts to the loading problem. If v_k is joint variability at stage k, we have

$$v_k = \alpha_U \sum_{i=1}^{n}(x_{ik}-kr_i)^2 + \alpha_L \sum_{i=1}^{n} t_i^2 (x_{ik}-kr_i)^2 = \sum_{i=1}^{n}(\alpha_U + \alpha_L t_i^2)(x_{ik}-kr_i)^2 = \sum_{i=1}^{n} T_i^2 (x_{ik}-kr_i)^2,$$

where $T_i^2 = \alpha_U + \alpha_L t_i^2$. Here, T_i is referred as the implied production time for product i. Hence, the *SDJIT* objective function to be minimized can be restated as

$$\text{minimize} \sum_{k=1}^{D}\sum_{i=1}^{n} T_i^2 (x_{i,k}-kr_i)^2 \qquad (4.10)$$

The *DP* procedure: Let the demand vector be $d = (d_1, d_2, \ldots, d_n)$ and the product vector be $X = (x_1, x_2, \ldots, x_n)$ in which each x_i is a non negative integer representing the production of exactly x_i units of product i, $x_i \le d_i, \forall i$. Let e_i be unit vector with n entries, all of which are zero except a single one in i^{th} position. Products in X can be produced in first k periods if $k = |X| = \sum_{i=1}^{n} x_i$. Let $f(X)$ be minimal total variation of any schedule where the products in X are produced during the first k stages and let $g(X) = \sum_{i=1}^{n} T_j^2 (x_j - kr_j)^2$. Then the following *DP* recursion process holds for $f(X)$:

$$f(X)=f(x_1,x_2,\dots,x_n)=\min\{f(X-e_i)+g(X): i=1,\dots,n; x_i-1\ge 0\}$$
$$f(X)=f(X: x_i=0; i=1,\dots,n)=f(0,0,\dots,0)=0.$$

Clearly, $f(X)\ge 0$ and $g(X: x_i=0; i=1,\dots,n)=0$. The following theorem states the time and the space complexities of the above procedure (see [49] for the proof).

Theorem 4.9 The DP recursion solves (4.10) in $O\left(n\prod_{i=1}^{n}(d_i+1)\right)$ time and $O\left(\prod_{i=1}^{n}(d_i+1)\right)$ space.

Note that total number of feasible schedule is $\frac{D!}{d_1!d_2!\dots d_n!}$, which is considerably larger than the number of stages in the DP recursion. Moreover, $\prod_{i=1}^{n}(d_i+1)\le\left(\frac{d_1+d_2+\dots+d_n+n}{n}\right)^n=\left(\frac{D+n}{n}\right)^n$.

Thus, the growth rate of the number of sets is polynomial in D, although it is exponential with n. This indicates that the DP procedure is effective for small n even with large D and hence it significantly reduces the space complexity. The optimization process that run in polynomial time in D for all products, produced over a given time horizon is presented for two instances of the single-level problem [49].

The DP procedure for general multi-level problem is developed in [39], which is also polynomial in D_l and consequently, seems to be effective for small number of products n_l even when the total product demand D_l is large. During the enumeration process, an excessive amount of time or space is reduced by using some fast heuristic as a filter which eliminates any states from DP's state space that would lead to no optimality. Two myopic heuristics to generate the filter are proposed in [39]. If the heuristics yield near-optimal sequences, then the state space size can be reduced.

5. METHODS OF APPORTIONMENT

5.1 Largest Reminder Method

This is the simplest method of apportionment proposed by A. Hamilton (known as method of *Hamilton*), used to apportion the House of Representatives in USA from 1850 to 1900 under the name of *Vinton* method of 1850. Computing the fair share q_{ih}, each state is given its lower quota of seats $(\lfloor q_{ih}\rfloor)$. Then the states are

listed in order beginning with the state having the largest fractional remainder $(q_{ih} - l_i)$ and continuing on down to the state with the smallest such remainder. The remaining seats are then assigned one each to the states ranking with the highest fractional part on the list, till the house is full. It is somehow based on quota system. Its mathematical model is: minimize $\sum_{i=1}^{s}(a_{ih} - q_{ih})^2$ s. t. $\sum_{i=1}^{s} a_{ih} = h$ and $a_{ih} \geq 1$. Clearly, it is a constrained integer programming problem seeking for integer allocations a_{ih} that are never less than unity and staying as close as possible to fair shares q_{ih}. Hamilton observed at absolute deviation, whose global min-max property is: $\min\max|a_{ih} - q_{ih}|$. This method is summarized as follows:

Algorithm 5.1 (The Largest Reminder Algorithm):

Step 1. Compute $q_{ih} = \frac{p_i h}{p}$, the ideal (fractional) value that gives perfect proportionality.

Step 2. Set $r_i = q_{ih} - \lfloor q_{ih} \rfloor$, the fractional remainder of q_{ih}.

Step 3. Assign $a_{ih} = \lfloor q_{ih} \rfloor$, for $i = 1, 2, \ldots, n$.

Step 4. Let $R = h - \sum_{i=1}^{s} a_{ih}$, be the number of seats that remain to be allocated.

Step 5. If $R > 0$, assign one more seat to states/parties having the largest fractional remainders r_i.

This method satisfies quota rule, however suffers Alabama, population and new states paradoxes. To avoid these shortcomings and motivated by the need for *house-monotone* methods, Huntington [29, 30] developed the divisor methods (Section 5.3). Any monotone solution can be characterized by identifying the sequence in which the states successively gain seats as the house size h increases. The divisor-based methods, known as the methods of highest averages, vary according to the form of divisors.

5.2 Quota Method

Balinski and Young [6] proved that there is no divisor method that satisfies quota; only the method of the smallest divisor satisfies upper quota (ceiling of exact representation) and only the method of the largest divisors satisfies lower quota (floor of exact representation). As a refinement of the Huntington methods, they devised Quota method of apportionment, which avoids both the Alabama paradox

and Quota paradox. Instead of comparing all the states in the minimization of shortchangedness, only states that are eligible to receive a seat or to lose a seat are considered. Eligibility means that they won't exceed upper quota or won't go below lower quota upon receiving or loosing a seat. However, this method is biased favoring large states, since Jefferson's method is used to compare the states. To avoid this flaw, in the uniqueness proof for their method, Balinski and Young proved that Quota method is the only method which satisfies three requirements: *satisfying quota, house monotonicity* and *mathematical consistency.* More precisely,

Theorem 5.1 The quota method is the unique apportionment method which is house monotone, consistent and satisfies quota.

Let $M(p,h)$ be set of apportionment methods. If $f \in M(p,h)$, and $f_i(p,h) = a_i$ and if $q_i(p,h)$ denotes the quota of state i, then the state i is eligible at $h+1$ for its $(a+1)^{th}$ seat if $a_i < q_i(p,h+1) = \frac{(h+1)p_i}{p}$.

Defining eligible set as $E(a,h+1) = \{i \in s : i \text{ is eligible for } a_i + 1 \text{ at } h+1\}$, the quota algorithm is:

Algorithm 5.2 (The Quota Algorithm):

Step 1. Start with $f(p,0) = 0$, that is, $a_i = 0,\ i = 1,2,\ldots,s$

Step 2. Find a state $t \in E(a,h+1)$ such that $\frac{p_t}{a_t+1} = \max\limits_{i \in E(a,h+1)} \frac{p_i}{a_i+1}$

Then $f_t(p_t, h+1) = a_t + 1$ for one such t and $f_j(p_j, h+1) = a_j$ for $j \neq t$

Step 3. Repeat step 2 until all seats are allocated.

Furthermore, a class of new apportionment methods (including Quota method) is defined in [58] that are also house monotone and satisfy quota. The first characteristic of this method is that all of them are defined *recursively* as follows: in the trivial case of a house of size 0, all states are assigned 0 seats. At all larger house sizes, the apportionment is the same as at the next lower house size, but with the additional seat assigned to one of the states according to specified rules. This sequential procedure assures house monotonicity. The second characteristic is the use of *eligibility set*: a set of those states which are eligible to receive the additional seat, denoted by $E(h)$, where h is house size. The eligibility set $E(h)$ at any house size $h > 0$ consists of all states i that satisfy the two tests- *The upper quota test:* state i satisfies upper quota test, if $a_i(h-1) < u_i(h)$, where u_i is the upper

quota. *The lower quota test:* let h_i be the house size at which state i first becomes entitled to obtain the next seat, i.e., h_i is the smallest house size $h' \geq h$ at which the lower quota of state i is greater or equal to $a_{i(h-1)}$ or $h_i = \left\lceil \frac{a_{i(h-1)+1}}{p_i} \sum_{i=1}^{n} p_i \right\rceil$.

For each house size g in the interval $h \leq g \leq h_i$, define $s_i(g,i) = a_{i(h-1)} + 1$ (the number of seats that state i has in a house of size h before an additional seat is assigned +1); for $j \neq i$, $s_i(g,i) =$ $\max\{a_i(h-1),\, l_g(h)\}$. If there is no house size g, $h \leq g \leq h_i$, for which $\sum_j s_j(g,i) > g$, then state i satisfies the lower quota test. The eligibility set $E(h)$ consists of all states which may receive the available seat without causing a violation of quota either at h or any larger house size.

That is, $E(h) = \{i^{th}$ state: i^{th} state passes the upper and the lower quota tests$\}$.

Still [58] proved that the eligibility set $E(h)$ contains at least one state, for $h > 0$; and all apportionment methods in the class are house monotone and satisfy quota. The states from $E(h)$ can be chosen in various ways, for example (i) by using *ranking functions* (population, land area, alphabetical order, percent of minorities or women in population etc), (ii) by using *random selection*, (iii) by using *quota-divisor methods*, which are based on divisor methods. The only difference is that the states in quota divisor methods must be from $E(h)$. This algorithm is defined as follows [33]:

(i) $M(p,0) = 0$, (ii) If $a \in M(p,h)$ and $t, i \in E(h)$ satisfies $\frac{p_t}{d(a_t)} = \max_i \frac{p_i}{d(a_i)}$

Then $b \in M(p, h+1)$ with $b_t = a_t + 1$ for $i = t$ and $b_i = a_i$ for $i \neq t$.

It is really difficult to find a perfect apportionment method meeting all the desired requirements. Even the quota methods for congressional apportionment are non-unique [44]. In this regard, Balinski and Young [8] established the following famous Impossibility Theorem:

Theorem 5.2 There are no perfect apportionment methods. Moreover, it is impossible for an apportionment method to be population monotone and stay within the quota at the same time for any reasonable instance of the problem $(s \geq 4 \text{ and } h \geq s+3)$.

5.3 Divisor Methods

The divisor methods comprise a family of monotone methods involving a notion of *rounding*, each of which is defined by a monotone increasing divisor function $d(a)$ such that $a \le d(a) \le a+1$. Huntington [30] made the systematic study of divisor methods based upon the *rank-index* $r(p, a) = \frac{p}{d(a)}$, $d(a) \neq 0$ and the *fairness measure* $\frac{a_{ih}}{p_i} > \frac{a_{jh}}{p_j}$, minimizing pairwise measures of inequity between two states i and j [7, 8], where p and a represent the population and the apportionment vectors. The state achieving maximum of $\frac{p}{d(a)}$ gains the $(h+1)^{th}$ seat. For given h, let $\frac{p_i}{a_i}$ and $\frac{a_i}{p_i}$ represent the average district sizes and the share of representatives respectively of state i, and an apportionment method M is said to be house monotone if for every apportionment solution $f \in M$, we have $f(p,h) \le f(p,h+1)$. Practically, there always exists a certain inequality between two states, which gives one of the states a slight advantage over the other. The state i is better off than state j, if $\frac{a_{ih}}{p_i} > \frac{a_{jh}}{p_j}$. An apportionment a is stable if no transfer of one seat from a better off state i to a less well off state j reduces the value of the inequality. The local measures of inequalities can be rearranged by cross-multiplication in $2^4 = 16$ different ways by taking different combinations of p_i, p_j, a_i, a_j [8]. The $d(a)$ and the rank-indices are not unique (see Table 5.1).

If a tie occurs between states with unequal populations (extremely unlikely), Huntington suggested that it should be broken in favor of the larger state. His approach made remarkable use of pairwise comparison of local measures of inequity to be minimized between two states i and j, which are not unique as below:

1. **Adam's Method:** absolute representative surplus $\rightarrow a_i - a_j\left(\frac{p_i}{p_j}\right)$.

2. **Dean's Method:** absolute difference in average district sizes $\rightarrow \left|\frac{p_i}{a_i} - \frac{p_j}{a_j}\right|$.

3. **Hill's Method:** relative differences in both district sizes and shares of a representative $\rightarrow \left|\frac{p_i}{a_i} - \frac{p_j}{a_j}\right| \Big/ \min\left(\frac{p_i}{a_i}, \frac{p_j}{a_j}\right)$ and $\left|\frac{a_i}{p_i} - \frac{a_j}{p_j}\right| \Big/ \min\left(\frac{a_i}{p_i}, \frac{a_j}{p_j}\right)$.

4. **Webster's Method:** absolute difference in shares of a representative $\rightarrow \left|\frac{a_i}{p_i} - \frac{a_j}{p_j}\right|$.

5. **Jefferson's Method:** absolute representative deficiency $\rightarrow a_i\left(\frac{p_j}{p_i}\right) - a_j$.

Table 5.1 The best known Divisor methods

Methods	**Alternative names**	**When used**	**Divisor** $d(a)$	**Rank-index** $r(p, a)$	**Pairwise comparison** $\frac{a_i}{p_i} > \frac{a_j}{p_j}$
Adams (**A**)	Smallest divisors	---	a	$\frac{p}{a}$	$a_i - a_j(\frac{p_i}{p_j})$
Dean (**D**)	Harmonic means	---	$\frac{2a(a+1)}{2a+1}$	$\frac{p}{\frac{2a(a+1)}{2a+1}}$	$\frac{p_j}{a_j} - \frac{p_i}{a_i}$
Hill (**H**)	Equal proportion	1940 - date	$\sqrt{a(a+1)}$	$\frac{p}{\sqrt{a(a+1)}}$	$\frac{a_i p_j}{a_j p_i} - 1$
Webster (**W**)	Major fractions	1840, 1910, 1930	$a+\frac{1}{2}$	$\frac{p}{a+\frac{1}{2}}$	$\frac{a_i}{p_i} - \frac{a_j}{p_j}$
Jefferson (**J**)	Largest divisors	1792-1830	$a+1$	$\frac{p}{a+1}$	$a_i(\frac{p_j}{p_i}) - a_j$

Huntington [29, 30] showed that method of equal proportion *(MEP)* is the best of the five divisor methods, since it relies on the most natural measure of inequality, the relative difference. He was supported by two selection committees which reported to the president of the National Academy of Sciences in 1929 [13] and in 1948 [53]. Both of these reports pleaded for *MEP* because it is unambiguous and house monotone yielding apportionments that are neutral with respect to emphasis on larger and smaller states. Moreover, *MEP* is consistent: If (p,a) and (p',a') are tied (two states having equal populations $p = p'$), then any method M should be *indifferent* between such states. That is, for some p, h, $f_i(p,h) = a$, and $f_j(p,h) = a'$, if a solution $f \in M$ gives the $(h+1)^{th}$ seat to state i, then there should be an alternative solution $g \in M$, identical with f up to h (i.e., $g_h = f_h$) that gives the $(h+1)^{th}$ seat to state j. Any method having this property is called *consistent.* Moreover, consistency means if $(p,a) \sim (p', a')$, then any two states with populations p, p' and apportionments a, a' equally deserve a seat. Divisor method based on $r(p,a)$ is

$$M(p,h)=\left\{a\geq 0:\sum\nolimits_{i=1}^{s}a_i=h,\ \max_i r(p_i,a_i)\leq \min_{a_j>0} r(p_j,a_{j-1})\right\} \tag{5.1}$$

Theorem 5.3 [7] An apportionment method M is a house monotone and consistent if and only if it is a Huntington method.

Huntington [30] examined 64 different measures of local inequity considering different combinations of p_i, p_j, a_i, a_j between two states including 32 relative and 32 absolute differences. All of the relative differences and two of the absolute differences lead to *MEP*. His noble approach is pairwise transfer of seats among states according to the priority basis to balance the apportionment. A transfer is made from the more favored state to less favored state if this reduces the inequity measure between two states.

Theorem 5.4 Between two states i and j, the assignment $(A1)$ a_i+1 and a_j is better assignment than $(A2)$ a_i and a_j+1 if and only if $\frac{p_i}{\sqrt{a_i(a_i+1)}}>\frac{p_j}{\sqrt{a_j(a_j+1)}}$.

Proof: We prove the theorem when the state i is more favored in $(A1)$ with $\frac{p_j}{a_j}-\frac{p_i}{a_i+1}>0$ and the state j is more favored in $(A2)$ with $\frac{p_i}{a_i}-\frac{p_j}{a_j+1}>0$. Now $(A1)$ is a better assignment than $(A2)$ if and only if

$$\frac{p_j/a_j-p_i/(a_i+1)}{p_i/(a_i+1)}<\frac{p_i/a_i-p_j/(a_j+1)}{p_j/(a_j+1)}\Leftrightarrow\frac{p_j(a_i+1)-p_ia_j}{p_ia_j}<\frac{p_i(a_j+1)-p_ja_i}{p_ja_i}\Leftrightarrow\frac{p_j^2}{a_j(a_j+1)}<\frac{p_i^2}{a_i(a_i+1)}$$

Note that some inequality measures are unworkable, which may lead to infinite cycling of solutions, e. g., $\frac{a_i}{a_j}-\frac{p_i}{p_j}$ and $\frac{p_ja_j}{p_i}-\frac{1}{a_i}$, where $\frac{p_i}{a_i}=\frac{p_j}{a_j}$. Keeping deep insights to the traditional divisor methods, Oyama [55] gave *ARPT* (average ratio pairwise transfer) rule, which implied larger stable region than Huntington's one. He viewed the apportionment methods from the angle of constrained optimization problem, restricting its application to the case when one state is over-represented absolutely and the other under-represented absolutely, that is, when $\frac{a_i}{p_i}\geq\frac{h}{\sum_i p_i}\geq\frac{a_j}{p_j}$ [3]. The main idea of divisor (rank-index) methods is:

Algorithm 5.3 (Rank-index Algorithm):

Step 1. Start with $f(p,0)=0$, that is, $a_i=0$, $i=1,2,\ldots,s$

Step 2. Find a state t such that $r(p_t,a_t)=\max_i r(p_i,a_i)$

Then $a_t'=a_t+1$ and $a_j'=a_j$ for $j\neq t$

Step 3. Repeat step 2 until all h seats are allocated.

5.3.1 Parametric Divisor Methods

A parametric divisor method denoted by ϕ^{δ}, is a divisor method ϕ^{d} based on $d(a) = a + \delta$, where $0 \le \delta \le 1$. The divisor methods *A, W, J* are parametric with $\delta = 0$, $\delta = 0.5$, $\delta = 1$ respectively. Saint-Lague favored *W* and d'Hondt favored *J*. Condorcet proposed slightly different parameter δ = 0.4. We have proposed $\delta = 0.7$ in Section 5.3.2. Being linear divisor functions, parametric methods are computationally very efficient. Moreover, they are cyclic generating cyclic *JIT* sequences: for two instances of the *JIT* sequencing $D_1 = d_1, d_2, \ldots\ldots, d_n$ and $D_2 = kD_1 = kd_1, kd_2, \ldots\ldots, kd_n$, the sequence for D_2 is obtained by k repetitions of the sequence for problem D_1. Note that as δ increases from 0 to 1, seats being given-up by the smaller states in favor of the larger states [4].

Lemma 5.1 A parametric method ϕ^{α} gives-up to another parametric method ϕ^{β} if and only if $\alpha < \beta$.

Thus parametric method ϕ^{δ} is most favorable to smaller states with $\delta = 0$ and most favorable to larger states with $\delta = 1$. The fundamental three properties of parametric method are as follows. *Anonymity*: the solutions must depend only on the values of the data, not on the order in which the data is presented. *Scale-invariancy*: $\phi(p,h) = \phi(\lambda p, h)$, for all $\lambda > 0$; *Exactness*: if p is integer valued and $\sum_i p_i = h$, then p is the unique solution $\phi(p,h) = p$. A method ϕ is balanced if $a \in \phi(p,h)$ and $p_i = p_j$ implies $|a_i - a_j| \le 1$.

Lemma 5. 2 A consistent, exact and anonymous method is balanced.

An apportionment method ϕ is cyclic, if $a \in \phi(p,h)$, p integer implies $a + p \in \phi(p, h + p_s)$, for an example, Hamilton method is cyclic.

Theorem 5. 5 [4] A divisor method ϕ is parametric if and only if it is cyclic.

There are infinitely many parametric divisor functions lying between a and $a + 1$ depending upon the value of δ s. t. $0 \le \delta \le 1$. We identify the two slightly new divisors, to which we call mean-based divisors.

5.3.2 Mean-based Divisor Methods

The two mean-based divisor (*MBD*) functions are computed from available five divisors, both of which are based on arithmetic mean. The first divisor is

calculated by the mean of all divisors whereas the second is calculated by the mean of Webster's and Jefferson's divisors. The first divisor falls between Hill's and Webster's divisors, immediate right to Hill's divisor capturing the properties of both. Obviously the second one lies between Webster's and Jefferson's divisors. The **first divisor (*MBD1)*** is computed as:

$$\begin{aligned} d(a) &= \frac{1}{5}\left[a + \frac{2a(a+1)}{2a+1} + \sqrt{a(a+1)} + a + \tfrac{1}{2} + a + 1\right] \\ &= \frac{1}{5}\left[3a + \tfrac{3}{2} + \frac{2a(a+1)}{2a+1} + \sqrt{a(a+1)}\right] \\ &= \frac{1}{5}\left[\frac{16a^2+16a+2(2a+1)\sqrt{a(a+1)}+3}{2(2a+1)}\right] \\ &= \frac{16a^2+16a+2(2a+1)\sqrt{a(a+1)}+3}{10(2a+1)} \\ &= \frac{\Psi(a)}{10(2a+1)}, \text{ where } \Psi(a) = 16a^2 + 16a + 2(2a+1)\sqrt{a(a+1)} + 3. \end{aligned}$$

The respective rank-index is $r(p,a) = \frac{p(2a+1)}{\Psi(a)}$. The **second divisor (*MBD2*)** is computed as:

$d(a) = \frac{1}{2}\left[a + \frac{1}{2} + a + 1\right] = a + \frac{3}{4}$, and the respective rank-index is $r(p,a) = \frac{4p}{4a+3}$.

Clearly the second divisor is parametric, whereas the first is not. The actual location of our divisors among the existing five divisors can be sketched from the following inclusion with the notations from Table 5.1:

Proposition 5.1

The inclusion holds true $A < D < H < MBD1 < W < MBD2 < J$ (5.2)

Thus, our divisors are positioned in the neighborhood of Hill's and Webster's divisors. As Hill's and Webster's methods are considered to be mathematically neutral with respect to emphasis on larger and smaller states, near to ideal fraction and consistent, the proposed new divisors yield the better results in apportioning the seats to states/parties. With this discussion, we establish the following theorem:

Theorem 5.6 The mean-based divisors $MBD1$ and $MBD2$ generate the apportionments, which are near to ideal, consistent, monotone and neutral.

In this sense, we claim that our divisors clearly point out the location of ideal apportionment, standing in the "middle" of the divisors of other methods, and so they are better than others. However, we agree that time complexity of our

methods is higher than other methods. The local measure of inequalities to be minimized under these two new divisors is considered with relative difference as given by Hill-Huntington, it is because the method yielding the smallest relative difference is taken as the best method. Together with our divisors, we argue that method of equal proportion is stable with the relative difference given by

$$T = \left|\frac{a_{ih}}{p_i} - \frac{a_{jh}}{p_j}\right| \Big/ \min\left\{\frac{a_{ih}}{p_i}, \frac{a_{jh}}{p_j}\right\} = \frac{a_i p_j}{a_j p_i} - 1, \text{ for } \frac{a_{ih}}{p_i} \geq \frac{a_{jh}}{p_j} \tag{5.3}$$

The ideal position is $T = 0$, which is very rare in practical. Therefore, the measures of inequality between two states can not be eliminated perfectly; and hence the fundamental objective is to minimize the measures of inequality as far as possible to reach the ideal position. To this point, we refer [64].

5.4 Other Approaches

There are other several methods of apportionment suggested by many mathematical scientists and political theorists. The **balanced method** is proposed in [57] to minimize the advantages of large states over small states. With the earlier notations, the following formula is used to find the apportionment for state i: $a_i = \frac{p_i h}{p(1+\Delta)}$, where $\Delta = \varepsilon \big/ 1 + \frac{p_i}{pC}$. The exact quota $\frac{p_i h}{p}$ of state i is multiplied by a proportion that is somewhat bigger than one. The epsilon $(0 \leq \varepsilon \leq 1)$ is balancing out the effect of truncation, but favoring large states. To reduce this effect and to satisfy upper quota, there is another number C, balancing that effect and making sure that the results satisfy quota. This method satisfies quota and is uniform for all states. However, it favors small states and admits the Alabama paradox. Compared to other methods, this provides a better alternative if one wishes to give an advantage to smaller states and stay within quota. The **generalized apportionment problem** (*GAP*), a more general formalization, is studied in [12] and solved via optimization procedure of a very broad class of discrepancy functions. In addition of presenting a synthesis of the classical approaches to the apportionment, the generalized divisor methods (*GDMs*) are defined, that optimized a family of general discrepancy functions for the *GAP*. A method for determining which discrepancy functions are optimized by any given *GDM*, is established in [12], including the classical divisor methods. The method is applied to resolve the minsum problem, indicating a possible line of research that the approach can be extended to the resolution of minmax problems.

6. LINKAGE: *JIT* SEQUENCING VERSES APPORTIONMENT

6.1 Transformations of the two Problems

Establishing the relation between *PRV* problem and apportionment, Bautista et al. [10] stated that the *JIT* sequencing problem can be seen as a constrained sequential apportionment problem. The monotone condition of *PRV* problem is equivalent to house monotonicity. They indicated that the algorithm of Inman and Bulfin [31] is the Webster divisor method of apportionment. Balinski and Shahidi [5] proposed a strong approach to *JIT* sequencing via axiomatics, which are originally developed for the apportionment problem.

The axiomatic method of apportionment depends on some socially desirable characteristics, such as satisfying quota, house and population monotonicity etc, which must be satisfied for the solution of apportionment problem. However, the *impossibility theorem* of Balinski and Young put a limitation that there is no perfect apportionment method satisfying all properties. The *PRV* problem in terms of parametric divisor methods is studied in [4]. Józefowska et al. [33] characterized some of the algorithms of *JIT* sequencing via apportionment theory with suitable transformation of the problems. Adding some similar properties, we present the notational interrelation of the two problems as follows [23, 64]:

Table 6.1 JIT sequencing verses apportionment

Number of products n	$\Leftrightarrow$	number of states s
Product i	$\Leftrightarrow$	state i
Vector of demands d	$\Leftrightarrow$	vector of populations p
Demand d_i for product i	$\Leftrightarrow$	population p_i of state i
Position in sequence k	$\Leftrightarrow$	size of house h
Actual production x_{ik}	$\Leftrightarrow$	a_{ih} to state i, for house size h
Ideal production kr_i	$\Leftrightarrow$	exact quota q_{ih}
Total demand $D=\sum_{i=1}^{n} d_i$	$\Leftrightarrow$	total population $p=\sum_{i=1}^{s} p_i$
Monotone condition in *JIT*	$\Leftrightarrow$	house monotone in apportionment

Thus, the two problems can be seen from the same window and handled in similar ways in most of the instances, such as the parametric divisor methods of apportionment generate cyclic just-in-time sequences. The further algorithmic characterizations and joint approaches are discussed in the following sections.

6.2 Algorithmic Characterization of the two Problems

It is observed in [10] that Inman-Bulfin (*IB*) algorithm [31] to minimize the sum deviation objective (4.3) is equivalent to Webster divisor method. The optimal value is found by applying *EDD* rule taking t_{ik} as due dates, and reducing into single machine scheduling problem. In *IB* algorithm, the units are sequenced according to the increasing order of the values $\frac{2k_i-1}{2r_i}$ and in Webster method, the rank-index is $\frac{2p_i}{2a_i-1}$. Thus both procedures are equivalent and so Webster method optimizes (4.3) with due dates $t_{ik_i}=\frac{2k_i-1}{2r_i}$.

Steiner and Yeomans [59, 60] proposed a graph theoretic polynomial time algorithm (say, *SY* algorithm) to minimize the *MDJIT* problem (4.4) with the targeted bound B. based on the following theorem:

Theorem 6.1 A *JIT* sequence with $\min_i \max_k |x_{ik}-kr_i|<B$ exists iff there exists a sequence that associates the part (i,j) in the interval $[E(i,j),L(i,j)]$, where $E(i,j)=\left\lceil \frac{1}{r_i}(j-B)\right\rceil$ and $L(i,j)=\left\lfloor \frac{1}{r_i}(j-1+B)+1\right\rfloor$ are the earliest and the latest starting times respectively of (i,j) in the final production sequence.

The *SY* algorithm tests the values of B from the list $B=\frac{D-d_{\max}}{D}, \frac{D-d_{\max}-1}{D}, \ldots, \frac{D-1}{D}$, in ascending order.

For each B, the algorithm calculates the $E(i,j)$ and $L(i,j)$ for each part (i,j), and assigns $k=1,2,\ldots,D$ starting with $k=1$ and ending with $k=D$ to the yet unassigned but still available at k. A pair (i,j) is available at k if and only if $E(i,j)\le k\le L(i,j)$. If some pairs cannot be assigned, then the value of B is rejected as infeasible. Brauner and Crama [15] proved that at least one of the values of B in the above list is feasible, and hence we have, $\min_i \max_k |x_{ik}-kr_i|<1-\frac{1}{D}$.

If B' is feasible, then all $B.B'\le B\le 1-\frac{1}{D}$ are feasible as well. The smallest feasible B is denoted by B^* and referred to as optimum. It is proved in [33] that *SY* algorithm is a quota-divisor method of apportionment:

Theorem 6.2 The *SY* algorithm with B, $B^* \le B < 1$ and a tie $L(i,j) = L(k,l)$ between i and k broken by choosing the one with $\min\left\{\frac{1}{r_i}(j-1+B), \frac{1}{r_k}(l-1+B)\right\}$ is a quota-divisor method with $d(a) = a + B$.

Moreover, *SY* algorithm is a quota-parametric method with $\delta = B$.

Keeping deep insights in Tijdeman's chairman assignment algorithm [65], Józefowska et al. [33] obtained the stronger upper bound $1 - \frac{1}{2n-2}$ than $1 - \frac{1}{D}$. Defining the set J_k, $k = 1,2,\ldots,D$ of eligible states, they proved that Tijdeman's algorithm is quota-divisor method with $d(a) = a + \Delta$, where $\Delta = 1 - \frac{1}{2n-2}$. Further narrowing the eligible set, they showed that Tijdeman algorithm is quasi quota-divisor method.

Józefowska et al. [33] characterized Kubiak-Sethi *(KS)* algorithm based on the following three lemmas:

Lemma 6.1 The *KS* algorithm does not stay within the quota.

Proof: Corominas and Moreno [20] observed that no solution minimizing the *SDJIT* problem subject to the constraints (3.3) to (3.6) stays within the quota. for instance of $n = 6$ products with their demands being $d_1 = d_2 = 23$ and $d_3 = d_4 = d_5 = d_6 = 1$. Since *KS* algorithm minimizes *SDJIT*, this proves the lemma.

Lemma 6. 2 The *KS* algorithm is house monotone. It is obvious due to the constraint (3.4).

Lemma 6. 3 The *KS* algorithm is not uniform, and hence is not population monotone.

Balinski and Shahidi [5] proposed other types of deviations for the products i and j, targeting to minimize the variation of production rates from product to products. In Section 6.3, we present its equivalency with the state to state variation of apportionment via local indices as well as the equivalency of global indices.

6.3 Equitably Efficient Frontiers

6.3.1 Global Indices

The global index $|x_{ik} - kr_i|$ of *SDJIT* sequencing problem is studied in [20] with oneness property. A sequence z is said to have oneness property if and only if

$-1 \le z_{ik} \le 1, \forall i, \forall k$. Considering absolute or squared deviation global indices, we can say that any sequence is feasible if the discrepancy of x_{ik} and kr_i lies between 0 and 1. Thus, main idea is to seek for the smaller bound nearer to 0. If $x_{ik} = kr_i$, then it is done, however it is very rare in practice. In view of *MBD1* defined in Section 5.3.2 for apportionment, we propose a stronger bound β^* for *SDJIT* sequencing problem based on arithmetic mean. Taking the intervals $[a, a+1]$ and $[0, 1]$ together, the key idea is as follows:

Step 1. Set $a = 0$ and $a + 1 = 0 + 1 = 1$.

Step 2. Make partition of the interval $[0, 1]$ by computing the harmonic mean $= 0$, geometric mean $= 0$, and arithmetic mean $= \frac{1}{2}$ of the two ends 0 and 1.

Step 3. Calculate the arithmetic mean of $0, 0, 0, \frac{1}{2}, 1$. That is, $\beta^* = \frac{1}{5}\left(0 + 0 + 0 + \frac{1}{2} + 1\right) = \frac{3}{10} = 0.3$.

To this end, we claim that $\beta^* = 0.3$ is the efficient bound yielding minimum deviation lying near to ideal. The absolute and squared global indices of both the *JIT* sequencing and apportionment problems are equivalent and have same set of optimal solutions with the proposed bound β^*. Moreover, we have

Proposition 6.1 The absolute *SDJIT* sequencing objective $\sum_{i=1}^{n}\sum_{k=1}^{D}|x_{ik} - kr_i|$ and absolute apportionment objective $\sum_{i=1}^{s}|a_{ih} - q_{ih}|$ are equivalent and have same set of optimal solutions with the upper bound β^*.

Proposition 6.2 The squared *SDJIT* sequencing objective $\sum_{i=1}^{n}\sum_{k=1}^{D}(x_{ik} - kr_i)^2$ and squared apportionment objective $\sum_{i=1}^{s}(a_{ih} - q_{ih})^2$ are equivalent and have same set of optimal solutions with the upper bound β^*.

Corollary 6.1 The complexities of *SDJIT* sequencing and apportionment problems are equivalent, that is, $O(nD)$ and $O(sh)$ respectively.

Clearly, the other divisor *MBD2*, defined in Section 5.3.2 for apportionment, generates the bigger bound $\frac{1}{4}$. Being parametric, we can say this bound generates cyclic *JIT* sequences, though it is not near to ideal.

6.3.2 Local Indices

The production rates of the products i and j are measured as $\frac{x_{ik}}{r_i}$ and $\frac{x_{jk}}{r_j}$ respectively. If $\frac{x_{ik}}{r_i} = \frac{x_{jk}}{r_j}$ for all i, j, then the perfection will be gained. However, it is very rare in practical. To this point, Balinski and Shahidi [6] proposed the following objective to be minimized over the vector x:

$$\min_{x} \max_{i,j} \left| \frac{x_{ik}}{r_i} - \frac{x_{jk}}{r_j} \right| \qquad (6.1)$$

For apportionment problem, we propose a similar objective for the fairness measure between two states i and j as the local measure of inequality

$$\min_{a} \max_{i,j} \left| \frac{a_{ih}}{p_i} - \frac{a_{jh}}{p_j} \right| \qquad (6.2)$$

We claim that the computational complexities of these two objectives are of same type, depending on the number of products and number of states. But the complexities given in corollary 6.1 do not work here.

In view of relative difference T defined in (5.3), we define the relative difference T^* for product to product rate variation problem and propose the equitably efficient *(EE)* solution:

$$T^* = \left| \frac{x_{ik}}{r_i} - \frac{x_{jk}}{r_j} \right| \Big/ \min\left(\frac{x_{ik}}{r_i}, \frac{x_{jk}}{r_j} \right) \qquad (6.3)$$

Theorem 6.3 If the state to state variation problem of the apportionment is stable with T, then product to product rate variation problem of the *JIT* sequencing is stable with T^*.

Proof: As discussed above, state to state variation problem *(i.e., MEP)* is mathematically neutral, near to ideal, monotone and consistent [7]. So it suffices to show that state to state variation and product to product variation problems are mathematically equivalent. Products i, j in *JIT* sequencing correspond with the states i, j among s states. The divisors r_i and r_j correspond with p_i and p_j in apportionment. The product rates of i^{th} and j^{th} products in period k are equivalent to the shares of representatives of i^{th} and j^{th} states for house size h. Thus in view of *MEP*, we conclude that the product to product rate variation problem with the relative difference T^* is near to ideal and mathematically neutral.□

Consequently, we propose the following theorem and corollary:

Theorem 6.4 The state to state variation problem with T has optimal solution if and only if the product to product rate variation problem with T^* has optimal solution.

Corollary 6.2 Balancing inter-state apportionments is equivalent to balancing sub-products of a product.

7. CONCLUSION

The *JIT* sequencing problem in mixed-model assembly process attempts to minimize the certain penalties concerned with the holding of inventories and the occurrences of the shortages. Perfect matching approach is workable for *MDJIT* whereas assignment approach is for *SDJIT* to obtain optimal sequences. Posing the literature of apportionment methods, we have proposed mean-based divisor methods claiming that they are better than existing ones. Linking up the *JIT* sequencing problem with the apportionment problem, we have proposed a stronger bound for *SDJIT* sequencing problem that falls near to ideal. The complexities of global indices of both the problems are same depending on products and states. Inequality measures in *JIT* sequencing and apportionment problems have been considered both in global and local senses, and hence compared to each other. State to state variation problem and product to product rate variation problem have been shown equivalent with respect to the relative differences and the *EE* solution is proposed for the combined approach. The further depth and the linkages of the problems are the directions of our study.

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Generalized Fixed Point Theorem in Fuzzy Metric Space

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Abstract: The main objective of the present paper is to establish a common fixed point theorem for pair of self fuzzy mappings in a fuzzy metric space which generalizes and improves various known results.

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Key Words: Fuzzy metric space, Compatible mappings, R-weakly commuting mappings, Reciprocal continuity.

Key Words: Fourier transform, Hilbert Schmidt norm, kernel function.

1. INTRODUCTION

The concept of fuzzy sets was initiated by Zadeh [14] in 1965. After that, a lot of works have been done regarding fuzzy sets and applications. Deng[3], Erceg [4], Kalva and Seikkala [7] introduced the concepts of fuzzy metric spaces in different ways. In 1975, Kramosil and Michalek [8] introduced the fuzzy metric space by generalizing the concept of probabilistic metric space to fuzzy situation. Grabiec [6] proved the contraction principle in the setting of the fuzzy metric space introduced by Kramosil and Michalek [8]. Grabiec's result was further

generalized by Subrahmanyam [12] for a pair of commuting mappings. Since then, a substantial literature has been developed on this topic. Also, George and Veermani [5] modified the notion of fuzzy metric spaces with the help of continuous t-norm, by generalizing the concept of probabilistic metric space to fuzzy situation. In 1999, Vasuki [13] introduced the concept of R-weak commutativity of mappings in fuzzy metric space and Pan[9] introduced the notion of reciprocal continuity of mappings in metric space and proved some common fixed point theorems. Balasubramaniam et. al. [1] proved a fixed point theorem, which generalizes a result of Pant [9] for fuzzy mappings in fuzzy metric space.

Pant and Jha [10] proved a fixed point theorem that gives an analogue of the results by Balasubramaniam et. al. [1] by obtaining a connection between the continuity and reciprocal continuity for four mappings in fuzzy metric space. Recently, Chugh and Kumar [2] proved a common fixed point theorem for four mappings in fuzzy metric space generalizing the result of Vasuki [13]. The present paper is aimed to prove a fixed point theorem assuming the reciprocal continuity of fuzzy mappings in fuzzy metric space that generalizes the results of Chugh and Kuamr [2], Vasuki [13] and improves various other similar results of fixed points. We also give an example to illustrate our main theorem.

We have used the following notions:

Definition 1.1 ([13]) Let X be any set. A fuzzy set A in X is a function with domain X and values in [0, 1].

Definition 1.2 ([11]) A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous t–norms if, ([0, 1], *) is an abelian topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for all a, b, c, d in [0, 1].

Examples of t-norms are $a * b = ab$, $a * b = \min \{a, b\}$.

Definition 1.3 ([8]) The triplet (X, M, *) is called a fuzzy metric space (shortly, a FM-space) if, X is an arbitrary set, * is a continuous t–norm and M is a fuzzy et on $X^2 \times [0, \infty)$ satisfying he following conditions: for all x, y, z in X, s, t > 0,

(i) $M(x, y, 0) = 0$, $M(x, y, t) > 0$;

(ii) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,

(iii) $M(x, y, t) = M(y, x, t)$,

(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(v) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous for all $x, y \in X$ and $s, t > 0$,

(vi) $\lim_{t\to\infty} M(x, y, t) = 1$, for all $x, y \in X$.

Definition 1.4 ([6]) A sequence $\{x_n\}$ is a fuzzy metric space (X, M, *) is called Cauchy sequence if, $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ for every $t > 0$ and for each $p > 0$. A fuzzy metric space (X, M, *) is complete if, every Cauchy sequence in X converges in X.

Definition 1.5 ([6]) A sequence $\{x_n\}$ in a fuzzy metric space (X, M, *) is said to be convergent to x in X if, $\lim_{n\to\infty} M(x_n, x, t) = 1$ for each $t > 0$.

Definition 1.6 ([9]) Two self mappings A and S of a metric space (X, d) are called compatible if, $\lim_{n\to\infty} d(Asx_n, SAx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some t in X.

Definition 1.7 ([1]) Two self mappings A and S of a fuzzy metric space (X, M, *) are called compatible if, $\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = p$ for some p in X.

Definition 1.8 ([9]) Two self mappings A and S of a metric space (X, d) are called R-weekly commuting at a point x in X if, $d(ASx, SAx) \leq Rd(Ax, Sx)$, for $R > 0$.

Definition 1.9 ([1]) Two self mappings A and S of a fuzzy metric space (X, M, *) are called weekly commutating if, $M(ASx, SAx, t) \geq A(Ax, Sx, t)$ for each $x \in X$ and $t > 0$.

Definition 1.10 ([1]) Two self mappings A and S of a fuzzy metric space (X, M, *) are called R-weekly commuting provided there exists some real number R such that $M(ASx, SAx, t) \geq M(Ax, Sx, t/R)$ form some $x \in X$ and $t > 0$.

Definition 1.11 ([1]) Two self mappings A and S of a fuzzy metric space (X, M, *) are called pointwise R-weakly commuting on X if, given x in (X, M, *), there exists $R > 0$ such that $M(ASx, SAx, t) \geq M(Ax, Sx, t/R)$.

It is noted that R-weakly commutativity in fuzzy metric space implies weak commutativity only when $R \leq 1$ (Chugh and Kumar [2]).

Definition 1.12 ([1]) Two self mappings A and S of a fuzzy metric space (X, M, *) are said to be reciprocally continuous if, $\lim_{n\to\infty} AS_n = Ap$ and $\lim_{n\to\infty} SAx_n = Sp$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Sx_n = p$ and $\lim_{n\to\infty} Ax_n = p$ for some p in X.

Note that in the metric setting if A and S are both continuous then they are obviously reciprocal continuous. But the converse need not be true (Pant [9]).

2. MAIN RESULTS

Theorem 2.1 Let (A, S) and (B, T) be pointwise R-weakly commuting pairs of self mappings of complete fuzzy metric space (X, M, *) such that

(i) $AX \subseteq TX$, $BX \subseteq SX$,

(ii) $M(Ax, By, t) \geq r(M(Sx, Ty, t))$,

for all $x, y \in X$, where $r : [0, 1] \to [0, 1]$ is continuous function such that $r(t) > t$ for each $0 < t < 1$. If the pair (A, S) or (B, T) is compatible pair of reciprocally continuous mappings, then A, B, S and T have a unique common fixed point.

Proof. Let x_0 be any point in X. We define sequences $\{x_n\}$ and $\{y_n\}$ in X given by the rule

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \text{ and } y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}, \text{ for } n = 0, 1, 2, 3, ... \quad (1)$$

This can be done by virtue of (i). Then, using (ii), we get

$$\begin{aligned} M(y_{2n}, y_{2n+1}, t) &= M(Ax_{2n}, Bx_{2n+1}, t) \\ &\geq r(M(Sx_{2n}, Tx_{2n+1}, t) = r(M(y_{2n-1}, y_{2n}, t)) \\ &> M(y_{2n-1}, y_{2n}, t), \end{aligned}$$

since $r(t) > t$ for $0 < t < 1$. Similarly, we have $M(y_{2n+1}, y_{2n+2}, t) > M(y_{2n}, y_{2n+1}, t)$. So, $\{M(y_{2n}, y_{2n+1}, t)\}$, for $n \geq 0$, is an increasing sequence of positive real numbers in [0, 1] and therefore, tends to a limit $\alpha \leq 1$. We claim that $\alpha = 1$. For this, if $\alpha < 1$, then on letting $n \to \infty$ in relation (2), we get $\alpha \geq r(\alpha) > \alpha$, a contradiction. Hence, we get $\alpha = 1$. Thus, for every $n \in N$,

$$M(y_n, y_{n+1}, t) > M(y_{n-1}, y_n, t) \text{ and } M(y_n, y_{n+1}, t) \to 1 \text{ as } n \to \infty, \text{ for } t > 0. \quad (3)$$

Now, for any positive integer p, we get

$$\begin{aligned} M(y_n, y_{n+p}, t) &\geq M(y_n, y_{n+1}, t/p) * M(y_{n+1}, y_{n+2}, t/p) * ... * M(y_{n+p-1}, y_{n+p}, t/p) \\ &\geq M(y_n, y_{n+1}, t/p) * M(y_n, y_{n+1}, t/p) * ... * M(y_n, y_{n+1}, t/p) \\ &\geq 1 * 1 * ... * 1, \text{ using (3)}. \end{aligned}$$

This implies that $M(y_n, y_{n+p}, t) \to 1$ as $n \to \infty$. Therefore, $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a point z in X such that $y_n \to z$ as $n \to \infty$. Moreover, we have

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \to z \text{ and } y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \to z.$$

Suppose A and B are compatible and reciprocally continuous mappings, then by definition, we have $ASx_{2n} \to Az$ and $SAx_{2n} \to Sz$. Also, compatibility of A and S yields that $\lim_{n\to\infty} M(ASx_{2n}, SAx_{2n}, t) = 1$, that is, $M(Az, Sz, t) = 1$. Hence, we

have $Az = Sz$. Since $AX \subset TX$, there exists a point w in X such that $Az = Tw$. So, using (ii), we get $M(Az, Bw, t) \geq r(M(Sz, Tw, t)) = r(M(Az, Tw, t)) = r(1) = 1$, since $r(t) = 1$ for $t = 1$. This implies that $Az = Bw$.

Thus, we have $Sz = Az = Tw = Bw$.

Again, the pointwise R-weakly commutativity of A and S implies that there exists $R > 0$ such that $M(ASx, SAz, t) \geq M(Az, Sz, t/R) = 1$. That is, $ASz = SAz$ and $AAz = ASz = SSz$. Similarly, the pointwise R-weakly commutativity of B and T implies that $BBW = BTW = TBw = TTw$. So that, using (ii), we have

$$M(Az, AAz, t) = M(Bw, AAz, t) \geq r\,(M(SAz, Tw, t)) > M(AAZ, Az, t).$$

That is, $M(Az, AAz, t) = 1$. Hence, we have $Az = AAz$ and $Az = AAz = SAz$. This implies that Az is a common fixed point of A and S. Similarly, by using (ii), we can show that $Bw(= Az)$ is a common fixed point of B and T. The uniqueness of a common fixed point of the mappings A, B, S and T be easily verified by using (ii). In fact, if u' be another fixed point for mappings A, B, S and T, then, we have $M(u, u', t) = M(Au, Bu', t) \geq r(M(Su, Tu', t)) = r(M(u, u', t)) > M(u, u', t)$, for $r(t) > t$ and hence, we get $u = u'$.

This completely establishes the theorem.

We now give an example to illustrate the above Theorem 2.1.

Example: Let $X = [2, 20]$ and M be the usual fuzzy metric on $(X, M, *)$. Define mappings A, B, S and $T : X \to X$ by

$A2 = 2$, $\quad Ax = 3$ if, $x > 2$;

$Bx = 2$ if, $\quad x = 2$ or > 5, $\quad Bx = 6$ if, $\quad 2 < x \leq 5$;

$S2 = 2$, $\quad Sx = 6$ if, $x > 2$;

$T2 = 2$, $\quad Tx = 12$ if, $2 < x \leq 5$, $\quad Tx = x - 5$ if, $x > 5$.

Also, we define $M(Ax, By, t) = \frac{t}{[t + d(x, y)]}$, for all x, y in X and for all $t > 0$. Then, A, B, S and T satisfy all the conditions of the above theorem with $r : [0, 1] \to [0, 1]$ by $r(t) = t^{1/2}$ for $0 < t < 1$ and $r(t) = 1$ for $t = 1$. So that, we have $r(t) > t$ for $0 < t < 1$. Also, $M(Ax, By, t) \geq r(M(Sx, Ty, t)$ for all x, y in X. Moreover, the pair (A, S) and (B, T) are R-weakly commuting and reciprocally continuous mappings on X. Thus, all the conditions of the above Theorem 2.1 are satisfied and $x = 2$ is a common fixed point of A, B, S and T.

Remarks: As Pant [9] has shown that the reciprocally continuous maps need not be continuous. so this result generalizes the results of Chugh and Kumar [2] and

Vasuki [13]. It also improves the results of Balasubramaniam et. al [1], Pant and Jha [10] and other similar results for fixed points.

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Operation Approaches on Fuzzy Pre-Open Sets

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Abstract: In this paper, the concepts of an operation γ on a family of fuzzy pre-open sets in a fuzzy topological spaces (X, T) is introduced. Using the operation γ on FPO (X) the concepts of fuzzy pre-γ-open sets, fuzzy pre-γ-border, fuzzy pre-γ-frontier, fuzzy pre-(γ, β)-continuous mappings, fuzzy pre-γ-normal spaces and fuzzy pre-γ-compact spaces are introduced. Some interesting properties and characterizations of them are investigated. Further, fuzzy pre-γ-R_o and fuzzy pre-γ-T_i (i = 0, 1/2, 1, 2) spaces are introduced and interrelations among the spaces are discussed with relevant examples.

Key Words

Fuzzy pre-γ-open set, fuzzy pre-γ-border, fuzzy pre-γ-frontier, fuzzy pre-(γ, β)-continuous mapping, fuzzy pre-γ-normal space, fuzzy pre-γ-compact space, fuzzy pre-γ-R_o space, fuzzy pre- γ-T_i (i = 0, 1/2 , 1, 2) space.

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1. INTRODUCTION AND PRELIMINARIES

The concept of fuzzy sets has invaded almost all branches of mathematics since the introduction of the concept by Zadeh [11]. Fuzzy sets have applications in many fields such as information [7] and control [8]. The theory of fuzzy topological spaces was introduced and developed by Chang [3]. The concept of

fuzzy pre-open sets and fuzzy pre–closed sets were introduced by Singal and Prakash [6]. The concept of fuzzy pre-continuity was introduced by Bin Shahna [1] and was studied by Uma, Roja and Balasubramanian [10]. By using the concepts of semi-γ-open sets introduced by Sai Sundara Krishnan, Ganster and Balachandran [4] and that of g-border and g-frontier introduced by Caldas, Jafari and Noiri [2], the concepts of fuzzy pre-γ-open set, fuzzy pre-γ-border, pre-γ-frontier, fuzzy pre-(γ, β)-continuous mappings, fuzzy pre-γ-normal spaces, fuzzy pre-γ-compact spaces, fuzzy pre-γ-T_i ($i = 0, 1, 2$) spaces and fuzzy pre-γ-R_o space are introduced and interrelations among the spaces are discussed with relevant examples.

Definition 1.1 [6]

Let (X, T) be a fuzzy topological space. A fuzzy set λ in (X, T) is said to be fuzzy pre-open if $\lambda \leq \text{int cl}(\lambda)$.

The complement of a fuzzy pre-open set is fuzzy pre-closed.

Definition 1.2 [4]

Let (X, T) be a fuzzy topological space. An operation γ on the topology T is a mapping from T into power set P(X) of X such that $V \subseteq V^\gamma$ for each $V \in T$, where V^γ denotes the value of γ at V. It is denoted by $\gamma : T \to P(X)$.

Definition 1.3 [4]

A subset A of a topological space is called a γ-open set of (X, T) if for each $x \in A$ there exists an open set U such that $x \in U$ and $U^\gamma \subseteq A$. The complement of a γ-open set is said to be γ-closed.

Notation 1.1 [4]

SO(X) denotes the family of all semi-open sets of (X, T).

Definition 1.4 [4]

Let (X, T) be a topological space. An operation γ on the SO(X) is a mapping from SO(X) into a power set P(X) of X such that $V \subseteq V^\gamma$ for each $V \in SO(X)$ and V^γ denotes the value of γ at V. It is denoted by $\gamma : SO(X) \to P(X)$.

Definition 1.5 [4]

Let (X, T) be a topological space and γ be an operation on SO(X). Then a subset A of X is said to be a semi-γ-open set if for each $x \in A$, there exists a semi-open set U such that $x \in U$ and $U^\gamma \subseteq A$. Also $SO(X)_\gamma$ denotes the family of semi-γ-open sets in X.

Definition 1.6 [2]

Let (X, T) be a topological space. For a subset A of (X, T), $b_g(A) = A - int_g(A)$ is said to be the g-border of A where $int_g(A)$ is the set of all g-interior points of A.

Definition 1.7 [2]

Let (X, T) be a topological space. For a subset A of (X, T), $Fr_g(A) = cl_g(A) - int_g(A)$ is said to be the g-frontier of A.

Definition 1.8 [9]

A topological space (X, T) is said to be a fuzzy pre-$T_{1/2}$ space if every gfpre-closed set in (X, T) is fuzzy closed in (X, T).

Definition 1.9 [5]

A fuzzy set λ is quasi-coincident with a fuzzy set μ, denoted by $\lambda\ q\ \mu$, if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$. Otherwise $\lambda\ \not{q}\ \mu$.

2. FUZZY PRE-γ-OPEN SETS

Definition 2.1

Let (X, T) be a fuzzy topological space. Let $\gamma : I^X \to T$ be an operation such that $\lambda^\gamma = \wedge\ \mu$ where $\lambda \le \mu$, for each fuzzy open set μ in (X, T), $\lambda \in I^X$ and λ^γ denotes the value of γ at λ. That is, $\lambda^\gamma = \gamma(\lambda)$.

Definition 2.2

Let (X, T) be a fuzzy topological space. Let $\gamma : I^X \to T$ be an operation. A fuzzy set δ is said to be fuzzy-γ-open if for a fuzzy set α with $\alpha \le \delta$, there exists a fuzzy open set λ such that $\alpha \le \lambda$ and $\lambda^\gamma \le \delta$.

The complement of a fuzzy γ-open-set is fuzzy-γ-closed.

Definition 2.3

Let (X, T) be a fuzzy topological space. Let $\gamma : I^X \to T$ be an operation. For any fuzzy set λ, fuzzy-γ-interior of λ (briefly, γ-int (λ)) is defined as $\gamma\text{-int}(\lambda) = \vee\ \{\mu : \mu \le \lambda$ and μ is fuzzy-γ-open$\}$.

Definition 2.4

Let (X, T) be a fuzzy topological space. Let $\gamma : I^X \to T$ be an operation. For any fuzzy set λ, fuzzy-γ-closure of λ (briefly, γ-cl (λ)) is defined as $\gamma\text{-cl}(\lambda) = \wedge\ \{\mu : \mu \ge \lambda$ and μ is fuzzy-γ-closed$\}$.

Remark 2.1

γ-int $(1 - \lambda) = 1 - (\gamma\text{-cl}(\lambda))$.

Notation 2.1

FPO (X) denotes the family of all fuzzy pre-open sets of (X, T).

Definition 2.5

Let (X, T) be a fuzzy topological space. Let γ : FPO (X) $\rightarrow$ T be an operation such that $\lambda^\gamma = \wedge\, \mu$, where $\lambda \leq \mu$, for each fuzzy open set μ in (X, T) and $\lambda \in$ FPO (X).

Definition 2.6

Let (X, T) be a fuzzy topological space. Let γ be an operation on FPO (X). A fuzzy set δ is called fuzzy pre-γ-open if for a fuzzy set α with $\alpha \leq \delta$, there exists a fuzzy pre-open set λ such that $\alpha \leq \lambda$ and $\lambda^\gamma \leq \delta$.

The complement of a fuzzy pre-γ-open set is fuzzy pre-γ-closed.

Definition 2.7

Let (X, T) be a fuzzy topological space. Let γ be an operation on FPO (X). The fuzzy pre-γ-interior of δ (briefly, γ-fp int (δ)) is defined by γ-fp int $(\delta) = \vee \{ \mu : \mu \leq \delta$ and μ is fuzzy pre-γ-open $\}$.

Definition 2.8

Let (X, T) be a fuzzy topological space. Let γ be an operation on FPO (X). The fuzzy pre-γ-closure of δ (briefly, γ-fp cl (δ)) is defined by γ-fp cl $(\delta) = \wedge \{ \mu : \mu \geq \delta$ and μ is fuzzy pre-γ-closed $\}$.

Remark 2.2

γ-fp int$(1 - \delta) = 1 - (\gamma\text{-fp cl}(\delta))$.

Remark 2.3

Fuzzy pre-open set and fuzzy pre-γ-open set are independent notions.

Example 2.1

Let X = { a, b }. Define T = $\{ 0, 1, \lambda_1, \lambda_2 \}$ where $\lambda_1, \lambda_2 : X \rightarrow [0, 1]$ are defined as λ_1 (a) = 0.3, λ_1 (b) = 0.2, λ_2 (a) = 0.45, λ_2 (b) = 0.4. Let γ : FPO (X) $\rightarrow$ T be an operation. Let $\mu, \delta, \eta : X \rightarrow [0, 1]$ be defined as μ (a) = 0.4, μ (b) = 0.3, δ (a) = 0.45, δ (b) = 0.3, η (a) = 0.55, η (b) = 0.65. Now int cl $(\mu) \geq \mu$. **Hence μ is fuzzy pre-open but not fuzzy pre-γ-open.** Now, for a fuzzy set α with $\alpha \leq \eta$, then $\alpha \leq \mu$ and $\mu^\gamma \leq \eta$. **Hence η is fuzzy pre-γ-open but not fuzzy pre-open.**

Proposition 2.1

Let (X, T) be a fuzzy topological space. Let λ and μ be any two fuzzy pre-γ-open sets in (X, T). Then $\lambda \vee \mu$ (resp. $\lambda \wedge \mu$) is also a fuzzy pre-γ-open set in (X, T).

Proposition 2.2

Let (X, T) be a fuzzy topological space. For any two fuzzy sets λ, μ, the following statements hold :

a. If λ is fuzzy-γ-open then λ is fuzzy pre-γ-open.

b. γ-int (λ) is fuzzy pre-γ-open.

c. γ-cl (λ) is fuzzy pre-γ-closed.

d. is fuzzy pre-γ-open iff $\lambda = \gamma$-fp int (λ).

e. is fuzzy pre-γ-closed iff $\lambda = \gamma$-fp cl (λ).

f. γ-int (λ) $\leq \gamma$-fp int (λ) $\leq \lambda \leq \gamma$-fp cl (λ) $\leq$ γ-cl (λ).

g. γ-cl (γ-fp cl (λ)) = γ-fp cl (λ).

h. γ-cl (γ-fp cl (λ)) = γ-fp cl (γ-cl (λ)) = γ-cl (λ).

i. (γ-fp int (λ)) $\wedge$ (γ-fp int (μ)) $\geq \gamma$-fp int ($\lambda \wedge \mu$).

j. (γ-fp int (λ)) $\vee$ (γ-fp int (μ)) $\leq \gamma$-fp int ($\lambda \vee \mu$).

Definition 2.9

Let (X, T) be a fuzzy topological space and let $\gamma : I^X \rightarrow T$ be an operation. For any fuzzy set λ, fuzzy-γ-border of λ (briefly, γ-fb (λ)) is defined as γ-fb (λ) = λ – (γ-int (λ)).

Definition 2.10

Let (X, T) be a fuzzy topological space and let γ be an operation on FPO (X) for any fuzzy set λ, fuzzy pre-γ-border of λ (briefly, γ-fpb (λ)) is defined as γ-fpb (λ) = λ – (γ-fp int (λ)).

Definition 2.11

Let (X, T) be a fuzzy topological space and let $\gamma : I^X \rightarrow T$ be an operation. For any fuzzy set λ, fuzzy-γ-frontier of λ (briefly, γ-f Fr (λ)) is defined as γ-f Fr (λ) = (γ-cl (λ)) – (γ-int (λ)).

Definition 2.12

Let (X, T) be a fuzzy topological space and let γ be an operation on FPO (X) . For any fuzzy set λ, fuzzy pre-γ-frontier of λ (briefly, γ-fp Fr (λ)) is defined as γ-fp Fr (λ) = (γ-fp cl (λ)) – (γ-fp int (λ)).

Proposition 2.3

Let (X, T) be a fuzzy topological space. For any two fuzzy sets λ, μ the following statements hold :

a. γ-fpb $(\lambda) \leq \gamma$-fp cl $(1 - \lambda)$.

b. γ-fpb $(\lambda \vee \mu) \leq (\gamma$-fpb $(\lambda)) \vee (\gamma$-fpb $(\mu))$.

c. γ-fpb $(\lambda \wedge \mu) \geq (\gamma$-fpb $(\lambda)) \wedge (\gamma$-fpb $(\mu))$.

d. $(\gamma$-int $(\lambda)) \vee (\gamma$-fb $(\lambda)) \geq \gamma$-int (λ).

e. $(\gamma$-int $(\lambda)) \wedge (\gamma$-fb $(\lambda)) \leq \gamma$-int (λ).

f. γ-fp Fr $(\lambda) = \gamma$-fp Fr $(1 - \lambda)$.

g. γ-fp Fr $(\gamma$-fp int $(\lambda)) \leq \gamma$-fp Fr (λ).

h. γ-fp Fr $(\gamma$-fp cl $(\lambda)) \leq \gamma$-fp Fr (λ).

i. $\lambda - (\gamma$-fp Fr $(\lambda)) \leq \gamma$-fp int (λ).

j. γ-fp Fr $(\lambda \vee \mu) \leq (\gamma$-fp Fr $(\lambda)) \vee (\gamma$-fp Fr $(\mu))$.

k. γ-fp Fr $(\lambda \wedge \mu) \geq (\gamma$-fp Fr $(\lambda)) \wedge (\gamma$-fp Fr $(\mu))$.

3. FUZZY PRE-γ-T_1 SPACES

Definition 3.1

A fuzzy topological space (X, T) is called

(a) a fuzzy pre-γ-T_o space iff for any two fuzzy sets λ, μ with $\lambda \bar{q} \mu$, there exists a fuzzy pre-γ-open set δ such that $\lambda \leq \delta, \mu \bar{q} \delta$ or $\mu \leq \delta, \lambda \bar{q} \delta$.

(b) a fuzzy pre-γ-T_1 space iff for any two fuzzy sets λ, μ with $\lambda \bar{q} \mu$, there exist fuzzy pre-γ- open sets δ, η such that either $\lambda \leq \delta$, $\mu \bar{q} \delta$ or $\mu \leq \eta$, $\lambda \bar{q} \eta$.

(c) a fuzzy pre-γ-T_2 space iff for any two fuzzy sets λ, μ with $\lambda \bar{q} \mu$, there exist fuzzy pre-γ-open sets δ, η such that $\lambda \leq \delta$, $\mu \leq \eta$ and $\delta \bar{q} \eta$.

(d) a fuzzy pre-γ-R_o space iff for any two fuzzy sets λ, μ, $\lambda \bar{q} (\gamma$-fp cl $(\mu))$ implies that $\mu \bar{q} (\gamma$-fp cl $(\lambda))$.

Definition 3.2

Let (X, T) be a fuzzy topological space and let γ be an operation on FPO(X). A fuzzy set λ is called fuzzy pre-γ-g closed if γ-fp cl $(\lambda) \leq \mu$ whenever $\lambda \leq \mu$ and μ is fuzzy pre-γ-open.

The complement of a fuzzy pre-γ-gclosed set is fuzzy pre-γ-g open.

Definition 3.3

A fuzzy topological space (X, T) is called fuzzy pre-γ-$T_{1/2}$ space if every fuzzy pre-γ-g closed set is fuzzy pre-γ-closed.

Remark 3.1

From the above definitions we have the following implications.

fuzzy pre-γ-T_2 space $\Rightarrow$ fuzzy pre-γ-T_1 space $\Rightarrow$ fuzzy pre-γ-$T_{1/2}$space $\Rightarrow$ fuzzy pre-γ-T_o space.

The converse statements need not be true, as shown in the following examples.

Example 3.1

Let X = {a, b}. Define T = {0, 1, λ_1, λ_2, λ_3} where $\lambda_1, \lambda_2, \lambda_3 : X \to [0, 1]$ are defined as λ_1 (a) = 0.51, λ_1 (b) = 0.7, λ_2 (a) = 0.57, λ_2 (b) = 0.78, λ_3 (a) = 0.63, λ_3 (b) = 0.83. Let γ : FPO (X) $\to$ T be an operation. Let $\alpha, \mu, \delta, \eta : X \to [0, 1]$ be defined as α (a) = 0.3, α (b) = 0.4, μ (a) = 0.55, μ (b) = 0.75, δ (a) = 0.6, δ (b) = 0.8, η (a) = 0.65, η (b) = 0.85. Clearly μ is a fuzzy pre–open set. Now $\alpha \leq \eta$, $\alpha \leq \delta$ and $\alpha \leq \mu$. Further $\mu^\gamma \leq \delta$ and $\mu^\gamma \leq \eta$. Therefore δ and η are fuzzy pre-γ-open sets. Let $\theta, \lambda : X \to [0, 1]$ be such that θ (a) = 0.3, θ (b) = 0.1, λ (a) = 0.2, λ (b) = 0. Then $\theta \not{q} \lambda$. Further $\theta \leq \delta$, $\lambda \not{q} \delta$ and $\lambda \leq \eta$, $\theta \not{q} \eta$. **Hence (X, T) is a fuzzy pre-γ-T_1 space but not a fuzzy pre-γ-T_2 space.**

Example 3.2

Let X = {a, b}. Define T = { 0, 1, λ_1, λ_2, λ_3 } where $\lambda_1, \lambda_2, \lambda_3 : X \to [0, 1]$ are defined as λ_1 (a) = 0.5, λ_1 (b) = 0.6, λ_2 (a) = 0.7, λ_2 (b) = 0.75, λ_3 (a) = 0.8, λ_3 (b) = 0.9. Let γ : FPO (X) $\to$ T be an operation. The space (X, T) is a fuzzy pre-γ- $T_{1/2}$ space but not a fuzzy pre-γ-T_1 space.

Example 3.3

Let X = { a, b }. Define T = { 0, 1, λ_1, λ_2 } where $\lambda_1, \lambda_2 : X \to [0, 1]$ are defined as λ_1 (a) = 0.3, λ_1 (b) = 0.2, λ_2 (a) = 0.45, λ_2 (b) = 0.4. Let γ : FPO (X) $\to$ T be an operation. Let $\alpha, \mu, \delta : X \to [0, 1]$ be defined as α (a) = 0.2, α (b) = 0.3, μ (a) = 0.4, μ (b) = 0.3, δ (a) 0.55, δ (b) = 0.65. Clearly μ is a fuzzy pre-open set. Now, $\alpha \leq \delta$, $\alpha \leq \mu$ and $\mu^\gamma \leq \delta$. Therefore δ is a fuzzy pre-γ-open set. Let $\theta, \rho : X \to [0, 1]$be defined as θ (a) = 0.3, θ (b) = 0.4, ρ (a) = 0.4, ρ (b) = 0.2. Then $\theta \not{q} \rho$. Now, $\theta \leq \delta$ and $\rho \not{q} \delta$. Hence (X, T) is a fuzzy pre-γ-T_o space. Let $\lambda : X \to [0, 1]$ be defined as λ (a) = 0.5, λ (b) = 0.45. Now, $\lambda \leq \delta$ and γ-

fp cl(λ) $\leq \delta$. Therefore λ is a fuzzy pre-γ-g closed set. But not a fuzzy pre-γ-closed set. Hence (X, T) is not a fuzzy pre-γ-$T_{1/2}$ space.

Proposition 3.1

Let (X, T) be a fuzzy topological space . Then

(a) for all fuzzy pre-γ-open set λ in (X, T), λ q μ iff λ q (γ-fp cl (μ)), where μ is any fuzzy set in (X, T).

(b) δ q (γ-fp cl (λ)) iff λ q μ, for all fuzzy pre-γ-open set μ in (X, T), with $\delta \leq \mu$.

Proof:

(a) Let λ be a fuzzy pre-γ-open set such that λ q μ. Then since $\mu \leq \gamma$-fp cl (μ), λ q (γ-fp cl (μ)). Conversely let λ be a fuzzy pre-γ-open set in (X, T) such that $\lambda \not{q}\, \mu$. Then $\mu \leq 1 - \lambda$ and so γ-fp cl (μ) $\leq \gamma$-fp cl $(1 - \lambda) = 1 - \lambda$. Thus $\lambda \not{q}$ (γ-fp cl (μ)).

(b) Let δ q (γ-fp cl (λ)) and let μ be a fuzzy pre-γ-open set in (X, T) such that $\delta \leq \mu$. Then μ q (γ-fp cl (λ)). By (a), μ q λ for all fuzzy pre-γ-open set μ with $\delta \leq \mu$. Conversely suppose that $\delta \not{q}$ (γ-fp cl (λ)). Then $\delta \leq 1 -$ (γ-fp cl (λ)). Let $\mu = 1 -$ (γ-fp cl (λ)). Then μ is a fuzzy pre-γ-open set with $\delta \leq \mu$. Since $\lambda \leq \gamma$-fp cl (λ), $\mu = 1 -$ (γ-fp cl (λ)) $\leq 1 - \lambda$. Therefore $\lambda \not{q}\, \mu$.

Proposition 3.2

Let (X, T) be a fuzzy topological space. For any two fuzzy sets δ, ρ in (X, T), the following statements are equivalent :

(a) (X, T) is a fuzzy pre-γ-R_o space.

(b) If $\delta \not{q}\, \lambda = \gamma$-fp cl ($\lambda$), where λ is any fuzzy set in (X, T), then there exists a fuzzy pre-γ-open set μ in (X, T), such that $\delta \not{q}\, \mu$ and $\lambda \leq \mu$.

(c) If $\delta \not{q}\, \lambda = \gamma$-fp cl ($\lambda$) then ($\gamma$-fp cl ($\delta$)) $\not{q}\, \lambda = \gamma$-fp cl ($\lambda$), where λ is any fuzzy set in (X, T).

(d) If $\delta \not{q}$ (γ-fp cl (ρ)) then (γ-fp cl (δ)) $\not{q}$ (γ-fp cl (ρ)).

Proof:

(a) $\Rightarrow$ (b) Let $\delta \not{q}\, \lambda = \gamma$-fp cl ($\lambda$). Since γ-fp cl (ρ) $\leq \gamma$-fp cl (λ), for each $\rho \leq \lambda$, $\delta \not{q}$ (γ-fp cl (ρ)). Then by (a), $\rho \not{q}$ (γ-fp cl (δ)). Then by (b) of Proposition 3.1, there exists a fuzzy pre-γ-open set η in (X, T), such that $\delta \not{q}\, \eta$ and $\rho \leq \eta$. Let $\mu = \vee \{\eta : \delta \not{q}\, \eta \}$. Then $\delta \not{q}\, \mu$ and $\lambda \leq \mu$, where μ is fuzzy pre-γ-open in (X, T).

(b)⇒ (c) Let $\delta \not{q}\, \lambda = \gamma$-fp cl ($\lambda$). Then by (b), there exists a fuzzy pre-γ-open set μ in (X, T), such that $\delta \not{q}\, \mu$ and $\lambda \leq \mu$. Since $\delta \not{q}\, \mu$, $\delta \leq 1 - \mu$. Therefore γ-fp cl (δ) $\leq \gamma$-fp cl ($1 - \mu$) $= 1 - \mu \leq 1 - \lambda$.

Hence (γ-fp cl (δ)) $\not{q}\, \lambda = \gamma$-fp cl ($\lambda$).

(c)⇒ (d) Let $\delta \not{q}$ (γ-fp cl (ρ)). since γ-fpcl (γ-fp cl (ρ)) $= \gamma$-fp cl (ρ), by (c), γ-fp cl (δ) $\not{q}$ (γ-fp cl (ρ)).

(d)⇒ (a) Let $\delta \not{q}$ (γ - fp cl (ρ)). Then by (d),

(γ-fp cl (δ)) $\not{q}$ (γ-fp cl (ρ)). Since $\rho \leq \gamma$-fp cl (ρ), $\rho \not{q}$ (γ-fp cl (δ)). Hence (X, T) is a fuzzy pre-γ-R_o space.

4. FUZZY PRE - (γ, β)-CONTINUOUS MAPPINGS

Let (X, T), (Y, S) and (Z, R) be any three fuzzy topological spaces and let γ : FP0 (X) $\rightarrow$ T, β : FP0 (Y) $\rightarrow$ T and η : FP0 (Z) $\rightarrow$ T be operations on FP0 (X), FPO (Y) and FPO (Z) respectively.

Definition 4.1

Let f : (X, T) $\rightarrow$ (Y, S) be a mapping. Then

(a) f is called fuzzy pre-(γ, β)-continuous iff for each fuzzy pre-β-open set μ in (Y, S), f^{-1} (μ) is fuzzy pre-γ-open.

(b) f is called fuzzy pre-(γ, β)-closed iff for each fuzzy pre-γ-closed set λ in (X, T), f (λ) is fuzzy pre-β-closed.

(c) f is called fuzzy pre-(γ, β)-g continuous iff for each fuzzy pre-β-g closed set μ in (Y, S), f^{-1} (μ) is fuzzy pre-γ-g closed.

(d) f is called fuzzy pre-(γ, β)-g closed iff for each fuzzy pre-γ-g closed set λ in (X, T), f (λ) is fuzzy pre-β-g closed.

Proposition 4.1

A mapping f : (X, T) $\rightarrow$ (Y, S) is fuzzy pre-(γ, β)-continuous iff f (γ-fp cl (λ)) $\leq \beta$-fp cl (f (λ)), for each fuzzy set λ in (X, T).

Proposition 4.2

A mapping f : (X , T) $\rightarrow$ (Y, S) is fuzzy pre-(γ, β)-continuous iff γ-fp cl (f^{-1} (λ)) $\leq$ f^{-1} (β-fp cl (λ)), for each fuzzy set λ in (Y, S).

Proposition 4.3

Let $f : (X, T) \rightarrow (Y, S)$ be a fuzzy pre-(γ, β)-continuous and $g : (Y, S) \rightarrow (Z, R)$ be a fuzzy pre-(β, η)-continuous mappings. Then $g \circ f : (X, T) \rightarrow (Z, R)$ is fuzzy pre-(γ, η)-continuous.

Proposition 4.4

Let $f : (X, T) \rightarrow (Y, S)$ be a mapping. Then f is a fuzzy pre-(γ, β)-closed mapping iff β-fp cl $(f(\lambda)) \leq f$ (γ-fp cl (λ)), for each fuzzy set λ in (X, T).

Definition 4.2

Let $f : (X, T) \rightarrow (Y, S)$ be a bijective mapping. If both f and f^{-1} are fuzzy pre-(γ, β)-continuous, then f is called a fuzzy pre-(γ, β)-homeomorphism.

Proposition 4.5

Let $f : (X, T) \rightarrow (Y, S)$ be a bijective mapping. Then the following statements are equivalent:

(a) f is a fuzzy pre-(γ, β)-homeomorphism.

(b) f is a fuzzy pre-(γ, β)-continuous and fuzzy pre-(γ, β)-open mapping.

(c) f is a fuzzy pre-(γ, β)-continuous and fuzzy pre-(γ, β)-closed mapping.

(d) f (γ-fp cl (λ)) = β-fp cl $(f(\lambda))$, for each fuzzy set λ in (X, T)

Proposition 4.6

Let $f : (X, T) \rightarrow (Y, S)$ be a fuzzy pre-(γ, β)-continuous, fuzzy pre-(γ, β)-g continuous and fuzzy pre-(γ, β)-g closed mapping. Then the following statements hold:

(a) If f is injective and (Y, S) is a fuzzy pre-β-$T_{1/2}$ space, then (X, T) is a fuzzy pre-γ-$T_{1/2}$ space.

(b) If f is surjective and (X, T) is a fuzzy pre-γ-$T_{1/2}$ space, then (Y, S) is a fuzzy pre-β-$T_{1/2}$ space.

Proof:

(a) Let λ be a fuzzy pre-γ-g closed set in (X, T). Since f is fuzzy pre-(γ, β)-g closed, $f(\lambda)$ is fuzzy pre-β-g closed. Since (Y, S) is a fuzzy pre-β-$T_{1/2}$ space, $f(\lambda)$ is fuzzy pre-β-closed. Since f is fuzzy pre-(γ, β)-continuous,

$f^{-1}(f(\lambda))$ is fuzzy pre-γ-closed. Hence (X, T) is a fuzzy pre-γ-$T_{1/2}$ space.

(b) Let μ be a fuzzy pre-β-g closed set in (Y, S). Since f is fuzzy pre-(γ, β)-g continuous, $f^{-1}(\mu)$ is a fuzzy pre-γ-g closed set. Since (X, T) is a fuzzy pre-γ-$T_{1/2}$ space, $f^{-1}(\mu)$ is fuzzy pre-γ-closed. Therefore $\mu = f(f^{-1}(\mu))$ is a fuzzy pre-β-closed set. Hence (Y, S) is a fuzzy pre-β-$T_{1/2}$ space.

Proposition 4.7

Let $f : (X, T) \to (Y, S)$ be a fuzzy pre-(γ, β)-continuous injective mapping. If (Y, S) is a fuzzy pre-β-T_2 (resp. fuzzy pre-β-T_1) space then (X, T) is a fuzzy pre-γ-T_2 (resp. fuzzy pre-γ-T_1) space.

Proof:

Let (Y, S) be a fuzzy pre-β-T_2 space. Let λ_1, λ_2 be any two fuzzy sets in (X, T) such that $\lambda_1 \not{q}\, \lambda_2$. Then there exist fuzzy pre-β-open sets λ, μ in (Y, S) with $f(\lambda_1) \leq \lambda$ and $f(\lambda_2) \leq \mu$ such that $\lambda \not{q}\, \mu$. Then $\lambda \leq 1 - \mu$, which implies that $f^{-1}(\lambda) \not{q}\, f^{-1}(\mu)$. Now, $\lambda_1 \leq f^{-1}(\lambda)$ and $\lambda_2 \leq f^{-1}(\mu)$. Since f is fuzzy pre-($\gamma$, β)-continuous, $f^{-1}(\lambda)$ and $f^{-1}(\mu)$ are fuzzy pre-γ-open sets such that $f^{-1}(\lambda) \not{q}\, f^{-1}(\mu)$. Hence (X, T) is a fuzzy pre-γ-T_2 space. Similarly we prove the case of fuzzy pre-β-T_1 space.

5. FUZZY PRE-γ-NORMAL AND FUZZY PRE-γ-COMPACT SPACES.

Definition 5.1 A fuzzy topological space (X, T) is said to be fuzzy pre-γ-normal if for every fuzzy pre-γ-closed set λ and fuzzy pre-γ-open set μ in (X, T) such that $\lambda \leq \mu$, there exists a fuzzy set δ such that $\lambda \leq \gamma\text{-fp int}(\delta) \leq \gamma\text{-fp cl}(\delta) \leq \mu$.

Proposition 5.1 For any fuzzy topological space (X, T) the following statements are equivalent :

(a) (X, T) is fuzzy pre-γ-normal.

(b) For each fuzzy pre-γ-closed set λ and each fuzzy pre-γ-open set μ in (X, T) such that $\lambda \leq \mu$, there exists a fuzzy pre-γ-open set δ in (X, T) such that $\gamma\text{-fp cl}(\lambda) \leq \delta \leq \gamma\text{-fp cl}(\delta) \leq \mu$.

(c) For each fuzzy pre-γ-g closed set λ and each fuzzy pre-γ-open set μ in (X, T) such that $\lambda \leq \mu$, there exists a fuzzy pre-γ-open set δ in (X, T) such that $\gamma\text{-fp cl}(\lambda) \leq \delta \leq \gamma\text{-fp cl}(\delta) \leq \mu$.

Proof (a) $\Rightarrow$ (b) The Proof is trivial.

(b) ⇒ (c) Let λ be any fuzzy pre-γ-g closed set and μ be any fuzzy pre-γ-open set in (X, T) such that $\lambda \leq \mu$. Since λ is fuzzy pre-γ-g closed, γ-fp cl $(\lambda) \leq \mu$. Now, γ-fp cl (λ) is fuzzy pre-γ-closed and μ is fuzzy pre-γ-open in (X, T). By (b), there exits a fuzzy pre-γ-open set δ in (X, T) such that γ-fp cl $(\lambda) \leq \delta \leq \gamma$-fp cl $(\delta) \leq \mu$.

(c) ⇒ (a) The proof is trivial.

Proposition 5.2 Let (X, T) and (Y, S) be any two fuzzy topological spaces. If f : (X, T) → (Y, S) is a fuzzy pre-(γ, β)-homeomorphism and (Y, S) is fuzzy pre-β-normal, then (X, T) is fuzzy pre-γ-normal.

Proposition 5.3 Let f : (X, T) → (Y, S) be a fuzzy pre-(γ, β)- homeomorphism from a fuzzy pre-γ-normal space (X, T) onto a fuzzy topological space (Y, S). Then (Y, S) is fuzzy pre-β-normal.

Proof: Let λ be any fuzzy pre-β-closed set and μ be any fuzzy pre-β-open set in (Y, S) such that $\lambda \leq \mu$. Since f is fuzzy pre-(γ, β)-continuous, $f^{-1}(\lambda)$ is fuzzy pre-γ-closed and $f^{-1}(\mu)$ is fuzzy pre-γ-open in (X, T). Since (X, T) is fuzzy pre-γ-normal, there exists a fuzzy set δ in (X, T) such that

$$f^{-1}(\lambda) \leq \gamma\text{-fp int}(\delta) \leq \gamma\text{- fp cl}(\delta) \leq f^{-1}(\mu).$$

Now, $f(f^{-1}(\lambda)) = \lambda \leq f(\gamma\text{-fp int}(\delta)) \leq f(\gamma\text{-fp cl}(\delta)) \leq f(f^{-1}(\mu)) = \mu$.

That is, $\lambda \leq \beta$-fp int $(f(\delta)) \leq \beta$-fp cl $(f(\delta)) \leq \mu$. Therefore, (Y, S) is fuzzy pre-β-normal.

Definition 5.2 A collection $\{\lambda_i\}_{i \in J}$ of fuzzy pre-γ-open sets (resp. fuzzy pre-β-open sets) of fuzzy topological space (X, T) is called fuzzy pre-γ (resp. fuzzy pre-β)-covering of (X, T) if $1_X \leq \vee \lambda_i$.

A fuzzy topological space (X, T) is called fuzzy pre-γ (resp. fuzzy pre-β)-compact if every fuzzy pre-γ (resp. fuzzy pre-β)-cover of (X, T) has a finite subcover.

A collection $\{\lambda_i\}_{i \in J}$ of fuzzy pre-γ(resp. fuzzy pre-β)-open sets in (X, T) is called fuzzy pre-γ(resp. fuzzy pre-β)-cover of a fuzzy set μ in (X, T) if $\mu \leq \vee \lambda_i$.

Proposition 5.4 Let f : (X, T) → (Y, S) be an fuzzy pre-(γ, β) -continuous surjective function of a fuzzy pre-γ-compact space (X, T) onto a fuzzy topological space (Y, S). Then (Y, S) is fuzzy pre-β-compact.

Proposition 5.5 Let f : (X, T) $\rightarrow$ (Y, S) be a fuzzy pre-(γ, β)-open bijective function and (Y, S) be a fuzzy pre-β-compact space. Then (X, T) is fuzzy pre-γ-compact.

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New subclass of univalent function defined by using generalized Salagean operator

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Abstract:

In this paper, we have introduced and studied a new subclass $TD_{\lambda}(\alpha,\beta,\xi;n)$ of univalent functions defined by using generalized Salagean operator in the unit disk $U=\{z:|z|<1\}$. We have obtained among others results like, coefficient inequalities, distortion theorem, extreme points, neighbourhood and Hadamard product properties.

Key Words

Univalent function, Distortion theorem, Neighbourhood, Hadamard product 2000 AMS subject classification. 30C45.

1. INTRODUCTION

Let A denote the class of functions of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k \tag{1.1}$$

which are analytic in the unit disk $U=\{z:|z|<1\}$.

In [4], Al-oboudi defined a differential operator as follows, for a function $f(z)\in A$,

$$\begin{aligned} D^0 f(z) &= f(z), \\ Df(z)=D^1 f(z) &= (1-\lambda)f(z)+\lambda z f'(z) \\ &= D_\lambda f(z), \qquad \lambda\geq 0 \end{aligned} \tag{1.2}$$

in general

$$D^n f(z)=D_\lambda\left(D^{n-1}f(z)\right). \tag{1.3}$$

If $f(z)$ is given by (1.1), then from (1.2) and (1.3) we observe that

$$D^n f(z)=z+\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n a_k z^k \tag{1.4}$$

when $\lambda=1$, we get Salagean differential operator [7].

Further, let T denote the subclass of A which consists of functions of the form

$$f(z)=z-\sum_{k=2}^{\infty}a_k z^k, \qquad a_k\geq 0. \tag{1.5}$$

A function $f(z)$ belonging to A is in the class $D_\lambda(\alpha,\beta,\xi;n)$, if and only if

$$\left|\frac{\left(D^n f(z)\right)'-1}{2\xi\left[\left(D^n f(z)\right)'-\alpha\right]-\left[\left(D^n f(z)\right)'-1\right]}\right|<\beta \tag{1.6}$$

where $0\leq\alpha<1/2\xi$, $0<\beta\leq 1$, $1/2\leq\xi\leq 1$, $n\in N\cup\{0\}$, $z\in U$.

Let $TD_\lambda(\alpha,\beta,\xi;n) = T \cap D_\lambda(\alpha,\beta,\xi;n)$ (1.7)

2. MAIN RESULTS

Theorem 2.1. Let $f(z)$ be defined by (1.5). Then $f(z) \in TD_\lambda(\alpha,\beta,\xi;n)$, if and only if

$$\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]a_k \le 2\beta\xi(1-\alpha) \tag{2.1}$$

$$0 \le \alpha < 1/2\xi,\ 0 < \beta \le 1,\ 1/2 \le \xi \le 1,\ n \in N \cup \{0\},\ \lambda \ge 0.$$

Proof. For $|z| = 1$, we get

$$\left|\left(D^n f(z)\right)' - 1\right| - \beta\left|2\xi\left[\left(D^n f(z)\right)' - \alpha\right] - \left[\left(D^n f(z)\right)' - 1\right]\right|$$

$$= \left|-\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n k\, a_k z^{k-1}\right| - \beta\left|2\xi(1-\alpha) - 2\xi\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n k\, a_k z^{k-1}\right.$$

$$\left. + \sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n k\, a_k z^{k-1}\right|$$

$$\le \sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]a_k - 2\beta\xi(1-\alpha) \le 0,$$

by hypothesis. Thus by maximum modulus theorem, we have $f(z) \in TD_\lambda(\alpha,\beta,\xi;n)$.

Conversely, suppose that $f(z) \in TD_\lambda(\alpha,\beta,\xi;n)$, hence the condition (1.6) gives us

$$\left|\frac{\left(D^n f(z)\right)' - 1}{2\xi\left[\left(D^n f(z)\right)' - \alpha\right] - \left[\left(D^n f(z)\right)' - 1\right]}\right|$$

$$= \left| \frac{-\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^{n} k\, a_k z^{k-1}}{2\xi(1-\alpha)-(2\xi-1)\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^{n} k\, a_k z^{k-1}} \right| < \beta.$$

Since $|\mathrm{Re}(z)| < |z|$ for all z, we obtain

$$\mathrm{Re}\left\{ \frac{\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^{n} k\, a_k z^{k-1}}{2\xi(1-\alpha)-(2\xi-1)\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^{n} k\, a_k z^{k-1}} \right\} < \beta.$$

Letting $z \to 1^{-}$ through real values, we get (2.1). The result is sharp for the function

$$f(z) = z - \frac{2\beta\xi(1-\alpha)}{\left[1+(k-1)\lambda\right]^{n} k\left[1+\beta(2\xi-1)\right]} z^{k}, \quad k \geq 2.$$

Corollary 2.1. Let $f(z) \in T$ belong to the class $TD_{\lambda}(\alpha,\beta,\xi;n)$, then

$$a_k \leq \frac{2\beta\xi(1-\alpha)}{\left[1+(k-1)\lambda\right]^{n} k\left[1+\beta(2\xi-1)\right]}, \quad k \geq 2. \tag{2.2}$$

Theorem 2.2. Let $f(z) \in T$ belong to the class $TD_{\lambda}(\alpha,\beta,\xi;n)$, then for $|z| \leq r < 1$, we have

$$r - r^{2}\frac{\beta\xi(1-\alpha)}{1+\beta(2\xi-1)} \leq \left|D^{n} f(z)\right| \leq r + r^{2}\frac{\beta\xi(1-\alpha)}{1+\beta(2\xi-1)} \tag{2.3}$$

$$1 - r\frac{2\beta\xi(1-\alpha)}{1+\beta(2\xi-1)} \leq \left|\left(D^{n} f(z)\right)'\right| \leq 1 + r\frac{2\beta\xi(1-\alpha)}{1+\beta(2\xi-1)}. \tag{2.4}$$

The bounds given by (2.3) and (2.4) are sharp.

Proof. By Theorem 2.1, we have

$$\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^{n}k\left[1+\beta(2\xi-1)\right]a_{k}\leq 2\beta\xi(1-\alpha)$$

then, we have

$$2(1+\lambda)^{n}\left[1+\beta(2\xi-1)\right]a_{k}\leq\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^{n}k\left[1+\beta(2\xi-1)\right]a_{k}\leq 2\beta\xi(1-\alpha),$$

thus,

$$\sum_{k=2}^{\infty}a_{k}\leq\frac{2\beta\xi(1-\alpha)}{2(1+\lambda)^{n}\left[1+\beta(2\xi-1)\right]}.$$

Hence

$$\begin{aligned}\left|D^{n}f(z)\right| &\leq|z|+\sum_{k=2}^{\infty}\left|\left[1+(k-1)\lambda\right]^{n}a_{k}z^{k}\right|\\ &\leq|z|+|z|^{2}(1+\lambda)^{n}\sum_{k=2}^{\infty}a_{k}\\ &\leq r+r^{2}(1+\lambda)^{n}\sum_{k=2}^{\infty}a_{k}\\ &\leq r+r^{2}\frac{\beta\xi(1-\alpha)}{1+\beta(2\xi-1)},\end{aligned}$$

and

$$\begin{aligned}\left|D^{n}f(z)\right| &\geq|z|-\sum_{k=2}^{\infty}\left|\left[1+(k-1)\lambda\right]^{n}a_{k}z^{k}\right|\\ &\geq|z|-|z|^{2}(1+\lambda)^{n}\sum_{k=2}^{\infty}a_{k}\\ &\geq r-r^{2}(1+\lambda)^{n}\sum_{k=2}^{\infty}a_{k}\\ &\geq r-r^{2}\frac{\beta\xi(1-\alpha)}{1+\beta(2\xi-1)}.\end{aligned}$$

thus (2.3) is true. Further,

$$\left|\left(D^n f(z)\right)'\right| \le 1 + 2r(1+\lambda)^n \sum_{k=2}^{\infty} a_k$$
$$\le 1 + r\frac{2\beta\xi(1-\alpha)}{1+\beta(2\xi-1)}$$

and

$$\left|\left(D^n f(z)\right)'\right| \ge 1 - 2r(1+\lambda)^n \sum_{k=2}^{\infty} a_k$$
$$\ge 1 - r\frac{2\beta\xi(1-\alpha)}{1+\beta(2\xi-1)}.$$

The result is sharp for the function $f(z)$ defined by

$$f(z) = z - \frac{2\beta\xi(1-\alpha)}{1+\beta(2\xi-1)} z^2, \qquad z = \pm r.$$

Theorem 2.3. Let $n \in N \cup \{0\}$, $\lambda \ge 0$, $0 \le \alpha_1 \le \alpha_2 < 1/2\xi$, $0 < \beta \le 1$, $1/2 \le \xi \le 1$. Then $TD_\lambda(\alpha_2, \beta, \xi; n) \subset TD_\lambda(\alpha_1, \beta, \xi; n)$.

Proof. By assumption we have

$$\frac{2\beta\xi(1-\alpha_2)}{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]} \le \frac{2\beta\xi(1-\alpha_1)}{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]}.$$

Thus, $f(z) \in TD_\lambda(\alpha_2, \beta, \xi; n)$ implies that

$$\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n a_k \le \frac{2\beta\xi(1-\alpha_2)}{k\left[1+\beta(2\xi-1)\right]} \le \frac{2\beta\xi(1-\alpha_1)}{k\left[1+\beta(2\xi-1)\right]}$$

then $f(z) \in TD_\lambda(\alpha_1, \beta, \xi; n)$.

Theorem 2.4. The set $TD_{\lambda}(\alpha,\beta,\xi;n)$ is the convex set.

Proof. Let $f_i(z)=z-\sum_{k=2}^{\infty}a_{k,i}z^k$ $(i=1,2)$ belong to $TD_{\lambda}(\alpha,\beta,\xi;n)$ and

let $g(z)=\zeta_1 f_1(z)+\zeta_2 f_2(z)$, with ζ_1 and ζ_2 non negative and $\zeta_1+\zeta_2=1$,

We can write

$$g(z)=z-\sum_{k=2}^{\infty}\left(\zeta_1 a_{k,1}+\zeta_2 a_{k,2}\right)z^k.$$

It is sufficient to show that $g(z)\in TD_{\lambda}(\alpha,\beta,\xi;n)$ that means

$$\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]\left(\zeta_1 a_{k,1}+\zeta_2 a_{k,2}\right)$$

$$=\zeta_1\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]a_{k,1}+\zeta_2\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]a_{k,2}$$

$$\le\zeta_1\left(2\beta\xi(1-\alpha)\right)+\zeta_2\left(2\beta\xi(1-\alpha)\right)=\left(\zeta_1+\zeta_2\right)\left(2\beta\xi(1-\alpha)\right)=2\beta\xi(1-\alpha).$$

Thus $g(z)\in TD_{\lambda}(\alpha,\beta,\xi;n)$.

We shall now present a result on extreme points in the following theorem.

Theorem 2.5. Let $f_1(z)=z$ and

$$f_k(z)=z-\frac{2\beta\xi(1-\alpha)}{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]}z^k$$

for all $k\ge 2$, $n\in N\cup\{0\}$, $\lambda\ge 0$, $0\le\alpha<1/2\xi$, $0<\beta\le 1$, $1/2\le\xi\le 1$.

Then $f(z)$ is in the subclass $TD_{\lambda}(\alpha,\beta,\xi;n)$, if and only if it can be expressed in the form $f(z)=\sum_{k=2}^{\infty}\gamma_k z^k$ where $\gamma_k\ge 0$ and $\sum_{k=2}^{\infty}\gamma_k=1$ or $1=\gamma_1+\sum_{k=2}^{\infty}\gamma_k$.

Proof. Let $f(z)=\sum_{k=2}^{\infty}\gamma_k z^k$ where $\gamma_k \geq 0$ and $\sum_{k=2}^{\infty}\gamma_k = 1$. Thus

$$f(z)=z-\sum_{k=2}^{\infty}\frac{2\beta\xi(1-\alpha)}{\left[1+(k-1)\lambda\right]^{n}k\left[1+\beta(2\xi-1)\right]}\gamma_k z^k$$

and we obtain

$$\sum_{k=2}^{\infty}\frac{\left[1+(k-1)\lambda\right]^{n}k\left[1+\beta(2\xi-1)\right]}{2\beta\xi(1-\alpha)}\gamma_k\times\frac{2\beta\xi(1-\alpha)}{\left[1+(k-1)\lambda\right]^{n}k\left[1+\beta(2\xi-1)\right]}$$

$$=\sum_{k=2}^{\infty}\gamma_k = 1-\gamma_1 \leq 1.$$

In view of Theorem (2.1), this show that $f(z)\in TD_\lambda(\alpha,\beta,\xi;n)$.

Conversely, suppose that $f(z)$ of the form (1.5) belong to $TD_\lambda(\alpha,\beta,\xi;n)$ then

$$a_k \leq \frac{2\beta\xi(1-\alpha)}{\left[1+(k-1)\lambda\right]^{n}k\left[1+\beta(2\xi-1)\right]}, \qquad k\geq 2.$$

Putting

$$\gamma_k=\frac{\left[1+(k-1)\lambda\right]^{n}k\left[1+\beta(2\xi-1)\right]}{2\beta\xi(1-\alpha)}$$

and $\gamma_1 = 1-\sum_{k=2}^{\infty}\gamma_k$, then we have $f(z)=\gamma_1 f_1(z)+\sum_{k=2}^{\infty}\gamma_k f_k(z)$.

This completes the proof.

3. NEIGHBOURHOOD AND HADAMARD PRODUCT PROPERTIES

Definition 3.1. [6]. Let $\gamma_k \geq 0$ and $f(z)\in T$ of the form (1.5).
The (k,γ) – neighbourhood of a function $f(z)$ defined by

$$N_{(k,\gamma)}(f)=\left\{g\in T: g(z)=z-\sum_{k=2}^{\infty}b_k z^k \text{ and } \sum_{k=2}^{\infty}k\left|a_k-b_k\right|\leq\gamma\right\}, \qquad (3.1)$$

For the identity function $e(z) = z$, we have

$$N_{(k,\gamma)}(e) = \left\{ g \in T : g(z) = z - \sum_{k=2}^{\infty} b_k z^k \ and \ \sum_{k=2}^{\infty} k|b_k| \le \gamma \right\}. \quad (3.2)$$

Theorem 3.1. Let $\gamma = \dfrac{2\beta\xi(1-\alpha)}{(1+\lambda)^n\left[1+\beta(2\xi-1)\right]}$. Then $TD_\lambda(\alpha,\beta,\xi;n) \subset N_{k,\gamma}(e)$.

Proof. Let $f(z) \in TD_\lambda(\alpha,\beta,\xi;n)$ then we have

$$2(1+\lambda)^n\left[1+\beta(2\xi-1)\right]\sum_{k=2}^{\infty} a_k \le \sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]a_k \le 2\beta\xi(1-\alpha),$$

therefore

$$\sum_{k=2}^{\infty} a_k \le \frac{\beta\xi(1-\alpha)}{(1+\lambda)^n\left[1+\beta(2\xi-1)\right]}, \quad (3.3)$$

also we have for $|z| < r$

$$|f'(z)| \le 1 + |z|\sum_{k=2}^{\infty} k a_k \le 1 + r\sum_{k=2}^{\infty} k a_k.$$

In view of (3.3), we have

$$|f'(z)| \le 1 + r\frac{2\beta\xi(1-\alpha)}{(1+\lambda)^n\left[1+\beta(2\xi-1)\right]}.$$

From above inequalities we get

$$\sum_{k=2}^{\infty} k a_k \le \frac{2\beta\xi(1-\alpha)}{(1+\lambda)^n\left[1+\beta(2\xi-1)\right]} = \gamma.$$

therefore, $f(z) \in N_{k,\gamma}(e)$.

Definition 3.2. The function $f(z)$ defined by (1.5) is said to be a member of the subclass $TD_\lambda(\alpha,\beta,\xi,\zeta,n)$ if there exits a function $g(z) \in TD_\lambda(\alpha,\beta,\xi;n)$ such that

$$\left|\frac{f(z)}{g(z)}-1\right| \le 1-\zeta,\ z\in U,\ 0\le\zeta<1.$$

Theorem 3.2. Let $g(z)\in TD_\lambda(\alpha,\beta,\xi;n)$ and

$$\zeta = 1-\frac{\gamma}{2}d(\alpha,\beta,\xi;n). \tag{3.4}$$

Then $N_{k,\gamma}(g)\subset TD_\lambda(\alpha,\beta,\xi,\zeta;n)$ where $n\in N\cup\{0\}$, $\lambda\ge 0$,

$0\le\alpha<1/2\xi$, $0<\beta\le 1$, $1/2\le\xi\le 1$, $0\le\zeta<1$ and

$$d(\alpha,\beta,\xi;n)=\frac{(1+\lambda)^n\left[1+\beta(2\xi-1)\right]}{(1+\lambda)^n\left[1+\beta(2\xi-1)\right]-\beta\xi(1-\alpha)}.$$

Proof. Let $f(z)\in N_{k,\gamma}(g)$, then by (3.3) we have $\sum_{k=2}^{\infty}k\left|a_k-b_k\right|\le\gamma$, then

$\sum_{k=2}^{\infty}\left|a_k-b_k\right|\le \gamma/2$.

Since $g(z)\in TD_\lambda(\alpha,\beta,\xi;n)$, we have

$$\sum_{k=2}^{\infty}b_k\le\frac{\beta\xi(1-\alpha)}{(1+\lambda)^n\left[1+\beta(2\xi-1)\right]},$$

therefore,

$$\left|\frac{f(z)}{g(z)}-1\right|<\frac{\sum_{k=2}^{\infty}\left|a_k-b_k\right|}{1-\sum_{k=2}^{\infty}b_k}$$

$$\le\frac{\gamma}{2}\left(\frac{(1+\lambda)^n\left[1+\beta(2\xi-1)\right]}{(1+\lambda)^n\left[1+\beta(2\xi-1)\right]-\beta\xi(1-\alpha)}\right)=\frac{\gamma}{2}d(\alpha,\beta,\xi;n)=1-\zeta.$$

Then by definition 3.2, we get $f(z)\in TD_\lambda(\alpha,\beta,\xi,\zeta;n)$.

Theorem 3.3. Let $f(z)$ and $g(z)\in TD_\lambda(\alpha_1,\beta,\xi;n)$ be of the form (1.5) such that

$f(z)=z-\sum_{k=2}^{\infty}a_k z^k$ and $g(z)=z-\sum_{k=2}^{\infty}b_k z^k$, where $a_k, b_k \geq 0$. Then the Hadamard product $h(z)$ defined by $h(z)=z-\sum_{k=2}^{\infty}a_k b_k z^k$ is in the subclass $TD_\lambda(\alpha_2,\beta,\xi;n)$ where

$$\alpha_2 \leq \frac{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]-2\beta\xi(1-\alpha_1)^2}{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]}.$$

Proof. By Theorem 2.1, we have

$$\sum_{k=2}^{\infty}\frac{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]}{2\beta\xi(1-\alpha_1)}a_k \leq 1 \tag{3.5}$$

and

$$\sum_{k=2}^{\infty}\frac{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]}{2\beta\xi(1-\alpha_1)}b_k \leq 1. \tag{3.6}$$

We have only to find the largest α_2 such that

$$\sum_{k=2}^{\infty}\frac{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]}{2\beta\xi(1-\alpha_2)}a_k b_k \leq 1.$$

Now, by Cauchy-Schwarz inequality, we obtain

$$\sum_{k=2}^{\infty}\frac{\left[1+(k-1)\lambda\right]^n\left[1+\beta(2\xi-1)\right]}{2\beta\xi(1-\alpha_1)}\sqrt{a_k b_k} \leq 1, \tag{3.7}$$

we need only to show that

$$\frac{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]}{2\beta\xi(1-\alpha_2)}a_k b_k \leq \frac{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]}{2\beta\xi(1-\alpha_1)}\sqrt{a_k b_k}$$

equivalently,

$$\sqrt{a_k b_k} \le \frac{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]}{2\beta\xi(1-\alpha_1)} \times \frac{2\beta\xi(1-\alpha_2)}{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]}$$

$$\le \frac{1-\alpha_2}{1-\alpha_1}.$$

But from (3.7), we have

$$\sqrt{a_k b_k} \le \frac{2\beta\xi(1-\alpha_1)}{\left[1+(k-1)\lambda\right]^n\left[1+\beta(2\xi-1)\right]}.$$

Consequently, we need to prove that

$$\frac{2\beta\xi(1-\alpha_1)}{\left[1+(k-1)\lambda\right]^n\left[1+\beta(2\xi-1)\right]} \le \frac{1-\alpha_2}{1-\alpha_1}.$$

or equivalently, that

$$\alpha_2 \le \frac{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]-2\beta\xi(1-\alpha_1)^2}{\left[1+(k-1)\lambda\right]^n k\left[1+\beta(2\xi-1)\right]}.$$

Theorem 3.4. Let $f(z) \in TD_\lambda(\alpha,\beta,\xi;n)$ be defined by (1.5) and c any real number with $c>-1$ than the function $G(z)$ defined as $G(z)=\frac{c+1}{z^c}\int_0^z s^{c-1}f(s)ds,\ c>-1$, also belongs to $TD_\lambda(\alpha,\beta,\xi;n)$.

Proof. By virtue of $G(z)$ it follows from (1.5) that

$$G(z)=\frac{c+1}{z^c}\int_0^z\left(s^c-\sum_{k=2}^{\infty}a_k s^{k+c-1}\right)ds$$

$$=z-\sum_{k=2}^{\infty}\left(\frac{c+1}{c+k}\right)a_k z^k.$$

But $$\sum_{k=2}^{\infty} \frac{\left[1+(k-1)\lambda\right]^{n} k\left[1+\beta(2\xi-1)\right]}{2\beta\xi(1-\alpha)}\left(\frac{c+1}{c+k}\right) a_k \leq 1,$$

Since $\frac{c+1}{c+k} \leq 1$ and by Theorem 2.1, so the proof is complete.

Theorem 3.5. Let $f(z) \in TD_{\lambda}(\alpha,\beta,\xi;n)$ be defined by (1.5) and

$$F_{\mu}(z) = (1-\mu)z + \mu \int_0^z \frac{f(s)}{s} ds \quad (\mu \geq 0, z \in U).$$

Then $F_{\mu}(z)$ is also in $TD_{\lambda}(\alpha,\beta,\xi;n)$ if $0 \leq \mu \leq 2$.

Proof. Let $f(z)$ defined by (1.5) then

$$F_{\mu}(z) = (1-\mu)z + \mu \int_0^z \left(\frac{s - \sum_{k=2}^{\infty} a_k s^k}{s} \right) ds$$

$$= z - \sum_{k=2}^{\infty} \frac{\mu}{k} a_k z^k.$$

By Theorem 2.1 and since $\left(\frac{\mu}{k} \leq 1\right)$ we have

$$\sum_{k=2}^{\infty} \frac{\left[1+(k-1)\lambda\right]^{n} k\left[1+\beta(2\xi-1)\right]}{2\beta\xi(1-\alpha)}\left(\frac{\mu}{k}\right) a_k$$

$$\leq \sum_{k=2}^{\infty} \frac{\left[1+(k-1)\lambda\right]^{n} k\left[1+\beta(2\xi-1)\right]}{2\beta\xi(1-\alpha)}\left(\frac{\mu}{2}\right) a_k \leq 1,$$

then $F_{\mu}(z)$ is in $TD_{\lambda}(\alpha,\beta,\xi;n)$.

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A Note Concerning the Invariance of Baire Spaces under Mappings

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Abstract: In this note we prove that under a semi-continuous and almost open mapping the image of a Baire space is also a Baire space and as a result improves a theorem of Dasgupta and Lahiri [3]. Furthermore, a theorem of Noiri [6] on irresolute mapping is improved in this process.

Keywords and Phrases: Semi-continuous and almost continuous mapping, almost open and feebly open mapping, Baire spaces.
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1. INTRODUCTION

It is well known that under a continuous and open mapping, the image of a Baire space is also a Baire space. Dasgupta and Lahiri [3] and Frolik [4] reached the same conclusion under weaker hypotheses. In this note we prove another generalization of this classical theorem which improves the result of Dasgupta and Lahiri [3] and is independent from that of Frolik [4]. In this process we also improve a theorem of Noiri (Theorem 3 of [6]) which tells when a mapping is irresolute. Throughout the paper X, Y denote topological spaces, ϕ the empty set, $\Re$ the set of real numbers and U the usual topology. The closure and the interior of a set $A \subset X$ is denoted by Int A and Cl A respectively.

2. PRELIMINARIES

Definition 1.1 Let $A \subset X$. Then

(a) A is said to be semi-open [5] iff there exists an open set O such that $O \subset A \subset ClO$, or equivalently, $A \subset Cl\text{Int } A$. The union of all semi-open sets contained in A is called the semi-interior [3] of A and is denoted by SInt A (b) A is said to be semi-closed [3] iff the complement of A, $X - A$ is semi-open. The intersection of all semi-closed sets containing A is called the semi-closure [3] of A and is denoted by SCl A.

Remark 1.1 It is known from [3] that SCl A is semi-closed and SInt A is semi-open.

Definition 1.2 Let $f: X \to Y$ be a mapping. Then

(a) f is called semi-continuous [5] if for each open set V in Y, $f^{-1}(V)$ is semi-open in X.

(b) f is called almost open (in the sense of Rose) [7] if $f(U) \subset \text{Int } Cl\, f(U)$ for every open set U in X.

(c) f is called almost continuous [4] if for every open subset V of Y, $f^{-1}(V) \subset Cl\,\text{Int} f^{-1}(V)$.

(d) f is called feebly open [4] if $A \subset X$, $\text{Int } A \neq \phi \Rightarrow f(\text{Int } A) \neq \phi$.

Remark 1.2

(a) The notions of semi-continuity [5] and almost continuity [4] are the same in view of Definition 1.1 (a).

(b) Every continuous mapping is semi-continuous but the converse is not necessarily true (see [5]).

(c) Every open mapping is almost open (feebly open) (see [7], ([4])), but the converse is not necessarily true (see Example 2.2 and (Example 2.1)).

Theorem 1.1 [3] A set $A \subset X$ is semi-open iff SInt A = A and A is semi-closed iff SCl A = A.

Theorem 1.2 [3] If $A \subset X$, then $\text{Int } A \subset \text{SInt } A \subset A \subset SCl\, A \subset ClA$.

Theorem 1.3 [5] If A is a semi-open subset of X and $B \subset X$ such that $A \subset B\; Cl\, A$, then B is semi-open in X.

Lemma 1.1 [3] A set $A \subset X$ is semi-closed iff there exists a closed set F such that $\text{Int } F \subset A \subset F$.

Lemma 1.2 [3] For any $A \subset X$, $\text{SInt } A = X - [SCl\,(X - A)]$

Theorem 1.4 [7] A mapping $f: X \to Y$ is almost open iff $f^{-1}(Cl V) \subset Cl f^{-1}(V)$, for every open $V \subset Y$.

Theorem 1.5 [3] Let $f: X \to Y$ be surjective, open and semi-continuous. If X is a Baire space then Y is also a Baire space.

Theorem 1.6 [4] Let $f: X \to Y$ be surjective, feebly open and almost continuous. If X is a Baire space then Y is also a Baire space.

3. IMPROVEMENT OF CERTAIN THEOREMS

We start this section from the following theorem of Noiri [6].

Theorem 2.1 (Theorem 3 of [6]. If $f: X \to Y$ is an open and semi-continuous mapping then the inverse image $f^{-1}(B)$ of each semi-open set B in Y is semi-open in X. Our purpose is to set this result in a more general context. More precisely, we shall prove that the openness of the mapping can be replaced by the almost openness of the mapping and henceforth making an improvement (in view of Remark 1.2 (c)) of the above theorem.

Theorem 2.2 If $f: X \to Y$ is an almost open and semi-continuous mapping, then the inverse image $f^{-1}(B)$ of each semi-open set B in Y is semi-open in X.

Proof. Let B be an arbitrary semi-open set in Y. Then there exists an open set V in Y such that $V \subset B \subset Cl\, V$. Since f is almost open, we have, $f^{-1}(Cl V) \subset Cl f^{-1}(V)$, by Theorem 1.4. Again since f is semi-continuous, $f^{-1}(V)$ is semi-open in X. Thus $f^{-1}(V) \subset f^{-1}(B) \subset Cl f^{-1}(V)$ and so $f^{-1}(B)$ is semi-open in X, by Theorem 1.3.

Before going to our main theorem (Theorem 2.6) we shall prove some ancillary results in the following.

Theorem 2.3 Let $f: X \to Y$ be surjective and almost open. Then if $B \subset Y$ is dense and open in Y, $f^{-1}(B)$ is dense in X.

Proof. Let B be dense and an open subset of Y. Then since f is almost open, by Theorem 1.4 we have,

$$f^{-1}(Cl\, B) \subset Cl f^{-1}(B) \Rightarrow f^{-1}(Y) \subset Cl f^{-1}(B) \Rightarrow X \subset Cl f^{-1}(B) \Rightarrow f^{-1}(B) \text{ is}$$

dense in X.

Theorem 2.4 $A \subset D$ is dense in X iff $SCl\, D = X$.

Proof. Let D be dense in X so that by Theorem 1.2, $SClD \subset ClD = X$. Since SCl D semi-closed by Remark 1.1, there is a closed set $F \subset X$ such that $Int\, F \subset SCl\, D$

$\subset$ F by Lemma 1.1. But by Theorem 1.2, $D \subset SCl\, D \subset F$ and so $X = Cl\, D \subset Cl\, F$. Thus F = X and we get $X = \text{Int}\, X \subset SCl\, D \subset X$, which implies that $SCl\, D = X$.

If $SCl\, D = X$, then because by Theorem 1.2, $SCl\, D \subset Cl\, D$, it follows that $Cl\, D$ = X and so D is dense in X.

Remark 2.1 The above theorem is an improvement of the requirement that D is dense iff $Cl\, D = X$.

Theorem 2.5 A set D is dense in X iff the complement of D has empty semi-interior.

Proof. If D is dense in X, then by Theorem 2.4, $SCl\, D = X$ and so in Lemma 1.2, replacing A by X – D we obtain, $\text{SInt}\,(X - D) = X - SCl\, D = \phi$. On the other hand if $\text{SInt}\,(X - D) = \phi$, then we obtain $X - SCl\, D = \text{SInt}\,(X - D) = \phi$ and so $SCl\, D = X$. Hence D is dense in X, by Theorem 2.4

Now we are in a position to show that a Baire space remains invariant under an almost open, semi-continuous surjection.

Theorem 2.6 Let $f : X \to Y$ be surjective, almost open and semi-continuous. If X is a Baire space then Y is also a Baire space.

Proof. Let $G = \cap_i D_i$ be a countable intersection of dense open sets in Y.

As f is almost open and semi-continuous by Theorem 2.2, f^{-1} (SInt (X – G)) is semi-open in X. Thus

$$\begin{aligned} f^{-1}(\text{SInt}(X - G)) &= \text{SInt}\, f^{-1}\,(\text{SInt}\,(X - G)), \text{ by Theorem 1.1} \\ &\subset \text{SInt}\, f^{-1}\,(X - G), \text{ by Theorem 1.2} \\ &= \text{SInt}\, f^{-1}\,(X - \cap_i D_i) \\ &= \text{SInt}\, f^{-1}\,[X - \cap_i f^{-1}\,(D_i)] \qquad (1) \end{aligned}$$

Since D_i is open for each i, semi-continuity of f implies that $f^{-1}\,(D_i)$ is semi-open in X for each i and so there is an open set V_i such that $V_i \subset f^{-1}\,(D_i) \subset Cl\, V_i$, for each i. Thus $X - \cap_i f^{-1}\,(D_i) \subset X - \cap_i V_i$. So from (1) we get,

$$f^{-1}\,[\text{SInt}\,(X - G)] \subset \text{SInt}\,[X - \cap_i V_i] \qquad (2)$$

Now by Theorem 2.3, $f^{-1}\,(D_i)$ is dense in X. So $ClV_i = X$. Consequently V_i is dense in X for each i. Thus $\cap_i V_i$ is a countable intersection of dense open set in X, and as X is a Baire space, $\cap_i V_i$ is dense in X so that $\text{SInt}\,[X - \cap_i V_i] = \phi$, by Theorem 2.5. Hence from (2) we get, $f^{-1}\,[\text{SInt}\,(X - G)] = \phi$, so that $\text{SInt}\,(X - G) = \phi$, which implies, by Theorem 2.5, that G is dense in X. Thus Y is a Baire space.

Remark 2.2 From Remark 1.2 (c) it follows that Theorem 2.6 is an improvement of Theorem 1.5. Furthermore, Theorem 2.6 of ours and Theorem 1.6 of Frolik does not imply any one from the other, because although by Remark 1.2 (a) the semi-continuity of Levine and almost continuity of Frolik are the same but feebly open mapping and almost open mapping of Rose are independent to each other as shown in the following examples.

Example 2.1

Let $f : (\Re, U) \to (\Re, U)$ be defined by $f(2) = 4$ and $f(x) = x$ otherwise. It is easy to verify that f is feebly open. But considering the open set $U = (1, 3)$ we see $f(U) = (1, 2) \cup (2, 3) \cup (2, 3) \cup \{4\}$. Hence $Cl\, f(U) = [1. 3] \cup \{4\}$ and so $\text{Int}\, Cl\, f(U) = (1, 3) \supseteq f(U)$. Therefore f is not almost open.

Example 2.2

Let N be the set of natural numbers and τ be the topology consisting of all sets O such that $O = \phi$ or $O = N$ or $O = \{1, 2, ..., n\}$ for each $n(> 1)$ in N. Let $f: (\Re, U) \to (N, \tau)$ be defined by

$$f(x) = 1, \text{ if x is rational}$$
$$= 3, \text{ if x is irrational and } 0 < |x| < 3$$
$$= n, \text{ if x is irrational and } n - 2 < |x| < n - 1 \text{ where } n \in N \text{ and} > 4.$$

Let O be any open set in $(\Re, U)$. Then $f(O)$ must contain 1. So $\text{Int}Cl\, f(O) = \text{Int}\, N = N \supset f(O)$. Hence f is almost open. Next we consider the open set $O = (0. 1)$ in $(\Re, U)$. Clearly $\text{Int}\, O \neq \phi$. But $f(O) = \{1, 3\}$ and so $\text{Int}\, f(O) = \phi$. Hence f is not feebly open.

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On the Approximation of Conjugate of Functions Belonging To Lip {ξ(t), p} Class By Generalized Nörlund Means

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Abstract: In this paper, the degree of approximation of conjugate of functions belonging to Lip$\{\xi(t),p\}$ class by generalized Nörlund means of conjugate series of **Key words:** model, indirect technique, ratio, parameters, mortality, deaths. Fourier series has been determined.

1. INTRODUCTION AND DEFINITION

Qureshi ([6]) has determined the degree of approximation of function $\tilde{f}(x)$, conjugate of a function $f \in \mathrm{Lip}\alpha$, $\mathrm{Lip}(\alpha,p)$ by Nörlund method .The purpose of this paper is to generalize above result in two ways and to determine the approximation of $\tilde{f}(x)$, conjugate of a function $f \in \mathrm{Lip}\{\xi(t),p\}$ class, by generalized Nörlund means .

Let f be periodic with period 2π and integrable over$(-\pi, \pi)$ in Lebesgue sense. Let its Fourier series be given by

$$f(t)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} A_n(x) \,. \tag{1}$$

The conjugate series of the Fourier series (1) is given by

$$\sum_{n=1}^{\infty}\left(a_n \sin nx - b_n \cos nx\right) = -\sum_{n=1}^{\infty} B_n(x). \qquad (2)$$

We define norm $\| \ \|_p$ by $\| f \|_p = \left(\int_0^{2\pi} |f(x)|^p \, dx \right)^{\frac{1}{p}}$, $p \geq 1$

and the degree of approximation $E_n(f)$ is given by (Zygmund [8])

$$E_n(f) = \min \| f - T_n \|_p$$

where $T_n(x)$ is a trigonometric polynomial of degree n.

A function $f \in Lip\alpha$ if $|f(x+t) - f(x)| = O(|t|^{\alpha})$, for $0 < \alpha \leq 1$.

$f(x) \in Lip(\alpha, p)$ for $0 \leq x \leq 2\pi$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^p \, dx \right)^{1/p} = O(|t|^{\alpha}), \quad 0 < \alpha \leq 1 \qquad \text{(McFadden [5]).}$$

Given a positive increasing function $\xi(t)$ and an integer $p \geq 1$,

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$f(x) \in Lip(\xi(t), p)$ if

$$\left(\int_0^{2\pi} |f(x+t) - f(t)|^p \, dx \right)^{\frac{1}{p}} = O(\xi(t)) \qquad \text{(Siddiqi [7]).} \qquad (3)$$

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series having its n^{th} partial sum $s_n = \sum_{v=0}^{n} u_n$.

Let $\{p_n\}$ and $\{q_n\}$ be two sequences of real numbers such that

$$R_n == \sum_{k=0}^{n} p_k q_{n-k} \neq 0 \ \forall n \geq 0 .$$

For any sequence $\{s_n\}$ we write

$$t_n^{p,q} = \frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k} s_{n-k}\,. \tag{4}$$

The generalized Nörlund transform of the sequence $\{s_n\}$ is the sequence $\{t_n^{p,q}\}$. If $t_n^{p,q} \to s$, as $n \to \infty$, then the series $\sum_{n=0}^{\infty} u_n$ or the sequence $\{s_n\}$ is said to be summable S by generalized Nörlund method (N,p,q) and is denoted by $S_n \to S(N,p,q)$.

(Borwein [1])

The necessary and sufficient conditions for a (N, p, q) method to be regular are

$$\sum_{k=0}^{n} |p_{n-k} q_k| = O(|R_n|)$$

and $p_{n-k} = o(|R_n|)$, as $n \to \infty$, for every fixed $k \geq 0$ for which $q_k \neq 0$.

The (N, p, q) method reduces to the Nörlund method if $q_n = 1$ for all n. The method (N, p, q) reduces to Riesz method $(\tilde{N}, q_n)$ if $p_n = 1$, for all n .When $p_n = \binom{n+\alpha-1}{\alpha-1}$, $\alpha > 0$, and $q_n = 1\ \forall\, n$, the method (N,p,q) reduces to (C, α).

We use following notations:

$$\psi(t) = f(x+t) - f(x-t), \qquad \tilde{f}(x) = -\frac{1}{2\pi}\int_0^{\pi} \psi(t)\cot\tfrac{t}{2}\,dt$$

2. MAIN THEOREM

We prove the following:

Theorem: Let the regular generalized Nörlund method (N, p, q) be defined by a non- negative, monotonic non-increasing sequence $\{p_n\}$ and a non-negative, monotonic non- decreasing sequence $\{q_n\}$ of real constants such that

$$q_n p_n = O(R_n \log n) \ \text{ with } n \geq n_0 > 1. \tag{5}$$

If $f: R \to R$ is a 2π periodic, Lebesgue integrable and belonging to Lip(ξ(t),p) class, ξ(t) is positive increasing function of t satisfying

$$\left\{\int_0^{\frac{1}{n}}\left(\frac{t|\psi(t)|}{\xi(t)}\right)^p dt\right\}^{\frac{1}{p}} = O\left(\frac{1}{n}\right) \tag{6}$$

and

$$\left\{\int_{\frac{1}{n}}^{\pi}\left(\frac{t^{-\delta}|\psi(t)|}{\xi(t)}\right)^p dt\right\}^{\frac{1}{p}} = O\left(n^{\delta}\right) \tag{7}$$

where δ is an arbitrary number such that $q(1-\delta)-1>0$, q the conjugate index of p and the condition (6) and (7) hold uniformly in x, then degree of approximation of $\tilde{f}(x)$, conjugate of $f \in Lip\{\xi(t), p\}$, by generalized Nörlund means $\tilde{t}_n^{p,q}(x)=\frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\tilde{s}_k$ of the conjugate series (2) is given by

$$\left\|\tilde{t}_n^{p,q}(x)-\tilde{f}(x)\right\|_p = O\left(n^{\frac{1}{p}}\xi\left(\frac{1}{n}\right)\log n\right) \tag{8}$$

3. LEMMAS

The following lemmas are required for the proof of our theorem

***Lemma* 1** (McFadden , 1942), If $\{p_n\}$ is a non-negative non-increasing sequence for $0 \le a \le b \le n$, $0 < t \le \pi$ then

$$\left|\sum_{k=0}^{n}\frac{p_k\cos\left(n-k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right| = O\left(\frac{p_\tau}{t}\right)$$

***Lemma* 2** If $\{p_n\}$ is a non-negative non-increasing and $\{q_n\}$ is a non-negative non- decreasing sequence then

$$\left|\frac{1}{2\pi R_n}\sum_{k=0}^{n}\frac{p_k q_{n-k}\cos\left(n-k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right| = O\left(\frac{q_n P_\tau}{R_n t}\right) \qquad \text{if} \quad \frac{1}{n}\le t\le\pi$$

Proof By Abel`s lemma, we have

$$\left|\frac{1}{2\pi R_n}\sum_{k=0}^{n}\frac{p_k q_{n-k}\cos\left(n-k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right| \le \frac{q_n}{\pi R_n}\max_{0\le m\le n}$$

$$\left|\sum_{k=0}^{n}\frac{p_k\cos\left(n-k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right|$$

$$= \quad O\left(\frac{q_n P_\tau}{R_n t}\right), \text{ by Lemma 1 .}$$

4. PROOF OF THE THEOREM

The n th partial sum of conjugate Fourier series is given by

$$\tilde{S}_n(x) = -\frac{1}{2\pi}\int_0^\pi \cot\frac{t}{2}\,\psi(t)\,dt + \frac{1}{2\pi}\int_0^\pi \frac{\cos\left(n-k\frac{1}{2}\right)t}{\sin\frac{t}{2}}\psi(t)\,dt$$

$$\tilde{S}_n(x) - \left(-\frac{1}{2\pi}\int_0^\pi \cot\frac{t}{2}\,\psi(t)\,dt\right) = \frac{1}{2\pi}\int_0^\pi \frac{\cos\left(n-k\frac{1}{2}\right)t}{\sin\frac{t}{2}}\psi(t)\,dt\,.$$

By taking (N, p, q) means of $\tilde{S}_n(x)$, we get

$$\bar{t}_n^{p,q}(x)-\bar{f}(x)=\frac{1}{2\pi R_n}\int_0^{\pi}\psi(t)\sum_{k=0}^{n}p_k q_{n-k}\frac{\cos\left(n-k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt$$

$$=\frac{1}{2\pi R_n}\left(\int_0^{\frac{1}{n}}+\int_{\frac{1}{n}}^{\pi}\right)\psi(t)\sum_{k=0}^{n}p_k q_{n-k}\frac{\cos\left(n-k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt$$

$$=I_1+I_2. \tag{9}$$

Applying Hölder's inequality and the fact that $\psi(t)\in Lip(\xi(t),p)$, we have

$$|I_1|\le\frac{1}{2\pi R_n}\left\{\int_0^{\frac{1}{n}}\left(\frac{t|\psi(t)|}{\xi(t)}\right)^p dt\right\}^{\frac{1}{p}}\left\{\int_0^{\frac{1}{n}}\left(\frac{\xi(t)}{t}\sum_{k=0}^{n}p_k q_{n-k}\left|\frac{\cos\left(n-k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right|\right)^q dt\right\}^{\frac{1}{q}}$$

$$\le O\left(\frac{1}{n}\right)\left\{\int_0^{\frac{1}{n}}\left(\frac{\xi(t)}{t^2}\right)^q dt\right\}^{\frac{1}{q}}\quad,\quad\text{by (6)}$$

$$=O\left(\frac{1}{n}\right)O\left(\xi\left(\frac{1}{n}\right)\right)\left(\int_{\epsilon}^{\frac{1}{n}}\frac{1}{t^{2q}}dt\right)^{\frac{1}{q}},\ \text{for some } 0<\epsilon<\tfrac{1}{n}$$

by second mean value theorem for integral.

$$=O\left(n^{\frac{1}{p}}\xi\left(\frac{1}{n}\right)\right) \tag{10}$$

Similarly, as above, we have

$$|I_2|=\left[\int_{\frac{1}{n}}^{\pi}\left(\frac{t^{-\delta}|\psi(t)|}{\xi(t)}\right)^p dt\right]^{\frac{1}{p}}\left[\int_{\frac{1}{n}}^{\pi}\left(\frac{\xi(t)}{2\pi R_n t^{-\delta}}\sum_{k=0}^{n}\frac{\cos\left(n-k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right)^q dt\right]^{\frac{1}{q}}$$

$$= O\left(n^{\delta}\right)\left[\int_{\frac{1}{n}}^{\pi}\left(\frac{\xi(t)q_n\,p_\tau}{t^{-\delta}t\,R_n}\right)^q dt\right]^{\frac{1}{q}} \quad \text{, by (7) and Lemma 2}$$

$$= O\left(n^{\delta}\right)O\left(\frac{q_n}{R_n}\right)\left[\int_{\frac{1}{n}}^{\pi}\left(\frac{\xi(t)\,p_\tau}{t^{1-\delta}}\right)^q dt\right]^{\frac{1}{q}}$$

$$= O\left(\frac{n^{\delta}q_n}{R_n}\right)\left[\int_{\frac{1}{\pi}}^{n}\left\{\frac{\xi\left(\frac{1}{y}\right)p_{[y]}}{\left(\frac{1}{y}\right)^{1-\delta}}\right\}^q \frac{dy}{y^2}\right]^{\frac{1}{q}} \text{, taking } t=\tfrac{1}{y}$$

$$= O\left(\frac{n^{\delta}q_n p_n}{R_n}\xi\left(\frac{1}{n}\right)\right)\left[\int_{\frac{1}{\pi}}^{n} y^{(1-\delta)q-2}dy\right]^{\frac{1}{q}} \text{, by mean value theorem}$$

$$= O\left(\frac{n^{\delta}q_n p_n}{R_n}\xi\left(\frac{1}{n}\right)\right)\left(\frac{n^{q(1-\delta)-1}-\left(\frac{1}{\pi}\right)^{q(1-\delta)-1}}{q(1-\delta)-1}\right)^{\frac{1}{q}}$$

$$= O\left(n^{1/p}\xi\left(\tfrac{1}{n}\right)\log n\right), \quad \text{by (5) and hypothesis of theorem}$$

(11)

Combining from (9) to (11), we have

$$\left\|\tilde{t}_n^{\,p,q}(x)-\tilde{f}(x)\right\|_p = O\left(n^{1/p}\xi\left(\tfrac{1}{n}\right)\log n\right)$$

5. COROLLARIES

Following Corollaries can be derived from the theorem.

***Corollary* 1** If $\xi(t)=t^{\alpha}$ then the degree of approximation of a function belonging to Lipα, $\frac{1}{p}<\alpha<1$ is given by

$$\left\| \tilde{t}_n^{\,p,q} - \tilde{f} \right\| = O\left(\frac{\log n}{n^{\alpha - \frac{1}{p}}} \right)$$

***Corollary* 2** If $p \to \infty$ in Cor.1 then we have for, $0 < \alpha < 1$,

$$\left\| \tilde{t}_n^{\,p,q}(x) - \tilde{f}(x) \right\| = O\left(\frac{\log n}{n^{\alpha}} \right)$$

***Remark*:-**An independent proof of Corollaries (1) and (2) can be developed along the same lines as the theorem.

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Mathematical Models to Estimate the Maternal Mortality

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Abstract: The main aim of this paper is to apply mathematical model to estimate maternal mortality ratio using two Demographic and Health Surveys data of Nepal. We applied Bhat et al. [7] model to estimate maternal mortality for Nepal. The modification was also made by assuming the sex-ratio of mortality rates (excluding maternal causes of death) specific to women aged between 15 and 49 years might be equal to some constant K. The proposed method provided consistent estimates of maternal mortality ratio of 588 deaths per 100000 live births in 1996 and 351 deaths per 100000 live births in 2006 against the observed value of 539 and 281 deaths per 100000 live births respectively.

Key words: model, indirect technique, ratio, parameters, mortality, deaths.

1. INTRODUCTION

Maternal mortality is an important health indicator of a country [1]. Childbearing is taken as highly valued as well as an inevitable part of a woman's life. Maternal deaths continue to be a leading cause of death during the reproduction process due to less access to quality health care [2]. Maternal deaths were mainly occurred :(i) within the pregnant period, (ii) at the time of pregnancy ended, and (iii) after giving birth of a child [3,4]. Maternal cases were often taken as hidden events and mostly were undercount. This situation is not mainly because of a lack of clarity

in defining a maternal death, but because of an inherent weakness in the health information and recording systems. Hence it is difficult to know the maternal mortality level in a country from the limited data. It is being rare occurrence events [2]. It needs a large population base and a sufficient number of observations for direct estimation of maternal deaths.

The level of maternal mortality in Nepal is known to be alarmingly high where it was 551 in 1991, 539 in 1996, 415 in 2001 and 281 per 100000 live births in 2006 [2,3,4]. However, figures show that maternal mortality ratio declined rapidly over time. Since, hospital-based studies tend to be highly localized and suffered from the problems of non-random case selection, inadequacies of sample size, and incomparable reference periods [1, 2]. Most of the births in Nepal do not take place in hospitals and therefore, the reported figures do not accurately reflect the number of deaths cases during pregnancy or childbirth [2].

Measurement of maternal mortality suffers seriously from under/over-reporting and misclassification of data [1]. Most of the surveys data were subject to wide variability and it needed to develop and graduate new technique to estimate the maternal mortality from limited data. Indirect techniques may be an appropriate tool to diagnose in such a situation. In fact, an indirect method of estimation has its origin and produce estimates of certain parameters on the basis of information, which is only related to its value indirectly [2].

Generally, estimation of demographic parameters has been done on the basis of data collected by census or by vital registration system or sample surveys [4]. Unfortunately, however, in many countries today, the data collection by these systems either do not exist or their quality is so poor [1, 2, 5]. The estimates based on such defective data yielded inconsistent results. Blum and Fergues [5] developed a technique to estimate maternal mortality ratio based on sex ratio differential and female age specific mortality rates. Devaraj *et al* [6] proposed a technique to estimate maternal mortality ratio through regression methodology by taking maternal mortality ratio as the dependent variable and infant mortality rate and total fertility rate as taking explanatory variables. Bhat *et al.* [7] proposed procedures to compute maternal mortality ratio based on sex differentials in mortality. Likewise, Aryal [1, 2] graduated the techniques to estimate the maternal mortality ratio from the data of sex differentials of mortality.

This paper tries to estimate maternal mortality ratio through different indirect techniques. The data were taken from Nepal Family Health Surveys 1996 and Nepal Demographic and Health Survey 2006 as well as other sources of data. The

suitability of model proposed by Bhat *et al.* [7] was tested to the Nepal data. Under some assumptions, the modification on this model has also been proposed.

2. MODELS

The force of female deaths rate at age x from all causes (maternal and non-maternal) of death can be expressed in the following relations.

(1) $D_f(x) = D_f^0(x) + D_f^a(x)$

Where $D_f^0(x)$ and $D_f^a(x)$ denote the female death rate at age x from obstetric causes (maternal related death rates) and other than maternity causes (non-maternal related death rates) respectively.

Since mortality risk at age x from obstetric causes of deaths would be the product of the risk of giving birth at that age and the risk of dying from giving the birth [1]. Symbolically it is given below.

(2) $D_f^0(x) = \omega(x)\, f(x)$

Where $\omega(x)$ and $f(x)$ denote the age pattern of maternal mortality and age specific fertility rate respectively.

It is well-established fact that maternal related deaths are most likely occurring at the early ages of life (before age 20 years and later ages of life (after age 35 years). Let the ratio of maternal deaths to live births at any conveniently chosen age of 20-24 be denoted as ω [1, 2, 7]. However, in 20-24 age group, maternal mortality ratio was observed minimum and out of this age interval, the risk of mortality increases linearly. In such a situation, the normalized maternal mortality ratio (as maternal mortality ratio of age 20-24 is made equal to 1) at age x would be written as:

$w(x) = \frac{\omega(x)}{\omega}$, *for* $\omega > 0$,, for ω>0, equivalently it is given as.

(3) $\omega(x) = \omega\, w(x)$

Where ω and w(x) denote the measure of the level of maternal mortality and the normalized maternal mortality ratio at age x respectively.

As from equations (2) and (3), we have the relation as: $D^0(x) = \omega\, w(x)\, f(x)$, then we substitute it to the equation (1), and we finally get the following equation.

(4) $D_f(x) = \omega\, w(x)\, f(x) + D_f^a(x)$

On simplification, we get the force of male mortality rate, $D_m(x)$, at age x from all causes of deaths as below.

$$\frac{D_f(x)}{D_m(x)} = \frac{\omega\ w(x)f(x)}{D_m(x)} + \frac{D_f^a(x)}{D_m(x)}, \quad \text{equivalently} \quad R_x = \frac{\omega\ w(x)f(x)}{D_m(x)} + \frac{D_f^a(x)}{D_m(x)} \text{ and}$$

finally we get the following model.

$$(5) \quad R_x = G(x) + \omega Z_x$$

Where, $R_x = \frac{D_f(x)}{D_m(x)}$ denotes the sex ratio of mortality risk, $G(x) = \frac{D_f^a(x)}{D_m(x)}$ denotes the sex ratio of mortality in the absence of maternal mortality and $Z_x = \frac{w_x f_x}{D_m(x)}$ {where $w_x = w(x)$ *and* $f_x = f(x)$}.

In model (5), G(x) is one of the functional parameter and ω is the slop parameter of the model and it is the level of maternal mortality at age interval 20-24.

Previous studies documented that the sex ratio of death rates among women of reproductive age in the absence of maternal mortality follows a linear function of age x [2, 7]. Then it follows the equation (5) as below.

$$(6) \quad R_x = \alpha + \beta x + \omega Z_x$$

Where α *and* β denote the regression coefficients of the model and other symbols have their usual meanings.

We can solve the equation (6) using least square principle, and once ω is estimated, an overall maternal mortality ratio (MMR) and maternal mortality rate would be computed from following relations.

$$(7) \quad MMR = \omega \frac{\sum w_x f_x n_x}{\sum f_x n_x} \quad \text{and} \quad \text{Maternal mortality rate} = \omega \frac{\sum w_x f_x n_x}{\sum n_x}$$

Where n_x denotes the number of women at age x.

Since maternal mortality is usually estimated as maternal mortality ratio (it is the number of maternal deaths from maternal causes, divided by the total number of live births and expressed per 100000 live births) and maternal mortality rates (it is the number of maternal deaths from maternal causes, divided by the number of women in the population aged 15 to 49 years and expressed per 1000 women).

Model (6) provides the better estimates of maternal mortality for Indian data [7] but it fails to provide consistent estimates of maternal mortality ratio for Nepal data [1, 2]. This may perhaps be due to the irregular pattern of sex ratio of

mortality or age specific death rate among females. Usually the sex ratio and female death rate corresponding to age interval (20-24) has been found significantly low in comparison with those of neighboring ages. Among others, a possible reason might be either the excess mortality among men due to accidents and violence or underreporting of the female deaths, and the assumption of a linear form for G(x) may be worthless [2]. Choosing G(x) is more critical for the estimation of maternal mortality ratio. Dummy variable, D, was introduced to balance such effects in switching the slope of G(x) beginning at the ages 20-24. [2, 7]. Hence the model (6) reduced to be:

$$(8) \quad R_x = \alpha + \beta x + \omega Z_x + D; \quad D = 0, \text{ for } 15-19 \text{ and } 20-24 \text{ and } D = x; \text{ otherwise.}$$

Since this model is also applied to Nepal as well as Indian data. Model gave better estimates for Indian data but it failed to give better estimates for Nepal data [1, 2, 7].

Finally keeping these limitations, we introduce the concept that the sex ratio of mortality rates (excluding the maternal causes of death) specific to women aged between 15 and 49 might be equal to some constant K.

Symbolically the sex ratio is, $\frac{D_f(x)}{D_m(x)} = G(x) = K$ and finally the model (6) reduces to the following equation.

$$(9) \quad R_x = K + \omega Z_x$$

This model may provide better estimate of MMR if there exist a different pattern in sex ratio of mortality by excluding maternal deaths of a population.

3. APPLICATION OF THE MODELS

We need the data for the application of the models (6) and (9) in order to estimate maternal mortality ratio (MMR). Data on age and sex specific death rates from all causes of deaths, and data on age specific fertility rate were taken from MOH [3,4]. The standardized age pattern of MMR was taken from Bhat *et al.* [7]. Since, this age pattern is universally accepted [1, 2, 5, 8]. The used data were presented in Tables 1 and 2.

Table 1: Data used for estimation of MMR (NFHS 1996)

Age group (x)	Re-scaled x	MDR/ $1000D_m$	FDR/ $1000D_f$	SR of D_f/D_m	ASFR f_x	No. of exposure n_x	Relative MMR* w_x	$Y=w_xf_x$	$Z=Y/D_m$

15-19	1	2.05	2.84	1.39	127	19627	1.9	241.33	117.70
20-24	2	2.37	2.29	0.97	266	20576	1.0	266.10	112.24
25-29	3	1.89	3.7	1.96	229	18107	1.2	274.80	145.40
30-34	4	2.4	3.12	1.3	160	14556	1.4	224.04	93.33
35-39	5	2.45	3.77	1.54	94	10818	1.8	169.20	69.06
40-44	6	4.52	5.13	1.13	37	6513	2.0	74.10	16.37
45-49	7	8.6	7.7	0.90	15	3964	2.4	36.03	4.19
Total					162	94,161			

MDR=male death rate, FDR=female death rate, SR=sex ratio ASFR=Age-specific fertility rate * *taken from Bhat et al.* [7]

Table 2: Data used for estimation of MMR (NDHS 2006)

Age group (x)	Re-scale x	MDR/ $1000D_m$	FDR/ $1000D_f$	SR of D_f/D_m	ASFR f_x	No. of exposure n_x	Relative MMR* w_x	$Y=w_xf_x$	$Z=Y/D_m$
15-19	1	3.03	3.44	0.88	98	22263	1.9	186.2	54.13
20-24	2	1.83	2.40	0.76	234	23427	1.0	234	97.50
25-29	3	3.16	2.43	1.30	144	20789	1.2	172.8	71.11
30-34	4	2.68	2.58	1.04	84	17953	1.4	117.6	45.58
35-39	5	2.92	2.52	1.16	48	13573	1.8	86.4	34.29
40-44	6	3.41	5.13	0.66	16	8625	2.0	32	6.24
45-49	7	3.90	5.70	0.68	2	4751	2.4	4.8	0.84
Total						111382			

* *taken from Bhat et al.*[7]

The observed values of MMR were 539 in 1996 and 281 deaths per 100000 live births (for 0-6 years before the survey) in 2006 [3,4]. MMR was derived by the maternal death cases from the survey data. The maternal deaths are defined as any death that occurred during pregnancy, childbirth, or within two months after the birth or termination of the pregnancy.

Now we fitted the models (6) and (9) to the data of two surveys of NDHS 1996 and NDHS 2006. First we take G(x) as linear form. Model (6) is fitted to the data of age intervals (15-19) to (45-49), with the sex ratio of mortality, R_x, as the dependent variable, and age x and Z_x as the explanatory variables and the fitted model is given below.

Using NFHS 1996 data in model (6), then we get the following estimated equation.

$R_x = -0.334 + 0.188\,x + 0.0112\,Z_x$; ($R^2$=0.89).

The coefficient of Z provides an estimate of MMR. Hence the estimated MMR was found to be 1120 for age interval of 20-24. As using, ω=1120, the overall MMR for women aged 15-49 came out to be 1490 deaths per 100000 live births. Model overestimates the MMR for NFHS 1996 as compared to the observed value of 539 deaths per 100000 live births [3].

Using NDHS 2006 data in model (6), then we get the following estimated equation.

$R_x = 0.738 + 0.013\,x + 0.003\,Z_x$; ($R^2$=0.38).

The coefficient of Z provides an estimate of maternal mortality level. Estimate of maternal mortality ratio was 300 for age interval of 20-24 i.e. assumed standardized age interval. With this estimate (ω = 300), the overall maternal mortality ratio for women aged 15-49 came out to be 389 deaths per 100000 live births for NDHS 2006 data. Model provides slightly overestimates of MMR for Nepal data as compared to the observed value of 281 deaths per 100000 live births [4].

Now, we fitted the model (9) and we get following estimated equation.

Using NFHS 1996 data in model (9), then we get the following estimated equation.

$R_x = 0.960 + 0.00442\,Z_x$; ($R^2$=0.41)

The model provided an estimate of MMR at age interval (20-24) of 442. Finally, an overall estimate of MMR was 588 deaths per 100000 live births for NFHS 1996 data, which is found to be very close to the observed value of MMR of 539 deaths per 100000 live births [3].

Using NDHS 2006 data in model (9), then we get the following estimated equation.

$R_x = 0.805 + 0.002701\,Z_x$; ($R^2$=0.98)

Model provided an estimate of MMR at age interval (20-24) of 270 and an overall estimate of MMR was 351 deaths per 100000 live births for NDHS 2006 data. The estimated value was also slightly over-estimates the MMR as compared to observed value of 281 deaths per 100000 live births [4]. The estimated MMR from different techniques and different data sets was given in Table 3.

Table 3: Estimated and observed MMR for the NDHS 1996 and 2006 data

Used data	Observed MMR*	Estimated MMR/100000 live births	
		Model: (6)	Model: (9)
NDHS 1996	539	1490	588
NDHS 2006	281	389	351

*MOH[3,4]

4. CONCLUSIONS

This paper discusses the model proposed by Bhat *et al.* [7]. The modification of the model was made by introducing the concept that the sex ratio of mortality rates (excluding the maternal causes of death) specific to women aged between 15 and 49 might be equal to some constant K. The suitability of the models was tested with the Nepal Demographic Surveys data of 1996 and 2006. The models provided an estimate of MMR ranges from 588 to 1490 deaths per 100000 live births for NFHS 1996 data while it ranges from 351 to 389 deaths per 100000 live births for NDHS 2006 data. However, the proposed model provided a very close estimate of MMR for both the data of Nepal. The estimated MMR was found to be 588 in 1996 and 351 deaths per 100000 live births in 2006. These estimates are consistent with the observed MMR of 539 in 1996 and 281 deaths per 100000 live births in 2006. Findings of this paper may help planners and policy-makers to design the policies for reducing maternal mortality as well as fertility in a country where maternal mortality is still a major cause of death.

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