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Number Theory as an Experimental Science

Donald G. Malm

The digital computer presents the mathematics teacher with the opportunity to approach various parts of mathematics in an experimental or empirical manner, quite different from the usual axiomatic approach. In particular, there are many advantages to teaching number theory as an experimental science.

In speaking of number theory as an "experimental science," I do not mean to imply that proofs are to be ignored. Proofs are still important. The experimental work should arise in the formulation and motivation of the theorems. Usually, in a standard mathematics course, the teacher and the textbook present the statements of the theorems to the students. By teaching number theory as an experimental science, I mean that the mathematical question or situation is presented to the students (rather than the answer). Then the experimental evidence is obtained, and the theorem formulated on the basis of the empirical facts. The theorem then requires proof. Often the experimental evidence is helpful for that too.

Number theory is an excellent preparation for modern algebra, which furnishes motivation for algebra as well as many concrete examples for later use. In addition, number theory is intrinsically interesting (and doubly so when combined with the computer). A computational-experimental approach to number theory also reflects the historical development of the subject, for number theory has always involved observation of phenomena about integers, followed by attempts at proofs.

Number theory, approached experimentally, is also an excellent vehicle for teaching problem solving. The axiomatic approach to mathematics has often led to the neglect of this important topic. There has recently been an increasing realization that students need problem-solving skills, and a number theory course of this type can help to provide them.

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*This article has appeared in ECMI/2 Resource Manual, which was produced as a follow up to the Second Conference on Educational Computing in Minority Institutions held in Atlanta, Georgia on March 9-12, 1976.

The teacher does not need to be a computer expert to use the computer in a number theory course. Indeed, it can be argued that the theory of numbers offers the easiest point of entry to computer use for a mathematics teacher. There are several reasons for this. To make a significant contribution to the course, the computed need only be used for numerical calculations, and the programs needed are relatively simple. Any general purpose programming language is suitable, and almost any computer facility is usable for a number theory course. Large installations are not needed. A time-sharing system is desirable, to help students debug their programs, but is not necessary. The instructor, even though inexperienced in programming, will find it easy to get involved with a computer and use it as an integral part of the course.

Because of this simple usage of the computer, relatively little programming knowledge is required of the students. It is quite feasible to start with students who have never programmed and teach them all the programming they need in the course, without diverting very much time from number theory, provided that a simple language, such as BASIC, is available. This means that a number theory course using the computer does not need any programming prerequisite. This is true even though the act of programming can be made to be an important part of the process of learning number theory.

To summarize, the objectives in using the computer in number theory are:

1) To provide illustrations of theorems.

2) To provide reinforcement and deeper understanding of theorems through the writing of programs which require thorough understanding of the proofs of the theorems.

3) To produce experimental evidence from which the students can discover facts and theorems and make further conjectures.

4) To develop problem-solving skills in the students.

These are the main goals. To successfully use the computer, especially with respect to goal number three, it is necessary to select and organize the experiments and homework assignments carefully, so that the students' learning will be maximized. These assignments and experiments will be discussed later.

A Description of Two Number Theory Courses Using the Computer

The author has taught number theory using the computer on two occasions, at Oakland University in 1972-73 and 1974-75, and has developed text material for an experimental approach. Our

number theory course is a junior-senior level (Diploma level) course, covering the standard topics: number theoretic functions, diophantine equations, prime number theorem, congruences, quadratic residues, etc. The percent of the course taken up by the computer work varied from week to week, as well as between courses, but probably averaged to about 30-40 percent. The students averaged about one hour of terminal time per week. Time-sharing was available to the student on a Hewlett-Packard 2000 C (in one course) and a Burroughs B5500 (in the other course). The language used was BASIC. The 2000 C has exact integer arithmetic for numbers only up to about 3,000,000; however, this did not prove to be a severe limitation.

The courses were small, with a lecture-discussion format. There were two tests; one for number theory and one for programming in BASIC. During the first three weeks or so, some time in each class period was devoted to programming in BASIC. This averaged to about one-third or one-fourth of each period. The students were given programming assignments related to number theory. After this period of time, programming was seldom discussed in class. The students could and did also get programming help from the person who monitored the terminal room. Though programming *per se* was seldom discussed after the beginning of the course, the use of the computer in each topic covered in number theory was discussed, and of course the students' homework assignments using the computer were discussed.

After all the students had learned to program, they were given more extensive assignments involving the computer. At the same time, the students regularly did homework in number theory not involving the computer.

Evaluation and Practical Problems

Every course using a computer generates some problems of access to the computer, hardware malfunction, etc. In our case these have never been serious. In the first course, the computer was only available in the evening and on weekends. If it had been available during week-days, the students' usage would probably have been heavier.

Another kind of problem which can arise is the student who does not like computers. This can come from several sources, including lack of experience. Almost without exception, students in the classes who were originally unenthusiastic because of lack of programming knowledge, etc., became enthusiastic computer users before the end of the term. One student did not take to the computer for quite different reasons. He was an excellent student with natural talent for abstract reasoning. The computer illustrations of theorems were unnecessary to him and did not illuminate the theorems in the same way as they did for other students. Pro-

gramming was difficult for him, as his programs always had trivial mistakes of detail. This student probably would have been better off in a more conventional number theory course.

The relatively small number of students and the lack of a referencing standard for an upper level undergraduate course such as number theory have made it impossible to do comparative statistical studies of the effectiveness of this computer-oriented approach to number theory. The students enjoyed the approach, with the exception of the one student noted earlier. A substantial number of them (about one-third) continued their work on their homework or projects (or work growing from their projects) after the term ended, an indication of their interest, as this could not affect their grade and was done only for their own satisfaction. Many students said that they got more sense of accomplishment from writing a program and discovering patterns than they obtained from reading a textbook and learning the proofs of theorems. In effect, they could do mathematical research in this computer setting. It is almost impossible for undergraduates to do research in mathematics in a conventionally organized course.

One pitfall that the teacher must be aware of is the necessity of carefully organizing the experiments or exercises in which the students are expected to observe patterns. It is all too easy to assume, knowing what the pattern is, that it is obvious and that each student will discover it with no guidance. In fact, pattern recognition is not encouraged in standard mathematics courses, and one of the virtues of the approach described herein is that it attempts to teach it. But the teacher should remember that the students have had no previous practice, and organize the experiments thoughtfully.

Algorithms and Experiments

There are some fundamental algorithms of number theory which are used over and over again in many computer programs. The books by D.E. Knuth and D.H. Lehmer listed in the bibliography are good sources of these algorithms. Among the most important algorithms are:

The Euclidean algorithm for calculating the g.c.d. of two integers, and the coefficients in the linear combination of it in terms of the two integers.

- An algorithm for factoring N, by searching for factors less than or equal to $\sqrt{N}$. This also serves as a primality test.

- An algorithm for calculating a^b (mod n), by repeatedly halving the exponent and squaring the base. This is much faster than multiplying a by itself b times.

- Algorithms for solving a linear congruence $ax \equiv b \pmod{n}$ and systems of linear congruences.

- Algorithms for calculating the continued fractions of rational numbers, real numbers, and quadratic irrationals.

All of the topics of elementary number theory are amenable to computer experiments. In some cases, it is highly desirable for the students to program the experiment as well as run it, while for other experiments the students would not learn a great deal from writing the program, and should run a program furnished by the teacher. Here is a sampling of computer experiments which have been used successfully by the author.

1) Write and run a program to factor a given number, list its factors, count the number of prime factors, and count the number of factors.

   This would normally be done through a search. It illustrates a number of theorems, including the unique factorization theorem.

2) Write and run programs to compute the greatest common divisor of two given numbers, both directly through a search and by means of the Euclidean algorithm.

3) Write and run a program to solve linear Diophantine equations in two variables by means of the Euclidean algorithm.

   This requires a thorough understanding of the proof that the Euclidean algorithm for finding the g.c.d. of two numbers works. Of course, one can go on to three or more variables.

4) Write and run a program to compute the first N Fibonacci numbers.

   This quickly leads to the problem of programming large number arithmetic. To do that requires a thorough understanding of the arithmetic algorithms.

5) Write and run programs to search for solutions to these Diophantine equations (for example):

a) $x^2 - 3y^2 = 6$

b) $x^4 + 9 = y^3$

c) $x^2 - 35y^2 = 3$

d) $x^2 + 2 = y^3$

e) $x^2 - 13y^2 = 1$

Some of these can be easily shown to have no solutions via congruence arguments. Others are not as tractable, but display interesting patterns in the solutions.

6) Write and run a program to solve simultaneous linear congruences.

This is an illustration and application of the Chinese Remainder Theorem.

7) Write and run a program to count the number of primes (or the density of primes) in a given interval. Compare this with x/lnx (or 1/lnx). Estimate the average distance between primes.

This is an illustration of the Prime Number Theorem.

8. Write and run a program to search for pseudo-primes. This is connected with Euler's Theorem and the Chinese Conjecture, which was widely believed for several hundred years, but turned out to be false. By using a computer, a student can settle the question very quickly now.

9) Write and run a program to represent an integer as a sum of four squares.

Lagrange's Theorem, which was proved in the course, states that is this always possible.

10) (Gauss' Circle Problem) For the first 500 or so integers compute the number of ways each can be represented as a sum of two squares. Make conjectures about which numbers can be written and which cannot. Compute the average number of ways the integers up through N can be so represented. Conjecture what the limit of this average is. In this problem, the conjectures made by the students can lead not only to the correct facts, but can also indicate the general nature of the proofs. This problem is a well-known theorem of geometric number theory.

11) Write and run a program to estimate the percent of pairs of integers which are relatively prime.

    A search or a Monte Carlo technique could be used. The The answer is $6/\pi^2$.

12) Write and run a program to find consecutive Pythagorean triples, i.e. integers u and v for which $u^2 + (u+1)^2 = v^2$. Observe patterns in the solutions, conjecture the recursion relation, and prove it. There are many fascinating patterns in a list of solutions, including a linear recurrence relation. This equation is equivalent to a Fermat-Pell equation.

13) Write and run a computer program which calculates the continued fraction for the square root of a positive integer.

14) Write and run a program which prints magic squares of various orders.

15) Write and run a program which uses the sieve of Eratosthenes to find all the primes less than 3000 (for example).

This is not a complete list of computer-related exercises and projects which have been used in the courses, but this sampling illustrates the contention that practically every topic in number theory can be illustrated by means of computer work, and that in a great many topics, the students can be motivated and guided to the facts via the computer.

Two Examples

As specific illustrations of these ideas, we give programs in BASIC and runs for two experiments.

The first experiment calculates (N-1) ! (mod N), and should lead the student to discover Wilson's theorem. (If the theorem has already been proved in the course, it is a striking illustration of the theorem.) The experiment can be introduced by observing that $N! \equiv 0 \pmod{N}$ of necessity, but that (N-1)! may not be congruent to 0 modulo N. A description, in English, of the program is:

Step 1. Read N, set X = 1 and set I = 2.

Step 2. Set X = X * I (mod N).

Step 3. Set I = I + 1. If I = N, stop, for X is (N-1)! (mod N).

Step 4. Go to step 2.

The Program:

```
100  PRINT  "N"," (N-1) FACTORIAL MOD N"
110  READ   N
120  DATA   2,3,4,5,6,7,8,9,10,11,15,16,17,20
125  DATA   23,41,82,137,141,341,565,1103
130  LET    X = 1
140  FOR    I = 2 TO N - 1
150  LET    X = X * I
160  LET    X = X - N * INT (X/N)
170  NEXT   I
180  PRINT  N, X
190  GOTO   110
200  END
```

The run:

N	(N - 1) FACTORIAL MOD N
2	1
3	2
4	2
5	4
6	0
7	6
8	0
9	0
10	0
11	10
15	0

16	0
17	16
20	0
23	22
41	40
82	0
137	136
141	0
341	0
565	0
1103	1102

The second experiment addresses itself to the question: for which P can we solve the congruence $X^2 \equiv -1 \pmod{P}$? This can also be stated: for which P is - 1 a quadratic residue?

The Program:

```
16  REM  LINE 110 PRINTS THE TRABLE HEADING.  THE COMPUTATION
18  REM  DETERMINING WHETHER A IS A QUADRATIC RESIDUE
20  REM  MODULO P IS DONE IN LINES 150-190. IT IS A DIRECT
22  REM  SEARCH TO DETERMINE IF THERE IS AN X FOR WHICH
24  REM  X*X IS CONGRUENT TO A MODULO P. AS HAS BEEN NOTED
26  REM  BEFORE, IT IS ONLY NECESSARY TO SEARCH X BETWEEN
28  REM  1 AND (P-1)/2.
30  REM  LINES 200-230 PRINT THE PROPER RESPONSE ACCORDING
32  REM  AS TO WHETHER OR NOT A IS A QUADRATIC RESIDUE
34  REM  MODULO P, AND RETURN CONTROL TO THE INITIAL
36  REM  READ STATEMENT
38  REM  WHEN THE FLAG AT THE END OF THE DATA LIST IS READ,
```

```
40   REM    CONTROL IS DIRECTED TO LINE 240.
52   REM    *********************************************
54   REM
100  LET    A = - 1
110  PRINT  "P", A; "IS QUAD. RES. OF P?"
120  READ   P
125  IF     P = - 10 THEN 240
130  DATA   17,41,73,89,5,13,29,37,3,11,19,43,7,23,31,47,-10
145  REM
148  REM
150  LET    B = A - P*INT(A/P)
160  FOR    X = 1 TO (P-1)/2
170  LET    S = X*X - P*INT(X*X/P)
180  IF     S = B THEN 220
190  NEXT   X
192  REM
195  REM    ************* PRINTOUT  **************
197  REM
200  PRINT  P," NO"
210  GOTO   120
220  PRINT  P,"  YES"
230  GOTO   120
240  END
```

The run:

P	-1 IS QUAD. RES. OF P?
17	YES
41	YES
73	YES
89	YES
5	YES
13	YES
29	YES
37	YES
3	NO
11	NO
19	NO
43	NO
7	NO
23	NO
31	NO
47	NO

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On Generalized Rice's Polynomials

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Abstract

In this apper, we have established some interesting results for the generalized Rice's polynomials such as recurrence relations, contiguous relations, generating function etc.

1. Introduction

Recently, Khandekar [2] has introduced and studied the generalized Rice's polynomials defined by

$$(1.1) \qquad H_n^{\alpha,\beta}(\xi,p;x) = \frac{(1+\alpha)_n}{n!}\, {}_3F_2\left[\begin{matrix}-n, n+\alpha+\beta+1, \xi \\ 1+\alpha, p\end{matrix}, x\right]$$

Further Monacha [3], Deshpande [4], Srivastava [5] and others have also found out various properties for the above polynomials. The purpose of this paper is to derive some more results for the Rice's polynomials.

We shall require the following results in our analysis:

$$(1.2) \qquad \sum_{n=o}^{\infty} \frac{(\lambda)_n}{n!} F_1 \;(\mu, -n, \lambda+n; \nu; x,y) z^n$$

$$= (1-z)^{-\lambda}\, {}_2F_1\left[\begin{matrix}\lambda, \mu; \\ \nu;\end{matrix} \frac{xz-y}{z-1}\right], \quad [6,(7.8)]$$

$$(1.3) \qquad \Delta_{\alpha} f(\alpha) = f(\alpha+1) - f(\alpha), \qquad [6]$$

and

$$(1.4) \quad \Delta_{\alpha}^{n} [f(\alpha)g(\alpha)] = \sum_{k=o}^{n} \binom{n}{k} [\Delta_{\alpha}^{n-k} f(\alpha+k)]\, \Delta_{\alpha}^{k} g(\alpha). \; [1]$$

2. Contiguous Relations

Starting from (1.1) and using (1.3), we obtain

$$(2.1) \quad \Delta_{\xi} H_n^{\alpha,\beta}(\xi,p;x) = -\frac{(n+\alpha+\beta+1)}{p} x\, H_{n-1}^{\alpha+1,\beta+1}(\xi+1,p+1;x),$$

$$(2.2)\quad \Delta_p\, H_n^{\alpha,\beta}(\xi,p;x) = \frac{(n+\alpha+\beta+1)}{p(p+1)}\, \xi x\, H_{n-1}^{\alpha+1,\,\beta+1}(\xi+1,p+2;x)$$

and

$$(2.3)\quad \Delta_\beta H_n^{\alpha,\beta}(\xi,p,x) = -\frac{\xi x}{p}\, H_{n-1}^{\alpha+1,\,\beta+1}(\xi+1,p+1;x)$$

(2.1) and (2.3) generalizes to

$$(2.4)\quad \Delta_\xi^r\, H_n^{\alpha,\beta}(\xi,p;x) = \frac{(n+\alpha+\beta+1)_r}{(p)_r}\,(-x)^r\, H_{n-r}^{\alpha+r,\,\beta+r}(\xi+r,p+r;x),$$

and

$$(2.5)\quad \Delta_\beta^r\, H_n^{\alpha,\beta}(\xi,p;x) = \frac{(\xi)_r}{(p)_r}\,(-x)^r\, H_{n-r}^{\alpha+r,\,\beta+r}(\xi+r,p+r:x)$$

respectively.

From (2.1), (2.2) and (2.3) we have

$$(2.6)\quad H_n^{\alpha,\beta}(\xi+1,p;x) - H_n^{\alpha,\beta}(\xi,p;x) = -\frac{(n+\alpha+\beta+1)}{p}\, x\, H_{n-1}^{\alpha+1,\beta+1}(\xi+1,p+1;x),$$

$$(2.7)\quad H_n^{\alpha,\beta}(\xi,p+1;x) - H_n^{\alpha,\beta}(\xi,p;x) = \frac{(n+\alpha+\beta+1)}{p(p+1)}\,\xi x\, H_{n-1}^{\alpha+1,\beta+1}(\xi+1,p+2;x),$$

$$(2.8)\quad H_n^{\alpha,\beta+1}(\xi,p+1;x) - H_n^{\alpha,\beta}(\xi,p;x) = \frac{-\xi x}{p}\, H_{n-1}^{\alpha+1,\,\beta+1}(\xi+1,p+1;x),$$

$$(2.9)\quad (n+\alpha+\beta-p+2)\, H_n^{\alpha,\beta}(\xi,p;x) = (n+\alpha+\beta+1)\, H_n^{\alpha,\,\beta+1}(\xi,p;x) - (p-1)\, H_n^{\alpha,\beta}(\xi,p-1;x)$$

and

$$(2.10)\quad (n+\alpha+\beta-\xi+1)\, H_n^{\alpha,\beta}(\xi,p;x) = (n+\alpha+\beta+1)\, H_n^{\alpha,\,\beta+1}(\xi,p;x) - \xi H_n^{\alpha,\beta}(\xi+1,p;x).$$

The repeated use of (2.8), will give

$$(2.11)\quad H_{n-k}^{\alpha+k,\,\beta+k}(\xi+k,p+k,x) = \frac{(p)_k}{(\xi)_k\, x^k}\sum_{r=o}^{k}(-)^r\binom{k}{r}\, H_n^{\alpha,\,\beta+k}(\xi,p;x).$$

Now using the famous result

$$f\ (\beta+\mu) = \sum_{r} \binom{\mu}{r}\ \Delta_{\beta}^{r}\ f\ (\beta)$$

and (2.5), we have

$$(2.12)\qquad H_n^{\alpha,\beta+\mu}(\xi,p;x) = \sum_{r} \binom{\mu}{r} \frac{(\xi)_r}{(p)_r} (-x)^r\ H_{n-r}^{\alpha+r,\beta+r}\ (\xi+r,p+r;x).$$

Similarly one more result can be find out with the help of (2.4).

3. Explicit Representation

Consider

$$\frac{(-)^n(1+\alpha)_n x^{-\alpha}}{n!} \Delta_{\alpha,\xi,p}^{n} \left[\frac{\Gamma(\xi)\ \Gamma(n+\alpha+\beta+1)}{\Gamma(\alpha+1)\ \Gamma(p)}\ x^{\alpha}\right]$$

$$= \frac{(1+\alpha)_n\ x^{-\alpha}}{n!}\ (1-E_{\alpha} E_{\xi} E_p)^n \left[\frac{\Gamma(\xi)\ \Gamma(n+\alpha+\beta+1)}{\Gamma(p)\ \Gamma(\alpha+1)}\ x^{\alpha}\right]$$

$$= \frac{(1+\alpha)_n\ x^{-\alpha}}{n!} \sum_{r=0}^{n} \binom{n}{r}\ (-)^r\ \frac{\Gamma(\xi+r)\ \Gamma(n+\alpha+\beta+r+1)}{\Gamma(p+r)\ \Gamma(\alpha+r+1)}\ x^{\alpha+r}.$$

Therefore

$$(3.1)\qquad H_n^{\alpha,\beta}(\xi,p;x) = \frac{(-)^n x^{-\alpha}\Gamma(p)\ \Gamma(1+\alpha+n)}{n!\ \Gamma(\xi)\ \Gamma(\alpha+\beta+n+1)} \Delta_{\alpha,\xi,p}^{n}\left[\frac{\Gamma(\xi)\ \Gamma(1+\alpha+\beta+n)}{\Gamma(p)\ \Gamma(1+\alpha)}\ x^{\alpha}\right]$$

where $\Delta_{\alpha,\xi,p} = E_{\alpha} E_{\xi} E_p - 1.$

Now taking the different combinations $\frac{\Gamma(\xi)\ \Gamma(1+\alpha+\beta+n)}{\Gamma(p)\ \Gamma(1+\alpha)}\ x^{\alpha}$ in the right-hand expression of (3.1) and using (1.4), we can show that

$$(3.2)\ H_n^{\alpha,\beta}(\xi,p;x) = \frac{(1+\alpha)_n}{n!} \sum_{r=0}^{n} \binom{n}{r} \frac{(p-\xi)_r(1+\alpha+\beta+n)_r}{(p)_r(1+\alpha)_r}\ x^r\ {}_2F_1\left[\begin{matrix} r-n,1+\alpha+\beta+n+r \\ 1+\alpha+r \end{matrix} ;x\right]$$

(3.3) $$= \frac{(1+\alpha)_n}{n!} \sum_{r=0}^{n} \binom{n}{r} \frac{(\xi)_r (x)^r (-n-\beta)_r}{(1+\alpha)_r \; (p)_r} \; {}_2F_1\left[\begin{matrix} r-n, \; +r \\ p+r \end{matrix} ; x\right],$$

(3.4) $$= \frac{(1+\alpha)_n}{n!} \sum_{r=0}^{n} \binom{n}{r} \frac{1}{(p)_r} \; {}_1F_1\,(r-n;p+r;1) \; {}_3F_1\left[\begin{matrix} -r, 1+\alpha+\beta+n, \\ 1+\alpha \end{matrix} ; x\right]$$

and

(3.5) $$= \frac{(1+\alpha)_n}{n!} \sum_{r=0}^{n} \binom{n}{r} (\xi)_r \; {}_2F_0\,(r-n, \xi+r;-;1) \; {}_2F_2\left[\begin{matrix} -r, 1+\alpha+\beta+n \\ 1+\alpha, \; p \end{matrix} ; x\right]$$

4. Generating Function

The result to be proved is

(4.1) $$(1-t)^{-p} \; {}_2F_1\left[\begin{matrix} p, 1+\alpha+\beta ; \\ 1+\alpha ; \end{matrix} \; x\,(1+t)\right]$$

$$= \sum_{n=0}^{\infty} \frac{(p)_n}{(1+\alpha)_n} t^n H_n^{\alpha, \beta-n} (p, 2p; x).$$

Proof: With the help of (3.2) the right-hand side of the above expression will become

$$= \sum_{n,r=0}^{\infty} \frac{(p)_{n+r} (1+\alpha+\beta)_r}{(1+\alpha)_r} \frac{(xt)^r}{r!} \frac{t^n}{n!} \; {}_2F_1\left[\begin{matrix} -n, 1+\alpha+\beta+r; \\ 1+\alpha+r; \end{matrix} \; x\right]$$

$$= \sum_{n=0}^{\infty} \frac{(p)_n}{n!} t^n \; F_1\,(1+\alpha+\beta, \; p+n, -n; \; 1+\alpha; \; xt, x)$$

now the use of (1.2) will give the above generating function.

Lastly from (4.1), we can deduce the following recurrence relations

(4.2) $$(\alpha+n+1)\;(p+n-1)\; H_n^{\alpha,\beta}(p,2p;x)$$

$$= (p+n-1)\;(p+n)\;(\alpha+\beta+n+1)\; H_n^{\alpha+1,\beta}(p+1,2p+2;x)$$

$$- (\alpha+n)\;(\alpha+n+1)\;(\alpha+\beta+n+1)\; H_{n-2}^{\alpha+1,\beta+2}(p+1,2p+2;x)$$

and

$$(4.3) \qquad (n+1)\ (p+n)\ (p+n-1)\ H_{n+1}^{\alpha,\beta-1}\ (p,2p;x)$$

$$= (p+n)\ (\alpha+n+1)\ (p-n-1)\ H_n^{\alpha,\beta}(p,2p;x) + x\ (\alpha+\beta+n+1)\ (p+n-1)$$

$$(p+n)\ H_n^{\alpha+1,\beta}(p+1,2p+2;x) - 2(\alpha+\beta+n+1)\ (p+n-1)\ (\alpha+n+1)$$

$$H_{n-1}^{\alpha+1,\beta+1}\ (p,2p+2;x) + (\alpha+\beta+n+1)\ (\alpha+n)\ (\alpha+n+1)\ H_{n-2}^{\alpha+,\beta+2}\ (p,2p;x).$$

I wish to express my sincere thanks to Dr. R.C.S. Chandel of D.V. College, Orai for his interest during the preparation of this paper.

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On Characterization of Hyperbolic (f, g, ξ, η)-Structure

K.K. Dube

Summary

In the present paper we shall give a classification of a differentiable manifold M^{n+1} with an almost contact hyperbolic structure [5] using the classification of differentiable manifold with an almost hyperbolic Hermitian structure [4].

1. Introduction

Let us consider an (n+1)-dimensional real differentiable manifold V^{n+1} of class C^{∞}. Let there exist a C^{∞} tensor field ϕ^i_j, a contravariant vector field ξ^i and covariant vector field η_j which satisfy the following:

$$\text{(a)} \quad \phi^i_j \phi^j_k = \delta^i_k + \xi^i \eta_k,$$

$$\text{(1.1)(b)} \quad \phi^i_j \xi^j = 0, \quad \phi^i_j \eta_i = 0,$$

$$\text{(c)} \quad \xi^i \eta_i = -1 \qquad (i,j,k=1,2,\ldots,n+1)$$

The structure satisfying the conditions (1.1) are called almost contact structure in product manifold.

Now every differentiable manifold with almost contact structure in product manifold admits a Riemannian metric g_{ij}[3] which satisfies the following conditions:

$$\text{(1.2)} \quad g_{ij}\xi^j = \eta_i, \quad g_{ij}\phi^i_h \phi^j_k = -g_{hk} - \eta_h \eta_k,$$

$$g_{jh}\phi^h_i = g_{ih}\phi^h_j = (\phi_{ij}).$$

The manifold V^n satisfying all the conditions from (1.1) to (1.2) yields a hyperbolic structure and we call it as hyperbolic almost contact metric structure or almost contact hyperbolic (ϕ , ξ , η , g)-structure [5].

2. Some tensors on hyperbolic (ϕ , ξ , η , g)-Structure

Let M^{n+1} be a differentiable manifold with almost contact hyperbolic metric structure and R be a real line. Let us consider the product manifold $M^{n+1} \times R$. Now we take a sufficiently fine open covering $\mathcal{U}$ of M^{n+1} by coordinate neighbourhoods. If we denote coordinate of U in $\mathcal{U}$ by X^i $(i,j,k,\ldots=1,2,\ldots,n+1)$ and a certain coordinate of R by X^{∞}, then (X^i, X^{∞}) can be considered as a set of coordinates of $U \times R$ and $\{U \times R \mid U \in \mathcal{U}\}$ constitute an open covering of $M^{n+1} \times R$ by coordinate neighbourhoods.

Now we suppose that U, U (U, $U \neq 0$) belong to $\mathcal{U}$ and X^i, $X^{i'}$ are their coordinates and let $X^{i'} = X^{i'}(X^1,\ldots,X^{n+1})$ be the coordinate transformation between $U \times R$ and $U' \times r$ by

$$(2.1) \qquad X^{i'} = X^{i'}(X^1,\ldots,X^{n+1}),\ X^{\infty'} = X^{\infty};$$

then with the help of lemma 1 [3], if we put

$$(2.2) \qquad F^i_j = \phi^i_j\,,\quad F^i_{\infty} = -\,\xi^i\,,\quad F^{\infty}_j = \eta_j\,,\quad F^{\infty}_{\infty} = 0$$

in coordinate neighbourhoods $\{U \times R \mid U \in \mathcal{U}\}$, then F^A_B $(A,B,C, = 1,2,\ldots,n+1, \infty)$ defines a field of mixed tensors on $M^{n+1} \times R$ with respect to the pseudo group of transformation of type (2.1) and F^A_B gives an almost product structure on $M^{n+1} \times R$.

Using the properties of (1.1), we know that the tensor F^A_B satisfies the relation

$$F^A_B\, F^B_C = \delta^A_C$$

Thus we obtain the following easily:

Proposition 2.1.

If we put $G_{ij} = g_{ij}$, $G_{\infty j} = G_{j\infty} = 0$, $G_{\infty\infty} = 1$, in coordinate neighbourhoods $\{U \times R \mid U \in \mathcal{U}\}$ then,

$G_{AB}(A,B,C = 1,2,\ldots,n+1, \infty)$ define a covariant tensor field with respect to the pseudo-group of transformation of type (2.1) and the Riemannian metric given by [3].

$$(2.3) \qquad G_{AB}\, F^A_{\ M}\, F^B_{\ N} = -\, G_{MN}\ ,$$

holds; the set $(F^A_{\ B}, G_{AB})$ defines an almost hyperbolic Hermitian structure on $M^{n+1} \times R$ [4].

If we define $F_{BC} = F^A_{\ B}\, G_{AC}$, then we obtain

$$F_{jk} = \phi_{jk},\quad F_{i\infty} = \eta_j\ ,\quad F_{\infty j} = -\ \eta_j\ ,\quad F_{\infty\infty} = 0.$$

We have also,

$$(2.4) \qquad \nabla_j F^h_{\ i} = \mathring{\nabla}_j\ \phi^h_{\ i}\ ,\quad \nabla_j F^h_{\ \infty} = -\ \mathring{\nabla}_j\ \xi^h,$$

$$\nabla_j F_i^{\ \infty} = \mathring{\nabla}_j\ \eta_i,$$

$$\nabla_j F^{\infty}_{\ \infty} = \nabla_{\infty} F^h_{\ i} = \nabla_{\infty} F^h_{\ \infty} = \nabla_{\infty} F_j^{\ \infty} = \nabla_{\infty} F^{\infty}_{\ \infty} = 0.$$

$$(2.5) \qquad \nabla_j F_{ih} = \mathring{\nabla}_j\ \phi_{ih},\quad \nabla_j F_{i\infty} = \mathring{\nabla}_j\ \eta_i\ ,$$

$$\nabla_j F_{\infty h} = -\ \mathring{\nabla}_j\ \eta_h\ ,$$

$$\nabla_j F_{\infty\infty} = \nabla_{\infty} F_{ih} = \nabla_{\infty} F_{\infty h} = \nabla_{\infty} F_{i\infty}$$

$$= \nabla_{\infty} F_{\infty\infty} = 0\ ,$$

where ∇_B and $\mathring{\nabla}_j$ be the covariant derivatives with respect to the connection $\left\{ {A \atop BC} \right\}$ and $\left\{ {i \atop jk} \right\}$ respectively.

Let $N_{BC}{}^A$ be the Nijenhine tensor of the product manifold $M^{n+1} \times R$ given by

$$(2.6) \qquad N_{CB}{}^A = F^E_C\, (\nabla_E F^A_B - \nabla_B F^A_E) - F^E_B (\nabla_E F^A_C - \nabla_C F^A_E).$$

In view of the transformation (2.1), if we calculate the components of (2.6) in two groups, we get the following by virtue of (2.4) and (2.5).

$$\text{(a)}\quad N_{kj}{}^{i} = \phi_k{}^{h}(\overset{\circ}{\nabla}_h \phi^i_j - \overset{\circ}{\nabla}_j \phi^i_h) -$$
$$- \phi^h_j(\overset{\circ}{\nabla}_h \phi^i_k - \overset{\circ}{\nabla}_k \phi^i_h) + \eta_k \overset{\circ}{\nabla}_j \xi^i - \eta_j \overset{\circ}{\nabla}_k \xi^i,$$

$$\text{(2.7)(b)}\quad N_{kj} = \phi^h_k(\overset{\circ}{\nabla}_h \eta_j - \overset{\circ}{\nabla}_j \eta_h) - \phi^h_j(\overset{\circ}{\nabla}_h \eta_k - \overset{\circ}{\nabla}_k \eta_h),$$

$$\text{(c)}\quad N^i_j = - \xi^h(\overset{\circ}{\nabla}_h \phi^i_j - \overset{\circ}{\nabla}_j \phi^i_h) + \phi^h_j \overset{\circ}{\nabla}_h \xi^i,$$

$$\text{(d)}\quad N_j = \xi^h(\overset{\circ}{\nabla}_h \eta_j - \overset{\circ}{\nabla}_j \eta_h).$$

3. Characterization of hyperbolic (ϕ, ξ, η, g)-Structure

In [4] we have given some kind of classification of an almost hyperbolic Hermitian manifolds. In this section using the classification of the almost hyperbolic Hermitian manifold $M^{n+1} \times R$, we shall give a classification of a manifold with an almost contact hyperbolic metric structure [5].

(1) Almost hyperbolic Aptian manifold

If on $M^{n+1} \times R$ the relation $\nabla_A F^A{}_B = 0$ holds, we call the manifold an almost hyperbolic Aptian manifold.

Theorem 3.1.

In order that $M^{n+1} \times R$ be an almost hyperbolic Aptian manifold, it is necessary and sufficient that

$$\text{(3.1)}\qquad \overset{\circ}{\nabla}_a \phi^a_i = 0, \quad \overset{\circ}{\nabla}_a \xi^a = 0.$$

Also ξ^i is incompressible vector field.

Proof. By virtue of (2.1), on calculating the components of $\nabla_A F^A_B = 0$ and using (2.4) and (2.5) we get (3.1). The converse is also true. Hence the theorem is proved.

(ii) Almost hyperbolic Kahlerian manifold

If on $M^{n+1} \times R$

$$F_{ABC} \overset{\text{def}}{=\!=\!=} \nabla_A F_{BC} + \nabla_B F_{CA} + \nabla_C F_{AB} = 0. \tag{3.2}$$

We call this manifold an almost hyperbolic Kahlerian manifold [4].

Theorem 3.2

The necessary and sufficient condition that $M^{n+1} \times R$ be an almost hyperbolic Kahlerian manifold is that

(3.3)

$$\text{(a)} \quad \phi_{jih} \overset{\text{def}}{=\!=\!=} \overset{\circ}{\nabla}_j \phi_{ih} + \overset{\circ}{\nabla}_i \phi_{hj} + \overset{\circ}{\nabla}_h \phi_{ji} = 0,$$

$$\text{(b)} \quad \overset{\circ}{\nabla}_j \eta_i = \overset{\circ}{\nabla}_i \eta_j ;$$

holds for M^{n+1}.

Also $N_{kj} = 0$, $N_j = 0$. In order that $N_j^i = 0$, it is necessary and sufficient that $\overset{\circ}{\nabla}_j \xi^i = 0$.

Proof. In view of (2.1), on calculating the components of (3.2) and using (2.4) and (2.5) we obtain (3.3) a, b. The converse also holds.

By virtue of (3.3) b and (2.7) b, d we get $N_{kj}=0$ and $N_j = 0$. Transvecting both sides of (3.3) a with ξ^i and using (3.3) b and (2.7) c, we get

$$N_j^i = 2 \phi_j^a \overset{\circ}{\nabla}_a \xi^i , \tag{3.4}$$

which on multiplication by ϕ_l^j leads to

$$\phi_l^j N_j^i = 2 \overset{\circ}{\nabla}_l \xi^i . \tag{3.5}$$

Let $\overset{\circ}{\nabla}_j \xi^i = 0$, then we get $\phi_l^j N_j^i = 0$. Hence $N_j^i = 0$. Conversely if $N_j^i = 0$, then by virtue of (3.5) we get $\overset{\circ}{\nabla}_j \xi^i = 0$. this proves the theorem.

(iii) Nearly hyperbolic Kahlerian manifold: on $M^{n+1} \times R$

Let

(3.6) $$\nabla_A F_{BC} + \nabla_B F_{AC} = 0,$$

then the manifold is called nearly hyperbolic Kahlerian manifold [4].

Theorem 3.3

The nase that $M^{n+1} \times R$ be a nearly hyperbolic Kahlerian manifold is that

(3.7) a) $$\overset{\circ}{\nabla}_j \phi_{ih} + \overset{\circ}{\nabla}_i \phi_{jh} = 0,$$

b) $$\overset{\circ}{\nabla}_j \eta_h = 0,$$

holds for M^{n+1}.

Also we have $\overset{\circ}{\nabla}_j \xi^i = 0$, $N_{kj} = 0$, $N^i_j = 0$ and consequently $N_j = 0$. In order that $N_{kj}{}^i = 0$, it is necessary and sufficient that $\overset{\circ}{\nabla}_j \phi^i_k = 0$.

Proof. Now, on calculating the components of (3.6) with respect to the transformation (2.1), we get (3.7) a, b. Conversely, if (3.7) a, b hold in M^{n+1}, then (3.6) holds for $M^{n+1} \times R$.

Now by firtue of (2.7) b and (3.7) b we obtain $N_{kj} = 0$, and consequently $N_j = 0$.

Also by firtue of (2.7) c, (3.7) a and $\overset{\circ}{\nabla}_j \xi^i = 0$, we have $N^i_j = 0$.

Moreover, in view of (2.7) a, (3.7) a, b and $\overset{\circ}{\nabla}_j \xi^i = 0$, we get

$$N_{kj}{}^i = 4 \phi^i_h \overset{\circ}{\nabla}_j \phi_k{}^h .$$

Hence

(3.8) $$\phi^l_i N_{kj}{}^i = 4 \overset{\circ}{\nabla}_j \phi_k{}^l .$$

Consequently, from (3.8), in order that $N_{kj}{}^{i} = 0$, it is necessary and sufficient that

$$\overset{\circ}{\nabla}_j \phi_k{}^i = 0.$$

This proves the theorem.

(iv) Hyperbolic Kahlerian manifold

If on $M^{n+1} \times R$ the relation

$$\nabla_A F_{BC} = 0 \tag{3.9}$$

then it is called a hyperbolic Kahlerian manifold [3].

Theorem 3.4.

In order that $M^{n+1} \times R$ be an hyperbolic Kahlerian manifold, it is necessary and sufficient that for M^{n+1}

$$\overset{\circ}{\nabla}_i \phi_{jk} = 0 \text{ and } \overset{\circ}{\nabla}_i \eta_j = 0. \tag{3.10}$$

Proof. If we calculate the components of (3.9) with respect to the transformation (2.1), we get (3.10). The converse also hold.

(v) Hyperbolic 0* -space

If on $M^{n+1} \times R$ the relation

$$*0^{BA}_{J.I} \nabla_B F^H_A = 0, \tag{3.11}$$

holds, then we call the manifold an hyperbolic 0*-manifold.

In view of (2.1), (2.2), (2.4), (2.5) and (3.11) we obtain the following relations:

$$\text{(3.12)} \quad \begin{aligned} &\text{(a)} \quad *0^{ba}_{ji} \overset{\circ}{\nabla}_b \phi^b_a + \phi^b_j \eta_i \overset{\circ}{\nabla}_b \xi^h = 0, \\ &\text{(b)} \quad 0^{bh}_{ja} \overset{\circ}{\nabla}_b \xi^a = 0, \\ &\text{(c)} \quad *0^{ba}_{ji} \overset{\circ}{\nabla}_b \eta_a = 0, \end{aligned}$$

$$\text{(d)} \quad \xi^b \phi_i^a \overset{\circ}{\nabla}_b \phi_a^h - \xi^b \overset{\circ}{\nabla}_b \xi^h = 0,$$

$$\text{(e)} \quad \xi^a \phi_j^b \overset{\circ}{\nabla}_b \eta_a = 0,$$

$$\text{(f)} \quad \xi^b \phi_a^h \overset{\circ}{\nabla}_b \xi^a = 0,$$

$$\text{(g)} \quad \xi^h \phi_i^a \overset{\circ}{\nabla}_b \eta_a = 0,$$

and

$$\text{(h)} \quad \xi^b \xi^a \overset{\circ}{\nabla}_b \eta_a = 0.$$

Theorem 3.5

In order that $M^{n+1} \times R$ be an hyperbolic 0*-space, it is necessary and sufficient that (3.12)a holds for M^{n+1}.

Proof. Now transvecting both sides of (3.12)a with ξ^i we get

$$(3.13) \quad \phi_i^h \overset{\circ}{\nabla}_j \xi^i - \phi_j^b \overset{\circ}{\nabla}_b \xi^h = 0.$$

Further transvecting both sides of (3.13) with ϕ_h^l we get (3.12)b and rearranging this we obtain (3.12)c. By using (3.12)b, we see that (3.12)f holds. In view of (3.12) c, we notice that (3.12)g holds. Transvecting both sides of (3.12)a with ξ^j, we have

$$(3.14) \quad \xi^j \overset{\circ}{\nabla}_j \phi_i^h = 0.$$

Making use of (3.12)b and (3.14), we observe that (3.12)d holds. Hence (3.12)a is equivalent to (3.11). The converse is also true. This proves the theorem.

Theorem 3.6.

For M^{n+1}, such that $M^{n+1} \times R$ is an hyperbolic 0*-space, $N_j = 0$. In order that $N_j^i = 0$, it is necessary and sufficient that $\overset{\circ}{\nabla}_j \xi^i = 0$. Also for $N_{kj} = 0$, it is necessary and sufficient that

$$\overset{\circ}{\nabla}_k \eta_j = \overset{\circ}{\nabla}_j \eta_k .$$

Proof. In view of (3.12)c we get $N_j=0$. Now by virtue of (2.8)c, (3.13) and (3.14), we obtain

$$N^i_j = 2 \; \phi^i_h \; \overset{\circ}{\nabla}_j \; \xi^h .$$

Hence from the above equation $N^i_j = 0$, it is necessary and sufficient that $\overset{\circ}{\nabla}_j \; \xi^h = 0$. Transvecting both sides of (3.12)a with η_h we get

$$(3.15) \qquad \phi^b_j \; \overset{\circ}{\nabla}_b \; \eta_i = \phi^b_i \; \overset{\circ}{\nabla}_j \; \eta_b .$$

Hence by means of (3.15) and rewriting N_{kj}, we obtain by means of (2.7)b the following:

$$(3.16) \qquad N_{kj} = 2 \; \phi^b_j \; (\overset{\circ}{\nabla}_k \; \eta_b - \overset{\circ}{\nabla}_b \; \eta_k).$$

Transvecting both sides of (3.16) with ϕ^j_l we get

$$N_{kj} = 2 \; (\overset{\circ}{\nabla}_l \; \eta_k - \overset{\circ}{\nabla}_k \; \eta_l).$$

From this we get the last part of the theorem. The converse is also true.

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On the Source and Sink Flow Past a Porous Spherical Shell

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Abstract

Slow flow past a porous spherical shell which is placed in a non-uniform stream generated by a source and sink of equal strength placed far apart, has been studied using the boundary conditions modified by Jones [3] for curved surface at the interface of the free fluid region and porous material. It is found that the expression for drag as obtained by Jones [3] remains unaltered.

1. Introduction

Jones [3] has considered the problem of Stokes flow past a porous spherical shell of finite thickness on the basis of Stokes equations in the free fluid region outside the spherical shell and in the cavity, and Darcy's equations inside the porous region. The boundary conditions given by Beavers and Joseph [2] for plane boundary has been generalized for curved surface and using these conditions the solutions of the equations have been obtained and several limiting cases of interest have been deduced. Following Jones [3] we have made an attempt to study the flow of viscous incompressible fluid past a porous spherical shell which is placed in a stream generated by a source and sink of equal strength placed at large distance apart and the expressions for velocity, pressure and drag have been obtained.

2. Basic Equations and Simplifications

Consider a porous spherical shell of external radius 'a' and internal radius 'b' immersed in a stream generated by a source and sink of equal strength Q situated at equal distance 'c' on either side of the centre of the shell which is taken as the origin of the coordinate system. The whole flow is devided into three regions: the external region outside the porous spherical shell, the porous region and the cavity and are called region I, II and III respectively. Let (r,θ,ϕ) be the spherical polar coordinates and q_r, q_θ and q_ϕ the corresponding velocity components. Since the flow considered is axisymmetric, $q_\phi = 0$ and derivatives with respect to ϕ vanish.

As in [3] the flow in the region I and III are governed by Stokes equations

(1) $$\frac{\partial p}{\partial r} = \mu\left(\nabla^2 q_r - \frac{2q_r}{r^2} - \frac{2}{r^2}\frac{\partial q_\theta}{\partial\theta} - \frac{2q_\theta\cot\theta}{r^2}\right),$$

(2) $$\frac{\partial p}{r\partial\theta} = \mu\left(\nabla^2 q_\theta + \frac{2}{r^2}\frac{\partial q_r}{\partial\theta} - \frac{\partial q_\theta}{r^2\sin^2\theta}\right)$$

and the equation of continuity

(3) $$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 q_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(q_\theta\sin\theta) = 0,$$

where p is the pressure, the viscosity of the fluid and

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right).$$

On the other hand the flow withen the porius region II is governed by Darcy's equation [3]

(4) $$\frac{\partial p}{\partial r} = -\frac{\mu}{k} Q_r,$$

(5) $$\frac{\partial p}{r\partial\theta} = -\frac{\mu}{k} Q_\theta$$

and the equation of continuity

(6) $$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 Q_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(Q_\theta\sin\theta) = 0,$$

where Q_r, Q_θ are the velocity components in r-, θ - directions respectively and k the permeability of the shell.

Boundary Conditions: The value of the stream function for the flow due to source and sink as obtained in [1] is

$$-Q\left\{(c - r\cos\theta)(c^2+r^2-2cr\cos\theta)^{-\frac{1}{2}} + (c+r\cos\theta)(c^2+r^2+2cr\cos\theta)^{-\frac{1}{2}}\right\}$$

This, for sufficiently large c, can be approximated by

(7) $$\tfrac{1}{2}Ur^2\sin^2\theta + \frac{3}{8}\frac{Ur^4}{c^2}\sin^2\theta\,(5\cos^2\theta - 1), \quad (a \leq r \ll c),$$

where $U = 2Q/c^2$ and the constant term has been absorbed in the stream function itself. Far away from the spherical shell the stream function tends to the expression given above. It should,

however, be noted that far away from the shell, although r is much greater than a it has to be taken smaller than c so that the above approximation (7) is valid. Keeping in view that Stokes theory is applicable in the vicinity of the shell and c is sufficiently large, there is no harm in making the above assumption.

Also, as in [3], the appropriate boundary conditions at the surfaces of the porous shell r=a, b are

(i) the continuity of pressure

(ii) $q_r = Q_r$, from conservation of mass

(iii) $\sigma_{r\theta} = (q_\theta - Q_\theta)$ at r=a,

$\sigma_{r\theta} = -\beta(q_\theta - Q_\theta)$ at r=b,

where the tangential component of stress $\sigma_{r\theta}$ is given by

$$\sigma_{r\theta} = \frac{r\partial}{\partial r}\left(\frac{q_\theta}{r}\right) + \frac{\partial q_r}{r\partial\theta}$$

and $\beta = \alpha/\sqrt{k}$, being a non-dimensional constant. This latter conditions is the generalization of the conditions suggested by Beavers and Joseph [1] which Jones [3] has proposed for a curved surface.

We find it convenient to keep the governing equations in terms of stream function and so expressing the velocity components in terms of respective stream functions ψ and Ψ as

(8) $$q_r = \frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta}, \quad q_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

for regions I and III and

(9) $$Q_r = \frac{1}{r^2 \sin\theta}\frac{\partial\Psi}{\partial\theta}, \quad Q_\theta = -\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r}$$

for porous region II and then eliminating pressure, we obtain respectively from the set of equations (1)-(3) and the set of equations (4)-(5) the following differential equations for ψ and Ψ :

(10) $E^4\psi = 0$ (for region I and III)

(11) $E^2\Psi = 0$, (for region II),

where

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right).$$

3. Solution

The solutions of equations (10) and (11) which satisfy the condition at large distance from the centre of the shell and which are bounded at the origin can be found by standard techniques and are given by:

for region I

$$\psi = \left(\frac{A_1}{r} - \frac{B_1 r}{2} + \frac{U}{2} r^2\right) \sin^2\theta \tag{12}$$

$$+ \left(\frac{A_2}{r^3} - \frac{B_2}{10r} + \frac{3U r^4}{8c^2}\right) \sin^2\theta \,(5\cos^2\theta - 1) \quad (r > a);$$

the corresponding velocity components and pressure are

$$(13)\quad \left[\begin{aligned}
q_r &= \left(\frac{2A_1}{r^3} - \frac{B_1}{r} + U\right)\cos\theta \\
&\quad + \left(\frac{4A_2}{r^5} - \frac{2B_2}{5r^3} + \frac{3U r^2}{2c^2}\right)\cos\theta\,(5\cos^2\theta - 3), \\
q_\theta &= \left(\frac{A_1}{r^3} + \frac{B_1}{2r} - U\right)\sin\theta \\
&\quad + \left(\frac{3A_2}{r^5} - \frac{B_2}{10r^3} - \frac{3U}{2c^2} r^2\right)\sin\theta\,(5\cos^2\theta - 1), \\
p &= p_\infty - \frac{\mu}{r^2} B_1\cos\theta - \frac{\mu}{r^4} B_2\cos\theta\,(5\cos^2\theta - 3);
\end{aligned}\right.$$

for region II

$$\psi = \left(\frac{C_1}{r} + D_1 r^2\right)\sin^2\theta \tag{14}$$

$$+ \left(\frac{C_2}{r^3} + D_2 r^4\right)\sin^2\theta\,(5\cos^2\theta - 1), \quad (b < r < a);$$

the corresponding velocity components and pressure are

$$(15)\quad \begin{cases} Q_r = \left(\dfrac{2C_1}{r^3} + 2D_1\right)\cos\theta + \left(\dfrac{4C_2}{r^5} + 4D_2 r^2\right)\cos\theta\,(5\cos^2\theta - 3), \\ Q_\theta = \left(\dfrac{C_1}{r^3} - 2D_1\right)\sin\theta + \left(\dfrac{3C_2}{r^5} - 4D_2 r^2\right)\sin\theta\,(5\cos^2\theta - 1), \\ p = p_\infty + \dfrac{\mu}{k}\left(\dfrac{C_1}{r^2} - 2D_1 r\right)\cos\theta + \dfrac{\mu}{3k}\left(\dfrac{3C_2}{r^4} - 4D_2 r^3\right)\cos\theta\,(5\cos^2\theta - 3); \end{cases}$$

and <u>for region III</u>

$$(16)\quad \psi = \left(E_1 r^2 + \frac{F_1 r^2}{10}\right)\sin^2\theta + \left(E_2 r^4 + \frac{F_2 r^6}{18}\right)\sin^2\theta\,(5\cos^2\theta - 1),$$

$$(0 \leq r < b),$$

the corresponding velocity components and pressure are

$$(17)\quad \begin{cases} q_r = \left(2E_1 + \dfrac{F_1 r^2}{5}\right)\cos\theta + \left(4E_2 r^2 + \dfrac{2}{9}F_2 r^4\right)\cos\theta\,(5\cos^2\theta - 3), \\ q_\theta = -\left(2E_1 + \dfrac{2F_1 r^2}{5}\right)\sin\theta - \left(4E_2 r^2 + \dfrac{F_2 r^4}{3}\right)\sin\theta\,(5\cos^2\theta - 1), \\ p = p_\infty + 2\mu F_1 r\cos\theta + \dfrac{4}{3}\mu F_2 r^3\cos\theta\,(5\cos^2\theta - 3). \end{cases}$$

In equations (12) - (17) p_∞ is the pressure in the fluid at large distance and $A_1, B_1, C_1, D_1, E_1, F_1$; $A_2, B_2, C_2, D_2, E_2, F_2$ are arbitrary constants.

The boundary conditions to be applied at the interfaces r=a, b give six conditions to determine the six unknown constants result in each of two systems of simultaneous equations. Each system has their solutions as

$$(18)\quad A_1 = \frac{a^4\beta U}{L_1}\left[\left(\frac{1}{4} - \frac{3k}{2a^2}\right)\left(\frac{3b^3}{20a^4\beta} + \frac{b^4}{20a^4}\right) - \frac{1}{4}\left(\frac{3k}{2ab} + \frac{3}{20a\beta} + \frac{b}{20a}\right)\right],$$

$$B_1 = -\frac{3aU}{L_1}\left(1 + \frac{a}{2}\beta\right)\left(\frac{3k}{2ab} + \frac{3}{20a\beta} + \frac{b}{20a} - \frac{3b^3}{20a^4\beta} - \frac{b^4}{20a^4}\right),$$

$$C_1 = -\frac{3kaU}{L_1}\left(1+\frac{a}{2}\beta\right)\left(\frac{3b^3}{20a^4\beta}+\frac{b^4}{20a^4}\right),$$

$$D_1 = -\frac{3kU}{2a^2L_1}\left(1+\frac{a}{2}\beta\right)\left(\frac{3k}{2ab}+\frac{3}{20a\beta}+\frac{b}{20a}\right),$$

$$E_1 = -\frac{3kU}{a^2L_1}\left(1+\frac{a}{2}\beta\right)\left(\frac{9}{2ab\beta}+\frac{3b}{20a}+\frac{3k}{4ab}\right),$$

$$F_1 = \frac{9kU}{4a^3bL_1}\left(1+\frac{a}{2}\beta\right),$$

where

$$L_1 = \left(3+a\beta-\frac{6k}{a^2}\right)\left(\frac{3b^3}{20a^4\beta}+\frac{b^4}{20a^4}\right)$$

$$-\left(3+a\beta+\frac{3k}{a^2}+\frac{3k}{2a}\beta\right)\left(\frac{3k}{2ab}+\frac{3}{20a\beta}+\frac{b}{20a}\right)$$

and

(19)
$$A_2 = \frac{15a^5U}{28c^2L_2}\left[\left(\frac{a^3}{4}+\frac{8k}{\beta}-\frac{7ak}{2}\right)\left(\frac{7b^3}{3a^4\beta}+\frac{b^4}{3a^4}-\frac{48bk}{a^4\beta}\right)\right.$$

$$\left.+\left(\frac{6k}{\beta}-\frac{a^3}{4}\right)\left(\frac{7a^3}{3b^4\beta}+\frac{a^3}{3b^3}+\frac{64a^3k}{b^6\beta}+\frac{28a^3k}{b^5}\right)\right],$$

$$B_2 = \frac{15a^5U}{4c^2L_2}\left(\frac{a}{2}+\frac{1}{\beta}\right)\left(\frac{7b^3}{3a^4\beta}+\frac{b^4}{3a^4}-\frac{48bk}{a^4\beta}-\frac{7a^3}{3b^4\beta}-\frac{a^3}{3b^3}\right.$$

$$\left.-\frac{64a^3k}{b^6\beta}-\frac{28a^3k}{b^5}\right),$$

$$C_2 = -\frac{15a^5Uk}{4c^2L_2}\left(\frac{a}{2}+\frac{1}{\beta}\right)\left(\frac{7b^3}{3a^4\beta}+\frac{b^4}{3a^4}-\frac{48bk}{a^4\beta}\right),$$

$$D_2 = -\frac{45Uk}{16c^2a^2L_2}\left(\frac{a}{2}+\frac{1}{\beta}\right)\left(\frac{7a^3}{3b^4\beta}+\frac{a^3}{3b^3}+\frac{64a^3k}{b^6\beta}+\frac{28a^3k}{b^5}\right),$$

$$E_2 = -\frac{105aUk}{16c^2b^5L_2}\left(\frac{a}{2}+\frac{1}{\beta}\right)\left(\frac{5b}{\beta}+b^2+12k\right),$$

$$F_2 = \frac{315aUk}{c^2b^6L_2}\left(\frac{a}{2}+\frac{1}{\beta}\right)\left(\frac{b}{4}+\frac{1}{\beta}\right),$$

where

$$L_2 = \left(\frac{1}{\beta}+\frac{a}{7}-\frac{150k}{7a^2\beta}\right)\left(\frac{7b^3}{3a^4\beta}+\frac{b^4}{3a^4}-\frac{48bk}{a^4\beta}\right) - \left(\frac{1}{\beta}+\frac{a}{7}+\frac{225k}{14a^2\beta}+\frac{15k}{4a}\right)\left(\frac{7a^3}{3b^4\beta}+\frac{a^3}{3b^3}+\frac{64a^3k}{b^6\beta}+\frac{28a^3k}{b^5}\right).$$

Expression for drag: Let σ_{rr} and $\sigma_{r\theta}$ be the normal and tangential components of stress on the surface choosen to be the sphere of radius $r > a$, then the expression for the drag is given by [3]

$$D = 2\pi a^2 \int_0^{\pi} \left(\sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta\right)_{r=a} \sin\theta \, d\theta$$

$$= 2\pi a^2 \int_0^{\pi} \left[\left(-p + 2\mu\frac{\partial q_r}{\partial r}\right)\cos\theta - \mu\sin\theta\left\{r\frac{\partial}{\partial r}\left(\frac{q_\theta}{r}\right) + \frac{\partial q_r}{r\partial\theta}\right\}\right]_{r=a} \sin\theta \, d\theta$$

Using equations (13) the above expression becomes

$$D = -4\pi\mu B_1$$

$$= -\frac{6\pi\mu aU}{L_1}(2+a\beta)\left(\frac{3k}{2ab}+\frac{3}{20a\beta}+\frac{b}{20a}-\frac{3b^3}{20a^4\beta}-\frac{b^4}{20a^4}\right).$$

This is the same as that obtained by Jones [3] in the case of uniform external stream which corresponds to $c \to \infty$ in this analysis. Thus we conclude that the first order non-uniformity i.e., the second term in (7) introduced for large c, does not affect the drag to this order.

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A Survey of Ordered Loops I

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1. To get a clear picture of the topic of this paper we shall give some definitions.

Def. 1: If on the set A is defined a binary relation, written $x \leq y$, among some pairs of elements of A, which satisfies the following properties:

ORD 1. $x \leq x \ \forall \ x \in A$ (reflexivity)

ORD 2. $x \leq y \ \& \ y \leq z \implies x \leq z$ (transitivity)

ORD 3. $x \leq y \ \& \ y \leq x \implies x = y$ (antisymmetry)

then the set A is called a partially ordered (p.o.) set and the relation $\leq$ is called a partial order.

The elements x & y of p.o. set A are said to be non-comparable, written $x \parallel y$, it neither $x \leq y$ nor $y \leq x$ takes place. We shall say that A is totally ordered set (or chain) if any two of its elements are comparable.

Def. 2: Algebra $\mathfrak{Q} = \langle Q, \cdot \rangle$ is called a loop if each of the equations $xa=b$ & $ay=b$ have the unique solution in Q and there exists an element $e \in Q$ such that $ex=xe=x \ \forall \ x \in Q$.

Def. 3: Primitive loop is the algebra $\mathfrak{Q} = \langle Q, \cdot, \backslash, / \rangle$ of the type $\langle 2,2,2 \rangle$ which satisfies the following conditions:

1) $(x / y)\, y = x \, , \ y\, (y \backslash x) = x \ \forall \ x,y \in Q$

2) $(xy) / y = x \, , \ y \backslash (yx) = x \ \forall \ x,y \in Q$

3) $x / x = y \backslash y \ \forall \ x,y \in Q$

By the symbols $\cdot$, $\backslash$, $/$ are devoted here the composition laws defined on Q which are called multiplication, left and right division respectively. Type $\langle 2,2,2 \rangle$ of the algebra $\mathfrak{Q}$ means that all the composition laws mentioned above are binary ones.

It must be mentioned that there exists an identical correspondence between the loops and primitive loops, but their algebraic properties are different, for instance, any subalgebra of primitive loop is a subloop, but subalgebra of usual loop is a subgroupoid with identity as well.

Def. 4: Algebraic system $\mathcal{Q} = \langle Q, \cdot , \backslash , /, \leq \rangle$ is called a partially ordered primitive loop if

i) it is a primitive loop;

ii) its main set Q is a partially ordered set with the order defined by $\leq$.

iii) $\mathcal{Q}$ satisfies the following monotone laws:

a) if $x \leq y$ then $cx \leq cy$ & $xc \leq yc$ $\forall c \in Q$,

b) if $x \leq y$ then $x/c \leq y/c$ & $c\backslash x \leq c\backslash y$ $\forall c \in Q$,

c) if $x \leq y$ then $c/x \geq c/y$ & $x\backslash c \geq y\backslash c$ $\forall c \in Q$.

According to this definition the usual loop $\mathcal{Q} = \langle Q, \cdot \rangle$ is called a partially ordered loop if it satisfies only the first monotone law.

The following relations in the partially ordered loop $\mathcal{Q} = \langle Q, \cdot . \leq \rangle$ are called the cancellation laws:

a) if $ac \leq bc \implies a \leq b$ $\forall c \in Q$

b) if $ca \leq cb \implies a \leq b$ $\forall c \in Q$

These relations cannot be proved using the properties of the loop and so must be postulated. Certainly, in the primitive partially ordered loop the cancellation laws can be proved; they are also true in the totally ordered loop.

We shall say that the composition law is compatible with the order if the composition law and order are defined on the same set, and the composition law satisfies the corresponding monotone law.

Proposition 1. There exists an identical correspondence between the partially ordered primitive loops and a partially ordered loops with cancellation laws.

It can be easily seen that every primitive p.o. loop is a p.o. loop with respect to multiplication. Really, the element $a\backslash b$ is the solution of the equation $ax=b$, because $ax=b \implies a \backslash (ax) = a\backslash b$ & 2) (see def. 3) implies that the element $x=a\backslash b$ is a unique one, similarly, the element $y=b/a$ is a unique solution of the equation $ya=b$. From the axioms 1) & 3) of the definition 3 it follows that the quotient $x/x=y\backslash y=e$ is the identity. Cancellation laws follow from 1) & 2).

And conversely, let $\mathcal{Q} = \langle Q, \cdot, \leq \rangle$ is usual p.o. loop with cancellation laws. Let the unique solutions of the equations ax=b & ya=b be denoted by a\b & b/a respectively. We can consider the symbols \,/ as the new composition laws on Q. Putting x=a\b into equation ax=b and y=b/a into equation ya=b, we get the identical relations on Q:

$$(b/a)a=b, \qquad a(a\backslash b) = b \ \forall a, b \in Q.$$

The elements b & a satisfy the equations ax=ab & yb=ab respectively, so we get that

$$a \backslash (ab) = b, \quad (ab)/b = a \ \forall a, b \in Q.$$

If e is an identity in $\mathcal{Q} = \langle Q, \cdot, \leq \rangle$ then

$$ex=x, \quad ye=y \ \forall x, y \in Q \Rightarrow x/x = e = y \backslash y \ \forall x, y \in Q.$$

From the cancellation laws the monotone laws b) and c) of the definition 4 follow.

So, the algebraic system $\mathcal{Q} = \langle Q, \cdot, \backslash, /, \leq \rangle$ is a primitive partially ordered loops.

This proposition shows that the most natural class of partially ordered loops is a class of primitive partially ordered loops. Hence forward for simplicity we shall call the primitive partially ordered loops just partially ordered loops.

If Q is a totally ordered set then $\mathcal{Q} = \langle Q, \cdot, \backslash, /, \leq \rangle$ is called totally ordered (t.o.) loop.

The loops with the right (left) monotone law are called right (left) ordered loops.

Subset M of the basic set Q of the given algebra is called an invariant subset if it satisfies the conditions

1. $xM=Mx \quad \forall \ x \in Q$
2. $M(xy) = (Mx)y = x(My) \ \forall x, y \in Q$

Putting y=e in 2, we get the condition 1, so in algebraic systems with the identity the first condition is a corrollary of the second one, but they are independent in general.

The elements $a, a' \in Q$ such that $\exists x \in Q: ax=xa'$ are called the conjugate elements of the first type. The elements $b, b', b'' \in Q$ such that $\exists x, y \in Q$: $b(xy) = (b'x)y = x(b''y)$ are called the conjugate elements of the second type. The invariant set consists of elements along with their conjugate elements.

2. We shall say that the element a ef p.o. loop $\mathcal{Q}$ is positive (negative), if $a \geqslant e$ ($a \leq e$). The positive cone of the p.o. loop $\mathcal{Q}$ is a set of all its positive elements, so it can be defined in the usual way: $P=\{x \in Q | x \geqslant e\}$. Similarly, the negative cone of a p.o. loop is a set defined by the condition: $N=\{x \in Q | x \leq e\}$.

<u>Theorem 1</u>. Subset P of the basic set Q of the given loop $\mathcal{Q} = \langle Q, \cdot, \backslash, / \rangle$ is a positive cone of some partial order on $\mathcal{Q}$ iff it satisfies the conditions:

a) $e \in P$,

b) $PP = P$,

c) if $xy = e (x,y \in P)$ then $x=y=e$,

d) $P(xy) = (Px)y = x(Py) \;\forall\, x,y \in Q$.

<u>Necessity</u>. Let $\mathcal{Q} = \langle Q, \cdot, \backslash, /, \leq \rangle$ be a p.o. loop with the positive cone P. Condition a) is a corollary of the reflexivity of the order. If $x \geqslant e$ & $y \geqslant e$ then $xy \geqslant y \geqslant e$ and $PP \subset P$, but $e \in P \Rightarrow PP \supset P$ so $PP = P$ and b) is proved. Now $x \geqslant e$, $y \geqslant e$ & $xy=e \Rightarrow e/y \leq e$, but $e/y=x$ is a positive element, so it $y \neq e$ we have got the contradiction which proves that $x=y=e$ and we get the condition c). To prove d) let $a \in P$ & $x,y \in Q$, then $(ax)y \geqslant xy$. Putting $(ax)y=c$, we shall find q from the equation $q\,(xy)=c$. We have now $q(xy) \geqslant xy \Rightarrow q \in P$. so $(ax)y=c \in P\,(xy)$. But the elements a, x & y are the arbitrary elements of P and Q respectively, so we get that

$$(Px)y \subset P(xy) \;\forall\, x,y \in Q \qquad (*)$$

On the other hand $\forall\, a \in P$ & $\forall\, x,y \in Q$ we have $a(xy) \geqslant xy$. Putting $a(xy)=c$, we shall find $q \in Q$: $qy=c \Rightarrow qy \geqslant xy \Rightarrow q \geqslant x$. Now let us find $\rho \in Q$: $px=q \Rightarrow px \geqslant x \Rightarrow p \in P$. From this $(px)y=qy=c \in (Px)y$, but $c=a(xy) \in P(xy)$ and the elements a,x & y are the arbitrary elements, so

$$P(xy) \subset (Px)y \;\forall\, x,y \in Q \qquad (**)$$

Similarly: let $a \in P$, $a \geqslant e \Rightarrow ay \geqslant y \Rightarrow x(ay) \geqslant xy$. Devoting $x(ay)=c$ we find $q \in Q$: $q(xy)=c \Rightarrow q(xy) \geqslant xy \Rightarrow q \in P \Rightarrow q(xy)=c \in P(xy)$, but $q(xy)=c=x(ay)$ and we get

$$x(Py) \subset P(xy) \;\forall\, x,y \in Q \qquad (***)$$

Furthermore, $\forall\, a \in P$ & $x,y \in Q$ we have $a(xy) \geqslant xy$, $a(xy)= =c \Rightarrow \exists!\, q \in Q$: $xq=c \Rightarrow xq \geqslant xy \Rightarrow q \geqslant y$, but $\exists!\, p \in Q$: $py=q \Rightarrow py \geqslant y \Rightarrow p \in P \Rightarrow py \in Py \Rightarrow py=q \in Py \Rightarrow xq \in x(Py) \Rightarrow x(py) =xq =c =a(xy) \in x(Py)$, i.e.

$$P(xy) \subset x(Py) \ \forall \ x,y \in Q \qquad (****).$$

Compairing (*), (**), (***) & (****) we get the condition d), and necessity is proved.

<u>Sufficiency</u>. Let $P \subset Q$ and P satisfies the conditions a) - d) of the theorem, then $\mathfrak{Q}$ is a p.o loop with the positive cone P.

Really, $a \leq b$ iff $b/a \in P$. So $a \leq a \ \forall a \in Q$ because $e=a/a \in P$, so our relation is reflexive. If $a \leq b$ & $b \leq c$ it means that $b \in Pa$, i.e. $b=xa$ where $x \in P$. Furthermore, $c \in Pb \Rightarrow c=yb$, where $y \in P$, but $c=yb=y(xa) \in P(xa)=(Px)a \subseteq Pa$, so $c \in Pa \Rightarrow a \leq c$ and the transitivity is proved. Let now $a \leq b$ & $b \leq a$ at the same time for some $a,b \in Q$, then $a=xb$, $x \in P$ & $b=ya$, $y \in P \Rightarrow a=xb=x(ya) \in P(ya) = (Py)a$, so $a = (zy)a \Rightarrow zy=e \Rightarrow y=e$ and so $b=a$, and the relation $\leq$ is antisymmetric. So the relation $\leq$ defined above being reflexsive, transitive and antisymmetric is a relation of ordering on Q.

$a \leq b$ means that $b/a \in P$. Let c be any element of Q then devoting $c\backslash a=x$ & $c\backslash b=y$ we get $a=cx$ & $b=cy \Rightarrow cx \leq cy \Leftrightarrow cy/cx= z \in P$. Now $cy/cx= z \Rightarrow cy= z(cx) \in P(cx) = c(Px) \Rightarrow y \in Px$ or $y/x \in P \Rightarrow x \leq y \Leftrightarrow c\backslash a \leq c\backslash b \ \forall \ c \in Q$. Similarly $a/c \leq b/c \ \forall c \in Q$. Again $a \leq b \Leftrightarrow b/a \in P$, and let $c \in Q$. Devoting $a\backslash c=x$ & $b\backslash c=y$ we get $c=ax$ & $c=by$ so $ax=by$ but $b=pa$ where $p \in P \Rightarrow ax=by=(pa)y \in (Pa)y= =a(Py) \Rightarrow x \in Py \Rightarrow x/y \in P \Leftrightarrow y \leq x \Leftrightarrow b\backslash c \leq a\backslash c$ or $a\backslash c \geq b\backslash c$ $\forall c \in Q$. Similarly $c/a \geq c/b \ \forall c \in Q$. Similarly $a \leq b \Leftrightarrow b/a \in P$ $\Rightarrow b=xa$ and we have $\quad \forall c \in Q$.

$bc=(xa)c \in (Pa)c=P(ac) \Rightarrow bc/ac \in P \Leftrightarrow ac \leq bc$ and similarly $ca \leq cb \ \forall c \in Q$. So all the monotone laws are satisfied on $\langle Q, \leq \rangle$ and $\mathfrak{Q} = \langle Q, \cdot, \backslash, / \rangle$ is a p.o. loop with the order $\leq$ defined above. The theorem is proved.

This theorem shows that the positive cone of any order perfectly defines this order. So instead of order with positive cone P we shall simply say "the order P".

It can be easily seen that the negative cone $N= \{x \in Q | x \leq e\}$ of the p.o. loop $\mathfrak{Q}$ also satisfies the conditions a) - d) of the theorem 1 and perfectly defines the corresponding order. The order defined by N is a dual order with respect to the order defined by P.

The following statements are evident:

1) $P \cap N = e$;

2) $a \in P \Rightarrow e/a \in N$ & $a\backslash e \in N$;

3) $y \in Px \Rightarrow x \in Ny$.

So the negative cone N consists of all converse elements of the elements of positive cone P. The converse is also true.

The properties a)-d) of the positive and negative cones show that P & N are groupoids which have the identity elements only in converse.

It is obvious that

Proposition 2. The p.o. loop $\mathfrak{Q} = \langle Q, \cdot, \backslash, /, \leq \rangle$ is a t.o. loop iff $P \cup N = Q$.

It can be easily proved that the necessary and sufficient conditions for the loop $\mathfrak{Q}$ to be left-(right-)-ordered are the following:

1) $e \in P$;

2) $PP=P$;

3) $xy=e \ (x,y \in P) \Rightarrow x=y=e$;

4) $xP=Px \ \forall x \in Q$;

5) $(xy)P=x(yP) \quad [P(xy)=(Px)y] \ \forall x,y \in Q$.

3. In this section we shall study the properties of the invariant groupoids in $\mathfrak{Q}$.

Def. Subgroupoid S of the loop $\mathfrak{Q}$ is called invariant subgroupoid in $\mathfrak{Q}$ if the set of its elements is an invariant set in Q.

It should be mentioned that the set of the invariant groupoids in $\mathfrak{Q}$ is not empty, because Q is invariant in Q and in p.o. loop positive and negative cones are also invariant groupoids.

If the mapping $f: Q \longrightarrow Q$ is a one-to-one mapping, we shall say that f is a substitution of Q. Considering the mappings: $x \longrightarrow xa$ & $x \longrightarrow ax$ on the basic set Q of the loop $\mathfrak{Q}$ we can say that they are the substitutions of Q. We shall devote the first and the second mapping by R_a & L_a respectively, so $R_a x \equiv xa$ & $L_a x \equiv ax$, and their inverse mappings will be devoted by R_a^{-1} & L_a^{-1}, so $R_a^{-1}x \equiv x/a$ & $L_a^{-1}x \equiv a\backslash x$.

Proposition 3. The invariant groupoids have the following properties:

a) $R_a^{-1}(S\,b) = S(R_a^{-1}b)$

$L_a^{-1}(b\,S) = (L_a^{-1}b).S$ $\quad \forall a, b \in Q$

b) if $R_a^{-1}b \in S$, then $L_a^{-1}b \in S$ and vice versa.

Really, the equation b=pa has the unique solution in q with respect to p, and applying the property 2) of the invariant sets we get

$$R_a^{-1}(S\,b)= R_a^{-1}[S\,(pa)]= R_a^{-1}[(S\,p)a]= R_a^{-1}[R_a(S\,p)]=Sp,$$

but $p=R_a^{-1}b$, so the required result holds:

$$R_a^{-1}(S\,b)= S\,.R_a^{-1}b \quad \forall a,b \in Q.$$

Similarly $L_a^{-1}(b\,S)= L_a^{-1}b.S \quad \forall a,b \in Q.$

The property 2) follows from the fact that the elements $R_a^{-1}b$ & $L_a^{-1}b$ are the conjugate elements:

$$R_a^{-1}b.\, a= a\,.\, L_a^{-1}b.$$

Particularly if b=e we shall get from a) & b) respectively:

a′) $R_a^{-1}(S\,)= S.\, R_a^{-1}e$ & $L_a^{-1}(S\,)= L_a^{-1}e.S \quad \forall a \in Q$

b′) if $R_a^{-1}e \in S$ then $L_a^{-1}e \in S$ and vice versa.

Using induction we can easily prove the following

<u>Lemma 1.</u> The product of the finite number of the invariant groupoids is also an invariant groupoid.

Devoting $S=\prod_{i}^{n} S_i$ (i=1,2,...n), where i is invariant groupoid, we shall get

1) if $e \in S_i$ (i=1,2,...,j-1, j+1,...,n) then $S \supseteq S_j$;

2) if $e \in S_i \; \forall$ i=1,...,n then $S \supset S_i \; \forall i$

3) if $e \in S$ then $S_e = e \cup S$ is an invariant groupoid.

It follows from the definition of the invariant groupoids that each invariant groupoid contains together with each of its element all the conjugate elements.

Let $a_1, a_2, \ldots, a_n$ ($a_i \neq e$) be any finite collection of elements of Q and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ (ε_i = 1 or - 1) be the collection of signs. The correspondence between the elements and signs, will be written as.

$$\begin{pmatrix} a_1, & a_2, & \ldots, & a_n \\ \varepsilon_1, & \varepsilon_2, & \ldots, & \varepsilon_n \end{pmatrix}, \qquad (1)$$

The finite collection of the type

$$\begin{pmatrix} a_1, & a_2, & \ldots, & a_n \\ -\varepsilon_1, & -\varepsilon_2, & \ldots, & -\varepsilon_n \end{pmatrix}, \qquad (1^*)$$

where a_i are the same as in (1) and $-\varepsilon_i$ are opposite to ε_i, we shall call (1*) the opposite collection with respect to (1).

Definition. A pair of groupoids A & B ($A \subseteq Q$, $B \subseteq Q$), with the properties:

I. $R_{a_i}^{\varepsilon_i} e \in A$, $R^{-1}_{R^{\varepsilon_i}_{a_i} e} e \in B$ $(i = 1,2,\ldots,n)$,

II. A & B are the invariant groupoids,

III. $b \in A \Rightarrow R_b^{-1} e \in B$ $(L_b^{-1} e \in B)$;

IV. $c \in B \Rightarrow L_c^{-1} e \in A$ $(R_c^{-1} e \in A)$

is called an invariant pair, corresponding to the collection (1) and devoted by (A,B).

The invariant pair, corresponding to the opposite collection (1*) denoted by (A*,B*).

The set $\{(A,B)\}$ is not empty, because the pair (Q,Q) always satisfies the conditions I– IV.

Let $S = \bigcap A$ & $T = \bigcap B$, where A & B are corresponding components of all the pairs from $\{(A,B)\}$. In this case the groupoids S & T form the invariant pair corresponding to the collection (1). This pair is called a minimal invariant pair. The minimal invariant pair, corresponding to the collection (1) is said to be generated by this collection. As before by (S*, T*) we shall devote the opposite pair to the pair (S, T).

Lemma 2. The opposite pairs (S,T) & (S*,T*) satisfy the conditions:

$$S = T^* \; ; \; T = S^*.$$

Proposition 4. In the p.o. loop $\mathcal{O}\!\!\!/$ the groupoids S & T of the pair (S,T) satisfy the following conditions:

a) $a \in P \Longrightarrow S_e\,(R_a e) \subseteq P;$

b) $a \in P \Longrightarrow P \cap T(R_a e) = \emptyset;$

c) $S_e(R_{a_1} e, \ldots, R_{a_n} e) = \prod_{i=1}^{n} S_e\,(R_{a_i} e)$

$T_e(R_{a_1} e, \ldots, R_{a_n} e) = \prod_{i=1}^{n} T_e(R_{a_i} e)$

d) $S^-(R_{a_1} e, \ldots, R_{a_n} e) = T\,(R_{a_1} e, \ldots, R_{a_n} e)$

where S^- devotes the set of all local converse elements to the elements of the groupoid S.

From the Lemma 2 and Proposition 4 it follows:

(i) $a \in P \Rightarrow T^*_e(R_a e) \subseteq P;$

$T_e(R_a e) \subseteq N.$

$S^*_e(R_a e) \subseteq N;$

(ii) $a \in P\ (a \neq e) \Longrightarrow P \cap S^*(R_a e) = \emptyset$

$N \cap S(R_a e) = \emptyset$

$N \cap T^*(R_a e) = \emptyset$

where $\emptyset$ devotes the empty set.

(iii) For every finite collection (1) the following equalities are true:

$$S_e(R_{a_1} e), \ldots, R_{a_n} e) = \prod_{i=1}^{n} T^*_e (R_{a_i} e)$$

$$T^*_e (R_{a_1} e, \ldots, R_{a_n} e) = \prod_{i=1}^{n} S_e (R_{a_i} e)$$

$$S^*_e (R_{a_1} e, \ldots, R_{a_n} e) = \prod_{i=1}^{n} T_e (R_{a_i} e)$$

$$T_e (R_{a_1} e, \ldots, R_{a_n} e) = \prod_{i=1}^{n} S^*_e (R_{a_i} e)$$

(iv) $(S, T) = (T^*, S^*) = (T^-, S^-)$

The proof of the Lemma 2 and Proposition 4 is given in [2].

The question about the extension of the partial order in p.o. loop $\mathfrak{L}$ will be considered in the next issue.

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Problems Section

1. Show that $\lim_{n\to\infty} n \sin (2\pi e n!) = 2\pi$. Show how to deduce from this result that e is irrational.

2. Let E be any collection of n integers, not necessarily distinct. Show that there exists a nonempty subcollection $F \subseteq E$ such that the sum of the integers contained in F is divisible by n.

3. Perfectly rigid cards are piled on the edge of a table with the pile slanting up away from the table. How far from the edge of the table can the pile be made to extend without falling to the floor?

4. Prove that the product of four consecutive positive integers can be neither a perfect swuare nor a perfect cube.

प्रस्तावित गणितीय शब्दावली

GLOSSARY OF MATHEMATICAL TERMS

(Proposed)

Aberration	-	विपथन
Ab initio	-	आदि नियमत:, आदित:
Absolute	-	अमूर्त, सार
Abstract	-	पूर्ण, निरपेक्ष, शुद्ध
Absurd	-	विसङ्गत, अर्थहीन
Absurdity	-	विसङ्गति, अर्थहीनता
Acceleration	-	त्वरण, द्रुतता, शीघ्रता
Accumulation	-	संचय, संग्रह, पुञ्ज
Act	-	काम गर्नु, कार्य गर्नु
Action	-	कार्य, क्रिया
Actual	-	वास्तविक
Acute	-	न्यून
Add	-	जोड्नु, योग
Addition	-	जोड, योग
Adfected	-	मिश्र
Ad infinitum	-	यावत् अनन्त, असीमित
Adjacent	-	आसन्न
Adjoin	-	संलग्न, आसन्न
affine	-	सजातीय, यथावत् रूपान्तरित
Affinity	-	सजातीयता, यथावत् रूपान्तरण
Aggregate	-	समाहार, संग्रह
Algebra	-	बीज गणित
Algebraic	-	बीज गणितीय, बीजीय
Algebraical	-	बीजीय
Algebraically	-	बीजीयानुरूप
Algorithm	-	विधि-विशेष
Alligation	-	मिश्रण
Aliquot	-	अशेष भाजक
Alpha	-	अल्फा (α)
Alternate	-	एकान्तर
Altitude	-	उच्चाई
Ambiguity	-	सन्दिग्धता, अस्पष्टता
Ambiguous	-	सन्दिग्ध, अस्पष्ट
Amount	-	मिश्रधन, राशि
Amplitude	-	विस्तार, कोणांक, आयाम
Analogy	-	सादृश्य
Analysis	-	विश्लेषण
Analytic	-	विश्लेषणात्मक
Analytical	-	विश्लेषणात्मक
Angle	-	कोण

यो शब्दावली त्रि०वि०, कीर्तिपुर बहुमुखी क्याम्पस, गणित तथा नेपाली शिक्षण समितिले संयुक्त रूपमा तयार गरिएको हो ।

1978

Angular - कोणीय, कोणात्मक
Annihilate - शून्य गर्नु
Annihilator - शून्यकारी
Annual - वार्षिक
Annuity - वार्षिकी
Annular - वलय, वलयाकार
Answer - उत्तर
Antecedent - पूर्ववर्ती
Anti-derivative - प्रति-अवकलन
Anti-logarithm - प्रति-लघु गुणाक, प्रति-लघु
Aperiodic - अनावर्ती
Apparent - दृष्ट
Appendix - परिशिष्ट
Applicable - प्रयोज्य
Application - प्रयोग
Applied - व्यावहारिक, प्रायोगिक
Apply - प्रयोग गर्नु, लागू गर्नु
Approximate - सन्निकट
Approximation - सन्निकटता
Apriori - निगम्य, प्रागनुभव
Arc - वृत्त-खण्ड, चाप
Area - क्षेत्र, क्षेत्रफल
Areal - क्षेत्रीय
Arithmetic - अंकगणित
Arithmetical - अंकगणितीय
Arm - बाहु, भुजा
Arrangement - प्रबन्ध
Array - विन्यास
Ascent - चढाइ, आरोहण
Assemblage - समुच्चय, पुंज
Associate - सहचारी
Association - साहचर्य
Associative - सहचर्यात्मक
Astronomy - खगोल
Astronomer - खगोल शास्त्री
Astronomical - खगोलीय
Asymmetric - असममित
Asymmetry - असममिति
Asymptote - अनन्त स्पर्शी
Asymptotic - अनन्त स्पर्शीय
Attribute - गुण
Auxiliary - सहायक
Average - औसत
Axial - अक्षीय
Axiomx - स्वत: तथ्य, स्वयम्सिद्ध
Axis - अक्ष