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Probability distributions to describe the pattern of child loss from a family

TIKA RAM ARYAL*

Abstract: This paper attempts to investigate tne distribution of child loss from a family through probability models. The suitability of the models has been tested with the real sets of sample survey data from Nepal, Libya and Brazil. Poisson, geometric and displaced geometric distributions have been fitted to describe the distribution of families according to the number of child deaths. The parameters have been estimated by maximum likelihood method. It was found that the proposed geometric and displaced geometric distributions more or less provided a suitable description of child loss pattern at micro level (family level). These distributions fitted satisfactorily well to all the sets of sample data, which may be utilised to predict the risk of child loss in a family in any society. Findings may help planners and policy makers for designing proper policies and programs in a country.

1. Introduction

The force of mortality is still high at the younger ages particularly during the infancy [1] Level and trend of early childhood mortality indicate the standard of development of a country. A high rate of child mortality in a society indicates the reproductive wastages of physical, economical and psychological potential of a woman, and consequently shows a low level of success of a country's health program [2]. The child mortality has been of interest to researchers because of its apparent relationship with fertility and indirect relationship with the acceptance of modern contraceptive means [3]. The distribution of deaths with respect to age during the infancy is usually not governed by any single universal law because there exist a number of distinct patterns, which might be changed over time. It has been increasingly realised for several reasons that child loss from a family needs to be examined besides infant deaths [4]. The reported data of deaths during infancy and childhood suffer from substantial degree of errors where vital registration system of such information is not available [5]. Usually errors occur due to recall lapse, which result in omission of events, misplacement of dates and the distortion of reports on the duration of vital events [2, 5]. To overcome such limitations, models served the purpose and are needed to be developed to remove such variations due to these biases from one age to another. In fact, a model may smooth the data and provides a reasonable distribution of deaths according to age.

A number of attempts have been made to study the age pattern of mortality by using models [2, 6, 7, 8, 9, 10, 11]. Initially, Keyfitz [12] used a hyperbolic function to study the

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infant and child mortality. Choe [13] used Weibull function and Hartman [14] applied a logarithmic function for the same purpose. Krishnan and Jin [15] utilised Pareto distribution to describe the distribution of infant deaths and found a reasonable fitting. Chauhan [16] suggested finite range model to study the deaths under age one by months. Finite range model was proposed by Mukherjee and Islam [17] in reliability analysis. Krishnamoorthy and Rajna [18] checked the suitability and modified it for graduating the survivorship function. Bhuyan and Deogratias [19] used a modified Polya Aeppli model to study the pattern of child loss in Northeast Libya. These reflect an increasing application of probability models for describing the pattern of mortality.

An adequate research work on the pattern of child loss from a family has not been done yet in Nepal, which may be due to lack of interest among researchers or lack of reliable data. Child loss from a family has an important indicator for the well-being of the country, in general, and, in particular, well-being of the women health. Thus, this paper attempts to study the distributional pattern of families according to the number of child loss (under 5 years) through some probability models. Various sets of data from Nepal and other countries have also been used to discuss the applicability of the procedures proposed in this paper.

2. Models

As mentioned earlier, some probability distributions have been discussed this section to study the distribution of families according to the number of child deaths (deaths within the first five years of life). The assumption of using probability distributions is that only those families have been considered in which at least one birth prior to the survey date (study point) has had occurred.

2.1. Poisson Distribution

Let X denote the number of child deaths in a family at the survey point which follow a Poisson distribution. The probability mass function of X is

(1) $$p(x=k)=\frac{e^{-\lambda k}\lambda^{k}}{k!} \quad \text{for } x = 0, 1, 2, \ldots$$

when λ is the risk of death of a child in a family. The maximum likelihood equation of (1) is

(2) $$L=\prod_{k=0}^{n}[p(x=k)]=\prod_{k=0}^{n}\left[\frac{e^{-\lambda k}\lambda^{k}}{k!}\right]$$

Taking logarithms of (2) and differentiating with respect to λ and equating it to zero, we get

(3) $$\frac{\delta \log L}{\delta\lambda}=-n+\frac{n\bar{k}}{\lambda}=0$$

On solving (3), the estimates of λ can easily be obtained as

(4) $$\hat{\lambda}=\bar{k}$$

2.2. Geometric Distribution

Let X denote the number of child deaths in a family at the survey point which follows a geometric distribution. The probability mass function of X is

(5) $$p(x=k)=q^{k}p \quad \text{for } k = 0, 1, 2, 3, \ldots$$

It involves a single parameter p to be estimated from the observed distribution of families according to the number of child deaths. The likelihood function can be expressed as

$$L = \prod_{k=0}^{n} [p(x=k)] = \prod_{k=0}^{n} [pq^k] \tag{6}$$

$$L = p^n q^{\sum_{k=0}^{m} k} \tag{7}$$

Taking logarithms of (7) and differentiating with respect to p and equating it to zero, we get

$$\frac{\delta \log L}{\delta p} = \frac{n}{p} - \frac{n\bar{k}}{1-p} = 0 \tag{8}$$

On solving (8), the estimate of p can easily be obtained

$$\hat{p} = \frac{1}{1+\bar{k}} \tag{9}$$

2.3. A Mixture of Two Displaced Geometric Distribution

Let X denote the number of child deaths in a family at the survey point. The distribution of X is derived under some assumptions as (i) only those families are considered in which at least one birth prior to the survey has had occurred, (ii) at the survey point, a family either has experienced a child loss or not and let β and $(1-\beta)$ be the respective proportions, (iii) out of β proportion of families, let ξ be the proportion of families in which only one child death has occurred, and (iv) remaining $(1-\xi)\beta$ proportion of families, experiencing multiple child deaths, which follows a displaced geometric distribution with parameter p according to the number of child deaths.

Hence with these assumptions, the probability distribution of X is written as

$$\begin{aligned} p(x=k) &= 1-\beta && \text{for } k=0 \\ &= \xi\beta && \text{for } k=1 \\ &= (1-\xi)\beta p q^{k-2} && \text{for } k=2,3,\ldots \end{aligned} \tag{10}$$

Model (10) involves three parameters ξ, β and p to be estimated from the observed distribution of families according to the number of child deaths. Let $x_1, x_2, \ldots, x_N$ denote a random sample of size N from the population (10). Further, suppose that n_k ($k = 0, 1, 2, \ldots, m$) be the number of observations corresponding to the value of k and $\sum_{k=0}^{m} n_k = N$. The likelihood function for the given sample can be expressed as

$$L = \prod_{k=0}^{m} [p(x=k)]^{n_k} = (1-\beta)^{n_0}(\xi\beta)^{n_1} \prod_{k=2}^{m} [(1-\xi)\beta p q^{k-2}]^{n_k}$$

$$L = (1-\beta)^{n_0} \xi^{n_1} \beta^{n-n_0} (1-\xi)^{n-n_0-n_1} p^{n-n_0-n_1} q^{\sum_{k=3}^{m}(k-2)n_k} \tag{11}$$

Taking logarithm of (11) and differentiating with respect to β, ξ and p respectively and equating it to zero, we get

(12) $$\frac{\delta \log L}{\delta \beta} = -\frac{n_0}{1-\beta} + \frac{n-n_0}{1-\beta} = 0$$

(13) $$\frac{\delta \log L}{\delta \xi} = \frac{n_1}{\xi} - \frac{n-n_0-n_1}{1-\xi} = 0$$

(14) $$\frac{\delta \log L}{\delta p} = \frac{n-n_0-n_1}{p} - \frac{\sum_{k=3}^{m}(k-2)n_k}{1-p} = 0$$

From (12), (13) and (14), the estimates of β, ξ and p can easily be estimated as

$$\hat{\beta} = \frac{n-n_0}{n},\ \xi = \frac{n_1}{n-n_0} \quad \text{and} \quad \hat{p} = \frac{n-n_0-n_1}{(n-n_0-n_1) + \sum_{k=3}^{m}(k-2)n_k}$$

Variances and covariances of the estimates of the parameters are obtained by taking second partial derivatives of $\log L$ as

(15) $$\frac{\delta^2 \log L}{\delta \beta^2} = -\frac{n_0}{(1-\beta)^2} - \frac{n-n_0}{\beta^2}$$

(16) $$\frac{\delta^2 \log L}{\delta \xi^2} = -\frac{n_1}{\xi^2} - \frac{(n-n_0-n_1)}{(1-\xi)^2}$$

(17) $$\frac{\delta^2 \log L}{\delta p^2} = -\frac{(n-n_0-n_1)}{p^2} - \frac{\sum_{k=3}^{m}(k-2)n_k}{(1-p)^2}$$

(18) $$\frac{\delta^2 \log L}{\delta \beta \delta \xi} = \frac{\delta^2 \log L}{\delta \beta \delta p} = \frac{\delta^2 \log L}{\delta \xi \delta p} = 0$$

Taking the ract that $E(n_0) = E[\sum_{i=1}^{n} 1_{(X_i=0)}] = \sum_{i=1}^{n} 1\, p(X_i = 0) = \sum_{i=1}^{n} (1-\beta) = n(1-\beta)$, in similar way we can write $E(n_1) = n\xi\beta$, $E(n_k) = n(1-\xi)\beta p q^{k-2}$ for $k = 2, 3, \ldots, m.$, $E(n-n_0) = np$, $E(n-n_0-n_1) = n\beta(1-\xi)$ and

$$E[\sum_{i=3}^{m}(k-2)n_k] = E[n_3 + 2n_4 + 3n_5 + \ldots + (m-2)n_m]$$

$$= n(1-\xi)\beta pq\,[1 + 2q + 3q^2 + \ldots + (m-2)q^m]$$

$$= n(1-\xi)\beta q[\frac{1-q^{m-2}}{p} - (m-2)q^{m-2}], \text{ for small } m$$

$$= \frac{n(1-\theta)\beta q}{p}, \text{ for large } m,$$

Using above facts, the expected values of the second partial derivatives can be obtained as,

(20) $$-E(\frac{\delta^2 \log L}{\delta \beta^2}) = \frac{E(n_0)}{(1-\beta)^2} - \frac{E(n-n_0)}{\beta^2} = \frac{n}{\beta(1-\beta)} = \phi_{11} \quad \text{(say)}$$

(21) $$-E(\frac{\delta^2 \log L}{\delta \xi^2}) = \frac{E(n_1)}{\xi^2} + \frac{E(n-n_0-n_1)}{(1-\xi^2)} = \frac{n\beta}{\xi(1-\xi)} = \phi_{22} \quad \text{(say)}$$

(22) $$-E(\frac{\delta^2 \log L}{\delta p^2}) = \frac{E(n-n_0-n_1)}{p^2} + \frac{E[\sum_{k=3}^{m}(k-2)n_k]}{(1-p)^2}$$

$$= \frac{n\beta(1-\xi)q + n(1-\xi)\beta p[1-q^{m-2}-(m-2)pq^{m-2}]}{p^2 q} \text{ and}$$

$$= \phi_{33}(a) \quad \text{(say)}, \qquad \text{for small } m.$$

(23) $$-E(\frac{\delta^2 \log L}{\delta p^2}) = \frac{n\beta(1-\xi)}{p^2 q} = \phi_{33}(b) \text{ (say)}, \qquad \text{for large } m.$$

Since $E(\frac{\delta^2 \log L}{\delta\beta\delta\xi}) = E(\frac{\delta^2 \log L}{\delta\xi\delta p}) = E(\frac{\delta^2 \log L}{\delta\beta\delta p}) = 0$, so the covariances between the estimators becomes zero. Hence asymptotic variances of the estimators can be obtained as,

$$V(\hat{\beta}) = \frac{1}{\phi_{11}}, \quad V(\hat{\xi}) = \frac{1}{\phi_{22}}, \quad \text{and}$$

$$V(\hat{p}) = \frac{1}{\phi_{11}(a)} \text{ when } m \text{ is small}$$

$$= \frac{1}{\phi_{22}(b)} \text{ when } m \text{ is large.}$$

3. Applications

The proposed distributions have been applied to the real sets of sample survey data of Demographic Survey of Fertility and Mobility in Rural Nepal (DSFM): A Study of Palpa and Rupandehi Districts, which was carried out in 2000 [2]. Besides, one set of data has been taken from a household sample survey of Brazil in 1996 [19]. Another set of data from Northeast Libya conducted under a sample survey in 1987 [20].

The parameters of the proposed models have been estimated by the method of maximum likelihood. The observed and expected number of families (along with the estimates of the parameters) according to the number of child deaths is presented in Tables 1 to 3. It is seen that the Poisson distribution does not give a good fit to the data sets whereas the geometric distribution and displaced geometric distribution provided a good fit to all the data sets. The estimated value of β that shows the proportion of families experienced a child loss, was found slightly higher for Libya (0.36) as compared to Brazil (0.27) and Nepal (0.21). However, the proportion of families having a single child death was found higher for Nepal as compared to the data of other countries.

The average number of child deaths per family $[\hat{\xi}\hat{\beta} + (1-\hat{\xi})\hat{\beta}(1+\frac{1}{\hat{p}})]$ for Libya, Brazil and Nepal were found to be 0.53, 0.41 and 0.30 respectively. One of the applications of analyzing such data through model is to estimate the number of child deaths if family size is known. For example, if on an average total fertility rate (TFR) in Nepal is 4.1 (according Nepal Demographic Health Survey, 2001), then expected child mortality would be around 73 per

1000 live births (against the reported mortality rate 65 per 1000 live births) during the preceding (0.4) years from the survey date of NDHS. This is a close estimate of IMR (73) to observed IMR (65). The lower observed value of IMR of NDHS data may be due to under reporting in infant deaths in the survey or two different sources of data, which are not comparable [2]. Thus through such models if the size (TRR) of the family be known, child mortality may be predicted for a given society.

4. Conclusions

It was found that the proposed geometric distribution and displaced geometric distribution provided a suitable description of child deaths at micro level (family level). The distribution fitted satisfactorily well to several sets of sample data of Nepal, Libya and Brazil. So these distributions may be utilized to predict the risk of child deaths from a family in any society of the country. The findings may help planners and policy makers for designing policies and programs of a country.

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Table 1 Observed And Expected Number Of Families According To The Number Of Child Deaths In Nepal (2000)

No. of dead children	Observed No. of families	Expected No. of families		
		Poisson distribution	Geometric	Displaced geometric
0	669	623.76	649.65	669.24
1	137	195.24	154.87	137.06
2	32	34.00	36.92	28.54
3	6		11.55	10.45
4	3			5.74
5	2			
6	2			
7.	0			
Total	851	851.00	851.00	851.00
χ^2		24.21	4.327	2.591
d.f		1	2	1
		λ=0.2973	p=0.7616	β=0.2139 ξ=0.7527 p=0.6338
				v($\hat{\beta}$)=0.00020 v($\hat{\xi}$)=0.00102 v ($\hat{p}$)=0.00327

Table 2 Table 1 Observed And Expected Number Of Families According To The Number Of Child Deaths In North East Libya (1996)

No. of dead children	Observed No. of families	Expected No. of families		
		Poisson distribution	Geometric	Displaced geometric
0	805	740.19	820.67	805.79
1	306	389.04	282.73	306.17
2	93	102.24	97.41	94.26
3	36	20.53	33.56	31.27
4	7		17.39	9.38
5	2			5.41
6	1			
7	2			
Total	1252	1252.00	1252.00	1252.00
χ^2		60.99	4.261	1.367
d.f		2	3	2
		λ=0.5256	p=0.6555	β=0.3570 ξ=0.6846 p=0.6682
				$v(\hat{\beta})$=0.00018 $v(\hat{\xi})$=0.00048 $v(\hat{p})$=0.00106

Table 3 Table 1 Observed And Expected Number Of Families According To The Number Of Child Deaths In North East Brazil (1987)

No. of dead children	Observed No. of families	Expected No. of families		
		Poisson distribution	Geometric	Displaced geometric
0	769	698.12	745.85	769.40
1	185	285.60	216.55	185.07
2	60	58.42	62.87	63.61
3	26	7.97	18.26	21.88
4	9	8.85	7.42	11.05
5	1			
6	1			
7	0			
Total	1051	1051.00	1051.00	1051.00
χ^2		132.16	10.465	0.981
d.f		4	4	1
		λ=0.4091	p=0.7096	β=0.2683 ξ=0.6560 p=0.6554
				$v(\hat{\beta})$=0.00019 $v(\hat{\xi})$=0.00033 $v(\hat{p})$=0.00158

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Matrix maps of the classes $(S_r(p,\Delta), S\ell_\infty(p))$ through the connection of the classes $(S_r(p,\Delta),\ell_1)$, $(S\ell_\infty(p),c)$ and (ℓ_1,c)

KAMALMANI BARAL

Abstract: In tnis paper we deal on the matrix maps of the classes $(S_r(p,\Delta), S\ell_\infty(p))$ thro' the connection of the classes. $(S_r(p,\Delta),\ell_1)$, (ℓ_1,c), and $(S\ell_\infty(p),c)$.

Keywords: Sequence spaces, difference sequence spaces, matrix maps, duals, weighted mean matrix.

AMS Subject Classification: 40C10, 40D25, 40GO5, 40H05, 46A45, 40C05.

A1.0 Introduction

In [6] A.K. Gaur and Mursaleen has defined the sequence space.

$$S_r(\Delta)=\{x=(x_k):(k^r\,|\Delta x_k|)_{k=1}^{\infty}\,\varepsilon\, c_0\}\ \text{ for } (r\ge 1)$$

is studied. The authors have generalized this space $S_r(\Delta)$ to $S_r(p,\Delta)$ for a sequence of strictly positive reals. They have also characterized the matrix maps in the classes $(S_r(p,\Delta),\ell_\infty)$ and $(S_r(p,\Delta),\ell_1)$. Since ℓ_∞, c, c_0, be the sets of all bounded, convergent and null sequences of $x=(x_k)_1^\infty$ respectively and ω denote the set of all complex sequences and ℓ_1 denote the set of all convergent and absolutely convergent series. If z be any sequence and Y be any sub set of ω then

$$z^{-1}Y=\{x\,\varepsilon\,\omega : zx=(z_k x_k)_1^\infty\,\varepsilon\, Y\}$$

For any sub set X of ω the sets

$$X^\alpha=\bigcap_{x\varepsilon X}(x^{-1},\ell_1)\ \text{and}\ X^\beta=\bigcap_{x\varepsilon X}(x^{-1},\ell_s)$$

are called the α-and β-duals of X.

The authors define the linear operator Δ,

$$\Delta^{-1}:\omega\to\omega \text{ by}$$

$$\Delta x=(\Delta x_k)_1^\infty=(x_k-x_{k+1})_1^\infty$$

and
$$\Delta^{-1}x=(\Delta^{-1}x_k)_1^\infty=\left(\sum_{j=1}^{k-1}x_j\right)_1^\infty$$

such that
$$\Delta^{-1}x_1=0$$

Let $S_r(\Delta) = \{x \varepsilon \omega : (k^r \mid \Delta x_k \mid)_{k=1}^{\infty} \varepsilon\, c_0\}$ see [1]. The authors Gaur and Mursaleen have extended the space $S_r(\Delta)\, \ell_\infty\, S_r(p,\Delta)$ in the same manner as c_0, c, ℓ_∞ were extended to $c_0(p)$, $c(p)$, $\ell_\infty(p)$ respectively (cf.[11],[10],[23]). They have determined the α-and β-duals of their new sequence space. Let $p = (p_k)_1^{\infty}$ be any arbitrary sequences of positive reals for $r \geq 1$ then they have defined.

$$S_r(p,\Delta) = \{x \varepsilon \omega : (k^r \Delta x_k)_1^{\infty} \varepsilon\, c_0(p)\}$$

Where $c_0(p)\{x \varepsilon \omega : \lim_{k\to\infty} |x_k|^{p_k} = 0\}$

If $p = e = (1,1,1 \ldots)$ then the set $S_r(p,\Delta)$ reduces to the set $S_r(\Delta)$. For $r = 0$, $S_r(p,\Delta)$ is the same as $\Delta c_0(p)$ (cf. [1], [9], [16]).

We will need the following two lemmas.

Lemma A.1 (Corollary 1 in [16]). *Let $(p_n)_{n=1}^{\infty}$ be a sequence of non-decreasing positive reals. Then $a \varepsilon (p_n)^{-1}, c_s$ implies $R = (R_n), \varepsilon (p_n)^{-1}, c_0$ where*

$$R_n = \sum_{k=n+1}^{\infty} a_k \quad (n = 1,2,\ldots).$$

Lemma A.2 (Lemma 1(b) in [17]). *Let $p = (p_k)_{k=1}^{\infty}$ be a strictly positive sequences such that $p \varepsilon \ell_\infty$ then $A \varepsilon (c_0(p), \ell_1)$ iff*

$$(*) \qquad B(M) = \sup_{\substack{N \subset \mathbb{N} \\ \mathbb{N}\text{ finite}}} \left(\sum_{k=1}^{\infty} \Big| \sum_{n \varepsilon \mathbb{N}} a_{nk} \Big| M^{-\frac{1}{p_k}} \right) < \infty$$

Where the right $\mathbb{N}$ represents the positive integers for some integer $M \geq 2$

A.2 The α-and β-duals of $S_r(p, \Delta)$

Theorem A(2.1). *Let $p = (p_k)_1^{\infty}$ be a strictly positive sequences and $r \geq 1$ then*

a) $[S_r(p,\Delta)]^{\alpha} = \bigcup_{N>1} D_r^{(1)}(p)$,

b) $[S_r(p,\Delta)]^{\beta} = C_r(p) = \bigcap_{v \varepsilon c_0^+} D_r^{(2)}(p) \cap \bigcup_{N>1} D_r^{(3)}(p)$,

Where, $D_r^{(1)}(p) := \left(\Delta_r^{-1} N^{-\frac{1}{p}} \right)^{-1} . \ell_1 = \{a \varepsilon \omega : \sum_{k=1}^{\infty} |a_k| \left| \sum_{j=1}^{k-1} \frac{N^{-1/p_j}}{j^r} \right| < \infty\}$,

$$D_r^{(2)}(p) = (\Delta r^{-1} v^{1/p})^{-1}, c_s = \{a \varepsilon \omega : \sum_{k=1}^{\infty} a_k \sum_{j=1}^{k-1} \frac{V_j^{1/p_j}}{j^r} \text{ converges}\},$$

$$D_r^{(3)}(p) = \{a\, \varepsilon\, \omega : R\, \varepsilon \left(\frac{N^{-1/p}}{k^r} \right)^{-1} . \ell_1\} = \{a \varepsilon \omega : \sum_{k=1}^{\infty} |\bar{R}_k| \frac{N^{-1/p_k}}{k^r} < \infty\},$$

$$\Delta_r x = (k^r \Delta x_k)_1^{\infty}, \ \Delta r^{-1} x = (k^r \Delta^{-1} x_k)_{k=1}^{\infty}$$

and c_0^+ is the set of all positive sequences is c_0.

A 3. 0 Matrix transformations

For an infinite complex matrix $A=(a_{nk})_{n,k=1}^{\infty}$ we write $A_n=(a_{nk})_{k=1}^{\infty}$ for the sequence in the n^{th} row of A. Let X and Y be two subsets of ω. By (X,Y) we denote the class of all matrices A such that the series $A_n(x)=\sum_{k=1}^{\infty} a_{nk}x_k$ converges for all $x \,\varepsilon\, X$ and each $n \,\varepsilon\, \mathbb{N}$(positive integer) and the sequence $A_x=(A_n(x))_{n=1}^{\infty} \,\varepsilon\, Y$ for all $x \,\varepsilon\, X$.

Theorem A 3.1 *Let* $p=(p_k)_1^{\infty}$ *be a strictly positive sequence and* $r \geq 1$. *Then* $A \,\varepsilon\, (S_r(p,\Delta),\ell_\infty)$ *iff*

i) $$D_r(v)=\sup_n \left| A_n(\Delta_r^{-1}v^{1/p})\right|$$

$$=\sup_n \left| \sum_{k=1}^{\infty} a_{nk}\sum_{J=1}^{k-1}\frac{v^{1/p_j}}{j^r}\right| < \infty \text{ for all } v \,\varepsilon\, c_0^+,$$

ii) $$D_r(M):=\sup_n \left(\sum_{k=1}^{\infty}|R_{nk}|\frac{M^{1/p_k}}{k^r}\right)<\infty \text{ for some integer } M \geq 2,$$

Where, $R_{nk}=\sum_{j=k+1}^{\infty} a_{nk}$ for all n and k, and

iii) $$D_\infty=\sup_n |A_n(e)|=\sup_n \left|\sum_{k=1}^{\infty} a_{nk}\right|<\infty.$$

Theorem A.3.2 *Let* $p=(p_k)_1^{\infty}$ *be a strictly sequence such that* $p \,\varepsilon\, \ell_\infty$ *and* $r \geq 1$. *Then* $A \,\varepsilon\, (S_r(p,\Delta),\ell_\infty)$ *iff*

i) $$C_r^{(1)}(v):=\sup_{\substack{N\subset \mathbb{N}\\ N \text{ finite}}}\left|\sum_{n\,\varepsilon\, N} A_n(\Delta_r^{-1}v^{1/p})\right|$$

$$=\sup_{\substack{N\subset \mathbb{N}\\ N \text{ finite}}}\left|\sum_{n\,\varepsilon\, N}\sum_{k=1}^{\infty} a_{nk}\sum_{j=1}^{k-1}\frac{v^{1/p_j}}{j^r}\right|<\infty$$

where the right $\mathbb{N}$ *is a positive integer for all sequences* $v \,\varepsilon\, C_0^+$,

ii) $$C_r^{(2)}(M):=\sup_{\substack{N\subset \mathbb{N}\\ N \text{ finite}}}\left(\sum_{k=1}^{\infty}\sum_{n\,\varepsilon\, N}|R_{nk}|\frac{M^{\frac{1}{p_k}}}{k^r}\right)<\infty, \text{ for some integer } M\geq 2 \text{ and}$$

iii) $$D_r^{(3)}=\sup_{\substack{N\subset \mathbb{N}\\ N \text{ finite}}}\left|\sum_{n\,\varepsilon\, N} A_n(e)\right|\leq\infty.$$

Note that the right $\mathbb{N}$ is a positive integer.

These are the works done by A. K. Gaur and Mursaleen in their respective paper [6].

Now we would like to go in the matrix maps of the classes (ℓ_1, c) accomplished by Jinlu Li.

In [8] Jinlu Li has characterized the matrix maps in the classes (ℓ_1, c) which is treated as general weighted mean summability methods (GWMSM) in which the results include a classical result by Hardy and another by Moricz and Rhoades as particular cases. Jinlu Li has treated as follows.

$$A \text{ series } \sum_{k=0}^{\infty} x_k \tag{B1.1}$$

of complex numbers is said to be summable (C,1) if the sequence

$$\frac{1}{n+1}\sum_{k=0}^{n}\sum_{i=1}^{k} x_i \quad (n=0,1,2,\ldots) \tag{B 1.2}$$

converges to a finite limit as $n \to \infty$.

In [7] Hardy proved the following theorem.

Theorem (B 1.1) *The series (B.1.1) is summable (C, 1) to a finite number L iff the series*

$$\sum_{n=0}^{c}\sum_{k=n}^{\infty}\frac{x_k}{n+1} \tag{B 1.3}$$

converges to the same limit.

For a sequence of positive numbers (p_n). Let $p_n = \sum_{k=0}^{n} p_n$. A weighted mean matrix $\overline{N}$ is an infinity lower triangular matrix with entries (see [20]).

$$a_{nk} := \frac{P_k}{P_n}, \quad (k = 0, 1,2, \ldots, n = 0, 1,2, \ldots,) \tag{B.1.4}$$

The series (B1.1) is said to be summable $\overline{N}$ if the following sequence:

$$\frac{1}{p_n}\sum_{k=0}^{\infty} P_k \sum_{i=0}^{k} x_i, \quad (n=0,1,2,\ldots,) \tag{B.1.5}$$

converges to a finite limit as $n \to \infty$

It is clear that summable (C.1) is a special case of summable $\overline{N}$ where

$$P_k = 1, \quad (k=0,1,2, \ldots) \tag{B.1.6}$$

Based on the above idea, Moricz and Rhoades (2) established a result for a broad class of summability methods, which include the method of summability (C.1) as a particular case.

Theorem B.1.2. *Let $\overline{N}$ be the weighted mean matrix determined by a sequence (P_n) of positive numbers such that the following conditions are satisfied:*

i) $\quad p_n \to \infty, \dfrac{P_n}{p_n} \to 0$ as $n \to \infty$

ii) $$\sup_{n\geq 0}\left\{\frac{P_{n+1}\,P_{n-1}}{p_n\,p_{n+1}} + P_n \sum_{k=n}^{\infty}\frac{1}{p_{n+1}}\left|\frac{P_{k+1}}{P_k} = \frac{P_{k+2}\,P_k}{P_{k+1}\,p_{k+2}}\right|\right\} \leq \infty, \tag{B.1.7}$$

iii) $$\sup_{n\geq 0}\left\{\frac{P_n}{P_{n+1}} + \frac{1}{P_n}\sum_{k=0}^{n}\left|\frac{P_k\,p_{k+1}}{P_{k+1}} - \frac{P_{k-1}\,P_{k-1}}{P_k}\right|\right\} < \infty,$$

with the agreement that

$$P_{-1} = p_{-1} = 0 \tag{B.1.8}$$

Then the series (B.1.1) is summable $\overline{N}$ to a finite number L iff the series

$$\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{P_n}{p_k}x_k \tag{B.1.9}$$

converges to the same limit L.

In this paper [8] the author is studied the matrix transformations from the space of absolutely convergent series of complex numbers ℓ_1 to the space of convergent sequence of complex number c that is in the classes (ℓ_1, c). Further the author have established a more general result for a broader class of weighted mean methods, which includes the method of summable $\overline{N}$ as a particular case if the series (B.1.1) is absolutely convergent.

B.2.0 Matrix transformations from ℓ_1 to c

Let $A = (a_{nk})$ be an infinite matrix with complex entries and let ℓ denote the linear space of complex numbers sequences. For a sequence $x = (x_n)\,\varepsilon\,\ell$, A_x is in ℓ and its entries are given by

$$(A_x)n = \sum_{k=0}^{\infty} a_{nk}\,x_k,\ (n = 0,1,2,\ldots,) \tag{B.2.1}$$

provided the series converges to a finite complex number. The following result is well known (cf. [25], [27]), we list it as a proposition.

Proposition (B.2.1) *Let $a = (a_k)$ be a sequence of complex numbers. It for every $x = (x_n)\,\varepsilon\,\ell_1$ the series*

$$\sum_{k=0}^{\infty} a_k x_k \tag{B.2.2}$$

converges to a finite complex number. Then

$$\sup_{k\geq 0}\ \{|\,a_k\,|\} < \infty \tag{B.2.3}$$

from the proposition (B.2.1) we have the following interesting result.

Proposition (B.2.2) *Let $a = (a_k)$ be a sequence of complex numbers. If for every $x = (x_n)\,\varepsilon\,\ell_1$, the series*

$$\sum_{k=0}^{\infty} a_k x_k \tag{B 2.4}$$

converges to a finite complex number, then the linear functional f_a defined on ℓ_1 by

$$f_a(x) = \sum_{k=0}^{\infty} a_k x_k \tag{B.2.5}$$

is continuous (bounded) linear functional on ℓ_1, such that

$$\|\,f_a\| = \sup_{k\geq 0}\ \{|\,a_k\,|\} \tag{B 2.6}$$

From proposition B2.1, we know that A is well defined as a mapping from ℓ_1 to ℓ, iff

$$\sup_{k\geq 0} \{|a_{nk}|\} < \infty, \quad \text{for } n = 0,1,2 \ldots \tag{B 2.7}$$

The following result has been proved in [26] by using functional analysis techniques. It is also provided by summability methods. We list the following theorem without proof.

Theorem (B 2.3) Let $A = (a_{nk})$ *be an infinity matrix with complex entries. Then A is a mapping from* ℓ_1 *to c iff the following conditions are satisfied that is* $A \varepsilon (\ell_1, c)$ *iff*

i) for every fixed $k = 0,1,2\ldots$ the sequence (a_{nk}) converges to a finite limit as $n \to \infty$.

ii) $\sup_{n,k\geq 0} \{|a_{nk}|\} < \infty$.

Furthermore, if $A = (a_{nk})$ satisfies the conditions (i) and (ii), then for every $x = (x_n) \varepsilon \ell_1$, we have

$$\lim_{n\to\infty} (Ax)_n = \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} x_k = \sum_{k=0}^{\infty} \left(\lim_{n\to\infty} a_{nk}\right) x_k \tag{B 2.8}$$

The following corollary follows from theorems (B 2.3) and (B 2.8).

Corollary (B 2.4) *Let* $A = (a_{nk})$ *be an infinity matrix with complex entries. If A is a mapping from* ℓ_1 *to c, then the linear operator A is continuous (bounded) linear operator such that*

$$\|A\| = \sup_{n,k\geq 0} \{|a_{nk}|\} \tag{B 2.9}$$

B 3.0: Applications to summable $(C,1)$ and summable $\overline{N}$

The following corollary comes immediately from Theorem (B2.3) which describes an equivalent reformulation of summability by more general weighted mean methods which are matrix transformations.

Corollary (B 3.1) *Let* $A = (a_{nk})$ *and* $B = (b_{nk})$ *be two infinity matrices with complex entries. Suppose A, B are mappings from* ℓ_1 *to c, that is A, B satisfying conditions (i), (ii) of theorem (B 2.3). Then for every* $x = (x_n) \varepsilon \ell_1$,

$$\lim_{n\to\infty} (Ax)_n = \lim_{n\to\infty} (Bx)_n \tag{B 3.1}$$

iff for every fixed $k = 0, 1, 2 \ldots$,

$$\lim_{n\to\infty} a_{nk} = \lim_{n\to\infty} b_{nk} \tag{B 3.2}$$

Hence we have the following corollary of theorem (B 2.3)

Corollary (B 3.2) *For any sequence of positive numbers* $(P_n), B = (b_{nk})$ *defined by*

$$a_{nk} = \begin{cases} 0 & \text{if } k > n \\ \dfrac{p_n - p_{k-1}}{p_n} & \text{if } k \leq n \end{cases} \tag{B 3.9}$$

$$b_{nk} = \begin{cases} \dfrac{P_n}{P_k} & \text{if } k > n \\ 1 & \text{if } k \leq n \end{cases} \tag{B 3.10}$$

(B 3.10) is always a mapping from ℓ_1 to c. If (P_n) satisfying

$$P_n \to \infty \text{ as } n \to \infty \tag{B 3.11}$$

then $A = (a_{nk})$ defined by (B 3.9) is a mapping from ℓ_1 to c.

The following corollary will give the Moricz and Rhoades's result, theorem (B 1.2), if the series (B 1.1) is absolutely convergent.

Corollary (B 3.3) *Let (P_n) be a sequence of positive numbers satisfying (B 3.11). Let $A = (a_{nk})$, $B = (b_{nk})$ be defined by (B 3.9) and (B 3.10). Then for every $x = (x_n) \varepsilon \ell_1$ we have*

$$\lim_{n\to\infty} (Ax)_n = \lim_{n\to\infty} (Bx)_n = \sum_{k=0}^{\infty} x_k \tag{B 3.12}$$

These are the works done by Jinlu Li in his respective paper [8].

Now we would like to go in the matirx maps of the classes $(S\ell_\infty(p), c)$ and $(S\ell_\infty(p), c_s)$ accomblished by S.K. Mishra and K.M. Baral. In [3], B. Choudhary and S.K. Mishra have defined the sequence space $S\ell_\infty(p)$, $Sc(p)$ and $Sc_0(p)$ and determined the Köthe-Toeplitz (α-dual) of $S\ell_\infty(p)$ and characterized the matrix maps in the classes $S\ell_\infty(p)$ to ℓ_∞ and c.

In [18], Mishra, S.K. and Baral K.M. have determined the β-duals of $S\ell_\infty(p)$, $Sc_0(p)$ and characterized the matrix maps in the classes $(S\ell_\infty(p), c)$ and $(S\ell_\infty(p), c_s)$.

The β-duals of $S\ell_\infty(p)$ and $Sc_0(p)$ determined by S.K Mishra and K.M. Baral are listed in the following theorems.

Theorem C(1.0) *Let $p_k > 0$, for every k, then*

$$[S\ell_\infty(p)]^\beta = \bigcap_{m=1}^{\infty} \left\{ a = (a_k) : \left[\sum_{k=1}^{\infty} N^{1/p_m} \right] \text{ converges and } \sum_{k=1}^{\infty} N^{\frac{1}{p_k}} |R_k| < \infty, N > 1 \right\}$$

Where $R_k = \sum_{v=k}^{\infty} a_v$

(We assume that $\sum_{m=1}^{k} z_m = 0(k > 1)$)

Theorem C (1.1) *Let $p_k > 0$ for every k then*

$[Sc_0(p)]^\beta = SM_0(p)$ where

$$SM_0(p) = \bigcup_{N>1} \left\{ a = (a_k) : \sum_{k=1}^{\infty} a_k \left[\sum_{m=1}^{k} N^{-\frac{1}{p_m}} \right] \text{ converges } \& \sum_{k=1}^{\infty} |R_k| N^{-\frac{1}{p_k}} < \infty, N > 1 \right\}$$

The characterizations of the matrix maps of the classes $(S\ell_\infty(p), c)$ and $(S\ell_\infty(p), c_s)$ determined by Mishra S.K and Baral, K.M, are compiled in the following theorems.

Theorem C(1.2) *Let $p_k > 0$, for every k then $A \varepsilon (S\ell_\infty(p), c)$ iff*

(i) $R\varepsilon(\ell_\infty(p),c)$

(ii) An $\left[\sum_{i=1}^{k} N^{1/p_i}\right] \varepsilon\, c\ (n,k=1,2,3,\ldots)$ *for all integers* $N>1$ *and*

(iii) $\lim_{n\to\infty} a_{n,k} = \alpha_k (k=1,2,3,\ldots)$

where $R=r_{nk}=\left[\sum_{v=k}^{\infty} a_{n,v}\right]\ (n,k=1,2,3,\ldots)$

Theorem C (1.3). *Let* $p_k>0$ *for every* k *then* $A\varepsilon(S\ell_\infty(p),c)$ *iff*

(i) $C\,\varepsilon(S\ell_\infty(p),c_s)$

(ii) $B_n\left[\sum_{i=1}^{k} N^{1/p_i}\right] \varepsilon\, c_s\ (n,k=1,2,3,\ldots,)\ (N>1)$

(iii) $\lim_{n\to\infty} b_{nk} = \lim_{n\to\infty} \sum_{i=1}^{n} a_{ik} = B_k\ \ (k=1,2,3,\ldots)$

where $C=C_{nk}=\left\{\sum_{i=1}^{n}\left[\sum_{v=k}^{\infty} a_{nv}\right]\right\}\ (n,k=1,2,3,\ldots)$

There are the works done by Mishra S.K. and Baral K.M. is their respective paper [18]. Now we would like to enter in the matrix maps in the classes $(S_r(p,\Delta),S\ell_\infty(p))$ through the connection of the classes $(S_r(p,\Delta),\ell_1),(\ell_1,c)$ and $(S\ell_\infty(p),c)$.

Theorem (4.0) $A\varepsilon(S_r(p,\Delta),S\ell_\infty(p))$ *iff*

(A) $$C_r^{(1)}(v) := \sup_{\substack{N\subset \mathbb{N}\\ N \text{ finite}}} \left|\sum_{n\varepsilon N} A_n\left(\Delta_r^{-1} v^{1/p}\right)\right| = \sup_{\substack{N\subset \mathbb{N}\\ N \text{ finite}}} \left|\sum_{n\varepsilon N}\sum_{k=1}^{\infty} a_{nk} \sum_{j=1}^{k-1} \frac{v^{1/p_j}}{j^r}\right| < \infty$$

where the right N *is a positive integer for all sequences* $v\,\varepsilon\, C_0^+$.

(B) $$C_r^{(2)}(M) := \sup_{\substack{N\subset \mathbb{N}\\ N \text{ finite}}} \left(\sum_{k=1}^{\infty}\sum_{n\varepsilon N} |R_{nk}|\, \frac{M^{-\frac{1}{p_k}}}{k^r}\right) < \infty$$

for some integer $M\geq 2$ and

(C) $$D_r^{(3)}(M) = \sup_{\substack{N\subset \mathbb{N}\\ N \text{ finite}}} \left|\sum_{n\varepsilon N} A_n(e)\right| \leq \infty.$$

(D) For every fixed $k=0,1,2\ldots$ the sequence (a_{nk}) converges to a finite limit as $n\to\infty$.

(E) $\sup_{n,k\geq 0} \{|a_{nk}|\} < \infty$.

(F) $R\varepsilon\,(\ell_\infty(p),c)$

(G) An $\left[\sum_{i=1}^{k} N^{1/p_i}\right] \varepsilon\, c\ (n,k=1,2,3,\ldots)$ for all integers $N>1$ and

(H) $\lim_{n\to\infty} a_{n,k} = \alpha_k\,(k=1,2,3,\ldots)$

where $R = r_{nk} = \left[\sum_{v=k}^{\infty} a_{n,v}\right]\ (n,k=1,2,3,\ldots)$

Note that the right $\mathbb{N}$ represents the set of positive integers for all sequences $v\varepsilon C_0^+$ and the characterizations D and H are same.

These characterizations are the combinations of the theorems (A 3.2), (B 2.3) and (C 1.2) which are seen the required characterizations of the classes $(S_r(p,\Delta),S\ell_\infty(c))$.

It is seen that the characterization of the classes through the connections of the different classes seems slightly different but they must be compatible (reducible) to each other and same approach holds for the matrix maps in the cycle classes such as $(S_r(p,\Delta),\ell_1)$, (ℓ_1,c), $(c,S\ell_\infty(p))$, $(S\ell_\infty(p),S_r(p,\Delta))$.

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Uniform version of Wiener-Tauberian theorem for equicontinuous subsets of subspaces of $L^1(X,\mu)$

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Abstract: The Wiener-Tauberian theorem for **R** says that the closed translation invariant subspace generated by an $f \in L^1(\mathbf{R})$ is $L^1(\mathbf{R})$ if and only if the Fourier transform $\hat{f}$ of f never vanishes. In this paper we consider Banach subspace of $L^1(X, m)$ and prove the uniform version of the result for $L^1(X)$ and Segal algebra $S(X)$ on hypergroup X, where X is locally compact hypergroup possessing Haar measure m.

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1. Introduction:

Let X be a locally compact Hausdorff topological space. Suppose that there is a continuous map $x \to \tilde{x}$ from X into X such that $(\tilde{x})^{\sim} = x$. Let μ be a regular Borel measure on X such that supp $\mu = X$.

Let $(B, \|\cdot\|_B)$ be a Banach space of functions on X contained in $L^1(X,\mu)$ satisfying $\|\cdot\|_B \geq \|\cdot\|_1$. Suppose that there is a linear isometric map $f \to f^*$ from B into B such that $(f^*)^* = f$. Let there be maps $\sigma, \tau : X \to L(B, B)$ satisfying

(B.1) $\|\sigma(x)\| \leq C$, $\|\tau(x)\| \leq C$ for all $x \in X$ and some $C \geq 1$.

(B.2) there exists $e \in X$ such that $\sigma(e) = I$

For $\phi \in B^*$, the dual space of B, we define $\phi^*(f) = \phi(f^*)$. It is clear that $\phi^* \in B^*$ and $\|\phi^*\| = \|\phi\|$. The maps σ and τ induces σ^* and $\tau^*: X \to L(B^*, B^*)$ defined by $\sigma^*(x)\phi(f) = \phi(\sigma(x)f)$ and $\tau^*(x)\phi(f) = \phi((\tau(x)f)$. It is clear that $\|\sigma^*(x)\| \leq C$, $\|\tau^*(x)\| \leq C$ and $\sigma^*(e) = 1$.

For $\phi \in B^*$ and $f \in B$, we define $f \odot \phi$ and $\phi \odot f$ by $f \odot \phi(x) = \phi^*(\sigma(x)f)$ and $f \odot \phi(x) = \phi(\sigma(\tilde{x})f^*)$.

Lemma 1.1. Let B be a Banach subspace of $L^1(X,\mu)$ satisfying (B.1)–(B.2). Suppose that the measure μ satisfy

(M.1.) The function $x \to f \odot \phi(x)$ and $x \to \phi \odot f(x)$ are measurable.

(M.2.) For each $f \in B^*$ and $f, g \in B$, we have

$$\int_X \phi(\sigma(\widetilde{x})f)\, g^*(x)\, d\mu(x) = \int_X \phi(\sigma(x)f)\, g(x)\, d\mu(x).$$

(M.3.) For each $f \in B$, $\phi \in B^*$ and $x \in X$, we have

$$(\tau^*(x)\phi)^*(f) = \phi^*(\sigma(\widetilde{x})f)$$

(M.4.) For each $\phi \in B^*$; $f, g \in B$ and $x \in X$, we have

$$\int_X \phi^*(\sigma(\widetilde{y})g)\, \sigma(x) f(y)\, d\mu(y)$$
$$= \int_X g(y)(\sigma^*(y)\phi)^*(\sigma(x)f)\, d\mu(y).$$

Then, we have, for $f, g \in B$, $\phi \in B^*$ and $x \in X$

(i) $f \circledcirc \phi,\ \phi \circledcirc f \in L^\infty(X, \mu) \subset B^*$

(ii) $(f \circledcirc \phi)^* = \phi^* \circledcirc f^*$

(iii) $f \circledcirc (\tau^*(x)\phi)(e) = f \circledcirc \phi(\widetilde{x})$

(iii) $f \circledcirc (g \circledcirc \phi)(x) = \int_X g(y)(f \circledcirc \sigma^*(y)\phi)(x)\, d\mu(y)$

Proof: The proof of (i) and (ii) are same as in ([3], Lemma 3.1). For (iii)

$$(f \circledcirc (\tau^*(x)\phi))(e) = (\tau^*(x)\phi)^*(\sigma(e)f)$$
$$= (\tau^*(x)\phi)^*(f) = \phi^*(\sigma(\widetilde{x})f) \quad \text{(using M.3)}$$
$$= f \circledcirc \phi(\widetilde{x}).$$

For (iv)

$$f \circledcirc (g \circledcirc \phi)(x) = \phi^* \circledcirc g^*(\sigma(x)f) = \int_X \phi^* \circledcirc g^*(y)(\sigma(x)f)(y)\, d\mu(y)$$
$$= \int_X \phi^*(\sigma(\widetilde{y})g)(\sigma(x)f)(y)\, d\mu(y)$$
$$= \int_X g(y)(\sigma^*(y)\phi)^*(\sigma(x)f)\, d\mu(y)$$
$$= \int_X g(y)(f \circledcirc \sigma^*(y)\phi)(x)\, d\mu(y). \qquad \square$$

Theorem 1.2. *Let X be a separable locally compact Hausdorff topological space. Suppose that B is a Banach space of functions on X satisfying (B.1.)–(B.2.) and μ a measure satisfying (M.1.)–(M.4.). Let $\mathcal{H} \subset B$ be such that $\{\Phi_h : h \in \mathcal{H}\}$ is uniformly equicontinuous. Suppose that there exists $h_0 \in S_1^B$ such that $|h(t)| \le |h_0(t)|$ and $\|h\|_B \le \|h_0\|_B$ for all $h \in \mathcal{H}$ and $t \in X$. Let $\mathcal{U} \subset S_1^{B^*}$ be such that $\tau^*(x)\phi \in \mathcal{U}$ for all $x \in X$ and $\phi \in \mathcal{U}$. If $g \in S_1^B \cap U$ and for any $x, y \in X$, $g \circledcirc \sigma^*(x)\tau^*(y)\phi$ vanishes at infinity for ϕ in $\mathcal{U}$ then $h \circledcirc \phi$ vanishes at infinity for ϕ in $\mathcal{U}$ and h in $\mathcal{H}$.*

Proof: Assume to the contrary that there exists $\delta > 0$ such that for every compact set K in X there exists $x_K \in X \sim K$, $h_K \in \mathcal{H}$ and $\phi_K \in \mathcal{U}$ satisfying $|h_K \circledcirc \phi_K(x_K)| > \delta$.

Since X is separable and locally compact so X is σ-compact. Thus there exists an increasing sequence $\{K_n\}_{n \in \mathbb{N}}$ of compact set with $K_n \subset \operatorname{int} K_{n+1}$ and for F any compact

subset in X there exists n_0 with $F \subset K_{n_0}$. Write $h_{K_n} = h_n$, $\phi_{K_n} = \phi_n$ and $x_{K_n} = x_n$. We define a sequence of functions on X by

$$s_n(x) = (h_n \circledcirc \tau^*(\widetilde{x}_n)\,\phi_n)(x)$$

$$\begin{aligned} |s_n(x)| &= |(\tau^*(\widetilde{x}_n)\phi_n)^*(\sigma(x)h_n| \\ &\leq \|(\tau^*(\widetilde{x}_n)\phi_n)^*\|_{B^*}\,\|(\sigma(x)h_n\|_B \\ &\leq C^2\|\phi_n\|_{B^*}\,\|h_n\|_B \leq C^2 \end{aligned}$$

Therefore $s_n \in L^\infty \subset B^*$.

Since $x \to \Phi_h(x)$ is uniformly equicontinuous so for given $\in > 0$ there exists a neighbourhood U_x of x in X such that for $y \in U_x$

$$\|\Phi_h(x) - \Phi_h(y)\|_B < \in/C \text{ for all } h \in \mathcal{H}.$$

Thus for $y \in U_x$, we have

$$\begin{aligned} |s_n(x) - s_n(y)| &= |(\tau^*(\widetilde{x}_n)\,\phi_n)^*(\sigma(x)h_n - (\sigma(y)h_n)| \\ &\leq \|(\tau^*(\widetilde{x}_n)\,\phi_n\|_{B^*}\,\|(\sigma(x)h_n - (\sigma(y)h_n)\|_B \\ &\leq C\|\phi_n\|_{B^*}\,\|\Phi_{h_n}(x)\| - \Phi_{h_n}(y)\|_B \\ &\leq C\|\Phi_{h_n}(x) - \Phi_{h_n}(y\|_B < \in. \end{aligned}$$

By Ascoli's theorem ([2], Theorem 1.3.2) there exists a pointwise convergent subsequence $\{Sn_j\}$ converging to a continuous function s on X. Thus for fixed x,y in X

$$(\sigma(x)g)^*(y)\,s_{n_j}(y) \to (\sigma(x)g)^*(y)\,s(y) \text{ as } j \to \infty$$

also
$$|(\sigma(x)g)^*(y)\,s_{n_j}(y)| \leq C^2|(\sigma(x)g)^*(y)|$$

and $(\sigma(x)g)^* \in B \subset L^1(X,\mu)$, so by Lebesgue dominated convergence theorem

$$\int_X (\sigma(x)g)^*(y)\,s_{n_j}(y)\,d\mu(y) \to \int_X (\sigma(x)g)^*(y)\,s(y)\,d\mu(y) \text{ as } j \to \infty$$

$$\Rightarrow \quad g \circledcirc s_{n_j}(x) \to g \circledcirc s(x) \text{ as } j \to \infty.$$

But
$$\begin{aligned} g \circledcirc s_{n_j}(x) &= (g \circledcirc (h_{n_j} \circledcirc \tau^*(\widetilde{x}_{n_j})\,\phi_{n_j}))(x) \\ &= \int_X h_{n_j}(y)\,g \circledcirc \sigma^*(y)\,\tau^*(\widetilde{x}_{n_j})\,\phi_{n_j}(x)\,d\mu(y) \\ &= \int_X V^x_{n_j}(y)\,d\mu(y) \quad \text{(using Lemma 1.1(iv))} \end{aligned}$$

Since $g \circledcirc \sigma^*(y)\,\tau^*(x)\,\phi$ vanishes at infinity uniformly for $\phi \in \mathcal{U}$ so there exists a compact set K_k such that

$$|g \circledcirc \sigma^*(y)\,\tau^*(\widetilde{x}_{n_j})\,\phi_{n_j}(x)| < \tfrac{1}{k}$$

whenever $x \notin K_k$

$$\begin{aligned} |V^x_{n_j}(y)| &\leq |h_0(y)|\,|g \circledcirc \sigma^*(y)\,\tau^*(\widetilde{x}_{n_j})\,\phi_{n_j}(x)| \\ &\leq |h_0(y)|\,\|\sigma^*(y)\,\tau^*(\widetilde{x}_{n_j})\,\phi_{n_j}\|_B * \|\sigma(x)g\|_B \\ &\leq C^3|h_0(y)|. \end{aligned}$$

Applying Lebesgue dominated convergence theorem

$$\int_X V_{n_j}^x(y)\, d\mu(y) \to 0 \text{ as } j \to \infty.$$

$$\Rightarrow g \odot s_{n_j}(x) \to 0 \text{ as } j \to \infty.$$

so $g \odot s = 0$ but $g \in U$ so $s = 0$.

But
$$\begin{aligned} s_n(e) &= n_0 \odot \tau^*(\tilde{x}_n)\, \phi_n(e) \\ &= (\tau^*(\tilde{x}_{n_j})\, \phi_n)^*(h_n) \\ &= \phi_n^*\,(\sigma(x_n)h_n) = h_n \odot \phi_n(x_n) \end{aligned}$$

so $|s_n(e)| > \delta$. Thus $|s(e)| \geq \delta$ which is a contradiction. □

We now note that (B.1) – (B.2) and (M.1) – (M.4) above are satisfied if X is a locally compact hypergroup possessing a left Haar measure μ (in particular, if X is a locally compact group) and $B = L^1(X, \mu)$. In this case $\sigma(x)f = {}_xf$.

Definition 1. 3 : A hypergroup is a locally compact space X and a binary mapping $(x,y) \to p_x * p_y$ of $X \times X$ into $M(X)$ satisfying the following:

(i) The mapping $(x,y) \to p_x * p_y$ extends to a bilinear associative operation $*$ from $M(X) \times M(X)$ into $M(X)$ such that

$$\int_X d\mu * \gamma = \int_X \int_X \int_X f\, d(p_x * p_y)\, d\mu(x)\, d\gamma(y) \text{ for all } f \in C_0(x).$$

(ii) For each $x, y \in X$, the measure $p_x * p_y$ is a probability measure with compact support.

(iii) The mapping $(\mu, \gamma) \to \mu * \gamma$ is continuous from $M^+(X) \times M^+(X)$ into $M^+(X)$ where $M^+(X)$ is given the weak topology with respect to the family $C_{00}^+(X) \cup \{1\}$.

(iv) There exists an element e in X such that $p_x * p_e = p_e * p_x$ for all $x \in X$.

(v) There exists a homeomorphic involution $x \to \tilde{x}$ of X onto X so that given $x, y \in X$, we have $e \in \operatorname{supp}(p_x * p_y)$ if and only if $y = \tilde{x}$ and $(p_x * p_y)^\sim = p_{\tilde{y}} * p_{\tilde{x}}$.

(vi) The map $(x,y) \to \operatorname{supp}(p_x * p_y)$ is continuous from $X \times X$ into the space $C(X)$ of compact subset of X, where $C(X)$ is given the topology studied by Michael, a sub basis for which is given by all $C_{U,V} = \{A \in C(X) : A \cap U \neq \phi \text{ and } A \subset V\}$ where U, V are open subsets of X.

We now note that (B.1) – (B.2) and (M.1) – (M.4) above are satisfied if X is a locally compact hypergroup possessing a left Haar measure μ (in particular, if X is a locally compact group) and $B = L^1(X, \mu)$. In this case $\sigma(x)f = {}_xf$.

Since $\|{}_xf\|_1 \leq \|f\|$ so $\|\sigma(x)\| \leq 1$ for all $x \in X$. The map τ on X is given by $\tau(y)f = \Delta(y)f_y$. For $f \in B$.

$$\|\tau(y)f\|_1 \le \Delta(y)\int_X |f|(x*y)\,d\mu(x)$$
$$= \Delta(y)\Delta(\widetilde{y})\int_X |f|(x)\,d\mu(x)$$
$$= \|f\|_1 \ ([1], 5.3B)$$

Thus $\|\tau(y)\| \le 1$ for all $y \in X$.

For $\quad f \in B, f^*(x) = \dfrac{f(\widetilde{x})}{\Delta(x)}$.

For $\phi \in B^* = L^\infty(X, \mu)$ and $f \in B$,

$$\phi^*(f) = \int_X \phi(x) f^*(x)\,d\mu(x) = \int_X \frac{\phi(x) f(\widetilde{x})}{\Delta(x)}\,d\mu(x)$$
$$= \int_X \frac{\phi(\widetilde{x}) f(x)}{\Delta(x)\Delta(\widetilde{x})}\,d\mu(x)$$
$$= \int_X \phi(\widetilde{x}) f(x)\,d\mu(x)$$

Thus $\phi^*(x) = \phi(\widetilde{x})$

$$\sigma^*(x)\,\phi = {}_{\widetilde{x}}\phi \text{ and } \tau^*(y)\phi = \phi_{\widetilde{y}}.$$

$$f \odot \phi(x) = \int_X \phi^*(y)\,{}_x f(y)\,d\mu(y)$$
$$= \int_X \phi(\widetilde{y})\,f(x*y)\,d\mu(y)$$
$$= f * \phi(x)$$

(M.1) – (M.2) are satisfied as in ([3], lemma 3.1). For (M.3), let $f \in B$, $\phi \in B^*$, $x \in X$,

$$(\tau^*(x)\phi)^*(f) = \int_X \phi_{\widetilde{x}}(y)\frac{f(\widetilde{y})}{\Delta(y)}\,d\mu(y)$$
$$= \int_X \frac{\phi(y*\widetilde{x}) f(\widetilde{y})}{\Delta(y)}\,d\mu(y)$$
$$= \int_X \phi(\widetilde{y}*\widetilde{x}) f(y)\,d\mu(y)$$
$$= \int_X \int_X \phi(\widetilde{u}) f(y)\,d\mu(y)\,dp_x * p_y(u)$$
$$= \int_X \phi^*(x*y) f(y)\,d\mu(y)$$
$$= \int \phi^*(x*y) f(y)\,d\mu(y)$$
$$= \int_X f(\widetilde{x}*y)\phi^*(y)\,d\mu(y) \quad ([1], 5.1\text{ D})$$
$$= \phi^*(\sigma(\widetilde{x})f)$$

For (M.4), let $\phi \in B^*$, $f, g \in B$ and $x \in X$

$$\begin{aligned}
&\int_X \phi^*(\sigma(\tilde{y})g)(\sigma(x)f)(y)\,d\mu(y)\\
&= \int_X \int_X \phi(\tilde{u})g(\tilde{y}*u)f(x*y)\,d\mu(y)\,d\mu(u)\\
&= \int_X \int_X \phi(\tilde{u})g(\tilde{y})\,{}_xf(u*y)\,d\mu(u)\\
&= \int_X g(\tilde{y})\int_X \frac{\phi^*(u*\tilde{y})}{\Delta(y)}\,{}_xf(u)\,d\mu(u)\,d\mu(y)\\
&= \int_X \int_X g(y)\phi^*(u*g)\,{}_xf(u)\,d\mu(y)\,d\mu(u)\\
&= \int_X \int_X g(y)\phi(\tilde{y}*\tilde{u})\,{}_xf(u)\,d\mu(y)\,d\mu(u)\\
&= \int_X g(y)(\sigma^*(y)\phi)^*(\sigma(x)f)\,d\mu(y) \qquad \square
\end{aligned}$$

Thus we have the following generalization from separable locally compact group G to separable locally compact hypergroup X ([3], Theorem 2.3)

Theorem 1.4. *Let X be a separable locally compact hypergroup possessing a left Haar measure μ. Let $\mathcal{H} \subset L^1(X)$ be such that the family $\{\Phi_h : h \in \mathcal{H}\}$ is left uniformly equicontinuous. Suppose that there exists $h_0 \in S_1$ such that $|h(t)| \le |h_0(t)|$ for all $h \in \mathcal{H}$ and $t \in X$. Let $\mathcal{U} \subset S_\infty$ be left translation invariant. If $g \in U_0$ and $g * a(x) \to 0$ as $x \to \infty$ uniformly for $a \in \mathcal{U}$ then $h * a(x) \to 0$ as $x \to \infty$ uniformly for $a \in \mathcal{U}$ and $h \in \mathcal{H}$.*

2. Segal Algebras on Hypergroups

Let X be a locally compact hypergroup possessing a left Haar measure μ. Segal algebras on locally compact hypergroups have studied and defined in [5] and [8] (For Segal algebras on groups see [4].

Definition 2.1. Let $S(X)$ be a subspace of $L^1(X)$ which is a Banach space under a norm $\|\cdot\|_S$ such that $\|\cdot\|_S \ge \|\cdot\|_1$ and

S(i) $S(X)$ is dense in $L^1(X)$.

S(ii) $S(X)$ is left translation invariant and for some $\eta > 0$, $\|{}_xf\|_S \le \eta\|f\|_S$ For each $f \in S(X)$ and $x \in X$.

S(iii) For each $f \in S(X)$, the mapping $x \to {}_xf$ of X into $S(X)$ is continuous.

Then $S(X)$ will be called a Segal algebra. $S(X)$ is said to be symmetric Segal algebra if for $f \in S(X)$, $f^* \in S(X)$ where

$$f^*(x) = \frac{\overline{f(\tilde{x})}}{\Delta(x)} \text{ and } \|f\|_S = \|f^*\|_S$$

In fact $S(X)$ is Banach algebra under convolution. This can be seen as in ([6], § 4) Using vector valued integrals as in ([6], § 11, Lemma 1), the following result follows

Lemma 2.2. For any $\phi \in (S(X))^*$, $f \in L^1(X)$ and $g \in S(X)$, the following hold.

(i) $\phi(f*g) = \int_X f(y)\phi(_{\tilde{y}}g)\,d\mu(y)$

If $S(X)$ is symmetric, then

(ii) $\phi(g*f) = \int_X f(y)\phi(g_{\tilde{y}})\,d\mu(y)$.

Let $B = S(X)$ be a symmetric Segal algebra. Taking $\sigma(x)f = {}_xf$ and $\tau(y)f = \Delta(y)\, f_y$ we have $\| \sigma(x) \| \leq 1$ and $\|\tau(x)\| \leq 1$ for all $x \in X$. Note that $(_{\tilde{y}}f)^* = \Delta(y)\,(f^*)_y$. Since $x \to {}_xf$ is continuous so $x \to f \odot \phi(x)$ and $x \to \phi \odot f(x)$ are measurable. Thus (M.1) is satisfied. For $\phi \in (S(X))^*$, $f, g \in S(X)$, we have

$$\int_X \phi(_{\tilde{x}}f)g^*(x)\,d\mu(x)$$

$$= \int_X \phi(_xf)\frac{g^*(\tilde{x})}{\Delta(x)}\,d\mu(x) = \int_X \phi(_xf)\,g(x)\,d\mu(x)$$

Thus (M.2) is satisfied

$$\begin{aligned}((\tau^*(x)\phi)^*(f) &= \phi(\tau(x)f^*) = \phi(\Delta(x)(f^*)_x \\ &= \phi((_{\tilde{x}}f)^*) = \phi^*(\sigma(\tilde{x})f)\end{aligned}$$

For (M.4), let $\phi \in S(X)^*$, $f, g \in S(X)$ and $x \in X$ we have

$$\begin{aligned}&\int_X \phi^*(\sigma(\tilde{y})g)\sigma(x)f(y)\,d\mu(y)\\
&= \int_X \phi^*((_{\tilde{y}}g))\,{}_xf(y)\,d\mu(y).\\
&= \phi(g^* * (_xf)^*)\\
&= \phi^*(_xf * g)\\
&= \int_X g^*(y)\phi(_{\tilde{y}}(_xf)^*)\,d\mu(y)\\
&= \int_X g(y)\phi(_y(_xf)^*)\,d\mu(y).\\
&= \int_X g(y)(\sigma^*(y)\phi)^*(\sigma(x)f)\,d\mu(y).\end{aligned}$$ □

Thus (M.4) is satisfied. Hence we have the following uniform version of the Wiener Tauberian Theorem for Segal algebras.

Theorem 2.3. *Let X be a locally compact hypergroup possessing a left Haar measure μ. Suppose that $S(X)$ is a symmetric Segal algebra on X. Let $\mathcal{H} \subset S(X)$ be such that the family $\{\Phi_h : h \in \mathcal{H}\}$ is left uniformly equicontinuous. Suppose that there exists $h_0 \in S_1$ (unit ball in $S(X)$) such that $|h(t)| \leq |h_0(t)|$ and $\|h\|_S \leq \|h_0\|_S$ for all $h \in \mathcal{H}$ and $t \in X$. Let $\mathcal{U} \subset S_\infty$ (unit ball in $S(X)^*$) be such that $\sigma^*(x) \in \mathcal{U}$ for all $\phi \in \mathcal{U}$. If $g \in U_0$ and $g \odot a(x) \to 0$ as $x \to \infty$ uniformly for $a \in \mathcal{U}$ then $h \odot a(x) \to 0$ as $x \to \infty$ uniformly for $a \in \mathcal{U}$ and $h \in \mathcal{H}$.*

3. Examples :

Let X be a unimodular locally compact hypergroup possessing a left Haar measure μ.

(a) $S(X) = L^1(X) \cap L^p(X)\ (1 \le p < \infty)$

$\|f\|_S = \|f\|_1 + \|f\|_p$

Then $S(X)$ is a Segal algebra

S (i) follows since $C_\infty(X)$ is dense in $S(X)$

S (ii) follows from ([1], 3.3 B)

S (iii) follows from ([1], 5.4, 2.2 B)

Clearly $S(X)$ is symmetric

(b) $S(X) = L^1(X) \cap C_0(X)$

$\| f \|_S = \| f \|_1 + \| f \|_\infty,\ f \in S(X)$

Then $S(X)$ is a Segal algebra

S (i) follows since $C_\infty(X)$ is dense in $S(X)$

S (ii) follows from ([1], 3.3 B)

S (iii) follows from ([1], 2.2B, 4.2F)

Note that $S(X)$ is symmetric since $\|f^*\|_\infty = \|f\|_\infty$.

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Fixed points in group invariant subspaces

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Abstract: We investigate the subspaces of fixed elements (also known as centralizers) of G-invariant subspaces of $V = \prod_1^n F$ where G is a group of $n \times n$ permutation matrices, F is the Galois field of order p^r for some $r \geq 1$ and $\prod_1^n F$ is the usual canonical vector space of dimension n over F. We are able to characterize these subspaces when $(p, |G|) = 1$. In the case, when p divides $|G|$ all we know is where to look for these subspaces, namely inside the kernel of $\beta = \sum_{g \in G} g$.

Key words : Fixed points, group-invariant subspace, idempotent, permutation matrices.

1. Introduction:

Let F be a finite field of order p^r for some prime p and $r \geq 1$. Then $V = \prod_1^n F$ is a vector space of dimension n over F with basis canonical so that a typical vector has the shape $x = (x_1, \dots, x_n)$, $x_i \in F$, $i = 1, \dots, n$. A $[n,s]$ subspace S over F is a space inside V of dimension s. The dual subspace $S^\perp$ is the subspace orthogonal to S under the usual scalar product on V. That is $S^\perp = \{x \in V \mid (u,x) = \sum_{i=1}^n u_i x_i = 0 \text{ for all } u \in S\}$. Then $S^\perp$ is a $[n, n-s]$ subspace because $\dim S + \dim S^\perp = \dim V$.

Let G be a group of permutation matrices of order n. A subspace S of V is called G-invariant if $(S)g \subseteq S$.

It is easy to check that if S is G-invariant, so is $S^\perp$. Let $s^\perp \in S^\perp$. Then $(s^\perp, s^\perp g) = (sg^t, s^\perp) = (sg^{-1}, s^\perp) = 0$, when $s \in S$ and g^t is transpose of g. Then $s^\perp g \in S^\perp$ and $S^\perp$ is a G-invariant subspace.

2. Characterization of Fixed Points when $(p, |G|) = 1$

Throughout this section we will assume that $F = GF(p^r)$ and $(p, |G|)$. Set $\alpha = \frac{1}{|G|} \sum_{g \in G} g$.

Since $(p, |G|) = 1$, $\frac{1}{|G|} = |G|^{-1}$ exists in F and therefore α exists in the group-ring FG. We now show that α is an idempotent. Let $v \in V = \prod_1^n F$. Then

$$v\alpha^2=(v\alpha)\alpha=\left(v\frac{1}{|G|}\sum_{g\in G}g\right)\left(\frac{1}{|G|}\sum_{g\in G}g\right)=v\frac{1}{|G|^2}\sum_{g\in G}g\left(\sum_{g\in G}g\right)=v\frac{1}{|G|^2}|G|\sum_{g\in G}g$$

$$=v\frac{1}{|G|}\sum_{g\in G}g=v\alpha \text{ and } \alpha \text{ is indeed an indempotent.}$$

Next we prove a couple of theorems.

Theorem 2.1 : *Let G be a group of $n\times n$ permutation matrices and $F=GF(p^r)$ with $(p,|G|)=1$. If S is a G-invariant subspace of $V=\prod_1^n F$, then $S\alpha=\mathrm{Fix}_S(G)$*

Proof: We show that $S\alpha\subseteq \mathrm{Fix}_S(G)$. Let $x\in S\alpha$. Then $x=s\alpha$ for some $s\in S$ and

$$s\alpha=s\left(\frac{1}{|G|}\sum_{g\in G}g\right)=\frac{1}{|G|}\sum_{g\in G}sg\in S. \text{ Thus } x\in S. \text{ Moreover for any } g\in G.$$

$$xg=s\alpha g=s\left(\frac{1}{|G|}\sum_{g\in G}g\right)g=s\frac{1}{|G|}\sum_{g\in G}g=s\alpha=x. \text{ Hence } x=\mathrm{Fix}_S(G).$$

We now prove the other containment i.e. $Fix_S(G)\subseteq S\alpha$. Let $s\in Fix_S(G)$. Then

$$s\alpha=s\left(\frac{1}{|G|}\sum_{g\in G}g\right)=\frac{1}{|G|}\sum_{g\in G}sg=\frac{1}{|G|}\sum_{1}^{|G|}s=\frac{1}{|G|}|G|s=s. \text{ Hence } s=s\alpha\in S\alpha. \quad \blacksquare$$

Theorem 2.2 : *Let G be a group of $n\times n$ permutation matrices and $F=GF(p^r)$ with $(p,|G|)=1$. If S is a G-invariant subspace of $V=\prod_1^n F$, then $(S\alpha)^{\perp}=\mathrm{Ker}\,\alpha\oplus(S^{\perp})\alpha$.*

Proof : We prove that $(S\alpha)^{\perp}\subseteq \mathrm{Ker}\alpha+(S^{\perp})\alpha$. Let $x\in(S\alpha)^{\perp}$. Then $x-s\alpha\in\mathrm{Ker}\,\alpha$ as $\alpha^2=\alpha$. Let us now check if $x\alpha\in S^{\perp}\alpha$. Since $x\in(S\alpha)^{\perp}$, we have $(x,s\alpha)=0$ for $\forall s\in S$. Then $0=(x,s\alpha)=(x\alpha^t,s)=(x\alpha,s)$ and $x\alpha\in S^{\perp}$. By applying α on both sides of $x\alpha\in S^{\perp}$ and using the idempotence, we obtain $x\alpha\in S^{\perp}\alpha$. Hence $x=(x-x\alpha)+\mathrm{Ker}\,\alpha$ belongs to $\mathrm{Ker}\,\alpha+(S^{\perp})\alpha$. We now want to show that $\mathrm{Ker}\,\alpha+(S^{\perp})\,\alpha\subseteq(S\alpha)^{\perp}$. Let $x\in\mathrm{Ker}\,\alpha+(S^{\perp})\alpha$. Then $x=k+s^{\perp}\alpha$ for some $k\in K$ and $s^{\perp}\in S^{\perp}$ and $(x,s\alpha)=(k+s^{\perp}\alpha,\,s\alpha)=(k\alpha^t+(s^{\perp}\alpha)\alpha^t,s)=$ $(k\alpha+(s^{\perp}\alpha)\alpha,s)=(0+s^{\perp}\alpha,\,s)=0$ as $s^{\perp}\alpha\in S^{\perp}$. Hence $x\in(S\alpha)^{\perp}$ and $\mathrm{Ker}\alpha+(S^{\perp})\alpha\subseteq(S\alpha)^{\perp}$. Finally, we want to check if $\mathrm{Ker}\,\alpha\cap(S^{\perp})\alpha=\{0\}$. Let $x\in\mathrm{Ker}\,\alpha\cap(S^{\perp})\alpha=\{0\}$. Then $x=s^{\perp}\alpha$. Applying α to both sides, we obtain $x\alpha=(s^{\perp})\alpha^2$. Since $x\in\mathrm{Ker}\,\alpha$ and $\alpha^2=\alpha$, the previous equality yields $0=s^{\perp}\alpha$, which in turn yields $0=x$. Thus $\mathrm{Ker}\,\alpha\cap(S^{\perp})\alpha=\{0\}$.

Theorem 2.3. *Let G be a group of $n\times n$ permutation matrices and $F=GF(p^r)$ with $(p,|G|)=1$.* If S is a G-invariant subspace of $V=\prod_1^n F$, then $\dim\ Fix_V(G)=\dim Fix_S(G)+\dim Fix_{S^{\perp}}(G)$

Proof: As $S\alpha\subseteq V\alpha$, we have $\dim V\alpha=\dim S\alpha+\dim((S\alpha)^{\perp}\cap V\alpha)$. By Theorem (2.2.), $(S\alpha)^{\perp}=\mathrm{Ker}\alpha\oplus(S^{\perp})\alpha$ which shows $(S\alpha)^{\perp}\cap V\alpha=(\mathrm{Ker}\alpha\cap V\alpha)\oplus((S^{\perp})\alpha\cap V\alpha)$.

Assume $x \in V\alpha \cap \mathrm{Ker}\alpha$. Then $x \in V\alpha$ for some $v \in V$. Thus $x = v\alpha = v\alpha^2 = (v\alpha)\alpha = x\alpha = 0$. This shows $(S\alpha)^{\perp} \cap V\alpha = (S^{\perp})\alpha \cap V\alpha = (S^{\perp})\alpha$. Thus $\dim V\alpha = \dim S\alpha + \dim S^{\perp}\alpha$. Now we apply Theorem (2.1) to obtain $\dim \mathrm{Fix}_V(G) = \dim Fix_S(G) + \dim Fix_{S^{\perp}}(G)$. □

Notice that the theorem above may not work if $(p,|G|) \neq 1$. Consider for example $G = <12 \ldots n)>$, a cyclic group of order n generated by permutation $(12 \ldots n)$ acting on $V = Z_2^n$. Notice that $Fix_V(G) = \{0 \ldots 0, 1 \ldots 1\}$ and $S = Fix_V(G)$ is a G-invariant subspace in V. If n is odd i.e. $(p,|G|) = 1$ then $\mathrm{Fix}_{S^{\perp}}(G)$ comprises of zero element only and $\dim Fix_V(G) = \dim Fix_S(G) + \dim Fix_{S^{\perp}}(G) = 1$. But when is even i.e. $(p,|G|) = 2$, $\dim Fix_V(G) = \dim Fix_S(G) = \dim Fix_{S^{\perp}}(G) = 1$ and the equality in Theorem (2.3) fails to hold. Since $\dim Fix_V(G) = \dim Fix_S(G) + \dim Fix_{S^{\perp}}(G)$, one wonders if $Fix_V(G) = Fix_S(G) \oplus Fix_{S^{\perp}}(G)$ holds under the conditions of Theorem (2.3). But one immediately notices that $Fix_S(G) \cap Fix_{S^{\perp}}(G)$ may not always be the zero space. For example, if we let $G = <(123)>, (4)$, $V = \prod_1^n GF(4)$ and $S = <111>$, then $Fix_S(G) = Fix_{S^{\perp}}(G) = S$, and hence $Fix_S(G) \cap Fix_{S^{\perp}}(G)$ is not the zero space. This raises the question : when is then $\mathrm{Fix}_V(G) = Fix_S(G) \oplus Fix_{S^{\perp}}(G)$? The following theorem tries to answer that question.

Theorem 2.4. *Let G be a group of $n \times n$ permutation matrices and $F = GF(p^r)$ with $(p,|G|) = 1$. If S is a G-invariant subspace of $V = \prod_1^n F$, such that $Fix_S(G) \cap Fix_{S^{\perp}}(G) = \{0\}$, then $\mathrm{Fix}_V(G) = Fix_S(G) \oplus Fix_{S^{\perp}}(G)$,*

Proof: Let $x \in Fix_S(G) + Fix_{S^{\perp}}(G)$. Then $x = s + s^{\perp}$ where $s \in Fix_S(G)$ and $s^{\perp} \in Fix_{S^{\perp}}(G)$. Hence $xg = (s + s^{\perp})g = sg + s^{\perp}g = s + s^{\perp} = x$ and $x \in Fix_V(G)$, which follows $Fix_S(G) + Fix_{S^{\perp}}(G) \subseteq Fix_V(G)$. As $(p,|G|) = 1$, by Theorem (2.3), we have $\dim Fix_V(G) = \dim Fix_S(G) + \dim Fix_{S^{\perp}}(G)$.

Since $Fix_S(G) \cap Fix_{S^{\perp}}(G) = \{0\}$, $\dim (Fix_S(G) + Fix_{S^{\perp}}(G)) = \dim Fix_S(G) + \dim Fix_{S^{\perp}}(G)$. Hence $\dim Fix_V(G) = \dim (Fix_S(G) + Fix_{S^{\perp}}(G)$, which yields the desired equality $Fix_V(G) = Fix_S(G) \oplus Fix_{S^{\perp}}(G)$. ■

Corollary (2.5). Lie G be a group of *of $n \times n$ permutation matrices and $F = GF(p^r)$ with* $(p,|G|) = 1$. If S is a G*-invariant subspace of $V = \prod_1^n F$, such that $S \cap S^{\perp} = \{0\}$, then $Fix_V(G) = Fix_S(G) \oplus Fix_{S^{\perp}}(G)$.*

Proof: Follows immediately from the theorem above. ■

Notice that condition if $(p,|G|) = 1$ in Theorem (2.4) is a sufficiency condition, not a necessary one. To see this, we consider $G = <(12)(3)(4)>$ acting on $V = Z_2^4$ and $S = \{0000, 0010\}$. One checks that in spite of $(p,|G|) = 2$, $Fix_V(G)$. is still $Fix_S(G) \oplus Fix_{S^{\perp}}(G)$.

3. Characterization of Fixed Points when $(p,|G|) \neq 1$

Finally we consider the case when p divides $|G|$. We set $\beta = \sum_{g \in G} g$ and produce the following theorem, which states that when p divides $|G|$, the fixed points reside inside Ker β.

Theorem. *Let G be a group $n \times n$ permutation matrices and $F = GF(p^r)$ with p dividing $|G|$. If S is a G- invariant subspace of $V = \prod_1^n F$, then $S\beta \subseteq Fix_s(G) \subseteq$ Ker β.*

Proof: We first show that $S\beta \subseteq Fix_S(G)$. Let $v \in S\beta$ i.e. $v = s\beta$ for some $s \in S$. Then $vg = s\beta g = s(\sum_{g \in G} g)g = s\sum_{g \in G} g = s\beta = v$. Hence $v = Fix_s(G)$. To see the other containment i.e. $Fix_S(G) \subseteq$ Ker β, we let $v \in Fix_s(G)$ and apply β on it. Then

$$v\beta = v\sum_{g \in G} g = \sum_{g \in G} vg = \sum_1^{|G|} v = |G|v = 0. \quad \blacksquare$$

Notice that the containment $Fix_S(G) \subseteq$ Ker β may not hold if $(p,|G|) = 1$. To see this we consider the following example

Let $G = <(123)(4)>$ and $V = Z_2^4$. Then one checks that

$$\beta = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ and for any $v = (v_1, v_2, v_3, v_4) \in V$, $v\beta = (\sum_{i=1}^3 v_i, \sum_{i=1}^3 v_i, \sum_{i=1}^3 v_i, v_4)$.

Hence $V\beta = \{0000, 0001, 1110, 1111\}$,

On the other hand, $|G| = 3$ and $3 = 1 \pmod 2$, so we have $\beta = \alpha$ and by Theorem (2.1) of the previous section, $v\beta = V\alpha = Fix_V(G)$. Thus $Fix_V(G) = \{0000, 0001, 1110, 1111\}$. But from $v\beta = (\sum_{i=1}^3 v_i, \sum_{i=1}^3 v_i, \sum_{i=1}^3 v_i, v_4)$, we learn that for a vector in V to be in Ker β, the last coordinate must be zero. So the vectors 0001,1111 in $Fix_V(G)$ with their last coordinate 1 can't be in Ker β. This proves the fact that the containment $s\beta \subseteq Fix_S(G) \subseteq$ Ker β for an arbitrary G-invariant subspace S is specific to the case when p divides $|G|$.

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Time sharing machine repair model with mixed spares and additional repairmen

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Abstract: This investigational deals with time sharing machine repair system consisting of M operating machines under the supervision of R permanent and in additional removable repairmen. To improve the system performance S warm and Y cold standby spaces are provided to replace the failed machines. In case when all spares are used, the additional repairmen are introduced one by one according to a prespecified rule to facilitate the repair of failed machines. At least L machines are required for the successful performance of the system. The repair time of failed machines follows exponential distribution. The optional values of the number of repairmen and the number of spares can be evaluated simultaneously by minimizing the expected cost per unit time constructed in terms of different cost elements. These results may be useful for the industrial management. Production managers and others who face difficulty in decision taking regarding the installation of the number of machines, spares and repairmen for the maintenance of multi component machining system.

Index terms: Time-sharing, Machine repair, Mixed spares, Additional repairmen, Queue size, Cost function.

1. Introduction

In the fast growing industries, the production may be interrupted due to failure of machines involved in the system. The service facility may be adjusted by providing spare part support and the machine repair instantly, so as to continue the operation successfully. The time-sharing of repair time is a concept to regulate the repair of machines and is of great importance in some critical applications of computer, communication, manufacturing and switching systems, etc. The response time of M/M/I time-sharing queues with limited number of service positions was studied by Avi-Izhak and Halfin (1988). Avi-Izhak (1991) suggested approximation for the moments of response time in the time in the time-sharing queues. Jain and Prem Lata (1993) considered a time-sharing queues with a limited number of service positions and limited number of waiting space. Jain and Prem Lata (1995) examined accumulated work process in a time-sharing queues with the finite population. Jain and Singh (2001) analyzed the effect implementing additional service positions in case of no passing time-sharing queueing model. Jain et al. (2002) incorporated additional service positions to analyze loss and delay queueing model for time-shared system with no passing restriction. Sharma et al. (2004) derived numerical solution of processor–sharing queueing model. Singh et al. (2005) discussed no passing M/M/ϕ (.) time-sharing queueing system.

Many researchers working in the area of queueing theory have contributed significantly towards machine repair problems with spares and additional repairmen. A profit model in machine repair problem for warm standby system was studied by Wang and Sivazilian (1989). Wang and Wu (1995) considered a cost analysis of M/M/R machine repair problem with spares and two modes of failures, Shawaky (1997) studied the single server machine interference model with balking, reneging and an additional server for longer queue. Jain et al. (2000) analyzed M/M/C/K/N model with spare by implementing the customer's balking and reneging behavior. Jain and Bhagel (2002) studied a multi-component repairable system with spares and state

dependent rates. Jain et al. (2002) gave a diffusion process for multi-repairmen machining system with spares and balking. Jain et al. (2003) considered a repairable system with spares, state dependent rates and additional repairmen. Queueing network model for a single operator, machining system with external operation was developed by Yang et al. (2002).

2. The Model

We consider the time-sharing machine repair problem with mixed (warm and cold) spares and a repair facility consisting of permanent and additional removable repairmen. For modeling propose, following notations and assumptions are taken into consideration:

Notation:

R	Number of permanent repairmen in the system.
m	Number of additional repairmen
M	Number of machines in operating state when system works in normal mode.
S	Number of warm spares in the system.
Y	Number of cold spares in the system.
L	Minimum number of machines required for system to be functioning.
K	$M+S+Y-L+1$
N	Total number of machines in the system so that $N=M+S+Y$.
k	Threshold level of extra work load to turn of the additional repairmen.
α	Failure rate of warm spare.
$\lambda(\lambda_d)$	Failure rate of operating machines when all (less than) M operating machines are working
$\phi_n(j)$	Time-sharing function when there are n number of failed machines present in the system and available permanent and additional repairmen in the system is $j\,(1\le j\le R+m)$.

Assumptions:

- The time to failure of a machine in operation or as a warm spare and repair time of a failed machine are exponentially distributed.
- The failed machines are repaired in a repair facility consisting of R permanent and m additional removable repairmen in FIFO fashion.
- When all the spares are used, the operating machines work in degraded mode.
- After repair, the machine works as good as new one and joins the standby group of machines in the system if there is no shortage of operating machines, otherwise joins the working machines.
- The additional repairmen are turned on one by one according to rule as specified below:
  - When there are less than k-failed machines, only permanent R servers are available.
  - If the number of failed machines is greater than jk and less than or equal to $(j+k)\,k$, $j\,(j=1,2,\ldots,m-1)$ additional repairmen are made available in addition to permanent repairmen.
  - When there are greater than mk failed machines, all permenent and additional repairmen are made available.
- The repair of failed machines is provided in time-sharing manner, i.e. times of all available repairmen are shared among all failed machines.

The failure rates and repair rates of birth death process are stated below:

$$\text{(1)} \qquad \lambda(n)=\begin{cases} M\lambda+(S-n)\alpha; & 0<n<S \\ M\lambda; & S\le n<Y+S \\ (N-n)\lambda_d; & Y+S\le n<K \end{cases}$$

$$\text{(2)} \qquad \mu(n)=\begin{cases} \mu\phi_n(n); & 1\le n<R \\ \mu\phi_n(R); & R\le n\le R+k \\ \mu\phi_n(R+j); & R+jk<n\le R+(j+1)k \quad j=1,2,\ldots m-1 \\ \mu\phi_n(R+m); & R+mk<n\le K \end{cases}$$

2. The Analysis

The governing steady state equations related to our model are given as follows:

CASE I: $R \le S$

(3) $[M\lambda+S\alpha]P_0=\mu\phi_1(1)P_1$

(4) $[M\lambda+(S-n)\alpha+\mu\phi_n(n)]P_n=[M\lambda+(S-n+1)\alpha]P_{n-1}+\mu\phi_{n+1}(n+1)P_{n+1}, \quad 1\le n<R-1$

(5) $[M\lambda+(S-n)\alpha+\mu\phi_n(R)]P_n=[M\lambda+(S-n+1)\alpha]P_{n-1}+\mu\phi_{n+1}(R)P_{n+1}, \quad R\le n<S$

(6) $[M\lambda+\mu\phi_S(R)]P_S=(M\lambda+\alpha)P_{S-1}+\mu\phi_{S+1}(R)P_{S+1}$

(7) $[M\lambda+\mu\phi_n(R)]P_n=M\lambda P_{n-1}+\mu\phi_{n+1}(R)P_{n+1}, \quad S<n<Y+S$

(8) $[M\lambda_d+\mu\phi_{Y+S}(R)]P_{Y+S}=M\lambda P_{Y+S-1}+\mu\phi_{Y+S+1}(R)P_{Y+S+1}$

(9) $[\{N-n\}\lambda_d+\mu\phi_n(R)]P_n=[(N-n+1)\lambda_d]P_{n-1}+\mu\phi_{n+1}(R)P_{n+1}, \quad Y+S<n<R+k$

(10) $[\{N-(R+k)\}\lambda_d+\mu\phi_{(R+k)}(R)]P_{(R+k)}=[(N-(R+k)+1)\lambda_d]P_{(R+k)-1}+\mu\phi_{(R+k)+1}(R+j)P_{(R+k)+1}$

(11) $[(N-n)\lambda_d+\mu\phi_n(R+j)]P_n=[(N-n+1)\lambda_d]P_{n-1}+\mu\phi_{n+1}(R+j)P_{n+1},$

$R+jk\le n<R+(j+1)k, \ j=1,2,\ldots,m-1.$

(12) $[(N-R-jk)\lambda_d+\mu\phi_{R+jk}(R+j)]P_{R+jk}=(N-R-jk+1)\lambda_d P_{R-jk+1}+\mu\phi_{R+jk+1}(R+j)P_{R+jk+1},$

$j=1,2,\ldots,m-1.$

(13) $[(N-n)\lambda_d+\mu\phi_n(R+m)]P_n=(N-n+1)\lambda_d P_{n-1}+\mu\phi_{n+1}(R+m)P_{n+1}, \quad R+mk\le n<K$

(14) $[\mu\phi_k(R+m)]P_k=(N-K+1)\lambda_d P_{k-1}$

CASE II: $S<R\le S+Y$

(15) $[M\lambda+S\alpha]P_0=\mu\phi_1(1)P_1$

(16) $[M\lambda+(S-n)\alpha+\mu\phi_n(n)]P_n=[M\lambda+(S-n+1)\alpha]P_{n-1}+\mu\phi_{n+1}(n+1)]P_{n+1}, \quad 1\le n<S$

(17) $[M\lambda+\mu\phi_S(S)]P_S=(M\lambda+\alpha)P_{S-1}+\mu\phi_{S+1}(S)P_{S+1}$

(18) $[M\lambda+\mu\phi_n(n)]P_n=M\lambda P_{n-1}+\mu\phi_{n+1}(n+1)P_{n+1}, \quad S<n<R$

(19) $[M\lambda+\mu\phi_n(R)]P_n=M\lambda P_{n-1}+\mu\phi_{n+1}(R)P_{n+1}, \quad R\le n<Y<S$

(20) $[M\lambda_d+\mu\phi_{Y+S}(R)]P_n=M\lambda P_{Y+S-1}+\mu\phi_{n+1}(R)P_{Y+S+1}$

(21) $[(N-n)\lambda_d+\mu\phi_n(R)]P_n=[(N-n+1)\lambda_d]P_{n-1}+\mu\phi_{n+1}(R)P_{n+1}, \quad Y+S<n<R+k$

(22) $[\{N-(R+k)\}\lambda_d+\mu\phi_{(R+k)}(R)]P_{(R+k)}=[(N-(R+k)+1)\lambda_d]P_{(R+k)-1}+\mu\phi_{(R+k)+1}(R+j)P_{(R+k)+1}$

(23) $[(N-n)\lambda_d+\mu\phi_n(R+j)]P_n=[(N-n+1)\lambda_d]P_{n-1}+\mu\phi_{n+1}(R+j)P_{n+1},$
$R+jk\le n<R+(j+1)k-1,\ j=1,2,\ldots,m-1$

(24) $[(N-R-jk)\lambda_d+\mu\phi_{R+jk}(R+j)]P_{R-jk}=(N-R-jk+1)\lambda_d P_{R-jk-1}+\mu\phi_{R+jk+1}(R+j)P_{R+jk+1},$
$j=1,2,\ldots,m-1.$

(25) $[(N-n)\lambda_d+\mu\phi_n(R+m)]P_n=(N-n+1)\lambda_d P_{n-1}+\mu\phi_{n+1}(R+m)P_{n+1},\quad R+mk\le n<K$

(26) $[\mu\phi_k(R+m)]P_k=(N-K+1)\lambda_d P_{k-1}$

CASE III: $R>S+Y$

(27) $[M\lambda+S\alpha]P_0=\mu\phi_1(1)P_1$

(28) $[M\lambda+(S-n)\alpha+\mu\phi_n(n)]P_n=[M\lambda+(S-n+1)\alpha]P_{n-1}+\mu\phi_{n+1}(n+1)P_{n+1},\qquad 1\le n<S$

(29) $[M\lambda+\mu\phi_S(S)]P_S=(M\lambda+\alpha)P_{S-1}+\mu\phi_{S+1}(S)P_{S+1}$

(30) $[M\lambda+\mu\phi_n(n)]P_n=M\lambda P_{n-1}+\mu\phi_{n+1}(n+1)P_{n+1},\qquad S<n<Y+S$

(31) $[M\lambda_d+\mu\phi_{Y+S}(Y+S)]P_n=M\lambda P_{Y+S-1}+\mu\phi_{Y+S+1}(Y+S+1)P_{Y+S+1}$

(32) $[(N-n)\lambda_d+\mu\phi_n(n)]P_n=[(N-n+1)\lambda_d]P_{n-1}+\mu\phi_{n+1}(n+1)P_{n+1},\qquad S+Y<n<R-1$

(33) $[(N-n)\lambda_d+\mu\phi_n(R)]P_n=[(N-n+1)\lambda_d]P_{n-1}+\mu\phi_{n+1}(R)P_{n+1},\qquad R\le n<R+k$

(34) $[\{N-(R+k)\}\lambda_d+\mu\phi_{(R+k)}(R)]P_{(R+k)}=[(N-(R+k)+1)\lambda_d]P_{(R+k)-1}+\mu\phi_{(R+k)+1}(R+j)P_{(R+k)+1}$

(35) $[(N-n)\lambda_d+\mu\phi_n(R+j)]P_n=[(N-n+1)\lambda_d]P_{n-1}+\mu\phi_{n+1}(R+j)P_{n+1},$
$R+jk\le n<R+(j+1)k-1,\ j=1,2,\ldots,m-1.$

(36) $[(N-R-jk)\lambda_d+\mu\phi_{R+jk}(R+j)]P_{R+jk}=(N-R-jk+1)\lambda_d P_{R-jk+1}+\mu\phi_{R+jk+1}(R+j)P_{R+jk+1},$
$j=1,2,\ldots,m-1.$

(37) $[(N-n)\lambda_d+\mu\phi_n(R+m)]P_n=(N-n+1)\lambda_d P_{n-1}+\mu\phi_{n+1}(R+m)P_{n+1},\quad R+mk\le n<K$

(38) $[\mu\phi_K(R+m)]P_K=(N-K+1)\lambda_d P_{K-1}$

For solution purpose, we employ general product type formulation for birth death process as given by (cf. Gross and Harris, 1985).

(39) $$P_n=\prod_{j=0}^{n-1}\frac{\lambda(j)}{\mu(j+1)}P_0$$

The steady state solutions for three cases are obtained as:

Case (I): $R\le S$

(40) $$P_n=\begin{cases} f_1(n); & 1\le n<R \\ f_2(n); & R\le n\le S \\ f_3(n); & S<n<Y+S \\ f_4(n); & Y+S\le n<R+k \\ f_5(n); & R+jk\le n<R+(j+1)k,\ j=1,2,\ldots,m-1 \\ f_6(n); & R+mk\le n\le K \end{cases}$$

Case (II): $S < R \le S + Y$

$$P_n = \begin{cases} g_1(n); & 1 \le n < S \\ g_2(n); & S \le n \le R \\ g_3(n); & R < n \le S+Y \\ g_4(n); & S+Y < n \le R+k \\ g_5(n); & R+jk \le n < R+(j+1)k,\ j=1,2,\dots,m-1 \\ g_6(n); & R+mk \le n \le K \end{cases} \tag{41}$$

Case (III): $R > S + Y$

$$P_n = \begin{cases} h_1(n); & 1 \le n < S \\ h_2(n); & S \le n \le Y+S \\ h_3(n); & Y+S \le n < R \\ h_4(n); & R \le n < R+k \\ h_5(n); & R+jk \le n < R+(j+1)k,\ j=1,2,\dots,m-1 \\ h_6(n); & R+mk \le n \le K \end{cases} \tag{42}$$

where

$$f_1(n) = g_1(n) = h_1(n) = \frac{A(j,n-1)}{B(n)} \tag{43}$$

$$f_2(n) = \frac{A(j,n-1)}{B(R)\prod_{i=R+1}^{n} \phi_i(R)} \tag{44}$$

$$f_3(n) = \frac{(M\psi)^{n-S}\, A(j,S)}{B(R)\prod_{i=R+1}^{n} \phi_i(R)} \tag{45}$$

$$f_4(n) = \frac{(M\psi)^{Y} \prod_{i=Y+S+1}^{n} (K-(i-1))(\psi')^{n-(Y+S)} A(j,S)}{B(R)\prod_{i=R+1}^{n} \phi_i(R)}, \tag{46}$$

$$f_5(n) = \frac{(M\psi)^{Y} \prod_{i=Y+S+1}^{n} (K-(i-1))(\psi')^{n-(Y+S)} A(j,S)}{B(R)\prod_{i=R+1}^{R+k} \phi_i(R)\left[\prod_{l=1}^{j-1}\prod_{i=R+lk+1}^{R+(l+1)k} \phi_i(R+j)\right]\prod_{i=R+jk+1}^{n} \phi_i(R+j)},\quad j=1,2,\dots,m\text{–}1 \tag{47}$$

$$f_6(n) = \frac{(M\psi)^{Y} \prod_{i=Y+S+1}^{n} (K-(i-1)(\psi')^{n-(Y+S)} A(j,S)}{B(R)\prod_{i=R+1}^{R+k} \phi_i(R)\left[\prod_{l=1}^{m-1}\prod_{i=R+lk+1}^{R+(l+1)k} \phi_i(R+j)\right]\prod_{i=R+mk+1}^{n} \phi_i(R+m)},\quad j=1,2,\dots,m\text{–}1 \tag{48}$$

(49) $$g_2(n)=\frac{(M\psi)^{n-S}A(j,S)}{B(n)}$$

(50) $$g_3(n)=\frac{(M\psi)^{n-S}A(j,S)}{B(R)\prod\limits_{i=R+1}^{n}\phi_i(R)}$$

(51) $$g_4(n)=\frac{(M\psi)^{Y}\prod\limits_{i=Y+S+1}^{n}(K-(i-1)(\psi')^{n-(Y+S)}A(j,S)}{B(R)\prod\limits_{i=R+1}^{n}\phi_i(R)},$$

(52) $$g_5(n)=\frac{(M\psi)^{Y}\prod\limits_{i=Y+S+1}^{n}(K-(i-1)(\psi')^{n-(Y+S)}A(j,S)}{B(R)\prod\limits_{i=R+1}^{R+k}\phi_i(R)\left[\prod\limits_{l=1}^{j-1}\prod\limits_{i=R+lk+1}^{R+(l+1)k}\phi_1(R+j)\right]\prod\limits_{i=R+jk+1}^{n}\phi_1(R+j)},\quad j=1,2,\ldots,m-1$$

(53) $$g_6(n)=\frac{(M\psi)^{Y}\prod\limits_{i=Y+S+1}^{n}(K-(i-1))(\psi')^{n-(Y+S)}A(j,S)}{B(R)\prod\limits_{i=R+1}^{R+k}\phi_i(R)\left[\prod\limits_{l=1}^{m-1}\prod\limits_{i=R+lk+1}^{R+(l+1)k}\phi_1(R+j)\right]\prod\limits_{i=R+mk+1}^{n}\phi_1(R+m)},\quad j=1,2,\ldots,m-1$$

(54) $$h_2(n)=\frac{(M\psi)^{n-S}A(j,S)}{B(n)}$$

(55) $$h_3(n)=\frac{(M\psi)^{Y}\prod\limits_{i=Y+S+1}^{n}(K-(i-1))(\psi')^{n-(Y+S)}A(j,S)}{B(n)}$$

(56) $$h_4(n)=\frac{(M\psi)^{Y}\prod\limits_{i=Y+S+1}^{n}(K-(i-1))(\psi')^{n-(Y+S)}A(j,S)}{B(R)\prod\limits_{i=R+1}^{n}\phi_i(R)}$$

(57) $$h_5(n)=\frac{(M\psi)^{Y}\prod\limits_{i=Y+S+1}^{n}(K-(i-1))(\psi')^{n-(Y+S)}A(j,S)}{B(R)\prod\limits_{i=R+1}^{R+k}\phi_i(R)\left[\prod\limits_{l=1}^{j-1}\prod\limits_{i=R+lk+1}^{R+(l+1)k}\phi_i(R+j)\right]\prod\limits_{i=R+jk+1}^{n}\phi_i(R+j)},\quad j=1,2,\ldots,m-1$$

(58) $$h_6(n)=\frac{(M\psi)^{Y}\prod\limits_{i=Y+S+1}^{n}(K-(i-1))(\psi')^{n-(Y+S)}A(j,S)}{B(R)\prod\limits_{i=R+1}^{R+k}\phi_i(R)\left[\prod\limits_{l=1}^{m-1}\prod\limits_{i=R+lk+1}^{R+(l+1)k}\phi_i(R+j)\right]\prod\limits_{R+mk+1}^{n}\phi_i(R+m)},\quad j=1,2,\ldots,m-1$$

Also we have used notations :

$$\psi = \lambda/\mu,\ \ \delta = \alpha/\mu,\ \ \psi = \lambda_d/\mu,\ \ B(n) = \prod_{i=1}^{n} \phi_i(i),$$

$$A(j,k) = P_0 \prod_{j=0}^{k} [M\psi + (S-j)\delta]$$

4. Some Performance Measures

By using the queue size distribution given by equations (40), (41) and (42), we obtain average number of failed machines in the system as

$$E[n] = \sum_{n=0}^{N} nP_n \tag{59}$$

Machine availability is given by

$$[MA] = 1 - E[n]/N$$

Expected number of operating machines is the system is obtained as

$$E[O] = M - \sum_{n=Y+S+1}^{K} (n - \overline{Y+S})P_n \tag{60}$$

$$(61) = \begin{cases} M - \left[\begin{aligned} &\sum_{n=Y+S+1}^{R+k-1} (n-\overline{Y+S}) + f_4(n) + \sum_{j=1}^{m-1} \sum_{n=R+jk}^{R+(j+1)k-1} (n-\overline{Y+S})\, f_5(n) \\ &+ \sum_{n=R+mk}^{K} (n-\overline{Y+S})\, f_6(n) \end{aligned} \right]; \text{Case I} \\ M - \left[\begin{aligned} &\sum_{n=S+Y+1}^{R+k} (n-\overline{Y+S}) g_4(n) + \sum_{j=1}^{m-1} \sum_{n=R+jk}^{R+(j+1)k-1} (n-\overline{Y+S}) g_5(n) \\ &+ \sum_{n=R+mk}^{K} (n-\overline{Y+S})\, g_6(n) \end{aligned} \right]; \text{Case II} \\ M - \left[\begin{aligned} &\sum_{n=S+Y+1}^{R-1} (n-\overline{Y+S})\, h_3(n) + \sum_{n=1}^{R+k-1} (n-\overline{Y+S})\, h_4(n) \\ &+ \sum_{j=1}^{m-1} \sum_{n=R+jk}^{R+(j+1)k-1} (n-\overline{Y+S})\, h_5(n) + \sum_{n=R+mk}^{K} (n-\overline{Y+S})\, h_6(n) \end{aligned} \right]; \text{Case III} \end{cases}$$

Expected number of spare parts functioning as standbys is

$$(62)\ \ E[S] = \begin{cases} \left[\sum_{n=0}^{R-1} (Y+S-n) f_1(n) + \sum_{n=R}^{S} (Y+S-n) f_2(n) + \sum_{n=S+1}^{Y+S} (Y+S-n) f_3(n) \right]; \text{Case I} \\ \left[\sum_{n=0}^{S} (Y+S-n) g_1(n) + \sum_{n=S+1}^{R} (Y+S-n) g_2(n) + \sum_{n=R+1}^{Y+S} (Y+S-n)\, g_3(n) \right]; \text{Case II} \\ \left[\sum_{n=0}^{S} (Y+S-n) h_1(n) + \sum_{n=S+1}^{Y+S} (Y+S-n)\, h_2(n) \right]; \qquad \text{Case III} \end{cases}$$

Expected number of idle permanent repairmen is

$$(63)\qquad E[I]=\begin{cases}\left[\sum_{n=0}^{R-1}(R-n)f_1(n)\right]; & \text{Case I}\\ \left[\sum_{n=0}^{S}(R-n)g_1(n)+\sum_{n=S+1}^{R}(R-n)g_2(n)\right]; & \text{Case II}\\ \left[\sum_{n=0}^{S}(R-n)h_1(n)+\sum_{n=S+1}^{Y+S-1}(R-n)h_2(n)+\sum_{n=Y+S}^{R-1}(R-n)h_3(n)\right]; & \text{Case III}\end{cases}$$

Expected number of permanent busy repairmen

$$(64)\qquad E[B]=R-E[1]$$

Expected number of additional busy repairmen is

$$(65)\qquad E[A]=\sum_{j=0}^{m-}\sum_{r=0}^{K-1} jP_{R+jk+r}+m\sum_{n=R+mk}^{K}P_n$$

5. Cost Analysis

For determining the expression for profit function per unit time, we assume the following cost variables:

- The cost per unit time of failed machine is C_E when all spares are used
- The cost per unit time of one warm spare machine is C_E.
- The cost per unit time of one cold spare machine is C_Y.
- The cost per unit time of C_1 when one permanent server is idle.
- The cost per unit time of C_B when one permanent server is in busy state.
- The cost per unit time of additional machines is C_A.
- The system availability is denoted by A_V.

The average cost per unit time $CF(R,S)$ is given by

Case I

$$CF(R,S)=C_E\left[\sum_{n=Y+S+1}^{R+k-1}(n-\overline{Y+S})f_4(n)+\sum_{j=1}^{m-1}\sum_{n=R+jk}^{R+(j+1)k-1}(n-\overline{Y+S})f_5(n)+\sum_{n=R+mk}^{K}(n-\overline{Y+S})f_6(n)\right]$$

$$C_S\left[\sum_{n=0}^{R-1}(Y+S-n)f_1(n)+\sum_{n=R}^{S}(Y+S-n)f_3(n)\right]+C_Y\left[\sum_{m=S}^{Y+S}(Y+S-n)f_3(n)\right]$$

$$(66)\qquad C_I\left[\sum_{n=0}^{R-1}(R-n)f_1(n)\right]+C_AE[A]+C_BE[B]$$

Case II

$$CF(R,S)=C_E\left[\sum_{n=S+Y+1}^{R-1}(n-\overline{Y+S})g_4(n)+\sum_{j=1}^{m-1}\sum_{n=R+jk}^{R+(j+1)k-1}(n-\overline{Y+S})g_5(n)+\sum_{n=R+mk}^{K}(n-\overline{Y+S})g_6(n)\right]$$

$$+C_S\left[\sum_{n=0}^{S}(Y+S-n)g_1(n)+\sum_{n=S+1}^{R}(Y+S-n)g_2(n)\right]+C_Y\left[\sum_{n=S}^{Y+S}(Y+S-n)g_3(n)\right]$$

(67) $$+C_I\left[\sum_{n=0}^{S}(R-n)g_1(n)+\sum_{n=S+1}^{R}(R-n)g_2(n)\right]+C_AE[A]+C_BE[B]$$

Case III

$$CF(R,S)=C_E\left[\sum_{n=S+Y+1}^{R-1}(n-\overline{Y+S})h_3(n)+\sum_{n=R}^{R+k-1}(n-\overline{Y+S})h_4(n)+\sum_{j=1}^{m-1}\sum_{n=R+jk}^{R+(j+1)k-1}(n-\overline{Y+S})h_5(n)\right.$$

$$\left.+\sum_{n=R+mk}^{K}(n-\overline{Y+S})h_6(n)\right]+C_S\left[\sum_{n=0}^{S}(Y+S-n)h_1(n)\right]+C_Y\left[\sum_{n=S}^{Y+S}(Y+S-n)h_2(n)\right]$$

(68) $$+C_I\left[\sum_{n=0}^{S}(R-n)h_1(n)+\sum_{n=S+1}^{Y+S-1}(R-n)h_2(n)+\sum_{n=Y+S}^{R-1}(R-n)h_3(n)\right]+C_AE[A]+C_BE[B]$$

Optimal values of (R, S) can be evaluated by following optimization problem Minimize Z = CF(R, S)

(70) Subject to $A_V=\sum_{0}^{S}P_n\geq A$

6. Special Cases

Now we consider some special cases:

6.1. Special time-sharing function:

(a) Putting $\phi_0(n)=1/n$, we obtain the queue size distribution for time sharing system as:

(71) $$f_1(n)=g_1(n)=h_1(n)=A(j,n-1)n!$$

(72) $$f_2(n)=A(j,n-1)R!\{R\}^{n-(R+1)}$$

(73) $$f_3(n)=(M\psi)^{n-(S+1)}A(j,S)R!\{R\}^{n-(R+1)}$$

(74) $$f_4(n)=(M\psi)^Y\prod_{i=Y+S+1}^{n}(K-(i-1))(\psi')^{n-(Y+S)}A(j,S)R!\{R\}^{n-(R+1)}$$

(75) $$f_5(n)=(M\psi)^Y\prod_{i=Y+S+1}^{n}(K-(i-1))(\psi')^{n-(Y+S)}A(j,S)R!\{R\}^{k-1}\prod_{i=1}^{j-1}(R+j)^k\{R+j\}^{n-(R+jk+1)},$$
$$j=1,2,\ldots,m-1$$

(76) $$f_6(n)=(M\psi)^Y\cdot\prod_{i=Y+S+1}^{n}(K-(i-1))(\psi')^{n-(Y+S)}A(j,S)R!\{R\}^{k-1}\prod_{i=1}^{m-1}\{R+j\}^k\{R+m\}^{n-(R+mk+1)},$$
$$j=1,2,\ldots,m-1$$

(77) $$g_2(n) = (M\psi)^{n-S} A(j,S)n!$$

(78) $$g_3(n) = (M\psi)^{n-S} A(j,S)R!\{R\}^{n-(R+1)}$$

(79) $$g_4(n) = (M\psi)^{Y} \prod_{i=Y+S+1}^{n} (K-(i-1))(\psi')^{n-(Y+S)} A(j,S)R!\{R\}^{n-(R+1)}$$

(80) $$g_5(n) = (M\psi)^{Y} \prod_{i=Y+S+1}^{n} (K-(i-1))(\psi')^{n-(Y+S)} A(j,S)R!\{R\}^{k-1} \prod_{i=1}^{j-1} (R+j)^k \{R+j\}^{n-(R+jk+1)},$$
$$j = 1, 2, \ldots, m-1$$

(81) $$g_6(n) = (M\psi)^{Y} \prod_{i=Y+S+1}^{n} (K-(i-1))(\psi')^{n-(Y+S)} A(j,S)R!\{R\}^{k-1} \prod_{i=1}^{m-1} (R+j)^k \{R+m\}^{n-(R+mk+1)},$$
$$j = 1, 2, \ldots, m-1$$

(82) $$h_2(n) = (M\psi)^{n-S} A(j,S)n!$$

(83) $$h_3(n) = (M\psi)^{Y} \prod_{i=Y+S+1}^{n} (K-(i-1))(\psi')^{n-(Y+S)} A(j,S)n!$$

(84) $$h_4(n) = (M\psi)^{Y} \prod_{i=Y+S+1}^{n} (K-(i-1))(\psi')^{n-(Y+S)} A(j,S)R!(R)^{n-(R+1)}$$

(85) $$h_5(n) = (M\psi)^{Y} \prod_{i=Y+S+1}^{n} (K-(i-1))(\psi')^{n-(Y+S)} A(j,S)R!\{R\}^{k-1} \prod_{i=1}^{j-1} (R+j)^k \{R+j\}^{n-(R+jk+1)},$$
$$j = 1, 2, \ldots, m-1$$

(86) $$h_6(n) = (M\psi)^{Y} \prod_{i=Y+S+1}^{n} (K-(i-1))(\psi')^{n-(Y+S)} A(j,S)R!(R)^{k-1} \prod_{i=1}^{m-1} (R+j)^k \{R+m\}^{n-(R+mk+1)},$$
$$j = 1, 2, \ldots, m-1$$

(b) When $\phi_n(n) = n$, we get

(87) $$f_1(n) = g_1(n) = h_1(n) = \frac{A(j,n-1)}{n!}$$

(88) $$f_2(n) = \frac{A(j,n-1)}{R!\{R\}^{n-(R+1)}}$$

(89) $$f_3(n) = \frac{(M\psi)^{n-S} A(j,S)}{R!\{R\}^{n-(R+1)}}$$

(90) $$f_4(n) = \frac{(M\psi)^{Y} \prod_{i=Y+S+1}^{n} (K-(i-1))(\psi')^{n-(Y+S)} A(j,S)}{R!\{R\}^{n-(R+1)}}$$

(91) $$f_5(n) = \frac{(M\psi)^{Y} \prod_{i=Y+S+1}^{n} (K-(i-1))(\psi')^{n-(Y+S)} A(j,S)}{R!\{R\}^{k-1} \prod_{i=1}^{j-1} (R+j)^k \{R+j\}^{n-(R+jk+1)}}, \quad j = 1,2,\ldots,m-1$$

(92) $$f_6(n) = \frac{(M\psi)^Y \prod_{i=Y+S+1}^{n} (K-(i-1))\,(\psi')^{n-(Y+S)} A(j,S)}{R!\{R\}^{k-1} \prod_{i=1}^{m-1} (R+j)^k \{R+m\}^{n-(R+mk+1)}}, \quad j=1,2,\ldots,m-1$$

(93) $$g_2(n) = \frac{(M\psi)^{n-S} A(j,S)}{n!}$$

(94) $$g_3(n) = \frac{(M\psi)^{n-S} A(j,S)}{R!\{R\}^{n-(R+1)}}$$

(95) $$g_4(n) = \frac{(M\psi)^Y \prod_{i=Y+S+1}^{n} (K-(i-1))\,(\psi')^{n-(Y+S)} A(j,S)}{R!(R)^{n-(R+1)}}$$

(96) $$g_5(n) = \frac{(M\psi)^Y \prod_{i=Y+S+1}^{n} (K-(i-1))\,(\psi')^{n-(Y+S)} A(j,S)}{R!\{R\}^{k-1} \prod_{i=1}^{j-1} (R+j)^k \{R+j\}^{n-(R+jk+1)}}, \quad j=1,2,\ldots,m-1$$

(97) $$g_6(n) = \frac{(M\psi)^Y \prod_{i=Y+S+1}^{n} (K-(i-1))\,(\psi')^{n-(Y+S)} A(j,S)}{R!\{R\}^{k-1} \prod_{i=1}^{m-1} (R+j)^k \{R+m\}^{n-(R+mk+1)}}, \quad j=1,2,\ldots,m-1$$

(98) $$h_2(n) = \frac{(M\psi)^{n-S} A(j,S)}{n!}$$

(99) $$h_3(n) = \frac{(M\psi)^Y \prod_{i=Y+S+1}^{n} (K-(i-1))(\psi')^{n-(Y+S+1)} A(j,S)}{n!}$$

(100) $$h_4(n) = \frac{(M\psi)^Y \prod_{i=Y+S+1}^{n} (K-(i-1))\,(\psi')^{n-(Y+S+1)} A(j,S)}{R!(R)^{n-(R+1)}}$$

(101) $$h_5(n) = \frac{(M\psi)^Y \prod_{i=Y+S+1}^{n} (K-(i-1))\,(\psi')^{n-(Y+S+1)} A(j,S)}{R!(R)^{k-1} \prod_{i=1}^{j-1} (R+j)^k \{R+j\}^{n-(R+jk+1)}}, \quad j=1,2,\ldots,m-1$$

(102) $$h_6(n) = \frac{(M\psi)^Y \prod_{i=Y+S+1}^{n} (K-(i-1))(\psi')^{n-(Y+S+1)} A(j,S)}{R!(R)^{k-1} \prod_{i=1}^{m-1} (R+j)^k \{R+m\}^{n-(R+mk+1)}}, \quad j=1,2,\ldots,m-1$$

(c) For single permanent server, by putting $\phi_n(n)=1$, our results for probabilities reduce to

(103) $$f_1(n) = g_1(n) = h_1(n) = A(j, n-1)$$

(104) $$f_2(n) = A(j, n-1)$$

(105) $$f_3(n) = (M\psi)^{n-S} A(j,S)$$

(106) $$f_4(n) = f_5(n) = f_6(n) = (M\psi)^Y \prod_{i=Y+S+1}^{n} (K-(i-1))(\psi')^{n-(Y+S+1)} A(j,S)$$

(107) $$g_2(n) = g_3(n) = (M\psi)^{n-S} A(j,S)$$

(108) $$g_4(n) = g_5(n) = g_6(n)(M\psi)^Y \prod_{i=Y+S+1}^{n} (K-(i-1))(\psi')^{n-(Y+S+1)} A(j,S)$$

(109) $$h_2(n) = (M\psi)^{n-S} A(j,S)n!$$

(110) $$h_3(n) = h_4(n) = h_5(n) = h_6(n)(M\psi)^Y \prod_{i=Y+S+1}^{n} (K-(i-1))(\psi')^{n-(Y+S+1)} A(j,S)$$

6.2. Hot standbys model: In this case by setting $\alpha = \lambda$, we have

(111) $$f_1(n) = g_1(n) = h_1(n) = \frac{\prod_{i=0}^{n} [(M+S-(i-1))\psi] P_0}{B(n)}$$

(112) $$f_2(n) = \frac{\prod_{i=0}^{n} [(M+S-(i-1))\psi] P_0}{B(\kappa) \prod_{i=R+1}^{n} \phi_i(R)}$$

(113) $$f_3(n) = \frac{\prod_{i=0}^{n} [(M+S-(i-1))\psi][M\psi]^{n-S} P_0}{B(R) \prod_{i+R+1}^{n} \phi_i(R)}$$

(114) $$f_4(n) = \frac{\prod_{i=0}^{n} [(M+S-(i-1))\psi][M\psi]^Y \prod_{i=Y+S+1}^{n} (K-(i-1))(\psi')^{n-(Y+S+1)} P_0}{B(R) \prod_{i=R+1}^{n} \phi_i(R)}$$

(115) $$f_5(n)=\frac{\prod_{i=0}^{n}[(M+S-(i-1))\psi][M\psi]^{Y}\prod_{i=Y+S+1}^{n}(K-(i-1))(\psi')^{n-(Y+S+1)}P_0}{B(R)\prod_{i=R+1}^{R+k}\phi_i(R)\left[\prod_{l=1}^{j-1}\prod_{i=R+lk+1}^{R+(l+1)k}\phi_i(R+j)\right]\prod_{i=R+jk+1}^{n}\phi_i(R+j)},\ j=1,2,\ldots,m-1$$

(116) $$f_6(n)=\frac{\prod_{i=0}^{n}[(M+S-(i-1))\psi][M\psi]^{Y}\prod_{i=Y+S+1}^{n}(K-(i-1))(\psi')^{n-(Y+S+1)}P_0}{B(R)\prod_{i=R+1}^{R+k}\phi_i(R)\left[\prod_{l=1}^{m-1}\prod_{i=R+lk+1}^{R+(l+1)k}\phi_i(R+j)\right]\prod_{i=R+mk+1}^{n}\phi_i(R+m)},\ j=1,2,\ldots,m-1$$

(117) $$g_2(n)=\frac{\prod_{i=0}^{n}[(M+S-(i-1))\psi][M\psi]^{n-S}P_0}{B(n)}$$

(118) $$g_3(n)=\frac{\prod_{i=0}^{n}[(M+S-(i-1))\psi][M\psi]^{n-S}P_0}{B(R)\prod_{i+R+1}^{n}\phi_i(R)}$$

(119) $$g_4(n)=\frac{\prod_{i=0}^{n}[(M+S-(i-1))\psi][M\psi]^{Y}\prod_{i=Y+S+1}^{n}(K-(i-1))(\psi')^{n-(Y+S)}P_0}{B(R)\prod_{i=R+1}^{n}\phi_i(R)}$$

(120) $$g_5(n)=\frac{\prod_{i=0}^{n}[(M+S-(i-1))\psi][M\psi]^{Y}\prod_{i=Y+S+1}^{n}(K-(i-1))(\psi')^{n-(Y+S+1)}P_0}{B(R)\prod_{i=R+1}^{R+k}\phi_i(R)\left[\prod_{l=1}^{j-1}\prod_{i=R+lk+1}^{R+(l+1)k}\phi_i(R+j)\right]\prod_{i=R+jk+1}^{n}\phi_i(R+j)},\ j=1,2,\ldots,m-1$$

(121) $$g_6(n)=\frac{\prod_{i=0}^{n}[(M+S-(i-1))\psi][M\psi]^{Y}\prod_{i=Y+S+1}^{n}(K-(i-1))(\psi')^{n-(Y+S+1)}P_0}{B(R)\prod_{i=R+1}^{R+k}\phi_i(R)\left[\prod_{l=1}^{m-1}\prod_{i=R+lk+1}^{R+(l+1)k}\phi_i(R+j)\right]\prod_{i=R+mk+1}^{n}\phi_i(R+m)},\ j=1,2,\ldots,m-1$$

(122) $$h_2(n)=\frac{\prod_{i=0}^{n}[(M+S-(i-1))\psi)][M\psi]^{n-S}P_0}{B(n)}$$

(123) $$h_3(n)=\frac{\prod_{i=0}^{n}[(M+S-(i-1))\psi][M\psi]^{Y}\prod_{i=Y+S+1}^{n}(K-(i-1))(\psi')^{n-(Y+S)}P_0}{B(n)}$$

(124) $$h_4(n)=\frac{\prod_{i=0}^{n}[(M+S-(i-1))\psi][M\psi]^{Y}\prod_{i=Y+S}^{n}(K-i)(\psi')^{n-(Y+S)}P_0}{B(R)\prod_{i=R+1}^{n}\phi_i(R)},\quad j=1,2,\ldots,m-1$$

(125) $$h_5(n)=\frac{\prod_{i=0}^{n}[(M+S-(i-1))\psi][M\psi]^{Y}\prod_{i=Y+S+1}^{n}(K-(i-1))(\psi')^{n-(Y+S+1)}P_0}{B(R)\prod_{i=R+1}^{R+k}\phi_i(R)\left[\prod_{l=1}^{j-1}\prod_{i=R+lk+1}^{R+(l+1)k}\phi_i(R+j)\right]\prod_{i=R+jk+1}^{n}\phi_i(R+j)},\quad j=1,2,\ldots,m-1$$

(126) $$h_6(n)=\frac{\prod_{i=0}^{n}[(M+S-(i-1))\psi][M\psi]^{Y}\prod_{i=Y+S+1}^{n}(K-(i-1))(\psi')^{n-(Y+S+1)}P_0}{B(R)\prod_{i=R+1}^{R+k}\phi_i(R)\left[\prod_{l=1}^{m-1}\prod_{i=R+lk+1}^{R+(l+1)k}\phi_i(R+j)\right]\prod_{i=R+mk+1}^{n}\phi_i(R+m)},\quad j=1,2,\ldots,m-1$$

6.3. Cold standby Model : When $\alpha=0$, our model provides the results for cold standby model

6.4. M/M/R Model: In this case by setting $\alpha=0$, $S=0$, $Y=0$ we have

(127) $$f_1(n)=\frac{M!\psi^n P_0}{(M-n)!\,B(n)},\quad 1\le n<R$$

(128) $$f_2(n)=\frac{M!\psi^n P_0}{(M-n)!B(n)\prod_{i=R+1}^{n}\phi_i(R)},\quad R\le n<R+k$$

(129) $$f_3(n)=\frac{M!\psi^{n-(R+k-1)}P_0}{(M-n)!B(R)\prod_{i=R+1}^{R+k}\phi_i(R)\left[\prod_{l=1}^{j-1}\prod_{i=R+lk+1}^{R+(l+1)k}\phi_i(R+j)\right]\prod_{i=R+jk+1}^{n}\phi_i(R+j)},$$
$$R+j\,k\le n<R+(j+1)\,k-1,\quad j=1,2,\ldots,m-1$$

(130) $$f_4(n)=\frac{M!\psi^{n-(R+k-2)}P_0}{(M-n)!B(R)\prod_{i=R+1}^{R+k}\phi_i(R)\left[\prod_{l=1}^{m-1}\prod_{i=R+lk+1}^{R+(l+1)k}\phi_i(R+j)\right]\prod_{i=R+mk+1}^{n}\phi_i(R+m)},$$
$$R+mk\le n<k$$

6.5. Without additional server: By setting $m=0$, we get the results which coincide with the model studied by Wang and Sivazilian (1992)

6.6 M/M/R machine repairmen problem: Substituting $m = 0$ in (127) and (130), we get

$$(131) \qquad P_n = \begin{cases} \dfrac{M!\,\psi^n P_0}{(M-n)!\,B(n)}, & 1 \le n < R \\[2ex] \dfrac{M!\,\psi^n P_0}{(M-n)!\,B(R)\prod_{i=R}^{n}\phi_i(R)} & R \le n \le N \end{cases}$$

This case coincides with classical machine repair model as discussed by Gross and Harris (1985)

7. Conclusion

The purpose of this investigation has been to study the time-sharing machine repair problem with additional repairmen and spares. The finite source time-sharing queueing system with additional service positions has potential and wide utility in computer systems, telecommunication systems and manufacturing systems, etc. We have provided the explicit formulae for the system characteristics, which can be employed to determine the optimal combination of repairmen and the spares simultaneously to minimized the expected cost incurred. Analytical results obtained show promise for the proposed methodology in helping to analyze industrial problems specifically is a wide range of production / manufacturing scenarios.

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Analytic solution for a system of KDV equations

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Abstract: We consider a system of coupled Korteweg-de Vries equations and prove well-posedness results in a space of functions analytic in a strip. The typical class of functions we consider to obtain analytic solution is the Gevery class introduced by Foias and Temam in [6].

1. Introduction

In this work we consider the initial value problem (IVP)

$$(1.1)\qquad \begin{cases} u_t + u_{xxx} + 2\alpha u u_x + v v_x + (uv)_x = 0 \\ v_t + v_{xxx} + 2\beta v v_x + u u_x + (uv)_x = 0 \\ u(x,0) = u_0(x), v(x,0) = v_0(x) \end{cases}$$

where α, β are constants with $\alpha + \beta = 1$ and $x, t \in \mathbb{R}$ This is a system studied by Nutku and Oğuz in [16] and has a structure of the Korteweg-de Vries (KdV) equations coupled in the nonlinear terms. This system has a bi-Hamiltonian structure. If the constants are such that $\alpha = \pm\beta$, then the equations in the system (1.1) can be decoupled.

The main interest of this work is to find solutions $(u(x,t), v(x,t))$ of the IVP (1.1) which admit an extension as an analytic function to a complex strip $S_\sigma := \{x + iy : |y| < \sigma\}$, at least for small values of σ. Analytic Gevrey class introduced by Foias and Temam [6] is a suitable function space for our purpose.

In recent literature, many authors have devoted much effort to get analytic solutions to several evolution equations. An early work in this direction is due to Kato and Masuda [12]. They considered a large class of evolution equations and developed a general method to obtain spatial analyticity of the solution. In particular, the class considered in [12] contains the KdV equation. The more recent results in this field can be found in the work of Hayashi [10], Hayashi and Ozawa [11], de Bouard, Hayashi and Kato [3], Kato and Ozawa [13], Bona, Grujić and Kalish [1, 2], Grujić and Kalish [8, 9] and references there in.

Let us move to introduce some notations and define space of functions in which we will concentrate our work. For $\sigma > 0$ and $s \in \mathbb{R}$, the analytic Gevrey class $G^{\sigma,s}$ is defined as the subspace of $L^2(\mathbb{R})$ with norm.

$$(1.2)\qquad \|f\|^2_{G^{\sigma,s}} = \int_{\mathbb{R}} \langle\xi\rangle^{2s} e^{2\sigma\langle\xi\rangle} |\hat{f}(\xi)|^2 \, d\xi,$$

where $\langle\,.\,\rangle = 1 + |\,.\,|$ and $\hat{f}$ denotes the Fourier transform of f defined by

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(1.3) $$\hat{f}(\xi) = \frac{1}{\sqrt{(2\pi)}} \int_{\mathbb{R}} e^{-ix\xi} f(x)\,dx,$$

whose inverse transform is given by

(1.4) $$f(x) = \frac{1}{\sqrt{(2\pi)}} \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi)\,d\xi.$$

If we define a Fourier multiplier operator A by

$$\widehat{Af}(\xi) = \langle \xi \rangle \hat{f}(\xi)$$

then Gevrey norm of order (σ, s) can be written in terms of the operator A as

$$\| f \|_{G^{\sigma,s}} = \| A^s e^{\sigma A} f \|_{L^2(\mathbb{R})}$$

Note that a function in the Gevrey class $G^{S,\sigma}$ is a restriction to the real axis of a function analytic on a symmetric strip of width 2σ. Hence, our interest is to prove well-posedness result for the IVP (1.1) for given data in $G^{\sigma,s} \times G^{\sigma,s}$ for appropriate s.

Before establishing well-posedness results in the analytic Gevrey class, we will prove the same in the usual Sobolev spaces $H^s \times H^s$. Recall that, H^s denotes the L^2-based Sobolev space of order s with norm

$$\| f \|_{H^s}^2 = \int_{\mathbb{R}} \langle \xi \rangle^{2s} |\hat{f}(\xi)|^2\, d\xi,$$

We denote by $L_t^p(L_x^q)$, $(1 < p < \infty)$ the Banach spaces $L^p(\mathbb{R} : L^q(\mathbb{R}))$ for variables t and x respectively. For $-1 < b < 1$, let $X_{s,b}$ denote the Hilbert space with the norm

$$\| f \|_{X_{s,b}} = \left(\int_{\mathbb{R}^2} (1 + |\tau - \xi^3|)^{2b} (1 + |\xi|)^{2s} |\hat{f}(\xi,\tau)|^2\, d\xi\, d\tau \right)^{1/2},$$

where $\hat{f}(\xi,\tau)$ is the Fourier transform of f in both x and t variables. This is the space introduced by Bourgain [4] in the KdV context to obtain well-posedness results for low regularity data.

Let us recall some properties of the space $X_{s,b}$ regarding the regularity. First, observe that for $f \in X_{s,b}$, one has,

$$\| f \|_{X_{s,b}} = \| (1 + D_t)^b U(t) f \|_{L_t^2(H_x^s)},$$

where $U(t) = e^{-t\partial_x^3}$ is the unitary group associated with the linear *KdV* flow. If $b > 1/2$, the previous remark and the Sobolev lemma imply,

$$X_{s,b} \subset C(\mathbb{R}; H_x^s(\mathbb{R})).$$

We use C to denote various constants whose exact values are immaterial. Also, we use the notation $A \lesssim B$ if there exists a constant $C > 0$ such that $A < C\,B$, $A \gtrsim B$ if there exists a constant $C > 0$ such that $A > C\,B$ and $A \sim B$ if $A \lesssim B$ and $A \gtrsim B$.

Now we state the local existence result for given data in the usual Sobolev space $H^s(\mathbb{R}) \times H^s(\mathbb{R})$.

Theorem 1.1. *For any $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > -3/4$ and $b \in (1/2, 1)$, there exist $T = T(\| u_0 \|_{H^s}, \| v_0 \|_{H^s})$ and a unique solution of (1.1) in the time interval $[-T, T]$ satisfying*

(1.5) $$u, v \in C([-T,T]; H^s(\mathbb{R})),$$

(1.6) $$u, v \in X_{s,b} \subseteq L^p_{x,loc}(\mathbb{R}; L^2_t(\mathbb{R})), \quad for \ 1 \le p \le \infty,$$

(1.7) $$(u^2)_x, (v^2)_x \in X_{s,b-1}$$

and

(1.8) $$u_t, v_t \in X_{s-3,b-1}.$$

Moreover, given $T' \in (0,T)$, *the map* $(u_0, v_0) \mapsto (u(t), v(t))$ *is smooth from* $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ *to* $C([-T',T']; H^s(\mathbb{R})) \times C([-T',T']; H^s(\mathbb{R}))$.

Note that $\int (u^2 + v^2)\, dx$ is conserved by the flow of (1.1). Using this conserved quantity we can obtain an a *priori* estimate in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ which leads to the following global well-posedness result.

Theorem 1.2. *The unique local solution to the initial value problem (1.1) obtained in Theorem 1.1 can be extended globally in time for given data in* $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, *whenever* $s \ge 0$.

Remark 1.3. Using I-method and almost conserved quantity introduced in the KdV *context by Colliander et al in* [5], *the global well-posedness result of the above theorem can be improved for* $s > -3/10$. *There is similar work in this direction by the author in collaboration with Linares in* [17]. *As our interest here is to obtain analytic solutions, we do not proceed in this direction.*

Before stating the main result of this work, let us introduce the function space $X^{\sigma,s,b}$, which is analogue of Bourgain's space $X_{s,b}$ introduced earlier.

For $\sigma > 0$ and $s \in \mathbb{R}$, $b \in [-1, 1]$ define $X^{\sigma,s,b}$ with the norm

$$\| f \|^2_{H^{\sigma,s,b}} = \iint \langle \tau - \xi^3 \rangle \langle \xi \rangle^{2s} e^{2\sigma\langle \xi \rangle} |\hat{f}(\xi,\tau)|^2 \, d\xi \, d\tau.$$

If we define the operator Λ^ρ, for $\rho \in \mathbb{R}$ by

$$\widehat{\Lambda^\rho f}(\xi,\tau) = \langle \tau \rangle^\sigma \hat{f}(\xi,\tau).$$

then we have

$$\| Uf \|_{X^{\sigma,s,b}} = \| A^s e^{\sigma A} \Lambda^b f \|_{L^2(\mathbb{R}^2)}.$$

Let us record that $C([0, T]; G^{\sigma,s})$ denotes the space of continuous functions defined on the interval $[0, T]$ that take values in $G^{\sigma,s}$. If we equip $C([0, T]; G^{\sigma,s})$ with the norm

$$\sup_{0 \le t \le T} \| f(.,.) \|_{G^{\sigma,s}}$$

then it becomes a Banach space.

For $b > 1/2$, using Sobolev embedding we have

$$\sup_{0 \le t \le T} \| f(.,.) \|_{G^{\sigma,s}} \le c \| u \|_{X^{\sigma,s,b}}.$$

Therefore the space $X^{\sigma,s,b}$ is embedded in $C([0, T]; G^{\sigma,s})$ whenever $b > 1/2$. Now we are in position to state the main result of this work which reads as follows.

Theorem 1.4. Let $s \geq 0$ *and* $\sigma > 0$ *then for any* $(u_0, v_0) \in G^{\sigma,s} \times G^{\sigma,s}$, *there exists a time* $T > 0$ *such that the IVP* (1.1) *is well- posed in the space* $C([0,T]; G^{\sigma,s}) \times C([0,T]; G^{\sigma,s})$.

2. Well-posedness result in usual Sovolev space

In this section we will prove well-posedness results in the usual Sobolev spaces. The idea of the proof is similar to the one employed for the Gear and Grimshaw system in the author's previous work in collaboration with Linares in [17]. For the sake of completeness, we just give sketch of the proof.

Proof of Theorem 1.1. Using Duhamel's principle, we study the following system of integral equations equivalent to the system (1.1),

$$\text{(2.1)} \qquad \begin{cases} u(t) = U(t)u_0 - \int_0^t U(t-t')\, F(u,v,u_x,v_x)(t')\,dt', \\ v(t) = U(t)v_0 - \int_0^t U(t-t')\, G(u,v,u_x,v_x)(t')\,dt', \end{cases}$$

where $U(t) = e^{(-t\partial_x^3)}$ is the unitary group that describes the linear KdV flow and F and G are respective nonlinearities.

To find a local solution to the IVP (1.1) we can replace the system (2.1) with the following system

$$\text{(2.2)} \qquad \begin{cases} u(t) = \psi_1(t)U(t)u_0 - \psi_1(t) \int_0^t U(t-t')\,\psi_T(t')\, F(u,v,u_x,v_x)(t')\,dt', \\ v(t) = \psi_1(t)U(t)v_0 - \psi_1(t) \int_0^t U(t-t')\,\psi_T(t')\, G(u,v,u_x,v_x)(t')\,dt', \end{cases}$$

where $\psi \in C_0^\infty(\mathrm{IR})$, $0 \leq \psi(t) \leq 1$ is a smooth function such that

$$\text{(2.3)} \qquad \psi(t) = \begin{cases} 1, & |t| \leq 1, \\ 0, & |t| \leq 2, \end{cases}$$

and $\psi_T(t) = \psi(\frac{t}{T})$, $0 < T \leq 1$.

Now, we consider the following function space where we seek a solution to the IVP (1.1). For given $(u_0, u_0) \in H^s(\mathrm{IR}) \times H^s(\mathrm{IR})$ and $b > 1/2$, let us define,

$$\mathcal{H}_{MN} := \{(u,v) \in X_{s,b} \times X_{s,b} : \|u\|_{X_{s,b}} \leq M,\ \|v\|_{X_{s,b}} \leq N\},$$

where $M = 2C_0 \|u_0\|_{H^s}$ and $N = 2C_0 \|v_0\|_{H^s}$ Then $\mathcal{H}_{MN}$ is a complete metric space with norm,

$$|||(u,v)|||_{\mathcal{H}_{MN}} := \|u\|_{X_{s,b}} + \|v\|_{X_{s,b}}.$$

Without loss of generality, we may assume that $M > 1$ and $N > 1$. For $(u,v) \in \mathcal{H}_{MN}$, let us define the maps,

$$\text{(2.4)} \qquad \begin{cases} \Phi_{u_0}[u,v] = \psi_1(t)U(t)u_0 - \psi_1(t) \int_0^t U(t-t')\,\psi_T(t') F(u,v,u_x,v_x)(t')\,dt' \\ \Psi_{v_0}[u,v] = \psi_1(t)U(t)v_0 - \psi_1(t) \int_0^t U(t-t')\,\psi_T(t') G(u,v,u_x,v_x)(t')\,dt', \end{cases}$$

We prove that $\Phi \times \Psi$ maps $\mathcal{H}_{MN}$ into $\mathcal{H}_{MN}$ and is a contraction. To achieve this goal we use the following estimates

(2.5) $$\| \psi_1 U(t) u_0 \|_{X_{s,b}} \le C \| u_0 \|_{H^s},$$

(2.6) $$\| \psi_T \int_0^t U(t-t') f(t') dt' \|_{X_{s,b}} \le C T^{1-b+b'} \| f \|_{X_{s,b'}}, \; b > \frac{1}{2}, \; b-1 < b' < 0$$

and

(2.7) $$\| \partial_x (uv) \|_{X_{s,b'}} \le c \| u \|_{X_{s,b}} \| v \|_{X_{s,b}}, \; s > -\frac{3}{4}, \frac{1}{2} < b < \frac{3}{4}, b-1 < b' < -\frac{1}{4},$$

Proof of estimates (2.5) and (2.6) is given in [14] and [7] and that of (2.7) is given in [15]. Now using estimates (2.5)-(2.7) we obtain

(2.8) $$\begin{cases} \| \Phi[u,v] \|_{X_{s,b}} \le C_0 \| u_0 \|_{H^s} + C_1 T^{\theta} \{ \| u \|^2_{X_{s,b}} + \| v \|^2_{X_{s,b}} + \| u \|_{X_{s,b}} \| v \|_{X_{s,b}} \} \\ \| \Psi[u,v] \|_{X_{s,b}} \le C_0 \| v_0 \|_{H^s} + C_2 T^{\theta} \{ \| u \|^2_{X_{s,b}} + \| v \|^2_{X_{s,b}} + \| u \|_{X_{s,b}} \| v \|_{X_{s,b}} \}, \end{cases}$$

where $\theta = 1 - b + b'$.

As $(u,v) \in \mathcal{H}_{MN}$, with our choice of M and N we get from (2.8),

(2.9) $$\begin{cases} \| \Phi[u,v] \|_{X_{s,b}} \le \frac{M}{2} + C_1 T^{\theta} \{ M^2 + N^2 + MN \} \\ \| \Psi[u,v] \|_{X_{s,b}} \le \frac{N}{2} + C_2 T^{\theta} \{ M^2 + N^2 + MN \}. \end{cases}$$

If we choose T such that,

$$T^{\theta} \le (2 \max \{C_1, C_2\} (M+N)^2)^{-1}$$

then the estimate (2.9) yields,

$$\| \Phi[u,v] \|_{X_{s,b}} \le M \quad \text{and} \quad \| \Psi[u,v] \|_{X_{s,b}} \le N.$$

Therefore,

$$(\Phi[u,v], \Psi[u,v]) \in \mathcal{H}_{MN}.$$

In an analogous manner we can show that $\Phi \times \Psi : (u,v) \mapsto (\Phi[u,v], \Psi[u,v])$ is a contraction.

Therefore the map $\Phi \times \Psi$ is a contraction map in the ball $\mathcal{H}_{MN}$. Hence, there exists a unique fixed point (u,v) that solves the IVP (1.1) for $T \le \delta$. The remainder of the proof follows a standard argument. □

3. Well-posednesss results in the analytic class

Now we proceed to establish the estimates that are fundamental in the proof of the main result of this work.

3.1.Linear estimates.

Lemma 3.1. *Let $s \in \mathbb{R}$, $\sigma > 0$, $u_0 \in G^{\sigma,s}$, $b > 1/2$ and $b - 1 < b' < 0$. Then there exists a constant C such that*

(3.10) $$\| \psi(t) U(t) u_0 \|_{X^{\sigma,s,b}} \le \| u_0 \|_{G^{\sigma,s}},$$

(3.11) $$\| \psi_T \int_0^t U(t-t') f(t') dt' \|_{X^{\sigma,s,b}} \le C T^{1-b+b'} \| f \|_{X^{\sigma,s,b'}}$$

Proof: For $\sigma = 0$ the estimates in (3.10) and (3.11) turn to be estimates (2.5) and (2.6) respectively. For $\sigma > 0$, we just need to replace u_0 by $e^{\sigma A} u_0$ and f by $e^{\sigma A} f$ and so the proof follows in analogous manner. □

3.2. Bilinear estimate

Lemma 3.2. *Let $u,v \in X^{\sigma,s,b}$, $s \geq 0$, $\sigma > 0$, $\frac{1}{2} < b < \frac{3}{4}$. If $b-1 < b' < -\frac{1}{4}$, then there exists a constant C depending only on s, b and b' such that*

$$\| \partial_x(uv) \|_{X^{\sigma,s,b'}} \leq C \| u \|_{X^{\sigma,s,b}} \| v \|_{X^{\sigma,s,b}}. \tag{3.12}$$

Proof: We give proof of (3.12) for $s = 0$, the general case $s > 0$ follows from it. So, our interest here is to prove

$$\| \partial_x(uv) \|_{X^{\sigma,0,b'}} \leq C \| u \|_{X^{\sigma,0,b}} \| v \|_{X^{\sigma,0,b}}. \tag{3.13}$$

Let us define

$$f(\xi,\tau) = \langle \tau - \xi^3 \rangle^b e^{\sigma \langle \xi \rangle} \hat{u}(\xi,\tau),$$
$$g(\xi,\tau) = \langle \tau - \xi^3 \rangle^b e^{\sigma \langle \xi \rangle} \hat{v}(\xi,\tau).$$

So that, $\| u \|_{X^{\sigma,0,b}} = \| f \|_{L^2_\xi L^2_\tau}$ and $\| v \|_{X^{\sigma,0,b}} = \| g \|_{L^2_\xi L^2_\tau}$. Also,

$$\| \partial_x(uv) \|_{X^{\sigma,0,b'}} = \| \langle \tau - \xi^3 \rangle^{b'} e^{\sigma\langle \xi \rangle} \xi (\hat{u} * \hat{v})(\xi,\tau) \|_{L^2_\xi L^2_\tau}$$

$$= \left\| \langle \tau - \xi^3 \rangle^{b'} e^{\sigma\langle \xi \rangle} \xi \iint \hat{u}(\xi_1,\tau_1)\, \hat{v}(\xi-\xi_1,\tau-\tau_1)\, d\xi_1 d\tau_1 \right\|_{L^2_\xi L^2_\tau} \tag{3.14}$$

$$= \left\| \frac{e^{\sigma\langle \xi \rangle}\xi}{\langle \tau - \xi^3 \rangle^{-b'}} \iint \frac{f(\xi_1,\tau_1) e^{-\sigma\langle \xi_1 \rangle}}{\langle \tau_1 - \xi_1^3 \rangle^b} \frac{g(\xi-\xi_1,\tau-\tau_1) e^{-\sigma\langle \xi-\xi_1 \rangle}}{\langle \tau - \tau_1 - (\xi-\xi_1)^3 \rangle^b} d\xi_1 d\tau_1 \right\|_{L^2_\xi L^2_\tau}$$

Now, the estimate (3.13) can be written in terms of f and g as

$$\left\| \frac{\xi e^{\sigma(\xi)}}{\langle \tau - \xi^3 \rangle^{-b'}} \iint \frac{e^{-\sigma\langle \xi_1 \rangle} f(\xi_1,\tau_1) e^{-\sigma\langle \xi-\xi_1 \rangle} g(\xi-\xi_1,\tau-\tau_1)}{\langle \tau_1 - \xi_1^3 \rangle^b \langle \tau - \tau_1 - (\xi-\xi_1)^3 \rangle^b} d\xi_1 d\tau_1 \right\|_{L^2_\xi L^2_\tau} \tag{3.15}$$

$$C \| f \|_{L^2_\xi L^2_\tau} \| g \|_{L^2_\xi L^2_\tau}.$$

Using Cauchy-Schwarz inequality and Fubiani's theorem, the LHS of (3.15) can be estimated as

$$\left\| \frac{\xi e^{\sigma\langle \xi \rangle}}{\langle \tau - \xi^3 \rangle^{-b'}} \iint \frac{e^{-\sigma\langle \xi_1 \rangle} f(\xi_1,\tau_1) e^{-\sigma\langle \xi-\xi_1 \rangle} g(\xi-\xi_1,\tau-\tau_1)}{\langle \tau_1 - \xi_1^3 \rangle^b \langle \tau - \tau_1 - (\xi-\xi_1)^3 \rangle^b} d\xi_1 d\tau_1 \right\|_{L^2_\xi L^2_\tau}$$

$$\leq \left\| \frac{\xi e^{\sigma\langle \xi \rangle}}{\langle \tau - \xi^3 \rangle^{-b'}} \iint \frac{e^{-\sigma\langle \xi_1 \rangle} e^{-\sigma\langle \xi-\xi_1 \rangle} d\xi_1 d\tau_1}{\langle \tau_1 - \xi_1^3 \rangle^b \langle \tau - \tau_1 - (\xi-\xi_1)^3 \rangle^b} \right\|_{L^\infty_\xi L^\infty_\tau} \| f \|_{L^2_\xi L^2_\tau} \| g \|_{L^2_\xi L^2_\tau}. \tag{3.16}$$

So, to obtain the desired estimate (3.15) and there by (3.13), we need to show

$$(3.17)\qquad \left\| \frac{\xi e^{\sigma\langle\xi\rangle}}{\langle\tau-\xi^3\rangle^{-b'}} \iint \frac{e^{-\sigma\langle\xi_1\rangle} e^{-\sigma\langle\xi-\xi_1\rangle}\, d\xi_1 d\tau_1}{\langle\tau_1-\xi_1^3\rangle^b \langle\tau-\tau_1-(\xi-\xi_1)^3\rangle^b} \right\|_{L_\xi^\infty L_\tau^\infty} < C.$$

Note that, by triangle inequality we have $|\xi| \le |\xi_1| + |\xi-\xi_1|$. So, $e^{\sigma\langle\xi\rangle} \le e^{\sigma\langle\xi_1\rangle} e^{\sigma\langle\xi-\xi_1\rangle}$ and the estimate (3.17) will be proved if we can show

$$(3.18)\qquad \left\| \frac{\xi}{\langle\tau-\xi^3\rangle^{-b'}} \iint \frac{d\xi_1 d\tau_1}{\langle\tau_1-\xi_1^3\rangle^b \langle\tau-\tau_1-(\xi-\xi_1)^3\rangle^b} \right\|_{L_\xi^\infty L_\tau^\infty} < C.$$

The expression in (3.18) is exactly the same term appeared in the proof of the usual bilinear estimate related to the KdV equation in [15]. So, the rest of the proof follows the same lines in [15]. This completes the proof of the lemma. □

3.3. Proof of the main result. Now we will use the linear and bi-linear estimates derived above to prove the main result of this work.

Proof of Theorem 1.4. The idea of proof is the similar to that of Theorem 1.1 presented earlier. We write the IVP (1.1) in its equivalent integral form as in (2.2). For given $(u_0, v_0) \in G^{\sigma,s}(\mathrm{IR}) \times G^{\sigma,s}(\mathrm{IR})$ and $b > 1/2$, let us define $M = 2C_0 \| u_0 \|_{G^{\sigma,s}}$ and $N = 2C_0 \| v_0 \|_{G^{\sigma,s}}$. Now define a ball

$$\mathcal{B}_{MN} := \{(u,v) \in X^{\sigma,s,b} \times X^{\sigma,s,b} : \| u \|_{X^{\sigma,s,b}} \le M, \| v \|_{X^{\sigma,s,b}} \le N\}.$$

Then $\mathcal{B}_{MN}$ is a complete metric space with norm,

$$|\| (u,v) \||_{\mathcal{B}_{MN}} := \| u \|_{X^{\sigma,s,b}} + \| v \|_{X^{\sigma,s,b}}$$

In this case also, without loss of generality, we may assume that $M > 1$ and $N > 1$. For $(u,v) \in \mathcal{B}_{MN}$, let us define the maps $\Phi \times \Psi$ as in (2.5). We will show that $\Phi \times \Psi$ is a contraction map in the ball $\mathcal{B}_{MN}$.

First, let us move to show that $\Phi \times \Psi$ maps the ball $\mathcal{B}_{MN}$ into itself. Using estimates (3.10) – (3.12) we obtain as in (2.9), for $\theta = 1 - b + b'$,

$$(3.19)\qquad \begin{cases} \|\Phi\|_{X^{\sigma,s,b}} \le \frac{M}{2} + C_1 T^\theta \{M^2 + N^2 + MN\} \\ \|\Psi\|_{X^{\sigma,s,b}} \le \frac{N}{2} + C_2 T^\theta \{M^2 + N^2 + MN\}. \end{cases}$$

Now, choosing T such that,

$$T^\theta \le (2 \max\{C_1, C_2\}(M+N)^2)^{-1}$$

we obtain from (3.19),

$$\|\Phi\|_{X^{\sigma,s,b}} \le M \text{ and } \|\Psi\|_{X^{\sigma,s,b}} \le N.$$

Therefore, $\Phi \times \Psi$ maps $\mathcal{B}_{MN}$ into $\mathcal{B}_{MN}$. One can easily prove that $\Phi \times \Psi$ is a contraction map in an analogous manner, so we skip it.

Hence, the map $\Phi \times \Psi$ has a unique fixed point (u,v) which solves the IVP (1.1) for $T \le \delta$ in the ball $\mathcal{B}_{MN}$. The rest of the proof follows a standard argument so we omit the details. This completes the proof of the theorem. □

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Derivation of a class of bilateral generating functions from a set of orthogonal polynomials

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Abstract: A new class of bilateral generating functions are obtained from a set of orthogonal polynomials $\{F^{m}_{mn+v}(x\,;\lambda v)\}$. Known bilateral generating functions are obtained as a particular case.

1. Introduction

In 1983, Datta and Manocha [2] introduced a new class of orthogonal Polynomials $\{F^{m}_{mn+v}(x\,;\lambda v)\}$ with the help of a relation

$$(1.1)\qquad F^{m}_{mn+v}(x\,;\lambda v)=x^{v}\,{}_1F_1\frac{(-n;2v+\lambda+m-1\,;x^{m})}{m}$$

The polynomials $\{F^{m}_{mn+v}(x\,;\lambda v)\}$ satisfies the following differential equation

$$(1.2)\qquad x^{2}y''-(mx^{m+1}-\lambda x)y'+\{m(mn+v)x^{m}-v(v+\lambda-1)\}y=0$$

where λ, m are fixed parameters, n is a variable parameter and v a non-negative integer $< m$. Aim of the present paper is to obtain a new class of bilateral generating functions for $\{F^{m}_{mn+v}(x\,;\lambda v)\}$ which can be written in the form of the following theorem:

Theorem: *If there exists a unilateral generating function of the form*

$$(1.3)\qquad G(x,t)=\sum_{n=0}^{\infty} a_n F^{m}_{mn+v}(x\,;\lambda v)\,t^{n}$$

then the following class of bilateral generating function will hold:

$$(1.4)\qquad (1-mty)^{-(m+v+\lambda-1)/m}\exp(-mx^{m}ty/(1-mty))\,.\,f\,[x/(1-mty)^{1/m},ty/(1-mty)]$$
$$=\sum_{p=0}^{\infty}\sum_{n=0}^{p} a_n m^{p-n}/(p-n)!(n+(2v+\lambda-1+m)/m)_{p-n}\,F^{m}_{mp+v}(x;\lambda v)(ty)^{p}$$

The importance of the theorem lies on the fact that all particular bilateral. Generating functions can be easily deduced by attributing suitable values to a_n and then making use of known linear generating functions involving the class of orthogonal polynomials.

2. Derivation of the generating functions

For $\{F^m_{mn+v}(x;\lambda v)\}$ we have the following partial differential operator [6]

(2.1) $$B = e^y\{x\partial/\partial x + m\partial/\partial y + (m+v+\lambda-1-mx^m)\}$$

such that

(2.2) $$B[F^m_{mn+v}(x;\lambda v)e^{ny}\} = \{m(n+1)+2v+\lambda-1\}\,F^m_{m(n+1)+v}(x;\lambda v)\,e^{(n+1)y}$$

The extended form of the transformation group generated by B is

(2.3) $$[T(\exp(bB)f](x,t) = (1-mtb)^{-(m+v+\lambda-1)/m}\exp(-mx^m\,tb/(1-mtb)).\;f[x/(1-mtb)^{1/m},\,t/(1-mtb)]$$

Let

(2.4) $G(x,y) = \sum_{n=0}^{\infty} a_n F^m_{mn+v}(x;\lambda v)\,y^n$, where a_n is arbitrary, selected in such a way that the left hand side of (2.4) gives known generating functions.

Replacing y by ty in (2.4)

(2.5) $$G(x,ty) = \sum_{n=0}^{\infty} a_n F^m_{mn+v}(x;\lambda v)\,t^n y^n$$

Operating $\exp(bB)$ on both sides of (2.5), left hand side becomes

$$(1-mtby)^{-(m+v+\lambda-1)/m}\exp(-mx^m\,tby/(1-mtby)).\,f[x/(1-mtby)^{1/m},\,ty/(1-mtby)]$$

On the other hand, right hand side reduces to

$$\sum_{p=0}^{\infty}\sum_{n=0}^{p} a_n\,b^{p-n}\,m^{p-n}/(p-n)!\{(n+(2v+\lambda-1+m)/m)_{p-n}\,F^m_{mn+v}(x;\lambda v)(ty)^p$$

Equating and putting $b=1$, we obtain (1.4).

3. Applications

Assuming $a_n = (-p)_n(-n)_n/p!$ and using $f(x,y) = y^n F^m_{mn+v}(x;\lambda v)$

We get

(3.1) $$(1-mty)^{-(m+v+\lambda-1)/m}\exp(-mx^m\,ty/(1-mty)).x^v/(1-mty)^{n+v/m}.$$
$$_1F_1(-n;(2v+\lambda-1+m)/m;x^m/(1-mty)) =$$
$$\sum_{p=0}^{\infty}\frac{|(2v+\lambda-1+m/m)|_p}{p!}\,{}_2F_1(-p,-n;(2v+\lambda-1+m)/m;1/mt)F^m_{mp+v}(x;\lambda v)(mt)^p(y)^{p-n}$$

Changing my by y and putting $t=1$, we obtain the bilateral generating relation

(3.2) $$(1-y)^{-(n+m+2v+\lambda-1)/m}\exp(-x^m\,y/(1-y)).x^v \times {}_1F_1(-n;(2v+\lambda-1+m)/m;x^m/(1-y))$$
$$=\sum_{p=0}^{\infty}\frac{((2v+\lambda-1+m)/m)_p}{p!}\,{}_2F_1(-p,-n;(2v+\lambda-1+m)/m;1/mt)\,F^m_{mp+v}(x;\lambda v)$$

This was derived by Bhattacharya and Rath [1]

Putting $m = 1$, $v = 0$ and $\lambda = 1 + \alpha$ and nothing that $F_n'(x;1+\alpha,0) = n!(1+\alpha)_n L_n^{(\alpha)}(x)$ the bilateral generating relation (3.2) reduces to

$$(1-y)^{-n-\alpha-1}\exp(-xy/(1-y))\,{}_1F_1(-n;1+\alpha;x/(1-y)) = \sum_{p=0}^{\infty} {}_2F_1(-p,-n;1+\alpha;1)\,L_p^{\alpha}(x)y^{p-n}$$

which can be compared with a result of Mcbride (p.37)

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Common fixed point theorems for four maps in D-metric space using certain continuity conditions

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Abstract: In this paper we prove two common fixed point theorems; for four self maps on Dhage metric space using certain orbitally lower semi continuity condition on four maps; which are some probable modifications of theorems of Dhage and Dhage et al.

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Key Words and Phrases: *D*-metric spaces, fixed point, coincidently commuting maps, orbitally lower semi-continuous maps.

Dhage [1] introduced the concept of *D*-metric space as follows.

Definition: A non empty set X, together with a function $D : X \times X \times X \to [\,0, \infty\,)$ is called a *D*-metric space with *D*-metric satisfies the following properties

(i) $D(x,y,z) = 0$ iff $x = y = z$

(ii) $D(x, y, z) = D(p\{x, y, z\})$ where p is a permutation function of x, y, z

(iii) $D(x,y,z) \le D(a,y,z) + D(x,a,z) + D(x,y,a)\ \forall x,y,z,a \in X.$

Definitions [1]: A sequence $\{x_n\} \subset X$ is said to be convergent to a point $x \in X$ if $\underset{m,n\to\infty}{\mathrm{Lt}} D(x_m, x_n, x) = 0.$

A sequence $\{x_n\} \subset X$ is called *D*-Cauchy if $\underset{m,n,p\to\infty}{\mathrm{Lt}} D(x_m, x_n, x_p) = 0$

A complete *D*-metric space is one in which every *D*-Cauchy sequence converges to a point of X. A subset E of a *D*-metric space X is called bounded if there exists a constant $k > 0$ such that $D(x,y,z) \le k\ \forall x,y,z \in$ E. A mapping $f : X \to X$ is continuous if and only if, for any sequence $\{x_n\} \subseteq X$, $x_n \to x$ implies $fx_n \to fx$.

Dhage [1] also claimed that *D*-metric is continuous in all its three variables.

Dhage [2] proved the following

Lemma 1 (Lemma 2.2 [2]) : Let $\{x_n\} \subseteq X$ be bounded with *D*-bound k satisfying $D(x_n, x_{n-1}, x_m) \le \phi^n(k)\ \forall m > n \in \mathrm{N}$ where $\phi : \mathrm{R}^+ \to \mathrm{R}^+$ satisfies $\sum_{n=1} \phi^n(t) < \infty$ for each $t \in \mathrm{R}^+$. Then $\{x_n\}$ is *D*-Cauchy.

Theorem 2 (Theorem 2.1 of [2]): *Let S and T be self maps on a D-metric space X, X be (S,T) - orbitally complete and (S,T) - orbitally bounded D-metric space.*

Suppose that

$D(Sx,Ty,z) \leq \phi(\text{Max}\{D(x,y,z), D(x,Sx,z), D(y,Ty,z), \beta D(x,Ty,z),$

$\beta D(y,Sx,z)\})$ for all $x,y \in X$ and $z \in \overline{O(S,T;x) \cup O(T,S;y)}$ where $0 \leq \beta \leq 1/3$ and

$\phi: R^+ \to R^+$ is continuous, non decreasing, $\phi(t) < t$ for $t > 0$ and $\sum_{n=1} \phi^n(t) < \infty \; \forall t \in R^+$.

Then S and T have a unique common fixed point.

We observed that in Theorem 2, the author used the continuity of D-metric in one variable in proving the existence of fixed point of T. But Naidu et.al. [4] observed that there are D-metrices which are not continuous even in a single variable. (See Example 3 of [4]). Hence the validity of Theorem 2 is doubtful.

Dhage et.al. [3] proved

Theorem 3 (Theorem 2.4 of [3]): *Let A,B,S T be four self maps of a D-metric space X satisfying*

(3.1) $A(X) \subseteq T(X)$, $B(X) \subseteq T(S)$

(3.2) $D(Ax,By,z) \leq \lambda \,\text{Max}\{D(Sx,Ty,z), D(Sx,Ax,z), D(Ty,By,z)\} \; \forall x,y,z \in X$ where $0 \leq \lambda < 1$

(3.3) $\overline{O_{A,B}(S,T,x)}$ is complete for each $x \in X$

(3.4) the pairs $\{A,S\}$ and $\{B,T\}$ are limit coincidentally commuting

(3.5) any one of A,B,S,T is continuous

Then A, B, S and T have a unique common fixed point.

In this theorem also Dhage et. al [3] used the continuity of D-metric in two variables. Hence the validity of this theorem is also doubtful. Even if this theorem is valid, the inequality (3.2) forces the space X to be a singleton set. Hence this theorem is insignificant.

Now we give some modifications of Theorems 2 and 3 without using the continuity of D-metric.

We first give the following definitions

Let A,B,S and T be four self maps on a D-metric space X.

Let $G(x) = \min\{D(Ax,Ax,Sx), D(Ax,Sx,Sx), D(Bx,Bx,Tx), D(Bx,Tx,Tx)\}$,

$G^*(x) = \max\{D(Ax,Ax,Sx), D(Ax,Sx,Sx), D(Bx,Bx,Tx), D(Bx,Tx,Tx)\}$,

$H_1(x) = \max\{D(Ax,Ax,Sx), D(Ax,Sx,Sx)\}$,

$H_2(x) = \max\{D(Bx,Bx,Tx), D(Bx,Tx,Tx)\}$.

We say that A, B, S and T are (G,H_1,H_2)- orbitally lower semi continuous at $u \in X$ if

$$G(u) \leq \underline{\lim_{n \to \infty}} \max\{H_1(x_{2n}), H_2(x_{2n+1})\}$$

whenever there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_0 = Sx_0,\; y_{2n+1} = Ax_{2n} = Tx_{2n+1},$$
$$y_{2n+2} = Bx_{2n+1} = Sx_{2n+2},\; n = 0, 1, 2, \ldots$$

and $\{y_n\}$ converges to u

We prove the following two lemmas

Lemma 4: Let A, B, S and T be four self maps on a D-metric space (X,D) satisfying

(4.1) $D(Ax,By,z) \leq \phi(\text{Max}\{D(Sx,Ty,z), D(Sx,Ax,z), D(Ty,By,z), D(Sx,By,z), D(Ty,Ax,z)\}\ \forall x, y \in X$ and $z = Az_1$ or $B z_2$ for some $z_1, z_2 \in X$ where

$\phi : R^+ \to R^+$ is a mapping such that $\phi(t) < t\ \forall t > 0$.

Further suppose that

(4.2) $A(X) \subseteq T(X),\ B(X) \subseteq S(X)$

(4.3) there exists $u \in X$ such that $Au = Su$ or $Bu = Tu$

(4.4) the pairs $\{A, S\}$ and $\{B,T\}$ are coincidentally commuting at u

Then A, B, S and T have a unique common fixed point.

Proof: Suppose $Au = Su$

Since $A(X) \subseteq T(X)$ there exists $v \in X$ such that $Au = Tv$.

Suppose $Au \neq Bv$

$$D(Au,Bv,Au) \leq \phi(\text{Max}\{D(Su,Tv,Au), D(Su,Au,Au), D(Tv,Bv,Au), D(Su,Bv,Au), D(Tv,Au,Au)\}$$

$= \phi(D(Au,Bv,Au)) < D(Au,Bv,Au)$. It is a contradiction.

Hence $Au = Bv$

Thus $Au = Su = Bv = Tv$ ... ... (I)

Since the pair (A,S) is coincidentally commuting we have

$$A(Au) = A(Su) = S(Au) = S(Su) \quad \text{... ... (II)}$$

Since the pair (B,T) is coincidentally commuting we have

$$B(Bv) = B(Tv) = T(Bv) = T(Tv) \quad \text{... ... (III)}$$

$$D(Au,Au,A^2u) = D(Av,Bv,A^2u) \text{ from (I)}$$

$$\leq \phi(\text{Max}\{D(Su,Tv,A^2u), D(Su,Au,A^2u), D(Tv,Bv.A^2u), D(Su,Bv,A^2u), D(Tv,Au,A^2u\} \text{ from (4.1)}$$

$$= \phi(D(Au,Au,A^2u)) \text{ from (I)}$$

$\therefore A^2u = Au$

Hence $S(Au) = A^2u = Au$ from (II)

Thus Au is a common fixed point of A and S.

$$D(Bv,Bv,B^2v) = D(Au,Bv,B^2v) \text{ from (I)}$$

$$\leq \phi(\text{Max}\{D(Su,Tv,B^2v), D(Su,Au,B^2v), D(Tv,Bv,B^2v), D(Su,Bv,B^2v), D(Tv,Au,B^2v)\} \text{ from (4.1)}$$

$$= \phi(D(Bv,Bv,B^2v)) \text{ from (I)}$$

$\therefore B^2v = Bv$

Hence $T(Bv) = B^2 v = Bv$ from (III)

Thus Bv is a common fixed point of B and T.

Since $Au = Bv$ it follows that Au is a common fixed point of A, B, S, and T.

Similarly if $Bu = Tu$ then Bu is a common fixed point of A, B, S, and T.

Uniqueness of common fixed point follows easily by applying (4.1) two times.

Lemma 5: Let A, B, S, and T be four self maps on a D-metric space X satisfying

(5.1) $D(Ax, By, z) \le \phi(\text{Max}\{D(Sx, Ty, z), D(Sx, Ax, z), D(Ty, By, z),$
$D(Sx, By, z), D(Ty, Ax, z)\}\ \forall x, y \in X$ and $z = x$ or y or Az_1 or Bz_2 for some $z_1, z_2 \in X$.

where $\phi: \mathbb{R}^+ \to \mathbb{R}^+$ is a function such that $\phi(t) < t \quad \forall t > 0$.

(5.2) Suppose $Au = Su$ and $Tu = Bu$ for some $u \in X$.

Then u is the common fixed point of A, B, S and T.

Proof: Write $w_1 = Au = Su, w_2 = Tu = Bu$

Suppose $w_1 \neq w_2$

$$\begin{aligned} D(w_1, w_2, w_1) &= D(Au, Bu, w_1) \\ &\le \phi(\text{Max}\{D(Su, Tu, w_1), D(Su, Au, w_1), D(Tu, Bu, w_1) \\ &\qquad D(Su, Bu, w_1), D(Tu, Au, w_1)\}, \\ &= \phi(\text{Max}\{D(w_1, w_2, w_1), D(w_2, w_2, w_1)\}) \end{aligned}$$

Also

$$\begin{aligned} D(w_1, w_2, w_2) &= D(Au, Bu, w_2) \\ &\le \phi(\text{Max}\{D(Su, Tu, w_2), D(Su, Au, w_2), D(Tu, Bu, w_2), \\ &\qquad D(Su, Bu, w_2), D(Tu, Au, w_2)\} \\ &= \phi(\text{Max}\{D(w_1, w_2, w_2), D(w_1, w_1, w_2)\}) \end{aligned}$$

$$\therefore\ \text{Max}\{D(w_1, w_2, w_2), D(w_1, w_2, w_2)\} \le \phi(\text{Max}\{D(w_1, w_2, w_2), D(w_1, w_1, w_2)\})$$

Hence $w_1 = w_2$.

Thus $Au = Su = Bu = Tu$

Now

$$\begin{aligned} D(Au, Au, u) &= D(Au, Bu, u) \\ &\le \phi(\text{Max}\{D(Su, Tu, u), D(Su, Au, u), D(Tu, Bu, u), \\ &\qquad D(Su, Bu, u), D(Tu, Au, u)\} \\ &= \phi(D(Au, Au, u)) \end{aligned}$$

$$\therefore Au = u$$

Thus $Au = Bu = Su = Tu = u$

Hence u is a common fixed point of A, B, S and T.

Uniqueness of common fixed point follows easily from (5.1) using two times.

Now we give our main theorems.

Theorem 6: *Let A, B, S and T be four self maps on a D-metric space X such that*

(6.1) $A(X) \subseteq T(X), B(X) \subseteq S(X)$

(6.2) $D(Ax,By,z) \leq \phi(\text{Max}\{D(Sx,Ty,z),D(Sx,Ax,z),D(Ty,By,z),$
$\beta D(Sx,By,z), \beta D(Ty,Ax,z)\})$

$\forall x,y \in X, z = Az_1$ or Bz_2 for some $z_1,z_2 \in X$ where $0 \leq \beta \leq 1/3$ and $\phi: R^+ \to R^+$ is non decreasing, $\phi(t) < t$ for $t > 0$ and $\sum_{n=1} \phi^n(t) < \infty \; \forall \; t \in R^+$

(6.3) Assume that for some $x_0 \in X$, there exist sequence $\{x_n\}$ and $\{y_n\}$ such that $y_0 = Sx_0, y_{2n+1} = Ax_{2n} = Tx_{2n+1}, y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}, n = 0,1,2\ldots$ and the set $D(y_a,y_b,y_m)/\; 0 \leq a \leq 1,\; 1 \leq b \leq 2, m \geq 2\}$ is bounded by k.

Then the sequence $\{y_n\}$ is *D*-Cauchy.

Further assume that

(6.4) $\{y_n\}$ converges to some $u \in X$

(6.5) the pairs $\{A,S\}$ and $\{B,T\}$ are coincidentally commuting at u

(6.6) A,B,S,T are (G,H_1,H_2)- orbitally lower semi continuous at u

Then *A, B, S* and *T* have a unique common fixed point.

Proof : For any $m \geq 2$,

$$\begin{aligned}
D(y_1,y_2,y_m) &= D(Ax_0,Bx_1,y_m)\\
&\leq \phi(\text{Max}\{D(Sx_0,Tx_1,y_m),D(Sx_0,Ax_0,y_m),D(Tx_1,Bx_1,y_m),\\
&\qquad \beta D(Sx_0,Bx_1,y_m),\beta D(Tx_1,Ax_0,y_m)\})\\
&= \phi(\text{Max}\{D(y_0,y_1,y_m),D(y_0,y_1,y_m),D(y_1,y_2,y_m),\\
&\qquad \beta D(y_0,y_2,y_m),\beta D(y_1,y_1,y_m)\})\\
&\leq \phi(k)\ldots \text{(i) from (6.3)}
\end{aligned}$$

For $m \geq 3$,

$$\begin{aligned}
D(y_2,y_3,y_m) = D(Bx_1,Ax_2,y_m) &= D(Ax_2,Bx_1,y_m)\\
&\leq \phi(\text{Max}\{D(Sx_2,Tx_1,y_m),D(Sx_2,Ax_2,y_m),D(Tx_1,Bx_1,y_m),\\
&\qquad \beta D(Sx_2,Bx_1,y_m),\beta D(Tx_1,Ax_2,y_m)\})\\
&= \phi(\text{Max}\{D(y_2,y_1,y_m),D(y_2,y_3,y_m),D(y_1,y_2,y_m),\\
&\qquad \beta D(y_2,y_2,y_m),\beta D(y_1,y_3,y_m)\})\\
&= \phi(\text{Max}\{D(y_2,y_1,y_m),\beta D(y_2,y_2,y_m),\beta D(y_1,y_3,y_m)\}) \ldots\ldots \text{(ii)}
\end{aligned}$$

Case : Suppose $D(y_2,y_3,y_m) \leq \phi(D(y_1,y_2,y_m))$

Then $D(y_2,y_3,y_m) \leq \phi(\phi(k))$ from (i)

$= \phi^2(k)$

Case : Suppose $D(y_2,y_3,y_m) \leq \phi(\beta D(y_2,y_2,y_m))$

$$\begin{aligned}
\beta D(y_2,y_2,y_m) &\leq \beta D(y_2,y_2,y_m)+\beta D(y_2,y_1,y_m)+\beta D(y_2,y_2,y_1)\\
&\leq \beta\phi(k)+\beta\phi(k)+\beta\phi(k) \text{ from (i)}\\
&\leq \phi(k)
\end{aligned}$$

$$\therefore\; D(y_2,y_3,y_m) \leq \phi^2(k)$$

Case : Suppose $D(y_2,y_3,y_m) \le \phi(\beta D(y_1,y_3,y_m))$ ...... (iii)

$$\beta D(y_1,y_3,y_m) \le \beta D(y_1,y_2,y_m) + \beta D(y_1,y_2,y_m) + \beta D(y_1,y_3,y_2)$$
$$\le \beta D(y_2,y_3,y_m) + \beta\phi(k) + \beta\phi(k) \text{ from (i)}$$
$$= 2\beta\phi(k) + \beta D(y_2,y_3,y_m) \quad \text{......} \quad \text{(iv)}$$

$$\therefore D(y_2,y_3,y_m) \le \phi[2\beta\phi(k) + \beta D(y_2,y_3,y_m)]$$
$$\le 2\beta\phi(k) + \beta D(y_2,y_3,y_m) \text{ since } \phi(t) < t \text{ for } t > 0.$$

$$\therefore D(y_2,y_3,y_m) < \frac{2\beta}{1-\beta}\phi(k)$$

Now (iv) becomes

$$\beta D(y_1,y_3,y_m) \le 2\beta\phi(k) + \frac{2\beta^2}{1-\beta}\phi(k) = \frac{2\beta}{1-\beta}\phi(k)$$

Hence (iii) becomes $D(y_2,y_3,y_m) \le \phi\left[\frac{2\beta}{1-\beta}\phi(k)\right] \le \phi^2(k)$

Thus in all three cases we have $D(y_2,y_3,y_m) \le \phi^2(k)$.

In general for $m \ge n+1$ we have $D(y_n,y_{n+1},y_m) \le \phi^n(k)$

Now from Lemma 1 (Lemma 2.2 of 2.2[2]) it follows that $\{y_n\}$ is a D-Cauchy sequence in X. Suppose that $\{y_n\}$ converges to some $u \in X$ and A,B,S,T are (G,H_1,H_2)- orbitally lower semi continuous at u.

Then

$$\text{Min}\,\{D(Au,Au,Su), D(Au,Su,Su), D(Bu,Bu,Tu), D(Bu,Tu,Tu)\}$$
$$\le \lim_{n\to\infty} \max\{\max\{D(Ax_{2n},Ax_{2n},Sx_{2n}), D(Ax_{2n},Sx_{2n},Sx_{2n})\},$$
$$\max\{D(Bx_{2n+1},Bx_{2n+1},Tx_{2n+1}), D(Bx_{2n+1},Tx_{2n+1},Tx_{2n+1})\}\}$$
$$= \lim_{n\to\infty} \max\{\max\{D(y_{2n+1},y_{2n+1},y_{2n}), D(y_{2n+1},y_{2n},y_{2n})\},$$
$$\max\{D(y_{2n+2},y_{2n+2},y_{2n+1}), D(y_{2n+2},y_{2n+1},y_{2n+1})\}\} = 0$$

since $\{y_n\}$ is D-Cauchy

$\therefore Au = Su$ or $Bu = Tu$

The rest follows from Lemma 4.

Theorem 7: *Let A,B,S and T be four self maps on a D-metric space (X,D) such that*

(7.1) $D(Ax,By,z) \le \phi(\text{Max}\,\{D(Sx,Ty,z), D(Sx,Ax,z), D(Ty,By,z),$
$\beta D(Sx,By,z), \beta D(Ty,Ax,z)\})$

$\forall x,y \in X, z = x$ or y or Az_1 or Bz_2 for some $z_1,z_2 \in X$ where $0 \le \beta \le 1/3$,

$\phi: \mathbb{R}^+ \to \mathbb{R}^+$ is non decreasing, $\phi(t) < t$ for $t > 0$ and $\sum_{n=1} \phi^n(t) < \infty$ for each $t \in T^+$.

(7.2) Assume that for some $x_0 \in X$, there exist sequence $\{x_n\}$ and $\{y_n\}$ such that $y_0 = Sx_0, y_{2n+1} = Ax_{2n} = Tx_{2n+1}, y_{2n+2} = Bx_{2n+2} = Sx_{2n+2}$, $n = 0, 1, 2 \ldots$ and the set $\{D(y_a, y_b, y_m) / 0 \le a \le 1, 1 \le b \le 2, m \ge 2\}$ is bounded by k.

Then $\{y_n\}$ is a D-Cauchy sequence.

Further assume that

(7.3) $\{y_n\}$ converges to some $u \in X$

(7.4) A, B, S, T are (G^*, H_1, H_2)- orbitally lower semi continuous at u.

Then A, B, S and T have a unique common fixed point.

Proof:- The first part of the proof follows as in Theorem 6.

The rest follows from (7.4) and Lemma 5.

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Certain transformation formulae of hypergeometric type

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Abstract: The object of this paper is to be established certain transformation formulae involving hypergeometric series by making use of Bailey's transformation.

Key words and phrases: Hypergeometric series, Kampe de Feriet hypergeometric function, summation formulae and transformation formulae.

1. Introduction

An infinite series of the form $\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{(1)_n}$ is known as ordinary hypergeometric series. It is denoted as ${}_2F_1[a,b\,;c\,;z]$, where a, b and c are real or complex parameters and z is an argument with $|z| < 1$. The hypergeometric series (GHS) have been the topic of a significant study by W.N. Bailey, R.P. Agarwal, L. J. Slater and more recently G. Gasper and M. Rahman in its general form also.

A generalized hypergeometric series is defined as

$$(1.1)\qquad {}_rF_s\left[\begin{matrix} a_1, a_2, \ldots, a_r\,; \\ b_1, b_2, \ldots, b_s, z \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n, \ldots, (a_r)_n\, z^n}{(b_1)_n (b_2)_n, \ldots, (b_s)_n (1)_n},$$

where

$$(a)_n = \begin{cases} a(a+1), \ldots, (a+n-1), \; n = 1, 2, \ldots \\ \qquad 0, \;\; n = 0. \end{cases}$$

The series (1.1) converges for all z when $r \le s$, at least when none of denominator parameters are zero or negative integer. It converges for $|z| < 1$ when $r = s + 1$ and only converges for $z = 0$ when $r > p + 1$ unless it reduces to a polynomial.

A kampe de Feriet hypergeometric function of two variables is defined as

$$(1.2)\qquad F^{\lambda;\mu;\mu'}_{\rho;\upsilon;\upsilon'}\left[\begin{matrix} (a_\lambda) : (b_\mu), (B'_{\mu'})\,; x, y \\ (c_\rho), (d_\upsilon), (d'_{\upsilon'}) \end{matrix}\right]$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{[(a_\lambda)]_{m+n}[(b_\mu)]_m[(b'_{\mu'})]_n\, x^m y^n}{[(c_\rho)]_{m+n}[(d_\upsilon)]_m[(d'_{\upsilon'})]_n\,(1)_m\,(1)_n},$$

where $(a_\lambda)_m$ stand for the sequence of parameters $a_1, a_2, \ldots, a_\lambda$ and $|x|<1$, $|y|<1$, and $\lambda+\mu'+\mu'<\rho+\upsilon+\upsilon'+1$ for convergence.

2. In order to establish certain transformation and summation formulae for generalized hypergeomtric series, we shall make use of the following Bailey's transformation:

If

(2.1) $$\beta_n = \sum_{r=0}^{n} \alpha_r\, U_{n-r}\, V_{n+r} \quad \text{and}$$

(2.2) $\gamma_n = \sum_{r=0}^{n} \delta_{r+n}\, u_r\, v_{r+2n}$, where α_r, δ_r, u_r and v_r are the functions of r only such that the series γ_n exists then under suitable convergence conditions:

(2.3) $$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n$$

Now, we shall be in need of the following known relations due to Slater [4]:

(2.4) $${}_7F_6\left[\begin{matrix} a, 1+\frac{a}{2}, b, c, d, 1+2a-b-c-d+n, -n; 1 \\ \frac{a}{2}, 1+a-b, 1+a-c, 1+a-d, b+c+d, 1+a+n \end{matrix}\right]$$

$$= \frac{(1+a)_n(1+a-b-c)_n(1+a-b-d)_n(1+a-c-d)_n}{(1+a-b)_n(1+a-c)_n(1+a-d)_n(1+a-b-c-d)_n}$$

(2.5) $${}_3F_2\left[\begin{matrix} b, -n; 1 \\ 1+a-b, 1+a+n \end{matrix}\right] = \frac{(1+a)_n\left(1+\frac{a}{2}-b\right)_n}{\left(1+\frac{a}{2}\right)(1+a-b)_n}$$

(2.6) $${}_4F_3\left[\begin{matrix} a, 1+\frac{a}{2}a, b, -n; 1 \\ \frac{1}{2}a, 1+a-b, 1+a+n \end{matrix}\right] = \frac{(1+a)_n}{(1+a-b)_n}$$

(2.7) $${}_3F_2\left[\begin{matrix} a, b, -n; 1 \\ 1+a-b, a+2b-n \end{matrix}\right] = \frac{(a-2b)_n\left(1+\frac{a}{2}-b\right)_n(-b)_n}{(1+a-b)_n\left(\frac{a}{2}-b\right)_n\left(\frac{a}{2}-b\right)_n(-2b)_n}$$

(2.8) $${}_4F_3\left[\begin{matrix} a, 1+\frac{a}{2}, b, -n; 1 \\ \frac{a}{2}, 1+a-b, 1+2b-n \end{matrix}\right] = \frac{(a-2b)_n(-b)_n}{(1+a-b)_n(-2b)_n}$$

(2.9) $${}_2F_1\left[\begin{matrix} a,b\,;1 \\ 1+a-b \end{matrix}\right] = \frac{(1+a)_n(1+b)_n}{(1+a+b)_n(1)_n}$$

(2.10) $${}_2F_1\left[\begin{matrix} a,b\,;1 \\ c \end{matrix}\right] = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

3. Main Results.

(3.1) If we take

$$u_n = \frac{(1+a-b-c-d)_n}{(1)_n},$$

$$v_n = \frac{(1+2a-b-c-d)_n}{(1+a)_n},$$

and $$\alpha_n = \frac{(a)_n\left(1+\frac{a}{2}\right)_n(b)_n(c)_n(d)_n}{(1)_n\left(\frac{a}{2}\right)_n(1+a-b)_n(1+a-c)_n(1+a-d)_n}$$

in (2.1), then we have

$$\beta_n = \sum_{r=0}^{n} \frac{(a)_r\left(1+\frac{a}{2}\right)_r(b)_r(c)_r(d)_r}{(1)_r\left(\frac{a}{2}\right)_r(1+a-b)_r(1+a-c)_r(1+a-d)_r} \times \frac{(1+a-b-d)_{n-r}\,(1+2a-b-d)_{n+r}}{(1)_{n-r}(1+a)_{n+r}}$$

$$= \frac{(1+a-b-c-d)_n(1+2a-b-c-d)_n}{(1)_n(1+a)_n}$$

$$\times \sum_{r=0}^{n} \frac{(a)_r\left(1+\frac{a}{2}\right)_r(b)_r(c)_r(d)_r(1+2a-b-c-d+n)_r}{(1)_r\left(\frac{a}{2}\right)_r(1+a-b)_r(1+a-c)_r(1+a-d)_r} \times \frac{(-n)_r}{(b+c+d-n)_r(1+a+n)_r}$$

$$= \frac{(1+a-b-c-d)_n(1+2a-b-c-d)_n}{(1)_n(1+a)_n}$$

(2.4) $$\times\, {}_7F_6\left[\begin{matrix} a,1+\frac{a}{2},b,c,d,1+2a-b-c+n,-n;1 \\ \frac{a}{b},1+a-b,1+a-c,1+a-d,b+c+d-n,1+a+n; \end{matrix}\right]$$

We shall now make use of (2.4) to obtain.

$$\beta_n = \frac{(1+2a-b-c-d)_n (1+a-b-c)_n (1+a-b-d)_n (1+a-c-d)_n}{(1)_n (1+a-b)_n (1+a-c)_n (1+a-d)_n}$$

Putting the values of u_n and v_n in (2.2) and taking $\delta_n = 1$, we have

$$\gamma_n = \sum_{r=0}^{\infty} 1, \frac{(1+a-b-c-d)_r (1+2a-b-c-d)_{r+2n}}{(1)_r (1+a)_{r+2n}}$$

$$= \frac{(1+2a-b-c-d)_{2n}}{(1+a)_{2n}} \times \sum_{r=0}^{\infty} \frac{(1+a+b-c-d)_r (1+2a-b-c-d+2n)_r}{(1)_r (1+a+2n)_r}$$

$$= \frac{(1+2a-b-c-d)}{(1+a)_{2n}} {}_2F_1 \left[\begin{matrix} 1+a-b-cd, 1+2a-b-c-d+2n ; 1 \\ 1+a+2n \end{matrix} \right]$$

Also, making use of (2.10), we get

$$\gamma_n = (1+2a-b-c-d)_{2n} \frac{\Gamma(1+a)\Gamma(2b+2c+2d-2a-1)}{\Gamma(b+c+d)\Gamma(b+c+d-a)},$$

provided $\operatorname{Re}(2b+2c+2d-2a-1) > 0.$

Putting the values of $\beta_n, \gamma_n, \alpha_n$ and δ_n in (2.3) , we get

$$\sum_{n=0}^{\infty} \frac{(a)_n \left(1+\frac{a}{2}\right)_n (b)_n (c)_n (d)_n}{(1)_n \left(\frac{a}{2}\right)_n (1+a-b)_n (1+a-c)_n (1+a-d)_n}$$

$$\times \frac{(1+2a-b-c-d)_{2n} \Gamma(1+a)\Gamma(2b+2c+2d-2a-1)}{(b+c+d)_{2n} \Gamma(b+c+d)\Gamma(b+c+d-a)}$$

$$= \sum_{n=0}^{\infty} \frac{(1+2a-b-c-d)_n (1+a-b-c)_n (1+a-b-d)_n (1+a=c=d)_n}{(1)_n (1+a-b)_n (1+a-c)_n (1+a-d)_n}$$

$$\times \frac{\Gamma(1+a)\Gamma(2b+2c+2d-2a-1)}{\Gamma(b+c+d)\Gamma(b+c+d-a)}$$

$$\times {}_7F_6 \left[\begin{matrix} a, 1+\frac{a}{2}, b, c, d-\frac{b}{2}-\frac{c}{2}-\frac{d}{2}, 1+a-\frac{b}{2}-\frac{c}{2}-\frac{d}{2} ; 1 \\ \frac{a}{2}, 1+a-b, 1+a-c, 1+a-d, \frac{b}{2}+\frac{c}{2}+\frac{d}{2}, \frac{1}{2}+\frac{b}{2}+\frac{c}{2}+\frac{d}{2} ; \end{matrix} \right]$$

$$= {}_4F_3 \left[\begin{matrix} 1+2a \quad b-c-d, 1+a-b-c, 1+a-b-d, 1+a-c-d ; 1 \\ 1+a-b, 1+a-c, 1+a-d \end{matrix} \right]$$

provided $\operatorname{Re}(2b+2c+2d-2a) > 0$

(3.2) If we take

$$u_n = \frac{1}{(1)_n},\ v_n = \frac{1}{(1+a)_n} \text{ and } \alpha_n = \frac{(a)_n (b)_n (-1)^n}{(1)_n (1+a-b)_n} \text{ in (2.1), then we have}$$

$$\beta_n = \sum_{r=0}^{n} \frac{(a)_r\,(b)_r (-1)^r}{(1)_r\,(1+a-b)_r\,(1)_{n-r}\,(1+a)_{n+r}}$$

$$= \frac{1}{(1)_n\,(1+a)_n} \sum_{r=0}^{n} \frac{(a)_r\,(b)_r (-n)_r}{(1)_r\,(1+a-b)_r\,(1+a+n)_r}$$

$$= \frac{1}{(1)_n\,(1+a)_n}\ {}_3F_2\left[\begin{matrix} a,b,-n\,; \\ 1+a-b, 1+a-n\,; \end{matrix}\right].$$

Now, making use of (2.5), we get

$$\beta_n = \frac{\left(1+\frac{a}{2}-b\right)_n}{n!\left(1+\frac{a}{2}\right)_n (1+a-b)_n}$$

Again, putting the values of u_n, v_n in (2.2) and taking $\delta_n = (\alpha)_n (\beta_n)$, we have

$$\gamma_n = \sum_{r=0}^{\infty} (\alpha)_{r+n}\,(\beta)_{r+n}\,\frac{1}{(1)_r} \times \frac{1}{(1+a)_{r+2n}}$$

$$= \frac{(\alpha)_n\,(\beta)_n}{(1+a)_{2n}} \sum_{r=0}^{\infty} \frac{(a+n)_r\,(\beta+n)_r}{(1)_r\,(1+a+2n)_r}$$

$$= \frac{(\alpha)_n\,(\beta)_n}{(1+a)_{2n}}\ {}_2F_1\left[\begin{matrix} \alpha+n\,, \beta+n\,; 1 \\ 1+a+2n\,; \end{matrix}\right]$$

Finally, using (2.10), we have

$$\gamma_n = \frac{\Gamma(1+a)\,\Gamma(1+a-\alpha-\beta)}{\Gamma(1+a-\alpha)\,\Gamma(1+a-\beta)} \cdot \frac{(\alpha)_n\,(\beta)_n}{(1+a-\alpha)_n (1+a-\beta)_n},$$

Provided $\operatorname{Re}(1+a-\alpha-\beta) > 0$

Putting the values of $\alpha_n, \beta_n, \gamma_n$ and δ_n in (2.3), we get

$$\frac{\Gamma(1+\alpha)\,\Gamma(1+a-\alpha-\beta)}{\Gamma(1+a-\alpha)\,\Gamma(1+a-\beta)} \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n (\alpha)_n\,(\beta)_n (-1)_n}{(1)_n (1+a-b)_n (1+a-\alpha)_n (1+a-\beta)_n}$$

$$= \sum_{n=0}^{\infty} \frac{\left(1+\frac{a}{2}-b\right)_n (\alpha)_n (\beta)_n}{(1)_n \left(1+\frac{a}{2}\right)_n (1+a-b)_n} \times {}_4F_3\left[\begin{matrix} a,b,\alpha,\beta\,; 1 \\ 1+a-b, 1+a-\alpha\,, 1+\alpha-\beta \end{matrix}\right]$$

$$= \frac{\Gamma(1+a-\alpha)\,\Gamma(1+a-\beta)}{\Gamma(1+a)\,\Gamma(1+a-\alpha-\beta)} \times {}_3F_2\left[\begin{matrix}\alpha, \beta, 1+\frac{a}{2}, -b\,; 1\\ 1+\frac{a}{2}, 1+a-b\end{matrix}\right]$$

provided $\operatorname{Re}(1+a-\alpha-\beta)>0$

Similarly, making use of (2.6) – (2.9), we can establish the other transformation formulae with the help of (2.10).

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Cartesian product of r hyperbolic Hermite manifolds

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Abstract: Cartesian Product of two manifolds has been defined and studied by Pandey [2]. In this paper we have taken Cartesian product of r Hyperbolic Manifold, where r is some finite integer and studied some properties of curvature and Ricci tensor of such a product manifold.

1. Introduction:

Let $M_1, M_2, \ldots, M_r$ be r Hyperbolic Hermite structure manifolds each of class C^∞ and of dimension $n_1, n_2, \ldots, n_r$ respectively. Suppose $(M_1)m_1, (M_2)\,m_2, \ldots, (M_r)\,m_r$ be their tangent spaces at $m_1 \in M_1$, $m_2 \in M_2, \ldots, m_r \in M_r$. Then the product space $(M_1)m_2 \times (M_2)m_2 \times, \ldots, \times (M_r)m_r$ contains vector fields of the form $(X_1, X_2, \ldots, X_r)$ where $X_1 \in (M_1)m_1$, $X_2 \in (M_2)m_2, \ldots, X_r \in (M_r)\,m_r$ [2]. Vector addition and scalar multiplication on above product space are defined as follows:

$$(1.1)\qquad (X_1, X_2, \ldots, X_r) + (Y_1, Y_2, \ldots, Y_r) = (X_1 + Y_2, X_1 + Y_2, \ldots, X_r + Y_r)$$

$$(1.2)\qquad \lambda(X_1, X_2, \ldots, X_r) = (\lambda X_1, \lambda X_2, \ldots, \lambda X_r)$$

Where $X_i, Y_i \in (M_i)m_i$, $i = 1, 2, \ldots, r$ and λ is a scalar.

Under these conditions the product space $(M_1)m_1 \times (M_2)m_2 \times \ldots \times (M_r)\,m_r$ forms a vector space.

Define a linear transformation F on the product space.

$$(1.3)\qquad F(X_1, X_2, \ldots, X_r) = (F_1X_1, F_2X_2, \ldots, F_rX_r)$$

Where $F_1, F_2, \ldots, F_r$ are linear transformations on $(M_1)m_1, (M_2)\,m_2, \ldots, (M_r)\,m_r$ respectively

If $f_1, f_2, \ldots, f_r$ be C^∞ functions over the spaces $(M_1)m_1, (M_2)\,m_2, \ldots, (M_r)\,m_r$ respectively, we define the C^∞ function $(f_1, f_2, \ldots, f_r)$ on the product as

$$(1.4)\qquad (X_1, X_2, \ldots, X_r)(f_1, f_2, \ldots, f_r) = (X_1f_1, X_2f_2, \ldots, X_rf_r)$$

Let If $D_1, D_2, \ldots, D_r$ be the connection on the Manifolds $M_1, M_2, \ldots, M_r$ respectively. We define the operator D on the product space as

$$(1.5)\qquad D_{(X_1X_2,\ldots,X_r)}(Y_1, Y_2, \ldots, Y_r) = D_{1X_1}, D_{2X_2}, \ldots, D_{rX_r})$$

Then D satisfies all four properties of a connection and thus it is a connection on the product manifold.

2. Some Results

Theorem 2.1. *The product manifold $M_1 \times M_2 \times \ldots \times M_r$ admits an almost Hyperbolic Hermite structure If and only if the manifolds $M_1, M_2, \ldots, M_r$ are almost Hyperbolic Hermite structure manifolds.*

Proof: Suppose $M_1, M_2, \ldots, M_r$ are Hyperbolic Hermite structure manifolds. Thus there exist tensor fields $F_1, F_2, \ldots, F_r$ each of type (1.1) on $M_1, M_2, \ldots, M_r$ respectively satisfying

$$F_i^2(X_i) = X_i \quad i = 1, 2, \ldots, r. \tag{2.1}$$

In view of equation (1.3) it follows that there exists a linear transformation F on $M_1 \times M_2 \ldots \times M_r$ satisfying

$$F^2(X_1, X_2, \ldots, X_r) = (F_1^2 X_1 F_2^2, X_2, \ldots, F_r^2 X_r) = (X_1, X_2, \ldots, X_r) \tag{2.2}$$

Let us define a Riemannian metric g on the product manifold $M_1 \times M_2 \times \ldots \times M_r$ as

$$g((X_1, X_2, \ldots, X_r), (Y_1, Y_2, \ldots, Y_r)) = g_1(X_1, Y_1) + g_2(X_2, Y_2) + \ldots + g_r(X_r, Y_r) \tag{2.3}$$

Where

$$g((FX_1, FX_2, \ldots, FX_r), (FY_1, FY_2, \ldots, FY_r)) = -g_1(X_1, Y_1) - g_2(X_2, Y_2) - \ldots - g_r(X_r, Y_r) - \{\eta(X_1)\eta(Y_1) + \eta(X_2)\eta(Y_2) + \ldots + \eta(X_r)\eta(Y_r)\} \tag{2.4}$$

and $g_1, g_2, \ldots, g_r$ are Riemannian metrics over the manifold $M_1 \times M_2 \times \ldots \times M_r$ respectively. Thus the product space admits an almost Hyperbolic Hermite structure [1].

If $\xi_1, \xi_2, \ldots, \xi_r$ be the vector fields and $\eta_1, \eta_2, \ldots, \eta_r$ be 1-form on the almost Hyperbolic Hermite structure manifolds $M_1, M_2, \ldots, M_r$ respectively then a vector field ξ and a 1-form η on the product manifold is defined as

$$\eta(X)\xi = (\eta_1(X_1)\xi_1, \eta_2(X_2)\xi_2, \ldots, \eta_r(X_r)\xi_r) \tag{2.5}$$

We now prove the following results.

Theorem 2.2. *The Hyperbolic product manifold $M_1 \times M_2 \times \ldots \times M_r$ admits Hyperbolic contact structure if and only if the manifold $M_1, M_2, \ldots, M_r$ possess the same structure.*

Proof: Let $M_1, M_2, \ldots, M_r$ are Hyperbolic almost contact manifolds. Thus there exists tensor fields F_i, of type (1,1), vector fields ξ_i and 1-forms. η_i, $i = 1, 2, \ldots, r$

$$F_i^2(X_i) = X_i + \eta_i(X_i)\xi_i \tag{2.6}$$

For product manifold $M_1 \times M_2 \times \ldots \times M_r$

$$F^2(X_1, X_2, \ldots, X_r) = (P_1^2 X_1, X_2, F_2^2 X_2, \ldots, F^2, X_r)$$

which is view of (2.5) and (2.6) takes the form

$$F^2(X_1, X_2, \ldots, X_r) = (X_1, X_2, \ldots, X_r) + (\eta_1(X_1)\xi_1, \eta_2(X_2)\xi_2, \ldots, \eta_r(X_r)\xi_r)$$

or $$F^2(X) = X + \eta(X)\xi$$

Hence product manifold admits a Hyperbolic almost contact metric structure [3].

Theorem 2.3. *The Hyperbolic product manifold $M_1 \times M_2 \times \ldots \times M_r$ admits Hyperbolic Kahler structure if and only if the manifolds $M_1, M_2, \ldots, M_r$ are Kahler manifolds.*

Proof: Suppose $M_1, M_2, \ldots, M_r$ are Kahaler manifolds. Then

(2.7) $$(D_{1x_1}\ F_1)(Y_1) = (D_{2x_2}\ F_2)(Y_1) = \ \ldots\ = (D_{rx_r}\ F_r)(Y_r) = 0$$

As D is a connection on the product manifold. Hence

(2.8) $$D_{(x_1x_2,\ldots,x_r)}F)(Y_1, Y_2, \ldots, Y_r) = D_{(x_1x_2,\ldots,x_r)}\{F(Y_1, Y_2, \ldots, Y_r) - $$
$$- F\{D_{(x_1x_2,\ldots,x_r)}(Y_1, Y_2, \ldots, Y_r)\}$$

Which in view of equation (1.3), equation (1.5) takes the form

$$D_{(x_1x_2,\ldots,x_r)}F)(Y_1, Y_2, \ldots, Y_r) = D_{(x_1x_2,\ldots,x_r)}\{F_1Y_1, F_2Y_2, \ldots, F_rY_r) -$$
$$- F(D_{1x_1}Y_1, D_{2x_2}Y_2, \ldots, D_{rx_r}Y_r)$$
$$= (D_{1x_1}F_1Y_1, D_{2x_2}F_2Y_2, \ldots, D_{rx_r}F_rY_r) - (F_1D_{1x_1}Y_1, F_2D_{2x_2}Y_2, \ldots, F_rD_{rx_r}Y_r)$$
$$= ((D_{1x_1}F_1), (D_{2x_2}F_2)(Y_2), \ldots, (D_{rx_r}F_r)(Y_r))$$
$$= 0.$$

Thus the product manifold is Hyperbolic Kahler structure manifold.

Theorem 2.4. *The product manifold $M_1 \times M_2 \times \ldots \times M_r$ of a Hyperbolic almost contact metric structure manifolds $M_1, M_2, \ldots, M_r$ is almost Tachibana if and only if the manifolds $M_1, M_2, \ldots, M_r$ are separately Tachibana manifolds.*

Proof: Let an almost Hyperbolic Hermite structure manifolds $M_1, M_2, \ldots, M_r$ are almost Tachibana manifolds. Then

(2.9) $$(D_{iX_i}F_i)(Y_i) + (D_{iX_i}F_i)(Y_i) = 0, i = 1, 2, \ldots, r$$

The result follows in view of the previous theorem (2.3)

3. Curvature and Ricci tensor

Let $X = (X_1, X_2, \ldots, X_r)$ and $Y = (Y_1, Y_2, \ldots, Y_r)$ be C^∞ vector fields on the product manifold $M_1 \times M_2 \times \ldots \times M_r$ and $f = (f_1, f_2, \ldots, f_r)$ be a C^∞ function. Then

$$[(X_1, X_2, \ldots, X_r), (Y_1, Y_2, \ldots, Y_r)](f_1, f_2, \ldots, f_r)$$
$$= (X_1, X_2, \ldots, X_r)\{(Y_1, Y_2, \ldots, Y_r)$$

(3.1) $$(f_1, f_2, \ldots; f_r)\} - (Y_1, Y_2, \ldots, Y_r)\{(X_1, X_2, \ldots, X_r), (f_1, f_2, \ldots, f_r)\}.$$
$$= ([X_1, Y_1]f_1, [X_1, Y_2]f_2, \ldots, [X_r, Y_r]f_r).$$

Suppose $K_i[X_i, Y_i, Z_i)$, $i = 1,2, \ldots, r$ be the curvature tensors of the almost Hyperbolic Hermite structure manifolds $M_1, M_2, \ldots, M_r$ respectively. If $K(X, Y, Z)$ be the Curvature tensor of the product manifold $M_1 \times M_2 \times \ldots \times M_r$.

Then we have

$$K(X, Y, Z) = [K_1(X_1, Y_1, Z_1], K_2(X_2, Y_2, Z_2), \ldots, K_r(X_r, Y_r, Z_r)]. \tag{3.2}$$

Let $W = (W_1, W_2, \ldots, W_r)$ be a vector field on the product manifold. Then

$$K'(X, Y, Z, W) = g(K(X, Y, Z)\, W) \tag{3.3}$$

$$K'(X, Y, Z, W) = K_1'(X_1, Y_1, Z_1, W_1) + K_2'(X_2, Y_2, Z_2, W_2) + \ldots + K_r'(X_r, Y_r, Z_r, W_r) \tag{3.4}$$

Then we have

Theorem 3.1. *The product manifold $M_1 \times M_2 \times \ldots \times M_r$ is of constant curvature if and only if almost Hyperbolic Hermite structure manifolds $M_1, M_2, \ldots, M_r$ are separately of constant curvature.*

Theorem 3.2. *The Ricci tensor of the product manifold $M_1 \times M_2 \times \ldots \times M_r$ is the sum of the Ricci tensor of the almost Hyperbolic Hermite structure manifolds $M_1, M_2, \ldots, M_r$.*

Theorem 3.3. *The product manifold $M_1 \times M_2 \times \ldots \times M_r$ is an Einstein space if and only if almost Hyperbolic Hermite manifolds $M_1, M_2, \ldots, M_r$ are separately Einstein spaces.*

Proof: Let the product manifold $M_1 \times M_2 \times \ldots \times M_r$ is an Einstein space. Thus

$$\mathrm{Ric}(X, Y) = Cg(X, Y) \tag{3.5}$$

Where $C = \frac{K}{n}$, K being the scalar curvature and n being the dimension of the product manifold. Then

$$\mathrm{Ric}(X_i, Y_i) = Cg_i(X_i\, Y_i), \quad i = 1,2, \ldots, r.$$

Therefore the manifolds $M_1, M_2, \ldots, M_r$ are also Einstein spaces.

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