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CONTENTS

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Asymptotic behavior of the solution of a Cauchy problem for a Sobolev type system

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Abstract: Asymptotic behavior of the solution of a Cauchy problem for a mathematical model of rotating inviscid compressible fluid is studied for a large time.

Key words: Sobolev type system, Cauchy problem, Asymptotic behavior, Bessel's function, Bessel's equation, Spherical co-ordinates, Taylor's series

1. Introduction

Our everyday life is full of examples of fluid motion, such as stirring a cup of tea, flows in rivers, waves in oceans, hurricanes and many others. The equations that describe the most fundamental behaviour of a fluid were first derived by Euler in 1755. The initial boundary value problems for the Euler equations are surprisingly difficult. Even the basic questions of existence and uniqueness of solutions in three dimensions still remain open.

The idea of wide application of mathematical models of rotating fluids to the study of atmospheric processes stems from A. Friedman, who in the early 20[th] century contributed a series of fundamental works in this direction. In connection with the investigation of a rotating body filled up with fluid, S.L. Šobolev initiated the study of the following system:

$$\frac{\partial \bar{v}}{\partial t} - [\bar{v}, \bar{\omega}] + \nabla p = \vec{F}(x,t); div\bar{v} = 0, x \in \Omega \subset \Re^3, t \geq 0.$$

Now-a-days, this system is known as Sobolev system. Different types of initial and initial-boundary value problems associated with Sobolev system have been studied by various mathematicians in the last five decades.

In a Cauchy problem, i.e. large volumes of rotating fluids (atmosphere, ocean etc.), even a numerical approach to the problem requires studying the solution's behavior as time $t \to \infty$, obtaining at least the leading term of the solution's asymptotic expansion with respect to the small parameter $1/t$ for large time t. In present work, the Cauchy problem for a Sobolev type system describing the motion of a rotating inviscid compressible fluid is taken under consideration and the asymptotic behavior of its solution is studied, as $t \to \infty$. Systems of such a kind are often employed in modeling the dynamics of atmosphere, ocean and environment protection. One of such hydrodynamic models was introduced by G. I. Marchhuk [6].
The following system of equations

(1) $$\frac{\partial \vec{v}}{\partial t} - [\vec{v}, \vec{\omega}] + \nabla p = \vec{F}(x,t); \alpha^2 \frac{\partial p}{\partial t} + div\vec{v} = \Psi(x,t)$$

is considered on the domain $D = \{x \in \Re^3, t \ge 0\}$, where

$\vec{v}(x,t) = (v_1, v_2, v_3)$ - velocity vector field of the fluid,
$\vec{\omega} = (0,0,1)$ - angular velocity of space,
$p(x,t) = (p_1, p_2, p_3)$ - hydrodynamic pressure,
$\vec{F}(x,t) = (F_1, F_2, F_3)$ - mass density of external forces,
$\alpha^2 = const$ - coefficient of compression, and
$[\vec{v}, \vec{\omega}]$ - the vector product of $\vec{v}$ and $\vec{\omega}$.

System (1) belongs to the hyperbolic type according to the classification of partial differential equations. S. L. Sobolev [1,2] was the first to study this system for the particular case $\alpha = 0$. The Cauchy problem and corresponding boundary value problems in certain domains for system (1) were studied by V. N. Maslennikova [3, 4, 5]. Various types of applications of the solutions of such types of system have been discussed by G. I. Marchhuk in his famous book [6].

2. Cauchy Problem and Explicit Solution

Let the initial conditions for the system (1) be

(2) $$\vec{v}(x,0) = \vec{v}^0(x), p(x,0) = p^0(x).$$

The explicit solution of the Cauchy problem (1),(2) was constructed in explicit form by V.N. Maslennikova. Assuming that the functions involved in the initial conditions (2) and the right hand side member of system (1) are sufficiently smooth, the solution of the system (1), (2) was obtained in the following explicit form [4]:

$$(3)\quad \vec{v}(x,t)=\frac{1}{4\pi}\iint\limits_{r=t/\alpha}\left\{-\frac{1}{r}\frac{\partial\vec{v}^0}{\partial n}+\left(\frac{1}{r^2}-\frac{\alpha^2\rho^2}{2r^2}\right)\vec{v}^0+\frac{\alpha}{r}\left([\vec{v}^0,\vec{\omega}]-\nabla p^0\right)\right\}ds_y$$

$$+\frac{1}{4\pi}\iiint\limits_{r\le t/\alpha}\left[G(x-y,t)\frac{\partial\vec{\Phi}^0(y)}{\partial y_3}+\sum_{k=1}^{3}\frac{\partial^k G(x-y,t)}{\partial t^k}\vec{\Phi}_k(y)\right]dy$$

$$+\frac{1}{4\pi}\iiint\limits_{r\le t/\alpha}\left[\int\limits_0^{t-\alpha r}G(x-y,t-\tau)\vec{f}(y,\tau)\,d\tau\right]dy$$

$$(4)\quad p(x,t)=\frac{1}{4\pi}\iint\limits_{r\le t/\alpha}\left\{-\frac{1}{r}\frac{\partial p^0(y)}{\partial n}-\frac{1}{\alpha r}div\vec{v}^0(y)+\left(\frac{1}{r^2}-\frac{\alpha^2\rho^2}{2r^2}\right)p^0(y)\right\}ds_y$$

$$+\frac{1}{4\pi}\iiint\limits_{r\le t/\alpha}\left\{G(x-y,t)\frac{\partial v_3^0}{\partial y_3}+\frac{\partial G}{\partial t}\left(\frac{\partial v_1^0}{\partial y_2}-\frac{\partial v_2^0}{\partial y_1}+\alpha^2 p^0\right)+\frac{\partial^2 G}{\partial t^2}div\vec{v}^0+\frac{\partial^3 G}{\partial t^3}\alpha^2 p^0\right\}dy$$

$$+\frac{1}{4\pi}\iiint\limits_{r\le t/\alpha}\left\{\int\limits_0^{t-\alpha r}G(x-y,t-\tau)f_4(y,\tau)\,d\tau\right\}dy.$$

The vector functions $\vec{\Phi}_k(y)$ in (3) are expressed in terms of initially given functions as follows:

$$\vec{\Phi}_0(y)=-rot\vec{v}^0+\alpha^2p^0\vec{\omega};$$
$$\vec{\Phi}_1(y)=-\Delta\vec{v}^0+\nabla div\vec{v}^0-\alpha^2[\nabla p^0,\vec{\omega}]+\alpha^2 v_3^0\vec{\omega};$$
$$\vec{\Phi}_2(y)=\alpha^2(\nabla p^0-[\vec{v}^0,\vec{\omega}]);$$
$$\vec{\Phi}_3(y)=\alpha^2\vec{v}^0.$$

The kernel G is given by $G(x-y,t)=\dfrac{1}{\rho}\displaystyle\int\limits_0^{\rho\sqrt{(t^2-\alpha^2r^2)}/r}\frac{\eta}{\sqrt{\eta^2+\alpha^2\rho^2}}J_0(\eta)\,d\eta$, where

$\rho^2=\sum\limits_{i=1}^{2}(x_i-y_i)^2$; $r^2=\rho^2+(x_3-y_3)^2$ and J_0 is the Bessel's function of order 0.

The functions $\vec{f}$ and f_4 are expressed in terms of $\vec{F}$ and ψ as follows:

$$\vec{f}(y,\tau)=-rot\frac{\partial\vec{F}}{\partial y_3}+\vec{\omega}\frac{\partial\Psi}{\partial y_3}+\Delta\frac{\partial\vec{F}}{\partial\tau}-\nabla div\left(\frac{\partial\vec{F}}{\partial\tau}\right)+$$
$$+[\nabla(\frac{\partial\Psi}{\partial\tau}),\vec{\omega}]-\alpha^2\vec{\omega}\frac{\partial F_3}{\partial\tau}+\nabla\frac{\partial^2\Psi}{\partial\tau^2}-\alpha^2\left[\frac{\partial^2\vec{F}}{\partial\tau^2},\vec{\omega}\right]-\alpha^2\frac{\partial^3\vec{F}}{\partial\tau^3};$$

$$f_4(y,\tau)=\frac{\partial F_3}{\partial y_3}+\frac{\partial^2 F_2}{\partial y_1\partial\tau}-\frac{\partial^2 F_1}{\partial y_2\partial\tau}-\frac{\partial\psi}{\partial\tau}+div\frac{\partial^2\vec{F}}{\partial\tau^2}-\frac{\partial^3\psi}{\partial\tau^3}.$$

It was proved by using the energy estimates [5] that the solution of the Cauchy problem (1), (2) is unique in $L_2(Q)$, where $Q = \{(x,t) : x \in \Re^3, 0 \le t \le T\}$.

3. Asymptotic Behavior of the Solution

In present work, the Cauchy problem (1), (2) for the corresponding homogeneous system is taken under consideration and the asymptotic behavior of its solution is determined, as $t \to \infty$. It should be noted that the solufion of the system automatically belongs to the space L_2. The representation (3), (4) permits us to obtain a series of interesting properties of solution of the Cauchy problem, mainly, the behavior of the Cauchy problem solution as $t \to \infty$.

Now, let us consider the solution (3) for the homogeneous system (1), taking $\vec{f}, f_4 \equiv 0$. The change of variable: $x - y = \xi$ produces $dy = -d\xi$ and then we integrate the first member in the second integral of (3) with respect to ξ_3 taking into account that $G = 0$ on the surface of the cone. Then, using the Bessel's equation $J_0(\eta) = -J_0''(\eta) - J_0'(\eta)/\eta$, the formula for the solution $\vec{v}(x,t)$ of our Cauchy problem corresponding to the homogeneous system can be explicitly written in the following form:

$$
(5)\quad \vec{v}(x,t) = \frac{1}{4\pi}\iint\limits_{r=t/\alpha}\left\{-\frac{1}{r}\frac{\partial \vec{v}^0}{\partial n} + \left(\frac{1}{r^2} - \frac{\alpha^2\rho^2}{2r^2}\right)\vec{v}^0 + \frac{\alpha}{r}([\vec{v}^0, \vec{\omega}] - \nabla p^0)\right\} ds_y
$$

$$
+ \frac{1}{4\pi}\iiint\limits_{r \le t/\alpha}\left\{\left(-\frac{t\xi_3}{r^3}J_0'' - \frac{t\xi_3}{r^2\xi\sqrt{t^2 - \alpha^2 r^2}}J_0'\right)\vec{\Phi}_0(x+\xi)\right.
$$

$$
+ \left(-\frac{1}{r}J_0'' - \frac{1}{\rho\sqrt{t^2 - \alpha^2 r^2}}J_0'\right)\vec{\Phi}_0(x+\xi) + \frac{\rho t}{r^2\sqrt{t^2 - \alpha^2 r^2}}J_0'\vec{\Phi}_2(x+\xi)
$$

$$
\left. + \left[\frac{\rho^2 t^2}{r^3(t^2 - \alpha^2 r^2)}J_0'' - \frac{\alpha^2\rho}{(t^2 - \alpha^2 r^2)^{3/2}}J_0'\right]\vec{\Phi}_3(x+\xi)\right\} d\xi .
$$

In (5), the omitted arguments of the Bessel's functions are each equal to $\rho\sqrt{t^2 - \alpha^2 r^2}/r$ and also $r^2 = \rho^2 + \xi_3^2, \rho^2 = \xi_1^2 + \xi_2^2$. The formula for $p(x,t)$ will have similar form as that of $\vec{v}(x,t)$ in (5). Now, we state and prove a theorem on asymptotic behavior of the solution (5), which is the main result of present work.

Basic Theorem

If the initial functions involved in (2) belong to the space C_0^∞, then the solutions $\vec{v}(x,t)$ and $p(x,t)$ of the Cauchy problem for the corresponding homogeneous system (1)

equipped with the initial conditions (2) decrease like $C(x)/t$ as $t \to \infty$, where $C(x)$ is a bounded function.

Proof of the Theorem

If the time t is sufficiently large and the given initial functions are finite, then the first integral in (5) will be zero. Let's take, into consideration, the first member in the second integral of formula (5).

We have:

$$-\frac{1}{4\pi}\iiint\limits_{r\le t/\alpha}\frac{t\xi_3}{r^3}J_0''\left(\frac{\rho\sqrt{t^2-\alpha^2r^2}}{r}\right)\vec{\Phi}_0(x+\xi)d\xi$$

$$=-\frac{1}{4\pi}\iiint\limits_{r\le t/\alpha}J_0'\left(\frac{\rho\sqrt{t^2-\alpha^2r^2}}{r}\right)\frac{\partial}{\partial\xi_3}\left(\frac{\sqrt{t^2-\alpha^2r^2}}{\rho t}\vec{\Phi}_0(x+\xi)\right)d\xi.$$

Then,

$$\frac{1}{4\pi}\iiint\limits_{r\le t/\alpha}G_0(\xi,t)\vec{\Phi}_0(x+\xi)d\xi$$

$$=-\frac{1}{4\pi}\iiint\limits_{r\le t/\alpha}\left[\frac{\partial\vec{\Phi}_0}{\partial\xi_3}\frac{\sqrt{t^2-\alpha^2r^2}}{\rho t}J_0'\left(\frac{\rho\sqrt{t^2-\alpha^2r^2}}{r}\right)\right.$$

$$\left.-\frac{\xi_3\sqrt{t^2-\alpha^2r^2}}{\rho t r^2}\vec{\Phi}_0J_0'\left(\frac{\rho\sqrt{t^2-\alpha^2r^2}}{r}\right)\right]d\xi$$

$$=Q_1+Q_2.\ \text{(Suppose)}.$$

Let us take Q_2 under consideration. Changing to spherical system of co-ordinates, we get:

$$Q_2=-\frac{1}{4\pi t}\int_0^{2\pi}d\varphi\int_0^{t/\alpha}dr\int_0^{\pi}\vec{\Phi}_0dJ_0$$

$$=\frac{1}{2t}\int_0^{t/\alpha}[\vec{\Phi}_0(x_1,x_2,x_3+r)-\vec{\Phi}_0(x_1,x_2,x_3-r)]dr+$$

$$+\frac{1}{4\pi t}\int_0^{2\pi}d\varphi\int_0^{t/\alpha}dr\int_0^{\pi}J_0\left(\sin\theta\sqrt{t^2-\alpha^2r^2}\right)\frac{\partial\vec{\Phi}_0}{\partial\theta}d\theta.$$

From this expression for Q_2, it follows that $Q_2 = C(x)/t$, where $C(x)$ is uniformly bounded as $|x| \to \infty$. Now, we take Q_1. Let the smoothly finite function $\dfrac{\partial \vec{\Phi}_0}{\partial \xi_3}$ be denoted by $\vec{\Phi}_{03}$ and we decompose it into Taylor's series in a neighborhood of $\xi_3 = 0$ with the remainder term in an integral form. To simplify the task, the first two arguments $x_1 + \xi_1$ and $x_2 + \xi_2$ of the function $\vec{\Phi}_{03}$ are omitted. We have:

$$\vec{\Phi}_{03}(x_3+\xi_3) - \vec{\Phi}_{03}(x_3) = \left(\frac{\partial \vec{\Phi}_{03}}{\partial \xi_3}\right)_{\xi_3=0} \xi_3 + \int\limits_{x_3}^{x_3+\xi_3} (x_3+\xi_3-\eta)\frac{\partial^2 \vec{\Phi}_{03}(\eta)}{\partial \eta^2} d\eta.$$

So, Q_1 can be expressed in the following form:

$$Q_1 = -\frac{1}{4\pi t}\iiint\limits_{r \le t/\alpha} \frac{\sqrt{t^2-\alpha^2 r^2}}{\rho} J_0'\left(\frac{\rho\sqrt{t^2-\alpha^2 r^2}}{r}\right)\left[\vec{\Phi}_{03}(x_3) + \left(\frac{\partial \vec{\Phi}_{03}}{\partial \xi_3}\right)_{\xi_3=0} \xi_3 + \int\limits_{x_3}^{x_3+\xi_3} (x_3+\xi_3-\eta)\frac{\partial^2 \vec{\Phi}_{03}(\eta)}{\partial \eta^2} d\eta \right] d\xi = \sum_{i=1}^{3} Q_{1i} \quad \text{(Suppose)}.$$

First, we consider the term Q_{11} for investigation.

In spherical co-ordinates, after making the substitution $\sin\theta = \gamma, d\theta = \dfrac{d\gamma}{\sqrt{1-\gamma^2}}$, we get:

$$Q_{11} = -\frac{1}{2\pi t}\int\limits_0^{2\pi} d\varphi \int\limits_0^{t/\alpha} r dr \int\limits_0^1 \frac{\sqrt{t^2-\alpha^2 r^2}\, J_0'(\gamma\sqrt{t^2-\alpha^2 r^2})}{\sqrt{1-\gamma^2}} \vec{\Phi}_{03}(x_1 + r\gamma\cos\varphi, x_2 + r\gamma\sin\varphi, x_3) d\gamma.$$

Since $\vec{\Phi}_{03}$ depends on γ, its Taylor series expansion in the neighborhood of $\gamma = 1$ gives:

$$\vec{\Phi}_{03}(\gamma) - \vec{\Phi}_{03}(1) = \left(\frac{\partial \vec{\Phi}_{03}}{\partial \gamma}\right)_{\gamma=1} (\gamma - 1) + o((\gamma-1)^2).$$

So, we can use the method of integration by parts in the improper integral involved in Q_{11} above w.r.t. the variable γ and get rid of the polynomial $\sqrt{t^2 - \alpha^2\gamma^2}$ in the numerator. The remaining integral with $\vec{\Phi}_{03}(1)$ can be obtained in explicit form. In fact,

$$Q_{111} = -\frac{1}{2\pi t}\int\limits_0^{2\pi} d\varphi \int\limits_0^{t/\alpha} r dr \int\limits_0^1 \frac{\sqrt{t^2-\alpha^2 r^2}\, J_0'\left(\gamma\sqrt{t^2-\alpha^2 r^2}\right)}{\sqrt{1-\gamma^2}} \vec{\Phi}_{03}(x_1 + r\cos\varphi, x_2 + r\sin\varphi, x_3) d\gamma.$$

But,

$$(2)\qquad \frac{1}{t}\int_0^1 \sqrt{t^2-\alpha^2r^2}\,\frac{J_0'(\gamma\sqrt{t^2-\alpha^2r^2})}{\sqrt{1-\gamma^2}}\,d\gamma=\frac{\cos\sqrt{t^2-\alpha^2r^2}-1}{t}$$

So, we have:

$$Q_{111}=\frac{1}{4\pi t}\int_0^{2\pi}d\varphi\int_0^{t/\alpha} r\left[1-\cos\sqrt{t^2-\alpha^2r^2}\,\right]\vec{\Phi}_{03}(x_1+r\cos\varphi,x_2+r\sin\varphi,x_3)\,dr.$$

In this way, we have obtained $Q_{11}=C(x)/t$. Note that for the case when $\alpha=0$ [7], it takes the form $Q_{11}=\dfrac{1-\cos t}{t}C(x)$, because of the fact that there is no delay (lag) of the argument.

Transferring to spherical co-ordinates and then integrating by parts with respect to the variable θ in the integrals Q_{12} and Q_{13}, we can see that these integrals can be also represented in the form $C(x)/t$, where in every case, $C(x)$ will be uniformly bounded with respect to $x\in\Re^3$. Conversion of the terms with potentials G_1-G_3 in similar way completes the proof of the theorem.

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Free and forced convective unsteady flow over an infinite porous surface in presence of magnetic field

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Abstract: The paper deals with the free convective oscillatory flow and heat transfer of an viscous incompressible and electrically conducting fluid past an infinite vertical porous plate in presence of magnetic field. The plate is moving in an oscillating free stream with constant suction and heat absorbing sinks. A magnetic field of uniform strength is applied in the direction normal to the plate. Using multi-parameter perturbation technique, approximate solutions have been derived for the velocity and temperature fields, mean skin-friction and mean rate of heat transfer. The findings are expected to through light on some problems of defence applications in the areas of aeronautical designs and also flow and heat transfer problems of a chemically reacting fluid.

Key Words and Phrases: Unstready Flow, MHD, Heat Transfer, Skin-friction, Free stream.

2000 Mathematical subject classification: 76D

1. Introduction

In unsteady boundary layer flow one area of study, which has received much attention, and increasing its importance in technological and physical problems, because of non-linearity of the governing equations. The study of such flow was initiated by lighthill [1] who studied the effects of free oscillations on the flow of a viscous incompressible fluid past an infinite plate. The theory was extended for free

convection boundary layers along a semi-infinite vertical plate by Nanda and Sharma [2].

The problem of unsteady free convection flow past an infinite plate with constant suction and heat sources have investigated by Pop and Soundalgekar [3]. Singh and Cowling [5] has considered the effect of magnetic field on free convective flow of electrically conducting fluids past a semi-infinite flat plate. An exact solution for the unsteady MHD problem has been derived by Sacheti, Chandran and Singh [6]. Raptis and Perdikis [4] studied the unsteady two-dimensional free convective flows through highly porous medium. Satter and Alom [7] has presented the MHD free convective flow with Hall current in a porous medium for electrolytic solution (viz. salt water). But they have neither considered the effect of constant suction nor included the heat absorbing sink and viscous dissipative term. The propagation of thermal energy through mercury and electrolytic solution in the presence of external, magnetic field and heat absorbing sinks has wide range of applications in chemical and aeronautical engineering, atomic propulsion, space science etc. In view of this Sahoo, Datta and Biswal [8] have analyzed the effects of MHD unsteady free convection flow past an infinite vertical plate with constant suction and heat sink. Recently, Ahmed *et.al.* [9] analyzed the effect of two-dimensional MHD oscillatory flow along a uniformly moving infinite vertical porous plate bounded by porous medium. Later on, the effects of unsteady free convective MHD flow through a porous medium bounded by an infinite vertical porous plate investigated by Ahmed [10].

The aim of the present paper is to investigate the effect of heat transfer in mercury ($P_r = 0.025$) and electrolytic solution ($P_r = 1.0$) past an infinite porous plate moving uniformly in an oscillating free stream with constant suction and heat sink.

2. Mathematical Analysis

Let us take the $\bar{x}$ axis along the infinite vertical plate in the upward direction and $\bar{y}$ axis is perpendicular to it into the fluid flowing with free stream velocity $\bar{U}$. In view of these, we consider that:

(i) all the fluid properties except density in the buoyancy force term are constant;

(ii) the influence of the density variations in other terms of the momentum and energy equations, and the variation of the expansion coefficient with temperature, is negligible;

(iii) the Eckert number E_c and the magnetic Reynolds number are small so that the induced magnetic field can be neglected.

(iv) all the physical variables are independent of $\bar{x}$, except possibly the pressure.

With foregoing assumptions, the equations governing the fluid flow and heat transfer are given by:

$$\frac{\partial \bar{v}}{\partial \bar{y}} = 0 \Rightarrow \bar{v} = -v_0 \text{ where } (v_0 > 0) \tag{2.1}$$

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = g\beta(\bar{T} - \bar{T}_\infty) + \frac{\partial^2 \bar{U}}{\partial \bar{t}^2} + v\frac{\partial B_0^2}{\rho}(\bar{U} - \bar{u}) \tag{2.2}$$

$$\frac{\partial \bar{T}}{\partial \bar{t}} + \bar{v}\frac{\partial \bar{T}}{\partial \bar{y}} = \kappa\frac{\partial^2 \bar{T}}{\partial \bar{y}^2} + \bar{S}(\bar{T} - \bar{T}_\infty) + \frac{v}{Cp}\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^2 \tag{2.3}$$

By Joulean Heat dissipation, the corresponding boundary conditions of the problem are:

$$\left.\begin{aligned} &\bar{u} = 0, \bar{v} = -v_0, \bar{T} = \bar{T} + \varepsilon(\bar{T}_w - \bar{T}_\infty)e^{iwt} && \text{at } \bar{y} = 0 \\ &\bar{u} \to \bar{U}, \bar{T} \to \bar{T}_\infty && \text{at } \bar{y} \to \infty \end{aligned}\right\} \tag{2.4}$$

3. Method of Solution

Introducing the following dimensionless quantities:

$$y = v_0\bar{y}/v, \quad t = \bar{t}v_0^2/4v, \quad w = 4\bar{w}v/v_0^2, \quad U = \frac{\bar{U}}{v_0}, \quad u = \bar{u}/v_0,$$

$$T = (\bar{T} - \bar{T}_\infty)/(\bar{T}_w - \bar{T}_\infty), P_r = v/\bar{\kappa}, \bar{\kappa} = \kappa/\rho C_p, S = 4v\bar{S}/v_0^2,$$

$$G = vg\beta(\bar{T} - \bar{T}_\infty)/v_0^3, \; v = \mu/\rho, \; M = \sigma B_0^2 v/\rho v_0^2, \; E = v_0^2/C_p(\bar{T} - \bar{T}_\infty),$$

where $(\bar{u}, \bar{v})$ the velocity components along $\bar{x}$ and $\bar{y}$ direction respectively, v_o the mean suction velocity, g the acceleration due to gravity, B_o the magnetic field, $\bar{t}$ the time, υ the kinematic viscosity, β the coefficient of volume expansion, μ the coefficient of viscosity, $\bar{U}$ the free stream velocity, ϖ the frequency parameter, $\bar{T}_w$ the temperature at the plate, $\bar{T}_\infty$ the free stream temperature, $\bar{T}$ the fluid temperature, P_r the Prandtl number, G the Grashoff number, S the Sink strength, and E_c the Eckert number.

On using the boundary conditions (2.4) and the dimensionless quantities, the equations (2.2) and (2.3) reduce to

$$4^{-1}u_t - u_y = 4^{-1}U_t + u_{yy} + GT + M(U - u) \tag{3.1}$$

$$P_r(4^{-1}T_t - T_y) = T_{yy} + 4^{-1}P_r ST + P_r E_c(u_y)^2 \tag{3.2}$$

The non-dimensional boundary conditions are:

$$
\left.\begin{aligned} &u=0\,, T=1+\varepsilon e^{i\omega t} && \text{at} \quad y=0 \\ &u\to U\,, T\to 0 && \text{at} \quad y\to\infty \end{aligned}\right\} \tag{3.3}
$$

To solve the equations (10.3.5) and (10.3.6) subject to the boundary conditions (10.3.7), the free stream velocity U, velocity u and temperature T in the neighbourhood of the plate are assumed to be of the form:

$$
\left.\begin{aligned} &U(t)=1+\varepsilon e^{i\omega t},\ u(y,t)=u_0(y)+\varepsilon e^{i\omega t}u_1(y) \\ &T(y,t)=T_0(y)+\varepsilon e^{i\omega t}\ T_1(y) \end{aligned}\right\} \tag{3.4}
$$

where u_0 and T_0 are respectively the mean velocity and mean temperature.

On using (3.4) into the equations (3.1) and (3.2), equating harmonic and non-harmonic terms and neglecting ε, the following set of equations are obtained:

$$u_0''+u_0'-Mu_0=-GT_0-M \tag{3.5}$$

$$T_0''+P_r\,T_0''+4^{-1}P_rS\,T_0=-P_rE_c(u_0')^2 \tag{3.6}$$

$$u_1''+u_1'-(M+4^{-1}i\omega)u_1=-GT_1-M-4^{-1}i\omega \tag{3.7}$$

$$T_1''+P_r\,T_1'+4^{-1}(S-i\omega)\,P_r\,T_1=-2P_rE_cu_0'u_1' \tag{3.8}$$

where desh denotes differentiations w.r.t.y.

The modified boundary conditions are:

$$
\left.\begin{aligned} &u_0=V\,, u_1=0\,, T_0=1\,, T_1=1 \quad \text{at} \quad y\to 0 \\ &u_0\to 1\,, u_1\to 1, T_0\to 0\,, T_1\to 0 \quad \text{at} \quad y\to\infty \end{aligned}\right\} \tag{3.9}
$$

With the help of multi-parameter perturbation technique, taking $\mathbf{E}_c \ll 1$ for all incompressible fluid and assumed that:

$$F(y)=F_0(y)+E_cF_1(y)+0(E_c^2) \tag{3.10}$$

where F stands for u_0, u_1, T_0 or T_1

On using (3.10) into the equations (3.5) to (3.8) and equating the like powers of E_c the following equations are obtained:

$$u_{00}''+u_{00}'-Mu_{00}=-GT_{00}-M \tag{3.11}$$

$$u_{11}''+u_{01}'-Mu_{01}=-GT_{01} \tag{3.12}$$

$$u_{10}''+u_{11}'-(M+i\omega/4)u_{10}=-i\omega/4-M-GT_{10} \tag{3.13}$$

(3.14) $$u_{11}'' + u_{11}' - (M + i\omega/4)\, u_{11} = -GT_{11}$$

(3.15) $$T_{00}'' + T_{00}' + 4^{-1} P_r\, S\, T_{00} = 0$$

(3.16) $$T_{01}'' + P_r T_{01}' + 4^{-1} P_r\, S\, T_{01} = -P_r (u_{00}')^2$$

(3.17) $$T_{10}'' + P_r T_{10}' + 4^{-1} P_r\, (S - i\omega)\, T_{10} = 0$$

(3.18) $$T_{11}'' + P_r T_{11}' + 4^{-1} P_r\, (S - i\omega)\, T_{11} = -2P_r u_{00}' u_{10}'$$

subject to the boundary conditions:

(3.19) $$u_{00} = 0, u_{01} = u_{10} = u_{11} = 0, T_{00} = 1, T_{01} = 0, T_{10} = 1, T_{11} = 0$$

(3.20) $$u_{00} = 1, u_{01} = 0, u_{10} = 1, u_{11} = 0, T_{00} = T_{01} = T_{10} = T_{11} = 0$$

In view of the boundary conditions (3.19) and (3.20), the solutions of the differential equations (3.11) to (3.18) are:

(3.21) $$T_{00} = e^{-A_1 y}$$

(3.22) $$u_{00} = 1 + (R_1 - 1)e^{-A_2 y} - R_1 e^{-A_1 y}$$

(3.23) $$T_{01} = R_5 e^{-A_1 y} - R_2 e^{-2A_1 y} + R_3 e^{-(A_1 + A_2) y} - R_4 e^{-2A_2 y}$$

(3.24) $$\left.\begin{aligned} u_{01} = G\Big[& R_{10}\, e^{-A_2 y} - R_6\, e^{-A_1 y} + R_7\, e^{-2A_1 y} \\ & - R_8\, e^{-(A_1 + A_2) y} + R_9\, e^{-2A_2 y} \Big] \end{aligned}\right\}$$

(3.25) $$T_{10} = e^{-A_3 y}$$

(3.26) $$u_{10} = 1 + (B_3 - 1)\, e^{-A_4 y} - B_3\, e^{-A_3 y}$$

(3.27) $$\left.\begin{aligned} T_{11} = & B_4\, e^{-A_3 y} - B_5\, e^{-(A_1 + A_3) y} + B_6\, e^{-(A_1 + A_4) y} \\ & + B_7\, e^{-(A_2 + A_3) y} - B_8\, e^{-(A_2 + A_4) y} \end{aligned}\right\}$$

(3.28) $$\left.\begin{aligned} u_{11} = & B_9\, e^{-A_4 y} - B_{10} e^{A_3 y} + B_{11}\, e^{-(A_1 + A_3) y} - B_{12}\, e^{-(A_1 + A_4) y} \\ & - B_{13}\, e^{-(A_2 + A_3) y} + B_{14}\, e^{-(A_2 + A_4) y} \end{aligned}\right\}$$

Separating real and imaginary parts of the velocity and temperature expressions (3.4) and taking only the real parts, the velocity and temperature fields in terms of the fluctuating parts are given by:

(3.29) $$u = u_0 + \varepsilon\, (M_r \cos wt - M_i \sin wt)$$

(3.30) $$T = T_0 + \varepsilon\, (T_r \cos wt - T_i \sin wt).$$

Where

$$
M_r = 1 + e^{-\lambda_2 y}\left[(P_1 + E_c P_{11} - 1)\cos \mu_2 y + (Q_1 + E_c Q_{11})\sin \mu_2 y\right]
+ e^{-\lambda_1 y}\left[(P_1 + E_c P_{12})\cos \mu_1 y + (Q_1 + E_c Q_{12})\sin \mu_1 y\right]
+ e^{-(A_1+\lambda_1)y}\left[P_{13}\cos \mu_1 y + Q_{13}\sin \mu_1 y\right]E_c
+ e^{-(A_1+\lambda_2)y}\left[P_{14}\cos \mu_2 y + Q_{14}\sin \mu_2 y\right]E_c
+ e^{-(A_2+\lambda_1)y}\left[P_{15}\cos \mu_1 y + Q_{15}\sin \mu_1 y\right]E_c
+ e^{-(A_2+\lambda_2)y}\left[P_{16}\cos \mu_2 y + Q_{16}\sin \mu_2 y\right]E_c \tag{3.31}
$$

$$
M_i = e^{-\lambda_2 y}\left[-(P_1 + E_c P_{11} - 1)\sin \mu_2 y + (Q_1 + E_c Q_{11})\cos \mu_2 y\right]
- e^{-\lambda_1 y}\left[(P_1 + E_c P_{12})\sin \mu_1 y - (Q_1 + E_c Q_{12})\cos \mu_1 y\right]
+ e^{-(A_1+\lambda_1)y}\left[-P_{13}\sin \mu_1 y + Q_{13}\cos \mu_1 y\right]E_c
+ e^{-(A_1+\lambda_2)y}\left[-P_{14}\sin \mu_2 y + Q_{14}\cos \mu_2 y\right]E_c
+ e^{-(A_2+\lambda_1)y}\left[-P_{15}\sin \mu_1 y + Q_{15}\cos \mu_1 y\right]E_c
+ e^{-(A_2+\lambda_2)y}\left[-P_{16}\sin \mu_2 y + Q_{16}\cos \mu_2 y\right]E_c \tag{3.32}
$$

$$
T_r = e^{-\lambda_1 y}\left[(1 + E_c P_2)\cos \mu_1 y + E_c Q_2 \sin \mu_1 y\right]
- e^{-(A_1+\lambda_1)y}\left[P_4\cos \mu_1 y + Q_4\sin \mu_1 y\right]E_c
+ e^{-(A_1+\lambda_2)y}\left[P_6\cos \mu_2 y + Q_6\sin \mu_2 y\right]E_c
+ e^{-(A_2+\lambda_1)y}\left[P_8\cos \mu_1 y + Q_8\sin \mu_1 y\right]E_c
- e^{-(A_2+\lambda_2)y}\left[P_{10}\cos \mu_2 y + Q_{10}\sin \mu_2 y\right]E_c \tag{3.33}
$$

$$
T_i = e^{-\lambda_1 y}\left[-(1 + E_c P_2)\sin \mu_1 y + E_c Q_2 \cos \mu_1 y\right]
- e^{-(A_1+\lambda_1)y}\left[-P_4\sin \mu_1 y + Q_4\cos \mu_1 y\right]E_c
+ e^{-(A_1+\lambda_2)y}\left[-P_6\sin \mu_2 y + Q_6\cos \mu_2 y\right]E_c
+ e^{-(A_2+\lambda_1)y}\left[-P_8\sin \mu_1 y + Q_8\cos \mu_1 y\right]E_c
- e^{-(A_2+\lambda_2)y}\left[-P_{10}\sin \mu_2 y + Q_{10}\cos \mu_2 y\right]E_c \tag{3.34}
$$

The transient velocity and temperature for $\omega t = \pi/2$ are in the form

(3.35) $$u = u_0 - \varepsilon M_i \quad \text{and} \quad T = T_0 - \varepsilon T_i$$

4. Skin-Friction and Rate of Heat Transfer

The skin-friction at the plate in dimensionless form is given by

(4.1) $$\tau_w = \left(\frac{\partial u}{\partial y}\right)_{y=0} = u_0'(0) + \varepsilon e^{iwt} u_1'(0)$$

Splitting the equation (4.1) into real and imaginary parts and taking real parts only:

(4.2) $$\tau_w = \tau_w^m + \varepsilon \,|\, \mathbf{B} \,|\, \cos(wt + \alpha)$$

where $$|\, B \,| = \sqrt{B_r^2 + B_i^2}, \alpha = \tan^{-1}(B_i/B_r), B_r = \mathrm{Re}(B) = M_r',$$

$$B_i = \mathrm{Im}\,(\mathrm{B}) = M_i' \text{ and } \tau_w^m = u_0'(0).$$

The rate of heat transfer at the plate in dimensionless form is given by

(4.3) $$q_w = \left(\frac{\partial T}{\partial t}\right)_{y=0} = T_0'(0) + \varepsilon e^{iwt}\, T_1'(0)$$

Splitting the equation (4.3) into real and imaginary parts and taking real parts only:

$$q_w = q_w^m + \varepsilon \,|\, H \,|\, \cos(wt + \beta)$$

where

$$q_w^m = T_0'(0) = -A_1(1 + E_c R_5) - E_c[-2A_1R_2 + (A_1 + A_2)R_3 - 2A_2R_4],$$

$$|\, H \,| = \sqrt{H_r^2 + H_i^2}\ , \beta = \tan^{-1}(H_i/H_r),\ H_r = \mathrm{Re}(H) = T_r', H_i = \mathrm{Im}(H) = T_i'$$

Expressions for B_r, B_i, H_r and H_i are not presented here for the sake of brevity.

5. Results and Discussion

The effects of Hartmann number (M), Grashoff number (**G**), Sink strength (**S**) and Prandtl number ($\mathbf{P_r}$) on the mean velocity ($\mathbf{u_0}$) and the transient velocity (**u**) are shown in the respective Figures **1** and **2** with other parameters are fixed. It is observed from Fig. 1 that an increase in **M** leads to an increase in the mean velocity and similar

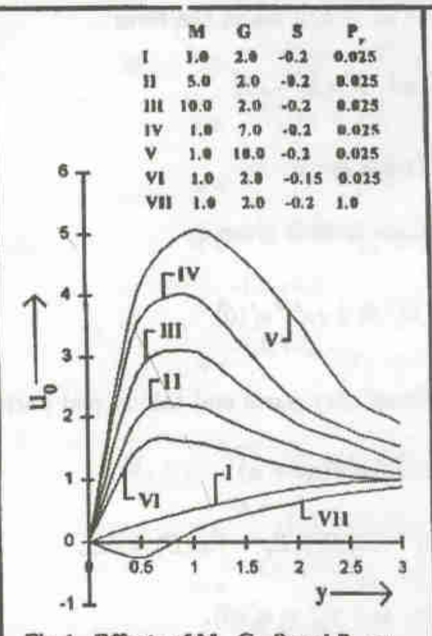

Fig.1. Effects of M , G , S and P_r on Mean velocity u_0 for $E_c = 0.005$

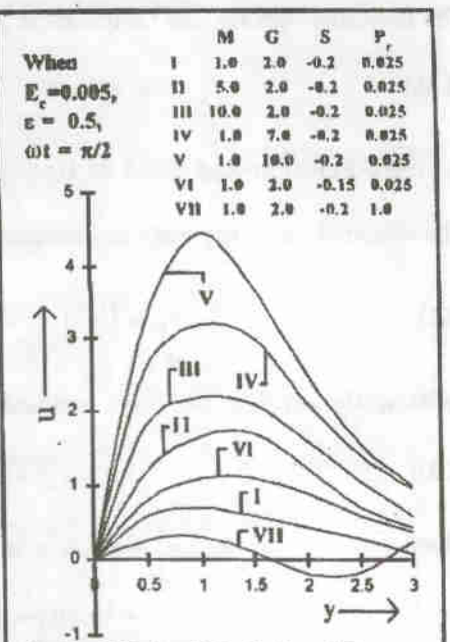

Fig.2. Effects of M , G , S and P_r on Transient velocity U

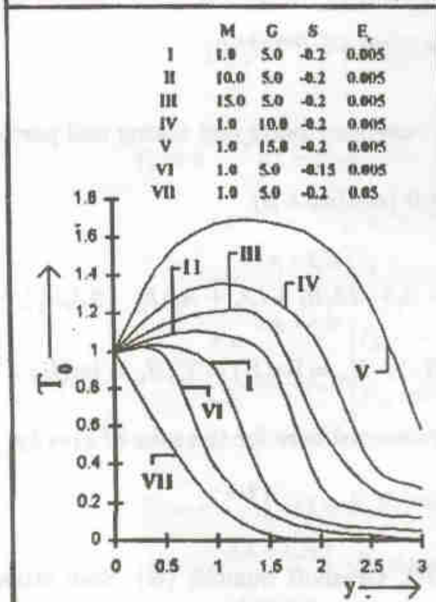

Fig.3. Effects of M , V , S and E_c on Mean Temperature T_0

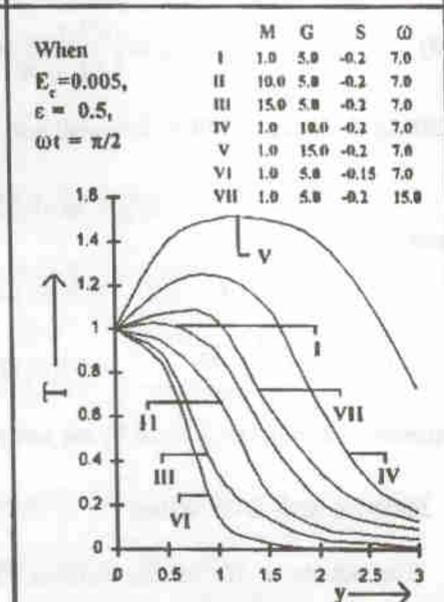

Fig.4. Effects of M , V , S and ω on Transient Temperature T

effect is marked in increasing the Grashoff number and Sink-strength; but increase in Prandtl number reduces the mean velocity. From Fig.2, it is noticed that the transient velocity increases with increase in G, S and M, whereas it decreases with increase in P_r. It is evident from Figs.1 and 2 that both mean velocity and transient velocity increase first near the plate and then the trend gets reversed as y increases.

The effects of M, G, S and E_c on the mean temperature (T_0) have been exhibited by the curves shown in Fig.3. It is noticed that increase in magnetic field and Grashoff number raises the mean temperature, whereas increase in Eckert number (E_c) and Sink strength reduces the mean temperature.

Fig. 4 displays the effects of M, G, S and ω on the transient temperature (T). The transient temperature is raising with the increase in Grashoff number (G) and frequency parameter (ω). It is also noticed that, the increase in magnetic field and Sink strength reduces T. The increase of mean velocity and transient velocity at the plate with G and M are the greatest. It is found that the mean temperature and transient temperature raises with the Grashoff number for both the cases of $P_r = 1.0$ and $P_r = 0.025$

The numerical values of mean Skin-friction (τ_w^m) and corresponding amplitude ($|B|$) and phase ($\tan\alpha$) are presented in the Table-1 for different values of M, G, S and P_r. A close study of Table-1 indicates that an increase in magnetic field increases τ_w^m, while it decreases with increase in Skin-strength, Prandtl number and Grashoff number. It is observed that, $|B|$ increases with increase in M and G, whereas increase in Skin-strength and Prandtl number decreases $|B|$. Also it is seen that, an increase in M and G decreases $\tan\alpha$; but decrease in S and P_r increases $\tan\alpha$.

The numerical values of mean heat transfer (q_w^m), amplitude $|H|$ and phase $\tan\beta$ are presented in the Table-2 for different values of M, G, S and P_r. It is observed that, the effects of increase in magnetic field, Sink-strength, Prandtl number and Grashoff number on q_w^m are reversed to τ_w^m. It is noticed that, $|H|$ increases with increase in G and S ; but increase in magnetic field and Prandtl number decreases $|H|$. The effects of increase in Grashoff number and Sink-strength on $\tan\beta$ are same as in case of amplitude $|H|$ and similar effect is noticed in increasing the Sink-strength; but decrease in M decreased $\tan\beta$.

TABLE - 1

Values of mean skin-friction (τ_w^m), amplitude ($|B|$) and phase ($\tan\alpha$) for $E_c = 0.005,\ \varepsilon = 0.5,\ \omega t = p/2,\ \omega = 3.0$:

P_r	S	G	M	t_w^m	$\|B\|$	Tanα
1.0	-0.15	5.0	1.0	0.23157	0.46228	2.38119
1.0	-0.15	5.0	10.0	0.50234	2.47351	1.57051
1.0	-0.15	5.0	15.0	0.70108	3.00117	1.32055
1.0	-0.15	10.0	1.0	0.03713	3.90164	1.82066
1.0	-0.15	15.0	1.0	-0.21057	4.61074	1.43281
1.0	-0.20	5.0	1.0	-1.19055	1.78059	2.78115
0.025	-0.15	5.0	1.0	-0.79271	1.01429	2.87014

TABLE - 1

Values of rate of heat transfer (q_w^m), amplitudes ($|H|$) and phase ($\tan\beta$) for $E_c = 0.005,\ \varepsilon = 0.5,\ \omega t = p/2,\ \omega = 3.0$:

P_r	S	G	M	q_w^m	$\|H\|$	$\tan\beta$
0.025	-0.2	2.0	1.0	1.80541	0.36324	-0.53170
0.025	-0.2	2.0	5.0	1.00245	0.18301	-0.84211
0.025	-0.2	2.0	10.0	0.68147	0.10005	-0.92391
0.025	-0.2	7.0	1.0	2.51190	0.60510	-0.20058
0.025	-0.2	12.0	1.0	2.95391	0.70221	0.12028
0.025	-0.15	2.0	1.0	0.25198	0.50322	0.61038
1.0	-0.2	2.0	1.0	-0.040231	0.27107	0.70251

Appendix

$$A_1=\frac{1}{2}\left[P_r+\sqrt{P_r^2-P_rS}\right]\ ,\quad A_2=\frac{1}{2}\left[-P_r+\sqrt{P_r^2-P_rS}\right],$$

$$B_1=\frac{1}{2}\left[1+\sqrt{1+4M}\right],\qquad B_2=\frac{1}{2}\left[-1+\sqrt{1+4M}\right],$$

$$R_1=\frac{G}{(A_1+B_1)\,(A_1-A_2)},\quad R_2=\frac{P_rA_1^2R_1^2}{(2A_1+B_1)\,A_1},$$

$$\lambda_1=\frac{P_r}{2}+\frac{1}{2}\left[\frac{1}{2}\left\{\sqrt{\left(P_r^2-P_rS\right)^2+P_r^2w^2}+\left(P_r^2-P_rS\right)\right\}\right]^{\frac{1}{2}},$$

$$\mu_1=\frac{1}{2}\left[\frac{1}{2}\left\{\sqrt{\left(P_r^2-P_rS\right)^2+P_r^2w^2}-\left(P_r^2-P_rS\right)\right\}\right]^{\frac{1}{2}},$$

$$C_1=\lambda_1^2-\mu_1^2-\lambda_1-M,\ D_1=2\mu_1\lambda_1-\mu_1-\frac{1}{4}w,$$

$$P_1=\frac{GC_1}{C_1^2+D_1^2}\ ,\ Q_1=\frac{-GD_1}{C_1^2+D_1^2},$$

$$P_2=2P_r\left[(R_1A_1-R_1A_2+A_2)\ (\lambda_1P_1-\pi_1Q_1)+(R_1A_2-R_1A_1-A_2)(\ \lambda_2P_1-\mu_2Q_1-\lambda_2)\right],$$

$$Q_2=2P_r\left[(R_1A_1-R_1A_1+A_2)\ (\mu_1P_1+\lambda_1Q_1)+(R_1A_2-R_1A_1-A_2)(\mu_2P_1-\lambda_2Q_1-\mu_2)\right],$$

and the other constants like A_1 to A_4,B_1 to B_{14},R_1 to R_{19},P_1 to P_{16},Q_1 to Q_{16},C_1 to C_8,D_1 to D_5,λ_1,λ_2, μ_1 and μ_2 are not presented here for the sake of brevity.

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Mathematical model to describe the distribution of female age at marriage

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Abstract: This paper has attempted to study the pattern of age at marriage by using a mathematical model. Model proposed by Mishra [1] has been applied and modified under some specified assumptions. The data are taken from a sample survey of Palpa and Rupandehi Districts. Other sources of data have also been utilized for testing the suitability of the model. The proposed model was found to be an appropriate model for describing the distribution of females according to age at marriage in the developing countries like Nepal and neighboring country India.

1. Introduction

Female age at marriage is an important demographic variable due to its influence on fertility especially in the developing countries with a low rate of contraceptive use. Marriage usually puts the foundation of family formation and, as such, is an important determinant of fertility associated with the duration of exposure for the risk of childbearing [2, 3]. Marriage in Nepal is universal and an early age at marriage is observed for both the males and females.

Several authors have used a number of probability models such as lognormal distribution[4], convolution of a normal and exponential distribution [5], linear function of the logarithm of a standard gamma distribution [6], two parameter log-logistic model, Gompertz curve, simple polynomials and logistic curve [7, 8, 9] convolution of two exponential distributions [10] type I extreme value distribution [11, 12, 13] to fit and graduate the distribution of females according to their age at

marriage. However, these models are conceptually difficult to understand and computation of parameters as well these models provide a large discrepancy between observed and expected values. Likewise a number of models were used in order to describe the distribution demographic parameters [14, 15, 16, 17, 18]. However a very simple mathematical model was proposed by Mishra [1] to describe the data on the age at first marriage. This simple model is applied here with some modification to study the distribution of females according to their age at marriage. In brief, it is given in the following section.

2. The data

This study is based on the data taken from a sample survey entitled "Demographic Survey on Fertility and Mobility in Rural Nepal (DSFM): A Study of Palpa and Rupandehi Districts" conducted between January and June 2000 [1]. A total of 811 households were surveyed. The data of NFHS 1996 has been utilized. More over, the data from India, UP [13] and Assam [12] have also been utilized for testing the suitability of the model.

3. Model

Let x be a variable which takes the value 0, if the female is married before attaining 12 years of age, 1, if she is married between 12–15 years, 2, if between 15–18 years, 3, if 18–21 years, and so on. The variable x can also be regarded as a number of failures preceding the first success. Researchers have not assumed constant probability of success and independence of trials by discussing the distribution of marriage and first birth [19, 20]. The probability p_i of getting a success in the $(i+1)^{\text{th}}$ trial when it is known that first i trials resulted in failure, increases as $i(i = 0, 1, 2, 3, \ldots k)$ moves from zero to a certain value s and decreases monotonically as i moves from s to a value $t(\geq s)$ and thereafter remains constant. That is, the probability that the female is married in i^{th} age group (i.e. $x = i$, for $i = 0, 1, 2, 3, \ldots k$) then,

$$P(x=i) = P(x \geq i) * P(x=i / x \geq i) \tag{1}$$

In other words, the probability can be expressed as the product of the probabilities that the same did not marry in the preceding i age groups i.e. failure $(1-p_i)$ and the same marries in the i^{th} age-group given that she did not marry in the preceding i^{th} age-groups i.e. success (p_i). i.e.

(2) $P(x=i/x\geq i)=p_i$ and $P(x=i)=1-p_i=q_i$

and $P(x=0)=p_0$; as $P(x\geq 0)=1$,

$$P(x=1)=q_0\,p_1$$
$$P(x=2)=q_0\,p_1\,p_2$$
$$P(x=3)=q_0\,q_1\,q_2\,p_3$$
$$\cdots$$
$$\cdots$$
$$P(x=k)=q_0\,q_1\,q_2\,\cdots\quad\cdots\,q_{k-1}\,p_k$$
$$P(x\geq k)=q_0\,q_1\,q_2\,\cdots\quad\cdots\,q_{k-1}\,q_k$$

The different probabilities $p_0, p_1, p_2, \ldots$ are defined in the following ways,

$$p_0=a$$
$$p_1=a+b$$
$$p_2=a+rb \text{ and}$$

(3) $p_i=a+(r+c)b$ for all $i=3, 4, 5, \ldots k$.

where a,b,c and r are the four parameters of the model.

However, in some societies the probability $p_i(i=3,4,5, \ldots k)$ is found not constant due to the occurrence of some marriages at the late ages and it decreases as i increases. For this purpose a number of decreasing factors have been tried for getting the decreasing probability p_i after $i=4,5, \ldots, k$ and it was found that an appropriate decreasing factor may be $c/(i-3)$. Thus replacing c by $c/(i-3)$ in equation (3) we get

(4) $p_i=a+\{r+c/(i-3)\}b$,

i.e. start decreasing from $i=5$, where $i=4,5, \ldots, k$ and $i\neq 3$ and the respective different probabilities are:

(4a) $p_4=a+(r+c)\,b$

(4b) $p_5=a+(r+c/2)\,b$

(4c) $p_6=a+(r+c/3)\,b$

(4d) $p_7=a+(r+c/4)\,b$, and so on.

It is observed that the model (4) has four parameters and the expected frequencies would be exactly equal to the observed frequencies for $x=0,1,2$. The difference between the observed and expected frequencies would start from x=3

onwards. Thus, for a small value of k, the model (4) may not be appropriate. Model (4) is now modified as

(5) $\quad p_i = a + (r + b)r$ for all $i = 3, 4, 5, \ldots k$.

This model has only three parameters, and these have been estimated by using iteration or the maximum likelihood method. For this, Mishra [1] has given a likelihood function as,

(6) $$L = a^{f_0}(1-a)^{N-f_0}(a+b)^{f_1}(1-a-b)^{N-f_0-f_1}(a+rb)^{f_2}(1-a-rb)^{N-f_0-f_1-f_2}$$
$$\{a+(r+c)b\}^{\Sigma f_i}\{1-a-(r+c)b\}^{T}$$

where $f_0, f_1, f_2, \ldots f_{k-1}, f_k$, are the respective frequencies for $i = 0, 1, 2, \ldots k-1, k$; $N = f_0 + f_1 + f_2 + \ldots + f_k$ and $T = f_4 + 3f_5$

Taking logarithms in (3) and solving, we get the following estimating equations:

(6a) $\quad f_0 / a - (N - f_0)/(1-a) = 0$

(6b) $\quad f_1 /(a+b) - (N - f_0 - f_1)/(1-a-b) = 0$

(6c) $\quad f_2 /(a+rb) - (N - f_0 - f_1 - f_2)/(1-a-rb) = 0$

(6d) $\quad \Sigma f_i /[a+(r+c)b] - T/[1-a-(r+c)b] = 0$

Solving above equations one can estimate the parameters (a, b, c and r) of the models (3) and (4). Similar procedure has also been used to estimate the parameters (a, b and r) of the model (5).

4. Applications

The proposed model has been fitted to the data on age at marriage for females residing in *Hills*, *Tarai*, and rural Nepal (Tables 1 and 2). Data of NFHS 1996 as well as the data from UP, India [13] and Assam, India [12] have also been tested (Tables 2 and 3). The chi-square values suggest that the models (3), (4) and (5) fit well to all the data set on age at marriage. Chi-square values suggested that the proposed model was more powerful to describe the distribution of females according to the age at marriage in Nepal and India. Mean age at marriage was found 17.0 and 16.5 years for females residing in *Hills* and *Tarai* respectively whereas it was 16.7 years for females residing in rural Nepal. The mean age at marriage was 16.4 years for the NFHS 1996 data.

Table 1 Observed and expected distribution of female age at marriage.

Re-scale	Hill, DSFM, 2000				Tarai, DSFM, 2000			
age	Obs	Exp.			Obs.	Exp.		
		(3)	(4)	(5)		(3)	(4)	(5)
0	10	10.00	10.00	10.00	26	26.00	26.00	26.00
1	82	82.00	82.00	82.00	104	104.00	104.00	104.00
2	197	197.00	197.00	205.86	206	206.00	206.00	205.60
3	103	114.58	115.11	106.45	166	166.88	167.31	163.92
4	50	42.56	42.76	41.19	35	36.20	36.28	38.09
5	19	15.81	14.53	15.94	8	7.85	6.41	8.85
6	6	5.87	5.99	6.17	2	1.70	2.15	2.06
7	3	2.18	2.61	2.39	2	0.37	0.85	0.48
Total	470	470.00	470.00	470.00	549	549.00	549.00	549.00
χ^2.		3.22	3.87	2.29		0.484	0.765	0.131
d.f.		2	2	3		1	1	2
Parameter								
a		0.021277		0.021277		0.047359		0.047359
b		0.156984		0.156984		0.151494		0.151494
r		3.184320		3.293200		2.932711		2.925300
c		0.683560		-		1.923783		-

Table 2 Observed and expected distribution of female age at marriage.

Rescale	Rural Nepal, DSFM, 2000				NEPAL, NFHS, 1996			
age	Obs.	Exp.				Exp		
		(3)	(4)	(5)	Obs.	(3)	(4)	(5)
0	36	36.00	36.00	36.00	309	309.00	309.00	309.00
1	186	186.00	186.00	186.00	2189	2189.00	2189.00	2189.00
2	403	403.00	403.00	422.90	3741	3741.00	3741.00	3701.74
3	269	277.84	279.05	241.78	1549	1568.09	1568.09	1598.01
4	85	82.36	82.72	86.44	450	445.89	446.15	453.12
5	27	24.41	21.07	30.89	139	126.80	119.34	128.44
6	8	7.24	7.88	11.04	39	36.05	38.18	36.43
7	5	2.15	3.28	3.95	6	10.25	12.88	10.33
8	-	-	-	-	7	2.92	4.45	2.93
Total	1019	1019.0	1019.0	1019.00	8429	8429.00	84296.00	8429.00
χ^2		4.52	2.98	5.63		9.22	8.87	9.89
d.f.		3	3	4		4	4	5
Parameter								
a		0.035329		0.035329		0.036666		0.03666
b		0.153888		0.153888		0.232922		0.23292
r		3.056233		3.209100		2.550610		2.52165
c		1.286165		-		0.36450		-

Table 3: Observed and expected distribution of female age at marriage

Rescale	INDIA, UP (Sinha,1998)				India, Assam (Nath & Talukdar, 1992)			
Age		Exp.				Exp.		
	Obs.	(3)	(4)	(5)	Obs.	(3)	(4)	(5)
0	247	247.00	247.00	247.00	221	221.00	221.00	221.00
1	564	564.00	564.00	564.00	694	694.00	694.00	694.00
2	962	962.00	962.00	960.86	240	240.00	240.00	235.28
3	776	772.93	772.95	768.85	105	102.57	102.81	111.03
4	182	195.07	194.02	196.74	21	25.50	25.56	19.83
5	56	47.43	44.79	50.37	8	6.34	5.88	7.17
6	15	12.09	14.23	12.88	2	1.58	1.75	2.69
7	2	3.48	5.01	3.30	-	-	-	-
Total	2804	2804.00	2804.00	2804.00	1291	1291.0	1291.00	1291.00
$\chi 2$		2.93	3.19	1.84		1.54	1.49	0.69
d.f.		2	2	3		1	1	2
Parameter								
a		0.088090		0.088090		0.171185		0.17118
b		0.132483		0.132483		0.477413		0.47741
r		2.978513		2.97420		0.978425		0.94455
c		1.991741		-		0.236867		-

5. Conclusions

The model proposed for describing the distribution of females according to age at marriage in Nepal was found an appropriate distribution. The proposed model is also fit well to the data of India. Hence the proposed model may be used to describe the distribution of females according to age at marriage in the developing countries like Nepal, India, Bangladesh, etc. The model also provided the average age at marriage of about 18 and 17 years for females residing in *Hills* and *Tarai* respectively and 17 years for females residing in rural Nepal, which was found very close with the median age at marriage while computed using all females (married as well as unmarried females) [2].

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On certain Köthe-Toeplitz duals

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Abstract In this chapter we deal on the results corresponding to Dutta, Srivastava, Gurusingh and others [6, 7]. Promising from this and based on these results we introduce a general sequence space X_t where X is any sequence space. We establish some inclusion relations, topological results and characterize $\alpha-$, $\beta-$ and $\gamma-$ duals of X_t in terms of the $\alpha-$, $\beta-$ and $\gamma-$ duals of X. Furthermore we characterize the Köthe Toeplitz duals and discuss the perfectness of these sequence spaces.

Key words: Duals, Kothe-Toeplitz duals, perfectness, separable space, sequence space.
AMS subject classification:40C10,40D25,40G05,40H05,40C05.

1. Introduction

The following definitions and notations will be useful in our discussion and presentation.

$$\ell_\infty = \text{The space of all bounded sequences}$$
$$= \left\{ x = (x_k) : \sup_k |x_x| < \infty \right\},$$
$$c = \text{The space of all convergent sequences}$$
$$= \left\{ x = (x_k) : |x_k - \ell| \to 0 \text{ for some } \ell \,\varepsilon\, C \right\},$$

$c_0 =$ The space of all null sequences

$= \left\{x = (x_k) : |x_k| \to 0 (k \to \infty)\right\}$, and

$\ell_p = (1 \le p < \infty)$. The space of sequences $x = (x_k)$ with absolutely

$p =$ summable series.

If $p = (p_k)$ is a bounded sequence of strictly positive real numbers then;

$$\ell_\infty(p) = \left\{x = (x_k) : \sup_k |x_x|^{p_k} < \infty\right\},$$

$$= \left\{x = (x_k) : |x_k|^{p_k} \to 0 \text{ as } k \to \infty\right\},$$

$$c(p) = \left\{x = (x_k) : |x_k - \ell|^{p_k} \to 0 \text{ for some } k \to C\right\},$$

$$\ell(p) = \left\{x = (x_k) : \sum_k |x_k|^{p_k} < \infty\right\},$$

$$c = \left\{x = \{x_k\} : |x_k| \to \ell(k \to \infty), \text{ for some } \ell \varepsilon C\right\}, \text{ and}$$

and $$c_s(p) = \left\{x = (x_k) = \frac{1}{n} \sum_{k=1}^{n} |x_k - \ell|^{p_k} \to 0 \text{ for some } \ell \varepsilon C\right\}.$$

Let $t = (t_k)$ be any fixed sequence of nonzero complex numbers satisfying

$$\operatorname{Lim}_k \inf (t_k)^{1/k} = r(0 < r \le \infty)$$

and let X be any sequence space. Then we define X_t by

$$X_t = \{x = (x_k) : (t_k x_k) \varepsilon X\}$$

For detailed discussion on these spaces we refer [1, 2, 3, 4, 5, 9, 10,12] to the reader. In this chapter we give some topological relations between X and (X_t), and also we give the $\alpha-$, $\beta-$ and $\gamma-$duals of X_t in terms of the $\alpha-$, β and $\gamma-$ duals of X.

2 Some Topological Properties of X_t

In this section we give some topological relations between X_t and X, and we discuss some properties of X_t.

Theorem 2.1 *If X is a complete paranormed space, then X_t is also a complete paranormed space.*

Proof: Since $(0)\,\varepsilon\, X_t, X_t \neq 0$. It is easy to check that X_t is a linear space. And also it is clear that the function g^* is defined by

$$g^*(x) = g(tx)$$

where g is the paranorm in X, is a paranorm because $g^*(0) = 0$, $g^*(x) = g^*(-x)$ and $g^*(x+y) \leq g^*(x) + g^*(y)$. Now clearly $\lambda_n \to \lambda$ in C and $g^*(x-x) \to 0$ as $n \to \infty$ imply that $g^*(\lambda_n x^n - \lambda x) \to 0$ as $(n \to \infty)$, where $x^n = (x_k^n)_k = (x_1^n, x_2^n, x_3^n, \ldots, x_k^n, \ldots)$ and $x = (x_k)$.

To show that X_t is complete, let (x^n) be a Cauchy sequence in X_t, where $x^n = (x_1^n, x_2^n, \ldots)\,\varepsilon\, X_t$. Then $(tx^n) = \left((t_k x_k^1), (t_k x_k^2), \ldots\right)$ is a Cauchy sequence in X. Since X is complete, it converges to (z_k) say. Let $z_k = t_k z_k$, so that $x_k = t_k^{-1} z_k$. Then (tx^n) converges to $(t_k x_k)$ in X.

Hence,

$$g(t_k x_k^n) = (t_k x_k) = g\left(t(x^n - x)\right) \to 0$$

$\left(g(t_k x_k^n) - (t_k x_k)\right) = g\left(t(x^n - x)\right) \to 0)$ as $n \to \infty$, which implies that

$$g^*(x^n - x) \to 0 \text{ as } n \to \infty$$

Therefore x^n is convergent, consequently X_t is a complete paranormed space. This completes the proof of the theorem.

Corollary 2.1: *If X is a Banach space, then so is X_t. Here the norm in X_t is defined by $\| x \|_{(t)} = \| (t_k x_k) \|$, where $\| \ \|$ is the norm in X.*

Corollary 2.1

(i) $(\ell_\infty)_t (c_0)_t (c)_t$ are BK space with the norm $\| x \|_t = \| t_k x_k \|$

Lemma 2.1 If $X \subset Y$ then $X_t \subset Y_t$ and $X_t^* \subset Y_t^*$, where $X_t^* = \{x = (x_k) : (x_k t_k^{-1})\,\varepsilon X\}$,

(i) $(\cup_i X_i)_t = \cup_i (X_i)_t$

(ii) $(\cap_i Y_i)_t = \cap_i (Y_i)_t$

The proof of this Lemma 1 is easy and hence omitted.

Theorem 2.2 *Let X be a complete paranormed space and let Z be a closed subset of X then Z_t is a closed subset of X_t.*

Proof: Since $Z \subset X$, $Z_t \subset X_t$ by Lemma-1. Now let $x \varepsilon (\overline{Z_t})$, then there exists a sequence $(x^n) \subset Z_t$ such that (x^n) converges to x. This implies that $g^*(x^n - x) = g^*((t_k x_k)^n - (x_k)) \to 0$ as $n \to \infty$ in Z_t. Thus $g((t_k x_k^n) - (t_k x_k)) \to 0$ as $n \to \infty$ in Z. Hence $(t_k x_k)$ is the limit of a sequence of points in Z.
Therefore $(t_k x_k) \varepsilon \overline{Z}$ which gives that $x \varepsilon (\overline{Z_t})$. Conversely if $x \varepsilon (\overline{Z})_t$ then $x \varepsilon (\overline{Z})_t$, since Z is closed that is $\overline{Z} = Z$. Therefore $(\overline{Z}_t) = (\overline{Z})_t = Z_t$, hence Z_t is closed in X_t.
This completes the proof of the theorem.

Corollary 2.3: *Let X be a Banach space and Z be a closed subset of X. Then Z_t is a closed subset of X_t.*

Theorem 2.3 *If X is a separable space, so is X_t.*

Proof: Let X be a separable space. Then there exists a countable subset Z of X such that $\overline{Z} = X$. Then $(\overline{Z}_t) = X_t$ by Theorem 4.2.2. Hence Z_t is dense in X_t. Let us define $f : Z_t \to Z$ by $f(x) = (t_k x_k)$. It is clear that f is bijective. Since Z is countable, Z_t is also a countable subset of X_t. Hence X_t is separable.
This completes the proof of the theorem.

Theorem 2.4 *If X is a Hilbert space then X_t is also a Hilbert space.*

Proof: Let X be a Hilbert space. If we define the inner product $\langle \ \rangle_t$ in X_t by

$$\langle x, y \rangle_t = \langle (t_k x_k), (t_k y_k) \rangle, \quad (x, y \varepsilon X_t)$$

Where $\langle \ \rangle$ denotes the inner product in X. It is easily seen that $\langle \ \rangle_t$ satisfies the conditions of inner product, so X_t is an inner product space and hence X_t is a Hilbert space.

Remarks: X_t need not to be a sequence algebra even if X is so. Indeed, it is known that c_0 is a sequence algebra. But $(c_0)_t$ is not a sequence algebra for $(t_k) = (1/k)$. For let $x = (\sqrt{k})$ and $y = (\lambda\sqrt{k})$, where $\lambda \varepsilon C$ is a constant. Then $x, y \varepsilon (c_0)_t$ but $z \notin (c_0)_t$, where $z = (x_k y_k)$.

3 Köthe Toeplitz Duals of X_t

In this section first we give the α-duals of $\ell_\infty(p), c_0(p), c(p)$ and $\ell(p)$, and we discuss the second duals and perfectness of some sequence spaces. Then we characterize the $\alpha-$, $\beta-$ and $\gamma-$ duals of X_t in terms of the $\alpha-$, $\beta-$ and $\gamma-$ duals of X respectively.

We also discuss the second duals and perfectness of X_t for various sequence spaces X. It may be noted here that $\beta-$duals of $\ell_\infty(p), c_0(p), c(p)$ and $\ell(p)$ have been characterized in [6,7].

Definition 3.1 Let X be a sequence space and define

(i) $X^\alpha = \left\{ a = (a_k) : \sum_k |a_k x_k| < \infty \text{ for all } x \varepsilon X \right\}$,

(ii) $X^\beta = \left\{ a = (a_k) : \sum_k a_k x_k \text{ converges for all } x \varepsilon X \right\}$,

(iii) $X^\gamma = \{ a = (a_k) : \sup_n \left| \sum_{k=1}^{\infty} a_k x_k \right| < \infty \text{ for all } x \varepsilon X \}$

Then X^α, X^β, and X^γ are called the $\alpha-, \beta-$and $\gamma-$dual spaces of X respectively. X^α is also called Köthe- Toeplitz dual space and X^β is called the generalized Köthe-Toeplitz dual space. It is easy to show that $\phi \subset X^\alpha \subset X^\beta \subset X^\gamma$. If $X \subset Y$, then $Y^\eta \subset X^\eta$ for $\eta = \alpha, \beta$ or γ. Also for a sequence space X it is clear that $X \subset (X^\eta)^\eta = X^{\eta\eta}$, where $\eta = \alpha, \beta$ or γ.

Definition 3.2 For a sequence space X, if $X = X^{\eta\eta}$, then X is called an η-space, where $\eta = \alpha, \beta$ or γ. In particular, an α-space is called Köthe space or a perfect sequence space.

Theorem 3.1 *Let η denote α, β or γ. Then*

(i) If the η-dual X^η exists, then $(X_t)^\eta$ exists and

$$(X_t)^\eta = \left\{ a = \langle a_k \rangle : \left\langle \frac{a_k}{t_k} \right\rangle \varepsilon X^\eta \right\} = \langle X^\eta \rangle_t$$

(ii) If $X^{\eta\eta}$ exists, then $(X_t)^{\eta\eta}$ exists and

$$(X_t)^{\eta\eta} = \left\{ x = (x_k) : (t_k x_k) \varepsilon X^{\eta\eta} \right\} = (X^{\eta\eta})_t$$

Proof: Let $\eta = \alpha$ and $D = \{ a = (a_k) : (a_k / t_k) \varepsilon X^\alpha \}$. We show that $(X_t)^\alpha = D$. Let $a \varepsilon (X_t)^\alpha$ then $\Sigma_k |a_k x_k| < \infty$ for every $x \varepsilon X_t$ so that

$$\sum_k \left| \frac{a_k}{t_k} t_k x_k \right| = \sum_k |a_k x_k| < \infty .$$

Since $(t_k x_k) \varepsilon X$ it follows that $(a_k / t_k) \varepsilon X^\alpha$ which implies that $a \varepsilon D$. Hence

$$(X_t)^\alpha \subset D.$$

Conversely, if $a \varepsilon D$ and $x \varepsilon X_t$, then $(a_k / t_k) \varepsilon X^\alpha$ and $(t_k x_k) \varepsilon X$ so that

$$\sum_k |a_k x_k| = \sum_k \left| \frac{a_k}{t_k} t_k x_k \right| < \infty$$

As $a \varepsilon X_t$ it follows that $a \varepsilon (X_t)^\alpha$. Hence $D \varepsilon (X_t)^\alpha$.

Consequently $(X_t)^\alpha = (X^\alpha)_t$.

For $\eta = \beta$ and $\eta = \gamma$ the proofs are similar. Therefore we omit them.

(i) Let $\eta = \alpha$ and let $X^{\alpha\alpha}$ exist. Then

$$\begin{aligned}(X_t)^{\alpha\alpha} &= [(X_t)^\alpha]^\alpha \\ &= [(X^\alpha)_t]^\alpha \\ &= (X^{\alpha\alpha})_t\end{aligned}$$

For $\eta = \beta$ and $\eta = \gamma$ the proof follows as in (i).

This completes the proof of the theorem.

Theorem 3.2 *X_t is an η-space if and only if X is an η-space, where $\eta = \alpha, \beta$ or γ.*

Proof: Let X be an η-space. Then $X^{\alpha\alpha} = X$. Now $(X_t)^{\alpha\alpha} = (X^{\alpha\alpha})_t$ by Theorem 4.3.1 and hence $(X_t)^{\eta\eta} = X_t$. Thus X_t is an η-space.

Conversely, if X_t is an η-space then $(X_t)^{\eta\eta} = X_t$ which implies that $(X^{\eta\eta})_t = X_t$ by Theorem 3.1 (ii). From Lemma (1) it follows that $X^{\eta\eta} = X$ that is, X is an η-space.

Theorem 3.3. *Let $0 < p_k \le 1$ for every k. Then the following statements are equivalent:*

(i) $\ell(p)$ is perfect,

(ii) $[\ell(p)]_t$ is perfect,

(iii) $\ell(p) = \ell_1$

Proof: (i) is equivalent to (ii) by Theorem 3.2. We show that (i) is equivalent to (iii). It is easy to show that (iii) implies (i). Now suppose that (i) holds, that is $\ell^{\alpha\alpha}(p) = \ell(p)$ since $\ell^\alpha(p) = \ell_\infty(p)$ then $\ell^{\alpha\alpha}(p) = \ell^{\alpha\alpha}_\infty(p) = M_\infty(p)$. We shall show that $M_\infty(p) = \ell(p)$ implies inf $p_k > 0$. Suppose that $M_\infty(p) = \ell(p)$ but inf $p_k = 0$. Then there exists a strictly increasing sequence (k_i) of positive integers such that $p_{k_i} < i^{-1}$. We put

$$a_k = \begin{cases} 0 & \text{if } k \neq k_i \\ i^{-1/p_k} & \text{if } k = k_i \end{cases} \quad (i = 1,2,\ldots) \qquad (*)$$

Then for every $N>1$ we have, for $i > 2N$, $|a_k|^{p_k} = i^{-1}$ and $|a_k| N^{1/p_k} < i^{-1}$ where $k = k_i$ by (*). Therefore $a \varepsilon M_\infty(p) - \ell(p)$, contrary to the assumption that $M_\infty(p) - \ell(p)$. Hence inf $p_k > 0$, which gives us $\ell(p) = \ell_1$. It is easy to check that $M_\infty(p) = \ell_1$ if and only if inf $p_k > 0$. This completes the proof of the theorem.

Theorem 3.4 *For every* $p = (p_k)$ *we have*

(i) $c_0^\alpha(p) = M_0(p)$, where

$$M_0(p) = \bigcup_{N>1} \left\{ a = (a_k) : \sum_k |a_k| N^{-1/p_k} < \infty \right\},$$

(ii) $$[(c_0(p))_t]^\eta = \bigcup_{N>1} \left\{ a = (a_k) : \sum_k \left| \frac{a_k}{t_k} \right| N^{-1/p_k} < \infty \right\}$$

Where $\eta = \alpha$ or β,

(iii) $$\left[c_0(p)_t \right]^{\eta\eta} = \bigcap_{N>1} \left\{ x = (x_k) : \sup_k | t_k x_k | N^{1/p_k} < \infty \right\},$$

where $c_0^{\alpha\alpha}(p) = E_0$, where

$$E_0 = \bigcap_{N>1} \left\{ x = (x_k) : \sup_k |x_k| N^{1/p_k} < \infty \right\}$$

for $\eta = \alpha$ or β,

The following conditions are equivalent:

(a) $c_0(p)$ **is perfect,**

(b) $[c_0(p)]_t$ **is perfect,**

(c) $p \varepsilon c_0$.

Proof

(i) Let $a \varepsilon M_0(p)$ and $x \varepsilon c_0(p)$. Then $\Sigma_k |a_k| N^{-1/p_k} < \infty$ for some $N > 1$ and $|x_k|^{p_k} < N^{-1}$ for sufficiently large k; whence for such k it follows that

$$| a_k x_k | \leq | a_k | N^{-1/p_k}$$

so $\Sigma_k | a_k x_k | \leq \Sigma_k | a_k | N^{-1/p_k} < \infty$ and hence $M(p) \subset c_0^\alpha(p)$.
Since $c_0^\alpha(p) \subset c_0^\beta(p)$ it follows that $c_0^\alpha(p) = M_0(p)$ by Theorem (3.2) in [6,7].

(ii) Proof follows from (i), Theorem (3.1(i)) and Theorem (3.2) in [6,7].

(iii) Let $a \varepsilon E_0$ and $x \varepsilon c_0^{\alpha}(p)$. Then for every $N>1$, $|a_k|N^{1/p_k} \leq K$ for all k and for some $K>0$, and $\Sigma_k |x_k| N^{1/p_k} < \infty$ for some $N>1$. Hence $|a_k x_k| \leq K|x_k|N^{-1/p_k}$ which implies that $\sum_k |a_k x_k| \leq K \sum_k |x_k| N^{-1/p_k} < \infty$

Consequently $a \varepsilon c_0^{\alpha\alpha}(p)$, whence $E_0 \subset c_0^{\alpha\alpha}(p)$. Since $c_0^{\alpha}(p) = c_0^{\beta}(p)$ by (i), it follows that $c_0^{\alpha\alpha}(p) \subset c_0^{\beta\beta}(p)$.

Then we have $c_0^{\alpha\alpha}(p) = E_0$, since $c_0^{\beta\beta}(p) = E_0$ by Theorem 3.2 and Theorem 3.1(ii) give us (iii). (iv) (a) is equivalent to (b) by Theorem (3.2). Since $c_0(p)$ is a β-space if and only if $p \varepsilon c_0$ and since $c_0^{\alpha\alpha}(p) = c_0^{\beta\beta}(p)$ by (c) the equivalence of (a) and (c) is immediate.

This completes the proof of the theorem.

Theorem 3.5. *For every* $p = (p_k)$ *we have*

(i) $\ell_0^{\alpha\alpha}(p) = M_\infty(p)$, where

$$M_\infty(p) = \bigcap_{N>1} \left\{ a = (a_k) : \sum_k |a_k| N^{1/p_k} < \infty \right\},$$

(ii) $$[\ell_\infty(p)_t]^{\eta} = \bigcap_{N>1} \left\{ a = (a_k) : \sum_k \left| \frac{a_k}{t_k} \right| N^{1/P_k} < \infty \right\},$$

where $[\ell_\infty(p)_t]^{\eta} = \bigcap_{N>1} \left\{ a = (a_k) : \sum_k |a_k| N^{1/P_k} < \infty \right\}$ for $\eta = \alpha$ or β

(iii) $\ell_\infty^{\alpha\alpha}(p) = E_\infty$, where $E_\infty = \bigcup_{N>1} \left\{ x = (x_k) : \sup_k |x_k| N^{-1/P_k} < \infty \right\}$,

(iv) $$[\langle \ell_\infty(p) \rangle_t]^{\eta\eta} = \bigcup_{N>1} \left\{ x = (x_k) : \sup_k |t_k x_k| N^{-1/P_k} < \infty \right\},$$

Where $\eta = \alpha$ or β

(v) The following conditions are equivalent:

(1) $p \varepsilon \ell_\infty$

(2) $\ell_\infty(p)$ is perfect,

(3) $[\ell_\infty(p)]_t$ is perfect

Proof (i): It is similar to the proof of Theorem 3.4(i), since $\ell_\infty^{\beta}(p) = M_\infty(p)$ Theorem 3.1(i) give us (ii).The proof of (iii) is similar to the proof of Theorem 3.4(iii), since

$\ell_{\infty}^{\beta\beta}(p) = E_{\infty}$. Theorem 3.1(ii) give us (iv). The proof of (v) is similar to the proof of Theorem 3.4(iv).

This completes the proof of the theorem.

Theorem 3.6 *For every* $p = (p_k)$ *we have*

(i) $c^{\alpha}(p) = c_0^{\alpha}(p) \cap \ell_1$,

(ii) $\{\langle c(p)\rangle_t]^{\alpha} = [M_0(p)]_t^* \cap (\ell_1)_t^*$,

(iii) $[\langle c(p)\rangle_t]^{\beta} = [M_0(p)]_t^* \cap \gamma_t^*$,

Where $\gamma = \left\{ a = (a_k) : \sum_k a_k \text{ converges} \right\}$.

Proof (i) : Let $a \varepsilon c^{\alpha}(p) \cap \ell_1$ and $x \varepsilon c(p), |x_k - \ell|^{p_k} \to 0 (k \to \infty)$. Then $\sum_k |a_k| < \infty$ and since $x \varepsilon c(p), (x_k - \ell) \varepsilon c_0(p)$ and hence

$$\sum_k |a_k(x_k - \ell)| < \infty.$$

Now from the inequality

$$|a_k x_k| \leq |a_k(x_k - \ell)| + |\ell a_k|$$

We obtain that $\sum_k |a_k x_k| < \infty$. Therefore $a \varepsilon c^{\alpha}(p)$.

Since $c_0(p) \subset c(p)$ it follows that $c_0(p) \subset c_0^{\beta}(p)$. Let $a \varepsilon c^{\alpha}(p)$.

Since $e = (1,1,\ldots) \varepsilon c(p)$, it follows that $\sum_k |a_k| < \infty$, so that $a \varepsilon \ell_1$. Hence $a \varepsilon c_0^{\alpha}(p) \cap \ell_1$. This completes the proof of (i). Theorem 3.6 (i) and theorem 3.1(i) give us (ii); theorem3.1(i) give us 3.6 (iii). This completes the proof.

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Uniform version of Wiener–Tauberian theorem for Wiener algebra on a real line

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Abstract: The Winner-Tauberian theorem for $\mathbf{R}$ says that the closed translation invariant subspace generated by $f \in L^1(\mathbf{R})$ is $L^1(\mathbf{R})$ if and only if the Fourier transform $\hat{f}$ of f never vanishes. In this paper we prove a uniform version of this result in Winer algebra for real line.

Key words: Weiner-Tauberian theorem, translation invariant subspace, Wiener algebra, Radon measure.

1. Introduction:

For $k = 0, \pm 1, \pm 2, \ldots$, let I_k denotes the closed interval $I_k = [k, k+1]$. Let $W(\mathbf{R})$ be the linear space of all complex valued functions f on $\mathbb{R}$ for which

$$\|f\|_{W(\mathbf{R})} = \sum_{k \in \mathbf{Z}} \max_{x \in I_k} |f(x)| \text{ is finite.}$$

Let $C_{00}(\mathbf{R})$ denotes the set of all complex-valued continuous functions f with compact support. Let $\mathscr{R}$ be the set of Radon measure on $\mathbb{R}$. Then $\mathscr{R}$ can be identified with a sequence $(\mu_n)_{n \in \mathbf{Z}}$, where for $n \in \mathbb{N}$. μ_n is a measure in $M([n, n+1))$. That is, for a Borel set $\mathscr{B}$, let $\mathscr{B}_n = \mathscr{B} \cap [n, n+1)$ so that $\mathscr{B} = \bigcup_{n \in \mathbf{Z}} \mathscr{B}_n$; $\mu B = \sum_{n \in \mathbf{Z}} \mu_n(B_n)$.

In other words, for $f \in C_{00}(\mathbf{R})$,

$$\mu(f) = \int_{\mathbf{R}} f d\mu = \sum_{n \in \mathbf{Z}} \int_{J_n} f d\mu_n = \sum_{n \in \mathbf{Z}} \mu_n (f / J_n)$$

where, $J_n = [n, n+1)$.

Let $\mathscr{R}^b = \{\mu = (\mu_n)_{n\in\mathbb{Z}}$ and $\|\mu\|_b = \sup\{\|\mu_n\| : n\in\mathbb{Z}\}\} < \infty$ is isomorphic with $W(\mathbb{R})^*$ via $\mu \in \mathscr{R}^b \leftrightarrow F_\mu \in W(\mathbb{R})^*$ and given by

$$F_\mu(f) = \int_{\mathbb{R}} f d\mu, \ f\in W(\mathbb{R})$$

The norm of the fuctional, which we write as $\|F_\mu\|$, satisfies the following inequalities:

$$\tfrac{1}{2}\|\mu\|_b \le \|F_\mu\| \le \|\mu\|_b$$

Let $\mathrm{U} = \{g\in W(\mathbb{R}) : \text{for } \mu\in\mathscr{R},\ g*\mu = 0 \Rightarrow \mu = 0\}$

Let W_1 be the unit ball of $W(\mathbb{R})$ and $\mathscr{R}_1^\infty$ be the unit ball of $\mathscr{R}^b$ i.e.

$\mathscr{R}_1^\infty = \{\mu\in\mathscr{R}^b : \|\mu\|_b \le 1\}$

For $f\in W(\mathbb{R})$ and $\mu\in\mathscr{R}^b$, the convolution is defined by,

$$f*\mu(x) = \int_{\mathbb{R}} f(x-y)\, d\mu(y)$$

and $\displaystyle\int_{\mathbb{R}} f(x-y)\,d\mu(y) = \int_{\mathbb{R}} f_x(-y)\,d\mu(y)$

$$y\to -y$$

$$= \int_{\mathbb{R}} f_x(y)\,d\mu(-y)$$

$$= \int_{\mathbb{R}} f_x(y)\,d\tilde{\mu}(y) \ \ (\because\ \tilde{\mu}(y) = \mu(-y))$$

$$= F_{\tilde{\mu}}(f_x)$$

Theorem: *Let* $\mathscr{H}\in W(\mathbb{R})$ *be such that*

(i) $\{\mathscr{S}_h : h\in\mathscr{H}\}$ *given by* $\mathscr{S}_h(x) = h_x, x\in\mathbb{R}$ *is uniformly equicontinuous.*

(ii) $\exists\tilde{h}\in W_1$ *with* $|h(t)| \le |\tilde{h}(t)|\ \forall\, h\in\mathscr{H}$ *and* $\forall\, t\in\mathbb{R}$.

Let $g\in W_1\cap U$. Let $\mathscr{U}\subset\mathscr{R}_1^\infty$. Suppose that $g*\mu(x)\to 0$ as $x\to\infty$ uniformly for μ in $\mathscr{U}$ then $h*\mu(x)\to 0$ as $x\to\infty$ uniformly for h in $\mathscr{H}$ and μ in $\mathscr{U}$.

Proof: Assume to the contrary that there exists $\delta > 0$ such that $\forall n$ there exists $x_n\in\mathbb{R}$ with $x_n > n$, $h_n\in\mathscr{H}$ and $\mu_{(n)}\in\mathscr{U}$ satisfying:

$$\left|\left(h_n * \mu_{(n)}\right)(x_n)\right| > \delta.$$

Let us consider the sequences

$$\begin{aligned}\mathcal{S}_n(x) &= \left(h_n * \mu_{(n)}\right)_{x_n}(x),\ n = 1,2,3\ ,\ \ldots\ ,\ x \in \mathbb{R}\\ &= \left(h_n * \mu_{(n)}\right)(x + x_n)\end{aligned}$$

$\mathcal{S}_n$ is measurable. Now we shall show that it is bounded and equicontinuous on $\mathbb{R}$.

$$\begin{aligned}\sup_{x\in\mathbb{R}} |s_n(x)| &= \sup_{x\in\mathbb{R}} \left| \left(h_n * \mu_{(n)}\right)_{x_n}(x) \right|\\ &= \sup_{x\in\mathbb{R}} \left| \left(h_n * \mu_{(n)}\right)(x + x_n) \right|\\ &= \sup_{x\in\mathbb{R}} \left| F_{\tilde{\mu}_{(n)}}\left((h_n)_{x+x_n}\right) \right|\\ &\le \left\| F_{\tilde{\mu}_{(n)}} \right\|_{W(\mathbb{R})^*} \left\| (h_n)_{x+x_n} \right\|_{W(\mathbb{R})}\\ &\le 2 \left\| F_{\tilde{\mu}_{(n)}} \right\|_b \left\| \bar{h} \right\|_{W(\mathbb{R})}\\ &\le 2\end{aligned}$$

$\therefore\ \{\mathcal{S}_n\}_{n\in\mathbb{N}}$ is uniformly bounded.

Since, $\{\mathcal{S}_n : h \in \mathcal{H}\} = \tau$ is uniformly equicontinuous, so for a given $\frac{\epsilon}{2} > 0$ there corresponds a $\delta > 0$ such that $\left\| h_x - h_y \right\|_{W(\mathbb{R})} = \left\| \mathcal{S}_h(x) - \mathcal{S}_h(y) \right\|_{W(\mathbb{R})} < \epsilon/2, \forall\ \mathcal{S}_h \in \tau$ and for all pair of point x, y with $|x - y| < \delta$.

For $x, y \in \mathbb{R}$,

$$\begin{aligned}|\mathcal{S}_n(x) - \mathcal{S}_n(y)| &= \left| \left(h_n * \mu_{(n)}\right)_{x_n}(x) - \left(h_n * \mu_{(n)}\right)_{x_n}(y) \right|\\ &= \left| \left(h_n * \mu_{(n)}\right)(x + x_n) - \left(h_n * \mu_{(n)}\right)(y + x_n) \right|\\ &= \left| \left(F_{\tilde{\mu}_{(n)}}(h_n)_{x+x_n}\right) - F_{\tilde{\mu}_{(n)}}\left((h_n)_{y+x_n}\right) \right|\\ &= \left| F_{\tilde{\mu}_{(n)}}\left\{ \left((h_n)_x\right)_{x_n} - \left((h_n)_y\right)_{x_n} \right\} \right|\\ &= \left| F_{\tilde{\mu}_{(n)}}\left\{ \left((h_n)_x - (h_n)_y\right)_{x_n} \right\} \right|\\ &\le \left\| F_{\tilde{\mu}_{(n)}} \right\|_{W(\mathbb{R})^*} \left\| \left((h_n)_x - (h_n)_y\right)_{x_n} \right\|_{W(\mathbb{R})}\\ &\le 2 \left\| F_{\tilde{\mu}_{(n)}} \right\|_{W(\mathbb{R})^*} \left\| (h_n)_x - (h_n)_y \right\|_{W(\mathbb{R})}\\ &\le 2 \left\| F_{\tilde{\mu}_{(n)}} \right\|_b \left\| (h_n)_x - (h_n)_y \right\|_{W(\mathbb{R})}\\ &\le 2 \left\| (h_n)_x - (h_n)_y \right\|_{W(\mathbb{R})}\\ &< \epsilon.\end{aligned}$$

and therefore, $|\mathcal{A}_n(x)-\mathcal{A}_n(y)|<\in$ for $n\in\mathbf{N}$ and $|x-y|<\delta$. Therefore, $\{\mathcal{A}_n\}_{n\in\mathbf{N}}$ is uniformly equicontinuous on $\mathbb{R}$.

Thus by Ascoli's Lemma [5] we now select from the sequence $\mathcal{A}_n(x)$ a subsequences $\mathcal{A}_{n_k}(x)$ which tends to limit $\mathcal{A}(x)$ pointwise as $k\to\infty$ and continuous on $\mathbb{R}$. For each fixed $x\in\mathbb{R}$ and $t\in\mathbb{R}$, $\mathcal{A}_{n_k}(x-t)\to\mathcal{A}(x-t),\ k\to\infty$ and therefore, $\mathcal{A}_{n_k}(x-t)g(t)\to\mathcal{A}(x-t)g(t),\ k\to\infty$.

Now $\forall t\in\mathbb{R}$,

$$
\begin{aligned}
|\mathcal{A}_{n_k}(x-t)g(t)| &= |\mathcal{A}_{n_k}(x-t)\ g(t)| \\
&\leq \left\|\mathcal{A}_{n_k}\right\|_\infty |g(t)| \\
&\leq 2|g(t)|
\end{aligned}
$$

Thus by Lebesgue dominated convergence therems, $\forall x\in\mathbb{R}$

$$
\begin{aligned}
\int_{\mathbb{R}}\mathcal{A}_{n_k}(x-t)\,g(t)\,dt &\to \int_{\mathbb{R}}\mathcal{A}(x-t)\,g(t)\,dt,,\quad k\to\infty \\
&\equiv(\mathcal{A}*g)(x)
\end{aligned}
$$

Now,
$$
\begin{aligned}
\int_{\mathbb{R}}\mathcal{A}_{n_k}(x-t)\,g(t)\,dt &= \int_{\mathbb{R}}\left(h_{n_k}*\mu_{(n_k)}\right)\left(x_{n_k}+x-t\right)g(t)\,dt \\
&= \left(h_{n_k}*\mu_{(n_k)}*g\right)\left(x_{n_k}+x\right) \\
&= \left(\left(g*\mu_{(n_k)}\right)*h_{n_k}\right)\left(x_{n_k}+x\right) \\
&= \int_{\mathbb{R}}\left(g*\mu_{(n_k)}\right)\left(x_{n_k}+x-t\right)h_{n_k}(t)\,dt
\end{aligned}
$$

Put $I_{k,x}(t)=\left(g*\mu_{(n_k)}\right)\left(x_{n_k}+x-t\right)h_{n_k}(t)$

Since we know, $(g*\mu)(z)\to 0$ as $z\to\infty$ uniformly for μ in $\mathcal{U}$. we have for given $\in>\exists\Delta:|(g*\mu)(z)|<\in\forall z\geq\Delta$ and μ in $\mathcal{U}$.

Therefore $\left|\left(g*\mu_{(n_k)}\right)\left(x_{n_k}+x-t\right)\right|<\in$ for $x_{n_k}+x-t\geq\Delta$

Thus for a fixed x and t in $\mathbb{R}$,

$$\left|\left(g*\mu_{(n_k)}\right)\left(x_{n_k}+x-t\right)\right|<\in \text{ for } x_{n_k}+x-t\geq\Delta.$$

Thus for a fixed x and t in $\mathbb{R}$,

$$\left(g*\mu_{(n_k)}\right)\left(x_{n_k}+x-t\right)\to 0 \text{ as } k\to\infty.$$

$$\begin{aligned}\left|I_{k,x}(t)\right| &= \left|\left(g*\mu_{(n_k)}\right)\left(x_{n_k}+x-t\right)h_{n_k}(t)\right| \\ &= \left|\left(g*\mu_{(n_k)}\right)\left(x_{n_k}+x-t\right)\right|\left|h_{n_k}(t)\right| \\ &\le \in \left|h_{n_k}(t)\right| \\ &\le \in \left|\tilde{h}(t)\right|\end{aligned}$$

Thus by applying Lebesgue dominated convergence theorem for each $x\in\mathbb{R}$,

$$\int_{\mathbb{R}} I_{k,x}(t)\,dt\to 0 \text{ as } k\to\infty$$

and therefore $\mathcal{S}*g(x)=0$. Since $g\in U$ we get $\mathcal{S}=0$.
But

$$\begin{aligned}|\mathcal{S}(0)| &= \lim_{k\to\infty}\left|\mathcal{S}_{n_k}(0)\right| \\ &= \lim_{k\to\infty}\left|\left(h_{n_k}*\mu_{(n_k)}\right)\left(x_{n_k}\right)\right| \\ &\ge \delta>0\end{aligned}$$

which is a contradiction and therefore

$$\int_{\mathbb{R}} h(x-t)d\mu(t)\to 0 \text{ as } x\to\infty \text{ uniformly for } h \text{ is } \mathscr{H} \text{ and } \mu \text{ in } \mathscr{U},$$

Example: Take $f(x)=e^{-|x|}$

$$\begin{aligned}\hat{f}(y) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)\exp(-iyx)\,dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp(-|x|)\exp(-iyx)\,dx \\ &= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0}\exp(x)\exp(-iyx)\,dx+\int_{0}^{\infty}\exp(-x)\exp(-iyx)\,dx\right. \\ &= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0}\exp\left((1-iy)x\right)dx+\int_{0}^{\infty}\exp\left(-(1+iy)x\right)dx\right]\end{aligned}$$

Putting $x\to -x$ in first integral

$$= \frac{1}{\sqrt{2\pi}}\left[\int_0^\infty \exp\big(-(1-iy)x\big)\,dx + \int_0^\infty \exp\big(-(1+iy)x\big)\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_0^\infty \exp(-x)\left[\exp(iyx)+\exp(-iyx)\right]dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\cdot\frac{1}{1+y^2}$$

$\hat{f}(y)\to 0$ as $y\to\pm\infty$.

f is continuous function on $\mathbb{R}$ and so is measurable

$$\sum_{n=-\infty}^{\infty}\max_{x\in[n,n+1]} e^{-|x|} = \sum_{n=-\infty}^{0}\max_{x\in[n,n+1]} e^{-|x|} + \sum_{n=0}^{\infty}\max_{x\in[n,n+1]} e^{-|x|}$$

$$= \sum_{n=-\infty}^{0} e^{-|n+1|} + \sum_{n=0}^{\infty} e^{-|n|}$$

put $n+1=-k,\ k>0$

$$= \sum_{k=\infty}^{0} e^{-|-k|} + \sum_{n=0}^{\infty} e^{-n}$$

$$= \sum_{n=0}^{\infty} e^{-n} + \sum_{n=0}^{\infty} e^{-n}$$

$$= 2\sum_{n=0}^{\infty} e^{-n}$$

$$= 2\sum_{n=0}^{\infty}\frac{1}{e^n} = \frac{2e}{1-e} < \infty$$

therefore $e^{-|x|}\in W(\mathbb{R})$

since $$\|f\|_{W(\mathbb{R})} = \sum_{n\in\mathbb{Z}}\max_{x\in[n,n+1]}|f(x)|$$

$$= \sum_{n\in\mathbb{Z}}^{\infty}\|f\|_{[n,n+1]}$$

But $$\sum_{n\in\mathbb{N}}\|f\|_{[n,n+1]} \le \sum_{n\in\mathbb{Z}}\|f\|_{[n,n+1]} < \infty$$

Thus applying the definition of convergence on $\mathbb{N}$. Let $\in>0$ then $n_0\in\mathbb{N}$.

Such that

$$T_{n_o} = \sum_{n \geq n_o} \|f\|_{[n,n+1]} \to 0 \text{ as } n_o \to \infty.$$

$[t]=$ The greater integer $\leq t$

$$T_{[t]} = \sum_{n \geq [t]} \|f\|_{[n,n+1]} = \sum_{n \geq t} \|f\|_{[n,n+1]}$$

$$\mu(f) = \sum_{n \in \mathbf{N}} f(n)$$

$$(\mu * f)(t) = \mu(\tilde{f}_t) = \sum_{n \in \mathbf{N}} f(t+n) = \sum_{n \in \mathbf{N}} e^{-|t+n|}$$

$$= e^{-|t+1|} + e^{-|t+2|} + \ . \ . \ .$$

But $\qquad |f(t)| = e^{-|t|} \leq \|f\|_{([t],[t]+1)}$

$$|f(t+1)| = e^{-|t+1|} \leq \|f\|_{([t]+1,[t]+2)}$$

$$|f(t+2)| = e^{-|t+2|} \leq \|f\|_{([t]+2,[t]+3)} \quad \text{and so on.}$$

Therefore $\qquad |f(t)| + \sum_{n \in \mathbf{N}} |f(t+1)| \leq \sum_{n \geq [t]} \|f\|_{[n,n+1]} = T_{[t]} \to 0 \quad t \to \infty.$

$\therefore \ (\mu * f)(t) \to 0$ as $t \to \infty$.

Now, $\qquad (\mu * f_s)(t) = \sum_{n \in \mathbf{N}} f(s+t+n) = \sum_{n \in \mathbf{N}} e^{-|s+t+n|}$ take $s = -t$

and therefore,

$$\mu * f_s(t) = \sum_{n \in \mathbf{N}} e^{-n} < \infty$$

$\therefore \ (\mu * f_s)(t) \nrightarrow 0$ as $t \to \infty$.

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Approximation of the conjugate of a function belonging to Lip (α,p) class by matrix means

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Abstract: The present paper deals with approximation of the conjugate of a function belonging to the $\mathrm{Lip}(\alpha, p)$ class by matrix summability method. A new estimate on the degree of approximation of conjugate function $\bar{f}$, conjugate to a function $f \in \mathrm{Lip}(\alpha, p)$, has been determined by matrix summability of conjugate series of a Fourier series.

Key words and phrases: Degree of approximation, matrix summability, Fourier series conjugate series of the Fourier series, $\mathrm{Lip}(\alpha, p)$ class.

Subject classification: 42B05, 42B08

§ 1. Let f be 2π-periodic, integrable over $(-\pi, \pi)$ in the sense of Legesgue, then its Fourier series is given by

$$f(t) \sim \frac{1}{2}a_o + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) = \frac{1}{2}a_o + \sum_{n=1}^{\infty} A_n(t) \tag{1}$$

with partial sum $S_n(x)$.

The conjugate series of the Fourier series (1) given by

$$\sum_{n=1}^{\infty}(a_n \sin nt - \cos nt) = -\sum_{n=1}^{\infty} B_n(t) \tag{2}$$

with partial sum $S_n(x)$.

Let $T = (a_{n,k})$ be an infinite lower triangular matrix satisfying the Silverman-Tŏeplitz [5] conditions of regularity i.e.

$$\sum_{k=0}^{n} a_{n,k} \to 1 \text{ as } n \to \infty,$$

$$a_{n,k} = 0, \text{ for } k > n$$

and $\sum_{k=0}^{n} | a_{n,k} | \leq M$, where, M is a finite positive constant.

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series whose n^{th} partial sum is given by

$$S_n = \sum_{v=0}^{\infty} u_v.$$

This sequence–to-sequence transformation

$$t_n = \sum_{k=1}^{n} a_{n,n-k}\ S_{n-k} \tag{3}$$

defines the sequence $\{t_n\}$ of matrix means of sequence $\{S_n\}$, generated by the sequence of coefficient $\{a_{n,k}\}$. If

$$t_n \to S \text{ as } n \to \infty,$$

then the series $\sum_{n=0}^{\infty} u_n$ or sequence $\{S_n\}$ is said to be summable by matrix method (T) to S. It is dentoed by

$$t_n \to S(T) \text{ as } n \to \infty \text{ (Zygmund [6]).}$$

The summability method (T) reduces to

(i) Harmonic means, when $a_{n,k} = \dfrac{1}{(n-k+1)\log n}\ \forall\ 0 \leq k \leq n.$

(ii) (H,p) means, when $a_{n,k} = \dfrac{1}{\log^{p-1}(n+1)} \prod_{q=0}^{p-1} \log^{q}(k+1).$

(iii) (N, p_n), when $a_{n,k} = \dfrac{p_{n-k}}{p_n}$, where $p_n = \sum_{k=0}^{n} p_k \neq 0.$

(iv) (N,p,q) means, when $a_{n,k} = \dfrac{p_{n-k}q_k}{R_n}$, where $R_n = \sum_{k=0}^{n} p_k q_{n-k} \neq 0.$

The L^p norms is defined by

$$\|f\|_p = \left(\int_0^{2\pi} |f(x)|^p\, dx \right)^{1/p},\ p \geq 1$$

and the degree of approximation $E_n(f)$ under norm $\|\ \|_p$ is given by (Zygmund [6])

$$E_n(f) = \min_{t_n} \| t_n(x) - f(x) \|_p,$$

$t_n(x)$ where $t_n(x)$ is a trigonometric polynomial of degree n.

A function $f \in \text{Lip}\,\alpha$, if

$$|f(x+t)-f(x)| = O(|t|^{\alpha}), \text{ for } 0<\alpha\leq 1 \text{ and}$$

$f \in \text{Lip}(\alpha, p)$, for $0 \leq x \leq 2\pi$, if

$$\left(\int_0^{2\pi} \left|f(x+t)-f(x)\right|^p dx\right)^{\frac{1}{p}} = O(|t|^{\alpha}),\ 0<\alpha\leq 1,\ p\geq 1.$$

We write

$$\psi(t) = f(x+t) - f(x-t) \tag{4}$$

$$\overline{K}(n,t) = \frac{1}{2\pi}\sum_{k=0}^{n} a_{n,n-k}\frac{\cos\left(n-k+\frac{1}{2}\right)t}{\sin\frac{t}{2}} \tag{5}$$

$$A_{n,n} = \sum_{k=0}^{n} a_{n,n-k} = O(1) \tag{6}$$

$$A_{n,\tau} = \sum_{k=0}^{\tau} a_{n,n-k} \tag{7}$$

$\tau = \left[\frac{1}{t}\right]$, where, τ denotes the greatest integer not greater then $\frac{1}{t}$.

§2 Bernstin [1], used $(C,1)$ means to obtain the degree of approximation of lip 1 function. Jackson [2] determined the degree of approximation by using (C,δ) method in lip α class, for $0<\alpha<1$. First time, the concept of the degree of approximation of the conjugate function $\overline{f}(x)$ has been introduced by Qureshi [4]. He used Lip α class functions by Nörlund method. The purpose of this paper is to obtain the approximation of $\overline{f}(x)$, the conjugate of a function f belonging to Lip (α, p) class, by matrix means of conjugate series of a Fourier series. In fact, I prove following theorem:

Theorem: *$T=(a_{n,k})$ be an infinite lower regular triangular matrix such that the element $(a_{n,k})$ be non-negative, non-decreasing with $k\leq n$, then the degree of approximation of function $\overline{f}(x)$, conjugate to a 2π-periodic function $f(x)\in Lip\ (\alpha,p)$, $0<\alpha\leq 1, p\geq 1$, by matrix means (T) of its conjugate series (2), is given by*

$$\|\overline{t}_n(x) - \overline{f}(x)\|_p = O\left(\frac{1}{n^{\alpha-\frac{1}{p}}}\right) \tag{8}$$

where $\overline{t}_n(x) = \sum_{k=0}^{n} a_{n,n-k}\, \overline{S}_{n-k}(x)$ is the matrix means of the series (2).

§ 3. For the proof of my theorem following lemmas are required.

Lemma 1: Let $\bar{K}(n,t)$ be given in (5), then

$$\bar{K}(n,t) = O\left(\frac{1}{t}\right), \quad \text{for} \quad 0 \le t < \frac{1}{n}.$$

Proof:
$$|\bar{K}(n,t)| = \frac{1}{2\pi}\sum_{k=0}^{n} a_{n,n-k}\left|\frac{\cos\left(n-k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right|$$
$$\le \frac{1}{2t}\sum_{k=0}^{n} a_{n,n-k}$$
$$= \frac{A_{n,n}}{2t}$$
$$= O\left(\frac{1}{t}\right)O(1)$$
$$= O\left(\frac{1}{t}\right).$$

Lemma 2: (Lal [3]), *If $a_{n,k}$ is non-negative and non-decreasing with $k \le n$ then for, $0 \le a < b \le \infty$, $0 \le t \le \pi$ and for any n.*

We have
$$\sum_{k=a}^{b}\left|a_{n,n-k}e^{i(n-k)t}\right| = O(A_{n,\tau}) \text{ where } \tau = \text{Integral part of } \frac{1}{t} = \left[\frac{1}{t}\right].$$

Lemma 3: *Let $\bar{K}(n,t)$ be given in (5) and under the condition of my theorem on $(a_{n,k})$, we have*

$$\bar{K}(n,t) = O\left(\frac{A_{n,\tau}}{t}\right), \quad \text{for} \quad \frac{1}{n} < t \le \pi.$$

Proof: It is well known that, $\sin\frac{t}{2} \ge \frac{t}{\pi}$ (since, $\sin\theta \ge \frac{2\theta}{\pi}, 0 < \theta < \pi$, Jordan's Lemma).

Now, for $t > 0$ and $\tau \le n$, we have

$$|\bar{K}(n,t)| = \left|\frac{1}{2\pi}\sum_{k=0}^{n} a_{n,n-k}\frac{\cos\left(n-k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right|$$
$$= \left|\frac{1}{2\pi\sin\frac{t}{2}} \text{ real part of } \sum_{k=0}^{n} a_{n,n-k}\, e^{i\left(n-k+\frac{1}{2}\right)t}\right|$$
$$\le \left|\frac{1}{2t} \text{ real part of } \sum_{k=0}^{n} a_{n,n-k}\, e^{i(n-k)t}\, e^{\frac{it}{2}}\right|$$

$$\leq \left| \frac{1}{2t} \sum_{k=0}^{n} a_{n,n-k}\ e^{i(n-k)t}\ | e^{\frac{it}{2}} | \right|$$

$$= O\left(\frac{1}{t}\right) \left| \sum_{k=0}^{n} a_{n,n-k}\ e^{i(n-k)t} \right| \qquad \left(\Theta\, | e^{\frac{it}{2}} | \leq 1\right)$$

$$= O\left(\frac{A_{n,\tau}}{t}\right) \text{ by Lemma 2.}$$

§ 4. n^{th} partial sum $\overline{S}(x)$ of the series (2) is given by

$$\overline{s}_n(x) - \overline{f}(x) = \frac{1}{2\pi} \int_0^{\pi} \frac{\psi(t) \cos\left(n + \frac{1}{2}\right)t}{\sin\frac{t}{2}}\, dt$$

then,

$$\sum_{k=0}^{n} a_{n,n-k} \left(\overline{S}_{n-k}(x) - \overline{f}(x)\right) = \frac{1}{2\pi} \int_0^{\pi} \psi(t) \sum_{k=0}^{n} a_{n,n-k} \frac{\cos\left(n - k + \frac{1}{2}\right)t}{\sin\frac{t}{2}}\, dt$$

or,

$$\overline{t}_n(x) - \overline{f}(x) = \int_0^{\pi} \psi(t)\, \overline{K}(n,t)\, dt$$

$$= \int_0^{\frac{1}{n}} \psi(t)\, \overline{K}(n,t)\, dt + \int_{\frac{1}{n}}^{\pi} \psi(t)\, \overline{K}(n,t)\, dt$$

$$= I_1 + I_2, \text{ say.} \tag{9}$$

Applying Hölder's inequality, Lemma 1 and fact that $\psi(t) \in \operatorname{Lip}(\alpha, p)$, we have

$$| I_1 | \leq \left[\int_0^{\frac{1}{n}} \left| \frac{t\psi(t)}{t^{\alpha}} \right|^p dt \right]^{\frac{1}{p}} \left[\int_0^{\frac{1}{n}} \left| \frac{\overline{K}(n,t)}{t^{1-\alpha}} \right|^q dt \right]^{\frac{1}{q}}$$

$$= O\left[\left(\int_0^{\frac{1}{n}} t^{p-1} dt \right)^{\frac{1}{p}} \right] O\left[\left(\int_0^{\frac{1}{n}} t^{(\alpha-2)q} dt \right)^{\frac{1}{q}} \right]$$

$$= O\left[\left\{ \left(\frac{t^p}{p} \right)_0^{\frac{1}{n}} \right\}^{\frac{1}{p}} \right] O\left[\left\{ \left(\frac{t^{(\alpha-2)q+1}}{(\alpha-2)q+1} \right)_0^{\frac{1}{n}} \right\}^{\frac{1}{q}} \right]$$

$$= O\left(\frac{1}{n}\right) O\left(\frac{1}{n^{\alpha-2+\frac{1}{q}}}\right)$$

$$= O\left(\frac{1}{n^{\alpha-\frac{1}{p}}}\right). \tag{10}$$

Using Hölder's inequality and Lemma 3, we have

$$|I_2| \leq \left[\int_{\frac{1}{n}}^{\pi}\left|\frac{t^{-\delta}\psi(t)}{t^{\alpha}}\right|^{p} dt\right]^{\frac{1}{p}}\left[\int_{\frac{1}{n}}^{\pi}\left|\frac{\overline{K}(t)}{t^{-\delta-\alpha}}\right|^{q} dt\right]^{\frac{1}{q}}$$

$$= O\left[\left(\int_{\frac{1}{n}}^{\pi}\left(t^{-\frac{1}{p}-\delta}\right)^{p} dt\right)^{\frac{1}{p}}\right] O\left[\left(\int_{\frac{1}{n}}^{\pi}\left(\frac{A_{n,\tau}}{t^{1-\delta-\alpha}}\right)^{q} dt\right)^{\frac{1}{q}}\right]$$

$$= O\left[\left\{\left(\frac{t^{-\delta p}}{-\delta p}\right)_{\frac{1}{n}}^{\pi}\right\}^{\frac{1}{p}}\right] O(A_{n,n})\, O\left[\left\{\left(\frac{t^{(\delta+\alpha-1)q+1}}{(\delta+\alpha-1)q+1}\right)_{\frac{1}{n}}^{\pi}\right\}^{\frac{1}{q}}\right]$$

$$= O(n^{\delta})\, O\left(\frac{1}{n^{\delta+\alpha-1+\frac{1}{q}}}\right)$$

$$= O\left(\frac{1}{n^{\alpha-1+\frac{1}{q}}}\right)$$

(11) $$= O\left(\frac{1}{n^{\alpha-\frac{1}{p}}}\right)$$

where δ is an arbitrary number such that $q(1-\delta)-1>0$ and q is the conjugate index of p. Combining the conditions (9)-(11), we have

$$\left|\overline{t}_n(x)-\overline{f}(x)\right| = O\left(\frac{1}{n^{\alpha-\frac{1}{p}}}\right).$$

Now, $$\left\|t_n(x)-\overline{f}(x)\right\|_p = \left[\int_0^{2\pi}\left|\overline{t}_n(x)-\overline{f}(x)\right|^p dx\right]^{\frac{1}{p}}$$

$$= O\left[\int_0^{2\pi}\left(\frac{1}{n^{\alpha-\frac{1}{p}}}\right)^p dx\right]^{\frac{1}{p}}$$

$$= O\left(\frac{1}{n^{\alpha-\frac{1}{p}}}\right)\left[\int_0^{2\pi} dx\right]^{\frac{1}{p}}$$

(9) $$= O\left(\frac{1}{n^{\alpha-\frac{1}{p}}}\right).$$

This completes the proof of the theorem.

§ 5. Following corollary can be derived from the main theorem.

Corollary 1. *If $p \to \infty$, the degree of approximation of $\overline{f}(x)$, the conjugate of a function $f \in \operatorname{Lip}\alpha$ class by matrix means is given by*

$$\left\| \overline{t}_n(x) - \overline{f}(x) \right\|_\infty = \sup_{0 \le x \le 2\pi} \left| \overline{t}_n(x) - \overline{f}(x) \right| = O\left(\frac{1}{n^\alpha}\right), \quad \text{for } 0 < \alpha < 1,$$

where, $\overline{t}_n(x) = \sum_{k=0}^{n} a_{n,n-k}\, \overline{S}_{n-k}(x)$ is the matrix (T) means of the series (2).

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Bottleneck product rate variation problem with absolute-deviation objective

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Abstract: The bottleneck product rate variation problem with absolute-deviation objective is pseudo-polynomially solvable. There always exists an optimal sequence with the property that the deviation for every product is no more than one unit. Only the standard instance has optimal value less than $\frac{1}{2}$ if and only if the demands are successive powers of two. In this paper, we establish that there exists no feasible solution for any instance with the deviation less than $\frac{1}{3}$.

Keywords: non-linear integer programming, mixed-model just-in-time production, balanced word.

1. Introduction

Mixed-model just-in-time production system aims to obtain a sequence of a number of different products with multiple copies that minimizes deviation throughout the time, between the actual and the desired production. Such a sequence maintains the final assembly line keeping the rate of usage of parts as constant as possible and affects the entire supply chain as all other levels are also inherently fixed due to the pull nature.

Minimization of the maximum variation in the rate at which different products are produced on the line is known as the bottleneck product rate variation problem (PRVP) [5]. The problem is formulated as a non-linear integer programming [7, 8]. The problem with absolute-deviation objective has been extensively studied in a great number of papers, for instance see [3].

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The problem is reducible to the release date/due date decision problem, which can be solved to optimality with a pseudo-polynomial algorithm. An optimal sequence always exists when the deviation for every product is never more than one unit [9]. It is important to observe instances that have as small optimal values as possible. It has been established that the problem has optimal absolute deviation less than a half if and only if the demands are successive powers of two [1]. In this paper, we establish that there exists no instance that has a feasible solution with the value less than $\frac{1}{3}$. The general version of the problem is *Co-NP* and is still open whether it is *Co-NP*-complete or polynomially solvable but is polynomially solvable when the number of products is fixed [1].

The plan of the paper is as follows. Section 2 reviews the mathematical model. In Section 3 perfect matching method and the bounds have been studied. Section 4 explains the bisection search and the last section concludes the paper.

2. Mathematical Model

Given $d_i \in N$ demand for a product i, $i = 1,\dots,n$, N being the set of positive integers, with total demand $D = \sum_{i=1}^{n} d_i$ and demand ratio $r_i = \frac{d_i}{D}$, let the time horizon be partitioned into D equal units and each product is produced in a unit time. There will be k complete units of various products during the first k, $k = 1,\dots,D$, time units. Let x_{ik} be the quantity of product i produced during the time units 1 through k. Consider f_i, $i = 1,\dots,n$ unimodal symmetric convex function with minimum 0 at 0.

The mathematical model of the bottleneck PRVP [7, 8] is

$$\text{(1)} \qquad \min \max f_i\,(x_{ik} - kr_i)$$

subject to

$$\text{(1.1)} \qquad \sum_{i=1}^{n} x_{ik} = k \qquad k = 1,\dots,D$$

$$\text{(1.2)} \qquad x_{i(k-1)} \le x_{ik} \qquad i = 1,\dots,n\,;\; k = 2,\dots,D$$

$$\text{(1.3)} \qquad x_{iD} = d_i\,;\; x_{i0} = 0 \quad i = 1,\dots,n$$

$$\text{(1.4)} \qquad x_{ik} \ge 0\,,\ \text{integer}$$

Constraint (1.1) shows the cumulative production during the time units 1 through k. Constraint (1.2) ensures that the total production of every product over k time units is a non-decreasing function of k. Constraint (1.3) guarantees that the demands for each product are met exactly. Constraint (1.2) and (1.4) ensure that exactly one unit of a

product is scheduled during one time unit. In this paper, we consider most studied absolute-deviation objective function $f_i(x_{ik} - kr_i) = |x_{ik} - kr_i|$.

3. Perfect Matching Method and the Bounds

A pseudo-polynomial algorithm, order-preserving perfect matching together with bisection search yields an optimal solution to the problem. The problem has been shown to be Co-NP but remains open whether it is *Co-NP-Complete* or polynomially solvable [1]. As the input size is $O(n \log D)$ and there are nD variables with $O(nD)$ constraints, existence of a polynomial time algorithm seems unlikely.

The problem is reduced to an order-preserving perfect matching [9]. The problem is constructed in a V_1-convex bipartite graph $G = (V_1 \cup V_2, E)$ with $V_1 = \{1,\dots,D\}$; $V_2 = \{(i,j) \mid i = 1,\dots,n; j = 1,\dots,d_i\}$; and $E = \{(k,(i,j)) \mid k \in [E(i,j),\ L(i,j)]\}$, where $E(i,j)$ and $L(i,j)$ are the earliest and the latest starting time, respectively, for (i,j), the j^{th} copy of product i.

For a bound (target value) B, $E(i,j)$ and $L(i,j)$ can be determined by the integral adjustment of the points where the bound B and the curves $|j - kr_i|$, $i = 1,\dots,n$; $j = 0,\dots,d_i$ intersect.

Lemma 1: [1] *For a given bound* B, *the earliest and the latest starting times are the unique integers* $E(i,j) = \left\lceil \frac{j-B}{r_i} \right\rceil$ *and* $L(i,j) = \left\lfloor \frac{j-1+B}{r_i} + 1 \right\rfloor$, *respectively.*

Proof: If (i,j) is produced in the time unit k, $|x_{ik} - kr_i| = |j - kr_i|$, $i = 1,\dots,n$; $j = 0,\dots,d_i; k = 1,\dots,D$. For $j = 0$, $x_{ik} = 0$. $E(i,j)$ must satisfy the inequalities $|j - (E(i,j)-1)r_i| > B$ and $|j - E(i,j)r_i| \le B$. This implies $\frac{j-B}{r_i} \le E(i,j) < \frac{j-B}{r_i} + 1$.

Therefore, $E(i,j) = \left\lceil \frac{j-B}{r_i} \right\rceil$

Similarly, $E(i,j)$ must satisfy the inequalities $|(L(i,j)-1)r_i - (j-1)| \le B$ and $|L(i,j)r_i - (j-1)| > B$. This implies $\frac{j-1+B}{r_i} < L(i,j) \le \frac{j-1+B}{r_i} + 1$.

Therefore, $L(i,j) = \left\lfloor \frac{j-1+B}{r_i} + 1 \right\rfloor$

Finally, $E(i,j) = 1$ and $L(i,j) = D$ hold if $|j - r_i| \le B$ and $|d_i - r_i - j + 1| \le B$, respectively. □

$E(i,j)$ and $L(i,j)$ can be calculated in $O(D)$ time [9].

A modified version of Glover's earliest due date (EDD) rule finds a perfect matching when applied in the V_1-convex bipartite graph with $B < 1$. The algorithm matches each ascending $k \in V_1$ to the unmatched (i,j) with the smallest $L(i,j)$ [9].

The necessary and sufficient condition for the existence of a perfect matching is the following.

Theorem 1: [1] *For $B < 1$, the graph $G = (V_1 \cup V_2, E)$ formed by the problem has a perfect matching if and only if* $\sum_{i=1}^{n}(\lfloor k_2 r_i + B\rfloor - \lceil (k_1-1)r_i - B\rceil) \geq k_2 - k_1 + 1$ *and*

$\sum_{i=1}^{n}(\lceil k_2 r_i - B\rceil - \lfloor (k_1-1)r_i - B\rfloor) \leq k_2 - k_1 + 1$ *for all* $k_1,\ k_2 \in V_1,\ k_1 \leq k_2$ *and*

$[E(i,j), L(i,j)] \cap [k_1, k_2] \neq \phi$

Proof: Let $K = [k_1, k_2] \subset V_1$. Then $(i,j) \in N(K)$, where

$N(K) = \{(i,j) : (i,j) \in V_2, \exists k \in K, (k,(i,j)) \in E\}$, the neighborhood of an interval K in V_1.

$\Leftrightarrow [E(i,j), L(i,j)] \cap [k_1, k_2] \neq \phi$

$\Leftrightarrow E(i,j) \leq k_2$ and $L(i,j) \geq k_1$

$\Leftrightarrow \dfrac{j-B}{r_i} \leq k_2$ and $\dfrac{j-1+B}{r_i} + 1 \geq k_1$

$\Leftrightarrow \lceil (k_1-1)r_i + 1 - B\rceil \leq j \leq \lfloor k_2 r_i + B\rfloor$

Therefore, $\sum_{i=1}^{n}(\lfloor k_2 r_i + B\rfloor - \lceil (k_1-1)r_i - B\rceil) \geq k_2 - k_1 + 1$

Let $N(K) = [k_1, k_2] \subset V_1$.

Then $(i,j) \in K \subset V_2$

$\Leftrightarrow [E(i,j), L(i,j)] \subset [k_1, k_2] \subset V_1$

$\Leftrightarrow k_1 \leq E(i,j)$ and $L(i,j) \leq k_2$

$\Leftrightarrow k_1 \leq \dfrac{j-B}{r_i}$ and $\dfrac{j-1+B}{r_i} + 1 \leq k_2$

$\Leftrightarrow (k_1-1)r_i + B < j < k_2 r_i + 1 - B$

$\Leftrightarrow \lfloor (k_1-1)r_i + 1 + B\rfloor \leq j \leq \lceil k_2 r_i - B\rceil$

Therefore, $\sum_{i=1}^{n}(\lceil k_2 r_i - B\rceil - \lfloor (k_1-1)r_i + B\rfloor) \leq k_2 - k_1 + 1$.

Since $E(i,j)$ and $L(i,j)$ are strictly monotonic and the *EDD* rule assigns the lower numbered copies to earlier time units, the perfect matching is order-preserving. The order-preserving perfect matching gives rise a bijection $(i,j) \to k$, $(i,j) \in V_2$ and $k \in V_1$,

$i=1,\ldots,n\,;\, j=1,\ldots,d_i$ and hence creates a feasible solution to an instance of the problem [9]. The lower and the upper bounds are $1-r_{\max}$ and $1-\frac{1}{D}$, respectively.

Theorem 2: **[9, 1]** *For any instance, the lower and the upper bounds are* $1-r_{\max}$ *and* $1-\frac{1}{D}$, *respectively.*

Proof: Given a bound B, a copy of some product i should be produced during $k=1$. Thus, $\min\,(1-r_i)\le B$, that is, $1-r_{\max}\le B$.

Let $B=1-\frac{1}{D}$.

Since $\lfloor k_2r_i+1-\frac{1}{D}\rfloor \ge k_2r_i$ as $\lfloor k_2r_i+1-\frac{1}{D}\rfloor = k_2r_i$ if k_2r_i is an integer and $\lfloor k_2r_i+1-\frac{1}{D}\rfloor > k_2r_i$ if k_2r_i is not an integer, we have

$$\sum_{i=1}^{n}(\lfloor k_2r_i+B\rfloor-\lceil (k_1-1)r_i-B\rceil)\ge \sum_{i=1}^{n}k_2r_i-\sum_{i=1}^{n}(k_1-1)r_i\ge k_2-k_1+1.$$

Likewise,

Since $\lceil k_2r_i-1+\frac{1}{D}\rceil \le k_2r_i$ as $\lceil k_2r_i-1+\frac{1}{D}\rceil = k_2r_i$ if k_2r_i is an integer and $\lceil k_2r_i-1+\frac{1}{D}\rceil < k_2r_i$ if k_2r_i is not an integer, we have

$$\sum_{i=1}^{n}(\lceil k_2r_i-B\rceil-\lfloor (k_2-1)r_i+B\rfloor)\le \sum_{i=1}^{n}k_2r_i-\sum_{i=1}^{n}(k_1-1)r_i\le k_2-k_1+1. \qquad \square$$

The lower bound is not always attained, however, the optimal value B^* of the problem often coincides with the lower bound for small size instances [4].

The upper bound is not tight. There exists another upper bound $B^*\le 1-\dfrac{1}{2(n-1)}$ [10].

Therefore, we can write, $B^*\le 1-\max\left\{\frac{1}{D},\frac{1}{2(n-1)}\right\}$ for $n\ge 2$.

It is interesting to investigate those instances with the optimal value that does not exceed $\frac{1}{2}$.

Lemma 2: *An instance has no optimal value less than* $\frac{1}{2}$ *if* $\Delta_i=\dfrac{D}{\gcd(d_i,D)}$, $i=1,\ldots,n$ *is even.*

Proof: We show that any instance with Δ_i even has lower bound $\frac{1}{2}$. Any feasible solution must satisfy $|[kr_i]-kr_i|\le |x_{ik}-kr_i|$ where $[kr_i]$ is the closest integer to kr_i.

If Δ_i is even, $\Delta_i=2k$ for some k.

$$|[kr_i] - kr_i| = \left|\left[\frac{\Delta_i}{2}\frac{d_i}{D}\right] - \frac{\Delta_i}{2}\frac{d_i}{D}\right| = \left|\left[\frac{\Delta_i}{2}\frac{\delta_i}{\Delta_i}\right] - \frac{\Delta_i}{2}\frac{\delta_i}{\Delta_i}\right| = \left|\left[\frac{\delta_i}{2}\right] - \frac{\delta_i}{2}\right| = \frac{1}{2} \text{ as}$$

$\gcd(\delta_i, \Delta_i) = 1$, δ_i is odd, $i = 1, \ldots, n$. □

It is clear that a standard instance i.e. instance with $\gcd(d_i, D) = 1$, $i = 1, \ldots, n$ has lower bound $\frac{1}{2}$ if D is even.

One expects that there may exist instances with $B < \frac{1}{2}$ for Δ_i odd since $|[kr_i] - kr_i| = \dfrac{\Delta_i - 1}{2\Delta_i}$.

For $n = 2$, infinitely many instances with $B^* < \frac{1}{2}$ exist. A sequence with distances $\left\lceil \frac{D}{d_1} \right\rceil$ and $\left\lfloor \frac{D}{d_1} \right\rfloor$ for product 1 with demand d_1 and $\left\lceil \frac{D}{d_2} \right\rceil$ and $\left\lfloor \frac{D}{d_2} \right\rfloor$ for product 2 with demand d_2 is optimal for two product case [6]. Thus, □

Theorem 3: [6] *For $n = 2$, the optimal value of the problem is less than $\frac{1}{2}$ if and only if one of the demands d_1 or d_2 is odd and the other even.*

For $n > 2$, it has been proven, though appeared as the small deviations conjecture that the standard instance has optimal value $B^* < \frac{1}{2}$ if and only if $d_i = 2^{i-1}$, $i = 1, \ldots, n$ and $B^* = \dfrac{2^{n-1} - 1}{2^n - 1}$ [1].

The sufficient condition of the statement is the following.

Theorem 4: [1] *The instance with $d_i = 2^{i-1}$, $i = 1, \ldots, n$, $n \geq 2$ has an optimal sequence with a bound $B^* < \frac{1}{2}$.*

Proof: Consider a standard instance $d_i = 2^{i-1}$, $i = 1, \ldots, n$, $n \geq 2$ and a bound $B^* = \dfrac{2^{n-1} - 1}{2^n - 1}$. Obviously, $B^* < \frac{1}{2}$. Let the copy (i, j) be sequenced at the ideal position $\left\lceil \dfrac{2j-1}{2r_i} \right\rceil = 2^{n-i}(2j-1)$, $i = 1, \ldots, n$.

The copies do not compete for the position. Let $\dfrac{2j-1}{2^{n-i}} = \dfrac{2j'-1}{2^{n-i'}}$ for some positions. Then since both $(2j-1)$ and $(2j'-1)$ are odd, neither 2^i divides $(2j-1)$ nor $2^{i'}$ divides $(2j'-1)$. This implies $i = i'$ and $j = j'$.

Now we show $2^{n-i}(2j-1)$ lies in between $E(i, j)$ and $L(i, j)$.

$$\frac{j-B}{r_i}=\frac{j-\frac{2^{n-1}-1}{2^n-1}}{\frac{2^{i-1}}{2^n-1}}=\frac{(2j-1)(2^n-1)+1}{2^i}=2^{n-i}(2j-1)+\frac{1-(2j-1)}{2^i}\leq$$

$$2^{n-i}(2j-1)\leq 2^{n-i}(2j-1)+1-\frac{2j}{2^i}=\frac{j-1+\frac{2^{n-1}-1}{2^n-1}}{\frac{2^i-1}{2^n-1}}+1=\frac{j-1+B}{r_i}+1$$

Since $2^{n-i}(2j-1)$ is an integer, $E(i,j)=\left\lceil\frac{j-B}{r_i}\right\rceil\leq 2^{n-i}(2j-1)\leq\left\lfloor\frac{j-1+B}{r_i}+1\right\rfloor=L(i,j)$.□

For the necessary case, there exist a geometric proof in [6] and a proof based on balanced word in [2].

The geometric proof exploits a natural symmetry of regular polygons inscribed in a circle of circumference D such that each polygon corresponds to a different product having $d_i, i=1,\ldots,n$ corners for product i at $\left\lceil\frac{2j-1}{2r_i}\right\rceil$ points i.e. the ideal positions on the perimeter of the circle. All the products with demands $d_i=2^{i-1}, i=1,\ldots,n$ are sequenced in the ideal positions [6].

The other proof is based on the concept of balanced word. A δ-balanced word on a finite set $\{1,\ldots,n\}$ is an infinite sequence $s=s_1s_2\ldots$ with $s_i\in\{1,\ldots,n\}$ such that every two subsequences of equal length consist of only those letters whose number of occurrences in each subsequence differ by at most a positive integer δ, (see [11]). Consider a finite word W on $\{1,\ldots,n\}$ of length D with d_i occurrences of a letter i and $r_i=\frac{d_i}{D}$, the rate of letter i with $r_1\leq\ldots\leq r_n$. W is said to be symmetric if $W=W^R$, a mirror reflection of W. An infinite word W is periodic if $w=WW\ldots$ for some W.

It is shown that a periodic, symmetric and balanced word with $r_1<\ldots<r_n, n>2$ exists if and only if $r_i=\frac{2^{i-1}}{2^n-1}$. This is known as Fraenkel's conjecture for symmetric case. Symmetry and balancedness imply that the rates are all different. The small deviations conjecture is shown to be true as a consequence of the Fraenkel's conjecture for symmetric case using the fact that a solution to the problem with $d_i=2^{i-1},\ i=1,\ldots,n,\ n>2$ is periodic, symmetric and balanced word [2].

We find that only the standard instance with $d_i=2^{i-1}$, $i=1,\ldots,n,\ n>2$ has optimal sequence when $B^*<\frac{1}{2}$. What happens for the instances with gcd $(d_i,D)>1$, $i=1,\ldots,n$ is still unresolved. In this paper, one fact is established that no feasible instance exists when $B<\frac{1}{3}$.

Theorem 5: *There is no instance* $(d_1,...,d_n)$ *with* $n \geq 2$ *that has feasible solution with* $B < \frac{1}{3}$.

Proof: Since tight lower bound is $1 - r_{\max}$, $1 - r_{\max} \leq B$ and $E(i,j) \leq L(i,j)$, that is, $\frac{j-B}{r_i} \leq \frac{j-1+B}{r_i} + 1$ for any feasible sequence.

Thus, $1 - r_i \leq 2B$, $\forall i, i = 1,...,n$ and $1 - r_{\min} \leq 2B$.

This implies $\sum_{i'=1}^{n} r_{i'} \leq 2B,\ r_{i'} \neq r_{\min}$.

Therefore, $r_{\max} \leq \sum_{i'=1}^{n} r_{i'} \leq 2B,\ r_{i'} \neq r_{\min}$.

That is, $1 - r_{\max} \geq 1 - 2B$ which is $1 - 2B \leq B$.
Hence, $\frac{1}{3} \leq B$.

4. The Bisection Search

A bisection search algorithm to find an optimal sequence for the problem must run in the interval $\left[1 - r_{\max}, 1 - \frac{1}{D}\right]$. The bisection search is performed using integer instead the rational. For the integral selection to solve the decision problems, the interval is $[D - d_{\max}, D - 1]$ that requires $O(\log D)$ time. The overall complexity of the problem is $O(D \log D)$ [9].

5. Conclusion

The bottleneck product rate variation problem with absolute-deviation objective is pseudo-polynomially solvable. The problem is *CO-NP* but remains open for its exact complexity. There always exists an optimal sequence when the deviation for every product is never more than one unit. However, only the standard instance has optimal absolute deviation less than $\frac{1}{2}$ if and only if the demands are successive powers of two and no instance is feasible when the deviation is less than $\frac{1}{3}$. Optimal value for the instance with gcd $(d_i, D) > 1$ is still unresolved.

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Continuity on a dense subset of a Baire space

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Abstract: It is proved, in particular, that every real valued quasi-continuous mapping on a Baire space X is continuous on a dense subset of X ; furthermore, we proved that every real valued mapping on a hyperconnected Baire space X is continuous on a dense subset of X if each point of discontinuity is of d_2-type. In fact, in the above results, the range space R of real numbers may be generalised to any secod countable space.

Key words and phrases: Quasi-continuity, Baire space, Hyperconnected space, d_1-point, d_2-point.

1. Introduction:

This note stems from the following theorem of Lin [5]:

Theorem 1.1. (Theorem 1 of [5]). *If $f : X \to Y$ is a mapping from a Baire space X to a second countable space Y, then the mapping f is almost continuous on a dense subset of X.*

Although it is easy to observe that under the hypothesis of the above theorem, a mapping may not be continuous on any dense subset of the domain space; and the required example might have been known, but we are unable to cite a source of print. So, we give the following example.

Example: 1.2. Let N be the set of natural numbers and let τ be the topology consists of all sets O such that $O = \phi$ or $O = N$ or $O = \{1, 2, \ldots, n\}$ for some $n\{> 1\}$ in N

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and let $\{R,\mathcal{U}\}$ be the usual topological space of real numbers. Let $f : N \to R$ be a mapping defined by $f(1) = 5$, $f(2) = 7$ and $f(n) = n$ otherwise. It is easy to verify that each point of N is a point of discontinuity of f and so, f is not continuous on any dense subset of N; alghough $\{N,\tau\}$ is a Baire space and $(R,\mathcal{U})$ is a second countable space.

It is then a pertinent and natural question as to under what additional restriction on the mapping or on the space under consideration, the mapping will be continuous on a dense subset of the domain space. Before working on this question, we state some known definitions and results which we need in the sequel.

Definition 1.3. ([4],[6]). A mapping $f : X \to Y$, from a topological space to another, is said to be quasi-continuous at $x \in X$ if for any U, V open such that $x \in U$ and $f(x) \in V$ there exists a non-empty open set $G \subset U$ such that $f(G) \subset V$; the mapping f is called quasi-continuous on $A \subset X$, if it is quasi-continuous at every point $x \in A$. Every continuous mapping is quasi-continuous, but the converse is not necessarily true.

Definition 1.4. [8]. The boundary of a set A in a topological space X is the set A sans its interior and is denoted by $Bd\,A$. The interior and the closure of a set A in X is denoted by $Int\,A$ and $Cl\,A$ respectively.

Definition 1.5. [1]. A Baire space is a topological space in which the intersection of each countable family of open dense subsets is dense. Every non-empty Baire space is a set of the second category, i.e., it is not the union of a countable family of sets E_n such that $\text{Int}\; ClE_n = \phi$ where ϕ denotes the empty set.

Definition 1.6. [2]. Let $f : X \to Y$ be a mapping from a topological space to another. A point $x \in X$ is called a d_1-point (or a point of d_1-type) of f, if there exists an open neighbourhood N of $f(x)$ such that $x \in B\,d f^{-1}(N)$ and $\text{Int}\, f^{-1}(N) = \phi$; and a point $x \in X$ will be called a d_2-point (or a point of d_2-type) of f, if for any open neighbourhood $N$of $f(x)$, there exists an open sub-neighbourhood O of $f(x)$ [i.e, O is a neighbourhood of $f(x)$ and $O \subset N$] such that $x \in Bdf^{-1}(O)$ and $\text{Int}\, f^{-1}(O) \neq \phi$. If each of these points exists then the set of points of discontinuity is partitioned by the set of d_1-points and the set of d_2-points.

Definition 1.7. [7]. A topological space X is called hyperconnected if every pair of non-empty open sets of X has non-empty intersection, or equivalently, every non-empty open set in X is dense in X.

Now, we have come up with the following result.

Theorem 1.8. If $f: X \to Y$ *is quasi-continuous mapping from a Baire space X to a second countable space Y, then the mapping f is continuous on a dense subset of X.*

We further observe that the assumption '*f is quasi-continuous*' can be dropped from Theorem 1.8 if the domain space in addition is hyperconnected provided that each point of discontinuity of f is of d_2-type. Precisely, we have.

Theorem 1.9. If $f: X \to Y$ *is mapping from a hyperconnected Baire space X to a second countable space Y such that the points of discontinuity of f (if any) are of d_2-type, then f is continuous on a dense subset of X.*

The proofs of Theorem 1.8 and Theorem 1.9 are given in the next section. A particular interesting special case of Theorem 1.8 (Theorem 1.9) is obtained by using the usual space R of real numbers in place of the space Y in Theorem 1.8 (Theorem 1.9). Thus

Corollary 1.10. *Every real valued quasi-continuous mapping on a Baire space X is continuous on a dense subset of X.*

Corollary 1.11. *Every real valued mapping on hyperconnected Baire space X is continuos on a dense subset of X if each point of discontinuity of f is of d_2-type.*

We conclude this section by demonstrating that the concepts of Baire space and hyperconnected space are independent with the help of the followng examples.

Example 1.12. The topological space (N, τ) of Example 1.2 is a Baire space as well as a hyperconnected space.

Example 1.13. Let N be the set of positive integers and τ^* be the cofinite topology. Then (N, τ^*) is a hyperconnected space but not a Baire space.

Example 1.14. The usual space R of real numbers is an obvious example of a Baire space which is not hyperconnected.

2. Proofs of main theorems

Before proving Theorem 1.8, we shall need the following lemma

Lemma 2.1. *A mapping $f : X \to Y$ is quasi-continuous at $x \in X$ if and only if $x \in Cl\ Int\ f^{-1}(V)$ for every open set V containing $f(x)$.*

Proof: First we suppose that f is quasi-continuous at $x \in X$. Let U and V be open such that $x \in U$ and $f(x) \in V$. Then there is a non-empty open set $G \subset U$ such that $f(G) \subset V$. So, $\phi \neq G \subset \text{Int}\ f^{-1}(V)$. Now, if $x \in G$ then $x \in Cl\ \text{Int}\ f^{-1}(V)$. Again, if $x \notin G$ then $\phi \neq G \subset (U \setminus \{x\}) \cap \text{Int}\ f^{-1}(V)$ and hence $x \in Cl\ \text{Int}\ f^{-1}(V)$.

Next, let $x \in Cl\ \text{Int}\ f^{-1}(V)$ for every open set V containing $f(x)$. Let U be any open set containing x. Then $U \cap \text{Int}\ f^{-1}(V) \neq \phi$. We take $G = U \cap \text{Int}\ f^{-1}(V)$. Hence G is a non-empty open set such that $G \subset U$ and $f(G) \subset f f^{-1}(V) \subset V$.

Proof of Theorem: 1.8. Let $B = \{B_n : n = 1, 2, \ldots\}$ be a countable basis for the open sets in Y. We select $B^* = \{B_{n_1}, B_{n_2}, \ldots\} \subset B$ such that $\text{Int}\ f^{-1}(B_{n_k}) \neq \phi$, $k = 1, 2, \ldots$. This sub-class B^* is non-empty ; for, if $x \in X$ then by the above lemma $x \in Cl\ \text{Int}\ f^{-1}(B_n)$ for each $B_n \in B$ ontaining $f(x)$ because f is quasi-continuous, and so, $\text{Int}\ f^{-1}(B_n) \neq \phi$. Now, for each $B_{n_k} \in B^*$, let us set $E_{n_k} = Cl\,\text{Int}\ f^{-1}(B_{n_k}) \setminus \text{Int}\ f^{-1}(B_{n_k})$. Then $\text{Int}\ Cl\ E_{n_k} = \text{Int}\ E_{n_k} = \text{Int}\ Cl\ \text{Int}\ f^{-1}(B_{n_k}) \setminus Cl\,\text{Int} f^{-1}(B_{n_k}) = \phi$ for such k, and thus the set $E = \cup_{k=1}^{\infty} E_{n_k}$ is a set of the first category. But if f is not continuous at x, then there exists a $B_{n_k} \in B^*$ such that $f(x) \in B_{n_k}$ and $x \notin \text{Int}\ f^{-1}(B_{n_k})$. But since f is quasi-continuous at $x, x \in Cl\ \text{Int}\ f^{-1}(B_{n_k})$ by the lemma stated earlier. Hence $x \in E_{n_k}$ for some k and so, $x \in E$. Therefore, f is continuous on $X \setminus E$, which as a complement of a first category subset of a Baire space, is dense in X.

Proof of Theorem 1.9. Let $B = \{B_n : n = 1, 2, \ldots\}$ be a countable basis for the open sets in Y. We select $B^* = \{B_{n_1}, B_{n_2}, \ldots\} \subset B$ such that $\text{Int}\ f^{-1}(B_{n_k}) \neq \phi$, $k = 1, 2, \ldots$ This sub-class B^* is non-empty, because if x is a point of discontinuity then x is a d_2-point and hence there exists a $B_n \in B$ such that $x \in Bd\, f^{-1}(B_n)$ and $\text{Int}\ f^{-1}(B_n) \neq \phi$. Now, for each $B_{n_k} \in B^*$, we set $E_{n_k} = f^{-1}(B_{n_k}) \setminus \text{Int}\ f^{-1}(B_{n_k})$. Since X is hyperconnected, for each $B_{n_k} \in B^*$, $Cl\ \text{Int}\ f^{-1}(B_{n_k}) = X$. So, for each k, $\text{Int}\ Cl\ E_{n_k} \subset \text{Int}\ Cl\ f^{-1}(B_{n_k}) \setminus Cl\ \text{Int}\ f^{-1}(B_{n_k}) = \phi$, i.e., E_{n_k} is nowhere dense for each k ; and thus the set $F = \cup_{k=1}^{\infty} E_{n_k}$ is a set of the first category. But if f is not continuous

at x, there exists a $B_{n_k} \in B^*$ such that $f(x) \in B_{n_k}$ and that $x \notin \operatorname{Int} f^{-1}(B_{n_k})$, consequently, $x \in E_{n_k}$ because $x \in f^{-1}(B_{n_k})$. Hence $x \in E$, and so, f is continuous on the dense subset $X \setminus E$ of X.

Remark 2.2. The condition '*f is quasi-continuous on X*' in Theorem 1.8 is sufficient but not a necessary one as shown in the following example.

Example 2.3. Consider the topological spaces (N, τ) and $(R, \mathcal{U})$ of Example 1.2. Let $f : N \to R$ be a mapping defined by $f(2) = 1$ and $f(n) = n$ otherwise. Clearly, f is not quasi-continuous at $n(\neq 1,2)$; but f is continuous on the dense subset $\{1,2\}$ of N.

Remark 2.4. Since each point of discontinuity of f, viz., $n(\neq 1,2)$ of Example 2.3 is a d_1-point, it is clear that the hypothesis '*points of discontinuity of f (if any) are of d_2-type*' of Theorem 1.9 is only a sufficient condition but not a necessary one.

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A note on common fixed point principle

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The study of common fixed point of mappings satisfying contractive type conditions has been a very active field of research activity during the last three decades. The most general of the common fixed point theorems pertain to four mappings, say A, B, S and T of a metric space (X, d), and use either a Banach type contractive condition of the form,

(1) $$d(Ax, By) \leq h\, m(x, y), \qquad \text{for } 0 \leq h < 1,$$

where $m(x,y) = \max\,\{d(Sx, Ty),\ d(Ax, Sx),\ d(By, Ty),\ [d(Sx, By) + d(Ax, Ty)]/2\}$, or, a Meir–Keeler type (ε, δ)-contractive condition of the form,

given $\varepsilon > 0$, there exists a $\delta > 0$ such that

(2) $$\varepsilon \leq m\,(x, y) < \varepsilon + \delta \Rightarrow \mathrm{d}(Ax, By) < \varepsilon,$$

or, a ϕ-contractive condition of the form,

(3) $$d\,(Ax, By) \leq \phi\,(m(xy)),$$

involving a contractive gauge function $\phi : \Re_+ \to \Re_+$ is such that $\phi(t) < t$ for each $t > 0$. Also, the weak form of contractive condition (2) is of the form,

given $\varepsilon > 0$, there exists a $\delta > 0$ such that

(4) $$\varepsilon < m(x,y) < \varepsilon + \delta \Rightarrow d(Ax, By) \leq \varepsilon.$$

Clearly, condition (1) is a special case of both conditions (2) and (3). Moreover, Jachymski [2] has shown that contractive condition (2) implies (4) but the contractive condition (4) does not imply the contractive condition (2). Pant [4] proved the following common fixed point theorem.

Theorem 2.1 [4]: *Let (A,S) and (B,T) be compatible pairs of self maps of a complete metric space (X,d) and such that $AX \subset TX$, $BX \subset SX$ and*

(i) *given* $\varepsilon > 0$, there exists a $\delta > 0$ *such that*

$$\varepsilon < \max\{d(Sx,Ty),\ d(Ax,Sx),\ d(By,Ty),\ [d(Sx,By) + d(Ax,Ty)]/2\} \leq \varepsilon + \delta$$
$$\Rightarrow d(Ax,By) \leq \varepsilon,$$

(i)* $d(Ax,By) < \max\{d(Sx,Ty),\ d(Ax,Sx),\ d(By,Ty),\ [d(Sx,By) + d(Ax,Ty)]/2\}$ *whenever the right hand side is positive,*

(ii) $d(Ax,By) \leq \max\{d(Sx,Ty),\ d(Ax,Sx),\ d(By,Ty),\ k[d(Sx,By) + d(Ax,Ty)]/2\}$ *where* $0 \leq k < 2$ *and* $\phi\colon \Re_+ \to \Re_+$ *is such that* $\phi(t) < t$ *for each* $t > 0$.

If one of the mappings A, B, S or T is continuous, then A, B, S and T have a unique common fixed point.

This theorem gives a more general approach of generalizing the known common fixed point theorems for four mappings by assuming slightly weaker form of a Meir-Keeler type contractive condition together with a Lipschitz type contractive condition of the form (ii).

The main objective of this note is to provide a correction to a minor error in the proof of Theorem 2.1 of [4]. On page 293 (line 16–20) of Pant [4], condition (ii) has been used to arrive at the contradiction $d(AAz, Bw) < d(AAz, Bw)$.

However, by using condition (ii), we can obtain the above contradiction only if, $k \leq 1$ and not for $k > 1$. The correct approach to arrive at the desired contradiction would be to use condition (i)* in place of condition (ii) and then replace lines 16 – 20 on page 293 by the following derivation.

If $Az \neq AAz$, then using (i)*, we get

$$\begin{aligned} d(Az, AAz) &= d(AAz, Bw) \\ &< \max\{d(SAz, Tw),\ d(AAz, SAz),\ d(Bw, Tw), \\ &\qquad [d(AAz, Tw) + d(Bw, SAz)]/2\} \\ &= d(AAz, Bw), \end{aligned}$$

which is a contradiction.

The remaining part of proof of Theorem 2.1 of [4] remains unaltered.

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Approximation of functions belonging to Lip α class by $(N, p_n)(E,1)$ means of its Fourier series

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Abstract: In this paper, the degree of approximation of a function belonging to Lip α class by $(N, p_n)(E,1)$ means of its Fourier series has been determined.

1. Definitions and Notations

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series whose n^{th} partial sum s_n is given by $s_n = \sum_{v=0}^{n} u_v$.

If

(1) $$E_n^1 = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}s_k \to s \quad \text{as } n \to \infty$$

then an infinite series $\sum_{n=0}^{\infty} u_n$ or $\{s_n\}$ is said to be summable to the definite number s by $(E,1)$ method (Hardy [3]).

Let $\{p_n\}$ be a sequence of real constants such that $p_0 > 0, p_n \geq 0 \ \forall \ n \geq 1$. The sequences to sequence transformation

(2) $$t_n^p = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}s_k = \frac{1}{P_n}\sum_{k=0}^{n} p_k s_{n-k}$$

defines the sequence $\{t_n\}$ of Nörlund means of the sequence $\{s_n\}$ generated by the coefficients $\{p_n\}$. If $t_n^p \to s$ as $n \to \infty$, the series $\sum_{n=0}^{\infty} u_n$ is said to be summable (N, p_n) to the sum s.

The (N,p_n) transform of the $(E,1)$ transform defines the $(N,p_n)(E,1)$ transform of the partial sum s_n of the series $\sum_{n=0}^{\infty} u_n$.

Thus if

$$t_n^{NE} = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}\, E_k^1 = \frac{1}{P_n}\sum_{k=0}^{\infty} p_k\, E_{n-k}^1 \quad \text{as} \quad n \to \infty$$

then the series $\sum_{n=0}^{\infty} u_n$ is said to be summable by $(N,p_n)(E,1)$ means or simply, summable $(N,p_n)(E,1)$ to s.

A function $f \in \text{Lip}\,\alpha$ if

(3) $$f(x+t) - f(x) = 0(|t|^{\alpha}), \quad 0 < \alpha \le 1.$$

Let $f(x)$ be periodic with period 2π and Lebesgue integrable on $[-\pi, \pi]$ and belonging to Lip α class. The Fourier series of $f(x)$ is given by

(4) $$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{n}(a_n \cos nx + b_n \sin nx).$$

The degree of approximation of a function $f : \mathrm{R} \to \mathrm{R}$ by a trigonometric polynomial t_n of order n is defined by

(5) $$\left\| t_n + f \right\|_{\infty} = \sup\left\{ |\, t_n(x) - f(x) \,| : x \in \mathrm{R} \right\}, \quad \text{(Zygmund [9])}.$$

We use the following notations throughout this paper:

$$\phi(t) = f(x+t) + f(x-t) - 2f(x),$$

(6) $$N_n(t) = \frac{1}{2\pi P_n}\sum_{k=0}^{n} P_k \frac{\operatorname{Cos}^{n-k}\left(\frac{t}{2}\right)\sin\,(n-k+1)\frac{t}{2}}{\sin\frac{t}{2}}$$

τ = Integral part of $\frac{1}{t} = \left[\frac{1}{t}\right]$.

2. Theorem: *The degree of approximation of a function f belonging to Lip α by Cesàro means and Nörlund means has been discussed by a number of researchers like Alexits [1], Sahney and Goel [8], Chandra [2], Qureshi [5], Qureshi [6] and Qureshi and Neha [7]. The purpose of this paper is to determine the degree of approximation of a*

function belonging to Lip α class by (N, p_n) (E, 1) product means of its Fourier series in the following form:

Theorem: *Let (N, p_n) be a regular Nörlund summability method generated by a positive, monotonic decreasing sequence $\{p_n\}$ of real constants. If $f: [-\pi, \pi] \to \mathrm{R}$ is 2π periodic, Lebesgue integrable on $[-\pi,\pi]$ and belonging to Lip α class then the degree of approximation of f by the (N, p_n) (E, 1) means $t_n^{p,E} = \dfrac{1}{p_n}\sum_{k=0}^{n} p_k E_{n-k}^1$ of its Fourier series* (4) *is given by*

$$\left\| t_n^{NE} - f \right\|_\infty = \begin{cases} O\left(\dfrac{1}{(n+1)^a}\right), 0 < a < 1 \\ O\left(\dfrac{\log(n+1)\pi e}{n+1}\right), a = 1. \end{cases}$$

3. Lemma: *The proof of the theorem requires following lemmas.*

Lemma 1. *Let $N_n(t)$ be given by (6), then*

$$N_n(t) = 0(n+1)\,,\ \text{for}\ \ 0 < t < \frac{1}{n+1}.$$

Proof:

$$\left| N_n(t) \right| \le \frac{1}{2\pi P_n}\sum_{k=0}^{n} p_k \left| \frac{\cos^{n-k}\left(\frac{t}{2}\right)\sin(n-k+1)\frac{t}{2}}{\sin\frac{t}{2}} \right|$$

$$\le \frac{1}{2\pi P_n}\sum_{k=0}^{n} p_k (n-k+1)\frac{\left|\sin\frac{t}{2}\right|}{\left|\sin\frac{t}{2}\right|}$$

$$= \frac{1}{2\pi P_n}\sum_{k=0}^{n} p_k (n-k+1)$$

$$= (n+1)\frac{1}{2\pi P_n}\sum_{k=0}^{n} p_k$$

$$= \left(\frac{n+1}{2\pi}\right)$$

$$= O\,(n+1)$$

which completes the proof of lemma 1.

Lemma 2. [*Mc Fadden* (4)].

If $\{p_n\}$ is a non-negative and non-increasing sequence, then for $0 \le a < b \le \infty$, $0 \le t \le \pi$ and for any n,

$$\left| \sum_{a}^{b} p_k e^{i(n-k)t} \right| = O(P_\tau)$$

where $P_\tau = P_{\left[\frac{1}{t}\right]}$ and $\tau = \left[\frac{1}{t}\right]$.

Lemma 3. $N_n(t) = O\left(\frac{P_t}{tP_n}\right)$, for $\frac{1}{n+1} < t < \pi$.

Proof: Since, for $\frac{1}{n+1} < t < \pi$, $\sin\frac{t}{2} > \frac{t}{\pi}$, therefore

$$\left| N_n(t) \right| = \frac{1}{2\pi P_n} \left| \sum_{k=0}^{n} p_k \frac{\cos^{n-k}\left(\frac{t}{2}\right) \sin(n-k+1)\frac{t}{2}}{\sin\frac{t}{2}} \right|$$

$$= \frac{1}{2\pi P_n} \left| \frac{I_m \sum_{k=0}^{n} p_k \cos^{n-k}\left(\frac{t}{2}\right) e^{i(n-k+1)\frac{t}{2}}}{\left|\sin\frac{t}{2}\right|} \right|$$

$$\le \frac{1}{2tP_n} \left| \sum_{k=0}^{n} p_k \cos^{n-k}\left(\frac{t}{2}\right) e^{i(n-k)\frac{t}{2}} \right| \left| e^{i\frac{t}{2}} \right|$$

$$\le \frac{1}{2tP_n} \left| \sum_{k=0}^{n} p_k e^{i(n-k)\frac{t}{2}} \right|$$

$$= O\left(\frac{P_\tau}{tP_n}\right), \text{ by lemma (2).}$$

4. Proof of the Theorem: The n^{th} partial sum $S_n(x)$ of the Fourier series (4) is given by

$$S_n(x) - f(x) = \frac{1}{2\pi} \int_0^\pi \phi_x(t) \frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}} dt.$$

Then

$$\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}(S_k(x)-f(x))=\frac{1}{2^{n+1}\pi}\int_0^{\pi}\frac{\phi_x(t)}{\sin\frac{t}{2}}\left[\sum_{k=0}^{n}\binom{n}{k}\sin\left(k+\frac{1}{2}\right)t\right]dt.$$

or

$$E_n^1(x)-f(x)=\frac{1}{2^{n+1}\pi}\int_0^{\pi}\frac{\phi_x(t)}{\sin\frac{t}{2}}I_m\left\{e^{\frac{it}{2}}(1+e^{it})^n\right\}dt$$

$$=\frac{1}{2^{n+1}\pi}\int_0^{\pi}\frac{\phi_x(t)}{\sin\frac{t}{2}}I_m\left\{e^{\frac{it}{2}}\left(1+\cos t+i\sin t\right)^n\right\}dt$$

$$=\frac{1}{2^{n+1}\pi}\int_0^{\pi}\frac{\phi_x(t)}{\sin\frac{t}{2}}I_m\left\{e^{\frac{it}{2}}2^n\cos^n\left(\frac{t}{2}\right)\left(\cos\frac{t}{2}+i\sin\frac{t}{2}\right)^2\right\}dt$$

$$=\frac{1}{2^{n+1}\pi}\int_0^{\pi}\frac{\phi_x(t)}{\sin\frac{t}{2}}2^n\cos^n\left(\frac{t}{2}\right)I_m\left(\cos\frac{t}{2}+i\sin\frac{t}{2}\right)\left\{\cos\frac{nt}{2}+i\sin\frac{nt}{2}\right\}dt$$

$$=\frac{1}{2\pi}\int_0^{\pi}\frac{\phi_x(t)}{\sin\frac{t}{2}}\cos^n\left(\frac{t}{2}\right)\sin(n+1)\frac{t}{2}dt.$$

Now,

$$\frac{1}{P_n}\sum_{k=0}^{n}p_k\left(E_{n-k}^1(x)-f(x)\right)=\int_0^{\pi}\left[\frac{1}{2\pi P_n}\sum_{k=0}^{n}p_k\frac{\cos^{n-k}\left(\frac{t}{2}\right)\sin(n-k+1)\frac{t}{2}}{\sin\frac{t}{2}}\right]\phi(t)dt.$$

or,

$$t_n^{NE}(x)-f(x)=\int_0^{\pi}N_n(t)\phi(t)dt$$

(7)
$$=\int_0^{\frac{1}{n+1}}N_n(t)\phi(t)dt+\int_{\frac{1}{n+1}}^{\pi}N_n(t)\phi(t)dt$$

$$=I_1+I_2 \text{ say.}$$

Now,

$$\begin{aligned}\phi(t)&=f(x+t)+f(x-t)-2f(x)\\&=\{f(x+t)-f(x)\}+\{f(x-t)-f(x)\}\\&=O(t^{\alpha})+O(t^{\alpha})\quad(\because f\in \mathrm{Lip}^{\alpha})\\&=O(t^{\alpha}).\end{aligned}$$

We have,

$$|I_1| \leq \int_0^{\frac{1}{n+1}} \left| N_n(t) \right| \left| \phi(t) \right| dt$$

$$= O(n+1) \int_0^{\frac{1}{n+1}} |\phi(t)|\, dt, \text{ by lemma 1}$$

$$= O(n+1) \int_0^{\frac{1}{n+1}} t^{\alpha} dt$$

$$= O(n+1) \left[\frac{t^{\alpha+1}}{\alpha+1} \right]_0^{\frac{1}{n+1}}$$

$$= O(n+1) \left[\frac{1}{(n+1)^{\alpha+1}} \right]$$

$$(8) \qquad = O\left[\frac{1}{(n+1)^{\alpha}} \right].$$

Next,

$$|I_2| \leq \int_{\frac{1}{n+1}}^{\pi} \left| N_n(t) \right| \left| \phi(t) \right| dt$$

$$= \int_{\frac{1}{n+1}}^{\pi} O\left(\frac{P_\tau}{tP_n} \right) |\phi(t)|\, dt, \text{ by lemma 3}$$

$$= O\left(\frac{1}{P_n} \right) \int_{\frac{1}{n+1}}^{\pi} \frac{P_\tau}{t} t^{\alpha} dt$$

$$= O\left(\frac{1}{P_n} \right) \int_{\frac{1}{\pi}}^{n+1} \frac{P_\tau}{u^{\alpha+1}} du$$

$$= O\left(\frac{P_{n+1}}{(n+1)P_n} \right) \int_{\frac{1}{\pi}}^{n+1} \frac{1}{u^{\alpha}} du \left(\because \frac{P_{[u]}}{u} \text{ is decreasing} \right)$$

$$= O\left(\frac{1}{(n+1)} \right) \int_{\frac{1}{\pi}}^{n+1} \frac{1}{u^{\alpha}} du.$$

$$= O\left(\frac{1}{n+1}\right)\begin{cases}\left(\dfrac{u^{-\alpha+1}}{-\alpha+1}\right)_{\frac{1}{\pi}}^{n+1}, & \alpha \neq 1\\ (\log u)_{\frac{1}{\pi}}^{n+1}, & \alpha = 1\end{cases}$$

$$(9) \qquad = O\left(\frac{1}{n+1}\right)\begin{cases}\dfrac{1}{(1-\alpha)}\left(\dfrac{1}{(n+1)^{\alpha-1}} - \pi^{\alpha-1}\right), & \alpha \neq 1\\ \log(n+1)\pi, & \alpha = 1.\end{cases}$$

By (7), (8) and (9), we have

$$t_n^{NE}(x) - f(x) = \begin{cases} O\left(\dfrac{1}{(n+1)^{\alpha}}\right) + O\left(\dfrac{1}{(n+1)}\right)\left[\dfrac{1}{(1-\alpha)(n+1)^{\alpha-1}} - \pi^{\alpha-1}\right], & \alpha \neq 1\\ O\left(\dfrac{1}{n+1}\right) + O\left(\dfrac{\log(n+1)\pi}{n+1}\right), & \alpha = 1\end{cases}$$

$$= \begin{cases} O\left(\dfrac{1}{(n+1)^{\alpha}}\right) + O\left(\dfrac{1}{(1-\alpha)(n+1)^{\alpha}}\right) + O\left(\dfrac{1}{(n+1)^{\alpha}\pi^{1-\alpha}}\right), & \alpha \neq 1\\ O\left(\dfrac{1}{(n+1)}\right) + O\left(\dfrac{\log(n+1)\pi}{n+1}\right), & \alpha = 1\end{cases}$$

$$= \begin{cases} O\left(\dfrac{1}{(n+1)^{\alpha}}\right), & 0 < \alpha < 1\\ O\left(\dfrac{\log(n+1)\pi e}{n+1}\right), & \alpha = 1.\end{cases}$$

Then

$$\left\| t_n^{NE} - f \right\|_{\infty} = \sup\left\{ |t_n^{NE}(x) - f(x)| : -\pi \leq x \leq \pi \right\}.$$

or,

$$\left\| t_n^{NE} - f \right\|_{\infty} = \begin{cases} O\left(\dfrac{1}{(n+1)^{\alpha}}\right), & 0 < \alpha < 1\\ O\left(\dfrac{\log(n+1)\pi e}{n+1}\right), & \alpha = 1.\end{cases}$$

Hence the theorem is completely established.

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On ϕ-recurrent Lorentzion para-Sasakian manifolds

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Abstract: In this paper we introduce the notion of ϕ-recurrent LP-Sasakian manifolds and show that such a manifold is an Einstein manifold we have also proved that the Charecteristic vector field ξ and the vector field ρ associated to the 1-form A are not co-directional.

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1. Introduction

In (1989) K. Mstsumoto [4] introduced the notion of LP-Sasakian manifold. I. Mihai and R. Rosca [3] define same notion independently and therefore many authors [4], [5] studied LP-Sasakian manifolds. In (2003) U. C. De, A. A. Shaikh and Sudipta Biswas introduced the notion of ϕ-recurrent Sasakian manifold [1].

In this paper we introduced the notion of ϕ-recurrent LP-Sasakian manifold locally symmeric LP-Sasakian manifold and obtained some interesting results.

2. Preliminaries

A differentiable (smooth) manifold of dimension n is called LP-Sasakian manifold. [2], [3] if it admists a tensor filed ϕ, of type (1.1) a contravariant vector field η, and a covariant vector field η and a Lorentzian metric g which satisfy.

(2.1) $\phi^2 = 1 + \eta \otimes \xi$

(2.2) (a) $\eta(\xi) = -1$ (b) $g(X,\xi) = \eta(X)$ (c) $\eta(\phi X) = 0$ (d) $\phi\xi = 0$

(2.3) $g(\phi X, \phi Y) = g(X,Y) + \eta(X)\,\eta(Y)$

(2.4) (a) $(\nabla_X \phi)(Y) = [\,g(X,Y) + \eta(X)\,\eta(Y)\,]\,\xi + [\,X + \eta(X)\,\xi\,]\,\eta(Y)$

(b) $\nabla_X \xi = \phi X$ (c) $(\nabla_X \eta)\,(Y) = g(X, \phi Y)$

(2.5) $R(\xi, X)\,Y = g(X,Y)\,\xi - \eta(Y)X$

(2.6) $R(X,Y)\,\xi = \eta(Y)X - \eta(X)Y$

(2.7) $R(X,\xi)\,Y = \eta(Y)X - g(X,Y)\,\xi$

(2.8) $\eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\,\eta(Y)$

(2.9) $S(X,\xi) = (n-1)\,\eta(X)$

(2.10) $S(\phi X, \phi Y) = S(X,Y) + (n-1)\,\eta(X)\,\eta(Y)$

for all vector field X, Y, Z where ∇ denotes the operator of covariant differentiation with respect to Lorentzian metric g, S is the Ricci tensor of type (0, 2) and R is the Riemanian Curvature tensor of the manifold.

Definition (I) *An LP-Sasakian manifold is said to be locally ϕ-symmetnic manifold if $\phi^2((\nabla_W R)(X,Y)Z) = 0$ for all vector field X, Y, Z, W orthogonal to ξ.*

Definition (II) *An LP-Sasakian manifold is said to be a ϕ-recurrent manifold if there exists a non-zero 1-form A such that,*

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)\; R(X,Y)Z.$$

If the 1–form A vanishes, then the manifold reduces to a ϕ-symmetnic manifold.

3. ϕ-recurrent Lorentzian Para-Sasakian manifolds

Let us consider a ϕ-recurrent LP-Sasakian manifolds then from (2.1) and (2.2) we have,

(3.1) $(\nabla_W R)\,(X,Y)Z + \eta((\nabla_W R)\,(X,Y)Z)\,\xi = A(W)\,R(X,Y)Z$

from which it follows that

(3.2) $g((\nabla_W R)\,(X,Y)Z, U) + \eta((\nabla_W R)\,(X,Y)Z)\,\eta(U) = A(W)\; g(R(X,Y)Z, U)$

let $\{e_i\}$, $i = 1, 2, 3 \ldots n$ be orthnoramal basis of the tangent space at a point of the manifold. Then putting $X = U = e_i$ in (3.2) and taking summation over $i, 1 \leq i \leq n$, we get

$$(3.3) \qquad (\nabla_W S)(Y,Z) + \sum_{i=1}^{n} \eta((\nabla_W R)(e_i, Y)Z)\eta(e_i) = A(W)\, S(Y,Z)$$

The second term of equation (3.3) by putting $Z = \xi$ takes the form

$$(3.4) \qquad g((\nabla_W R)(e_i, Y)Z, \xi)\,\eta(e_i) = g((\nabla_W R)(e_i, Y)\xi, \xi)\, g(e_i, \xi)$$

$$= -g((\nabla_W R)(e_i, Y)\xi, \xi)$$

Since $\qquad g((\nabla_W R)(e_i, Y)\xi, \xi) = g(R(e_i \nabla_W Y)\xi, \xi) - g(R(e_i, Y)\nabla_W \xi, \xi)$

$$- g(R(e_i, Y)\xi, \nabla_W \xi)$$

and $\{e_i\}$ is an orthonormal basis $\nabla_X e_i = 0$ above equation reduces to

$$(3.5) \quad g((\nabla_W R)(e_i, Y)\xi, \xi) = -g(R(e_i, Y) - g(R(\nabla_W \xi . \xi) - g(R(e_i, Y)\xi, \nabla_W \xi)$$

using (2.4) and applying the skew symmetric of R, we get

$$(3.6) \qquad g((\nabla_W R)(e_i, Y)\xi, \xi) = -g(R(\phi W, \xi)Y, e_i) - g(R(\xi, \phi W)Y, e_i)$$

putting this result in (3.4), we get

$$(3.7) \qquad -g((\nabla_W R)(e_i, Y)\xi, \xi) = g(R(\phi W, \xi)Y, e_i) + g(R(\xi, \phi W)Y, e_i)$$

from (3.4), (3.7) and putting $Z = \xi$, we get

$$(3.8) \qquad (n-1)\, g(W, \phi Y) - S(Y, \phi W) = A(W)\eta(Y)$$

replacing Y by ϕY in (3.8) and using (2.1) (2.2) in (3.8), we get

$$S(Y, W) = (n-1)\, g(Y, W)$$

This leads to the following.

Theorem 3.1: *A ϕ-recurrent Lorentzian para-Sasakian manifold M^n is an Einstein manifold.*

Further from (3.1), we have

$$(3.9) \qquad (\nabla_W R)(X,Y)Z = -\eta((\nabla_W R)(X,Y)Z)\xi + A(W)\, R(X,Y)Z$$

from (3.9) and the Bianchi identify we have

$$(3.10) \qquad A(W)\,[g(Y,Z)\,\eta(X) - g(X,Z)\eta(Y)] + A(X)\,[g(W,Z)\,\eta(Y)$$

$$-g(Y,Z)\,\eta(W)] + A(Y)\,[g(X,Z)\,\eta(W) - g(W,Z)\,\eta(X)] = 0$$

putting $Y = Z = e_i$ in (3.10) and taking summation over $i, 1 \le i \le n$, we have $A(W)\eta(X) = A(X)\eta(W)$ for all vector field X, W replacing X by ξ in (3.11)

$$(3.11) \qquad A(W) = -\rho(\xi)\,\eta(W).$$

where $A(\xi) = g(\xi, \rho) = r(\rho)$, ρ being vector field associated to the I-form A, that is $g(X, \rho) = A(X)$.

Theorem 3.2: *In a ϕ–recurrent Lorentzian para–Sasakian manifold the charecteristics vector field ξ and the vector field ρ associated to the I-form A are not co-directional and I-form A is given by $A(W) = -\eta(\rho)\eta(W)$.*

Since

$$R(X,Y)\phi W = \nabla X \nabla Y \phi W - \nabla Y, \nabla X \phi W - \nabla [X,Y]\phi W. \tag{3.12}$$

Also from $(\nabla_X \phi)(Y) = \nabla_X \phi(Y) - \phi(\nabla_X Y)$ {we get}

$$(\nabla_X \phi)(Y) = (\nabla_X \phi)(Y) + \phi(\nabla_X Y) \tag{3.13}$$

using $(\nabla_X \phi)(Y) = \eta(Y)X + g(X,Y)X + 2\eta(X)\eta(Y)\xi$ and (3.13) in (3.12) we obtained

$$\begin{aligned} R(X,Y)\phi W = {} & \phi R(X,Y)W + g(Y,W)\phi X - g(X,W)\phi Y + g(X,\phi W)Y \\ & - g(Y,\phi W)X + 2\{g(X,\phi W)\eta(Y) - g(Y,\phi W)\eta(X)\xi \\ & + 2\{\eta(Y)\phi X - \eta(X)\phi Y\}\eta(W). \end{aligned} \tag{3.14}$$

We now consider that LP-Sasakian manifold (M^n, g) is ϕ-recurrent then from (3.9), we get

$$(\nabla_W R)(X,Y,Z) = -\eta(\nabla_W R)(X,Y,Z)\xi + A(W)R(X,Y,Z) \tag{3.15}$$

and from (3.14) we have

$$\begin{aligned} g((\nabla_W R)(X,Y)\xi, Z)\xi = {} & \{g(X,W)g(\phi, Y,Z) - g(Y,W)g(\phi X,Z)X \\ & + g(\phi R(X,Y)W,Z)\xi \end{aligned}$$

using above this result in (3.15), we get

$$\begin{aligned} g((\nabla_W R)(X,Y,Z) = {} & \{g(X,W)g(\phi Y,Z) - g(Y,W)g(\phi X,Z)\}\xi \\ & + g(\phi R(X,Y)W,Z)\xi + A(W)R(X,Y)Z \end{aligned} \tag{3.16}$$

This leads to the following.

Theorem 3.2: *If an LP-Sasakian manifold is ϕ-recurrent then the relation holds.*

$$\begin{aligned} (\nabla_W R)(X,Y,Z) = {} & \{g(X,W)g(\phi Y,Z) - g(Y,W)g(\phi X,Z)\}\xi \\ & + g(\phi R(X,Y)W,Z)\xi + A(W)R(X,Y)Z \end{aligned}$$

we have more over if the relation (3.16) holds in LP-Sasakian manifold than,

$$\phi^2((\nabla_W R)(X,Y)Z) = \phi^2\{A(W)R(X,Y)Z\} \tag{3.17}$$

using (2.1), (2.8) in (3.17) which yeilds

$$\phi^2\left((\nabla_W R)(X,Y)Z\right) = A(W)R(X,Y)Z)$$

if X and Y are orthogonal to ξ.

We can state the theorem.

Theorem 3.3: *In an LP-Sasakian manifold (M^n, g) satisfying the relation (3.16) is ϕ–recurrent provided that X and Y orthogonal to ξ.*

Let us suppose that in a ϕ-recurrent LP-Sasakian manifold the sectional curvature of a plane $\pi_C T_p(M)$ is defined by

$$k_p(\pi) = g(R(X,Y)Y,X) \tag{3.18}$$

is a non zero constant k, where X, Y is any orthonormal basis of π then from (3.18), we get

$$g((\nabla_z R)(X,Y)Y,X) = 0 \tag{3.19}$$

from (3.9) we get

$$g((\nabla_z R)\ (X,Y)Y,\xi)\eta(X) = A(Z)\ g(R(X,Y)Y,X). \tag{3.20}$$

Since in a ϕ-recurrent LP-Sasakian manifold the relation (3.16) holds, using (3.16) in (3.20) we get

$$\begin{aligned}\eta(X)[g(Y,Z)g(\phi X,Y) - g(X,Z)g(\phi Y,X) - g(\phi R(X,Y)Z,Y)]\\ + A(Z)\ [g(Y,Y)\eta(X) - g(X,Y)\eta(Y)] = kA(Z).\end{aligned} \tag{3.21}$$

putting $Z = \xi$ in (3.21) we obtain

$$A(\xi)\ [g(Y,Y)\eta(X) - g(X,Y)\eta(Y) - k] = 0$$

$$\eta(\rho)\ [g(Y,Y)\eta(X) - g(X,Y)\eta(Y) - k = 0.$$

Which implies that $\eta(\rho) = 0$ then from (3.11) and definition (II) we have .

$$\phi^2(\nabla_W R)(X,Y)Z = 0.$$

We state the theorem.

Theorem 3.4: *If a ϕ-recurrent LP-Sasakian has a non zero constant sectional curvature, then it reduces to a locally ϕ-symmetric LP-Sasakian manifold*

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Related fixed point theorems for set-valued mappings on two complete and compact metric spaces

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Abstract: A related fixed point theorem for set-valued mappings on two complete metric spaces is obtained. A generalization for two compact metric spaces is also obtained.

Keywords and Phrases: Fixed point, set-valued mappings, complete metric space, compact metric space.

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1. Introduction

The following theorem was proved by Namdeo, Tiwari, Fisher and Tas [3].

Theorem 1.1: *Let (X, d) and (Y, ρ) be complete metric spaces, let T be mapping of X into Y and let S be a mapping of Y into X satisfying the inequalities*

$$d(Sy, Sy')\, d(STx, STx') \leq c \max \{d(Sy, Sy')\, \rho(Tx, Tx'),$$
$$d(x', Sy)\, \rho(y', Tx),\ d(x, x')\, d(Sy, Sy'),\ d(Sy, STx)\, d(Sy', STx')\}$$

$$\rho(Tx, Tx')\, \rho(TSy, TSy') \leq c \max \{d(Sy, Sy')\, \rho(Tx, Tx'),$$
$$d(x', Sy)\, \rho(y', Tx),\ \rho(y, y')\, \rho(Tx, Tx'),\ \rho(Tx, TSy)\, \rho(Tx', TSy')\}$$

for all x, x' in X and y, y' in Y where $0 \le c < 1$. If either T or S is continuous then ST has a unique fixed point z in X and TS has a unique fixed point v in Y. Further $Tz = w$ and $Sw = z$.

The following theorem was proved by Fisher and Turkoglu [4].

Theorem 1.2: *Let (X, d_1) and (Y, d_2) be complete metric spaces. Let S be mapping of X into $B(Y)$ and R be mapping of Y into $B(X)$ satisfying the following inequalities,*

$$\delta_1(RSx, RSx') \le c \max\{d_1(x,x'),\ \delta_1(x, RSx),\ \delta_1(x', RSx'),\ \delta_2(Sx, Sx)\}$$

$$\delta_2(RSy, RSy') \le c \max\{d_2(y,y'),\ \delta_2(y, SRy),\ \delta_2(y', SRy'),\ \delta_1(Ry, Ry')\}$$

for all x, x' in X and y, y' in Y, where $0 \le c < 1$. If S is continuous then RS has a unique fixed point u in X and SR has a unique fixed point v in Y.

The aim of our work is to generalize theorem 1.1 by considering two set-valued mappings and two complete metric spaces.

Before comming to our main result we recall the following from Fisher [1, 2].

(i) The function $\delta(A, B)$ with A and B in $B(X)$ is defined by
$\delta(A, B) = \sup\{d(a, b) : a \in A,\ b \in B\}$.

(ii) If A consists of a single point a we write $\delta(A, B) = \delta(a, B)$

(iii) If B also consists of a single point b we write $\delta(A, B) = \delta(a, b) = d(a, b)$
If follows easily from the definition that $\delta(A, B) = \delta(B, A) \ge 0$,
$\delta(A, B) \le \delta(A, C) + \delta(C, B)$ for all A, B and C in $B(X)$.

Now let $\{A_n : n = 1, 2, 3, \ldots\}$ be sequence of non-empty subsets of X. We say that the sequence $\{A_n\}$ converges to the subset A of X if

(iv) Each point a in A is the limit of a convergent sequence $\{a_n\}$, where a_n is in A_n for $n = 1, 2, 3, \ldots$

(v) For arbitrary $\varepsilon > 0$, there exists an integer N such that $A_n \subset A_\varepsilon$ for $n > N$, where A_ε is the union of all open spheres with centres in A and radius ε. A is then said to be the limit of the sequence $\{A_n\}$.

The following lemma was proved in Fisher [1].

Lemma 1.3: *If $\{A_n\}$ and $\{B_n\}$ are sequences of bounded subsets of a complete metric space (X, d) which converge to the bounded subsets of A and B respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A, B)$.*

Now, let F be a mapping of X into $B(X)$. We say that the mapping F is continuous at a point in X if whenever $\{x_n\}$ is a sequence of points in X converging to x, the sequence $\{Fx_n\}$ in $B(X)$ converges to Fx in $B(X)$. We say that F is continuous mapping of X into $B(X)$ if F is continuous at each point x in X. We say that a point z in X is a fixed point of F if z is in Fz. If a is in $B(X)$, we define the set $FA = U_{a \in A}\ Fa$.

Main Results:

We prove the following theorems.

Theorem 2.1: *Let (X, d_1) and (Y, d_2) be two complete metric spaces, let T be a mapping of X into $B(Y)$ and let S be a mapping of Y into $B(X)$ satisfying inequalities,*

$$\begin{aligned} &\delta_1(Sy,Sy')\ \delta_1(STx,STx') \le c \max \{\delta_1(Sy,Sy')\ \delta_2(Tx,Tx'),\ \delta_1(x',Sy)\ \delta_2(y',Tx), \\ (1)\qquad &d_1(x,x')\ \delta_1(Sy,Sy'),\ \delta_1(Sy',STx)\delta_1(Sy',STx)\} \end{aligned}$$

$$\begin{aligned} &\delta_2(Tx,Tx')\ \delta_2(TSy,TSy') \le c \max \{\delta_1(Sy,Sy')\ \delta_2(Tx,Tx'),\ \delta_1(x',Sy)\ \delta_2(y',Tx), \\ (2)\qquad &d_2(y,y')\ \delta_2(Tx,Tx'),\ \delta_2(Tx,TSy)\ \delta_2(Tx',TSy')\} \end{aligned}$$

for all x, x' in X and y, y' in Y where $0 \le c < 1$. If T is continuous, then ST has a unique fixed point x in X and TS has a unique fixed point w in Y.

Proof: *Let x_1 be an arbitrary point in X. Define sequence $\{x_n\}$ and $\{y_n\}$ in X and Y respectively as follows. Choose a point y_1 in Tx_1 and a point x_2 in Sy_1. In general, having choosen x_n in X and y_n in Y, choose x_{n+1} in Sy_n and then y_{n+1} in Tx_{n+1} for* $n = 1, 2, 3, \ldots.$

Then,

$$\begin{aligned} d_1(x_{n-1}, x_n)\ d_1(x_n, x_{n+1}) &= \delta_1(Sy_{n-2}, Sy_{n-1})\ \delta_2(STx_{n-1}, STx_n) \\ &\le c \max \{d_1(x_{n-1},x_n)\ d_2(y_{n-1},y_n),\ d_1(x_n,x_{n-1})\ d_2(y_{n-1},y_{n-1}), \\ &\qquad d_1(x_{n-1},x_n)\ d_1(x_{n-1},x_n),\ d_1(x_{n-1},x_n)\ d_2(x_n,x_{n+1})\}. \end{aligned}$$

From which is follows that

$$(3)\qquad d_1(x_n, x_{n+1}) \le c \max \{d_2(y_{n-1},y_n), d_1(x_{n-1},x_n)\}$$

Applying inequality (2), we get

$$\begin{aligned} [d_2(y_{n-1},y_n)]^2 &= \delta_2(Tx_{n-1}, Tx_n),\ \delta_2(TSy_{n-2}, TSy_{n-1}) \\ &\le c \max \{d_2(x_{n-1},x_n)\ d_2(y_{n-1},y_n),\ d_1(x_n,x_{n-1})\ d_2(y_{n-1},y_{n-1}),\ d_2(y_{n-2},y_{n-1}) \\ (4)\qquad &\qquad d_2(y_{n-1},y_n),\ d_2(y_{n-1},y_{n-1})\ d_2(y_{n-1},y_n)\} \end{aligned}$$

from which is follows that $d_2(y_{n-1},y_n) \le c \max \{d_1(x_{n-1},x_n),\ d_2(y_{n-2},y_{n-1})\}$.

It now follows easily by induction that

$$d_1(x_n, x_{n+1}) \leq c^n \max\{d_1(x, x_1),\ d_2(y_1, y_2)\}$$
$$d_2(y_{n-1}, y_n) \leq c^{n-2} \max\{d_1(x_1, x_2),\ d_2(y, y_1)\}$$

for $n = 1, 2, \ldots$. Since $c < 1$, it follows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences with limits z in X with w in Y.

Applying inequality (1), we have

$$\delta_1(Sw, x_{n+1})\ \delta_1(STz, x_{n+1}) = \delta_1(Sw, Sy_n)\ \delta_1(STz, STx_n)$$
$$\leq c \max\{\delta_1(Sw, x_n)\ \delta_2(Tz, y_n),\ \delta_1(x_n, Sw)\ \delta_2(y_n, Tz),\ d_1(z, x_n)\ \delta_1(Sw, x_n),$$
$$\delta_1(Sw, STz)\ \delta_1(x_{n+1}, x_{n+1})\}$$

Letting n tends to infinity we have

$$\delta_1(Sw, z)\delta_1(STz, z) \leq c \max(\delta_1(Sw, z)\ \delta_2(Tz, w),$$
$$\delta_1(z, Sw)\ \delta_2(w, Tz),\ d_1(z, z)\ \delta_1(Sw, z)\}$$

From which it follows that

(5) $$\delta_1(Sw, z)\ \delta_1(STz, z) \leq c\{\delta_1(Sw, z)\ \delta_2(Tz, w)\}$$

and so either $Sw = z$ or

(6) $$\delta_1(STz, z) \leq c\ \delta_2(Tz, w)$$

Applying inequality (2), we have

$$\delta_2(Tz, y_n)\ \delta_2(TSw, y_{n+1}) = \delta_2(Tz, Tx_n)\ \delta_2(TSw, TSy_n)$$
$$\leq c \max\{\delta_1(Sw, x_{n+1})\ \delta_2(Tz, y_n),\ \delta_1(z, Sw)\ \delta_2(w, Tz),$$
$$d_2(w, y_n)\ \delta_2(Tz, y_n),\ \delta_2(Tz, TSw)\ \delta_2(y_n, TSw)\}$$

Letting n tends to infinity we have

(7) $$\delta_2(Tz, w)\ \delta_2(TSw, w) \leq c\delta_1(Sw, z)\ \delta_2(Tz, w)$$

and so either $Tz = w$

Theorem 2.3: *Let (X, d) and (Y, ρ) be compact metric spaces, let T be a continuous mapping of X into Y and let S be a continuous mapping of Y into X satisfying the inequalities*

$$d(Sy, Sy')\ d(STx, STx') < \max\{d(Sy, Sy')\ \rho(Tx, Tx'),\ d(x', Sy)\ \rho(y', Tx),$$
$$d(x, x')\ d(Sy, Sy'), d(Sy, STx)\ d(Sy', STx')\}$$

$$\rho(Tx, Tx')\ \rho(TSy, TSy') < \max\{d(Sy, Sy')\ \rho(Tx, Tx'), d(x', Sy)\ \rho(y', Tx),$$
$$\rho(y, y')\ \rho(Tx, Tx'),\ \rho(Tx, TSy)\ \rho(Tx', TSy')\}$$

for all x, x' in Y and y, y' in Y. Then ST has a unique fixed point z in X and TS has a **unique fixed point w in Y. Further,** $Tz = w$ and $Sw = z$.

We now prove the following fixed point theorem for compact metric spaces.

Theorem 2.4: *Let (X, d_1) and (Y, d_2) be two compact metric spaces, let T be a continuous mapping of X into $B(Y)$ and let S be a continuous mapping of Y into $B(X)$ satisfying the inequalities*

$$\delta_1(Sy, Sy')\ \delta_1(STx, STx') < \max\{\delta_1(Sy,Sy')\ \delta_2(Tx,Tx'),\ \delta_1(x',Sy)\ \delta_2(y',Tx),$$
$$d_1(x,x')\ \delta_1(Sy,Sy'),\ \delta_1(Sy,STx)\ \delta_1(Sy',STx')\} \tag{11}$$

$$\delta_2(Tx,Tx')\ \delta_2(TSy,TSy') < \max\{\delta_1(Sy,Sy')\ \delta_2(Tx,Tx'),\ \delta_1(x',Sy)\ \delta_2(y',Tx),$$
$$d_2(y,y')\ \delta_2(Tx,Tx'),\ \delta_2(Tx,TSy)\ \delta_2(Tx',TSy')'' \tag{12}$$

for all x, x' in X and y, y' in Y. Then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further $Tz = w$ and $Sw = z$.

Proof: Firstly, let us assume that there is no $a < 1$ such that

$$\delta_1(Sy,STSy)\ \delta_1(STx,STSTx) \le a \max\{\delta_1(Sy,STSy)\ \rho(Tx,TSTx),$$
$$\delta_1(STx, Sy)\ \delta_2(TSy,Tx),\ \delta_1(x,STx)\ \delta_1(Sy,STSy),$$
$$\delta_1(Sy,STx)\ \delta_1(STSy,STSTx)\} \tag{13}$$

for all x in X and y in Y. Then there exist sequences $\{x_n\}$ in X and $\{y_n\}$ in Y such that

$$\delta_1(Sy_n,\ STSy_n)\ d(STx_n,\ STSTx_n) > (1-n^{-1}) \max\{\delta_1(Sy_n,\ STSy_n)\ \delta_2(Tx_n,TSTx_n),$$
$$\delta_1(STx_n,Sy_n)\ \delta_2(TSy_n,Tx_n),\ \delta_1(x_n,STx_n)\ \delta_1(Sy_n,STSy_n),$$
$$\delta_1(Sy_n,STx_n)\ \delta_1(STSy_n,STSTx_n)\} \tag{14}$$

for $n = 1, 2, \ldots$. Since X and Y are compact, and by relabelling if necessary, we may suppose that the sequence $\{x_n\}$ converges to z' in X and the sequence $\{y_n\}$ converges to w' in Y. Letting n tends to infinity in inequality (14), it follows that

$$\delta_1(Sw',STSw')\ \delta_1(STz',STSTz') \ge \max\{\delta_1(Sw',STSw')\ \delta_2(Tz',TSTz'),$$
$$\delta_1(STz',Sw')\ \delta_2(TSw',Tz'),\ \delta_1(z',STz')\ \delta(Sw',STSw'),$$
$$\delta_1(Sw',STz')\ \delta_1(STSw',STSTz')\} \tag{15}$$

This is only possible if the right hand side of inequalities (15) is zero. It follows that either

$$STz' = STSTz' \text{ or } Sw' = STSw'.$$

If $STz' = STSTz'$, then $STz' = z$ is a fixed point of ST and it follows that $Tz = w$ is a fixed point of TS.

If $Sw' = STWw'$, then $Sw' = z$ is a fixed point of ST and it again follows that $Tz = w$ is a fixed point of TS.

Secondly, let us assume that there exists no $b < 1$ such that

$$\delta_2(Tx, TSTx)\ \delta_2(TSy, TSTy) \le b \max\{\delta_1(Sy, STSy)\ \delta_2(Tx, TSTx),$$
$$\delta_1(STx, Sy)\ \delta_2(TSy, Tx),\ \delta_2(y, TSy)\ \delta_2(Tx, TSTx),$$
$$\delta_2(Tx, TSy)\ \delta_2(TSTx, STSTy)\} \tag{16}$$

for all x in X and y in Y. Then it follows that ST has a fixed point z and TS has a fixed point w.

Finally, suppose that there exist $a, b < 1$ satisfying (13) and (16). Then with $c = \max\{a, b\}$, it follows that if the sequences $\{x_n\}$ and $\{y_n\}$ are defined as in the proof of Theorem 2.1, inequalities (3) and (4) will hold. It then follows as in the proof of theorem 2.1 that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences with limits z in X and w in Y. Since ST and TS are continuous, it now follows that z is a fixed point of ST and w is a fixed point of TS.

To prove the uniqueness, suppose that ST has a second distinct common fixed point z'. Then applying inequality (11) we have

$$[d_1(z, z')]^2 = [\delta_1(STz, STz')]^2 < \max\{d_1(z, z')\ d_2(Tz, Tz'),\ [d_1(z, z')]^2\}$$

which implies that

$$d_1(z, z') < \delta_2(Tz, Tz') \tag{17}$$

Further, applying inequalities (12) we have

$$[\delta_2(Tz, Tz')]^2 = \delta_2(Tz, Tz')\ \delta_2(TSTz, TSTz') < \max\{d_1(z, z')\ \delta_2(Tz, Tz'),$$
$$[\delta_2(Tz, Tz')]^2\}$$

which implies that

$$\delta_2(Tz, Tz') < d_1(z, z') \tag{18}$$

It now follows from inequalities (17) and (18) that

$$d_1(z, z') < \delta_2(Tz', Tz) < d_1(z, z')$$

which is a contradiction and hence the fixed point z must be unique.

Similarly, we can prove the uniqueness of w. This completes the proof of the theorem.

Corollary 2.5: *Let (X,d) be a compact metric space and let T be a continuous mapping of X into $B(X)$ satisfying the inequality*

$$\delta(Ty,Ty')\,\delta(T^2x,T^2x') < \max\{\delta(Ty,Ty')\,\delta(Tx,Tx'),\ \delta(x',Ty)\,\delta(y',Tx),$$
$$d(x,x')\,\delta(Ty,Ty'),\ \delta(Ty,T^2x)\,\delta(Ty',T^2x')\}$$

for all x, x', y, y' in X for which the right hand side of the inequality is positive. Then T has a unique fixed point z in X.

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Hypergeometric series involving Ramanujan's mock–theta functions

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Abstract: In this paper, we shall deal with certain aspects of hypergeometric series involving Ramanujan's mock-theta functions. In the concluding sections, certain generating relations of mock-theta functions of order three and five have also been deduced.

Keywords: Bailey's transformations, Generating relations, Hypergeometric series, Mock-theta functions, Saalschutzian series, Theta functions.

1.1 Introduction (A Historical Survey):

S. Ramanujan (1887–1923) was an Indian mathematician whose originality and natural ability for calculation enabled him to develop innovative mathematical concepts enough during the early 20th century. Ramanujan, three months before his death, had written his last letter to G. H. Hardy, a mathematician at the University of Cambridge, England, discovered much interesting functions which I recently call mock-theta functions unlike the false theta functions due to L. J. Rogers, they enter into mathematics as beautifully as the ordinary theta functions." The mock-theta function is the last gift of Ramanujan to the mathematical world.

Due to Hardy [4] a mock-theta function is a function defined by a q-series convergent $|q|<1$ for which we may be able to calculate asymptotic formulae when q tends to a rational point of the unit circle of the same degree of precision as those furnished for the ordinary theta functions by the theory of linear transformations.

1.2 Notations and definitions:

The generalized hypergeometric series, both ordinary and basic have been a very significant tool in the derivation of the generating relations for mock theta functions. The

usual hypergeomtric notation shall be followed throughout, in what follows. As usual, let for any positive integer n,

$$(a)_n = a(a+1)\ldots\ldots\ldots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, n>0, (a)_0 = 1, a \neq 0, (a)_{-n} = \frac{(-1)^n}{(1-a)_n}$$

Then the generalized hypergeometric series is defined by

$$(1) \quad F_1(q) = \sum_{k=0}^{\infty} \frac{q^{2k^2+2k}}{\left[q;q^2\right]_{k+1}}$$

The series 1.2(1) converges for all z if $r \leq s$ while it converges for only $z \neq 0$ if $r > s+1$, for $r = s+1$ and also when $z=1$ provided that $R_e\left[\sum(b) - \sum(a)\right] > -1$, When $r = s+1$, the series is called Saalschutzian when $R_e\left[\sum(b) - \sum(a)\right] = 1$ and well poised when $1 + a_1 = b_1 + a_2 = \ldots = b_s + a_{s+1}$.

If any one of the numerator parameters in 1.2(1) is zero or a negative integer than ${}_rF_s$ reduces to a polynomial but if any b parameter is a negative integer-N (say) where $N = 1,2,3,\ldots$ (unless any of the a parameters is also a negative integer-M say, where $M = N, N+1, N+2, \ldots$, or zero), the ${}_rF_s$ series is not defined.

Let, for $\left|q^k\right| < 1$,

$$\left(a;q^k\right)_n = (1-a)\left(1-aq^k\right)\ \ldots\ \left(1-aq^{k(n-1)}\right), n > 0,$$

$$\left(a;q^k\right)_0 = 1$$

and $\left(a;q^k\right)_\infty = \prod_{n=0}^{\infty}\left(1-aq^{kn}\right)$

Then a basic hypergeometric series is defined by

$$(2) \quad {}_{r+1}\Phi_r^{q^k}\left[\begin{matrix} a_1, \ldots, a_{r+1}; z \\ b_1, \ldots, b_r \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{\left(a_1;q^k\right)_n \ \ldots\ \left(a_{r+1};q^k\right)_n z^n}{\left(b_1;q^k\right)_n \ldots \left(b_r;q^k\right)_n \left(q^k;q^k\right)_n}$$

When $k=1$ in the above symbols, including that in ${}_{r+1}\Phi_r$, it shall be omitted from the symbols, so $(a;q)_n = (a)_n$ etc. G.N.Watson [7] made use of the basic hypergeometric series to get new definitions of mock theta functions. For this, he used a limiting case of a transformation connecting a terminating well poised ${}_4\Phi_3$ which Watson himself discovered many years back.

We shall often use the abbreviated notation $\left(a_1, a_2, \ldots a_m; q^k\right)_n$ to denote $\left(a_1;q^k\right)_n \left(a_2;q^k\right)_n, \ldots \left(a_m;q^k\right)_n$ for all non-negative integers n, where $\left(a;q^k\right)_n = \prod_{r=0}^{n-1}\left(1-aq^{kr}\right)$ and $\left(a;q^k\right)_\infty = \prod_{r=0}^{n-1}\left(1-aq^{kr}\right)$, if $k=1$ and there is no chance of any confusion, then q^k shall be omitted from the $\left(a;q^k\right)_n$ symbol.

So, let $(a)_n=(1-a)(1-aq)\ \ldots\ \left(1-aq^{n-1}\right)$, $(a)_0=1$, $\left(a;q^k\right)_n$ $(a)_\infty=(1-aq^n)$, $|q|<1$, we define a generalized basic hypergeometric series by

$$(3)\qquad {}_r\Phi_s\left[\begin{matrix}(a_r);z\\(\beta_s)\end{matrix}\right]=\sum_{n=0}^{\infty}\frac{[(a_r)]_n\,z^n}{(q)_n\,[(\beta_s)]_n},$$

where r is the number of parameters (a_r) and s is the number of parameters (β_s). We shall use the notation ${}_r\Phi_s\,(q^k)\,(z)$ to denote a ${}_r\Phi_s$ series that all the terms are on the base q^k.

If q^k is not written in Φ symbol, the transformation formula is then deduced by

$$(4)\qquad {}_8\Phi_7\left[\begin{matrix}a,q\sqrt{a},-q\sqrt{a},c,d,e,f,g,\dfrac{a^2q^2}{cdefg}\\ \sqrt{a},-\sqrt{a},aq/c,aq/d,aq/e,aq/f,aq/g\end{matrix}\right]$$

$$=\prod_{n=1}^{\infty}\left[\frac{(1-aq^n)(1-aq^n/fg)(1-aq^n/ge)(1-aq^n/ef)}{(1-aq^n/e)(1-aq^n/f)(1-aq^n/g)(1-aq^n/efg)}\right]\times{}_4\Phi_3\left[\begin{matrix}aq/cd,e,f,g;q\\ efg/a,aq/c,aq/d\end{matrix}\right],$$

provided e, f, or g is of the term q^{-N}, where N is a positive integer. Assuming $a\rightarrow 1$, $e\rightarrow\infty$, $g\rightarrow\infty$ and taking $c=\exp(i\theta)$, $d=\exp(-i\theta)$, 1.2 (4) becomes

$$(5)\qquad 1+\sum_{n=1}^{\infty}\frac{(-)^n\,(1+q^n)2-\cos\theta)q^{n(2n+1)}/2}{1-2q^n\cos\theta+q^{2n}}=\prod_{r=1}^{\infty}(1-q^r)\left[1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-2q^m\cos\theta+q^{2m})}\right]$$

1.2(5) is the most important relation to obtain the new alternative definition of the Ramanujan's mock theta functions. Substituting $\theta=\Pi$, $\theta=\dfrac{\Pi}{2}$, $\theta=\dfrac{\Pi}{3}$ respectively in 1.2(5), we have

$$(6)\qquad f(q)\prod_{r=1}^{\infty}(1-q^r)=1+4\sum_{n=1}^{\infty}\frac{(-)^n q^{n(3n+1)/2}}{1+q^n},$$

$$(7)\qquad \phi(q)\prod_{r=1}^{\infty}(1-q^r)=1+2\sum_{n=1}^{\infty}\frac{(-)^n(1+q^n)\,q^{n(3n+1)/2}}{1+q^{2n}}$$

$$(8)\qquad \chi(q)\prod_{r=1}^{\infty}(1-q^r)=1+\sum_{n=1}^{\infty}\frac{(-)^n(1+q^n)q^{n(3n+1)/2}}{1-q^n+q^{2n}}$$

These are the relations providing alternative definitions of $f(q)$, $\phi(q)$ and $\chi(q)$ respectively

1.3 Mock theta functions associated partially:

Let $F(q)=\sum_{n=0}^{\infty}\Psi n(q)$ be a mock theta function. Then the corresponding partial mock theta function is defined by $F_m(q)=\sum_{n=0}^{\infty}\Psi_n(q)$. Thus the partial mock theta functions of order three and five are defined below:

Mock theta function of Partial mock theta function order three:of order three :

(i) $F(q)=\sum_{k=0}^{\infty}\frac{q^{k^2}}{[-q;q]_k^2}$ $\qquad f_n(q)=\sum_{k=0}^{\infty}\frac{q^{k^2}}{[-q;q]_k^2}$

(ii) $\Phi(q)=\sum_{k=0}^{\infty}\frac{q^{k^2}}{[-q^2;q^2]_k}$ $\qquad \Phi_n(q)=\sum_{k=0}^{\infty}\frac{q^{k^2}}{[-q^2;q^2]_k}$

(iii) $\Psi(q)=\sum_{k=0}^{\infty}\frac{q^{k^2}}{[-q;q]_k}$ $\qquad \Psi_n(q)=\sum_{k=0}^{\infty}\frac{q^{k^2}}{[-q;q]_k}$

(iv) $\chi(q)=\sum_{k=0}^{\infty}\frac{q^{k^2}}{[-wq;-w^2q;q]_k}$, $\qquad \chi_n(q)=\sum_{k=0}^{\infty}\frac{q^{k^2}}{[-wq;-w^2q;q]_k}$

where w is the cube root of unity.

(v) $w(q)=\sum_{k=0}^{\infty}\frac{e^{2k(k+1)}}{[-q;q^2]_{k+1}^2}$ $\qquad w_n(q)=\sum_{k=0}^{\infty}\frac{e^{2k(k+1)}}{[-q;q^2]_{k+1}^2}$

(vi) $u(q)=\sum_{k=0}^{\infty}\frac{q^{k(k+1)}}{[-q;q^2]_{k+1}}$ $\qquad u_n(q)=\sum_{k=0}^{\infty}\frac{q^{k(k+1)}}{[-q;q^2]_{k+1}}$

(vii) $p(q)=\sum_{k=0}^{\infty}\frac{q^{2k(k+1)}}{[wq;w^2q;q^2]_{k+1}}$ $\qquad p_n(q)=\sum_{k=0}^{\infty}\frac{q^{2k(k+1)}}{[wq;w^2q;q^2]_{k+1}}$

Mock theta functions Partial mock theta functions of order five : of order five :

(i) $f_0(q)=\sum_{k=0}^{\infty}\frac{q^{k^2}}{[-q;q]_k}$ $\qquad f_{0m}(q)=\sum_{k=0}^{\infty}\frac{q^{k^2}}{[-q;q]_k}$

(ii) $F_0(q)=\sum_{k=0}^{\infty}\frac{q^{2k^2}}{[q;q^2]_k}$ $\qquad F_{0,n}(q)=\sum_{k=0}^{\infty}\frac{q^{2k^2}}{[q;q^2]_k}$

(iii) $1+2\Psi_0(q)=\sum_{k=0}^{\infty}[-1;q]_k\, q^{\binom{k+1}{2}}$ $\qquad 1+2\Psi_{0,n}(q)=\sum_{k=0}^{\infty}[-1;q]_k\, q^{\binom{k+1}{2}}$

(iv) $\Psi_0(q)=\sum_{k=0}^{\infty}\left[-1;q^2\right]_k q^{k^2}$ $\qquad \Psi_{0,n}(q)=\sum_{k=0}^{\infty}\left[-1;q^2\right]_k q^{k^2}$

(v) $f_1(q)=\sum_{k=0}^{\infty}\frac{q^{k^2+k}}{[-q;q]_k}$ $\qquad f_{1,n}(q)=\sum_{k=0}^{\infty}\frac{q^{k^2+k}}{[-q;q]_k}$

(vi) $F_1(q)=\sum_{k=0}^{\infty}\frac{q^{2k^2+2k}}{\left[q;q^2\right]_{k+1}}$ $\qquad F_{1,n}(q)=\sum_{k=0}^{\infty}\frac{q^{2k^2+2k}}{\left[q;q^2\right]_{k+1}}$

(vii) $\Psi_1(q)=\sum_{k=0}^{\infty}[-q;q]_k\, q^{\binom{k+1}{2}}$ $\qquad \Psi_{1,n}(q)=\sum_{k=0}^{\infty}[-q;q]_k\, q^{\binom{k+1}{2}}$

(viii) $\Phi_1(q)=\sum_{k=0}^{\infty}\left[-q;q^2\right]_k q^{(k+1)^2}$ $\qquad \Phi_{1,n}(q)=\sum_{k=0}^{\infty}\left[-q;q^2\right]_k q^{(k+1)^2}$

(ix) $\chi_0(q)=\sum_{k=0}^{\infty}\frac{q^k}{\left[q^{k+1};q\right]_k}$ $\qquad \chi_{0,n}(q)=\sum_{k=0}^{\infty}\frac{q^k}{\left[q^{k+1};q\right]_k}$

(x) $\chi_1(q)=\sum_{k=0}^{\infty}\frac{q^k}{\left[q^{k+1};q\right]_{k+1}}$ $\qquad \chi_{1,n}(q)=\sum_{k=0}^{\infty}\frac{q^k}{\left[q^{k+1};q\right]_{k+1}}$

1.4 Generating relations for mock theta function of order three:

In 1944, Bailey [3] established the following transformations:

If $\quad \beta_n=\sum_{r=0}^{n}a_r U_{n-r}V_{n+r}$ and $\gamma_n=\sum_{r=0}^{n}\delta_{r+n}u_r v_{r+2n}$

then under suitable convergence conditions:

(1) $$\sum_{n=0}^{\infty}\alpha_n\gamma_n=\sum_{n=0}^{\infty}\beta_n\delta_n\ ,$$

where u_r, v_r, α_r and δ_r are the functions of r only and the series β_n and γ_n shall be convergent. In this section, we shall make use of the above so called Bailey's transformation in order to establish the generating relations for partial mock theta functions of order three defined in 1.3(i)–1.3 (vii). If we take $u_r=v_r=1$ and $\delta_r=z^r$ in 1.4(1), the Bailey's transformation yields

(2) $$(1-z)^{-1}\sum_{n=0}^{\infty}a_n Z^n=\sum_{n=0}^{\infty}\beta_n Z^n \text{ whenever } \beta_n=\sum_{r=0}^{n}a_r\ ,\ \gamma_n=\frac{Z^n}{1-z}.$$

We shall now make use of 1.4(2) for the derivation of the mock theta functions of order three and five in the following sections:

1.5. Main generating results involving partial Mock-theta functions of order three:

(i) Taking $\alpha_r = \dfrac{q^{r^2}}{[-q;q]_r^2}$, we find

$$\beta_n = \sum_{r=0}^{n} \frac{q^{r^2}}{[-q;q]_r^2} = f_n(q)$$

Putting these values of and in 1.4 (2), we get

$$\text{(1)} \quad \sum_{n=0}^{\infty} f_n(q) z^n = (1-z)^{-1} \sum_{n=0}^{\infty} \frac{q^{n^2}}{[-q;q]_r^2} z^n$$

(ii) Taking $\alpha_r = \dfrac{q^{r^2}}{[-q^2;q^2]_r}$, we find

$$\beta_n = \sum_{r=0}^{n} \frac{q^{r^2}}{[-q;q^2]} = \Phi_n(q)$$

Putting these values of α_n and β_n in 1.4(2), we get

$$\text{(2)} \qquad \sum_{n=0}^{\infty} \Phi_n(q) z^n = (1-z)^{-1} \sum_{n=0}^{\infty} \frac{q^{n^2}}{[-q^2;q^2]_n}$$

(iii) Taking $\alpha_r = \dfrac{q^{n^2}}{[q;q^2]_r}$, we find

$$\beta_n = \sum_{r=0}^{n} \frac{q^{r^2}}{[q;q^2]_r} = \Psi(n)$$

Putting these values of α_n and β_n in 1.4(2), we get

$$\text{(3)} \qquad \sum_{n=0}^{n} \Psi(n) z^n = (1-z)^{-1} \sum_{n=0}^{\infty} \frac{q^{n^2}}{[q;q^2]_n} z^n$$

(iv) Taking $\alpha_r = \dfrac{q^{r^2}}{[-wq;-w^2q;q]_r}$, we find

$$\beta_n = \sum_{r=0}^{n} \frac{q^{r^2}}{[-wq;-w^2q;q]_r} = \chi_n(q),$$

where w is the cube root of unity.

Putting these values of α_n and β_n in 1.4(2), we get

$$\text{(4)} \qquad \sum_{n=0}^{\infty} f_n(q) z^n = (1-z)^{-1} \sum_{n=0}^{\infty} \frac{q^{r^2}}{[-wq;-w^2q;q]_n} z$$

(v) Taking $\alpha_r = \dfrac{q^{2r(r+1)}}{[q;q^2]^2_{r+1}}$, we find

$$\beta_n = \sum_{r=0}^{n} \frac{q^{2r(r+1)}}{[q;q^2]_{r+1}} w_n(q)$$

Putting these values of α_n and β_n in 1.4(2), we get

$$(5) \qquad \sum_{n=0}^{\infty} w_n(q) z^n = (1-z)^{-1} \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{[q;q^2]^2_{n+1}} w_n(q)$$

(vi) Taking $\alpha_r = \dfrac{q^{r(r+1)}}{[-q;q^2]_{r+1}}$, we find

$$\beta_n = \sum_{r=0}^{n} \frac{q^{r(r+1)}}{[-q;q^2]_{r+1}} = u_n(q)$$

Putting these values of α_n and β_n in 1.4(2), we get

$$(6) \qquad \sum_{n=0}^{\infty} v_n(q) z^n = (1-z)^{-1} \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{[-q;q^2]_{n+1}} z^n$$

(vii) Taking $\alpha_r = \dfrac{q^{2r(r+1)}}{[wq;w^2q;q^2]_{r+1}}$, we find

$$\beta_n = \sum_{n=0}^{\infty} \frac{q^{2r(r+1)}}{[wq;w^2q;q^2]_{r+1}},$$

where w is the cube root of unity

Putting these values of α_n and β_n in 1.4(2), we get

$$(7) \qquad \sum_{n=0}^{\infty} \rho_n(q) z^n = (1-z)^{-1} \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{[-wq;-w^2q;q]_{n+1}} z^n$$

1.6 Main generating results involving partial mock theta functions of order five:

(i) Taking $a_r = \dfrac{q^{r^2}}{[-q;q]_r}$, we find $\beta_n = \sum_{r=0}^{n} \dfrac{q^{r^2}}{[-q;q]_r} = f_{0,n}(q)$.

Putting these values of α_n and β_n in 1.4(2), we get

$$(1) \qquad \sum_{n=0}^{\infty} f_{0,n}(q) z^n = (1-z)^{-1} \sum_{n=0}^{\infty} \frac{q^{n^2}}{[-q;q]_n} z^n$$

(ii) Taking $a_r = \dfrac{q^{2r^2}}{\left[-q;q^2\right]_r}$, we find

$$\beta_n = \sum_{r=0}^{n} \frac{q^{2r^2}}{[-q;q2]_r} = F_{0,n}(q)$$

Putting these values of α_n and β_n in 1.4(2), we get

$$(2) \qquad \sum_{n=0}^{\infty} F_{0,n}(q) z^n = (1-z)^{-1} \sum_{n=0}^{\infty} = \frac{q^{2n^2}}{\left[-q;q^2\right]_n} z^n$$

Similarly, making use of 1.4(2), we can establish the generating results of mock theta functions 1.3(iii) – 1.3(x) of order five also.

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Contact CR-warped product submanifolds in locally conformal almost cosymplectic manifolds

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Abstract: B. Y. Chen [2] studied warped products, which are CR-submanifolds in Kaehler manifolds and established sharp inequalities for CR-warped products in Kaehler manifolds. In this article, we establish the inequality for the squared norm of the second fundamental form in terms of warping function for contact CR-warped products isometrically immersed in locally conformal almost cosymplectic manifold.

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1. Introduction

Let $\bar{M}$ be a $(2m+1)$-dimensional almost contact manifold equipped with an almost contact structure (ϕ, ξ, η), that is, ϕ is a $(1, 1)$ tensor field, ξ is a vector field and η is a 1-form such that $\phi^2 X = -X + \eta(X)\xi$ and $(\xi) = 1$, then $\phi(\xi) = 0$ and $\eta \, o \, \phi = 0$. The almost contact structure is said to be normal if the induced almost complex structure J on the product manifold $\bar{M} \times \mathbb{R}$ defined by

$$J\left(X, \lambda \frac{d}{dt}\right) = \left(\phi X - \lambda \xi x, \eta(X) \frac{d}{dt}\right) \tag{1.1}$$

is integrable, where X is tangent to $\bar{M}$, t the coordinate of $\mathbb{R}$ and λ is a smooth function on $\bar{M} \times \mathbb{R}$. The condition for being normal is equivalent to vanishing of the torsion tensor

$$[\phi, \phi] + 2d\eta \otimes \xi, \tag{1.2}$$

where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ

Let g be a compatible Riemannian metric with (ϕ, ξ, η), that is

$$g(\phi X, \phi Y) = g(XY) - \eta(X)\eta(Y). \tag{1.3}$$

Let Φ be the fundamental 2-form of $\bar{M}$, defined by

$$\Phi(X,Y) = g(X, \phi Y) = -g(\phi X, Y) \tag{1.4}$$

for all $X, Y \in T\bar{M}$..

If Φ and 1-form η are closed, then $\bar{M}$ is said to be almost cosymplectic manifold. A normal almost cosymplectic manifold is cosymplectic. $\bar{M}$ is called a locally conformal almost cosymplectic if there exists a 1-form ω such that

$$d\Phi = 2\omega \wedge \Phi, \ d\eta = \omega \wedge \eta \ \text{ and } \ d\omega = 0. \tag{1.5}$$

A necessary and sufficient condition for an almost contact structure to be normal locally conformal almost cosymplectic is [6]

$$(\bar{\nabla}_X \phi)Y = u(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{1.6}$$

where $\bar{\nabla}$ is the Levi-Civita connection of the Riemannian metric g and $\omega = u\eta$.

From (1.6) it follows that

$$\bar{\nabla}_X \xi = u\{X - \eta(X)\xi)\}. \tag{1.7}$$

A plane section π in $T_p\bar{M}$ is called a ϕ-section if it is spanned by X and ϕX, where X is the unit tangent vector orthogonal to ξ. The sectional curvature of the ϕ-section is called a ϕ-sectional curvature. A locally conformal almost cosymplectic manifold $\bar{M}$ of dimension ≥ 5 is of point wise constant ϕ-sectional curvature if and only if its curvature tensor $\bar{R}$ is of the form [8]

$$\begin{aligned} \bar{R}(X,Y,Z,W) = {} & \frac{c-3u^2}{4}\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\} \\ & + \frac{c+u^2}{4}\{g(X,\phi W)g(Y,\phi Z) - g(X,\phi Z)g(Y,\phi W) - 2g(X,\phi Y)g(Z,\phi W)\} \\ & - \left(\frac{c+u^2}{4} + u\right)\{g(X,W)\eta(Y)\eta(Z) - g(X,Z)\eta(Y)\eta(W) + g(Y,Z)\eta(X)\eta(W) \\ & - \{g(Y,W)\eta(X)\eta(Z)\}, \end{aligned} \tag{1.8}$$

where u is the function such that $\omega = u\eta$ and $u = \xi u$ and c is the point wise constant ϕ-sectional curvature of $\bar{M}$.

Let M be a n-dimensional submanifold of a manifold $\bar{M}$ equipped with a Riemannian metric g. The Gauss and Weingarten formulae are given respectively by

(1.9) $\quad \overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \quad \text{and} \quad \overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N,$

for all $X, Y \in TM$ and $N \in T^{\perp}M$, where $\overline{\nabla}, \nabla$ and $\nabla^{\perp}$ are respectively the Riemannian, induced Riemannian and induced normal connections in $\bar{M}, M$ and normal bundle $T^{\perp}M$ of M respectively, and h is the second fundamental form related to the shape operator A by $g\{h(X,Y),N\} = g(A_N X, Y)$. Then the equation of Gauss is given by

(1.10) $$R(X,Y,Z,W) = \bar{R}(X,Y,Z,W) + g(h(X,W), h(Y,Z)) - g(h(X,Z), h(Y,W)),$$

for any vector fields X, Y, Z and W tangent to M.

Let $\{e_1, \ldots, e_n, \ldots, e_{2m+1}\}$ be an orthonormal basis of the tangent space $T_p\bar{M}$, such that $e_1, e_2, \ldots, e_n$ are tangent to M at p. The mean curvature vector $H(p)$ at $p \in M$ is

(1.11) $$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i).$$

The submanifold M is totally geodesic in $\bar{M}$ if $h = 0$, and minimal if $H = 0$. We set

(1.12) $$h_{ij}^r = g(h(e_i, e_j), e_r), \quad \text{and} \quad \|h\|^2 = \sum_{i,j=1}^{n} \left(g(h(e_i, e_j), h(e_i, e_j)\right)$$

Let M be a Riemannian manifold of dimension n and $'a'$ a smooth function on M. Now, we recall

(i) ∇_a, the gradient of a is defined by

$$g(\nabla_a, X) = X(a),$$

for all vector field X on M.

(ii) Δ_a, the Laplacian of a is defined by

$$\Delta_a = \sum_{j=1}^{n}\left\{(\nabla_{e_j} e_j)a - e_j e_j(a)\right\} = -\operatorname{div} \nabla_a,$$

where ∇ is the Levi-Civita connection on M and $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal frame on M.

Consequently, we have

$$\|\nabla_a\|^2 = \sum_{j=1}^{n}\left(e_j(a)\right)^2$$

For submanifolds tangent to the structure vector field ξ, there are different classes of submanifolds:

(i) A submanifold M tangent to ξ is called an invariant submanifold if $\phi(T_pM) \subset T_pM$ for all $p \in M$, i.e. ϕ-preserves the tangent space of M.

(ii) A submanifold M tangent to ξ is called an anti-invariant submanifold if $\phi(T_pM) \subset T_p^{\perp}M$ for all $p \in M$, where T_pM and $T_p^{\perp}M$ denote tangent and normal space at $p \in M$, respectively.

(iii) A submanifold M tangent to ξ is called a contact CR - submanifold if it admits an invariant distribution D whose orthogonal complimentary distribution $D^{\perp}$ is anti-invariant, that is, $T_pM = D_p \oplus D_p^{\perp}$, with $\phi(D_p) \subset D_p$ and $\phi(D_p^{\perp}) \subset T_p^{\perp}M$, for every $p \in M$.

2. Contact CR-warped product submanifolds:

B.Y. Chen established a sharp relationship between the warping function f of a warped product CR-submanifold $M_1 \times {}_f M_2$ of a Kaehler manifold $\bar{M}$ and the squared norm of the second fundamental form $\|h\|^2$ (see [2]).

We prove a similar inequality for contact CR-warped product submanifolds in locally conformal almost cosymplectic manifold.

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds of positive dimension n_1 and n_2 respectively and f a positive differentiable function on M_1. The warped product of M_1 and M_2 is the Riemannian manifold $M_1 \times {}_f M_2 = (M_1 \times M_2, g)$, where $g = g_1 + f^2 g_2$ (see [3] and [4]).

We recall the following general formulae on a warped product

$$\nabla_U V = \nabla_V U = (U \ln f)V, \tag{2.1}$$

for any vector fields U tangent to M_1 and V tangent to M_2.

In this section, we investigate warped products $M = M_1 \times {}_f M_2$ which are contact CR-submanifolds of a locally conformal almost cosymplectic manifold $\bar{M}(c)$ of point wise constant ϕ-sectional curvature c. Such submanifolds are tangent to the structure vector field ξ. We distinguish two cases:

(a) ξ is tangent to M_1.

(b) ξ is tangent to M_2..

In case (a), we consider two sub cases:

(1) M_1 is an anti-invariant submanifold and M_2 is an invariant submanifold of $\bar{M}$.

(2) M_1 is an invariant submanifold and M_2 is an anti-invariant submanifold of $\bar{M}$.

We start with the sub case (1)

Theorem 2.1 *Let $\bar{M}(c)$ be a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold of point wise constant ϕ-sectional curvature c. Then there do*

not exist warped product submanifold $M = M_1 \times_f M_2$ such that M_t is an anti-invariant submanifold tangent to ξ and M_2 an invariant submanifold of $\bar{M}$.

Proof : Let $M = M_1 \times_f M_2$ be a warped product submanifold of a locally conformal almost cosymplectic manifold $\bar{M}(c)$ of point wise constant ϕ-sectional curvature c,such that M_1 is an anti-invariant submanifold tangent to ξ and M_2 an invariant submanifold of $\bar{M}$. From equation (2.1) we have

$$\nabla_X Z = \nabla_Z X = (Z \ln f)X, \tag{2.2}$$

for any vector fields Z and X tangent to M_1 and M_2 respectively.

In particular, for $z = \xi$, we get $\xi f = 0$. Using (1.7), (1.9) and (2.2), we have

$$u(X - \eta(X)\xi) = \bar{\nabla}_X \xi = \nabla_X \xi = (\xi \ln f)X = 0.$$

Thus M_2 cannot exist.

Now for sub case (2), we have

Theorem 2.2 *Let $\bar{M}(c)$ be a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold of point wise constant ϕ-sectional curvature c and $M = M_1 \times_f M_2$ an n-dimensional warped product submanifold such that M_1 is a $(2\alpha+1)$-dimensional invariant submanifold tangent to ξ and M_2 a β-dimensional C-totally real submanifold of $\bar{M}(c)$.*

Then

(i) The squared norm of the second fundamental form of M satisfies

$$\|h\|^2 \geq 2\beta\Big[\big\|\nabla(\ln f)\big\|^2 - \Delta(\ln f)\Big] + \alpha\beta(c + u^2 + 4) \tag{2.3}$$

where Δ denotes the Laplace operator on M_1.

(ii) The euqlity of (2.3) holds identicallyif M_1 is a totally geodesic submanifold of $\bar{M}(c)$. Hence M_1 is a locally conformal almsot cosymplectic manifold of point wise constant ϕ-sectioanl curvature c.

Proof: Let $M = M_1 \times_f M_2$ be a contanct CR-warped product submanifold in locally conformal almost cosymplectic manifold $\bar{M}(c)$ of point wise constant ϕ-sectional curvature c such that dim $M_2 = (2\alpha+1)$ and dim $M_2 = \beta$. Let $\{X_0 = \xi, X_1, \ldots, X_\alpha, X_{\alpha+1} = \phi X_1, \ldots, X_{2\alpha} = \phi X_\alpha, Z_1, \ldots, Z_\beta\}$ be a local orthonormal frame on M such that $X_0, \ldots, X_{2\alpha}$ are tangent to M_1 and $Z_1, \ldots, X_\beta$ tangent to M_2. For any unit vector fields X tangent to M_1 and Z, W tangent to M_2 respectively, we have

$$(2.4)\qquad g(h(\phi X,Z),\phi Z)=g(\overline{\nabla}_Z\,\phi X,\phi Z)=g(\phi\,\overline{\nabla}_Z X,\phi Z)$$
$$=g(\overline{\nabla}_Z X,Z)=g(\overline{\nabla}_Z X,Z)=X\ln f.$$

On the other hand, since the ambient manifold $\bar{M}(c)$ is a locally conformal almost cosymplectic manifold, it is easily seen that

$$(2.5)\qquad h(\xi,Z)=0$$

We denote by $h_{\phi D^{\perp}}(X,Z)$ the component of $h(X,Z)$ in $\phi D^{\perp}$. Therefore from (2.4) and (2.5) we have

$$g(h(\phi X,Z),\phi W)=g(A_{\phi W}Z,\phi X)=g(\overline{\nabla}_Z\,\phi W,\phi X)$$
$$(2.6)\qquad =g(\overline{\nabla}_Z W,X)=(X\ln f)\,g(Z,W)$$

Putting $X=\phi X, W=\phi W$ in (2.6) we have

$$(2.7)\qquad g(h(X,Z),W)=\phi X(\ln f)\,g(Z,\phi W)=-\,\phi X(\ln f)\,g(\phi Z,W).$$

Using (2.7) we have

$$h(X,Z)=-\phi X(\ln f)\phi Z.$$

Therefore for $X\in TM_1,\ Z\in TM_2$,

$$(2.8)\qquad \|h(X,Z)\|^2=\left(\phi X(\ln f)\right)^2 g(\phi Z,\phi Z)=\left(\phi X(\ln f)\right)^2 g(Z,Z)$$
$$=\left(\phi X(\ln f)\right)^2.$$

Let ν be the normal sub bundle orthogonal to $\phi D^{\perp}$. Obviously, we have

$$T^{\perp}M=\phi D^{\perp}\oplus v,\qquad \phi v=v.$$

Let $\{e_i\}_{i=0,\ldots,2\alpha}$ and $\{Z_i\}_{i=1,\ldots,\beta}$ are local orthonormal frame on M_1 and M_2 respectively. On M_1, we consider a ϕ-adapted orthonormal frame namely $\{e_i,\phi\,e_i,\xi\}_{i=1,\ldots,\alpha}$. We calculate $\|h(X,Z)\|^2$ for $X\in D$, and $Z\in D^{\perp}$. Since, we know that

$$h(X,Z)=h_{\phi D^{\perp}}(X,Z)+h_{v}(X,Z),$$

where $h_{\phi D^{\perp}}(X,Z)\in\phi D^{\perp}$ and $h_v(X,Z)\in v$.

For $X\in TM_1,\ Z\in TM_2$, we have

$$\|h(X,Z)\|^2=\sum_{i=1}^{2\alpha}\sum_{t=1}^{\beta}\left\{\|h(e_i,Z_t)\|^2+\|h(fe_i,Z_t)\|^2\right\}+\sum_{t=1}^{\beta}\|h_{\phi D^{\perp}}(\xi,Z_t)\|^2$$

Now from (2.8), we have

$$\|h_{\phi D^{\perp}}(e_i,Z_t)\|^2=(\phi\,e_i(\ln f))^2$$
$$\|h_{\phi D^{\perp}}(\phi\,e_i(\ln f)\|^2=(\phi^2 e_i(\ln f))^2=(e_i(\ln f))^2$$

Since

$$\| \nabla_a \|^2 = \sum_{i=1}^{2a} (e_i(a))^2.$$

We have

$$(2.9) \qquad \|\nabla(\ln f)\|^2 = \sum_{i=1}^{2a}(e_i(\ln f))^2 + \sum_{i=1}^{2a}(\phi e_i(\ln f))^2$$
$$= \sum_{i=1}^{2a}\sum_{t=1}^{b}\left(\left\| h_{\phi D^\perp}(\phi e_i, Z_t)\right\|^2 + \left\| h_{\phi D^\perp}(e_i, Z_1)\right\|^2\right).$$

Therefore from (2.5) and (2.9), we have

$$\sum_{i=1}^{2a}\sum_{t=1}^{b}\left\| h_{\phi D^\perp}(X_i,Z_t)\right\|^2 = \sum_{i=1}^{2a}\sum_{t=1}^{\beta}\left(\left\| h_{\phi D^\perp}(X_i,Z_t)\right\|^2 + \left\| h_{\phi D^\perp}(\mathrm{e},Z_t)\right\|^2\right)$$
$$+ \sum_{t=1}^{\beta}\left\| h_{\phi D^\perp}(\xi,Z_1)\right\|^2 = \sum_{t=0}^{\beta}\left(\|\nabla(\ln f)\|^2\right).$$

From above equation, we have

$$(2.10) \qquad \sum_{i=0}^{2a}\sum_{t=0}^{\beta}\left\| h_{\phi D^\perp}(X_i,Z_t)\right\|^2 = \sum_{t=0}^{\beta}\|\nabla(\ln f)\|^2 = \beta\left(\|\nabla(\ln f)\|^2\right).$$

For any vector field X tangent to M_1 and orthogonal to ξ and Z tangent to M_2, equation (1.8) gives

$$(2.11) \quad \bar{R}(X,\phi X,Z,\phi Z) = \frac{c-3u^2}{4}\{g(\phi X,Z)g(X,\phi Z) - g(X,Z)g(\phi Y,\phi Z)\} +$$
$$+\frac{c+u^2}{4}\{g(X,\phi Z)g(\phi^2 X,\phi Z) - g(\phi X,\phi Z)g(\phi X,\phi Z)\} +$$
$$+ 2g(X,\phi^2 X)g(\phi Z,\phi Z)\}$$
$$+\left(\frac{c+u^2}{4}+u\right)\{g(\phi X,\phi Z)\eta(X)\eta(Z) - g(X,\phi Z)\eta(\phi X)\eta(Z)$$
$$+ g(X,Z)\eta(\phi X)\eta(\phi Z) - g(\phi X,Z)\eta(X)\eta(\phi Z)\}.$$
$$= 2\left(\frac{c+u^2}{4}\right)\{g(X,\phi^2 X)g(\phi Z,\phi Z)\}.$$
$$= -\left(\frac{c+u^2}{2}\right)$$

On the other hand, by Codazzi equation, we have

$$(2.12) \qquad \bar{R}(X,\phi X,Z,\phi Z) = -g\left(\nabla_X^\perp h(\phi X,Z) - h(\nabla_X \phi X,Z) - h(\phi X,\nabla_X Z),\phi Z\right)$$
$$+ g\left(\nabla_{\phi X}^\perp h(X,Z) - h(\nabla_{\phi X} X,Z) - h(X,\nabla_{\phi X} Z),\phi Z\right)$$

By using equations (1.6) and (2.1), we get

$$\begin{aligned} g\left(\nabla^{\perp}_{X}h(X,Z),\phi Z\right) &= Xg\left(h(\phi X,Z),\phi Z\right)-g\left(h(\phi X,Z),\bar{\nabla}_{X}\phi Z\right)\\ &= Xg(\nabla_{Z}X,Z)-g\left(h(\phi X,Z),\phi\bar{\nabla}_{X}Z\right)=X(X\ln f)\,g(Z,Z)\\ &\quad -(X\ln f)\,g\left(h(\phi Y,Z)\phi Z\right)-g\left(h(\phi X,Z),\phi h_{v}(X,Z)\right)\\ &= (X^{2}\ln f)\,g(Z,Z)+(X\ln f)^{2}g(Z,Z)-\|h_{v}(X,Z)\|^{2}, \end{aligned}$$

where $h_{\nu}(X,Z)$ denotes the ν-component of $h(X,Z)$. Also, we have

$$\begin{aligned} g(h(\nabla_{X}\phi X,Z),\phi Z) &= g\left(\bar{\nabla}_{Z}\nabla_{X}\,\phi X,\phi Z\right)\\ &= g\left(\bar{\nabla}_{Z}\bar{\nabla}_{Z}\,\phi X,\phi Z\right)-g\left(\bar{\nabla}_{Z}\,h(X,\phi X),\phi Z\right)\\ &= -g(X,X)\,g(Z,Z)+\left((\bar{\nabla}_{X}X)\ln f\right)g(Z,Z)\\ g\left(h(\phi X,\nabla_{X}Z),\phi Z\right) &= (X\ln f)\,g\left(h(\phi X,Z),\phi Z\right)=(X\ln f)^{2}g(Z,Z) \end{aligned}$$

Substituting the above relation in (2.12) we have

$$\begin{aligned} \bar{R}(X,\phi X,Z,\phi Z) &= 2\|(h_{v}(X,Y))\|^{2}-(X^{2}\ln f)\;g(Z,Z)\\ &\quad +((\nabla_{X}X)\ln f)\,g(Z,Z)-2g(X,X)\,g(Z,Z)\\ &\quad +\left((\phi X)^{2}\ln f\right)g(Z,Z)+\left(\left(\nabla_{\phi X}\,\phi X\right)\ln f\right)\right)g(Z,Z). \end{aligned} \tag{2.13}$$

Now by summing the equation (2.13) and using (2.11) we get

$$\sum_{i=1}^{2\alpha}\sum_{t=1}^{\beta}\|h_{v}(X,Z)\|^{2}=2\alpha\beta\left(\frac{c+u^{2}}{2}+1\right)-\beta\,\Delta(\ln f) \tag{2.14}$$

Next, inequality (2.3) follows from (2.10) and (2.14).

Let h'' be the second fundamental form of M_2 in M. Then, we get

$$g(h''(Z,W),X)=g\left(\nabla_{Z}W,X\right)=-\left(X\ln f\right)g(Z,W),$$

or equivalently

$$h''(Z,W)=-g(Z,W)\,\nabla(\ln f). \tag{2.15}$$

If the equality sign of (2.3) hold identically then we obtain

$$h(D,D)=0,\,h(D^{\perp},D^{\perp})=0,\,h(D,D^{\perp})\subset\phi D^{\perp} \tag{2.16}$$

The first condition of (2.16) implies that M_1 is totally geodesic on M. On the other hand, we have

$$g(h(X,\phi Y),\phi Z)=g(\bar{\nabla}_{X}\phi Y,\phi Z)=g(\bar{\nabla}_{X}Y,Z)=0.$$

Thus M_1 is totally geodesic in $\bar{M}(c)$ and hence is a locally conformal almost cosymplectic manifold with constant ϕ-sectional curvature c. The second condition of

(2.16) and (2.15) imply that M_2 is a totally umbilical in $\bar{M}(c)$. Moreover, by (2.16), it follows that M is minimal submanifold of $\bar{M}(c)$.

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CONTENTS

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