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CONTENTS

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Finite range model to describe the distribution of infant death by age

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Abstract: This paper proposes the Finite range model to describe the distribution of infant death by age. Nepal Demographic and Health Surveys data were utilized for this purpose. Finite range model was found to be an appropriate model for the approximation of the distribution of infant deaths by age in months. Misreported death cases were found to be 0.46 and 1.56 per cent respectively for the surveys of 1996 and 2001 data. It is believed that the finding of this paper may help planners and policy-makers for designing proper policy in reducing the infant and child mortality of a country.

1. Introduction

The level of infant and child mortality reflects the level of socio-economic development and health status of a country [1]. Despite the fact that infant mortality has reduced to a very low level among well-off and industrialized countries but developing countries still experienced a very high level of early childhood mortality [2]. The consistency of mortality estimates is entirely depends on the extent and nature of data to which date of birth and age at death are accurately reported and recorded, and along with the completeness of such data [3, 4].

In most of the developing countries, the vital registration system including births and deaths suffers from the problems of omission, over or under counts the events [5]. The omissions and misreporting of the data are more often appeared in the retrospective surveys, which either may be due to memory lapse or may be due to digit preference of the respondents [6]. Generally, events are over-reported at some preferred ages like 3 months, 6 months, 9 months, etc., particularly in the developing countries [1,7], which affects the estimation of infant mortality rate and child mortality rate of a country.

Theile [8] proposed a seven-parameter model to study the age pattern of mortality; however, Heligman and Pollard [9] introduced an eight-parameter model in order to improve the fitting of mortality data. Mitra and Denny [10] proposed a model to graduate mortality pattern after the age of 30 years. A number of researchers had developed and/or graduated the survival function for the study of mortality pattern at early ages using hyperbolic, logarithmic and Weibull distributions [11,12]. Krishnan and Jin [13] applied Pareto distribution to describe the age pattern of early childhood mortality. Chauhan [7] used finite range model to describe the age pattern of infant death.

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In Nepal, the infant mortality rate was 182 in 1964, 156 in 1974, 113 in 1984, 79 in 1996, 64 in 2001 and it came down to 51 per 1000 live births in 2006 [1, 14, 15, 16]. Under-five mortality rate declined from 297 in 1960 to 202 in 1986 to 118 in 1996 to 91 in 2001 to 51 per 1000 live births in 2006 [14, 16]. However, total fertility rate has declined from 6.3 in 1971 to 4.1 in 2001 to 3.1 per woman in 2006 [14, 16]. These figures show a declining trend in early childhood mortality and fertility in a country. Though, it is still very high level of infant and child mortality and fertility in a country as compared to Asian developing countries [1].

On this background of high mortality and fertility in a country, a thorough study on the distribution of infant and child deaths is important for achieving to the reduction in fertility and early mortality as well as to improve the quality of health of the people. No doubt a concise model description of past mortality patterns provides the basis for projections [17]. It has been realized that the theory and practice of forecasting mortality have evolved rapidly in current decades [18, 19, 20, 21].

Adequate work on the distribution of infant deaths by age has not intensively been carried out yet in Nepal. This may perhaps be due to the lack of reliable data or the lack of interest among the researchers. Therefore, the main aim of this article is to propose a Finite range model to describe the distribution of infant death by age. The parameters involved of the model have been estimated through the least square and iteration methods. The data are taken from the Nepal Family Health Survey (NFHS 1996) 1996 and the Nepal Demographic and Health Survey (NDHS 2001) 2001 [4, 14]. In both the surveys, each woman was asked about the number of sons and daughters living with her, elsewhere and the number who had died, and the number of pregnancies that did not end in a live birth along with their name, sex, age (if alive) or age at death (if dead), etc. The details about the surveys are found in MOH [4, 14].

2. Finite Range Model

Finite range model has been proposed to study the distribution of infant deaths by age. This model has been chosen because of its simplicity in nature and having less number of parameters involved in the model. Finite range model was proposed by Mukherjee and Islam [22] for the study of reliability. Chauhan [7] introduced it to graduate the infant deaths. A brief description of the model is given below.

Let x be the age in months at death of the infant, then the probability density function of x is,

(1) $$f(x) = \frac{p}{x}\left(\frac{x}{\theta}\right)^{p} \; ; \; 0 < x < \theta \; ; \; \theta > 0 \text{ and } 0 < p < 1$$

where θ and p are scale and shape parameters of the distribution respectively.
The corresponding cumulative distribution function is given by

(2) $$F(X < x) = \left(\frac{x}{\theta}\right)^{p} \; ; \; 0 < x < \theta \; ; \; \theta > 0 \text{ and } 0 < p < 1$$

The mean, median and variance of the Finite range model are given as below.

$$\text{Mean} = \bar{m} = \left(\frac{p}{p+1}\right)\theta, \; \text{Median} = M_d = \left[\left(\frac{1}{2}\right)^{\frac{1}{p}}\right]\theta$$

$$\text{and Variance} = \sigma_x^2 = \left[\frac{p}{(p+1)\,(p+2)^2}\right]\theta^2$$

3. Estimation of Parameters

Least square technique has been proposed to estimate the parameters of the model. The parameters are also estimated by using iteration method, and then compared it with least square method. It is hypothesize that the least square technique may provide better estimate than that of iteration method because in iteration method the scale parameter is taken as fixed.

Iteration Method: The estimated value of shape parameter, p, can be obtained by fixing the scale parameter θ. Let D and D_0 are the total number of infant deaths (deaths up to 12 months) and neonatal deaths (deaths up to 30 days) respectively. Then the proportion of neonatal deaths is given as,

$$R = \frac{D_0}{D} \tag{3}$$

Thus the proportion of infant deaths up to the first month of age by fixing θ=12, is given as,

$$F(X < x) = \left(\frac{1}{12}\right)^p \tag{4}$$

On solving equations (3) and (4), we get,

$$R = \left(\frac{1}{12}\right)^p \tag{5}$$

By solving equation (5), we get the estimated value of shape parameter, p, as

$$\hat{p} = \frac{\text{Log R}}{\text{Log (1/12)}} \tag{6}$$

Using estimated value p given by expression (6), the corresponding cumulative proportion of death up to age of 1st month, 2nd month, 3rd month, 4th . . . , 12th month can be calculated by using equation (4). The proportion of deaths during particular months of age can also be calculated by subtracting the successive values of cumulative proportion of deaths.

Symbolically, proportion of deaths during month x is expressed as

$$F(X < x+1) - F(X < x)\text{ ; for } x > 1 \tag{7}$$

and the proportion of deaths during 1st month is $F(X < 1)$.

Thus the expected deaths can be obtained as

$$E[D_x] = DF(X < x),\ \text{for } x = 1, 2, 3, \ldots 12. \tag{8}$$

Least-Square Method: It is quite difficult to fix the scale parameter θ of the distribution and sometimes it would lead mis-interpretations of the parameters. Thus the least square method can be used to estimate both the parameters involved in the model. By taking log both sides in equation (2), we get the following linear equation.

$$\log F(x) = -p \log\theta + p \log x \tag{9}$$

The equation (9) can be re-written as,

$$Y = A + pX \tag{10}$$

where, $\log F(x) = Y$, $-p \log \theta = A$ and $\log x = X$

$F(x)$ is the sample cumulative distribution function of age x. Since, expression (10) is in a linear form. So that using least square principle, both the parameters p and θ can easily be estimated.

4. Application of the Model

Table 1 shows the observed and expected distribution of infant deaths by age in months (deaths during 0–4 years prior to the survey date) for NFHS 1996 data. The parameters of the model are estimated by using iteration as well as least square methods. The estimated parameters, chi-square values and their degrees of freedom are also presented in the same table. The summary measures like mean, median and variance of the model are also computed and are presented in the same table for both the data sets.

Table 1 Observed And Expected Number Of Infant Deaths By Age (NFHS 1996)

Age at death (in months)	Observed Deaths	Iteration method		Least square method	
		$F(X \le x)$	Expected deaths	$F(X \le x)$	Expected deaths
0–1	352	0.6629	352.00	0.65087	352.69
1–2	32	0.7435	42.77	0.7296	42.64
2–3	30	0.7950	27.39	0.7799	27.29
3–4	15	0.8338	20.58	0.8178	20.50
4–5	19	0.8652	16.65	0.8484	16.58
5–6	12	0.8917	14.06	0.8742	14.01
6–7	14	0.9147	12.23	0.8967	12.17
7–8	9	0.9351	10.85	0.9166	10.80
8–9	12	0.9535	9.76	0.9346	9.72
9–10	12	0.9703	8.90	0.9509	8.86
10–11	10	0.9857	8.19	0.9660	8.15
11–12	14	1.0000	7.58	0.99997	7.55
Total	531		531.00		531.00
	$\hat{p}$	0.16545		0.1647	
	$\hat{\theta}$	12.00		13.57	
	χ^2	13.48		12.07	
	d.f.	9		9	
Mean		1.70		1.83	
Median		0.18		0.20	
Variance		4.36		5.56	

The shape parameter, p, was found to be 0.1655 for iteration method whereas it was 0.1647 for least-square method, which is almost identical for the NFHS data. It is to be noted that the scale parameter, θ (=12), was kept fixed in iteration method whereas it was estimated in the least square method. Least square method provides the estimated value of scale parameter of θ=13.58, which is higher than that of the iteration method. The chi-square value suggested that the Finite range model fitted well to the data of NFHS 1996. The fitting of the distribution was found very close by the least square method than that of iteration method. Moreover, the estimated mean age of infant deaths provided by the model was found to be 1.8 months while median age of infant deaths was around 6 days with a variance of 5.6 months.

Table 2 Observed And Expected Number Of Infant Deaths By Age (NDHS 2001)

Age at deaths (in months)	Observed Deaths	Iteration method		Least square method	
		$F(X<x)$	Expected deaths	$F(X<x)$	Expected deaths
0–1	264	0.6168	264.00	0.6242	267.166
1–2	38	0.7058	38.09	0.7119	37.53
2–3	19	0.7637	24.78	0.7688	24.35
3–4	26	0.8077	18.81	0.8119	18.45
4–5	10	0.8435	15.33	0.8470	15.02
5–6	9	0.8739	13.03	0.8768	12.75
6–7	11	0.9005	11.38	0.9028	11.13
7–8	8	0.9242	10.14	0.9260	9.91
8–9	13	0.9456	9.16	0.9469	8.95
9–10	11	0.9652	8.38	0.9660	8.18
10–11	9	0.9832	7.73	0.9836	7.54
11–12	10	1.0000	7.18	0.9998	7.01
Total	428		428.00		428.00
	$\hat{p}$	0.1944		0.1896	
	$\hat{\theta}$	12.00		12.97	
	χ^2	12.41		11.13	
	d.f.	9		9	
Mean		1.95		2.07	
Median		0.34		0.34	
Variance		4.87		5.59	

Table 2 presents the observed and expected distribution of infant deaths by age in months (deaths during 0-4 years prior to the survey date) for NDHS 2001 data. The shape parameter, p, and the scale parameter, θ, was found to be 0.1944 and 12 respectively by iteration method whereas it was respectively 0.1896 and 12.97 by least-square method. The value of scale parameter was found to be slightly higher by least square method than that of iteration method. The insignificant chi-square value also suggested that the Finite range model fitted well to the data of NDHS 2001. Here too, the small value of chi-square produced by the least square method confirms the better fits to the data of infant deaths by age than that of iteration method. The estimated mean age of infant deaths provided by the model was found to be 2.1 months and median age of infant death was 10 days with variance of 5.6 months.

Further, Finite range model has been proposed to graduate the death patterns up to 24 months. The suitability of this graduated model has also been tested by the same data of NFHS 1996 and NDHS 2001. Table 3 displays the observed and expected distribution of infant deaths by age up to 24 months for NFHS 1996 data. The estimated shape parameter, p, was found to be 0.1673 and 0.1805 for iteration and least-square method respectively. The scale parameter, θ(=24), was kept fixed in iteration method whereas it was estimated in least square method and it was found to be lower (θ=23.36) than that of the iteration method.

Similarly, Table 4 displays the observed and expected values of infant deaths by age up to 24 months (deaths during 0-4 years prior to the survey date) for the NDHS 2001 data. The shape parameter, p, was found to be 0.1881 and 0.1970 for iteration and least-square method respectively. The scale parameter, θ (=24), was kept fixed in iteration method whereas it was 21.98 obtained by least square method, which is quite lower than that of the iteration method.

The least square method was found to be more appropriate technique to estimate the parameters of the model than iteration method and that confirms by fitting the distributions (Tables 1 to 4).

Table 3 Estimation Of Extent Of Misreporting Of Infant Deaths (NFHS 1996)

Age at deaths (in months)	Observed Deaths	Iteration method		Least square method	
		$F(X \le x)$	Expected deaths	$F(X \le x)$	Expected deaths
0–1	352	0.5876	352.00	0.5662	337.50
1–2	32	0.6599	43.27	0.6417	44.98
2–3	30	0.7062	27.74	0.6904	29.04
3–4	15	0.7410	20.85	0.7272	21.94
4–5	19	0.7692	16.88	0.7571	17.81
5–6	12	0.7930	14.27	0.7824	15.10
6–7	14	0.8137	12.41	0.8045	13.16
7–8	9	0.8321	11.01	0.8241	11.69
8–9	12	0.8487	9.92	0.8418	10.51
9–10	12	0.8638	9.04	0.8580	9.63
10–11	10	0.8777	8.32	0.8729	8.87
11–12	14	0.8905	7.71	0.8867	8.23
12–13	15	0.9025	7.19	0.8996	7.69
13–14	7	0.9138	6.74	0.9117	7.22
14–15	6	0.9244	6.35	0.9231	6.81
15–16	7	0.9344	6.01	0.9339	6.44
16–17	6	0.9439	5.71	0.9442	6.12
17–18	4	0.9530	5.43	0.9540	5.83
18–19	10	0.9617	5.19	0.9634	5.57
19–20	2	0.9699	4.96	0.9723	5.34
20–21	3	0.9779	4.76	0.9809	5.12
21–22	2	0.9856	4.57	0.9892	4.93
22–23	2	0.9929	4.40	0.9972	4.75
23–24	4	1.0000	4.25	0.9999	4.58
Total	599		599.00		599.00
	$\hat{p}$	0.16728		0.1805	
	$\hat{\theta}$	24.00		23.36	
	χ^2	22.82		21.01	
	d.f.	15		15	
Reported infant deaths		531.00		531.00	
Calculated infant deaths		533.42		531.13	
No. of deaths under reported		2.42		0.13	
Percent under reported		0.46		0.03	

Thus, the values of chi-square indicate that the graduated model fitted reasonably well to the data of Nepal for describing the distribution of infant deaths by age up to 24 months. So this model may be utilized to estimate the extent of misreporting of infant deaths by age during the infancy.

It has been observed that the death patterns during infancy usually biased either under-reporting or over-reporting of death events. It is also noted that the infant mortality rate is apparently affected by such biased reporting during the ages of 12 months [1]. The extent of

misreporting of infant deaths may be estimated by taking the expected deaths from the graduated model.

Tables 3 and 4 (the last row) show the extent of misreporting at 12 months, when age at death is graduated up to 24 months. Since the model was fitted reasonably well to both the data sets. Hence the corrected values of infant deaths can be obtained by taking the sum of expected deaths up to 12 months from graduated model. This may provide the nature and extent of misreported deaths at the infancy. The differences between the distributions of the fittings of infant death by age up to 12 months and up to 24 months may be attributed the extent of misreporting of the events. It was found that around 0.46 per cent infant deaths were misreported i.e., under-reported for the data of NFHS 1996 (Table 3). Similarly, the extent of misreporting was found to be 1.56 per cent for the data of NDHS 2001 (Table 4).

Table 4 Estimation Of Extent Of Misreporting Of Infant Deaths (NDHS 2001)

Age at deaths (in months)	Observed Deaths	Iteration method		Least square method	
		$F(X < x)$	Expected deaths	$F(X < x)$	Expected deaths
0–1	264	0.5500	264.00	0.5346	256.63
1–2	38	0.6266	36.77	0.6129	37.55
2–3	19	0.6763	23.84	0.6639	24.47
3–4	26	0.7139	18.05	0.7026	18.58
4–5	10	0.7445	14.69	0.7341	15.16
5–6	9	0.7704	12.47	0.7610	12.89
6–7	11	0.7931	10.88	0.7845	11.26
7–8	8	0.8133	9.68	0.8054	10.04
8–9	13	0.8315	8.75	0.8243	9.08
9–10	11	0.8482	7.99	0.8416	8.30
10–11	9	0.8635	7.37	0.8575	7.66
11–12	10	0.8778	6.84	0.8723	7.12
12–13	8	0.8911	6.39	0.8862	6.66
13–14	11	0.9036	6.00	0.8992	6.26
14–15	6	0.9154	5.67	0.9116	5.91
15–16	2	0.9266	5.37	0.9232	5.60
16–17	2	0.9372	5.10	0.9343	5.32
17–18	4	0.9473	4.86	0.9449	5.08
18–19	10	0.9570	4.65	0.9550	4.86
19–20	7	0.9663	4.45	0.9647	4.66
20–21	1	0.9752	4.28	0.9740	4.47
21–22	0	0.9838	4.11	0.9830	4.30
22–23	0	0.9920	3.97	0.9916	4.15
23–24	1	1.0000	3.83	0.9999	4.01
Total	480		480.00		480.00
$\hat{p}$		0.1881		0.1970	
$\hat{\theta}$		24.00		21.98	
χ^2		21.68		20.51	
d.f.		13		13	
Reported infant deaths		428.00		428.00	
Calculated infant deaths		421.32		418.73	
No. of deaths under reported		6.68		9.27	
Percent of under reported		1.56		2.17	

Hence, the model also confirms that the infant deaths under-reported in the surveys. This result is consistent with the other findings [1,21]. These figures clearly indicate that the misreporting of infant deaths by age was higher for the NDHS 2001 data than that of the NFHS 1996 data. Similar fashion has been adopted for computing probable bias occurred in successive months during infancy.

Finding suggests that infant mortality rate directly affects by the reporting of deaths during infancy. Therefore due attention should be paid by considering deaths underreported during infancy while computing the infant mortality rates. This study further suggests that Finite range model may graduate the number of deaths at different ages during early childhood for the developing country like Nepal.

5. Conclusions

This study suggests that Finite range model may be an appropriate model for the approximation of the distribution of infant deaths by age in months in the developing countries like Nepal. With the help of this model, the extent of misreporting of age at deaths can also be estimated. The descriptive statistics are also computed with the help of the fitted model. The estimated probable biased value was found to be 0.46 per cent for the NFHS 1996 data while it was 1.56 per cent for the NDHS 2001 data. Findings may help policy-makers and planners for designing proper policy to reduce infant and child mortality and to improve mother's health of a country.

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Uniform version of Wiener–Tauberian theorem for Wiener algebra

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Abstract: The Wiener-Tauberian theorem for $\mathbb{R}$ says that the closed translation invariant subspace generated by $f \in L^1(\mathbb{R})$ is $L^1(\mathbb{R})$ if and only if the Fourier transform $\hat{f}$ of f never vanishes. In this paper we prove a uniform version of this result in Wiener algebra.

Keywords: Wiener-Tauberian theorem, locally compact abelian group, translation invariant subspace, Wiener algebra.

1.Introduction:

Let G be a separable locally compact abelian group with left Haar measure λ. Let K be a nonempty compact subset of G which is the closure of its interior. If f is measurable function on G, we define

$$f^{\#} : G \to [0,\infty] \text{ by}$$
$$f^{\#}(x) = \| f \xi_{xK} \|_{\infty}$$

Let $\mathcal{N}_1$ be the set of Measurable functions g on G for which $g^{\#} \in L_1$. $\mathcal{N}_1$ is a Banach space with norm $\| f \|_1^{\#} = \| f^{\#} \|_1$. In fact $\mathcal{N}_1$ is a Banach algebra under Convolution.

The Wiener algebra $W(G)$ is defined as follows:

$$W(G) = \{ f \in \mathcal{N}_1 \,;\, f \text{ is continuous} \}$$

Let $\mathscr{R}$ be the set of all Radon measures on G. For $\mu \in \mathscr{R}$ we define

$$\mu^{\#} : G \to [0,\infty] \text{ by}$$
$$\mu^{\#}(x) = \| \xi_{xK} \mu \| = |\mu|(xK)$$

also $\mu^{\#} = |\mu| * \xi_{K^{-1}}$

$$W(G)^* = \{ \mu \in \mathscr{R} \,;\, \mu^{\#} \in L_\infty \} \text{ with the norm } \|\mu\| = \|\mu^{\#}\|_\infty$$

$$(f, \psi\mu) = \psi(\mu) f = \int_G f \, d\mu, \; f \in W(G) \,;\, \mu \in W(G)^*$$

For $h \in \mathscr{H}$, $\Phi_h : G \to L^1(G)$ by

$$\Phi_h(x) = {}_x h, \; x \in G$$

Let $S_1^{W(G)}$ and $S_\infty^{W(G)^*}$ be the unit ball of $W(G)$ and $W(G)^*$ respectively.

$$\mathrm{U} = \{ g \in W(G) \; ; \text{ for } \mu \in \mathscr{R}, \; g * \mu = 0 \Rightarrow \mu = 0 \}$$

For $f \in C_{00}(G)$

$$\begin{aligned}
\text{(1.1)} \qquad \| ({}_x f)^{\#} \|_1 &= \int_G |({}_x f)^{\#}(y)| \, d\lambda(y) \\
&= \int_G \| {}_x f \xi_{yK} \|_\infty \, d\lambda(y) \\
&= \int_G \sup_{z \in G} | f(xz) | \, \xi_{yK}(z) d\lambda(y) \\
&= \int_G \sup_{z \in G} | f(z) | \, \xi_{yK}(x^{-1} z) \, d\lambda(y) \\
&= \int_G \sup_{z \in G} | f(z) | \, \xi_K(y^{-1} x^{-1} z) \, d\lambda(y) \\
&= \int_G \sup_{z \in G} | f(z) | \, \xi_K(y^{-1} z) \, d\lambda(y) \\
&= \int_G \sup_{z \in G} | f(z) | \, \xi_{yK}(z) \, d\lambda(y) \\
&= \int_G \sup_{z \in G} | f \xi_{yK}(z) | \, d\lambda(y) \\
&= \int_G \| f \xi_{yK} \|_\infty \, d\lambda(y) \\
&= \| f \|_{W(G)}
\end{aligned}$$

since $C_{00}(G)$ is dense in $W(G)$,

so $\| {}_x f \|_{W(G)} = \| f \|_{W(G)}$ for all $f \in W(G)$.

Let $h \in C_{00}(G)$,

$$\begin{aligned}
\| h \|_{W(G)} = \| h^{\#} \|_1 &= \int_G | h^{\#}(x) | \, d\lambda(x) \\
&= \int_G \sup_{y \in G} \| h \xi_{xK} \|_\infty \, d\lambda(x) \\
&= \int_G \sup_{y \in G} | h \xi_{xK}(y) | \, d\lambda(x) \\
&= \int_G \sup_{y \in G} | h(y) | \, \xi_{xK}(y) \, d\lambda(x)
\end{aligned}$$

$$\leq \int_G \operatorname{Sup}_{y\in G} |\tilde{h}(y)|\, \xi_{xK}(y)\, d\lambda(x)$$

$$= \int_G |\tilde{h}^{\#}(x)|\, d\lambda(x) = \|\tilde{h}\|_{W(G)}$$

Let us take, $h \in C_{00}(G)$

1.3. $$\int h(x)\tilde{\mu}^{\#}(x)\, d\lambda(x) = \int_G \int_G h(x)\xi_{xK}(y)\, d|\sim|\tilde{(y)}\, d\lambda(x)$$

$$= \int_G \int_G h(x)\xi_K(x^{-1}y^{-1})\, d|\mu|(y)\, d\lambda(x)$$

$$= \int_G \int_G h(x)\xi_{K^{-1}}(yx)\, d|\mu|(y)\, d\lambda(x)$$

$$= \int_G \int_G \xi_{K^{-1}}(x)\, h(y^{-1}x)\, d|\mu|(y)\, d\lambda(x)$$

$$= \int_G \int_G h(y^{-1}x)\xi_K(x^{-1})\, d\lambda(x)\, d|\mu|(y)$$

$$= \int_G (h * \xi_K)(y^{-1})\, d|\mu|(y)$$

$$= \langle \xi_{K^{-1}} * \tilde{h}, |\mu| \rangle$$

and hence $\tilde{\mu}^{\#} \in L_\infty$.

1.4. For $f \in W(G)$ and $\mu \in W(G)^*$

$$\mu * f(x) = \int_G f(y^{-1}x)\, d\mu(y)$$

$$= \int_G f(xy^{-1})\, d\mu(y)$$

$$= \int_G {}_x f(y^{-1})\, d\mu(y)$$

$$= \int_G {}_x f(y)\, d\mu(y^{-1})$$

$$= \int_G {}_x f(y)\, d\tilde{\mu}(y)$$

$$= \psi(\tilde{\mu})({}_x f)$$

1.5. Theorem : *Let G be a separable locally compact abelian group. Let $\mathscr{H} \subset W(G)$ be such that the family $\{\Phi_h : h \in \mathscr{H}\}$ is uniformly equicontinuous. Suppose there exist $\tilde{h} \in \mathscr{H}$ with $|h(t)| \leq |\tilde{h}(t)|$ and $\|h\|_{W(G)} \leq \|\tilde{h}\|_{W(G)}$ for all $h \in \mathscr{H}$ & $t \in G$.*

Let $g \in U$ be fixed and $\mathcal{U} \subset S_\infty^{W(G)^*}$. Suppose that $\mu * g(x) \to 0$ as $x \to \infty$ uniformly for μ in $\mathcal{U}$ then $h * \mu(x) \to 0$ as $x \to \infty$ uniformly for h in $\mathscr{H}$ and μ in $\mathcal{U}$.

Proof: Assume to the contrary. So there exists $\delta > 0$ such that for every compact set K in G there exists $x_K \in G \sim K$, $h_K \in \mathscr{H}$ and $h_K \in \mu_K \in \mathcal{U}$ satisfying :

$$|(h_K * \mu_K)(x_K)| > \delta.$$

Since G is separable and locally compact so G is σ-compact. Thus there exists an increasing sequence say $\{K_n\}_{n\in\mathbb{N}}$ of compact sets with $K_n \subset \operatorname{Int} K_{n+1}$ and if F is any compact set in G thus there exists n_0 with $F \subset K_{n_0}$. We write

$$h_{K_n} = h_n,\ \mu_{K_n} = \mu_n \text{ and } x_{K_n} = x_n.$$

Define a sequence of functions on G by

$$\mathcal{S}_n(x) = {}_{x_n}(h_n * \mu_n)(x)$$

$$\begin{aligned}
\sup_{x\in G} |\mathcal{S}_n(x)| &= \sup_{x\in G} |{}_{x_n}(h_n * \mu_n)(x)| \\
&= \sup_{x\in G} |(h_n * \mu_n)(x_n x)| \\
&= \sup_{x\in G} \left| \psi(\tilde{\mu}_n)\left({}_{x_n x}(h_n)\right) \right| \\
&\le \|\psi(\tilde{\mu}_n)\|_{W(G)^*} \|{}_{x_n x} h_n\|_{W(G)} \\
&= \|\tilde{\mu}_n\|_{W_\infty} \|h_n\|_{W(G)} \\
&= \|\tilde{\mu} n^{\#}\|_\infty \|\bar{h}\|_{W(G)} \\
&\le 1
\end{aligned}$$

Since $\{\Phi_h : h \in \mathscr{H}\}$ is uniformly equi continuous, so for given $\in > 0$ there exists a nbd $U_\in$ of e in G s.t. for $t \in U_\in$, for $x \in G$, $y = tx$ we have

$$\left\| {}_x h - {}_y h \right\|_{W(G)} < \in.$$

For $y = tx$, $t \in U_\in$.

$$\begin{aligned}
|\mathcal{S}_n(x) - \mathcal{S}_n(y)| &= \left| {}_{x_n}(h_n * \mu_n)(x) - {}_{x_n}(h_n * \mu_n)(y) \right| \\
&= |(h_n * \mu_n)(x_n x) - (h_n * \mu_n)(x_n y)| \\
&= \left| \psi(\tilde{\mu}_n)\left({}_{x_n x}(h_n)\right) - \psi(\tilde{\mu}_n)\left({}_{x_n y}(h_n)\right) \right| \\
&= \left| \psi(\tilde{\mu}_n)\left({}_{x_n x}(h_n)\right) - {}_{x_n y}(h_n) \right| \\
&= \left| \psi(\tilde{\mu}_n)\left\{ {}_{x_n}\left({}_x(h_n) - {}_y(h_n)\right) \right\} \right| \\
&\le \left\| \psi(\tilde{\mu}_n) \right\|_{W(G)^*} \left\| {}_{xn}\left({}_x(h_n) - {}_y(h_n)\right) \right\|_{W(G)}
\end{aligned}$$

$$= \| \tilde{\mu}_n \|_{W_\infty} \left\| {}_x(h_n) - {}_y(h_n) \right\|_{W(G)}$$

$$= \left\| \tilde{\mu}_n^{\#} \right\|_\infty \left\| {}_x(h_n) - {}_y(h_n) \right\|_{W(G)}$$

$$\leq \left\| {}_x(h_n) - {}_y(h_n) \right\|_{W(G)}$$

$$< \epsilon$$

By Ascoli's theorem [5] there exists a pointwise Convergent subsequence $\{ \Delta_{n_j} \}$ converging to a continuous function Δ on G. Thus for a fixed x and t in G.

$$\Delta_{n_j}(xt^{-1}) \to \Delta(xt^{-1}),\ j \to \infty$$

$$\left| \Delta_{n_j}(xt^{-1})\, g(t) \right| = \left| \Delta_{n_j}(xt^{-1}) \right| |g(t)|$$

$$\leq \| \Delta_{n_j} \|_\infty |g(t)|.$$

$$\leq |g(t)|$$

Thus by Lebesgue dominated Convergence theorem

$$\int_G \Delta_{n_j}(xt^{-1})\, g(t)\, d\lambda(t) \to \int_G \Delta(xt^{-1})\, g(t)\, d\lambda(t)$$

i.e. $\Delta_{n_j} * g(x) \to s * g(x)\ \forall x \in G$

Now, $$\Delta_{n_j} * g(x) = \int_G \Delta_{n_j}(xt)\, g(t^{-1})\, d\lambda(t)$$

$$= \int_G {}_{x_{n_j}}(h_{n_j} * \mu_{n_j})(xt)\, g(t^{-1})\, d\lambda(t)$$

$$= \int_G (h_{n_j} * \mu_{n_j})(x_{n_j} xt)\, g(t^{-1})\, d\lambda(t)$$

$$= \left((h_{n_j} * \mu_{n_j}) * g \right)(x_{n_j} x)$$

$$= \left((h_{n_j} * (\mu_{n_j} * g) \right)(x_{n_j} x)$$

$$= (\mu_{n_j} * g) * h_{n_j}(\mu_{n_j} x)$$

$$= \int_G \bigcup_{n_j}^{x}(y)\, d\lambda(y)$$

Where $\bigcup_{n_j}^{x}(y) = h_{n_j}(y)(\mu_{n_j} * g)(y^{-1} x_{n_j} x)$. Since $\mu * g$ vanishes at infinity uniformly for $\mu \in \mathcal{U}$. We have for any $n \in \mathbb{N}$ there is a compact set $\tilde{K}_n$ s.t $|\mu * g(z)| < 1/n$ for $z \in G \sim \tilde{K}_n$ and $\mu \in \mathcal{U}$.

For $y \in G$, let $L_n^{x,y} = y \tilde{K}_n x^{-1}$ which is compact so there exists $j_n^{x,y}$ such that $L_n^{x,y} \subset K_j$ for $j \geq j_n^{x,y}$ and as $x_{n_j} \notin K_j$, we have,

$$= \| \tilde{\mu}_n \|_{W_\infty} \| {}_x(h_n) - {}_y(h_n) \|_{W(G)}$$
$$= \| \tilde{\mu}_n^{\#} \|_\infty \| {}_x(h_n) - {}_y(h_n) \|_{W(G)}$$
$$\le \| {}_x(h_n) - {}_y(h_n) \|_{W(G)}$$
$$< \in$$

By Ascoli's theorem [5] there exists a pointwise Convergent subsequence $\{ \mathcal{S}_{n_j} \}$ converging to a continuous function $\mathcal{S}$ on G. Thus for a fixed x and t in G.

$$\mathcal{S}_{n_j}(xt^{-1}) \to \mathcal{S}(xt^{-1}),\ j \to \infty$$
$$\left| \mathcal{S}_{n_j}(xt^{-1})\, g(t) \right| = \left| \mathcal{S}_{n_j}(xt^{-1}) \right| |g(t)|$$
$$\le \| \mathcal{S}_{n_j} \|_\infty \ |g(t)|.$$
$$\le |g(t)|$$

Thus by Lebesgue dominated Convergence theorem

$$\int_G \mathcal{S}_{n_j}(xt^{-1})\, g(t)\, d\lambda(t) \to \int_G \mathcal{S}(xt^{-1})\, g(t)\, d\lambda(t)$$

i.e. $\mathcal{S}_{n_j} * g(x) \to s * g(x)\ \forall x \in G$

Now, $$\mathcal{S}_{n_j} * g(x) = \int_G \mathcal{S}_{n_j}(xt)\, g(t^{-1})\, d\lambda(t)$$
$$= \int_G x_{n_j}(h_{n_j} * \mu_{n_j})(xt)\, g(t^{-1})\, d\lambda(t)$$
$$= \int_G (h_{n_j} * \mu_{n_j})(x_{n_j} xt)\, g(t^{-1})\, d\lambda(t)$$
$$= \left((h_{n_j} * \mu_{n_j}) * g \right)(x_{n_j} x)$$
$$= \left((h_{n_j} * (\mu_{n_j} * g) \right)(x_{n_j} x)$$
$$= (\mu_{n_j} * g) * h_{n_j}(\mu_{n_j} x)$$
$$= \int_G \cup_{n_j}^{x}(y)\, d\lambda(y)$$

Where $\cup_{n_j}^{x}(y) = h_{n_j}(y)(\mu_{n_j} * g)(y^{-1} x_{n_j} x)$. Since $\mu * g$ vanishes at infinity uniformly for $\mu \in \mathcal{U}$. We have for any $n \in \mathbb{N}$ there is a compact set $\tilde{K}_n$ s.t $|\mu * g(z)| < 1/n$ for $z \in G - \tilde{K}_n$ and $\mu \in \mathcal{U}$.

For $y \in G$, let $L_n^{x,y} = y \tilde{K}_n x^{-1}$ which is compact so there exists $j_n^{x,y}$ such that $L_n^{x,y} \subset K_j$ for $j \ge j_n^{x,y}$ and as $x_{n_j} \notin K_j$, we have,

$$\left|(\mu_{n_j} * g)(y^{-1}x_{n_j}x)\right| < 1/n$$

$$|\mathcal{U}n_j^x(y)| \le |\bar{h}(y)|\, \mu_{n_j} * g(y^{-1}x_{n_j}x)$$

$$\le |(\bar{h}(y)|\, 1/n\,,\ \ j \ge j_n^{x,y}$$

$$\Rightarrow \mathrm{U}_{n_j}^x(y) \to 0 \text{ as } j \to \infty$$

Thus by Lebesgue dominated convergence theorem $\int_G u_{n_j}^x(y)\, d\lambda(y) \to 0$ as $j \to \infty$

$$\text{i.e. } \mathcal{S}_{n_j} * g(x) \to 0 \ \text{ i.e. } \ \mathcal{S} * g(x) \to 0$$

$$\Rightarrow \mathcal{S} = 0 \text{ since } g \in U.$$

But $\mathcal{S}_n(e) = (h_n * \mu_n)(x_n)$ so $|\mathcal{S}_n(e)| > \delta$

$\Rightarrow |\mathcal{S}(e)| \ge \delta$. Which is contradiction.

This completes the proof.

Let S_∞ be the unit ball of $W(G)^*$ and S_1 be the unit ball of $W(G)$ and $U = \{g : G \to \mathbb{C}$ measurable, $g \in W(G)$; for $a \in B \cap C$, $a * g = 0 \Rightarrow a = 0\}$.

Where B and C are sets of bounded and continuous functions respectively. The following theroem may also holds.

1.6. Theorem: *Let $\mathcal{U} \subset S_\infty$ be such that the family $\{\phi_\mu : \mu \in \mathcal{U}\}$ is uniformly equicontinuous. Let $\mathcal{H} \subset W(G)$, suppose that there exist $\bar{h} \in S_1$ s.t $|h(t)| \le |\bar{h}(t)|$ for all $h \in \mathcal{H}$ and all $t \in G$. If $g \in S_1 \cap U$ and $\mu^\# * g$ vanishes at infinity uniformly for $\mu \in \mathcal{U}$ then $\mu^\# * h$ vanishes at infinity uniformly for every $\mu \in \mathcal{U}$ and $h \in \mathcal{H}$.*

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On double Nörlund summability of Fourier–Jacobi series

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Abstract: In this paper we have proved a theorem on double Nörlund summability of Furier-Jacobi series, which generalizes various known results. However our theorem is as follows:

Theorem: *Let* (N, p_m, q_n) *be a double Nörlund method defined by a real non-negative, non-increasing sequenc* $\{p_m\}$ and a real-non-negative, non-decreasing sequence $\{q_n\}$.

Let $\psi(t)$ and $\lambda(t)$ be non-negative monotonic increasing functions of t, such that

$$\psi(n) \log n = O[\lambda(P_n)]$$

$$q_n P_n = O[(p*q)_n \log n]$$

$$\sum_{k=2}^{n} \frac{P_k}{k^{(2\alpha+1)/2} \log k} = 0\left(\frac{(p*q)_n^{1-c}}{q_n\, n^{(2\alpha+1)/2}}\right)$$

as $n \to \infty$, where c is a parameter with the restriction that $0 \le c \le 1$. If

$$F_1(t) = \int_0^t |F(\phi)|\, d\phi = 0\left(\frac{t^{(2\alpha+2)}\psi(1)}{\lambda(P_\tau)}\right)$$

as $t \to$ o, where $\tau = [\,1/t\,]$ then the Fourier-Jabobi series is summable (N, p_m, q_n) at the point $x = +1$ to the sum A, provided that the condition

$$-1/2 \le \alpha < 1/2, \; \beta > -1/2$$

and the antipole condition

$$\int_{-1}^{b} (1+x)^{(\beta-\alpha-1)/2} |f(x)|\, dx < \infty$$

are satisfied, where b is fixed.

Definitions and Notations: Let $f(x)$ be a function defined on the interval $-1 \le x \le 1$ such that the integral.

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} f(x)\,dx \tag{1.1}$$

exists in the sense of Lebesgue for $\alpha > -1$, $\beta > -1$. The Fourier-Jacobi series corresponding to the function $f(x)$ is given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x) \tag{1.2}$$

where

$$a_n = g_n \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x) f(x)\,dx \tag{1.3}$$

with

$$g_n = \frac{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}$$

and $P_n^{(\alpha,\beta)}(x)$ are the Jacobi polynomials defined by the generating function.

$$2^{\alpha+\beta}(1-2xt+t^2)^{-1/2}[1-t+(1-2xt+t^2)^{1/2}]^{-\alpha}\,[1+t+(1-2xt+t^2)^{1/2}]^{-\beta}$$

$$= \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\,t^n \tag{1.5}$$

Let us write

$$F(\phi) = \{f(\cos\phi) - A\}(\sin\phi/2)^{2\alpha+1}(\cos\phi/2)^{2\beta+1}$$

A being a fixed constant.

Let $\{s_n\}$ be the sequence of partial sums of a given infinite series $\sum a_n$. Let $\{p_m\}$ and $\{q_n\}$ be any two sequences of constants with P_m and Q_n as their partial sums respectively and let

$$(p*q)_n = \sum_{k=0}^{n} p_{n-k}\,q_k = \sum_{k=0}^{n} p_k\,q_{n-k} \tag{1.6}$$

tends to infinity as $n \to \infty$

If the sequence-to sequence transformation defined by

$$t_n^{p,q} = \frac{1}{(p*q)^n} \sum_{k=0}^{n} p_{n-k}\,q_k\,s_k \tag{1.7}$$

tends to a fixed limit s as $n \to \infty$, then the sequence $\{s_n\}$ or the series $\sum a_n$ is said to be summable by double Nörlund method (N, p_m, q_n) to s, Borwein [2].

2. Introduction: The study of summability of Fourier-Jacobi series by ordinary Nörlund summability method has been made by several workers ([1],[5],[9],[16]). In the present paper we study the summability of Fourier-Jacobi series by double Nörlund summability method.

Dealing with the Nörlund summability of Fourier-Jacobi series, Prasad and Saxena [10] have established the follwing :

Theorem A : If

$$F_1(t) = \int_0^t |F(\phi)|\, d\phi = 0\left(\frac{\psi(t)\, t^{2\alpha+2}}{0(P_n)}\right) \quad \text{as } t \to 0$$

where

$$F(\phi) = \{f(\cos\phi) - A\}(\sin\phi/2)^{2\alpha+1}(\cos\phi/2)^{2\beta+1},$$

$\psi(t)$ and $\theta(t)$ are non-negative monotonic increasing functions of t such that

(2.2) $$\psi(n)\log n = \mathrm{O}(\theta(P_n)) \quad \text{as } n \to \infty$$

(2.3) $$n^{(2\alpha+1)/2} = \mathrm{o}(P_n) \qquad \text{as } n \to \infty$$

and

(2.4) $$\sum_{k=2}^{n} \frac{P_k}{k^{(2\alpha+1)/2}\log k} = 0\left(\frac{P_n}{n^{(2\alpha+1)/2}}\right) \text{as } n \to \infty$$

then the Fourier-Jacobi series (1.2) is summable (N, p_n) at the point $x = +1$, to sum A, provided that the condition

$$-1/2 \le \alpha < 1/2, \quad \beta > -1/2 \quad \text{and the antipole condition}$$

(2.5) $$\int_{-1}^{b} (1+x)^{(2\beta-3)/4}\, |f(x)|\, dx < \infty$$

are satisfied, where b is fixed and (N, p_n) is regular Nörlund method defined by the real non-negative and non-increasing sequence $\{p_n\}$ such that $p_n \to \infty$ as $n \to \infty$.

The object of this paper is to generalize the above theorem to a more general class of double Nörlund summability of Fourier-Jacobi series.

2. We establish our result in the form of the following theorem.

Theorem: *Let* (N, p_m, q_n) *be a double* Nörlund *method defined by a real non-negative, non-increasing sequence* $\{p_m\}$ *and a real non-negative, non decreasing sequenc* $\{q_n\}$.

Let $\psi(t)$ and $\lambda(t)$ be non-negative monotonic increasing functions of t such that

(3.1) $$\psi(n)\log n = \mathrm{O}[\lambda(P_n)]$$

(3.2) $$q_n P_n = \mathrm{O}[(p * q)_n \log n]$$

(3.3) $$\sum_{k=2}^{n} \frac{P_k}{k^{(2\alpha+1)/2} \log k} = O\left(\frac{(p*q)_n^{1-c}}{q_n \, n^{(2\alpha+1)/2}} \right)$$

as $n \to \infty$, where c is a parameter with the restriction that $0 \le c \le 1$.

(3.4) $$\text{If } F_1(t) = \int_0^t |F(\phi)| \, d\phi = 0 \left(\frac{t^{(2\alpha+2)} \psi(t)}{\lambda(P_\tau)} \right)$$

as $t \to 0$, where $\tau = [\, 1/t \,]$ then the Fourier-Jacobi series (1..2) is summable (N, p_m, q_n) at the point $x = +1$ to the sum A, provided that the condition

$$-1/2 \le \alpha < 1/2, \quad \beta > -1/2$$

and the antipole condition

(3.5) $$\int_{-1}^{b} (1+x)^{(\beta-\alpha-1)/2} \, | f(x) | \, dx < \infty$$

are satisfied, where b is fixed.

4. The following lemmas are needed for the proof of our theorem

Lemma 1 : ([15] p. 167 & 196) : For $\alpha > -1, \beta > -1$

$$P_n^{(\alpha,\beta)}(\cos\phi) = \begin{cases} O(n^\alpha), \text{ when } 0 \le \phi \le 1/n \\ O(n^\beta), \text{ when } \pi - 1/n \le \phi \le \pi \\ \dfrac{1}{(n\pi)^{1/2}} (\sin\phi/2)^{-(2\alpha+1)/2} (\cos\phi/2)^{-(2\beta+1)/2} \\ \quad \times \left(\cos\left\{ \dfrac{(2n+\alpha+\beta+1)}{2}\phi - (2\alpha+1)\dfrac{\pi}{4} \right\} + \dfrac{O(1)}{n \sin\phi} \right) \\ \qquad \text{when } 1/n \le \phi \le \pi - 1/n \end{cases}$$

Lemma 2 : The antipole condition

$$\int_{-1}^{b} (1+x)^{(\beta-\alpha-1)/2} \, | f(x) | \, dx < \infty$$

is equivalent to

$$\int_{-1}^{b} (1+x)^{(\beta-\alpha-1)/2} \, | f(x) - A | \, dx < \infty$$

which is further

$$\int_{n}^{\pi} | F(\phi) | (\cos\phi/2)^{-\alpha-\beta-1} \, d\phi < \infty, \; 0 < n < \pi .$$

Lemma 3. [7]. If $\{p_n\}$ is a non-negative, non-increasing sequence then for large n, uniformly in $0 < \phi \le \pi$, $0 \le a \le b \le n$,

$$\left| \sum_{k=a}^{n} p_k \cos\{(n-k+p)\phi - \gamma\}(n-k)^{(2\alpha+1)/2} \right| = O\left(n^{(2\alpha+1)/2}\rho(1/\phi)\right)$$

where

$$\rho = \frac{\alpha+\beta+2}{2},\ \gamma = \frac{(2\alpha+3)\pi}{4},\ \alpha \ge -1/2$$

Lemma 4. [7]: If $\{p_n\}$ is a non-negative, non-increasing and $\{q_n\}$ is a non-negative, non-decreasing sequence then

$$\sum_{k=0}^{n-1} p_k\, q_{n-k}\, (n-k)^{(2\alpha-1)/2} = O\left((p * q)_n\, n^{(2\alpha+1)/2}\right)$$

Lemma 5: Let

$$N_n(\phi) = \frac{2^{\alpha+\beta+1}}{(p*q)_n} \sum_{k=0}^{n-1} p_k\, q_{n-k}\, \delta_{n-k}\, P_{n-k}^{(\alpha+1,\beta)}(\cos\phi)$$

where

$$\delta_n = \frac{2^{-\alpha-\beta-1}\,\Gamma(n+\alpha+\beta+2)}{\Gamma(n+1)\,\Gamma(n+\beta+1)} \simeq \frac{2^{-\alpha-\beta-1}}{\Gamma(n+1)},\ n^{\alpha+1}$$

Then for $-1/2 \le \alpha < 1/2$, $\beta > -1/2$ and $\{p_m\}, \{q_n\}$ satisfying the conditions of the theorem, we have

$$N_n(\phi) = \begin{cases} O(n^{2\alpha+2}), \text{ when } 0 \le \phi \le 1/n & (4.1) \\ O(n^{2\alpha+\beta+1}), \text{ when } \pi - 1/n \le \phi \le \pi & (4.2) \\ O\left(\dfrac{q_n\, n^{(2\alpha+1)/2}}{(p*q)_n}(\sin\phi/2)^{-(2\alpha+3)/2}(\cos\phi/2)^{-(2\beta+1)/2}\right) & (4.3) \\ \quad + O\left[n^{(2\alpha-1)/2}(\sin\phi/2)^{-(2\alpha+5)/2}(\cos\phi/2)^{-(2\beta+3)/2}\right] & \\ \quad \text{when } 1/n \le \phi \le \pi - 1/n & \end{cases}$$

Proof: Using Lemma 1 for $0 \le \phi \le 1/2$ together with Lemma 4, the required estimate in (4.1) follows. For the estimate in (4.2), we use Lemma 1 for $\pi - 1/n \le \phi < \pi$ together with Lemma 4.

For $1/n \le \phi \le \pi - 1/n$, we have from Lemma 1.

$$N_n(\phi) = \frac{O(1)}{(p*q)_n} \sum_{k=0}^{(n-1)} p_k\, q_{(n-k)}\, (n-k)^{(2\alpha+1)/2}$$

$$\times (\sin\phi/2)^{-(2\alpha+1)/2} (\cos\phi/2)^{-(2\beta+1)/2}$$

$$\times \left(\cos\{(n-k+p)\phi - \gamma\} + \frac{O(1)}{(n-k)\sin\phi} \right)$$

Since for fixed $n, \{q_{n-k}\}$ is non-increasing, we can deal with the first term of the right by first using the second mean value theorem and then applying Lemma 3 to deal with the second term on the right, we apply the result of Lemma 4 and the required estimate follows.

Lemma 6. The condition

$$q_n\, n^{(2\alpha+1)/2} = O(p*q)_n^{1-c}$$

For $0 \le c \le 1$ under the hypothesis of the theorem.

Proof: The expression on the left of (3.3) is increasing and hence greater than or equal to a positive constant. Hence (3.4) implies that for some positive constant A.

$$\sum_{k=2}^{n} \frac{P_k}{k^{(2\alpha+1)/2} \log k} \ge \sum_{k=a}^{n} \frac{P_k}{k^{(2\alpha+1)} \log k}$$

where $a > 2$

(note that $k\,p_k \le p_k \Rightarrow p_k \ge k\,p_k$, by the condition on $\{p_n\}$)

$$= \sum_{k=a}^{n} \frac{P_k\, q_{n-k}}{q_{n-k}\, k^{(2\alpha+3)/2}\, \log k}$$

$$> \frac{(p*q)_n}{q_n\, n^{(2\alpha+1)/2}} \text{ by the condition on } \{q_n\}$$

$$> \frac{(p*q)_n^{1-c}}{q_n\, n^{(2\alpha+1)/2}} \quad \{\therefore (p*q)_n > (p*q)_n^{1-c} \text{ for } 0 \le c \le 1\}$$

$$> A$$

From where the result in Lemma 6 follows.

5. **Proof of the theorem:** Following Obrechkoff ([8, p. 99. and Rao [11] the n[th] partial sum of the series (1.2) at the point $x = 1$ is given by

$$S_n(1) = 2^{\alpha+\beta+1}\, \delta_n \int_0^{\pi} (\sin\phi/2)^{2\alpha-1} (\cos\phi/2)^{2\beta+1} . f(\cos\phi)\, P_n^{(\alpha+1,\beta)}(\cos\phi)\, d\phi$$

Consequently,

$$S_n(1) - A = 2^{\alpha+\beta+1}\ \delta_n \int_0^{\pi} F(\phi)\, P_n^{(\alpha+1,\beta)}(\cos\phi)\, d\phi \tag{5.1}$$

Using (1.7), the (N, P_m, q_n) mean of the series (1.2) is given by

$$t_n^{p,q} - A = \frac{1}{(p * q)_n} \sum_{k=0}^{n} p_k\, q_{n-k}\, \{S_{n-k}(1) - A\}$$

$$= \int_0^{\pi} F(\phi)\, N_n(\phi)\, d\phi$$

$$= I, \text{ say}$$

$$= \left(\int_0^{1/n} + \int_{1/\eta}^{\eta} + \int_{\eta}^{\pi-1/n} + \int_{\pi-1/n}^{\pi} \right) F(\phi)\, N_n(\phi)\, d\phi$$

$$t_n^{p,q} - A = I_1 + I_2 + I_3 + I_4 \tag{5.2}$$

say, where η is a suitable constant such that $0 < \eta < \pi$.

Now in order to prove our theorem, we have to show that

$$I = 0\,(1), \quad \text{as } n \to \infty \tag{5.3}$$

For which we need to prove that

$$I_j = 0\,(1), \quad \text{as } n \to \infty \tag{5.4}$$

for $j = 1, 2, 3, 4, \ldots$

Let us first consider I_1, we have

$$I_1 = \mathrm{O}\left(\int_0^{1/n} |F(\phi)||N_n(\phi)|\, d\phi \right)$$

$$= \mathrm{O}\,(n^{2\alpha+2}) \int_0^{1/n} |F(\phi)|\, d\phi \quad \text{by (4.1) of Lemma 5.}$$

$$= O\,(n^{2\alpha+2})\,.\,0 \left(\frac{\psi(n)\, n^{-2\alpha-2}}{\lambda(P_n)} \right) \text{ by (3.4)}$$

$$= 0 \left(\frac{1}{\log n} \right) \text{ by (3.1)}$$

$$I_1 = \mathrm{o}(1)\,, \quad \text{as } n \to \infty \tag{5.5}$$

Considering I_2, we have

$$I_2 = 0\left(\frac{q_n\, n^{(2\alpha+1)/2}}{(P*q)_n}\right)\int_{1/n}^{\eta}\frac{|F(\phi)|\,P_{[1/\phi]}}{\phi^{(2\alpha+3)/2}}\,d\phi$$

$$+\,0\,(n^{(2\alpha-1)/2})\int_{1/n}^{\eta}\frac{|F(\phi)|}{\phi^{(2\alpha+5)/2}}\,d\phi$$

using (4.3) of Lemma 5.

(5.6) $$I_2 = I_{2.1} + I_{2.2},\ \text{say}$$

Given $\varepsilon > 0$, let η be chosen so that

$$|F_1(\phi)| \le \frac{\varepsilon\,\phi^{(2\alpha+2)}\,\delta_{(1/\phi)}}{\phi(P_{(1/\phi)})},\ 0 \le \phi \le \eta$$

Then

$$|I_{2.1}| \le \frac{Kq_n\,n^{(2\alpha+1)/2}}{(P*q)_n}\int_{1/n}^{\eta}\frac{|F(\phi)|\,P_{[1/\phi]}}{\phi^{(2\alpha+3)/2}}\,d\phi$$

$$\le \frac{Kq_n\,n^{(2\alpha+1)/2}}{(P*q)_n}\left\{\left(\frac{|F_1(\phi)|\,P_{(1/\phi)}}{\phi^{(2\alpha+3)/2}}\right)_{1/n}^{\eta} - \int_{1/n}^{\eta}F_1(\phi)\,d\left(\frac{P_{(1/\phi)}}{\phi^{(2\alpha+3)/2}}\right)\right\}$$

(5.7) $$|I_{2.1}| = I_{2.1.1} + I_{2.1.2}\ \text{(say)}$$

whre K is an absolute constant, not necessarily same at each occurrence. If $K(\eta)$ denotes a constant depending on η, we see that, for fixed η,

$$|I_{2.1.1}| = K(\eta)\,\frac{q_n\,n^{(2\alpha+1)/2}}{(p*q)_n} + \mathrm{O}\left(\frac{q_n\,P_n\,\psi(n)}{(p*q)_n\,\lambda(P_n)}\right)$$

$$= K(\eta)\ \mathrm{o}\left(\frac{1}{(p*q)_n^{c}}\right) + \mathrm{o}(1)$$

(5.8) $$|I_{2.1.1}| = \mathrm{o}\,(1),\ \text{as}\ n \to \infty$$

by using Lemma 6, (3.4), (3.1) and (3.2).

Further

$$I_{2.12} \le \frac{K\,\varepsilon\, q_n\, n^{(2\alpha+1)/2}}{(p*q)_n} \int_{1/n}^{\eta} \frac{\phi^{2\alpha+2}\,\psi_{(1/\phi)}}{\lambda(P_{[1/\phi]})} \left| d\left(\frac{P_{[1/\phi]}}{\phi^{(2\alpha+3)/2}} \right) \right|$$

$$\le \frac{K\,\varepsilon\, q_n\, n^{(2\alpha+1)/2}}{(p*q)_n} \int_{1/\eta}^{n} \frac{x^{-2\alpha-2}}{\log x}\, d\,|\,P_{[x]}\, x^{(2\alpha+3)/2}\,|$$

Using (3.1)

$$= \frac{K\,\varepsilon\, q_n\, n^{(2\alpha+1)/2}}{(p*q)_n} \left(\int_{1/\eta}^{n} \frac{x^{-(2\alpha+1)/2}}{\log x}\, d\,P_{[x]} + \frac{(2\alpha+3)}{2} \int_{1/\eta}^{n} \frac{x^{(-2\alpha+3)/2}}{\log x}\, P_{[x]}\, dx \right)$$

$$= \frac{K\,\varepsilon\, q_n\, n^{(2\alpha+1)/2}}{(p*q)_n} \left(L + \frac{(2\alpha+3)}{2} M \right), \text{ say}$$

Now,

$$L = \sum_{k=2}^{n} \frac{P_k}{k^{(2\alpha+1)/2}\log k}$$

$$L = \mathrm{O}\left(\sum_{k=2}^{n} \frac{P_k}{k^{(2\alpha+3)/2}\log k} \right)$$

and

$$M \le \sum_{k=1}^{n-1} P_k \int_{k}^{k+1} \frac{x^{-(2\alpha+3)/2}}{\log x}\, dx$$

$$= \mathrm{O}\left(\sum_{k=1}^{n-1} \frac{P_k}{k^{(2\alpha+3)/2}\log k} \right)$$

Hence

(5.9) $$|\,I_{2.12}\,| \le \frac{K\,\varepsilon}{(p*q)_n^{c}} = \mathrm{o}(1), \text{ as } n \to \infty$$

by using (3.3) from (5.7), (5.8) and (5.9) it follows that

(5.10) $$I_{2.1} = \mathrm{o}(1), \text{ as } n \to \infty$$

Next, considering $I_{2.2}$, we have

$$|I_{2.2}| \le K\, n^{(2\alpha-1)/2} \int_{1/n}^{\eta} |F(\phi)|\, \phi^{-(2\alpha+5)/2}\, d\phi$$

$$= n^{(2\alpha-1)/2} \left\{ K\,[F_1(\phi)\,\phi^{-(2\alpha+5)/2}]_{1/n}^{\eta} + K \int_{1/n}^{\eta} F_1(\phi)\,\phi^{-(2\alpha+7)/2}\, d\phi \right\}$$

(5.11) $$|I_{2.2}| = I_{2.2.1} + I_{2.2.2}, \text{ say}$$

Hence

$$|I_{2.2.1}| = K(\eta)\, n^{(2\alpha+1)/2} + o(1)$$

(5.12) $$|I_{2.2.1}| = o(1), \text{ as } n \to \infty$$

and

$$|I_{2.2.2}| \le K\,\varepsilon\, n^{(2\alpha-1)/2} \int_{1/n}^{\eta} \frac{\phi^{(2\alpha+2)}\, \psi_{(1/\phi)}\, \phi^{-(2\alpha+7)/2}}{\lambda(P_{[1/\phi]})}\, d\phi$$

$$= K\,\varepsilon\, n^{(2\alpha-1)/2} \int_{1/n}^{\eta} \frac{x^{-(2\alpha+1)/2}}{\log x}\, dx$$

(5.13) $$|I_{2.2.2}| \le K\varepsilon\,; \text{ since } \alpha < 1/2.$$

Hence from (5.11), (5.12) and (5.13), it follows that

(5.14) $$I_{2.2} = o(1), \text{ as } n \to \infty$$

Thus from (5.6), (5.10) and (5.14), we have

(5.15) $$I_2 = o(1), \text{ as } n \to \infty$$

Considering I_3, we have

$$I_3 = O\left(\frac{(p*q)_n\, n^{(2\alpha+1)/2}}{(p*q)_n} \right) \int_{\eta}^{\pi-1/n} |F(\phi)|\,(\sin\phi/2)^{-(2\alpha+3)/2} (\cos\phi/2)^{-(2\beta+1)/2}\, P_{[1/\phi]}\, d\phi$$

$$+ O(n^{(2\alpha+1)/2}) \int_{\eta}^{\pi-1/n} |F(\phi)|\,(\sin\phi/2)^{-(2\alpha+5)/2} (\cos\phi/2)^{-(2\alpha+3)/2}\, d\phi$$

(5.16) $\quad I_3 = I_{3.1} + I_{3.2}$, say

Since $(\sin\phi/2)^{-(2\alpha+3)/2}$ is bounded for $\eta \le \phi \le \pi$ and since $P_{[1/\phi]}$ is bounded and $-\beta - 1/2 > -\beta - \alpha - 1$, , we have

$$I_{3.1} = 0\left(\frac{q_n\, n^{(2\alpha+1)/2}}{(p*q)_n}\right)\int_{\eta}^{\pi-1/n} |F(\phi)|\,(\cos\phi/2)^{-\alpha-\beta-1}\,d\phi$$

$$= 0\left(\frac{q_n\, n^{(2\alpha+1)/2}}{(p*q)_n}\right) \text{ by lemma 2.}$$

$$= 0\left(\frac{1}{(p*q)_n^c}\right) \text{ by lemma 6.}$$

(5.17) $\qquad I_{3.1} = \mathrm{o}(1), \text{ as } n\to\infty.$

Again

(5.18) $\qquad I_{3.2} = \mathrm{O}\,(n^{(2\alpha-1)/2})\left(\int_{\eta}^{\eta'} + \int_{\eta'}^{\pi-1/n}\right)$, say

Given $\varepsilon' > 0$, we can choose η' so that

$$\int_{\eta'}^{\pi} (\cos\phi/2)^{-\alpha-\beta-1}\,|F(\phi)|\,d\phi \le \varepsilon'.$$

The contribution of $I_{3.2}$ of the range $(\eta', \pi-1/n)$ is

$$\le K\,n^{(2\alpha-1)/2}\int_{\eta'}^{\pi-\frac{1}{n}} |F(\phi)|\,(\cos\phi/2)^{-(2\beta+3)/2}\,d\phi$$

$$\le Kn^{(2\alpha-1)/2}\int_{\eta'}^{\pi-1/n} |F(\phi)|(\cos\phi/2)^{-\alpha-\beta-1}\,(\cos\phi/2)^{-(2\alpha-1)/2}\,d\phi$$

(5.19) $\qquad \le K\varepsilon'$;

Since, in the range considered,

$$(\cos\phi/2)^{-(2\alpha-1)} = \mathrm{O}\left(\frac{1}{n^{(2\alpha-1)/2}}\right)$$

Thus the lim sup of the contribution of this range can be made arbitrarily small by suitable choice of ε'. Thus it is enough to prove that for fixed η', the contribution in the range (η,η') is zero. For fixed η',

$$\int_{\eta}^{\eta'} |F(\phi)|\,(\sin\phi/2)^{-(2\alpha+5)/2}\,(\cos\phi/2)^{-(2\beta+3)/2}\,d\phi$$

is a constant, so that the contribution

$$= O[n^{(2\alpha-1)/2}]$$

(5.20) $$= o(1), \text{ as } n \to \infty \text{ for } \alpha < 1/2$$

From (5.18), (5.19) and (5.20), it is obtained that

(5.21) $$I_{3.2} = o(1), \text{ as } n \to \infty.$$

Hence from (5.16), (5.17) and (5.21), it is obtained that

(5.22) $$I_3 = o(1), \text{ as } n \to \infty.$$

Finally, considering I_4, we see that

$$I_4 = o(n^{\alpha+\beta+1}) \int_{\pi-1/n}^{\pi} |F(\phi)|\, dx,$$

by (4.2) of Lemma 5.

But $$n^{\alpha+\beta+1} = o[(\cos\phi/2)^{-\alpha-\beta-1}]$$

uniformly in $\pi - 1/n \le \phi \le \pi$; whence by the use of Lemma 2, it follows immediately that

(5.23) $$I_4 = o(1), \text{ as } n \to \infty.$$

Collecting (5.5), (5.15), (5.22) and (5.23) the required result in (5.4) is established, which, in turn, proves the result in (5.3).

This completes the proof of the theorem.

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The $(E,1)(C,1)$ summability of the conjugate series of a Fourier series

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Abstract: In this paper, a new theorem on $(E, 1)$, $(C, 1)$ summability of the conjugate series of a Fourier series has been proved.

Key words and phrases: $(E,1)$ $(C,1)$ Summability means, Fourier series, Conjugate series of a Fourier series, periodic function, n^{th} partial sum.

1. Definitions and Notations

Let $f(t)$ be 2π periodic function and integrable over $(-\pi,\pi)$ in the sense of Lebesgue. Then its "Fourier series" is given by

$$f(t) = \frac{1}{2}a_o + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} A_n(t) \tag{1.1}$$

The series

$$\sum_{n=1}^{\infty}(a_n \sin nt - b_n \cos nt) = -\sum_{n=1}^{\infty} B_n(t) \tag{1.2}$$

is called the "conjugate series" of the Fourier series (1.1),
where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

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and

$$\left.\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt \end{aligned}\right\} n = 1, 2, 3, \ldots$$

Let $\sum_{n=0}^{\infty} u_n$ be the infinite series whose n^{th} partial sum is given by $S_n = \sum_{i=0}^{\infty} u_i$.

We write

$$\sigma_n = \frac{1}{n+1}\sum_{k=0}^{n} S_k = \text{Cesàro means (C, 1) of sequence } \{S_n\}.$$

If

$$\sigma_n \to S, \text{ as } n \to \infty.$$

where S is a finite number, then sequence $\{S_n\}$ or the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable by Cesåro means method $(C, 1)$ to S. It is denoted by

$$\sigma_n \to S\,(C,1), \text{ as } n \to \infty \text{ (Hardy, 1913).}$$

Next,

$$E_n^1 = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} S_k = \text{Euler means } (E, 1) \text{ of sequences } \{S_n\}.$$

If

$$E_n^1 \to S \quad \text{as } n \to \infty.$$

then sequence $\{S_n\}$ or infinite series $\sum_{n=0}^{\infty} u_i$ is said to be summable by Euler means method $(E, 1)$ to S. It is denoted by

$$E_n^1 \to S\,(E, 1) \quad \text{as } n \to \infty \text{ (Hardy, 1949, p. 180).}$$

The E_n^1 trsnsformation of $\{\sigma_n\}$, denoted by $t_n^{E_1,C_1}$, is defined by

$$E_n^{E_1,C_1} = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\sigma_k.$$

$$= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}\sum_{r=0}^{k} S_r.$$

If $t_n^{E_1,C_1} \to S$, as $n \to \infty$.

then sequence $\{S_n\}$ or infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable by $(E, 1$ $(C, 1)$ means method to S. It is denoted by

$$t_n^{E_1,C_1} \to S\ (E,1)(C,1) \text{ as } n \to \infty.$$

Thus, if $(E, 1)$ transform is superimposed on $(C, 1)$ transform of sequence $\{S_n\}$, a new transformation $(E, 1)(C, 1)$ is obtained.

Here

$$S_n \to S \ \Rightarrow \sigma_n(S_n) \to S, \text{ as } n \to \infty \text{ since } (C, 1) \text{ method is regular.}$$

$$\Rightarrow E_n^1(\sigma_n) \to S, \text{ as } n \to \infty \text{ since } (E, 1) \text{ method is regular.}$$

$$\Rightarrow (E,1)(C,1) \text{ method is regular.}$$

We use following notations

$$\psi(t) = f(x+t) - f(x-t) \tag{1.3}$$

$$\Psi(t) = \int_0^t |\psi(u)|\, du \tag{1.4}$$

$$N_n^{E_1,C_1}(t) = \frac{1}{2^{n+1}\pi} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} \sum_{r=0}^{k} \frac{\cos\left(r+\frac{1}{2}\right)t}{\sin\frac{t}{2}}. \tag{1.5}$$

2. Introduction

Quite a good amount of works are known on Summability of a Fourier series and its allied series. Versaney (1959) has dicussed $(H,1)\,C_1$ summability on sequence of the Fourier coefficient. Naturally, we have to consider other product summability method of the form $(E,1)$ $(C, 1)$.

Recently, Dhakal & Lal (2007) have proved a theorem on $(E,1)$ $(C,1)$ summability of a Fourier series in the following form.

Theorem A: If

$$\Phi(t) = \int_0^t |\phi(u)|\, du = o\left(\frac{t\xi\left(\frac{1}{t}\right)}{\log\frac{1}{t}}\right), \quad \text{as } t \to +0. \tag{2.1}$$

provided $\xi(t)$ is a prositive monotonic decreasing function of t such that $\dfrac{t\xi\left(\frac{1}{t}\right)}{\log\frac{1}{t}}$ increases monotonically as $t \to +0$, then the Fourier series (1,1) is summable by $(E,1)$ $(C,1)$ method to $f(x)$ at $t = x$.

3. Theorem: *The purpose of this paper is to study the conjugate series of the Fourier series by $(E,1)$ $(C,1)$ summability method. In fact, we prove following theorem:*

Theorem: If

$$\Phi(t) = \int_0^t |\phi(u)|\, du = o\left(\frac{t\xi\left(\frac{1}{t}\right)}{\log\frac{1}{t}} \right), \text{ as } t \to +o. \tag{3.1}$$

then the conjugate series of the Fourier series (1.2) is summable by $(E,1)\,(C,1)$ method at $t = s$, to

$$\frac{1}{2\pi} \int_0^{\pi} \psi(t) \cot\tfrac{t}{2}\, dt,$$

provided this integral exist in sense of Lebesgue.

4. Proof of the theorem

Following Lal (1997) and using Roemann-Lebesgue theorem, nth partial sum $S(x)$ of conjugate series (1.2) of Fourier series at $t = x$ is given by

$$\tilde{S}_n(x) = \frac{1}{2\pi} \int_0^{\pi} \psi(t) \frac{\cos\left(n+\frac{1}{2}\right)t - \cos\frac{t}{2}}{\sin\frac{t}{2}}\, dt$$

$$\tilde{S}_n(x) = \frac{1}{2\pi} \int_0^{\pi} \psi(t) \frac{\cos\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\, dt - \frac{1}{2\pi} \int_0^{\pi} \psi(t) \cot\tfrac{t}{2}\, dt$$

$$\tilde{S}_n(x) - \left(-\frac{1}{2\pi} \int_0^{\pi} \psi(t) \cot\tfrac{t}{2}\, dt \right) = \frac{1}{2\pi} \int_0^{\pi} \psi(t) \frac{\cos\left(n+\frac{t}{2}\right)t}{\sin\frac{t}{2}}\, dt$$

$(C,1)$ transform of $\tilde{S}(x)$ i.e. $\sigma(x)$

$$\sigma_n(x) - \left(-\frac{1}{2\pi} \int_0^{\pi} \psi(t) \cot\tfrac{t}{2}\, dt \right) = \frac{1}{2(n+1)\pi} \int_0^{\pi} \psi(t) \sum_{k=0}^{n} \frac{\cos\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\, dt$$

$(E,1)$ transform of $\tilde{\sigma}_n(x)$ i.e., $\tilde{t}_n^{E_1,C_1}$

$$\tilde{t}_n^{E_1,C_1} - \left(-\frac{1}{2\pi} \int_0^{\pi} \psi(t) \cot\tfrac{t}{2}\, dt \right) = \int_0^{\pi} \psi(t) \frac{1}{2^{n+1}\pi} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} \sum_{r=0}^{k} \frac{\cos\left(r+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\, dt$$

$$= \int_0^{\pi} \psi(t)\, \tilde{N}_n^{E_1,C_1}(t)\, dt$$

$$= \left(\int_0^{\frac{1}{n}} \psi(t)\, \tilde{N}_n^{E_1,C_1}(t)\, dt + \int_{\frac{1}{n}}^{\delta} \psi(t)\, \tilde{N}_n^{E_1,C_1}(t)\, dt + \int_{\delta}^{\pi} \psi(t)\, \tilde{N}_n^{E_1,C_1}(t)\, dt \right)$$

$$= (I_1 + I_2 + I_3). \tag{4.1}$$

Since conjugate function exist, therefore

$$\frac{1}{2\pi}\int_0^{\frac{1}{n}}\psi(t)\cot\tfrac{t}{2}\,dt = o(1) \quad \text{as } n\to\infty .$$

(4.2)
$$\frac{1}{2^n\pi}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}\left(\frac{1}{2\pi}\int_0^{\frac{1}{n}}\psi(t)\cot\tfrac{t}{2}\,dt\right) = o\,(1) \quad \text{as } n\to\infty.$$

We have, for $0<t<\dfrac{1}{n}$

$$\begin{aligned}
&\frac{1}{2^{n+1}\pi}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}\sum_{r=0}^{k}\frac{\cos\left(r+\frac{1}{2}\right)t-\cos\frac{t}{2}}{\sin\frac{t}{2}}\\
&= -\frac{1}{2^n\pi}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}\sum_{r=0}^{k}\frac{\sin\,(r+1)\,\frac{t}{2}\sin\frac{rt}{2}}{\sin\frac{t}{2}}\\
&= -\frac{1}{2^n\pi}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}\sum_{r=0}^{k}\sum_{m=1}^{r}\sin mt\\
&\le \frac{1}{2^n\pi}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}\sum_{r=0}^{k}\sum_{m=0}^{r}1\\
&= \frac{1}{2^n\pi}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}\sum_{r=0}^{k}r\\
&= \frac{1}{2^{n+1}\pi}\sum_{k=0}^{n}\binom{n}{k}k\\
&= \frac{n}{4\pi}
\end{aligned}$$

(4.3)
$$= O\,(n)$$

Using (1.5), (3.1), (4.2) and (4.3) we have

$$\begin{aligned}
|I_1| &= \int_0^{\frac{1}{n}}|\psi(t)||\tilde{N}^{E_1,C_1}(t)|\,dt\\
&= \int_0^{\frac{1}{n}}|\psi(t)|\left|\frac{1}{2^{n+1}\pi}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}\sum_{r=0}^{k}\frac{\cos\left(r+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right|dt
\end{aligned}$$

$$= \int_0^{\frac{1}{n}} |\psi(t)| \left| \frac{1}{2^{n+1}\pi} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} \sum_{r=0}^{k} \frac{\cos\left(r+\frac{1}{2}\right)t - \cos\frac{t}{2}}{\sin\frac{t}{2}} \right| dt$$

$$+ \int_0^{\frac{1}{n}} |\psi(t)| \frac{1}{2^{n+1}\pi} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} \sum_{r=0}^{k} \cot\frac{t}{2}\, dt$$

$$= \int_0^{\frac{1}{n}} |\psi(t)|\, O(n)\, dt + \frac{1}{2^n \pi} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} \left(\frac{1}{2\pi} \int_0^{\frac{1}{n}} \psi(t) \cot\frac{t}{2}\, dt \right)$$

$$= O(n)\, \Psi\left(\frac{1}{n}\right) + o(1)$$

$$= O(n)\, o\left(\frac{\xi(n)}{n \log n}\right) + o(1)$$

$$= o\left(\frac{\xi(n)}{\log n}\right) + o(1)$$

(4.4)1.1. $= 0\,(1)$, by the hyperthesis of the Theorem

Also, for $\frac{1}{n} < t < \delta$

$$\frac{1}{2^{n+1}\pi} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} \sum_{r=0}^{k} \frac{\cos\left(r+\frac{1}{2}\right)t}{\sin\frac{t}{2}}$$

$$= \frac{1}{2^{n+2}\pi} \sum_{k=0}^{n} \binom{n}{k} \frac{\sin(r+1)t}{(k+1)\sin^2\frac{t}{2}}$$

$$\leq \frac{1}{2^{n+2}\pi} \sum_{k=0}^{n} \binom{n}{k} \frac{|\sin(r+1)t|}{(k+1)\left|\sin^2\frac{t}{2}\right|}$$

$$= \frac{1}{2^{n+2}\pi} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{(k+1)\left|\sin^2\frac{t}{2}\right|}$$

$$= \frac{\pi}{2^{n+2}t^2} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{(k+1)}$$

$$= \frac{\pi}{2^{n+2}t^2} \left(\frac{2^{n+1}-1}{n+1}\right)$$

$$= \frac{\pi}{2(n+1)t^2}\left(1-\frac{1}{2^{n+1}}\right)$$

$$\leq \frac{\pi}{nt^2}$$

$$= O\left(\frac{1}{nt^2}\right) \tag{4.5}$$

Using (3.1) and (4.5), we have

$$|I_2| = \int_{\frac{1}{n}}^{\delta} |\psi(t)| \left| \tilde{N}_n^{E_1 C_1}(t) \right| dt$$

$$= O\left(\frac{1}{n}\right) \int_{\frac{1}{n}}^{\delta} \frac{|\psi(t)|}{t^2} dt$$

$$= O\left(\frac{1}{n}\right)\left[\left(\frac{\Psi(t)}{t^2}\right)_{\frac{1}{n}}^{\delta} - 2\int_{\frac{1}{n}}^{\delta} \frac{\Psi(t)}{t^3} dt\right]$$

$$= O\left(\frac{1}{n}\right)\left[o\left(\left(\frac{\xi\left(\frac{1}{t}\right)}{t \log \frac{1}{t}}\right)_{\frac{1}{n}}^{\delta}\right) + o\left(\int_{\frac{1}{n}}^{\delta} \frac{t\,\xi\left(\frac{1}{t}\right)}{t^3 \log \frac{1}{t}} dt\right)\right]$$

$$\leq O\left(\frac{1}{n}\right)\left[o\left(\frac{\xi\left(\frac{1}{\delta}\right)}{\delta \log \frac{1}{\delta}}\right) + o\left(\frac{n\xi(n)}{\log n}\right) + o\left(\frac{\xi(n)}{n \log n}\right) \int_{\frac{1}{n}}^{\delta} \frac{1}{t^3} dt\right]$$

by Second Mean value theroem for integral calculas.

$$= o\left(\frac{\xi\left(\frac{1}{\delta}\right)}{n\delta \log \frac{1}{\delta}}\right) + o\left(\frac{\xi(n)}{\log n}\right) + o\left(\frac{\xi(n)}{\delta^2 n^2 \log n}\right)$$

$$= o(1) + o(1) + o(1) \quad \text{as } n \to \infty.$$

$$= o(1), \quad \text{as } n \to \infty. \tag{4.6}$$

Lastly, By the Riemann-Lebesgue theorem and the regularity conditions of (E,1), (C,1) summability, we have

$$|I_3| = \int_{\delta}^{\pi} |\psi(t)| \left| \tilde{N}_n^{E_1 C_1}(t) \right| dt$$

$$= \int_{\delta}^{\pi} |\psi(t)| \frac{1}{2^{n+1}\pi} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} \sum_{r=0}^{k} \frac{\cos\left(r+\frac{1}{2}\right)t}{\sin\frac{t}{2}} dt$$

$$= \frac{1}{2^{n+2}\pi} \int_{\delta}^{\pi} \psi(t) \sum_{k=0}^{n} \binom{n}{k} \frac{\sin(r+1)t}{(k+1)\sin^2 \frac{t}{2}} dt$$

$$= o(1) \text{ as } n \to \infty. \tag{4.7}$$

Collecting (4.1), (4.4), (4.6) and (4.7), we have

$$\bar{t}_n^{E_1, C_1} - \frac{1}{2\pi} \int_0^{\pi} \psi(t) \cot \tfrac{t}{2} \, dt = o(1) \quad \text{as} \quad n \to \infty \tag{4.8}$$

This completes the proof of the theorem

Remark: It is remarkable that our theorem is analogous to theorem Dhakal & Lal (2007) for a Fourier series.

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Associated polynomials to Dirichlet and Fejér kernels

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Abstract: We show that the Fejér kernel generates the fifth-kind Chebyshev polynomials.

Key words: Kernels in Fourier series; Chebyshev polynomials

Introduction

In the original approach to Fourier series, it is convenient to consider the following partial sums for the interval $[-\pi,\pi]$:

$$f_n(y) = \tfrac{1}{2}a_0 + a_1\cos y + \ldots + a_n\cos(ny) + b_1\sin(y) + \ldots + b_n\sin(ny), \tag{1}$$

assuming for a_r, b_r the values:

$$a_n = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(mt)\,dt,\ \ b_r = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(rt)\,dt, \tag{2}$$

and investigate what happens if n increases to infinity. From (1) and (2) we obtain:

$$f_n(y) = \int_{-\pi}^{\pi} f(t)\,\underset{D}{K_n}(t-y)\,dt, \tag{3}$$

with the Dirichlet kernel [1–3]:

$$\underset{D}{K_n}(t-y) = \frac{1}{2\pi}\,\frac{\sin\left[\left(n+\frac{1}{2}\right)(t-y)\right]}{\sin\left(\frac{t-y}{2}\right)} \tag{4}$$

Then we hope that with n increasing to infinity, $f_n(y)$ approaches $f(y)$ with an error which can be made arbitrarily small. This requires a very strong focusing power of $\underset{D}{K_n}(t-y)$, that is, we would like to have the strict property:

$$\lim_{n\to\infty}\underset{D}{K_n}(t-y) = \delta(t-y), \tag{5}$$

however, (4) simulates a Dirac delta only until certain approximation, then the convergence:

$$\lim_{n\to\infty} f_n(y) = f(y) \tag{6}$$

has to be restricted to a definite class of functions $f(y)$ which are conveniently smooth to counteract the insufficient focusing power of $\underset{D}{K_n}(t-y)$; the corresponding restrictions on $f(y)$ are the known Dirichlet conditions [1–3] for infinite convergent Fourier series.

From (4) we see that $\underset{D}{K_n}(\theta)$ is an even function, then here we consider it for $\theta \in [\,0,\pi\,]$:

$$\underset{D}{K_n}(\theta) = \frac{1}{2\pi}\,\frac{\sin\left(n+\frac{1}{2}\right)\theta}{\sin\left(\frac{\theta}{2}\right)}, \tag{7}$$

thus

$$\underset{D}{K_0}(\theta) = \frac{1}{2\pi},\quad \underset{D}{K_1}(\theta) = \frac{1}{2\pi}(1+2\cos\theta),\quad \underset{D}{K_2}(\theta) = \frac{1}{2\pi}(-1+2\cos\theta+4\cos^2\theta), \tag{8}$$

$$\underset{D}{K_3}(\theta) = \frac{1}{2\pi}(-1-4\cos\theta+4\cos^2\theta+8\cos^3\theta), \quad \text{etc.}$$

then it is natural to introduce the polynomials:

$$W_n(x) = W_n(\cos\theta) = 2\pi\,\underset{D}{K_n}(\theta), \quad x\in[-1,1] \tag{9}$$

which were named "fourth-kind Chebyshev polynomials" by Gautschi [4,5]. Therefore, see Fig. 1:

$$W_0(x)=1,\quad W_1(x)=2x+1,\quad W_2(x)=4x^2+2x-1, \tag{10}$$

$$W_3(x)=8x^2+4x^2-4x-1,\quad W_4(x)=16x^4+8x^3-12x^2-4x+1,\quad \text{etc.}$$

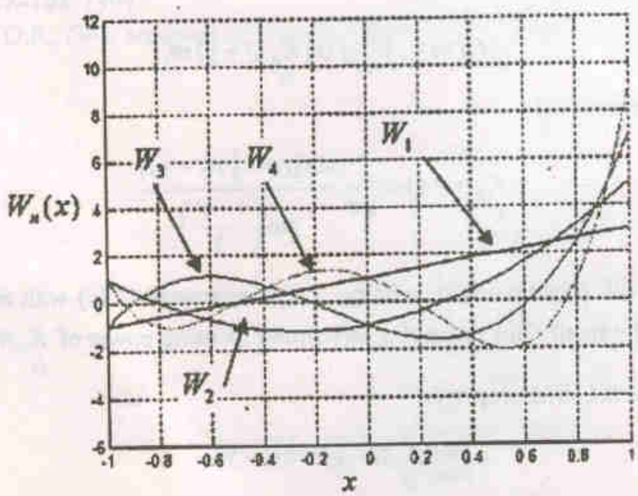

Fig. 1-Some fourth-kind Chebyshev polynomials

In the next Section we exhibit a set of associated polynomials to Fejér kernel [1-3]

Chebyshev- Fejér polynomials

Fejér [5] invented a new method of summing the Fourier series by which he greatly extended the validity of the series. Using the arithmetic means of the partial sums (1), instead of the $f_n(y)$ themselves, he could sum series which were divergent. The only condition the function still has to satisfy is the natural restriction that $f(y)$ shall be absolutely integrable.

Then, in the Fejér approach we construct the sequence:

$$(11)\qquad g_1(y)=f_0(y),\ g_2(y)=\tfrac{1}{2}[(f_0(y)+f_1(y)],\ g_3(y)=\tfrac{1}{3}[(f_0(y)+f_1(y)+f_2(y)],\ldots,$$
$$g_n(y)=\tfrac{1}{n}[(f_0(y)+f_1(y)+\ldots+f_{n-1}(y)],$$

accepting the expressions (1) and (2), therefore,

$$(12)\qquad g_n(y)=\int_{-\pi}^{\pi} f(t)\,\underset{F}{K_n}(t-y)\,dt,$$

thus we see that Fejér results come about by the fact that his method is related with the following kernel [1–3]:

$$(13)\qquad \underset{F}{K_n}(t-y)=\frac{1}{2\pi n}\,\frac{\sin^2[\frac{n}{2}(t-y)]}{\sin^2\frac{t-y}{2}},$$

which possesses a strong focusing power, that is, it satisfies (5), then a $f(y)$ absolutely integrable in $[-\pi,\pi]$ guarantees the convergence of $g_n(y)$ towards $f(y)$.

Now we consider the Fejér kernel:

$$(14)\qquad \underset{F}{K_n}(\theta)=\frac{1}{2\pi n}\,\frac{\sin^2\left(n\frac{\theta}{2}\right)}{\sin^2\frac{\theta}{2}},\quad \theta\in[\,0,\pi\,]$$

that is

$$(15)\qquad \underset{F}{K_0}(\theta)=0,\ \ \underset{F}{K_1}(\theta)=\frac{1}{2\pi},\ \ \underset{F}{K_2}(\theta)=\frac{1}{2\pi}(1+\cos\theta),$$
$$\underset{F}{K_3}(\theta)=\frac{1}{6\pi}(1+4\cos\theta+4\cos^2\theta),\ \text{etc.}$$

then it is natural the introduction of the functions:

$$(16)\qquad \bar{W}_n(x)=\bar{W}_n(\cos\theta)=\frac{2\pi}{n+1}\,\underset{F}{K_{n+1}}(\theta),\ \ x\in[\,-1,1\,]$$

that we name "fifth-kind Chebyshev polynomials", which are not explicitly in the literature.

Therefore:

$$\tilde{W}_0(x)=1,\ \tilde{W}_1(x)=\frac{1}{2}(x+1),\ \tilde{W}_2(x)=\frac{1}{9}(4x^2+4x+1),$$

(17)

$$\tilde{W}_3(x)=\frac{1}{2}(x^3+x^2),\ \tilde{W}_4(x)=\frac{1}{25}(16x^4+16x^3-4x^2-4x+1),\ \text{etc.}$$

Thus $\tilde{W}_n(1)=1$, see the following Figure:

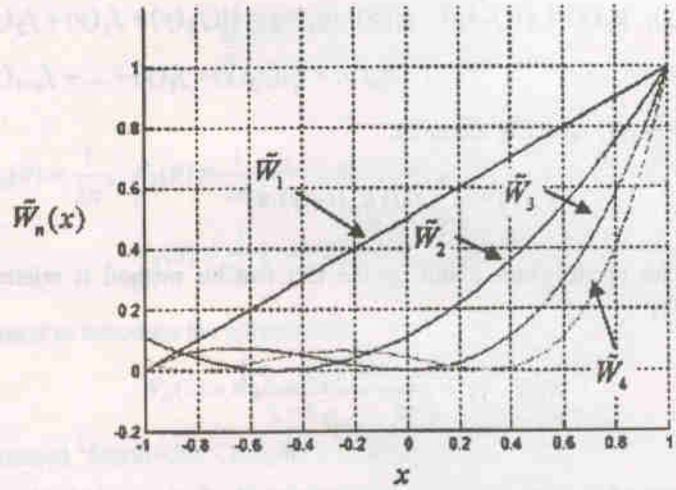

Fig. 2- Some fifth-kind Chebyshev polynomials

which are solutions of the non-homogeneous differential equation:

$$(1-x)[(1-x^2)\tilde{W}_n''-(3x+2)\tilde{W}_n'+(n+1)^2\tilde{W}_n]+x\tilde{W}_n=1. \tag{18}$$

In other paper we will study topics as recurrence, Rodrigues formula, interpolation properties, orthogonality, generating function, etc., for fifth-kind Chebyshev polynomials introduced in this work.

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Transient analysis of *M/M/R* machine repair model with mixed spares

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Abstract: In this paper markov model is considered for the analysis of machine repair problem consisting of M operating machines under the care of two types of repairmen and mixed spares. There is provision of Y cold standbys and S warm standbys to replace the failed units. When all spares are being used, the failure of units occur in degraded mode. To cope up with the increased load of failed units, there is facility of additional repairmen. The purpose of our study is to establish various performance measures in terms of transient probabilities. The expressions for system reliability, availability and mean time to system failure are facilitated in terms of transient probabilities. Computation scheme based on matrix technique is facilitated to obtain the numerical results, which are displayed graphically and in tabular form.

Key Words: Transient analysis, Machine repair, Mixed spares, Queue size, Additional repairmen, Matrix technique.

1. Introduction

In many fast growing industries, the operation of the machining system may be interrupted due to failure of machines involved in the system. The service facility therefore is to be so adjusted such that the failed machines are sent for repair instantly and the operation of the system is continued by using proper combination of spare part support, without much delay. The failed machines wait for repair until the repair of other failed machines is completed. In case of several repairmen, if the machines fail, repairmen repair these failed machines and the excess number of failed machines beyond the number of repairmen wait until at least one repairman is available. This affects the production and results into production loss.

In the present investigation we develop a model for the machine repair problem with mixed spares in which cold spares are first used to replace the failed units and when all cold spares are exhausted, the warm spares are used. Since the repair of failed units plays a central role in any machining system, the provision of repair facility consisting of permanent and additional removable repairmen has been made.

The provision of additional repairmen may be helpful to ensure the desirable reliability with a limited number of spares at reasonable cost of failed units in case of heavy work load.

Sivazlian and Wang (1989) gave economic analysis of the M/M/R machine repair problem with warm standby. Analysis of an M/M/R queue with servers vacations was developed by Kau and Naryana (1991). Wang (1994) provided profit analysis of machine repair problem with a single service station subject to breakdown. Jain and Premlata (1994) considered *M/M/R* machine repair problem with reneging and spares. Hsieh and Wang (1995) discussed reliability of a repairable system with spares and a removable repairman. Wang and Wu (1995) discussed cost analysis of the M/M/R machine repair problem with spares and two modes of failure. M/M/R machine repair problem with spares and additional servers was analyzed by Jain (1998). The cost analysis of the M/M/R machine repair problem with balking, reneging and server breakdown was done by Ke and Wang (1999). Wang et al. (2002) examined profit analysis of M/M/R machine problem with balking, reneging and standby switching failures. Jain et al. (2003) discussed *M/M/R* machine interference model with balking, reneging, spares and two modes of failure.

A repairable system with spares, state dependent rates and additional repairman was explored by Jain and Baghel (2003).The analysis of R out of N systems with several repairmen, exponential life times and phase type repair times using an algorithmic approach was given by Barron et al. (2005), Rafael and Delia (2006) considered a multiple warm standby system with operation and repair times following phase type distributions. Wang et al. (2007) have done profit analysis of the M/M/R machine repair problem with balking, reneging and standby switching failure.

In this paper, we have performed transient analysis of M/M/R machine repair model with mixed spares. The organization of the rest of the paper is as follows. In section 2, we develop model for M/M/R machine repair problem with spare part support and maintained by a pool of repairmen. The transient analysis using matrix method is given in section 3. Some performance indices are calculated in section 4. A sensitivity analysis is facilitated in section 5 to validate the analytical results. We conclude our investigations in final section 6 by highlighting the scope of the work done.

2. Model Descriptions

Consider a machine repair model consisting of $K = M$ (operating)+Y (cold standby)+S (warm standby) units under the care of R permanent and r additional repairmen. The operating and warm standby units fail in Poisson fashion with failure rate λ and α $(0 \le \alpha \le \lambda)$, respectively. Here the failure of units refers to arrival of machines to get repair from the service facility on a FCFS basis. The model is developed by making the following assumptions:

- When a spare moves into an operating state its failure characteristics will be that of operating unit.
- The operating machine as well as spare units fail independently.
- The life time and repair time of operating and warm units are assumed to be exponentially distributed.
- For normal operation of system, M operating units are required but system may work in short mode also with at least m units $(m < M)$.
- Whenever an operating units or a warm spare unit fails it is immediately sent to service facility where it is repaired according to the first come first served (FCFS) discipline.
- The switchover times from standby state to operating state or from repair to standby state are instantaneous.
- Each repairman can repair only one failed unit a time.

- If an operating unit fails, the failed unit at one goes for repair and a spare (first cold, then warm) if available, is put into operation.
- Once an unit is repaired, it is as good as new one. The repair unit goes to standby or operating state depending upon whether some standbys are left or all are exhausted.

The following notations are used to formulate the mathematical model.

λ Failure rate of operating units.
S Number of warm spare units in the system.
Y Number of cold spare units in the system.
α Failure rate of warm standby units.
λ_d Degraded failure rate of operating unit when all spares are being used.
R Number of permanent repairmen in the system.
r Number of additional removable repairmen in the system.
T Threshold increment value of the queue length, to turn on additional repairmen, one by one.
μ Repair rate of permanent repairmen.
μ_j Repair rate of j^{th} additional repairman, $j = 1, 2, \ldots, r$.
$P_n(t)$ Probability that there are n failed units in the system at time t.

3. The Transient Analysis

3.1. The State Transition Rates

The state dependent failure and repair rates of the units are given by

$$\lambda_n = \begin{cases} M\lambda + S\alpha, & 0 \le n < Y \\ M\lambda + (Y+S-n)\alpha, & Y \le n < Y+S \\ (M+Y+S-n)\lambda_d, & Y+S \le n < K = M+Y+S+1. \\ 0, & \text{Otherwise} \end{cases}$$

The mean repair rate is given by

$$\mu_n = \begin{cases} n\mu, & 1 \le n \le R \\ R_\mu & R < n \le T \\ R\mu + \sum_{i=1}^{j} \mu_i, & jT < n \le (j+1)T, \quad j = 1,2,\ldots,r-1 \\ R\mu + \sum_{i=1}^{r} \mu_i, & rT < n \le K \end{cases}$$

3.2. Governing Equations

The differential-difference equations governing the model are given by

Case-I : For $R \le Y$.

(1) $$P_0'(t) = -\left[M\lambda + S\alpha \right] P_0(t) + \mu P_1(t),$$

(2) $$P_n'(t) = -\left[M\lambda + S\alpha + n\mu \right] P_n(t) + \left[M\lambda + S\alpha \right] P_{n-1}(t) + \left[(n+1)\mu \right] P_{n+1}(t), \quad 1 \le n \le R$$

(3) $$P_n'(t) = -\left[M\lambda + S\alpha + n\mu \right] P_n(t) + \left[M\lambda + S\alpha \right] P_{n-1}(t) + R\mu P_{n+1}(t), \quad R < n < Y$$

(4) $$P_n'(t) = -\left[M\lambda + (Y+S-n)\alpha + R\mu \right] P_n(t) + \left[M\lambda + (Y+S-n+1)\alpha \right] P_{n-1}(t) + {} + R\mu\, P_{n+1}(t), \quad Y \le n < Y+S$$

(5) $$P_n'(t) = -\left[(M+Y+S-n)\lambda_d + R\mu \right] P_n(t) + (M+Y+S-n+1)\lambda_d\, P_{n-1}(t) + {} + R\mu\, P_{n+1}(t), \quad Y+S \le < n \le T$$

(6) $$P_n'(t) = -\left[(M+Y+S+n]\,\lambda_n + R\mu + \sum_{i=1}^{j} \mu_i \right] P_n(t) + (M+Y+S-n+1)\lambda\, P_{n-1} + \left[R\mu + \sum_{i=1}^{j} \mu_i \right] P_{n+1}(t), \quad jT < n \le (j+1)T, \quad J = 1, 2 \ldots r-1$$

(7) $$P_n'(t) = -\left[(M+Y+S-m+1-n)\lambda_d + R\mu + \sum_{i=1}^{r} \mu_i \right] P_n(t) + {} + (M+Y+S-m+1-n+1)\,\lambda P_{n-1}(t) + \left[R\mu + \sum_{i=1}^{r} \mu_i \right] P_{n+1}(t), \quad rT < n < K$$

(8) $$P_n'(t) = -\left[R\mu + \sum_{i=1}^{r} \mu_i \right] P_K(t) + \lambda_d\, P_{K-1}(t)$$

Case-II: For $Y < R \le Y+S$.

(9) $$P_0'(t) = -\left[M\lambda + S\alpha \right] P_0(t) + \mu P_1(t),$$

(10) $$P_n'(t) = -\left[M\lambda + S\alpha + n\mu \right] P_n(t) + \left[M\lambda + S\alpha \right] P_{n-1}(t) + \left[(n+1)\mu \right] P_{n+1}(t), \quad 1 \le n \le Y$$

(11) $$P_n'(t) = -\left[M\lambda + (Y+S-n)\alpha + n\mu \right] P_n(t) + \left[M\lambda + (Y+S-n+1)\alpha \right] P_{n-1}(t) + \left[(n+1)\mu \right] P_{n+1}(t), \quad Y < n \le R$$

(12) $$P_n'(t) = -\left[M\lambda + (Y+S-n)\alpha + R\mu \right] P_n(t) + \left[M\lambda + (Y+S-n+1)\alpha \right] P_{n-1}(t) + R\mu\, P_{n+1}(t), \quad R < n \le Y+S$$

$$(13)\qquad P_n'(t) = -\left[\left(M+Y+S-n\right)\lambda_d + R\mu\right]P_n(t) + \left[\left(M+Y+S-n+1\right)\lambda\right]P_{n-1}(t) + R\mu\, P_{n+1}(t), \qquad Y+S<N\le T$$

$$(14)\qquad P_n'(t) = -\left[\left(M+Y+S-n\right)\lambda_d + R\mu + \sum_{i=1}^{j}\mu_i\right]P_n(t) + \left[(M+Y+S-n+1)\lambda\right]P_{n-1}(t) + \left[R\mu + \sum_{i=1}^{r}\mu_i\right]P_{n+1}(t), \qquad jT<n\le(j+1)T, \quad J=1,2\ldots r-1$$

$$(15)\qquad P_n'(t) = -\left[(M+Y-S+n)\lambda_d + R\mu + \sum_{i=1}^{r}\mu_i\right]P_n(t) + \left[(M+Y+S-n+1)\lambda\right]P_{n-1}(t) + \left[R\mu + \sum_{i=1}^{r}\mu_i\right]P_{n+1}(t), \qquad rT<n<M+Y+S-m+1$$

$$(16)\qquad P_K'(t) = -\left[R\mu + \sum_{i=1}^{r}\mu_i\right]P_K(t) + \lambda_d\, P_{K-1}(t)$$

Case-III: For $Y+S<T$

$$(17)\qquad P_0'(t) = -\left[M\lambda + S\alpha\right]P_0(t) + \mu P_1(t),$$

$$(18)\qquad P_n'(t) = -\left[M\lambda + S\alpha + n\mu\right]P_n(t) + \left[M\lambda + S\alpha\right]P_{n-1}(t) + \left[(n+1)\mu\right]P_{n+1}(t), \qquad 1\le n\le Y$$

$$(19)\qquad P_n'(t) = -\left[M\lambda + (Y+S-n)\alpha + n\mu\right]P_n(t) + \left[M\lambda + (Y+S-n+1)\alpha\right]P_{n-1}(t) + \left[(n+1)\mu\right]P_{n+1}(t), \qquad Y<n\le Y+S$$

$$(20)\qquad P_n'(t) = -\left[(M+Y+S-n)\lambda_d + n\mu\right]P_n(t) + \left[(M+Y+S-n+1)\lambda_d\right]P_{n-1}(t) + \left[(n+1)\mu\right]P_{n+1}(t), \qquad Y+S<n\le R$$

$$(21)\qquad P_n'(t) = -\left[(M+Y+S-n)\lambda_d + R\mu\right]P_n(t) + \left[(M+Y+S-n+1)\lambda\right]P_{n-1}(t) + R\mu\, P_{n+1}(t), \qquad R<n\le T$$

$$(22)\qquad P_n'(t) = -\left[(M+Y+S-n)\lambda_d + R\mu + \sum_{i=1}^{j}\mu_i\right]P_n(t) + \left[(M+Y+S-n+1)\lambda\right]P_{n-1}(t) + \left[R\mu + \sum_{i=1}^{j}\mu_i\right]P_{n+1}(t), \qquad jT<n\le(j+1)T, \quad J=1,2\ldots r-1$$

$$(23)\qquad P_n'(t) = -\left[(M+Y+S-n)\lambda_d + R\mu + \sum_{i=1}^{r}\mu_i\right]P_n(t) + \left[(M+Y+S-n+1)\lambda_d\right]P_{n-1}(t) +$$

$$+\left[R\mu + \sum_{i=1}^{r}\mu_i\right]P_{n+1}(t), \qquad rT < n < M+Y+S-m+1$$

$$(24)\qquad P_K'(t) = -\left[R\mu + \sum_{i=1}^{r}\mu_i\right]P_K(t) + \lambda_d\, P_{K-1}(t)$$

3.3. Matrix Method

After taking Laplace transformation of above set of equations in each case, we find matrix equation in terms of Laplace transform of probabilities as

$$A(s)\,\overline{P}(s) = P(0)$$

where

$$A(s) = \begin{bmatrix} -(s+\lambda_0)\ \lambda_0 \\ \mu_1\ -(s+\lambda_1+\mu_1)\ \lambda_1 \\ \quad \mu_2\ -(s+\lambda_2+\mu_2)\ \lambda_2 \\ \quad\quad \cdots\cdots \\ \quad\quad\quad \cdots\cdots \\ \quad\quad\quad\quad \cdots\cdots \\ \quad\quad\quad\quad \mu_R\ -(s+\lambda_R+\mu_R)\ \lambda_R \\ \quad\quad\quad\quad\quad \cdots\cdots \\ \quad\quad\quad\quad\quad \cdots\cdots \\ \quad\quad\quad\quad\quad\quad \mu_{jT}\ -(s+\lambda_{jT}+\mu_{jT})\ \lambda_{jT} \\ \quad\quad\quad\quad\quad\quad\quad \cdots\cdots \\ \quad\quad\quad\quad\quad\quad\quad \cdots\cdots \\ \quad\quad\quad\quad\quad\quad\quad\quad \mu-(s+\lambda_{rT}+\mu_{rT})\ \lambda_{rT} \\ \quad\quad\quad\quad\quad\quad\quad\quad\quad \cdots\cdots \\ \quad\quad\quad\quad\quad\quad\quad\quad\quad \cdots\cdots \\ \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \mu_K\ -(s+\mu_K) \end{bmatrix} \qquad (25)$$

$$\overline{P}(s) = \left[\ \overline{P}_0(s), \ldots, \overline{P}_Y(s), \ldots, \overline{P}_{Y+s}(s), \ldots, \overline{P}_R(s), \ldots, \overline{P}_{jT}(s), \ldots, \overline{P}_{rT_r}(s), \ldots, \overline{P}_K(s)\right]^T$$

or

$$A(s)\,P(s) = I, \text{ where } I = [1,0,0,\ldots,0]^T$$

Here A(s) is the coefficient matrix of order $(K+1)\times(K+1)$.

Using Cramer's rule $\overline{P}(s)$ can be determined as follows:

$$\overline{P}_n(s) = \frac{|A_{n+1}(s)|}{|A(s)|}, \; n = 0, 1, 2, \ldots, K$$

where $|A(s)|$ is the determinant of coefficient matrix $A(s)$ and $|A_{n+1}(s)|$ ($n = 0, 1, \ldots, K$) is the determinant of the matrix obtained by replacing $(n+1)^{\text{th}}$ column of matrix $A(s)$ with initial vector $I = [1,0,0,\ldots,0]^T$.

It is noted $s = 0$ is a roof of $|A(s)| = 0$

If we take $s = (-\phi)$, then we get

$A(-\phi) = A - \phi I$, where I is the identity matrix and $A = A(0)$ is $(K+1) \times (K+1)$ matrix.

Purring the value of $A(-\phi)$ in equation (25), we get

$$A(-\phi)\,\overline{P}(s) = (A - \phi I)\,\overline{P}(s) = P(0)$$

The distinct eigen values $\phi_n (\phi_n \neq 0$ where $n = 0, 1, 2, \ldots, K)$ of the matrix $A = \phi I$, can be obtained by equating its determinant equals to zero. Now we assume that the other K real eigen values including 0, are denoted by $(\phi_1, \phi_2, \ldots, \phi_K)$ then $|A(s)|$ can be written as

$$|A(s)| = s \sum_{j=1}^{K} (S + \phi_j)$$

and

$$\overline{P}_n(s) = \frac{|A_{n+1}(s)|}{s \sum_{j=1}^{K} (S + \phi_j)} j$$

(26) $$= \frac{a_{0,n}}{s} + \sum_{j=1}^{K} \frac{a_{j,n}}{s + \phi_j}, \quad n = 0, 1, 2, \ldots, K$$

where $$a_{0,n} = \frac{A_{n+1}(0)}{\left[\prod_{j=1}^{K} \phi_j\right]}$$

$$a_{j,n} = \frac{|A_{n+1}(-\phi_j)|}{(-\phi_j)\left[\prod_{\substack{j=1 \\ i \neq j}}^{K} (\phi_i - \phi_j)\right]}; \; j = 1, 2, \ldots, K$$

where $a_{0,n}$ and $a_{j,n}$ $(j = 1, 2, \ldots, K)$ are all real numbers.

The inverse Laplace transform of equation (26) is given by

$$P_n(t) = a_{0,n} + \sum_{j=1}^{K} a_{j,n}\, e^{-\phi_j t},\ n = 0, 1, 2, \ldots, n \tag{27}$$

4. Some Performance Indices

In this section, we provide some measures of performance in terms of probabilities, which can be determined using matrix method discussed in previous section.

➢ Expected number of spare units in the system at time t.

$$E\{S(t)\} = S\sum_{n=1}^{Y} P_n(t) + \sum_{n=Y+1}^{Y+S} (Y+S-n)\,P_n(t)$$

➢ Expected number of operating units in the system at time t.

$$E\{O(t)\} = M - \sum_{n=Y+S+1}^{K} (n-\overline{Y+S})\,P_n(t)$$

➢ Expected number of idle repairmen servers in the system at time t.

$$E\{I(t)\} = \sum_{n=0}^{R-1} (R-n)\,P_n(t)$$

➢ Expected number of busy permanent servers in the system at time t.

$$E\{B(t)\} = R - E\{I(t)\}$$

➢ Machine availability i.e. rate of production per machine at time t is

$$A(t) = 1 - \frac{E\{O(t)\}}{M+Y+S}$$

➢ Expected number of busy additional repairmen in the system at time t.

$$E\{BR(t)\} = \sum_{j=1}^{r-1} \sum_{n=jT+1}^{(j+1)T} jP_n(t) + r\sum_{n=rT+1}^{K} P_n(t)$$

➢ Expected number of failed units in the system at time t

$$E\{N(t)\} = \sum_{n=1}^{K} nP_n(t)$$

5. Sensitivity Analysis

In this section, we obtain numerical results by taking an illustration for default parameters $M=10$, $Y=5$, $S=3$, $T=8$, $R=2$, $r=2$, $\lambda=1$, $\lambda_d=1.8$, $\mu=7$. $\mu_1=1$, $\mu_2=1$. For computation purpose, we develop program in MATLAB software. The numerical results are summarized in tables 1–2. The graphical representation of numerical results has also been done in figures 1–4. In table 1, we present numerical results for the expected number of spare units $E(S)$ and expected number of permanent idle repairmen $E(\mathrm{I})$ by varying the failure rates of operating units (λ) and warm standby (α). We note that for a particular value of t, both $E(S)$ and $E(\mathrm{I})$ decrease as λ increases. Table 2 summarizes results for $E(\mathrm{O})$ and $E(B)$. We notice that for fixed value of t $E(\mathrm{O})$ decreases but $E(B)$ increases by increasing failure spare (α) and arrival rate (λ). As expected $E(S)$, $E(I)$ decrease but $E(B)$ increases as time grows.

Figure 1-4 display the expected average number of failed unit $E(N)$ vs time t. In figure 1, we illustrate the effect of failure rate λ of the operating units on the average queue length $E(N)$. We see from graph that the queue length increases initially sharply then after becomes almost constant when t increases for different value of λ. As expected, $E(N)$ increases as failure rate λ increases.Figure 2 depicts the effect of degraded failure rate λ_d on $E(N)$; the queue length first increases sharply then gradually becomes constant, on increasing time t. We also notice that $E(N)$) increases on increasing λ_d.

The effect of repair rate μ of permanent repairmen on the average number of failed units $E(N)$ is displayed in figure 3. We observed that the average number of failed units decreases as μ increases. The increasing trend of $E(N)$ with respect to time t is also seen. It is clear from fig. 4 that $E(N)$ increases on increasing the failure rate of warm standbys α.

Finally, we conclude that the expected number of spares, idle permanent repairmen and operating units decrease while that of busy servers increase on increasing either the failure rate of operating units o the failure rate of warm standby units. The expected number of spares, idle permanent repairmen, operating units, and busy servers decrease on increasing time t for different values of λ and α. The average number of failed units increases with the increase in failure rate of operating units, degraded failure rate, failure rate of warm standby units and time while it decrease on increasing the service rate and with the increase in time t. This is as per our expectation.

6. Conclusion

In this paper, we have studied the transient analysis of $M/M/R$ machine repair problem with mixed standby. To avoid heavy workland of failed machines, the repair crew consists of additional removable repairmen and permanent repairmen. The threshold policy developed may be advantageous for large complex systems wherein only spare part support is not sufficient to achieve desired efficiency and reliability/availability. The cost analysis may be helpful in determining the optimal combination of cold/warm standbys and the number of additional repairmen. The provision of mixed standbys in a machining system has additional advantages from the economic point of view.

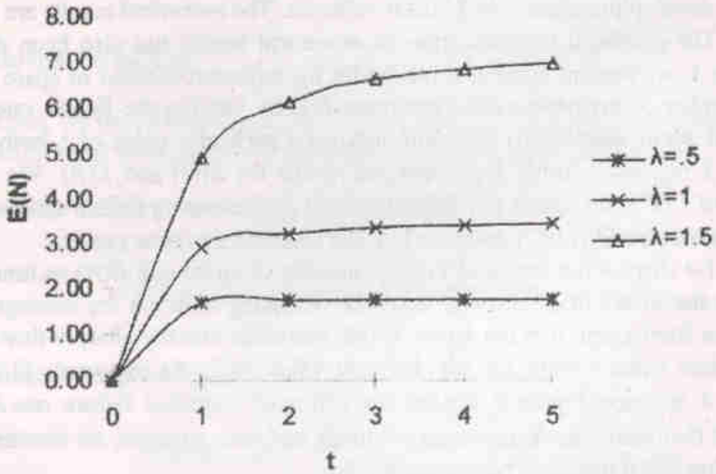

Fig. 1: Average number of failed units $E(N)$ by varying time t for different values of λ.

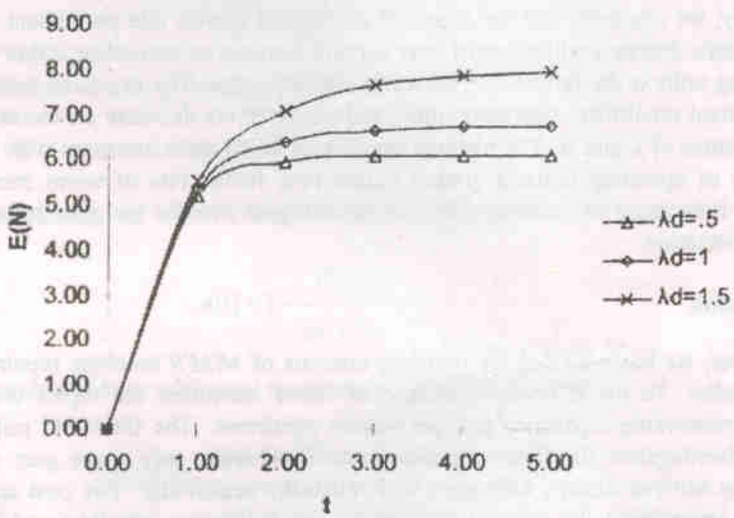

Fig. 2: Average number of failed units $E(N)$ by varying time t for different values of λ_d.

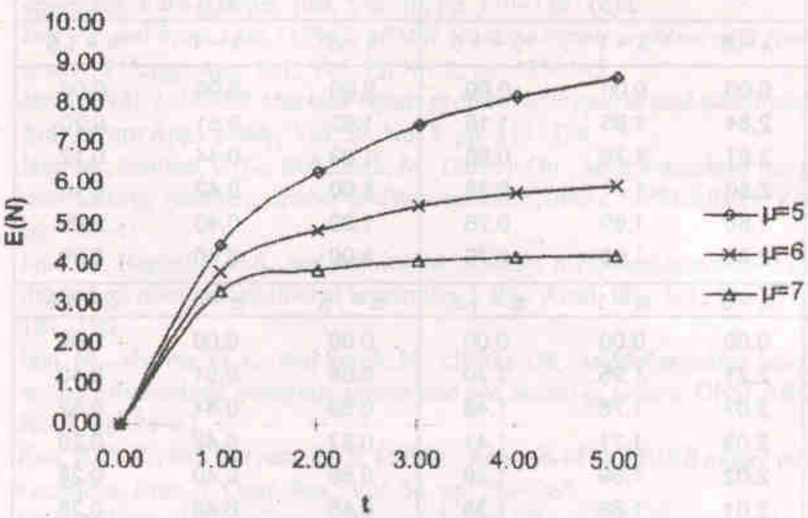

Fig. 3 : Average number of failed units E(N) vs t for different values of λ.

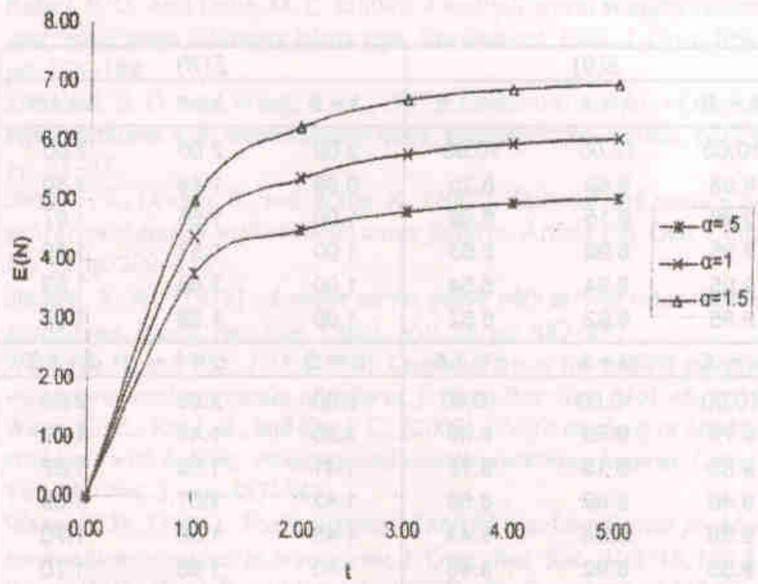

Fig. 4 : Average number of failed units E(N) vs t for different values of α.

t	$E(S)$			$E(I)$		
	$\lambda = .5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = .5$	$\lambda = 1$	$\lambda = 1.5$
0	0.00	0.00	0.00	0.00	0.00	0.00
1	2.84	1.95	1.16	1.02	0.51	0.20
2	2.81	1.76	0.86	1.00	0.44	0.12
3	2.80	1.71	0.78	1.00	0.42	0.10
4	2.80	1.69	0.76	1.00	0.40	0.09
5	2.80	1.68	0.76	1.00	0.40	0.09
t	$\alpha = .5$	$\alpha = 1$	$\alpha = 1.5$	$\alpha = .5$	$\alpha = 1$	$\alpha = 1.5$
0	0.00	0.00	0.00	0.00	0.00	0.00
1	2.21	1.95	1.70	0.65	0.51	0.40
2	2.07	1.76	1.48	0.59	0.44	0.32
3	2.03	1.71	1.41	0.57	0.42	0.29
4	2.02	1.69	1.39	0.56	0.40	0.28
5	2.01	1.68	1.38	0.55	0.40	0.28

Table 1: Expected number of spares and expected number of idle permanent repairman in the system for different values of λ and α.

t	$E(0)$			$E(B)$		
	$\lambda = .5$	$\lambda = 1$	$\lambda = 1.5$	$\lambda = .5$	$\lambda = 1$	$\lambda = 1.5$
0	10.00	10.00	10.00	2.00	2.00	2.00
1	9.98	9.59	8.25	0.98	1.49	1.80
2	9.96	9.15	6.98	1.00	1.55	1.87
3	9.95	8.99	6.63	1.00	1.57	1.89
4	9.95	8.94	6.54	1.00	1.58	1.89
50	9.95	8.92	6.52	1.00	1.58	1.89
t	$\alpha = .5$	$\alpha = 1$	$\alpha = 1.5$	$\alpha = .5$	$\alpha = 1$	$\alpha = 1.5$
0	10.00	10.00	10.00	2.00	2.00	2.00
1	9.78	9.59	9.35	1.35	1.49	1.60
2	9.50	9.15	8.72	1.41	1.55	1.67
3	9.40	8.99	8.50	1.42	1.57	1.69
4	9.38	8.94	8.43	1.43	1.58	1.70
5	9.35	8.92	8.40	1.43	1.58	1.70

Table 2: Expected numbers of operating units and expected number of busy servers in the system for different values λ and α.

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Köthe–Toeplitz duals

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Abstract: We define sequence space $(\overline{\ell(p)})_t$. We also establish Köthe-Toeplitz Duals of $(\overline{\ell(p)})_t$.

1. Introduction.

The following definitions and notations will be useful in our discussion: ℓ_p = the space of sequences $x = (x_k)$ with absolutely p-summable series $(1 \le p < \infty)$. If $p = (p_k)$ is a bounded sequence of strictly positive real numbers, then

$$\ell(p) = \left\{ x = (x_k) : \sum_k |x_k|^{p_k} < \infty \right\}$$

Let $t = (t_k)$ be any fixed sequence of non-zero complex numbers satisfying $\liminf (t_k)^{1/k} = \gamma\,(0 < \gamma \le \infty)$ and let

$$\overline{\ell(p)} = \left\{ x = (x_k) : \sum_{k=1}^{\infty} |t_k(x)|^{p_k} < \infty \right\}$$

where $t_k(x) = \sum_{i=1}^{k} x_i$, then we define

$$\left(\overline{\ell(p)}\right)_t = \left\{ x = (x_k) : (t_k x_k) \in \overline{\ell(p)} \right\}$$

2. Topological properties of $(\overline{\ell(p)})_t$.

Theorem 2.1. *$(\overline{\ell(p)})_t$ is a complete paranormed space.*

Proof : As $(0) \in (\overline{\ell(p)})_t$, $(\overline{\ell(p)})_t \neq 0$, it is easy to verify that it is a linear space. And also it is clear that the function defined by $g^*(x) = g(tx)$ where g is the paranorm in $\overline{\ell(p)}$ satisfying that $g^*(0) = 0$, $g^*(x) = g^*(-x)$ and $g^*(x+y) \leq g^*(x) + g^*(y)$. Clearly, $\lambda_n \to \lambda$ in C and $g^*(x^n - x) \to 0$ as $n \to \infty$ imply that $g^*(\lambda_n x^n - \lambda x) \to 0$ as $(n \to \infty)$ where $x^n = (x_k^n) = \left(x_1^n, x_2^n, x_3^n, \ldots, x_k^n, \ldots\right)$ and $x = (x_k)$, hence g^* is a paranorm in $(\overline{\ell(p)})_t$. To show that $(\overline{\ell(p)})_t$ is complete, let (x^n) be a Cauchy sequence in $(\overline{\ell(p)})_t$ where

$$x^n = \left(x_1^n, x_2^n, x_3^n, \ldots, x_k^n, \ldots \right) \in (\overline{\ell(p)})_t$$

Then $(tx^n) = \left((t_k x_k^1), (t_k x_k^2) \ldots \right)$ is a Cauchy sequence in $\overline{\ell(p)}$. As $\overline{\ell(p)}$ is complete, so it converges to (z_k) (say). Let $z_k = t_k x_k$ so that $x_k = t_k^{-1} z_k$. Then (tx^n) converges to $(t_k x_k) \in \overline{\ell(p)}$. Hence, $g\left(t_k x_k^n - t_k x_k\right) = g\left(t(x^n - x)\right) \to 0 \ (n \to \infty)$ implies that $g^*(x^n - x) \to 0$ as $n \to \infty$. Therefore x^n is convergent, and $(\overline{\ell(p)})_t$ is a complete paranormed space.

Corollary 2.1. *$\overline{\ell}_p$ is a Banach space for $(1 \leq p < \infty)$, normed by*

$$\| x \| = \left(\sum_{k=1}^{\infty} | t_k(x) |^p \right)^{\frac{1}{p}}$$

Here the norm in $\left(\overline{\ell(p)}\right)_t$ is defined by $s \| x \|_t = \| (t_k x_k) \|$. So it is also a Banach space.

Corollary 2.2. *$\overline{\ell}_2$ is a Hilbert space with inner product $\langle x, y \rangle = \sum_{k=1}^{\infty} t_k(x) \overline{t_k(y)}$*

Here the inner product in $(\overline{\ell}_2)_t$ is defined by $\langle x, y \rangle_t = \langle (t_k x_k), (t_k y_k) \rangle$ Hence it is a Hilbert space.

Theorem 2.2. *If z be a closed subset of $\overline{\ell(p)}$, then z_t is a closed subset of $(\overline{\ell(p)})_t$,*

Proof. Since $z \subset (\overline{\ell(p)})$, $z_t \subset (\overline{\ell(p)})_t$ (obvious). Now let $x \in (\overline{z_t})$ where $\overline{z_t}$ stands for the closure of z_t, then there exists a sequence $(x^n) \subset z_t$ such that (x^n) converges to x. This implies that $g^*(x^n - x) = g^*\left\{(t_k x_k^n) - (t_k x_k)\right\} \to 0 (n \to \infty)$ in z_t. Thus,

$$g^*\left\{(t_k x_k^n) - (t_k x_k)\right\} \to 0 \ (n \to \infty) \text{ in } z.$$

Hence $(t_k x_k)$ is the limit of a sequence of points in z and $(t_k x_k) \in (\bar{z})$ which yields $x \in (\bar{z})_t$. Conversely, if $x \in (\bar{z})_t$ then $x \in (\bar{z_t})$, since z is closed, that is $\bar{z} = z$. Therefore $(\bar{z}_k) = (\bar{z})_t = z_t$, hence z_t is closed in $(\overline{\ell(p)})_t$.

3. Köthe-Toeplitz Duals of $(\ell(p))_t$.

Definition: *Let X be a sequence space. We define:*

(i) $$X^\alpha = \left\{ a = (a_k) : \sum_k |a_k x_k| < \infty \text{ for all } x \in X \right\}$$

(ii) $$X^\beta = \left\{ a = (a_k) : \sum_k |a_k x_k| \text{ converges for all } x \in X \right\}$$

(iii) $$X^\gamma = \left\{ a = (a_k) : \sup \left| \sum_{k=1}^{n} a_k x_k \right| < \infty \text{ for all } x \in X \right\}$$

Then $X^\alpha, X^\beta, X^\gamma$ are called the α^-, β^- and γ^- dual spaces of X respectively. In [3] the author has established the β dual of $\overline{\ell(p)}$. Here we establish the α^-, β^- and γ^- duals of $(\overline{\ell(p)})_t$.

Theorem 3.1. *Let $\lambda = \alpha, \beta, \gamma$. Then*

(i) $$\left(\left(\overline{\ell(p)}\right)_t\right)^\lambda = \left\{ a = (a_k) : \left(\frac{a_k}{t_k}\right) \in \ell(p) \right\} = \left(\left(\overline{\ell(p)}\right)^\lambda\right)_t$$

(ii) $$\left(\left(\overline{\ell(p)}\right)_t\right)^{\lambda\lambda} = \left\{ x = (x_k) : (t_k x_k) \in \left(\left(\overline{\ell(p)}\right)^{\lambda\lambda}\right)_t \right\}$$

Proof: (i) Let $\lambda = \beta$ and $a \in \alpha$ and $D = \left\{ a = (a_k) : \left(\frac{a_k}{t_k}\right) \in \left(\overline{\ell(p)}\right)^\alpha \right\}$

We show that $\left(\left(\overline{\ell(p)}\right)_t\right)^\alpha = D$

Let $a \in \left(\left(\overline{\ell(p)}\right)_t\right)^\alpha$ then $\sum_{k=1}^{\infty} |a_k x_k| < \infty$ for every $x \in \left(\overline{\ell(p)}\right)_t$, so that

$$\sum_{k=1}^{\infty} \left| \frac{a_k}{t_k} t_k x_k \right| = \sum_{k=1}^{\infty} |a_k x_k| < \infty$$

Since $(t_k x_k) \in \overline{\ell(p)}$ it follows that $\left(\frac{a_k}{t_k}\right) \in \left(\overline{\ell(p)}\right)^\alpha$ implies that $a \in D$.

Hence, $\left(\left(\overline{\ell(p)}\right)_t\right)^\alpha \subset D$. Conversely, if $a \in D$ and $x \in \left(\overline{\ell(p)}\right)_t$, then, $\left(\frac{a_k}{t_k}\right) \in \left(\overline{\ell(p)}\right)^\alpha$

and $(t_k x_k) \in \overline{\ell(p)}$ so that. $\sum_{k=1}^{\infty} |a_k x_k| = \sum_{k=1}^{\infty} \left| \frac{a_k}{t_k} t_k x_k \right| < \infty$..

Since, $x \in \left(\overline{\ell(p)}\right)_t$, it follows that $a \in \left(\overline{\ell(p)}\right)^\alpha$. Hence $D \subset \left(\left(\overline{\ell(p)}\right)_t\right)^\alpha$

Thus, $\left(\left(\overline{\ell(p)}\right)_t\right)^\alpha = \left(\left(\overline{\ell(p)}\right)^\alpha\right)_t$.

Similar results hold for $\lambda = \beta$ or γ as well.

(ii) Let $\lambda = \alpha$ and let $\left(\overline{\ell(p)}\right)^{\alpha\alpha}$ exist. Then

$$\left(\left(\overline{\ell(p)}\right)_t\right)^{\alpha\alpha} = \left[\left(\left(\overline{\ell(p)}\right)_t\right)^\alpha\right]^\alpha$$

$$\left[\left(\left(\overline{\ell(p)}\right)^\alpha\right)_t\right]^\alpha = \left(\left(\overline{\ell(p)}\right)^{\alpha\alpha}\right)_t$$

For $\lambda = \beta$ or γ, the proof is same.

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Remarks on fixed point theorem under implicit relations

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Abstract: In this paper, a common fixed point theorem satisfying an implicit relation, is established by removing the reciprocal continuity, relaxing the compatibility partly and replacing the completeness of the space with a set of four alternative natural conditions. Some related results and illustrative examples are also discussed.

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1. Introduction

Sessa [15] initiated the tradition of improving commutativity conditions in fixed point theorems by introducing the notion of 'weakly commuting mappings' which asserts that a pair of self-mappings (S, I) of a metric space (X, d) is said to be ***weakly commuting*** if, $d(SIx, ISx) \le d(Ix, Sx)$ for all x in X. It is noted that every commuting pair is weakly commuting but not conversely as shown in Sessa [15]. Jungck [6] also enlarged this class of weakly commuting mappings by defining 'compatible mappings' which asserts that a pair of self-mappings (S,I) is said to be ***compatible*** if, $\lim_{n\to\infty} d(SIx_n, ISx_n) = 0$ whenever $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ix_n = t \in X$.

Recently, Jungck and Rhoades [7] (also Dhage [2]) termed a pair of self-mappings to be coincidentally commuting (or weakly compatible) if they merely commute at their coincidence points. One may note that this definition never needs to involve metric of the underlying set. The following one-way implication is obviously true but not conversely.

Commuting maps $\Rightarrow$ Weakly commuting maps $\Rightarrow$ Compatible Maps $\Rightarrow$ Coincidentally commuting maps.

Very recently Popa [11, 12, 13, 14] proved interesting fixed point theorems satisfying suitable implicit relations. For proving such results, Popa considers Φ to be the set of all continuous functions $F : (R^+)^6 \to R$ satisfying the following conditions:

F_1: F is non-increasing in t_5 and t_6.

F_2: there exists $h \in (0, 1)$ such that for $u,v \geq 0$ with

$F_{2(a)}$: $F(u, v, v, u, u+v, 0) \leq 0$ or

$F_{2(b)}$: $F(u, v, u, v, 0, u+v) \leq 0$

implies that $u \leq hv$.

The following examples of such functions F satisfying F_1 and F_2 appear in Popa [11, 12].

Example 1.1.[12] $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a_1(t_3^2 + t_4^2)/(t_3 + t_4) - a_2 t_2 - a_3(t_5 + t_6)$ where $a_i \geq 0$ $(i = 1, 2, 3)$ with at least one a_i non zero and $a_1 + a_2 + 2a_3 < 1$.

Example 1.2.[12] $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - (c\, t_3 t_4 + b\, t_5 t_6)/(t_3 + t_4) - a\, t_2$ where $a, b, c \geq 0$ and $1 < c + 2a < 2$.

Example 1.3.[12] $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^3 + t_2^2 + t_1 - a(t_3^3 + t_4^3 + t_2 t_5 t_6)/(t_3^2 + t_4^2)$ where $a \in (0, 1)$.

Here, we give some natural examples of implicit condition & functions satisfying the conditions F_1 and F_2, which further strengthen the significance of employing implicit functions as it improves a class of contractive conditions.

Example 1.4. $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - (a_1 t_3 t_4)/(t_3 + t_4) - a_2 t_2 - a_3(t_5 + t_6)$ with a_i positive and with at least one a_i $(i = 1, 2, 3)$ is not zero satisfying $a_1 + 2a_2 + 4a_3 < 2$.

Then,

F_1: Obvious

F_2: For $v > 0$ and $F(u, v, v, u, u+v, 0) = u - (a_1 u v)/(u+v) - a_2 v - a_3(u+v) \leq 0$, then $u(u+v) - a_1 uv - a_2 v(u+v) - a_3(u+v)^2 \leq 0$.

If we set $f(t) = (1 - a_3)t^2 + (1 - a_1 - a_2 - 2a_3)t - a_2 - a_3$, where $t = u/v$. Then, since $f(0) = -a_2 - a_3 < 0$ and $f(1) = 2 - a_1 - 2a_3 - 4a_3 > 0$, so there exists a positive root 'h' of the equation $f(t) = 0$ with $h \in (0, 1)$. Then $f(t) \leq 0$ for $0 < t \leq h$. Thus, we have $u \leq hv$ which establishes $F_{2(a)}$. In case $u = 0$, we have $u \leq hv$. Similarly, one can also establishes $F_{2(b)}$.

Example 1.5. $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a_1(t_5 + t_6)t_2/(t_3 + t_4) - a_2(t_2 + t_4)$ with a_i positive and with at least one a_i $(i = 1, 2, 3)$ is non zero satisfying $a_1 + 2a_2 < 1$.

Then,

F_1: Obvious

F_2: For $v > 0$ and $F(u, v, v, u, u+v, 0) = u - a_1 v - a_2(u+v) \leq 0$, then $(1 - a_2)u - (a_1 + a_2)v \leq 0$.

If we set $f(t) = (1 - a_2)t - (a_1 + a_2)$, where $t = u/v$. Then, since $f(0) = a_1 - a_2 < 0$ and $f(1) = 1 - a_1 - 2a_2 > 0$, so as earlier, it can be shown that $u \leq hv$ with $h \in (0, 1)$ which establishes $F_{2(a)}$. Also, if $u = 0$, then $u \leq h\, v$. Similarly, one can also establishes $F_{2(b)}$.

The following fixed point theorems are proved in [12, 13]

Theorem 1.1. [5] *Let S, T, I, J be self mappings of a complete metric space (X,d) satisfying $S(X) \subset J(X)$ and $T(X) \subset I(X)$ and for each x, y in X, either*

$$d(Sx, Ty) \le \alpha\,[\{d(Sx, Ix)\}^2 + \{d(Ty, Jy)\}^2]/[\,d(Sx, Ix) + d(Ty, Jy)] + \beta\, d(Ix, Jy)$$

if, $d(Sx, Ix) + d(Ty, Jy) \neq 0$, $\alpha, \beta > 0$ and $\alpha + \beta < 1$ or

$d(Sx, Ty) = 0$ if, $d(Sx, Ix) + d(Ty, Jy) = 0$.

If either (*a*) (S, I) are compatible, S or I is continuous and (T, J) are weakly compatible or (*b*) (T, J) are compatible, T or J is continuous and (S, I) are weakly compatible, then S, T, I and J have unique fixed point.

Recently, in an attempt to improve Theorem 1.1, Popa proved the following fixed point theorem via implicit relations.

Theorem 1.2.[13] *Let (S, I) and (T, J) be a weakly compatible pair of self-mappings of a complete metric space (X, d) satisfying $S(X) \subset J(X)$ and $T(X) \subset I(X)$ and for each x, y in X,*

$$F(d(Sx, Ty), d(Ix, Jy), d(Ix, Sx), d(Jy, Ty), d(Ix, Ty), d(Jy, Sx)) \le 0,$$

with $d(Sx, Ix) + d(Ty, Jy) \neq 0$, where $F \in \Phi$, or

$$d(Sx, Ty) = 0 \text{ if, } d(Sx, Ix) + d(Ty, Jy) = 0.$$

Then, If (S, I) or (T, J) is a compatible pairs of reciprocally continuous mappings, then S, T, I and J have unique fixed point.

The main purpose of this paper is to improve Theorem 1.2 besides discussing related results and illustrative examples to demonstrate the utility of the results as remarks. Pant [10] introduced the concept of reciprocal continuity and it is important to note that continuity implies reciprocal continuity but not conversely. So, in this theorem, we relax the reciprocal continuity and compatibility conditions of the maps completely, weaken the completeness condition of the space to four alternative natural conditions and also deduce some important corollaries.

2. Main Results

Theorem 2.1. *Let (S, I) and (T, J) be a weakly compatible pair of self-mappings of a metric space (X, d) such that*

(i) *$S(X) \subset J(X)$ and $T(X) \subset I(X)$ and*

(ii) *$F(d(Sx, Ty), d(Ix, Jy), d(Ix, Sx), d(Jy, Ty), d(Ix, Ty), d(Jy, Sx)) \le 0$,*
for each x, y in X, with $d(Ix, Sx) + d(Jy, Ty) \neq 0$, where $F \in \Phi$, or
$d(Sx, Ty) = 0$ if, $d(Ix, Sx) + d(Jy, Ty) = 0$.

If one of $S(X)$, $T(X)$, $I(X)$ and $J(X)$ is a complete subspace of X, then S, T, I and J have unique fixed point.

Proof. *Let x_0 be an arbitrary point in X, then since* (i) *holds, so we can inductively define sequences $\{x_n\}$ and $\{y_n\}$ by*

$$(1) \qquad y_{2n} = Sx_{2n} = Jx_{2n+1}\,;\, y_{2n+1} = Tx_{2n+1} = Ix_{2n+2} \text{ for } n = 0, 1, 2, \ldots$$

If $d(Ix_{2n}, Sx_{2n}) + d(Jx_{2n+1}, Tx_{2n+1}) \neq 0$ for $n = 0, 1, 2, \ldots$, then using inequality (ii), we have successfully

$$F(d(Sx_{2n}, Tx_{2n+1}), d(Ix_{2n}, Jx_{2n+1}), d(Ix_{2n}, Sx_{2n}), d(Jx_{2n+1}, Tx_{2n+1}), d(Ix_{2n}, Tx_{2n+1}),$$
$$d(Jx_{2n+1}, Sx_{2n})) \le 0.$$

That is, $F(d(Sx_{2n}, Tx_{2n+1}), d(Tx_{2n+1}, Sx_{2n}), d(Tx_{2n-1}, Sx_{2n}), d(Sx_{2n}, Tx_{2n+1}),$

$$d(Tx_{2n+1}, Sx_{2n+1}) + d(Sx_{2n+1}, Tx_{2n+1}), 0) \leq 0.$$

By (F_a), we have $d(Sx_{2n}, Tx_{2n+1}) \leq h\, d(Sx_{2n}, Tx_{2n+1})$.

Similarly, if $d(Ix_{2n}, Sx_{2n}) + d(Jx_{2n+1}, Tx_{2n+1}) \neq 0$, for $n = 0, 1, 2, \ldots$, then by (F_b), we have $d(Sx_{2n}, Tx_{2n-1}) \leq hd(Sx_{2n-2}, Tx_{2n-1})$ and so $d(Sx_{2n}, Tx_{2n+1}) \leq h^{2n} d(Sx_0, Tx_1)$.

By a routine calculation, it follows that $\{y_n\}$ is a Cauchy sequence.

Now, suppose $J(X)$ is a complete subspace of X, then the subsequence $Jx_{2n+1} = Sx_{2n}$ is contained in $J(X)$ and hence there exists a limit u. Let $v \in J^{-1}u$, then $Jv = u$. Also, the subsequence $Ix_{2n+2} = Tx_{2n+1}$ converges to u. We prove that $Tv = u$.

Suppose on the contrary that $d(u, Tv) > 0$. Then setting $x = x_{2n}$ and $y = v$ in (ii), we get

$$F(d(Sx_{2n}, Tv), d(Ix_{2n}, Jv), d(Ix_{2n}, Sx_{2n}), d(Jv, Tv), d(Ix_{2n}, Tv), d(Jv, Sx_{2n})) \leq 0,$$

which, on letting $n \to \infty$, reduces to $F(d(u, Tv), 0, 0, d(u, Tv), d(u, Tv), 0) \leq 0$. This implies that $d(u, Tv) \leq 0$. Thus, we have $u = Tv$. Hence, J and T have a point of coincidence. Since $T(X) \subset I(X)$, $Tv = u$ implies that $u \in I(X)$. Let $w \in I^{-1}u$, then $Iw = u$. Now, using the same argument, we can prove that $Sw = u$. Thus, S and I have a point of coincidence. If one assumes that $I(X)$ is complete, then analogous arguments establish the earlier conclusions.

If $S(X)$ is complete then by (i), we have $u \in S(X) \subset J(X)$. Similarly, if $T(X)$ is complete, then $u \in T(X) \subset I(X)$. Since the pairs (S, I) and (T, J) are weakly compatible and so coincidentally commuting at w and u and therefore, we have

$$u = Tv = Jv = Sw = Iw\,;\; Su = SIw = ISw = Iu \text{ and } Tu = TJv = JTv = Ju.$$

If $Tu \neq u$, then $d(Tu, u) > 0$ and hence,

$F(d(Sw, Tu), d(Iw, Ju), d(Iw, Sw), d(Ju, Tu), d(Iw, Tu), d(Ju, Sw)) = F(d(u, Tu), d(u, Tu), 0, 0, d(u, Tu), d(Tu, u)) > 0$, which is a contradiction (ii) and hence $d(u, Tu) = 0$, that is $u = Tu$. Similarly, we can prove that $Su = u$. Therefore, u is a common fixed point of S, T, I and J. The uniqueness of the common fixed point is obvious due to implicit condition (ii).

This completes the prove of Theorem 2.1.

As an application of this Theorem 2.1, we have the following common fixed point theorem for four families of mappings.

Theorem 2.2. Let $\{S_1, S_2, \ldots, S_m\}$, $\{T_1, T_2, \ldots, T_n\}$, $\{I_1, I_2, \ldots, I_p\}$, and $\{J_1, J_2, \ldots, J_q\}$ be four *families of self-mappings of a metric space* (X, d) *with* $S = S_1S_2 \ldots S_m$, $T = T_1T_2 \ldots T_n$, $I = I_1I_2 \ldots I_p$, *and* $J = J_1J_2 \ldots J_q$ *satisfying the following conditions*:

(iii) $S(X) \subset J(X)$ *and* $T(X) \subset I(X)$ *and*

(iv) $F(d(Sx, Ty), d(Ix, Jy), d(Ix, Sx), d(Jy, Ty), d(Ix, Ty), d(Jy, Sx)) \leq 0$,

for each x, y *in* X.

If one of $S(X)$, $T(X)$, $I(X)$ and $J(X)$ is a complete space of X, then (S, I) or (T, J) have a point of coincidence.

Moreover, if $S_i S_j = S_j S_i$; $I_k I_l = I_l I_k$; $T_r T_s = T_s T_r$; $J_t J_u = J_u J_t$; $S_i I_k = I_k S_i$;

$$I_k T_r = T_r I_k\,;\; T_r J_t = J_t T_r\,;\; S_i J_t = J_t S_i\,;\; S_i T_r = T_r S_i \text{ and } J_t I_k = I_k J_t,$$

for all $i, j \in I_1 = \{1, 2, \ldots, m\}$, $k, l \in I_2 = \{1, 2, \ldots, p\}$, $r, s \in I_3 = \{1, 2, \ldots, n\}$, and

$t, u \in I_4 = \{1, 2, \ldots, q\}$, then, for all $i \in I_1$, $k \in I_2$, $r \in I_3$ and $t \in I_4$, the mappings S_i, I_k, T_r and J_t have a common fixed point.

Proof: Since the mappings S, T, J and I satisfy all the required relevant conditions of Theorem 2.1, so the pair (S, I) or (T, J) have a point of coincidence. Also, appealing to component wise commutativity of various pairs, we can prove that $SI = IS$ and $TJ = JT$ and hence obviously both the pairs (S, I) and (T, J) are coincidentally commuting. Moreover, all the conditions of Theorem 2.1 (for mappings S, T, I and J) are satisfied ensuring the existence of unique common fixed point z.

Now, we need to show that z remains the fixed point of all component mappings. For this, we consider

$$\begin{aligned} S(S_i z) &= ((S_1 S_2 \ldots S_m) S_i) z \\ &= (S_1 S_2 \ldots S_{m-1})((S_m S_i) z) = (S_1 S_2 \ldots S_{m-1})(S_i S_m z) \\ &= (S_1 S_2 \ldots S_{m-2})(S_{m-1} S_i (S_m z)) = (S_1 S_2 \ldots S_{m-2})(S_i S_{m-1}(S_m z)) \\ &= (S_1 S_2 \ldots S_{m-3})(S_{m-2} S_i (S_m z)) = \ldots = (S_i S_1 S_2 \ldots S_m) z \\ &= S_i (Sz) = S_i z. \end{aligned}$$

Similarly, we can prove

$$S(I_k z) = I_k(Sz) = I_k z;$$

$$I(I_k z) = I_k(Iz) = I_k z; \quad S(T_r z) = T_r(Sz) = T_r z; \quad S(J_t z) = J_t(Sz) = J_t z;$$

$$I(T_r z) = T_r(Iz) = T_r z; \quad I(J_t z) = J_t(Iz) = J_t z; \quad T(S_i z) = S_i(Tz) = S_i z;$$

$$T(T_r z) = T_r(Tz) = T_r z; \quad T(I_k z) = I_k(Tz) = I_k z; \quad T(J_t z) = J_t(Tz) = J_t z;$$

$$J(S_i z) = S_i(Jz) = S_i z; \quad J(T_r z) = T_r(Jz) = T_r z; \quad J(I_k z) = I_k(Jz) = I_k z;$$

$$J(J_t z) = J_t(Jz) = J_t z; \text{ and } I(S_i z) = S_i(Iz) = S_i z;$$

This shows that (for all i, r, k and t), $S_i z$, $T_r z$, $I_k z$ and $J_t z$ are other fixed points of S, I, T and J. Now, appealing to the uniqueness of common fixed points of S, T, I, and J, we have, for all i, r, k and t,

$$z = S_i z = T_r z = I_k z = J_t z,$$

which shows that z is a common fixed point of S_i, T_r, I_k and J_t for all i, r, k and t.

This completes the proof of theorem 2.2.

We now have the following corollaries related to above theorems.

Corollary 2.1. *By choosing S, T, I and J suitably and modifying the remaining hypotheses accordingly, the derived conclusions of Theorem* 2.1 *remain true if for all x, y in X and $F \in \Phi$, the implicit condition* (ii) *is replaced by any one of the following conditions*:

(A) $F(d(Sx, Sy), d(Ix, Jy), d(Jx, Sx), d(Jy, Sy), d(Ix, Sy), d(Jy, Sx)) \leq 0$,
(derived by setting $S = T$)

(B) $F(d(Sx, Ty), d(Ix, Iy), d(Ix, Sx), d(Iy, Ty), d(Ix, Ty), d(Iy, Sx)) \leq 0$,
(derived by setting $I = J$)

(C) $F(d(Sx, Sy), d(Ix, Iy), d(Ix, Sx), d(Iy, Sy), d(Ix, Sy), d(Iy, Sx)) \leq 0$,
(derived by setting $S = T$ and $I = J$)

(D) $F(d(Sx, Sy), d(x, y), d(x, Sx), d(y, Sy), d(x, Sy), d(y, Sx)) \leq 0$,

(derived by setting $S = T$ and $I = J =$ an identity map)

(E) $F(d(Sx, Ty), d(x, y), d(x, Sx), d(y, Ty), d(x, Ty), d(y, Sx)) \leq 0$,

(derived by setting $I = J =$ an identity map)

Also, by setting $m = n = p = q$ and for all i, r, k and t, consider $S_i = T_r = I_k = J_t = F$ (say), then we have the following corollary as a variant of Bryant's Theorem [1].

Corollary 2.2.[1] *Let F be a self-mapping of a metric space (X, d) such that there exists some $n \in N$ satisfying*

$$F(d(F^n x, F^n y), d(x, y), d(x, F^n x), d(y, F^n y), d(x, F^n y), d(y, F^n x)) \leq 0,$$

for all x, y in X and $F \in \Phi$. If $F^n(X)$ is a complete subspace of X, then F has a unique fixed point.

[1] Examples

We now give the following examples to illustrate the above theorems.

Example 3.1. Consider $X = [0,6]$ with the usual metric. Define self-mappings S, T, I and J on X as

$$S\,0 = 0, \quad Sx = 1, \text{ for } 0 < x \leq 6,$$
$$T\,0 = 0, \quad Tx = 3, \text{ for } 0 < x \leq 6,$$
$$I\,0 = 0, \quad Ix = 5, \text{ for } 0 < x < 6, \quad I\,6 = 1,$$

and
$$J\,0 = 0, \quad Jx = 6, \text{ for } 0 < x < 6, \quad J\,6 = 1.$$

Then all four maps S, T, I and J are discontinuous, even at their unique common fixed point at $x = 0$. Also, the pairs (S, I) and (T, J) commute at $x = 0$ which is their common point of coincidence. Clearly, $S(X) = \{0, 1\} \subset \{0, 1, 6\} = J(X)$ and $T(X) = \{0, 3\} \subset I(X) = \{0, 3, 5\}$. If we define a continuous function $F : (R^+)^6 \to R$ by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - k \max\{t_2, t_3, t_4, (t_5 + t_6)/2\},$$ where $k \in (0,1)$ and F satisfies

F_1, F_2. Also, F satisfies the implicit contractive condition (ii) for $k = \frac{1}{10}$.

Moreover, the pairs (S, I) and (T, J) are weakly commuting [15] and hence compatible [12] because,

$$|SI6 - IS6| = |1 - 5| > 0 = |I6 - S6| \text{ whereas}$$
$$|TJ6 - JT6| = |3 - 6| > |1 - 3| = |J6 - T6|.$$

Example 3.2. *Let $X = \{0, 1, ½, 1/2^2, 1/2^3, \ldots, \ldots\}$ be a metric space with the usual metric $d(x, y) = |x - y|$ for all x, y in X. For $n = 0, 1, 2, 3, \ldots$, define mappings $S, I : X \to X$ by*

$$S(0) = 1/2^2, \qquad S(1/2^n) = 1/2^{n+2},$$
$$I(0) = ½, \qquad I(1/2^n) = 1/2^{n+1} \text{ respectively.}$$

Also, we set $S = T$ and $I = J$. Then, clearly $S(X) = \{1/2^2, 1/2^3, \ldots\} \subset \{1/2, 1/2^2, 1/2^3, \ldots\} = I(X)$. Define a continuous function $F : (R^+)^6 \to R$ by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - a t_2^2 - b(t_3^2 + t_4^2 + 1),$$ with $a = ½$ and $b = ¼$, then F satisfies

F_1 and F_2. Furthermore,

$$F(d(S0, S1), d(I0, I1), d(I0, S0), d(I1, S1), d(I0, S1), d(I1, S0))$$
$$= F(0, 0, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}) = -1/72 < 0.$$

Similarly, we can show that

$$F(d(S0, S1/2), d(I0, I1/2), d(I0, S0), d(I1/2, S1/2), d(I0, S1/2), d(I1/2, S0)) < 0,$$
$$F(d(S0,S1/4), d(I0,I1/4), d(I0,S0), d(I1/4,S1/4), d(I0,S1/4), d(I1/4,S0))\ 0, \text{ and so on.}$$

Also, for $x = 1/2^n$ and $y = 1/2^m$, for $n, m = 0,1,2, \ldots$ and $n \neq m$, we have

$$F(d(S1/2^n, S1/2^m), d(I1/2^n, I1/2^m), d(I1/2^n, S1/2^n), d(I1/2^m, S1/2^m),$$
$$d(I1/2^n, S1/2^m), d(I1/2^m, S1/2^n)) \leq 0.$$

Hence, all the conditions of Theorem 2.1 are satisfied except the completeness of the subspaces $S(X)$ and $T(X)$. Note that the mappings S and I have no point of coincidence. and even they are not continuous at the origin. This example shows that the completeness of the space is not sufficient for the existence of coincidence point, as the space X is complete with the usual metric.

Remarks: Theorem 2.2 is the application of Theorem 2.1. Also, the main result of Jeong and Rhoades [5] is a particular case of Theorem due to Imdad and Khan [3], which is established for six mappings. Since a variant of fixed point theorems corresponding to implicit conditions (D) and (E) appear in Popa, so our results extends the results of Popa[12, 13], improves the results of S. Kumar [8] & Popa [14], and also generalizes the result of Bryant [1] with respect to corollary 3.2.

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On *APST* Riemannian manifold with second order generalised structure

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In this present paper, we have studied some properties of a differentiable manifold and also studied the Almost para sasakian type (APST)-Riemannian manifold.

Introduction :

Let an n-dimensional Riemannian manifold M_n, on which there are defined a tensor field F of type (1,1) a tensor field T, a 1-form A and metric tensor g satisfying for arbitrary vector field X, Y, Z and a is any complex number (non-zero).

(1.1) $$F^2X = a^2X - A(X)T$$

(1.2) $$\overline{X} = F(X)$$

(1.3) $$A(T) = -a^2$$

(1.4) $$A(FX) = 0$$

(1.5) $$F(T) = 0$$

(1.6) $$g(T,X) = A(X)$$

(1.7) $$g(FX,FY) = -a^2\, g(X,Y) + A(X)\,A(Y)$$

then structure (F, T, A, g) is called almost para contact metric structure and manifold M_n will be called Almost para contact metric Riemannian manifold.

Let us call such a structure a generalized almost contact metric structure

Let us define

(1.8) $$'F(\bar{X},Y)=g(FX,Y)$$

And barring X in (1.8) we have

(1.9) $$'F(\bar{X},Y)=g(F^2X,Y)$$

Which by virtue of (1.1) yields

(1.10) $$'F(X,Y)=a^2g(X,Y)-A(X)A(Y)$$

Now barring Y in (1.8) we have

(1.11) $$'F(X,\bar{Y})=g(FX,FY)$$

This with the help of (1.7) becomes

(1.12) $$'F(X,\bar{Y})=-\{a^2g(X,Y)-A(X)A(Y)\}$$

Thus from the relation (1.10) and (1.12) we have

(1.13) $$'F(X,\bar{Y})='F(\bar{X},Y)$$

Replacing X by T in equation (1.18) and making use of (1.5), we obtain-

(1.14) $$'F(T,Y)=0$$

Barring X in equation (1.12) and making use of (1.1) and (1.14) we get

(1.15) $$'F(\bar{X},\bar{Y})=-a^2F(X,Y)$$

Now barring Y in (1.7) and making use of (1.4) and (1.5) in the resulting equation, we obtain

(1.16) $$g(FX,Y)=-g(X,F,Y)$$

Thus from the equation (1.8) and (1.16), we have

(1.17) $$'F(FX,Y)=-'F(Y,X)$$

2, Nijenhuis Tensor

Nijenhuis Tensor is given by

(2.1) $$N(X,Y)=[\bar{X},\bar{Y}]+\overline{[X,Y]}-\overline{[\bar{X},Y]}-\overline{[X,\bar{Y}]}$$

Making use of (1.1) in (2.1), we get-

(2.2) $$N(X,Y)=[\bar{X},\bar{Y}]+a^2[X,Y]-A([X,Y])T-\overline{[\bar{X},Y]}-\overline{[X,\bar{Y}]}$$

Now let us put

(2.3) $$P(X,Y)=[\bar{X},\bar{Y}]-\overline{[X,\bar{Y}]}$$

(2.4) $$Q(X,Y)=[\bar{X},\bar{Y}]-\overline{[\bar{X},Y]}$$

(2.5) $$H(X,Y)=[\bar{X},\bar{Y}]+a^2[X,Y]$$

Theorem (2.1) *The Nijenhuis tensor and P(X, Y) are related as-*

$$a^2 P(X,Y) - \overline{P(X,\bar{Y})} = a^2 N(X,Y) - A(Y) - \overline{[\bar{X},T]} + a^2 A(Y)[X,T] + A(Y)A([X,T])T \tag{2.6}$$

Proof : Barring Y in (2.3) and using (1.1), we obtain-

$$P(X,\bar{Y}) = a^2 [\bar{X},Y] + A(Y)[\bar{X},T] - a^2 \overline{X,Y]} - \overline{A(Y)[X,T]} \tag{2.7}$$

Again barring the above equation and making use of (1.1)

$$\overline{P(X,\bar{Y})} = a^2\overline{[\bar{X},Y]} + A(Y)\overline{[\bar{X},T]} - a^2\{a^2[X,Y] - A[X,Y]T\} - \{a^2 A(Y)[X,T] - A(Y)A([X,T])T\}$$

$$P[\bar{X},\bar{Y}] = a^2\overline{[\bar{X},Y]} + A(Y)\overline{[\bar{X},T]} - a^4[X,Y] + a^2[X,T]T - a^2 A(Y)[X,T] + A(Y)A([X,T])T \tag{2.8}$$

Now from the equation (2.3) and (2,8), we obtain

$$a^2 P(X,Y) - \overline{P(X,\bar{Y})} = a^2[\bar{X},\bar{Y}] - a^2\overline{[\bar{X},Y]} + a^4[X,Y] - A(Y)\overline{[\bar{X},T]} - a^2 A([X,Y])T + a^2 A(Y)[X,T] - A(Y)A([X,T])T \tag{2.9}$$

Making use of (2.2) in (2.9) we get the result.

Corollary (2.1): *In a differentiable manifold* M^n.

We have

$$a^2 P[X,T] = a^2 N(X,T) + a^2\overline{[\bar{X},T]} - a^4[X,T] - a^2 A([X,T])T \tag{2.10}$$

Proof: Putting T for Y in (2.6) and using (1.5)and (1.3), we get the result.

Theorem (2.2): *In a differentiable manifold* M^n,

Weh have

$$a^2 Q(X,Y) - \overline{Q(\bar{X},Y)} = a^2 N(X,Y) - A(X)\overline{[T,\bar{Y}]} + a^4 A(X)[T,Y] + A(X)A([T,Y])T \tag{2.11}$$

Proof: Barring X in (2.4) and making use of (1.1), we get

$$Q(\bar{X},Y) = a^2[X,\bar{Y}] + A(X)[T,\bar{Y}] - a^2\overline{[\bar{X},Y]} + A(X)\overline{[T,Y]} \tag{2.12}$$

Now barring the whole equation (2.12) and making use of (1.1), we get-

$$\overline{Q(\bar{X},Y)} = a^2\overline{[X,\bar{Y}]} + A(X)\overline{[T,\bar{Y}]} - a^2\{a^2[X,Y] - A([X,Y])T\} + A(X)\{a^2(T,Y] - A([T,Y)T\}$$

$$\overline{Q(\bar{X},Y)} = a^2\overline{[X,\bar{Y}]} + A(X)\overline{[T,\bar{Y}]} - a^4[X,Y] + a^2 A([X,Y])T + a^2 A(X)[T,Y] - A(X)A([T,Y)T \tag{2.13}$$

Now from (2.4) and (2.13) we get

$$(2.14)\qquad a^2Q(X,Y)-Q\overline{[\bar{X},Y]}=a^2[\bar{X},\bar{Y}]-a^2\overline{[X,\bar{Y}]}-a^2\overline{[\bar{X},Y]}+a^4[X,Y]$$
$$-a^2A([X,Y])T-A(X)\overline{[T,\bar{Y}]}-a^4A(X)[T,Y]-A(X)\,A([T,Y])T.$$

Thus from (2.2) and (2.14) we obtain the required result.

Corollary (2.2): *In a generalized almost contact metric manifolds M^n we have*

$$(2.15)\qquad a^2Q(T,Y)=a^2N(T,Y)+a^4\overline{[T,\bar{Y}]}-a^4[T,Y]=-a^2A([T,Y])T$$

Proof: Replacing X by T in (2.11) and using (1.3) and (1.5), we get the equation (2.15)

Theorem (2.3). *In a generalized almost contact metric structure manifold M^n*

$$(2.16)\qquad a^2H(X,Y)-H\overline{[\bar{X},Y]}=a^2N[X,Y]-a^2A([X,Y])T-A(X)\overline{[T,\bar{Y}]}$$

Proof: Barring X in (2.5) and making use of (1.1)

$$(2.17)\qquad H(\bar{X},Y)=a^2[X,\bar{Y}]+A(X)\,[T,\bar{Y}]-a^2(\bar{X},Y)$$

Now barring the whole equation (2.17) and making use of (1.1)

$$(2.18)\qquad H\overline{[\bar{X},Y]}=a^2\overline{[X,\bar{Y}]}-A(X)\overline{[T,\bar{Y}]}+a^2\overline{[\bar{X},Y]}$$

Thus with the help of (2.2), (2.5) and (2.18) we get (2.16)

Corollary (2.3): *The equation* (2.16) *is equivalent to*

$$(2.19)\qquad a^2H(T,Y)=a^2N[T,Y]-a^2A([T,Y])+a^2\overline{[T,\bar{Y}]}$$

Proof: Replacing X by T in (2.16) and using the equation (1.3) and (1.5), we get the result.

Theorem (2.4): *In a generalized almost contact metric structure manifold M^n, we have*

$$(2.20)\qquad H(T,Y)-Q[T,Y]=a^2[T,Y]$$

Proof: Equation (2.20) follows directly wih the help of equation (2.15) and (2.19)

Theorem (2.5): *In a generalized almost contact metric manifold M^n, we have*

$$(2.21)\qquad a^2H(X,Y)-H\overline{[\bar{X},Y]}=\{a^2P[X,Y]-P\overline{[X,\bar{Y}]}+A(Y)\overline{[\bar{X},T]}-a^2A(Y)[X,T]$$
$$-A(Y)\,A[(X,T)T\}-a^2T([X,Y])T-A(X))\overline{[T,Y]}$$

Proof: Proof follows with the help of equation (2.6) and (2.16)

Theorem (2.6): *In order that a generalized almost contact metric manifold by completely integrable it is necessary that*

$$A[\bar{X},\bar{Y}]T = 0 \tag{2.22}$$

Proof: Barring X in (2.2) and with the help of equation (1.1), we get

$$N(\bar{X},Y) = a^2(X,\bar{Y}) - A(X)[T,\bar{Y}] + a^2[\bar{X},Y] - A([\bar{X},Y])T - \overline{[\bar{X},Y]} - a^2\,\overline{[X,Y]} + A(X)\overline{[T,Y]} \tag{2.23}$$

Now barring the whole equation and using (1.1), we obtain

$$\overline{N[\bar{X},Y]} = a^2[\bar{X},\bar{Y}] - A(X)\overline{[T,\bar{Y}]} + a^2\overline{[\bar{X},Y]} - a^2[\bar{X},\bar{Y}] + A([\bar{X},\bar{Y}])T - a^4[X,Y] + a^2A([X,Y])T + a^2A(X)[T,Y] + A(X)\,A([T,Y])T \tag{2.24}$$

From the equation (2.2) and (2.24), we have

$$\overline{N[\bar{X},Y]} + a^2N(X,Y) = -A(X)([T,\bar{\bar{Y}}]) + A[\bar{X},\bar{Y}]T + a^2A(X)[T,Y] + A(X)A([T,Y])T \tag{2.25}$$

$$N(T,Y) = a^2[T,Y] + A([T,Y])T - \overline{[T,\bar{Y}]} \tag{2.26}$$

Using (2.26) in (2.25) we obtain

$$\overline{N[\bar{X},Y]} + a^2N(X,Y) = A(X)N(T,Y) + A([\bar{X},\bar{Y}])T \tag{2.27}$$

For completely integrable manifold equation (2.27) reduces to equation (2.22)

Theorem (2.7) *In a completely integrable generalized almost contact metric structure manifold M^n, we have the following result.*

$$A(X)\{[T,\bar{Y}] - \overline{[T,Y]}\} + A([\bar{X},Y])T = A(Y\{[\bar{X},T] - \overline{[X,\bar{T}]}\} + A([X,\bar{Y}])T \tag{2.28}$$

Proof: Barring X in equation (2.2) and making use of (1.1), we get

$$N(\bar{X},Y) = a^2(X,\bar{Y}) - A(X)[T,\bar{Y}] + a^2[\bar{X},Y] - A([\bar{X},Y])T - \overline{[\bar{X},Y]} - a^2\overline{[X,Y]} + A(X)\overline{[T,Y]} \tag{2.29}$$

Again barring Y in equation (2.2) and making use of (1.1), we get

$$N(X,\bar{Y}) = a^2[\bar{X},Y] - A(Y)[\bar{X},T] + a^2[X,\bar{Y}] + A([X,\bar{Y}])T - a^2\overline{[X,Y]} + A(Y))\overline{[X,T]} \tag{2.30}$$

Now from these two equation (2.29) and (2.30) and using $N(X, Y)$, we have the required result (2.28)

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Mathematical estimation of unsteady state burn damage due to hot temperature

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Abstract: Considering three natural layers of human dermal parts, an attempt has been made to estimate one dimensional unsteady state burn damage in the region due to high surrounding temperature. We consider two samples of skin and subcutaneous tissue (SST) of various thicknesses and constructed a mathematical model. Solving the same numerically we have obtained the temperature distribution that causes burn in the outer layer to inner layers. The temperature profiles have been further used to estimate the damage in the region.

Key words: variational finite element method, damage function, human skin.2000 Mathematical Subject Classification: 92.

1. Introduction

Burn injury in human dermal parts due to high environmental temperatures results in a clinical problem. Burns cause a range of physiological derangements including denaturation of macromolecular structures, leakage of cell membranes, activation of cytokines, and cessation of blood flow. This leads ultimately to tissue damage and circulatory disruption in skin and subcutaneous tissue. The various degrees of burn measure the extent and volume of the damage. This paper attempts to estimate the effect of outer high temperatures and consequently the disturbance in the thermal balance.

Thermal burns occur as a consequence of the elevation of tissue temperature above a threshold value. Guyton [6] mentioned that human dermal parts attain the threshold value if outer skin surface temperature reaches 45^0C. It is assumed that the resulting injury is governed by the chemical rate processes in terms of standard Arrhenius function as proposed by Henriques and Moritz [7]. This damage rate function describes the rate of tissue damage. By integrating the damage rate function, we find non-dimensional number Ω called the damage function. This damage function can be expressed as

(1) $$\Omega = A \int_0^t \exp\left[\frac{-\Delta E}{R(T+273)}\right] dt$$

where A, ΔE, R, and T are respectively, the frequency factor, activation energy, universal gas constant and tissue temperature in degree Celsius.

The value of Ω equal to 0.53, 1, and 10^4 represent injury threshold for first, second and third degree burns respectively. To a limited extent the experimental results can be compared with the theoretical estimation.

Human dermal parts consist of three distinct natural layers – epidermis, dermis and subcutaneous tissue. The epidermis is a complex multiple layered membrane having no blood vessels; hence there is no blood flow and metabolic activity. However, there are some blood vessels and metabolic activity at the junction of epidermis and dermis. The density of blood vessels increases from this junction toward the junction of dermis and subcutaneous tissue. Unlike in dermis the blood vessels are uniformly distributed in subcutaneous tissue.

Henriques and Moritz [7] demonstrated that time and temperature is responsible to produce a specified level of burn injury. Buetnner [1] have studied the effects of extreme heat and cold on human skin. Saxena and Yadav [15] studied steady state case for burn injury due to hot atmospheric temperatures. Dennis et al. [4] studied heat injury to cells in perfused system. Saxena et al. [11] used variational finite element approach to estimate burn damage in steadystate case.

The numerical computation has been carried out using MATLAB program and the results are discussed numerically and graphically.

2. Mathematical Model

The partial differential equation for heat transfer in peripheral layers of human body due to Perl [9] is

$$\rho c \frac{\partial T}{\partial t} = div(K\, gradT) + \rho_b c_b (\phi_A T_A - \phi_v T_v) + S \tag{2}$$

where

ρ = Tissue density (g/cm^3)
c = Tissue specific heat (cal/g^{0}C)
K = Tissue thermal conductivity (cal/cm-min^{0}C)
ρ_b = Blood density (g/cm^3)
c_b = Blood specific heat (cal/g^{0}C)
ϕ_A = Tissue perfusion due to arterial blood (/min)
ϕ_v = Tissue perfusion due to venous blood (/min)
T_A = Arterial blood temperature (^{0}C)
T_v = Venous blood temperature (^{0}C)
S = Metabolic heat generation rate (cal/cm^3-min)

There is no significant difference in the values of ϕ_A and ϕ_v in micro level, so $\phi_A = \phi_v$. Also T_v is dominated by tissue temperature T. Hence $T_v \approx T$. So equation (2) can rewrite as

$$\rho c \frac{\partial T}{\partial t} = div(K\, gradT) + M\,(T_A - T) + S \tag{3}$$

where $m_b = \rho_b \phi_A$ = Blood mass flow rate (g/cm^3-min), and $M = m_b c_b$.

The terms on the right side of equation (3) denote respectively, Fourier's laws of conduction, Fick's perfusion principle and the rates of metabolic heat generation.

The thickness of epidermis, dermis, and subcutaneous tissue have been considered as $a, b-a$, and $c-b$ respectively, and T_0, T_1, T_2, and $T_3 = T_b$ are respectively the nodal temperatures at distances $x = 0$, $x = a$, $x = b$, and $x = c$ (Figure - 1). The body core temperature T_b is almost 37^0C.

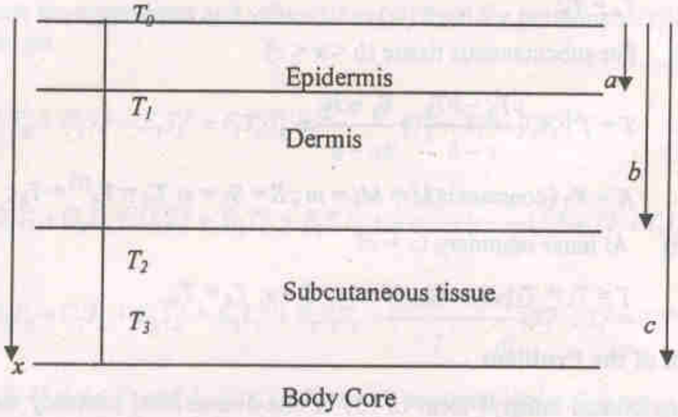

Figure - 1 Schematic diagram of Skin and Subcutaneous tissue

In this model it is assumed that heat transfer to the outer skin surface takes place due to convection, radiation and sweat evaporation. Consequently at the skin surface we have

(4) $$-K\left.\frac{\partial T}{\partial x}\right|_{\text{at } x=0} = h(T - T_a) + LE$$

where h, T_a, L, and E are respectively heat transfer coefficient, atmospheric temperature, latent heat, and rate of sweat evaporation.

The biological structure of human dermal parts makes it reasonable to consider M and S zeros in epidermis. As a whole, the assumptions and conditions can be summed up in the following forms

(i) For epidermis ($0 < x < a$)

$$T = T^{(1)} = T_0 + \frac{T_1 - T_0}{a}x;$$

$K = K_1$(constant); $M = M_1 = 0$; $S = S_1 = 0$; $T_A = T_A^{(1)} = 0$;

(ii) At interface - I ($x = a$)

$T = T^{(1)} = T^{(2)} = T_1$; $K = K_1 = K_2$; $M = M_1 = M_2 = 0$; $S = S_1 = S_2 = 0$;

$T_A = 0$;

(iii) For dermis ($a < x < b$)

$$T = T^{(2)} = \frac{bT_1 - aT_2}{b-a} + \frac{T_2 - T_1}{b-a}x \;;\; K = K_2 \text{ (constant)}$$

$$M = M_2 = \left(\frac{x-a}{b-a}\right)m;\ S = S_2 = \left(\frac{x-a}{b-a}\right)s;\ T_A = T_A^{(2)} = \left(\frac{x-a}{b-a}\right)T_b$$

(iv) At interface - II ($x = b$)

$$T = T^{(2)} = T^{(3)} = T_2;\ K = K_2 = K_3;\ M = M_2 = M_3;\ S = S_2 = S_3;$$

$$T_A = T_b;$$

(v) For subcutaneous tissue ($b < x < c$)

$$T = T^{(3)} = \frac{cT_2 - bT_3}{c-b} + \frac{T_3 - T_2}{c-b}x\ ;$$

$$K = K_3 \text{ (constant)};\ M = M_3 = m;\ S = S_3 = s;\ T_A = T_A^{(3)} = T_b;$$

(vi) At inner boundary ($x = c$)

$$T = T_3 = T_b;\ K = K_3;\ M = m;\ S = s;\ T_A = T_b;$$

3. Solution of the Problem

The variational integral form of (3) in one-dimensional unsteady state case together with outer skin boundary condition (4) is given by [8]

(5) $$I = \frac{1}{2}\int_0^c \left[K\left(\frac{dT}{dx}\right)^2 + M\left(T_A - T\right)^2 - 2ST + \rho c\frac{\partial T^2}{\partial t} \right]dx + \frac{1}{2}\left[h\left(T - T_a\right)^2 + 2LET\right]$$

provided I is optimized. We rewrite I separately for the three layers, i.e.,

(6) $$I = \sum_{i=1}^{3} I_i$$

where

$$I_1 = \frac{1}{2}\int_0^a \left[K_1\left(\frac{dT^{(1)}}{dx}\right)^2 + M\left(T_A^{(1)} - T^{(1)}\right)^2 - 2S_1T^{(1)} + \rho c\frac{\partial T^{(1)2}}{\partial t} \right]dx + \frac{1}{2}\left[h\left(T^{(1)} - T_a\right)^2 + 2LET^{(1)}\right]$$

$$I_2 = \frac{1}{2}\int_a^b \left[K_2\left(\frac{dT^{(2)}}{dx}\right)^2 + M\left(T_A^{(2)} - T^{(2)}\right)^2 - 2S_2T^{(2)} + \rho c\frac{\partial T^{(2)2}}{\partial t} \right]dx$$

$$I_3 = \frac{1}{2}\int_b^c \left[K_3\left(\frac{dT^{(3)}}{dx}\right)^2 + M\left(T_A^{(3)} - T^{(3)}\right)^2 - 2S_3T^{(3)} + \rho c\frac{\partial T^{(3)2}}{\partial t} \right]dx$$

Generally under normal condition the temperature decreases from body core towards skin surface. Hence we consider the initial condition for T_0, T_1, T_2 and T_3 in linear order towards body core. Thus we assume the following initial condition for the problem

$$T(x, 0) = T_0 + qx$$

where $T_0 = 22.87^0$C and q is a constant to be determined.

We substitute the expressions and values (i) to (vi) from the previous section to compute I_1, I_2 and I_3. We then get

$$I_1 = A_1 + B_1T_0 + D_1T_0^2 + E_1T_1^2 + F_1T_0T_1 + \frac{\rho ca}{2}\frac{d}{dt}(T_0^2 + T_1^2 + T_0T_1)$$

$$I_2 = A_2 + B_2T_1 + C_2T_2 + D_2T_1^2 + E_2T_2^2 + F_2T_1T_2 + \frac{\rho c(b-a)}{2}\frac{d}{dt}(T_1^2 + T_2^2 + T_1T_2)$$

$$I_3 = A_3 + B_3T_2 + C_3T_3 + D_3T_2^2 + E_3T_3^2 + F_3T_2T_3 + \frac{\rho c(c-b)}{2}\frac{d}{dt}(T_2^2 + T_3^2 + T_2T_3)$$

where A_i, B_i, D_i, E_i, F_i $(1 \le i \le 3)$ and C_j $(2 \le j \le 3)$ are all constants depending upon the physical and physiological parameters and are defined in Appendix.

As a next step of finite element method, we minimize I by differentiating it with respect to the nodal temperatures T_0, T_1, and T_2. Since $T_3 = T_b$ (the body core temperature) is known, we obtain the following system of equations

$$C\dot{T} + PT = W \tag{7}$$

where

$$C = \begin{bmatrix} 2\alpha & \alpha & 0 \\ \alpha & 2(\alpha+\beta) & \beta \\ 0 & \beta & 2(\beta+\mu) \end{bmatrix} \quad P = \begin{bmatrix} 2D_1 & F_1 & 0 \\ F_1 & 2(E_1+D_2) & F_2 \\ 0 & F_2 & 2(E_2+D_3) \end{bmatrix};$$

$$W = \begin{bmatrix} -B_1 \\ -B_2 \\ -(C_2 + B_3 + F_3T_b) \end{bmatrix}; \quad \dot{T} = \begin{bmatrix} \frac{dT_0}{dt} \\ \frac{dT_1}{dt} \\ \frac{dT_2}{dt} \end{bmatrix}; \quad T = \begin{bmatrix} T_0 \\ T_1 \\ T_2 \end{bmatrix}$$

with

$$\alpha = \frac{\rho ca}{2}; \quad \beta = \frac{\rho c(b-a)}{2}; \quad \mu = \frac{\rho c(c-b)}{2}$$

To solve the system of ordinary differential equations (7) we use Crank-Nicolson method. According to this method, the system of equations (7) can be written as

$$(8) \qquad \left(C + \frac{\Delta t}{2} P \right) T^{(i+1)} = \left(C - \frac{\Delta t}{2} P \right) T^{(i)} + \Delta t W$$

where Δt is the time interval and $T^{(0)}$ is the 3×1 matrix for initial nodal temperatures.

4. Numerical Results

To solve the system of equations (8) coupled with damage function (1) the following values have been taken [[3], [11], [15]].

$$K_1 = 0.030 \text{ cal/cm-min}^0\text{C}$$
$$K_2 = 0.045 \text{ cal/cm-min}^0\text{C}$$
$$K_3 = 0.060 \text{ cal/cm-min}^0\text{C}$$
$$m = M_{\max} = 0.0315 \text{ cal/cm}^3\text{-min}^0\text{C}$$
$$s = S_{\max} = 0.018 \text{ cal/cm}^3\text{-min}$$
$$L = 579 \text{ cal/g}$$
$$h = 0.18 \text{ cal/cm}^2\text{-min}^0\text{C}$$
$$E = 0.0096 \text{ g/cm}^2\text{-min}$$
$$\Delta E = 6.3 \times 10^8 \text{ J/Kg mol}$$
$$R = 8.3136 \times 10^3 \text{ J/Kg mol K}$$
$$A = 18 \times 10^{99} \text{ /min}$$
$$\rho = 1.05 \text{ g/cm}^3$$
$$c = 0.83 \text{ cal/g}$$

We consider the following two sets of sample SST of different thicknesses as shown in Table– 1.

Table – 1

Sample Skin	a (cm)	b (cm)	c (cm)
Set - I	0.10	0.35	0.50
Set - II	0.10	0.50	0.75

The temperature of heat source is assumed to be of 100^0C. It is also assumed that skin surface is exposed to this source for a time of 60 seconds. The system of equations (8) is then solved by subdividing this time of exposure in different time intervals of 3 seconds. Accordingly, we get the values of nodal temperatures at these time intervals for Set-I and Set-II as shown in Table-2. The graphs for these nodal temperatures for Set-I and Set-II are plotted as shown in Figures-2 and 3 respectively. Using the Table-2, we fitted the cubic splines for T_0, and then these cubic splines for T_0 are used to obtain the damage function at different times within the total time of exposure of the skin. The corresponding values of damage function at different times for Set - I and Set - II are plotted in Figure-4.

Table - 2

Nodal temp. Times (Seconds)	Set I T_0			Set II		
	T_0	T_1	T_2	T_0	T_1	T_2
$t_0 = 0$	22.87	25.70	32.76	22.87	24.75	32.29
$t_1 = 3$	34.15	25.10	33.07	33.79	24.27	32.50
$t_2 = 6$	38.22	26.77	32.72	37.86	25.40	32.23
$t_3 = 9$	40.49	28.67	32.51	39.91	26.86	31.92
$t_4 = 12$	42.15	30.43	32.51	41.30	28.32	31.68
$t_5 = 15$	43.51	32.01	32.67	42.42	29.69	31.52
$t_6 = 18$	44.68	33.43	32.95	43.41	30.97	31.46
$t_7 = 21$	45.71	34.72	33.29	44.30	32.14	31.46
$t_8 = 24$	46.63	35.89	33.68	45.11	33.23	31.51
$t_9 = 27$	47.45	36.97	34.08	45.86	34.24	31.62
$t_{10} = 30$	48.20	37.96	34.48	46.55	35.18	31.76
$t_{11} = 33$	48.89	38.88	34.88	47.19	36.05	31.93
$t_{12} = 36$	49.52	39.73	35.27	47.79	36.88	32.13
$t_{13} = 39$	50.10	40.52	35.65	48.35	37.65	32.34
$t_{14} = 42$	50.65	41.26	36.01	48.87	38.38	32.57
$t_{15} = 45$	51.15	41.95	36.35	49.36	39.06	32.82
$t_{16} = 48$	51.62	42.60	36.68	49.82	39.71	33.07
$t_{17} = 51$	52.06	43.20	36.98	50.25	40.33	33.32
$t_{18} = 54$	52.46	43.77	37.27	50.66	40.92	33.58
$t_{19} = 57$	52.85	44.29	37.54	51.05	41.47	33.84
$t_{20} = 60$	53.21	44.79	37.80	51.42	42.00	34.10

4. Discussion

From Table – 2, we observe that Set – I attains about 46^{0}C for T_0 in 21 seconds. So, the thermal disturbance in Set – I starts at about 20 seconds. But the same process in case of Set – II starts at about 24 seconds.

From Figure–4, we find that graphs of damage function for both sets increase exponentially as time of exposure of the skin to the source temperature increases. These graphs represent that both second and third degree burn for Set – I occur earlier than Set – II.

The graph of Figure – 2 are rising faster than the graphs of Figure – 3 in first few minutes and then attaining the lines parallel to t-axis earlier than the graphs of Figure – 3. So, Set – I reaches its steady state case earlier than Set – II.

Above discussion is only an example that how the mathematical model can help in many clinical cases of burn injury. This approach can in fact give comprehensive approach of damage estimation based on individual characteristics, climatic conditions and internal body mechanism.

APPENDIX

$$X_1 = -\frac{ma}{2(b-a)^2};\ X_2 = \frac{m(a+b)}{4(b-a)^2};\ X_3 = \frac{m(a^2+ab+b^2)}{6(b-a)^2};\ X_4 = \frac{m(a+b)(a^2+b^2)}{4(b-a)^2};$$

$$X_5 = -\frac{s}{(b-a)};\ V = \frac{K_3}{2(c-b)};\ Y_1 = \frac{m}{2(c-b)};\ Y_2 = -\frac{m(b+c)}{2(c-b)};\ Y_3 = \frac{m(b^2+bc+c^2)}{6(c-b)};$$

$$A_1 = \frac{hT_a^2}{2};\ B_1 = LE - hT_a;\ D_1 = \frac{1}{2}\left(\frac{K_1}{a} + h\right);\ E_1 = \frac{K_1}{2a};\ F_1 = -\frac{K_1}{a};$$

$$A_2 = T_b^2 a^2 X_1 + 3T_b^2 a^2 X_2 - 3T_b^2 a X_3 + T_b^2 X_4;$$

$$B_2 = 2T_b abX_1 + 2T_b a(2b+a)X_2 - 2T_b(b+2a)X_3 + 2T_b X_4 + (a-1)(a-b)X_5;$$

$$C_2 = -2T_b a^2 X_1 - 6T_b a^2 X_2 + 6T_b aX_3 - 2T_b X_4 + (a-1)(a-b)X_5;$$

$$D_2 = \frac{K_2}{2(b-a)} + b^2 X_1 + (b^2 + 2ab)X_2 - (2b+a)X_3 + X_4;$$

$$E_2 = \frac{K_2}{2(b-a)} + a^2 X_1 + 3a^2 X_2 - 3aX_3 + X_4;$$

$$F_2 = -\frac{K_2}{(b-a)} - 2abX_1 - 2a(2b+a)X_2 + (4a+2b)X_3 - 2X_4;$$

$$A_3 = T_b^2 (c-b)^2 Y_1;$$

$$B_3 = -2T_b c(c-b)Y_1 - T_b(c-b)Y_2 - \frac{s(c-b)}{2};$$

$$C_3 = 2T_b b(c-b)Y_1 + T_b(c-b)Y_2 - \frac{s(c-b)}{2};$$

$$D_3 = c^2 Y + cY_2 + Y_3 + V;$$

$$E_3 = b^2 Y_1 + bY_2 + Y_3 + V;$$

$$F_3 = -2bcY_1 - (b+c)Y_2 - 2Y_3 - 2V$$

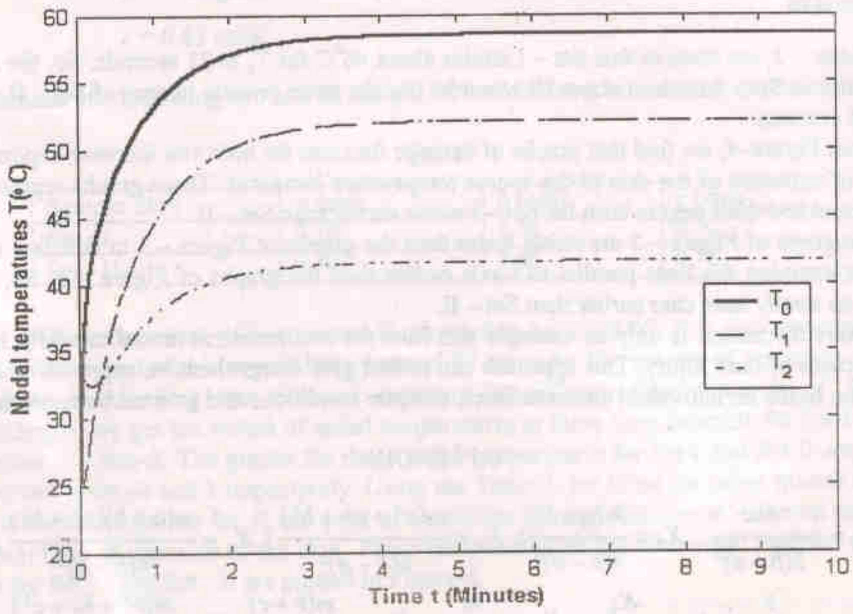

Figure - 2 Nodal Temperatures for Set - I.

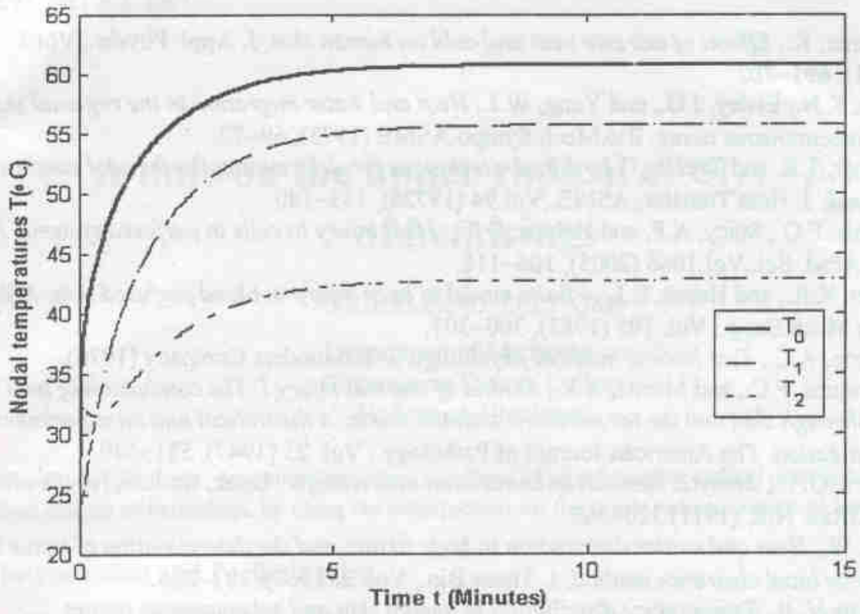

Figure - 3 Nodal Temperatures for Set - II.

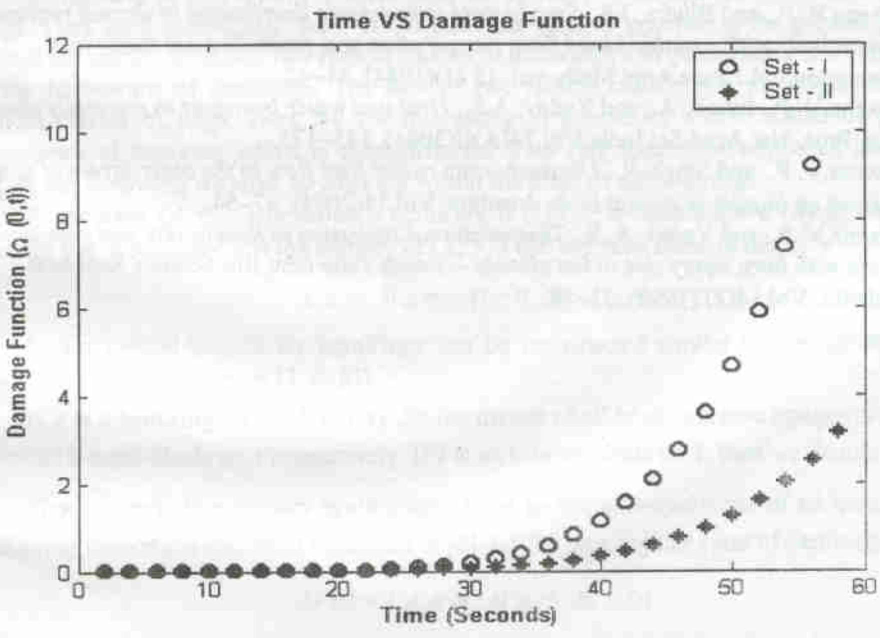

Figure - 4 Damage Function for different values of time when $x = 0$.

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A note on the upper radical of $S(\rho_1+\rho_2)$ of hemirings

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Abstract: In this paper, we generalize a few results of [4] for the upper radical classes of rings for radical classes of hemirings, by using the construction for the upper radical classes of hemirings.

1. Introduction and Preliminaries

The notion of radical classes of hemirings was introduced by D. M. Olson and T. L. Jenkins [7], as an extension of radical classes of rings (see [4]). The theory was further enriched by many authors (see [7, 8]).

Y. Lee and R. E. Propes [4] introduced the concept of the sum of two radical classes of rings. They have shown that 'sum' is not a radical class in general. In the present paper, we extend the notion of sum of two radical classes of hemirings and generalize a few results of [4] in the framework of hemirings. The sum of two radical classes was investigate by [4] for radical classes of rings. Here we are interesting to generalize a few results of [4] in the framework of hemiring which is quite different from ring theoretical approach discussed in [4]. In the following we shall be working within the class of all hemirings.

A semiring $(A,+,\,.\,)$ is called a hemiring if (i) '+' is commutative (ii) there exists an element $0 \in A$ such that 0 is the identity of $(A,+,)$ and the zero element of $(A,.)$.

i. e. $\quad 0\,a = a\,0 = 0,\ \forall\ a \in A$

Lower radical classes for hemirings can be constructed similar to the construction of lower radicals for rings (see [3, 6, 8]).

If A is a hemiring then $HA, K_1(A)$ denote the set of all homomorphic images of A and the set of all k-semi-ideals of A respectively. If I is an k-semi-ideals of A, then we denote $I \underset{K}{\leq} A$.

First we include necessary preliminary, let ω be the universal class of all hemirings. By using ring theoretical approach discussed in [6], let $\mathcal{M}$ be a regular class of hemiring, define

$$U\,\mathcal{M} = \{\, A \in \omega : H\,A \cap \mathcal{M} = 0\}$$

then the class $U\mathcal{M}$ is a radical class and is called the upper radical class determined by the class $\mathcal{M}$. For undefined terms of hemirings we may refer (see [1, 2, 5, 6]).

2. Radical and Semisimple Classes

We extend the result of [4] by using the above construction of upper radical for hemiring which is indeed provides an excellent and different approach to handle the many results of [4] in the framework of hemiring.

The following definition is taken from S. M. Yusuf and M. Shabir [8]. The semisimple class Sρ of a radical class ρ is defined as the class of all hemirings having zero ρ-radical.

This can be rephrases in the following form.

Definition 2.1 [8]. *Let $S \subseteq \omega$, S is said to be a semisimple class, if the following two axioms are satisfied:*

S_1) $A \in S \Rightarrow HI \cap S \neq 0$, $\forall\ (0 \neq I) \in K_1(A)$

S_2) *Let $A \in \omega$ such that $HI \cap S \neq 0$, $\forall\ (I \neq 0)\ I \underset{K}{\leq} A$. then $A \in S$.*

A subclass of hemirings ω satisfying the condition (S_1) is called a regular class.

Definition 2.2 [8]. *Let ρ be a radical class of hemirings. Then we define a class $S\rho$ as follows:*

$$S\rho = \{\ A \in \omega : \rho(A) = 0\}.$$

Theorem 2.3. *Every hereditary class is regular.*

Theorem 2.4. *$S\rho$ is hereditary.*

Theorem 2.5. *Let ρ be a radical class and $S\rho = \{\ A \in \omega : \rho(A) = 0\}$. Then $S\rho$ is a semisimple class.*

Proof: *By Theorem 2.3 and Theorem 2.4, $S\rho$ is regular class i.e. (S_1) is satisfied.*

S_2) *Let $A \in \omega$ such that $HI \cap S\rho \neq 0$, $\forall\ (I \neq 0) \underset{K}{\leq} A$, we claim that $A \in S\rho$.*

Assume on contrary $A \in S\rho$, therefore $\rho(A) \neq 0$. Now $(\rho(A) \neq 0) \underset{K}{\leq} A$, let $I = \rho(A)$. Let $I/J \in HI \cap S\rho = H(\rho(A)) \cap S\rho$. This implies that $\rho(A)/J \in S\rho$. Since ρ is a radical class, therefore ρ *is* homomorphically closed and $\rho(A) \in \rho$, therefore $\rho(A)/J \in \rho$ and we have $\rho(\rho(A)/J) = \rho(A)/J$. As $\rho(A)/J \in S\rho$. Thus $\rho(\rho(A)/J) = 0$. This implies that $\rho(A)/J = 0$ and $\rho(A) \subseteq J$. This implies that $\rho(A) = J\ (\because J \subseteq \rho(A))$ and hence $I/J = \rho(A)/J = 0$. As I/J is an arbitrary element of $HI \cap S\rho$ such that $I/J = 0$, therefore we have $HI \cap S = 0$, for some k-semi-ideal $I = \rho(A) \neq 0$. This contradicts the fact $HI \cap S\rho \neq 0, \forall\ (I \neq 0) \underset{K}{\leq} A$. Consequently, $\rho(A) = 0$ and hence $A \in S\rho$ and (S_2) is satisfied.

Definition 2.6. *Let ρ_1, ρ_2 be radical classes of hemirings, then we define their sum*

$$\rho_1 + \rho_2 = \{A \in \omega : \rho_1(A) + \rho_2(A) = A\}.$$

We write $(\rho_1 + \rho_2)(A) = \rho_1(A) + \rho_2(A)$ for all $A \in \omega$.

Definition 2.7. *Let $\rho_1 + \rho_2$ be radical classes of hemirings. Then*

$$S(\rho_1 + \rho_2) = \{A \in \omega : (\rho_1 + \rho_2)(A) = 0\}.$$

We now investigate conditions under which $\rho_1 + \rho_2$ will be a radical class.

In the case of hemirings one can easily prove all the standard result concerning radical classes, sum of two radical classes, k-semi-ideal and semisimple classes. Here we mention only few of them, which can be obtained on the line of rings theoretical approach.

Theorem 2.8. *If ρ_1, ρ_2 are radical classes of hemirings, then*

$$S(\rho_1+\rho_2) = S\rho_1 \cap S\rho_2$$

Theorem 2.9. *If ρ_1 and ρ_2 are radical classes of hemirings, then*

$$(\rho_1+\rho_2) \cap S(\rho_1+\rho_2) = 0.$$

Theorem 2.10. *If ρ_1 and ρ_2 are radical classes, then $\rho_1(A)+\rho_2(A)$ is the largest $\rho_1+\rho_2$ semi-ideal of the hemirings A.*

Theorem 2.11. *Let ρ_1, ρ_2 be radical classes of hemirings and $I \underset{K}{\leq} A$, then*

$$(\rho_1+\rho_2)(I) \subseteq (\rho_1+\rho_2)(A) \cap I.$$

Theorem 2.12. *Let ρ_1, ρ_2 be radical classes of hemirings, then $S(\rho_1+\rho_2)$ is a semi-simple class of hemiring.*

Proof: Let $A \in S(\rho_1+\rho_2) = S\rho_1 \cap S\rho_2$ (by Theorem 2.9). This implies that $A \in S\rho_1$ and $A \in S\rho_2$ and hence $\rho_1(A)=0$ and $\rho_2(A)=0$. Let $I \underset{K}{\leq} A$, then by hereditary of $S\rho_1$ and $S\rho_2$, $I \in S\rho_1$ and $I \in S\rho_2$. This implies $I \in S\rho_1 \cap S\rho_2$ and hence $I \in S(\rho_1+\rho_2)$. Thus $S(\rho_1+\rho_2)$ is hereditary. This implies that if $A \in S(\rho_1+\rho_2)$ and I is a non-zero k-semi-ideal of A, then its non-zero homomorphic image in $S(\rho_1+\rho_2)$ is I itself. Therefore (S_1) of the definition 2.1 is satisfied. Suppose every non-zero k-semi-ideal of a hemiring A has a non-zero homomorphic image in $S(\rho_1+\rho_2) = S\rho_1 \cap S\rho_2$. This means every non-zero k-semi-ideal of a hemiring A has a non-zero homomorphic image in $S\rho_1$ and also in $S\rho_2$. This implies $A \in S\rho_1$ and $A \in S\rho_2$ because $S\rho_1, S\rho_2$ are semisimple classes. Consequently, we have $A \in S\rho_1 \cap S\rho_2 = S(\rho_1+\rho_2)$. This shows that (S_2) of the definition 2.1 is also satisfied.

Theorem 2.13. Let ρ_1, ρ_2 *be radical classes of hemirings. Then $\rho_1+\rho_2$ is hereditary if and only if $(\rho_1+\rho_2)(I) = (\rho_1+\rho_2)(A) \cap I, \forall A \in \omega, \forall I \underset{K}{\leq} A$.*

Proof: Let $\rho_1+\rho_2$ be hereditary and $I \underset{K}{\leq} A$. Now $(\rho_1+\rho_2)(A) \in \rho_1+\rho_2$ and $(\rho_1+\rho_2)(A) \cap I \underset{K}{\leq} (\rho_1+\rho_2)(A)$. By hereditary of $\rho_1+\rho_2$, we have $(\rho_1+\rho_2)(A) \cap I \in \rho_1+\rho_2$. As $(\rho_1+\rho_2)(A) \cap I \leq I$, therefore, we have $(\rho_1+\rho_2)(A) \cap I \subseteq (\rho_1+\rho_2)(I)$. Also we have $(\rho_1+\rho_2)(I) \subseteq (\rho_1+\rho_2)(A) \cap I$ (by theorem 2.12). Consequently, we have $(\rho_1+\rho_2)(A) \cap I = (\rho_1+\rho_2)(I)$.

Conversely, assume that $(\rho_1+\rho_2)(I) = (\rho_1+\rho_2)(A) \cap I$, $\forall A \in \omega$, $\forall I \underset{K}{\leq} A$. Let $A \in \rho_1+\rho_2$ and $I \underset{K}{\leq} A$, then $(\rho_1+\rho_2)(A) = A$. Thus $(\rho_1+\rho_2)(I) = (\rho_1+\rho_2)(A) \cap I = A \cap I = I$. This shows that $I \in \rho_1+\rho_2$ and hence $\rho_1+\rho_2$ is hereditary.

Corollary 2.14. *Let ρ_1 and ρ_2 be hereditary radical classes of hemirings. Then the class $\rho_1 + \rho_2$ is hereditary if and only if*

$$I \cap \rho_1(A) + I \cap \rho_2(A) = I \cap (\rho_1(A) + \rho_2(A)), \forall\, A \in \omega, \forall\, I \underset{K}{\leq} A.$$

3. The upper radical of $\mathsf{S}(\rho_1+\rho_2)$

The notion of upper radical classes or upper radicals was originally introduced by F. A. Szasz ([5, see [6]) for rings.

In the case of hemirings one can easily prove all the standard result concerning lower radicals, upper radicals and regular classes. Here we mention only few of them, which can be obtained on the line of rings theoretical approach.

Theorem 3.1. *Let $\mathcal{M}$ be a regular class of hemirings, define $\mathsf{U}\mathcal{M} = \{A \in \omega : HA \cap \mathcal{M} = 0\}$ then the class $\mathsf{U}\mathcal{M}$ is a radical class and is called the upper radical class determined by the class $\mathcal{M}$.*

Theorem 3.2. *If $\mathcal{M}$ is regular class of hemirings such that $\mathcal{M} \subseteq \mathsf{S}(\rho_1 + \rho_2)$, then* $(\rho_1 + \rho_2) \subseteq \mathsf{U}\mathcal{M}$.

Theorem 3.3. *If ρ_1 and ρ_2 are radical classes of hemirings, then*

$$\mathsf{US}(\rho_1 + \rho_2) = \{A \in \omega : HA \cap \mathsf{S}\rho_1 + \mathsf{S}\rho_2 = 0\}.$$

Proof: *Since $\mathsf{S}(\rho_1 + \rho_2)$ is a semisimple class, so by definition, we have*

$$\mathsf{US}(\rho_1 + \rho_2) = \{A \in \omega : HA \cap \mathsf{S}(\rho_1 + \rho_2) = 0\}.$$

Since $\mathsf{S}(\rho_1 + \rho_2) = \mathsf{S}\rho_1 \cap \mathsf{S}\rho_2$ (by Theorem 2.9). Therefore we have

$$\mathsf{US}(\rho_1 + \rho_2) = \{A \in \omega : HA \cap \mathsf{S}\rho_1 + \mathsf{S}\rho_2 = 0\}.$$

Theorem 3.4. *Let ρ_1 and ρ_2 be radical classes of hemirings. Then*

$$\mathsf{S}(L(\rho_1 + \rho_2)) = \mathsf{S}(\rho_1 + \rho_2).$$

Proof : Let $A \in \mathsf{S}(L(\rho_1 + \rho_2))$, we will show that $A \in \mathsf{S}(\rho_1 + \rho_2)$. Assume on contrary that $A \notin \mathsf{S}(\rho_1 + \rho_2)$ but $A \in \mathsf{S}(L(\rho_1 + \rho_2))$ implies that $[L(\rho_1 + \rho_2)](A) = 0$. Since $A \notin \mathsf{S}(\rho_1 + \rho_2)$ $= \mathsf{S}\rho \cap \mathsf{S}\rho$, therefore $A \notin \mathsf{S}\rho_1 \cap \mathsf{S}\rho_2$. This implies that $A \notin \mathsf{S}\rho_1$ or $A \notin \mathsf{S}\rho_2$. Now $A \notin \mathsf{S}\rho_1$. This implies that $\rho_1(A) \neq 0$. As $0 \neq \rho_1(A) \in \rho_1 \subseteq \rho_1 \cup \rho_2 \subseteq L(\rho_1 \cup \rho_2) = L(\rho_1 + \rho_2)$. Since $A \in \mathsf{S}(L(\rho_1 + \rho_2))$ and $\mathsf{S}(L(\rho_1 + \rho_2))$ is heriditary, we have $\rho_1(A) \in \mathsf{S}(L(\rho_1 + \rho_2))$ and hence $\rho(A) \neq 0) \in \mathsf{S}(L(\rho_1+\rho_2)) \cap (L(\rho_1+\rho_2)$. We have a contradiction that $[L(\rho_1+\rho_2)] \cap \mathsf{S}(L(\rho_1+\rho_2)) = 0$. This proves that $A \in \mathsf{S}(\rho_1+\rho_2)$. Now we have $A \in \mathsf{S}(L(\rho_1+\rho_2))$ implies that $A \in \mathsf{S}(\rho_1+\rho_2)$. This shows that $\mathsf{S}(L(\rho_1+\rho_2)) \subseteq \mathsf{S}(L(\rho_1+\rho_2))$.

For reverse inclusion, let $A \notin \mathsf{S}(L(\rho_1+\rho_2))$. We shall show that $A \notin \mathsf{S}(\rho_1+\rho_2)$. Assume on contrary that $A \in \mathsf{S}(\rho_1+\rho_2)$ but $A \notin \mathsf{S}(L(\rho_1+\rho_2))$ implies that $[L(\rho_1+\rho_2)](A) \neq 0$. Let $I = [L(\rho_1+\rho_2)](A)$. Then I has a non zero accessible $\rho_1+\rho_2$ -sub-hemiring, say, T. Now

$A \in S(\rho_1+\rho_2) = S\rho_1 \cap S\rho_2$ and $I \underset{\kappa}{\leq} A$. By the heriditary of $S(\rho_1+\rho_2)$, $I \in S(\rho_1+\rho_2)$. Now T is an accessible sub-hemiring of I and so $T \in S(\rho_1+\rho_2)$. This implies that $(\rho_1+\rho_2)(T) = 0$. Since $T \in \rho_1+\rho_2$, therefore $(\rho_1+\rho_2)(T) = T$ and hence $T = 0$, which contradicts that $T \neq 0$. Thus $S(\rho_1+\rho_2) \subseteq S(L(\rho_1+\rho_2))$. Consequently, we have $S(L(\rho_1+\rho_2)) = S(\rho_1+\rho_2)$. This completes the proof.

Corollary 3.5. *If ρ_1, ρ_2 are radical classes of hemirings, then*

$$L(\rho_1 \cup \rho_2) = U[S(\rho_1+\rho_2)]$$

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CONTENTS

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