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CONTENTS

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A Mixture of Two Displaced Geometric Distributions for Describing the Distribution of the Total Number of Migrants at Micro-Level

TIKA RAM ARYAL

Abstract: The aim of this paper is to study the distribution of the total number of rural outmigrants from a household. Probability model has been used for this purpose. The maximum likelihood estimation technique has been proposed to estimate the parameters involved in the model. The data has been taken from a sample survey of Palpa and Rupandehi Districts of western Nepal. The proportion of households having only one migrant was found to be 0.52381 whereas the average number of migrants from the household was 0.57831. The asymptotic variances and co-variances of the estimators have also been obtained. A mixture of two displaced geometric distribution fit the data sets satisfactorily well. Thus proposed model describe the distribution of the total number of rural out-migrants from rural areas of Rupandehi and Palpa districts of western Nepal.

Key words: model, probability, frequency, maximum likelihood, risk, parameter, logarithm.

1. Introduction

Researchers have given their due attention on the formulation of models and their applications due to its usefulness and applicability in social sciences [1]. The macro-level migration studies have their own importance [2]. These approaches describe aggregate flow/rate of migration, and recognize the factors motivating out-migration [3]. Micro-level (i.e. at the level of household or individual) studies help to classify the behavioural parameter of migration process. There is a rare study in out-migration at micro-level, which may be due to the lack of interest of the researcher as well as unavailability of the data.

Several attempts have been made to documents the pattern of rural out-migration through probability models [2, 4, 7, 11]. The idea of cluster was integrated in the model by Yadava and Singh [10]. It was found that Thomas distribution is well suited to describe the number of migrants from a household.

Yadava and Yadava [9] further extended by assuming the occurrence of migration in cluster varies from household to household and the number of migrants to a cluster follows truncated displaced geometric distribution. Under such assumptions, probability model fitted well to the distribution of male migrants aged 15 years and above. However, these models do not fit the distribution of total number of migrants including their wife and children from a household.

Sharma [5,6] proposed a probability model with some assumptions : (i) the number of male migrants aged 15 years and above follows negative binomial distribution and (ii) the distribution of alive children to a couple be known. However, the prior knowledge about these two distributions is difficult since the distribution of children alive to a couple has not yet been derived theoretically. Singh [8] proposed a probability model for the total number of migrants under the assumption that there are two types of households i.e. the households from where male members aged 15 years and above migrate singly leaving their wives and children at home, and the households where male members migrate with their wives, children and other dependent relatives. Yadav [12] proposed a probability model to describe the distribution of households according to total number of migrants.

Moment techniques or mean-zero frequency method have been used to estimate the parameters involved in their proposed models. About 80 to 85 percent variation in migration is equated through zero'th cell frequencies i.e. all the non-migrant households are counted. About 15 to 20 percent variations are explained by the estimated parameters when mean-zero frequency method is applied [2]. Moment estimates are generally consistent, but they are often less efficient. By taking such limitations, the maximum likelihood estimation technique is applied to estimate the parameters involved in the proposed model. Maximum likelihood method provides standard error of the estimators as well as measures the total variation of the distribution.

The main aim of this paper is to study the distribution of the total number of rural out-migrants through probability model. Maximum likelihood estimate technique has been proposed to estimate the parameter involved in the model. The asymptotic variances and co-variances of the estimators have also been discussed. The suitability of the model has been tested to the data of Palpa and Rupandehi districts of western Nepal.

2. Model

Yadava [12] proposed a probability model to describe the distribution of households according to total number of migrants. Mean-zero frequency method

was used to estimate the parameters involved in the model. Here we used maximum likelihood estimation technique to estimate the parameters involved in the model by avoiding the limitation of mean-zero frequency method. In brief, the model along with their assumptions is given below :

(i) At the survey point, let β be the proportion of households which poses at-least one migrants

(ii) Out of β proportion of households, let ξ be the proportion of households, which poses only one migrant at the survey point.

(iii) Out of $(1-\xi)\beta$ proportion of households, let π be the proportion of households from which only males $\geq$ 15 years migrate and $(1-\pi)$ be the proportion of households which poses both types of migrants (males $\geq$ 15 years as well as males with their families).

(iv) The number of migrants from a household follows a mixture of two displaced geometric distributions with π proportion of households from which only males aged 15 years migrates and $(1-\pi)$ be the proposition of households from which both type of migration occur.

(v) Let p_1 and p_2 be the probability of migration of a person from π and $(1-\pi)$ proportions of households respectively.

Under these assumptions, the probability distribution for the total number of migrants, X, is given as

$$
\begin{aligned}
p(X=k) &= 1-\beta \text{ if } k=0 \\
&= \xi\beta \text{ if } k=1 \qquad (1)\\
&= (1-\xi)\beta\{\pi p_1 q_1^{k-2} + (1-\pi) p_2 q_2^{k-2}\} \text{ if } k=2,3,\ldots
\end{aligned}
$$

This model involves $\beta, \xi, \pi, p_1, p_2$ parameters, which is difficult to estimate from the observed data set. Assume that $p_1 = p_2 = p$, which is equivalent to the probability of migration from both types of households is same and (1) becomes

$$
\begin{aligned}
p(X=k) &= 1-\beta \text{ for } k=0 \\
&= \xi\beta \text{ for } k=1 \qquad (2)\\
&= (1-\xi)\beta\, pq^{k-2} \text{ for } k=2,3,\ldots
\end{aligned}
$$

3. Estimation of Parameters

Model (2) involves ξ, β and p parameters that are estimated by using observed data. Let $X_1, X_2, \ldots X_n$ denote a random sample of size n from the expression (2). Suppose, n_k $(k=0,1,2,\ldots,m)$ be the number of observations corresponding to the value of k such that $\sum_{k=0}^{m} n_k = n$. Likelihood function for the given sample is

$$L=\prod_{k=0}^{m}[p(X=k)]^{n_k}=(1-\beta)^{n_0}\ (\xi\beta)^{n_1}\prod_{k=2}^{m}[(1-\xi)\beta\, pq^{k-2}]^{n_k}$$

(3) $$=(1-\beta)^{n_0}\,\xi^{n_1}\,\beta^{n-n_0}\,(1-\xi)^{n-n_0-n_1}\,p^{n-n_0-n_1}\,q^{\sum_{k=2}^{m}(k-2)n_k}$$

Taking log in (3) and differentiating w.r.t. β, ξ and p respectively and by equating it to zero, then we get,

(4) $$\frac{\delta\log L}{\delta\beta}=-\frac{n_0}{1-\beta}+\frac{n-n_0}{\beta}=0$$

(5) $$\frac{\delta\log L}{\delta\xi}=\frac{n_1}{\xi}-\frac{n-n_0-n_1}{1-\xi}=0$$

(6) $$\frac{\delta\log L}{\delta p}=\frac{n-n_0-n_1}{p}-\frac{\sum_{k=2}^{m}(k-2)n_k}{1-p}=0$$

By solving (4),(5) and (6), the estimators of β, ξ and p is obtained

$$\hat{\beta}=\frac{n-n_0}{n},\ \hat{\xi}=\frac{n_1}{n-n_0}\ \text{ and }\ \hat{p}=\frac{n-n_0-n_1}{(n-n_0-n_1)+\sum_{k=3}^{m}(k-2)n_k}$$

The second partial derivations of $\log L$ is given as

(7) $$\frac{\delta^2\log L}{\delta\beta^2}=-\frac{n_0}{(1-\beta)^2}-\frac{n-n_0}{\beta^2}$$

(8) $$\frac{\delta^2\log L}{\delta\xi^2}=-\frac{n_1}{\xi^2}-\frac{(n-n_0-n_1)}{(1-\xi)^2}$$

(9) $$\frac{\delta^2\log L}{\delta p^2}=-\frac{(n-n_0-n_1)}{p^2}-\frac{\sum_{k=3}^{m}(k-2)n_k}{(1-p)^2}$$

(10) $$\frac{\delta^2\log L}{\delta\beta\delta\xi}=\frac{\delta^2\log L}{\delta\beta\delta p}=\frac{\delta^2\log L}{\delta\beta\delta p}=0$$

Here, $E(n_0)=E[\sum_{i=1}^{n}1_{\{X_i=0\}}]=\sum_{i=1}^{n}1p\{X_i=0\}=\sum_{i=1}^{n}(1-\beta)=n(1-\beta)$, in similar manner we can write $E(n_1)=n\xi\beta$, $E(n_k)=n(1-\xi)\ \beta\, pq^{k-2}$ for $k=2,3,\ldots,m$, $E(n-n_0)=np$, $E(n-n_0-n_1)=n\beta(1-\xi)$ and

$$E\left[\sum_{k=3}^{m}(k-2)n_k\right] = E[n_3+2n_4+3n_5+\ldots+(m-2)n_m]$$

$$= n(1-\xi)\beta pq\,[1+2q+3q^2+\ldots+(m-2)\,q^m]$$

(11) $$= n(1-\xi)\beta q\left[\frac{1-q^{m-2}}{p}-(m-2)q^{m-2}\right] \text{ for small } m$$

(12) $$= \frac{n(1-\theta)\beta q}{p} \text{ for large } m$$

By using these facts, the expected values of the second partial derivatives are

(13) $$-E\left(\frac{\delta^2\log L}{\delta\beta^2}\right)=\frac{E(n_0)}{(1-\beta)^2}-\frac{E(n-n_0)}{\beta^2}=\frac{n}{\beta(1-\beta)}=\phi_{11} \text{ (say)}$$

(14) $$-E\left(\frac{\delta^2\log L}{\delta\xi^2}\right)=\frac{E(n_1)}{\xi^2}+\frac{E(n-n_0-n_1)}{(1-\beta)^2}=\frac{n\beta}{\xi(1-\xi)}=\phi_{22} \text{ (say)}$$

$$-E\left(\frac{\delta^2\log L}{\delta p^2}\right)=\frac{E(n-n_0-n_1)}{p^2}+\frac{E\left[\sum_{k=3}^{m}(k-2)n_k\right]}{(1-p)^2}$$

$$=\frac{n\beta(1-\xi)q+n(1-\xi)\beta p[1-q^{m-2}-(m-2)pq^{m-2}]}{p^2q}$$

(15) $$=\phi_{33}\text{ (a) (say), for small } m$$

and

(16) $$-E\left(\frac{\delta^2\log L}{\delta p^2}\right)=\frac{n\beta(1-\xi)}{p^2q}=\phi_{33}\text{ (b) (say) for large } m$$

Covariance between the estimators equal zero since

$$E\left(\frac{\delta^2\log L}{\delta\beta\delta\xi}\right)=E\left(\frac{\delta^2\log L}{\delta\xi\delta p}\right)=E\left(\frac{\delta^2\log L}{\delta\beta\delta p}\right)=0$$

and the asymptotic variances of the estimators can be obtained as:

$$V(\hat\beta)=\frac{1}{\phi_{11}},\quad V(\hat\xi)=\frac{1}{\phi_{22}},\text{ and}$$

(17) $$V(\hat p)=\frac{1}{\phi_{11}(a)}\text{ when } m \text{ is small}$$

$$=\frac{1}{\phi_{22}(b)}\text{ when } m \text{ is large,}$$

4. Applications

The probability model discussed in this paper for describing the total number of migrants from a household is fitted using the maximum likelihood estimators to the data collected from rural areas of Palpa and Rupandehi districts of Nepal. These data were collected under a sample survey "Demographic Survey on Fertility and Mobility in Rural Nepal (DSFM, 2000) : A Study of Palpa and Rupandehi Districts" during January-June, 2000. The detail about the sample survey is discussed in Aryal [1].

Table 1: Distribution of Observed and Expected Number of Households According to the Total Number of Male Migrants in Rural, Nepal.

Number of migrants per household	Observed	Expected
0	622	622.00
1	99	99.00
2	27	30.98
3	17	21.02
4	13	14.26
5	11	9.68
6	7	6.57
7	4	7.49
8	11	
Total	811	811.00

χ^2	9.13
d.f.	4
$\hat{\beta}$	0.23305
$\hat{\xi}$	0.52381
$\hat{p}$	0.32143
$V(\hat{\beta})$	0.00022
$V(\hat{\xi})$	0.00132
$V(\hat{p})$	0.00079
Covariances	0.00000
Average Number of migrants per households	0.57831

The observed and expected number of households (along with the variances and co-variances between the estimators) according to the total number of migrants is presented in Table 1. The proportion of households that poses at-least one migrants ($\hat{\beta}$) was 0.23305 and out of which the proportion of households having only one migrant ($\hat{\xi}$) was found to be 0.52381. The probability of migration of a person from households ($\hat{p}$) was found to be 0.32143. The higher proportion of households having only one migrant may be due to higher cost on travel and higher cost of living at the place of destination along with their families. Moreover, most of the migrants move for a certain period of time and after completion of their tenure they have to return back to their home. Further, the higher value of $\hat{\beta}$ indicated that a higher rate of migration was observed in the rural migrant's households.

The average number of migrants per household can be obtained as $\hat{\xi}\hat{\beta} + (1-\hat{\xi})\hat{\beta}\left(1+\frac{1}{\hat{p}}\right)$. The average number of migrants from the household was found to be 0.57831. The chi-square value was found insignificant which confirms that the model fits the data sets reasonably well. Thus the model is a reasonable approximation to describe the distribution of the total number of migrants from a household for Nepal.

5. Conclusions

A mixture of two displaced geometric distribution was found a reasonable approximation to describe the distribution of the total number of migrants from a household at least at the micro-level. The exact variances and co-variance of the estimators for the model has also been computed. The proportion of households having only one migrant was found to be 0.52381. Whereas the average number of migrants from the household was 0.57831. For the development of a more effective and equitable rural and urban policies in the developing countries like Nepal, the policy planners and social researchers may get an idea from this study.

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Uniform Version of the Wiener-Tauberian Theorem for Real Line

CHET RAJ BHATTA

Abstract: The wiener-Tauberian theorem for $\mathbb{R}$ says that the closed translation invariant subspace generated by an $f \in L^1(\mathbb{R})$ is $L^1(\mathbb{R})$ if and only if the Fourier transform $\hat{f}$ of f never vanishes. In this paper we prove a uniform version of this result for $\mathbb{R}$.

Key words : Wiener-Tauberian theorem, locally compact abelian groups, translation invariant subspace.

1. Introduction :

The general Tauberian theorem proved by N.Wiener [11] says that if $g \in L^1(\mathbb{R})$ is a uniqueness function in the sense that its Fourier transform $\hat{g}$ vanishes nowhere on $\mathbb{R}$ (and thus the closed translation invariant subspace generated by g is $L^1(\mathbb{R})$ and $\phi \in L^\infty(\mathbb{R})$ is such that $g * \phi(x) \to A\hat{g}(0)$ (A is a complex number) as $x \to \infty$ then for every $f \in L^1(\mathbb{R}), f * \phi(x) \to A\hat{f}(0)$ as $x \to \infty$. Here the Fourier transform of a function $f \in L^1(\mathbb{R})$ is defined by $\hat{f}(y) = \int_{-\infty}^{\infty} f(x) \exp(-2\pi\, ixy)\, dx$ for $y \in \mathbb{R}$. For a generalization of the above result for locally compact abelian group see [1], [4] and [7]. Various analogues of the Wiener-Tabuerian theorem for non-abelian groups are given by Ehrenpreis and Mautner [2], Kaniuth and Steiner [5], Hauenschild, Kaniuth and Kumar [3], Sitaram [9] and others, see [8] for survey.

In section 2, we obtain uniform version of the Wiener-Tauberian theorem for locally compact abelian group $\mathbb{R}$ replacing $\{g\},\{f\}$ by suitable subsets of $L^1(\mathbb{R}),\{\phi\}$ by suitable subsets of $L^\infty(\mathbb{R})$. The techniques include figuringout equicontinuous subsets of $L^\infty(\mathbb{R})$ in $\| \ \|_\infty$ – topology.

We continue these investigations and prove a uniform version of the wiener-Tauberian theorem as given in Reiter and Stegeman.

2. Uniform Version

Let $\mathbb{R}$ be a locally compact abelian group with Haar measure μ. Here in this case Haar measure is ordinary Lebesgue measure. For basic notations and terminology, we refer to [4]. For $x \in \mathbb{R}$ and $f \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$, let f_x, be the translate of f. Let $\phi_f : \mathbb{R} \to L^\infty(\mathbb{R})$ be defined by $\phi_f(x) = f_x$, $x \in \mathbb{R}$, $f \in L^\infty(\mathbb{R})$. We denote by S_1 and S_∞, the unit balls of $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$ respectively.

Define $\mathsf{U} = \{g \in L^1(\mathbb{R}) : \text{for } a \in B \cap C, g * a = 0 \Rightarrow a = 0)$.

Where B and C are the sets of bounded and continuous function on $\mathbb{R}$ respectively.

The following result 2.3 may be thought of as a uniform version of the Wiener-Tauberian theorem as given in Widder ([11], theorem 3.1).

Definition 2.1. Let $\mathcal{F}$ be a collection of functions on a metric space X with metric ρ to a metric space Y with metric ρ^1. We say that $\mathcal{F}$ is uniformly equicontinuous if to every $\in > 0$ corresponds a $\delta > 0$ such that $\rho^1(f(x), f(y)) < \in$ for every $f \in \mathcal{F}$ and for all pairs of points x, y with $\rho(x,y) < \delta$.

Example 2.2 : If $X = \mathbb{R}$, $Y = L^1(\mathbb{R})$, $\mathcal{F}$ a set of functions on X to Y which is

(i) Equicontinuous at some point and
(ii) Translation-invariant i.e., for $f \in \mathcal{F}, x \in \mathbb{R}, f_x \in \mathcal{F}$, then $\mathcal{F}$ is uniformly equicontinuous.

For $x, y \in \mathbb{R}$ with $|y - x| < \delta$ we have

$$\|f(x) - f(y)\| = \|f_x(0) - f_x(y - x)\|$$
$$= \|f_{x-z}(z) - f_{x-z}(z + y - x)\| < \in$$

So equicontinuity of $\mathcal{F}$ at any point is equivalent to uniform equicontinuity of $\mathcal{F}$,

Theorem 2.3 : *Let $H \subset L^1(\mathbb{R})$ be such that*

(i) $\{\varphi_h : h \in H\}$ given by $\varphi_h(x) = h_x$, $x \in \mathbb{R}$ is uniformly equicontinuous.
(ii) There exists $h_0 \in S_1$ with $|h(t)| \leq |h_0(t)|$ for all $h \in H$ and for all $t \in \mathbb{R}$

Let $g \in S_1 \cap \mathsf{U}$. Let $\mathcal{U} \subset S_\infty$ be a family of bounded continuous function on $\mathbb{R}$.

Suppose that $g * a(x) \to 0$ as $x \to \infty$ uniformly for a in $\mathcal{U}$ then $h * a(x) \to 0$ as $x \to \infty$ uniformly for h in H and a in $\mathcal{U}$.

Proof: Assume on the contrary. Then there must exists $\delta > 0$ such that for every n there exists $x_n \in \mathbb{R}$ with $x_n > n$, $h_n \in H$ and $a_n \in \mathcal{U}$ satisfying $|h_n * a_n(x_n)| > \delta$.

Now consider, $g * (h_n * a_n) = (g * h_n) * a_n$

Let us consider the sequence

$$s_n(x) = (h_n * a_n)\,(x + x_n),\ n = 1, 2, 3, \ldots,\ -\infty < x < \infty.$$

We shall show that it is bounded and equicontinuous on $\mathbb{R}$.

Now,
$$\begin{aligned}\|s_n\|_\infty &= \|(h_n * a_n)_{x_n}\|_\infty \\ &= \|h_n * a_n\|_\infty \\ &\le \|h_n\|_1\ \|a_n\|_\infty \le \|h_0\|_1 \le 1.\end{aligned}$$

$\therefore$ s_n is bounded on $\mathbb{R}$.

Also for $x, y \in \mathbb{R}$.

$$\begin{aligned}|s_n(x) - s_n(y)| &= |(h_n * a_n)(x + x_n) - (h_n * a_n)(y + x_n)| \\ &= \left| \int_{-\infty}^{\infty} \{h_n(x + x_n - t)\, a_n(t) - h_n(y + x_n - t)\, a_n(t)\} \right| \\ &\le \int_{-\infty}^{\infty} |h_n(x + x_n - t) - h_n(y + x_n - t)|\,dt\end{aligned}$$

Since $x \to h_x$ is uniformly equicontinuous on $\mathbb{R}$ to $L^1(\mathbb{R})$, so given $\in > 0\ \exists \delta' > 0$ such that for $|r_1 - r_2| < \delta'$, $\|h_{r_1} - h_{r_2}\|_1 < \in$ (by taking $r_1 = x + x_n$, $r_2 = y + x_n$). So, $|s_n(x) - s_n(y)| < \in$ for $|y - x| < \delta' \Rightarrow s_n$ is equicontinuous on $\mathbb{R}$

Thus by Ascoli's Lemma [6] we now select from the sequence $s_n(x)$ a subsequence $s_{n_k}(x)$ which tends to a limit $s(x)$ pointwise as $k \to \infty$ and continuous on $\mathbb{R}$.

For each fixed $x \in \mathbb{R}$ and $t \in \mathbb{R}$,

$$s_{n_k}(x - t) \to s(x - t),\ k \to \infty, \text{ and therefore}$$
$$s_{n_k}(x - t)\, g(t) \to s(x - t)\, g(t),\ k \to \infty,$$

Now for each $t \in \mathbb{R}$, $|s_{n_k}(x - t)\, g(t)| \le |g(t)|$

Thus by Lebesgue dominated convergence theorem for each $x \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} s_{n_k}(x-t)\, g(t)\, dt \to \int_{-\infty}^{\infty} s(x-t)\, g(t)\, dt, \ k \to \infty$$
$$= s * g(x)$$

Now,

$$\begin{aligned}\int_{-\infty}^{\infty} s_{n_k}(x-t)\, g(t)\, dt &= \int_{-\infty}^{\infty} \left(h_{n_k} * a_{n_k}\right)\left(x_{n_k}+x-t\right) g(t)\, dt \\ &= \left(h_{n_k} * a_{n_k} * g\right)\left(x_{n_k}+x\right) \\ &= \left(\left(g * a_{n_k}\right) * h_{n_k}\right)\left(x_{n_k}+x\right) \\ &= \int_{-\infty}^{\infty} \left(g * a_{n_k}\right)\left(x_{n_k}+x-t\right) h_{n_k}(t)\, dt\end{aligned}$$

Put $$J_{k,x}(t) = \left(g * a_{n_k}\right)\left(x_{n_k}+x-t\right) h_{n_k}(t)$$

Since we know, $(g * a)(z) \to 0$ as $z \to \infty$ uniformly for a in $\mathcal{U}$, we have for a given $\in > 0 \ \exists \ \Delta : |g * a(z)| < \in \ \forall z \geq \Delta$, and a in $\mathcal{U}$. Therefore

$$\left|\left(g * a_{n_k}\right)\left(x_{n_k}+x-t\right)\right| < \in \text{ for } x_{n_k}+x-t \geq \Delta.$$

Thus for fixed x and t, $g * a_{n_k}\left(x_{n_k}+x-t\right) \to 0$ as $k \to \infty$ and therefore

$$J_{k,x}(t) \to 0 \text{ as } k \to \infty.$$

Now for each k, $\left\|g * a_{n_k}\right\|_\infty \leq \|g\|_1 \left\|a_{n_k}\right\|_\infty \leq 1$ and therefore

$$\begin{aligned}|J_{k,x}(t)| &= \left|g * a_{n_k}\left(x_{n_k}+x-t\right) h_{n_k}(t)\right| \\ &\leq \left|g * a_{n_k}\left(x_{n_k}+x-t\right)\right|\left|h_{n_k}(t)\right| \\ &\leq \left|h_{n_k}(t)\right| \leq |h_0(t)|\end{aligned}$$

Thus by applying Lebesgue dominated convergence theorem for each $x \in \mathbb{R}$, $\int_{-\infty}^{\infty} J_{k,x}(t)\, dt \to 0$ as $k \to \infty$ and therefore $s * g(x) = 0$. Since $g \in \mathrm{U}$, we get $s = 0$.

But $$|s(0)| = \lim_{k\to\infty} \left|s_{n_k}(0)\right| = \lim_{k\to\infty} \left(h_{n_k} * a_{n_k}\right)\left(x_{n_k}\right) \geq \delta > 0$$

which is a contradiction. Therefore

$\int_{-\infty}^{\infty} h(x-t)\, a(t)\, dt \to 0$ as $x \to \infty$ uniformly for h in H & a in $\mathcal{U}$.

This completes the proof.

Corollary 2.4. Let $g \in S_1 \cap U$ be fixed.

(i) $\mathcal{U}_1$ be a family of bounded continuous functions such that $g * a(x) \to A_a \int_{-\infty}^{\infty} g(t)dt$ as $x \to \infty$ uniformly for a in $\mathcal{U}_1$ and $M = \sup\,(\|a\|_\infty + |A_a|) < \infty$.

(ii) For $H_1 \subset L^1(\mathbb{R})$ suppose there exists $h_0 \in L^1(\mathbb{R})$ s.t. $|h_1(t)| \leq |h_0(t)|$ for all $h_1 \in H_1$ and $t \in \mathbb{R}$ and rest is as above in the theorem, then $h * a(x) \to A_a \int_{-\infty}^{\infty} h(t)dt$ as $x \to \infty$ uniformly for h in H and a in $\mathcal{U}_1$.

Proof : Let $H \subset L^1(\mathbb{R})$ be such that

$$H = (\|h_0\|_1 + 1)^{-1} \; H_1 = \{(\|h_0\|_1 + 1)^{-1} \, h_1 : h_1 \in H_1\}$$

Let $h \in H$ be arbitrary then $h = (\|h_0\|_1 + 1)^{-1} \, h_1$

Since $|h_1(t)| \leq |h_0(t)|$ for all $t \in \mathbb{R}$

$$\Rightarrow (\|h_0\|_1 + 1)\; h(t) \leq |h_0(t)| \text{ for all } t \in \mathbb{R}$$

$$\Rightarrow |h(t)| \leq \left| \frac{h_0(t)}{\|h_0\|_1 + 1} \right| \text{ for all } t \in \mathbb{R}$$

$$= |h_0'(t)|, \text{ where } h_0' = \frac{h_0}{\|h_0\|_1 + 1}$$

Therefore, $$\|h_0'\|_1 = \left\| \frac{h_0}{\|h_0\|_1 + 1} \right\| = \frac{\|h_0\|}{1 + \|h_0\|_1} < 1$$

Thus $h_0' \in S_1$.

Taking $$\mathcal{U} = \frac{1}{M+1}\, \{a - A_a I : a \in \mathcal{U}_1\}$$

Now for every $f \in \mathcal{U}$,

$$f = \frac{a - A_a I}{M+1} \text{ for some } a \in \mathcal{U}_1$$

$$f(x) = \frac{a(x) - A_a}{M+1}$$

$$|f(x)| \leq \frac{1}{M+1}\,(|a(x)| + |A_a|) \leq \frac{M}{M+1} < 1.$$

Therefore, $\|f\|_\infty \leq 1$ implies $f \in S_\infty$. Hence $\mathcal{U} \subset S_\infty$.

Now for $a \in \mathcal{U}_1$,

$$g * \frac{a - A_a I}{M+1} = \frac{1}{M+1}\left[g * a - A_a \int_{-\infty}^{\infty} g(u)\,du \right] \text{ for every } \frac{a - A_a I}{M+1} \in \mathcal{U}$$

But we know that $g * a(z) \to A_a \int_{-\infty}^{\infty} g(u)du$ as $z \to \infty$ uniformly for $a \in \mathcal{U}_1$, so $\left(g * \frac{a - A_a I}{M+1}\right)(z) \to 0$ as $z \to \infty$ uniformly for $a \in \mathcal{U}_1$

That is $(g * f)(z) \to 0$ as $z \to \infty$ uniformly for $a \in \mathcal{U}_1$

So by the above theorem applied to H & $\mathcal{U}$,

$(h * f)(z) \to 0$ as $z \to \infty$ uniformly for f in $\mathcal{U}$ and h in H ..... (1)

Now for any $h_1 \in H_1$ and $a \in \mathcal{U}_1$, we have

$$\frac{h_1}{\|h_0\|_1 + 1} \in H \text{ and } \frac{a - A_a I}{M+1} \in \mathcal{U}$$

Thus from equation (1) we have

$$\left(\frac{h_1}{\|h_0\|_1 + 1} * \frac{a - A_a I}{M+1}\right)(z) \to 0 \text{ as } z \to \infty \text{ uniformly for } a \in \mathcal{U}_1 \text{ and } h_1 \in H_1.$$

i.e., $(h_1 * (a - A_a I))(z) \to 0$ as $z \to \infty$ uniformly for a in $\mathcal{U}_1$ and h_1 in H_1.

i.e., $(h_1 * a)(z) \to A_a \int_{-\infty}^{\infty} h_1(u)du,$ as $z \to \infty$ uniformly for a in $\mathcal{U}_1$ and h_1 in H_1.

This completes the proof.

If we take $\mathbb{R} = G$, a locally compact abelian group with Haar measure μ, the following result can be proved.

Theorem 2.5. *$\mathcal{U} \subset S_\infty$ such that $\{\phi_a : a \in \mathcal{U}\}$ given by $\phi_a(x) = a_x$, $x \in G$ is uniformly equicontinuous from G to $L^\infty(G)$.*

$H \subset L^1(G)$, there exists $h_0 \in S_1$, $|h(t)| \le |h_0(t)|$ for all $h \in H$ and $t \in G$. Let $g \in S_1 \cap \mathrm{U}$. Suppose that if $g * a(x) \to 0$ as $x \to \infty$ in G uniformly for a in $\mathcal{U}$, then $h * a(x) \to 0$ as $x \to \infty$ in G uniformly for a in $\mathcal{U}$ and h in H.

Proof: The proof of this theorem follows from theorem 2.3.

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Normality of the Hypersurface of Almost Hyperbolic Hermite Manifolds

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Hyperbolic Hermite manifold have been studied by Dube [2]. Hypersurfaces of almost hyperbolic Hermite manifold have been studied by Pal and Mishra [7] and Dube and Mishra [3] Bhatt [1] Dube [4]. The propose of the present paper is to study the normality of the hypersurfaces of almost hyperbolic Hermite manifolds.

1. Introduction :

An even dimensional differentiable manifold V_{2m}, on which there are defined tensor field F of type (1,1) and a metric tensor g, satisfying for arbitrary vector field $\lambda,\mu,\nu\ \ldots \in V_{2m}$

(1.1)a $$F^2 = I_{2m},$$

(1.1)b $$'F(\lambda,\mu) = g(\lambda,\mu) = -'F(\mu,\lambda)$$

is called an almost hyperbolic Hermite manifold with the almost hyperbolic Hermite structure $\{F,g\}$ [2]. Let V_{2m-1} be the hypersurface V_{2m} with the immersion map b and the corresponding Jacobian map B. Let h be the induced metric tensor and E be the induced Riemmian connection on V_{2m-1}, then we can write the arbitrary vector fields

$$X, Y, Z, \ldots \in V_{2m-1}.$$

(1.2)a $$D_{BX}BY = BE_X Y + 'H(X,Y)N,$$

(1.2)b $$D_{BX}N = -BHX$$

(1.2)c $$g(BX,BY)ob = h(X,Y),$$

where N is unit normal vector to $V_{2m-1}, 'H$ is second fundamental tensor of V_{2m-1} and H is associate to $'H$ and D be the Riemannian Connexion on V_{2m},

(1.3) $$'H(X,Y) = h(HX,Y) = 'H(Y,X).$$

Let us write

(1.4)a $$FBX = BfX + U(X)N$$

(1.4)b $$FN = -B\,\mathrm{v}$$

where f is a tensor of type (1,1) V is a vector field and U is a 1-form, then we have

(1.5)
(a) $f^2 - I_{2m-1} + U(X)V$
(b) $fV = 0$
(c) $U\,o\,f = -1$
(d) $U(V) = -1$
(e) $'f(X,Y) = h(fX,Y) = -'f(Y,X)$,

An $(2m-1)$ dimensional differentiable manifold V_{2m-1} on which there are defined a tensor field f of type (1,1),.a vector field V, a 1-form U and a metric tensor field h, satisfying for arbitrary vector fields $X, Y, Z, \ldots \in V_{2m-1}$.

(1.5) a,c and (1.5)b,c is called an almost hyperbolic contact metric manifold with almost hyperbolic contact metric structure $\{f, u, \mathrm{v}, h\}$ [3]. Thus from above we see that the hypersurface of an almost hyperbolic Hermite manifold is an almost hyperbolic contact metric manifold [3].

The author have proved the following :

The hyper-surface of a hyperbolic Kahler manifold is an almost hyperbolic contact metric manifold satisfying

(1.6)a $$(E_X f)Y = U(Y)HX - 'F(X,Y)V,$$

where

(1.6)b $$(E_X V) = fHX$$

(1.6)c $$(E_U f)Y = U(Y)HV - U(H,Y)V.$$

The equation of the hypersurface of a nearly hyperbolic Kaehler manifold is given by [1].

(1.7)a $$(E_X f)Y + (E_Y f)\mathrm{x} = U(X)HY + U(Y)HX - 2'H(X,Y)V.$$

On this hypersurface we have

(1.7)b $$(E_V\, 'f)(Y,Z)U(Y)U(HZ) + U(Z)V(HY) = (E_Y U)(fZ) + 'H(Yf^2Z) = (E_Z U)(fY)$$

(1.7)c $$(E_X U)(Y) + (E_Y U)(X) = -'H(X,fY) - 'H(fX,Y)$$

The equation of the hypersurface of hyperbolic Hermite manifold is given by [1].

(1.8)a $$(E_{fX} f)fY + (E_X f)(Y) - U(Y)E_{fX}V + U(Y)(f^2Hf^2 + fHf)X + '\{H(fX,fY) - 'H(f2X, f2Y)\}V = 0.$$

Thus we have

(1.8)b $$(E_{fX} U)(fY) + (E^2_{fX} U)(Y) + 'H(fX,fY) - 'H(f^2X, f^2Y) = 0$$

The equation of hypersurface of a Quasi hyperbolic Kaehler manifold is given by [1]

(1.9) $$(E_{fX} f)fY + (E^2_{fX} f)Y - U(Y)E_{fX}\, V + (fHf - f^2Hf^2)X + \{'H(fX,fY) + 'H(f2X,f2Y)\} = 0.$$

On this hypersurface we have

(1.10) $$(E_{fX} U)fY - (E^2_{fX} U)Y + 'H((f^2X, fY) - 'H(fX, f^2Y) = 0$$

Definition : *An almost hyperbolic contact metric manifold is said to be normal if*

(1.10)a $$(E_{fX} f)fY - (E_X f)Y - U(Y)\,(E_{fX} V) = 0.$$

On this manifold we have

(1.10)b $$(E_{fX} U)\,(fY) = (E_X U)Y \Leftrightarrow (E_{fX} U)Y + (E_X U)FY = 0.$$

and $E_V f = 0$,

2. Some Results

Theorem 2.1. *If the hypersurface of hyperbolic kaehler manifold is normal, then we have*

$$fH = Hf \quad \text{and} \quad 'H(fX,Y) = -'H(X,fY)$$

Proof: From equation (1.10)b and (1.6)c, we have

(2.1)a $$U(Y)HV = U(HY)V \Rightarrow U(HY) = U(Y)U(HV)$$

(2.1)b $$HV = U(HV)Y$$

Substituting from equation (1.6)a,b in equation (1.10)a, we obtain

$$U(Y)\,(fHf + H)X + \{\,'H(fX,fY) - 'H(X,Y)\}\,V = 0.$$

Now putting V for Y in the above equation, we get

$$FHf + Hf = U(HX)V,$$

which in a consequence of equation (2.1) yields

(2.2)a
$$fHf + H - U(HV)\,U \otimes V = 0,$$

(2.2)b
$$'H(fX,fY) = 'H(X,Y) - U(X)U(Y)U(HV).$$

Equation (2.2)a and (2.2)b are equivalent to

(2.2)c
$$fH = Hf$$

(2.2)d
$$'H(fX,Y) = -'H(X,fY)$$

This completes the proof of the theorem.

Theorem 2.2. *If the hypersurface of nearly Hyperbolic Kaehler manifold is normal, then we have*

$$U(hfX) = 0$$

Proof: Replacing X and Y by fX and fY in equation (1.7)c we get

(2.3)
$$(E_{fX}\,U)(fY) + (E_{fY}\,U)fX = -\,'H(fX, f^2Y) - 'H(f^2X, fY).$$

Using equation (1.10)c in equation (1.7)c and equation (2.3) we get

(2.4)a
$$'H(fX,Y) = 0.$$

Putting V for X in the above equation, we obtain

(2.4)b
$$U(HfX) = 0.$$

Which gives the desired result.

Theorem 2.3. *If the hypersurface of nearly hyperbolic kaehler manifold is normal, it is the hypersurface of hyperbolic kaehler manifold.*

Proof: The equation (1.7)a of the hypersurface of a Nearly hyperbolic Kaehler manifold is equivalent to

(2.5)
$$(E_X\,'f)\,(Y,Z) - U(Y)\,'H(X,Z) + U(Z)\,'H(X,Y) = (E_Z\,'f)\,(X,Y) - U(X)\,'H(Y,Z) + U(Y)\,'H(X,Z).$$

Substituting equation (2.5) in equation (1.10)a, we get

(2.6) $$-U(Z)\,'H(fX,fY) + (E_Z{}'f)(fX,fY) - (E_X{}'f)(Y,Z) - U(Y)(E_{fX}U)(Z) = 0.$$

On an almost hyperbolic Hermite manifold, we have

(2.7)a $$(E_Z{}'f)(fX,fY) + (E_X{}'f)(X,Y) = U(X)(E_Z U)(fY) - U(Y)(E_Z U)f(X),$$

Using equation (2.5) in equations (2.6) and (2.7)a, we get

$$2(E_X{}'f)(Y,Z) - U(X)\{(E_Z U)(fY) - {}'H(Y,Z)\} = -U(Y)(E_Z U)(fX) + (E_{fX}U)(Z) - 2\,'H(X,Z) - U(Z)\{'H(fX,fY) + {}'H(X,Y)\}.$$

In view of equation (1.7)c, and equation (2.2)a, the above equation takes the form

(2.7)b $$2(E_X{}'f)(Y,Z) - U(X)\{(E_Z U)(fY) - {}'H(Y,Z)\} - 2U(Y)\,'H(X,Z) - 2U(Z)\,'H(X,Y)\}.$$

On hypersurfaces of nearly hyperbolic Kaehler manifold

(2.7)c $$2(E_V{}'f)(Y,Z) - U(Y)(HZ) + U(Z)\,U(HY) = (E_Z V)\,fY + {}'H(Y,Z),$$

Now when the hypersurfaces is normal, we have $E_V f = 0$.

Thus from equation (2.2)a and (2.7)c, we have

$$U(HX) = U(X)\,U(HV);$$

Hence equation (2.7)b becomes

(2.7)d $$(E_X{}'f)(Y,Z) = U(Y)\,'H(X,Z) - U(Z)\,'H(X,Y)$$

which is the equation of hypersurfaces of hyperbolic Kaehler manifold.

This proves our assertions.

Theorem 2.4. *If the hypersurfaces of hyperbolic Hermite manifold is normal, then we have*

$$'H(fX,Y) - {}'H(X,fY) = 0$$

Proof : Substituting from equation (1.10)a,b in (1.8)a, we get

(2.8) $$U(Y)(fHf + f^2Hf^2)X + \{'H(fX,fY) - {}'H(f^2X,f^2Y)\} = 0.$$

The above equation holds, if

(2.8)b $$fHf + f^2Hf^2 = 0 \Rightarrow {}'H(fX,fY) = {}'H(fX,fY)\,'H(f^2X,f^2Y),$$

which gives the required result

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Matrix Elements for the One–Dimensional Harmonic Oscillator and Morse's Radial Wave Function

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Abstract: We determine in a simple form the matrix elements for the one-dimensional harmonic oscillator and the radial wave functions of the Morse potential. The approach is performed using the known expression of the associated Laguerre polynomials in terms of an integral over the Hermite polynomials.

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PACS No. 03.65. Ge: Solutions of wave equations: bound states.

1. Introduction

We will use the notation and quantities of refs. [1,2,3]. In the literature [3,4] there is a direct integral relation between the associated Laguerre L_b^a and Hermite H_c Polynomials, given by $(m \geq n)$:

$$(1) \qquad L_n^{m-n}(-z) = \frac{(-1)^{m-n}}{(2z)^{\frac{m-n}{2}}\, 2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(x+\sqrt{\frac{z}{2}})^2} H_m(x)\, H_n(x) dx$$

Now we show that this integral representation is useful in quantum mechanics to calculate in a very simple way the matrix elements

$$(2) \qquad f(\beta) \equiv \left\langle m \left| e^{-\beta x} \right| n \right\rangle = \int_{-\infty}^{\infty} \psi_m^*(x)\, e^{-\beta x} \psi_n(x)\, dx$$

for the one-dimensional harmonic oscillator (HO), $\beta \geq 0$ is an arbitrary parameter. This is accomplished in section 2, and in section 3 we show that $f(\beta)$ allows to resolve the Schrödinger radial equation for the Morse potential.

2. Determination of $f(\beta)$.

In [2,5–8] are reported special methods to evaluate integrals over HO quantum states ; this type of calculations are improved with the use of eq. (1), thus avoiding the requirement of special techniques.

We employ natural units such that $\hbar = m = w = 1$. The normalized wave functions of the HO are [7,9]:

$$\psi_n(x) = \left(2^n \sqrt{\pi}\, n!\right)^{-1/2} e^{-x^2/2} H_n(x), \quad n = 0, 1, 2, \ldots \tag{3}$$

We select $m \geq n$ without loss of generality, following the symmetry of eq. (2) in indices m and n. The substitution of (3) into (2) with the use of (1) implies immediately that:

$$f(\beta) = \sqrt{\frac{n!}{m!}} \left(-\frac{\beta}{\sqrt{2}}\right)^{m-n} e^{\beta^2/4}\, L_n^{m-n} \left(-\frac{\beta^2}{2}\right) \tag{4}$$

which is an expression in accordance with the ones reported [2,5,6].
In ref. [2] it is shown that with eq. (4) it is a simple matter to obtain the matrix elements for the x^k, $k = 0, 1, 2, \ldots$, for $x^{-\gamma x^2}$, etc.
We now need to mention that the 2th order differential equation defining to L_q^p (see [1] page 781) can be used to prove that $f(\beta)$, as given in eq. (4), satisfies the differential equation :

$$\frac{d^2 f}{d\beta^2} + \frac{1}{\beta}\frac{df}{d\beta} - \frac{1}{4\beta^2}(\beta^4 + 4A\beta^2 + 4Q) f = 0 \tag{5}$$

where $A = m + n + 1$ and $Q = (m-n)^2$; that is, (4) is a solution of (5). It is also possible to deduce (5) using the hypervirial theorem [2, 7, 10 –12] and parametric differentiation.

3. Morse's Radial Wave Function.

Morse [3, 7, 13–20] proposed the potential :

$$V(r) = D\left[\, e^{-2a(r-r_0)} - 2e^{-a(r-r_0)} \,\right] \tag{6}$$

as an approximation to the vibrational motion of a diatomic molecule, where D is the dissociation energy (well depth), r_0 is the nuclear equilibrium separation and a is a parameter associated to the well width. The combination $a\sqrt{2D}/2\pi$ gives the frequency of the small classical vibrations around r_0. If we make the change of variable $u = r - r_0$ and use natural units, the corresponding Schrödinger equation is

$$\frac{d^2}{du^2}\psi_M + 2\left[E - D\left(e^{-2au} - 2e^{-au}\right)\right]\psi_M = 0 \tag{7}$$

where $\frac{1}{r}\underset{M}{\psi}$ is the Morse radial wave function. If now we introduce a new independent variable β at (7), given by :

$$\beta = i\sqrt{2K}e^{-\frac{au}{2}}, \quad i=\sqrt{-1}, \quad K=\frac{2}{a}\sqrt{2D} \tag{8}$$

and note that the constant $K>0$ is not necessarily an integer, then we see that (7) acquires the form

$$\frac{d^2}{\beta^2}\underset{M}{\psi}+\frac{1}{\beta}\frac{d}{d\beta}\underset{M}{\psi}-\frac{1}{4\beta^2}\left(\beta^4+4K\beta^2-\frac{32E}{a^2}\right)\underset{M}{\psi}=0 \tag{9}$$

with the same structure that (5)!

Therefore, by formal comparison of (5) with (9) we have :

$$K=m+n+1, \quad E_n=-\frac{a^2}{8}(m-n)^2=-\frac{a^2}{8}(k-2n-1)^2 \tag{10}$$

which implies that $m=n$ is not a possible choice, because it is forbidden the energy value $E=0$ for the bounded states of the Morse's potential. From eq. (10) it results the condition $K>1$ for the existence of a discrete spectrum energy [13]. Besides, the conditions $E_n \neq 0$ and $K>1$ give from (10) the inequality $(k-2n-1)>0$, that is :

$$0 \le 2n < (k-1) \tag{11}$$

which means a finite number of bounded states [16].

From (5) and (9) it is clear that $\underset{M}{\psi}$ is proportional to the $f(\beta)$ given by (4), therefore :

$$\underset{M}{\psi_n}(r)=\left[\frac{abn!}{\Gamma(k-n)}\right]^{\frac{1}{2}} q^{b/2}\, e^{-q/2}\, L_n^b(q) \tag{12}$$

where $\underset{M}{\psi}$ is normalized to unity.

$$q=Ke^{-a(r-r_0)}, \quad b=m-n=k-2n-1$$

Thus we see that the Schrödinger equation has been easily resolved for the vibrational Morse oscillator thanks to the matrix elements $\langle m|e^{-\beta x}|n\rangle$ for the one dimensional HO. The exhibited scheme is another example of the multiple correspondences between the Morse and harmonic oscillators [3,18,21].

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Semi Symmetric Non-Metric Connection on a Manifold with Generalised HSU-Structure

RAM NIVAS AND GEETA VERMA

Abstract : Semi symmetric metric connection have been studied by various mathematician including Yano [2], Mishra [3], Imai [4], S. I. Hussain [5]. M. D. Upadhaya and Jaya Pant [7], etc., manifold. Recently Nirmala S. Agashe and others [1] have defined the notion of semi symmetric non-metric connection on a Riemannian manifold. Singh and Nivas [6] studied semi-symmetric non-metric connection on almost para contact metric manifold. In this paper we study semi-symmetric non-metric connection on manifold with generalised Hsu-structure. In the first section, I have studied Nijenhuis tensor and integrability conditions of such manifolds. Some interesting results have been established in other sections of the paper

1. Preliminaries

Let an n-dimensional differentiable manifold M^n of class C^∞ admits a C^∞ tensor field F of the type (1,1) a C^∞ a vector field T and C^∞ 1-form A such that

$$
(1.1)\quad
\begin{aligned}
&\text{(i)} && \overline{\overline{X}} = a^r X + A(X)T,\\
&\text{(ii)} && \overline{X} = F(X),\\
&\text{(iii)} && A(T) = -a^r,\\
&\text{(iv)} && A(FX) = 0,\\
&\text{(v)} && FT = 0,\\
&\text{(vi)} && g(T,X) = A(X),
\end{aligned}
$$

and

$$\text{(vii)}\quad g(\overline{X},\overline{Y}) = -a^r\, g(X,Y) - A(X)\,A(Y)$$

where g is a non-singular metric tensor and 'a' is any non-zero complex number. Let us call such a structure a generalised almost contact metric structure

It follows from (1.1). (vii)

$$g(\overline{\overline{X}},\overline{\overline{Y}}) = -a^r\, g(\overline{X},\overline{Y})$$

Let us define

(1.2) $$F(X,Y) = g(FX,Y)$$

barring X in (1.2) we have

(1.3) $$F(\overline{X},Y) = g(F^2X,Y)$$

which by virtue of equation (1,1) (vii) yields

(1.4) $$F(\overline{X},Y) = a^r g(X,Y) + A(X)\,A(Y)$$

Now barring Y in (1.3) we have

$$F(X,\overline{Y}) = -\,a^r g(X,Y) - A(X)\,A(Y)$$

or

(1.5) $$F(X,\overline{Y}) = -\,(a^r g(X,Y) + A(X)\,A(Y))$$

Thus from (1.4) and (1.5) we have

$$F(X,\overline{Y}) + F(\overline{X},Y) = 0$$

Replacing X by T in equation (1.2) and making use of equation (1.1) (v) we obtain

(1.6) $$F(T,Y) = 0$$

A linear connection ∇ is said to be semi-symmetric connection if its torsion tensor

$$S(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

satisfied the formula

(1.7) $$S(X,Y) = A(Y)X - A(X)Y$$

∇ is said to be semi-symmetric non-metric with respect to the associated metric g if

(1.8) $$\nabla_X\, g(Y,Z) = -\,A(Y)\,g(X,Z) - A(Z)\,g(X,Y)$$

We define ∇ to be semi-symmetric non-metric F-connection if in addition (1.7), (1.8) ∇ satisfies

(1.9) $$(\nabla_X\, F) = 0$$

Suppose ∇ is a Riemannian connection on M^n then we can always put [1]

(1.10) $$\overset{*}{\nabla}_X Y = \nabla_X Y + U(X,Y)$$

U being tensor of type (1.2) satisfying

(1.11) $$g(U(X,Y)Z) + g(U(X,Z)Y) = A\,(Y)\,g(X,Z) + A(Z)\,g(X,Y)$$

obviously we have

(1.12) $$S(X,Y) = U(X,Y) - U(Y,X)$$

Nirmala S. Agashe and others expressed the value of $U(X,Y)$ in terms of S and S'

(1.13) $$U(X,Y) = \tfrac{1}{2}\,[S(X,Y) + S'(X,Y) + S'(Y,X)] + g(X,Y)T$$

where

(1.14) $$g(S(Z,X),Y) \underset{=}{\text{def}}\ g(S'(X,Y),Z)$$

It can be verified that

$$S'(X,Y) = A(X)Y - g(X,Y)T$$

and

(1.15) $$U(X,Y) = A(Y)X.$$

Thus we get

(1.16) $$\nabla_X Y = D_X Y + A(Y)X$$

It is easy to verify that

(1.17)

$$\begin{aligned}
&\text{(i)} && S'(Y,X) = U(X,Y) - g(X,Y)T,\\
&\text{(ii)} && g(S(X,Y),T) = 0,\\
&\text{(iii)} && S(X,T) = -\overline{\overline{X}},\\
&\text{(iv)} && S'(T,X) = U(X,T) + A(X)T,
\end{aligned}$$

and

$$\text{(v)} \quad S'(X,Y) - S'(Y,X) = S(X,Y)$$

Theorem 1.1. *In a generalised Hsu-structure manifold M^n the torsion tensor of the semi-symmetric non-metric connection satisfies the following identities.*

(1.18)

$$\begin{aligned}
&\text{(i)} && S(X,T) = -\overline{\overline{X}},\\
&\text{(ii)} && S(\overline{X},T) = -a^r\,\overline{X},\\
&\text{(iii)} && S(\overline{\overline{X}},Y) = a^r A(Y)X + A(X)A(Y)T,\\
&\text{(iv)} && S(\overline{\overline{X}},Y) + S(X,\overline{\overline{Y}}) = a^r S(X,Y),\\
&\text{(v)} && S(\overline{\overline{X}},T) = a^r S(\overline{X},T) = a^{2r}\,\overline{X},\\
&\text{(vi)} && A(S(X,Y)) = 0,\\
&\text{(vii)} && \overline{\overline{S(X,Y)}} = a^r S(X,Y).
\end{aligned}$$

Now we will establish certain identities among the (0,3) type tensor defined as follows

(1.19)

$$\begin{aligned}
&\text{(i)} && S'(X,Y,Z) \underset{=}{\text{def}}\ g(S(X,Y),Z)\\
&\text{(ii)} && U'(X,Y,Z) \underset{=}{\text{def}}\ g(U(X,Y),Z)
\end{aligned}$$

or equivalently

$$S'(X,Y,Z) = \begin{vmatrix} g(Y,T) & g(X,T) \\ g(Y,Z) & g(X,Z) \end{vmatrix}$$

and

$$U'(X,Y,Z) = \begin{vmatrix} g(Y,T) & g(Z,T) \\ g(X,Y) & g(X,Z) \end{vmatrix}$$

Theorem 1.2. *The following relations hold in a generalised Hsu-structure non-metric manifolds*

$$\begin{aligned}
&\text{(i)} && S(X,Y,\overline{\overline{Z}}) = a^r S'(X,Y,Z)\\
&\text{(ii)} && U'(\overline{\overline{X}},Y,Z) = -a^r U'(X,Y,Z) = 0\\
(1.20)\quad &\text{(iii)} && U'(\overline{\overline{X}},\overline{Y},\overline{Z}) = S'(\overline{X},\overline{Y},\overline{\overline{Z}}) = 0\\
&\text{(iv)} && S'(Z,Y,\overline{\overline{X}}) = a^r U'(X,Y,Z) = 0\\
&\text{(v)} && U'(\overline{\overline{Z}},Y,X) = a^r S'(X,Y,Z)\\
&\text{(vi)} && S'(X,Y,\overline{\overline{Z}}) = U'(\overline{\overline{Z}},Y,X)
\end{aligned}$$

2. Nijenhuis Tensor

The Nijenhuir tensor is given by

$$(2.1)\qquad N(X,Y) = [\overline{X},\overline{Y}] + \overline{\overline{[X,Y]}} - \overline{[\overline{X},Y]} - \overline{[X,\overline{Y}]}$$

Making use of (1.1) (i) in (2.1) we get

$$(2.2)\qquad N(X,Y) = [\overline{X},\overline{Y}] + a^r [X,Y] + A([X,Y])T - \overline{[\overline{X},Y]} - \overline{[X,\overline{Y}]}$$

We now put

$$(2.3)\qquad B(X,Y) = [\overline{X},\overline{Y}] - \overline{[X,\overline{Y}]},$$

$$(2.4)\qquad H(X,Y) = [\overline{X},\overline{Y}] - \overline{[\overline{X},Y]},$$

$$(2.5)\qquad W(X,Y) = [\overline{X},\overline{Y}] - a^r [X,Y].$$

Theorem 2.1. *The Nijenhuis tensor and $B(X,Y)$ are related as*

$$(2.6)\qquad a^r B(X,Y) - B(\overline{X,\overline{Y}}) = a^r N(X,Y) - A(Y)\,\overline{[X,T]} + a^r A(Y)\,[X,T] + A(Y)A([X,T])T$$

Proof: Barring Y in (2.3) and making use of (1.1) (i) we get

$$(2.7) \qquad B(X,\overline{Y}) = a^r(\overline{X},Y) + A(Y)[\overline{X},T] - a^r\overline{[\overline{X},T]} + \overline{A(Y)[X,T]}$$

Again barring above equation and making use of (1.1) (i) we get

$$(2.8) \qquad \overline{B[X,\overline{\overline{Y}}]} = a^r\overline{[\overline{\overline{X}},Y]} + A(Y)\overline{[\overline{\overline{X}},T]} - a^{2r}[X,Y]$$
$$- a^r A([X,Y])T - a^r A(Y)[X,T]$$
$$- A(Y)A([X,T])T$$

Now from equation (2.3) and (2.8) we obtain

$$(2.9) \qquad a^r B(X,Y) - B\overline{(X,\overline{\overline{Y}})} = a^r[\overline{X},\overline{Y}] - a^r\overline{(X,\overline{\overline{Y}})}$$
$$- a^r\overline{[\overline{\overline{X}},Y]} - a^{2r}[X,Y] - A(Y)\overline{[\overline{\overline{X}},T]}$$
$$+ a^r A([X,Y])T + a^r A(Y)[X,T]$$
$$+ A(Y)A([X,T])T$$

Making use of (2.2) in (2.9) we get the result putting T for Y in (2.6) and making use of (1.1) (iii) and (1.1) (v) we have in a differentiable manifold.

$$(2.10) \qquad a^r B(X,T) = a^r N[X,T] + a^r\overline{[\overline{\overline{X}},T]}$$
$$-a^{2r}[X,T] - a^r A([X,T])T$$

Theorem 2.2. *In a differentiable manifold M^n we have*

$$(2.11) \qquad a^r H(X,Y) - H\overline{[\overline{\overline{X}},Y]} = a^r N(X,Y) - A(X)\overline{[T,\overline{\overline{Y}}]}$$
$$+ a^r A(X)[T,Y]$$
$$+ A(X)A([T,Y])T$$

Proof : Barring X in (2.4) and making use of (1.1) (i) we get

$$(2.12) \qquad H(\overline{X},\mathrm{Y}) = a^r[X,\overline{Y}] + A(X)[T,\overline{Y}]$$
$$- a^r\overline{[X,Y]} - A(X)\overline{[T,Y]}$$

Now barring the whole equation (2.12) and making use of (1.1). (i) we have

$$(2.13) \qquad \overline{H(\overline{X},Y)} = a^r\overline{[X,\overline{Y}]} - A(X)\overline{[T,\overline{Y}]} - a^{2r}[X,Y]$$
$$- a^r A([X,Y])T - a^r A(X)[T,Y]$$
$$- A(X)A([T,Y])T$$

Now from (2.4) and (2.13) we have

$$\text{(2.14)} \quad a^r\, H(X,Y) - \overline{H(\overline{X},Y)} = a^r [\overline{X},\overline{Y}] - a^r\, \overline{[X,\overline{Y}]}$$
$$- a^r [\overline{\overline{X},Y}] + a^{2r} [X,Y]$$
$$+ a^r A([X,Y])T - A(X)\, \overline{[T,\overline{Y}]}$$
$$+ a^r A(X) [T,Y] + A(X) A[T,Y] T$$

Thus, from (2.2) and (2.14) we obtained the required result.
Replacing X by T in (2.11) and using (1.1) (iii) and (1.1) (v) we get in generalised Hsu-structure almost contact metric manifold.

$$\text{(2.15)} \quad a^r H(T,Y) = a^r N(T,Y) + a^r\, \overline{[T,\overline{Y}]}$$
$$- a^{2r} [T,Y] - a^r A([T,Y])T$$

Theorem 2.3. *In a generalised Hsu-structure almost contact metric structure manifold M^n we have*

$$\text{(2.16)} \quad a^r W(X,Y) - \overline{W(\overline{X},Y)} = a^r N(X,Y) - a^r A([X,Y])T$$
$$- A(X)\, \overline{[T,\overline{Y}]}$$

Proof : Barring X in (2.5) and making use of (1.1) (i) we get

$$\text{(2.17)} \quad W(\overline{X},Y) = a^r (X,\overline{Y}) + A(X)(T,\overline{Y})$$
$$+ a^r [\overline{X},Y]$$

Again barring (2.1) and making use of equation (1.1) (i) we have

$$\text{(2.18)} \quad \overline{W(\overline{X},Y)} = a^r\, \overline{[X,\overline{Y}]} + A(X)\, \overline{[T,\overline{Y}]}$$
$$+ a^r [\overline{\overline{X},\overline{Y}}]$$

Thus with the help of (2.2), (2.5) and (2.18) we get (2.16). Replacing X by T in (2.16) and using the equation (1.1) (iii) and (1.1) (v) we can show that equation (2.16) is equivalent to

$$\text{(2.19)} \quad a^r W(T,Y) = a^r N(T,Y) - a^r A([T,Y])\, T + a^r\, [\overline{T,\overline{Y}}]$$

3. The Curvature Tensor

The curvature tensors of semi-symmetric non-metric connection ∇ and the Riemanian connection D are respectively represented by R & K as follows

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$
$$K(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z$$

Thus we state the following theorem

Theorem 3.1. *The two curvature tensor are related to the following equations*

$$R(X,Y)Z = K(X,Y)Z + A(D_Y\ Z)X - A(D_X Z)Y - A(Z)\,S(X,Y)$$
$$+ X(A(Z))Y - Y(A(Z))X$$

4. Integrability Condition

Theorem 4.1. *In order that a generalised almost contact metric manifold to be completely integrable it is necessary that*

$$A\,([\overline{X}, \overline{Y}])T = 0$$

Proof: Barring X in (2.2) and with the help of equation (1.1) (i) we get

From equation (2.2) and (4.2) we have

$$N(\overline{\overline{X}},Y) + a^r N(X,Y) = A\,(X)\,\overline{[T,\overline{Y}]} - A\,[\overline{X},\overline{Y}]T \tag{4.3}$$
$$-a^r A(X)\,[T,Y] - A(X)\,([T,Y])T$$

Using

$$N\,[T,Y] = a^r\,(T,Y) + A\,([T,Y])T - \overline{[T,\overline{Y}]} \tag{4.4}$$

in (4.3) we have

$$N\,(\overline{\overline{X}},Y) + a^r N\,(X,Y) = -A\,(X)\,N\,(T,Y) \tag{4.5}$$
$$-A\,([\overline{X},\overline{Y}])T$$

For completely integrable manifold equation (4.5) reduces to give result in theorem.

Theorem 4.2. *For a completely integrable generalised almost contact metric structure manifold we have*

$$A(X)\,\{[T,\overline{Y}] - \overline{[\overline{T},Y]}\} + A([\overline{X},Y])T \tag{4.6}$$
$$= A(X)\,\{[\overline{X},T] - \overline{[X,\overline{T}]}\} + A\,([X,\overline{Y}])T$$

Proof: Barring X and Y in equation (2.2) and making use of (1.1) (i) we get the following equations

$$N\,(\overline{X},Y) = a^r\,[X,\overline{Y}] + A\,(X)\,[T,\overline{Y}] \tag{4.7}$$
$$+\, a^r\,[\overline{X},Y] + A\,([\overline{X},Y])\,T - \overline{[\overline{X},\overline{Y}]}$$
$$-\, a^r\,\overline{[X,Y]} - A(X)\,\overline{[T,Y]}$$

$$
\begin{aligned}
(4.8) \qquad N(X,\overline{Y}) = a^r[\overline{X},Y] + A(Y)[\overline{X},T] \\
+ a^r[X,\overline{Y}] + A([X,\overline{Y}])T \\
- a^r\overline{[X,Y]} - [\overline{\overline{X},\overline{Y}}] - A(Y)\overline{[X,T]}
\end{aligned}
$$

Now from these two equations (4.7) and ((4.8) and using $N(X,Y)$ we have then required result (4.6).

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On Complete and Horizontal Lifts from a Manifold with HSU–(4,2) Structure to its Cotangent Bundle

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Abstract : Manifolds with $f(4,2)$-structure have been defined and studied by Yano, Houh and Chen [3] and others. Geometry of tangent and cotangent bundles in a differentiable manifold has been studied by Yano and Ishihara [4]. Hsu-structure had been defined by Prof. Mishra [2]. The purpose of this chapter is to study complete and horizontal lifts from a manifold with Hsu-(4,2) structure to its cotangent bundle.

1. Preliminaries

Let M be a differentiable manifold of class C^∞ and dimension n and let C_{TM} denote the cotangent bundle of M. Then C_{TM} is also a differentiable manifold of class C^∞ and dimension $2n$ [4]. Throughout this chapter, the following notations and conventions will be used :

(i) The map $\pi : C_{TM} \to M$ is the projection map of C_{TM} onto M.

(ii) Suffixes $a, b, c, \ldots i, j, \ldots$ Take the value 1 to n and $\bar{i} = i + n$. Suffixes $A, B, C, \ldots$ take the values 1 to $2n$

(iii) $\mathfrak{J}_s^r(M)$ denotes the set of tensor fields of class C^∞ and of type (r,s) on M. Similarly $\mathfrak{J}_s^r(C_{TM})$ denotes the set of tensor fields of class C^∞ and of type (r, s) in C_{TM}.

(iv) Vectors in M are denoted by $X, Y, Z, \ldots$ and the Lie derivative by $\mathcal{L}_X$. The Lie product of X,Y is denoted by $[X,Y]$.

If A is a point in $M, \pi^{-1}(A)$ is a fibre over A. Any point $p \in \pi^{-1}(A)$ is the ordered pair (A, p_A) where p is 1-form in M and p_A is the value of p at A. Let U be a

coordinate neighbourhood in M such that $A \in U$. Then U induces a coordinate neighbourhood $\pi^{-1}(U)$ in C_{TM} and $p \in \pi^{-1}(U)$.

2. Complete Lift of *HSU*–(4,2) Structure

Let M be an n-dimensional differentiable manifold of class C^∞. Suppose there exist on M a tensor field $f\,(\neq 0)$ of type (1,1) satisfying

$$f^4 - \lambda^r f^2 = 0 \tag{2.1}$$

where λ is complex number not equal to zero and r some finite integer. In such a manifold M, let us put

$$\lambda = f^2/\lambda^r \text{ and } m = I - f^2/\lambda^r \tag{2.2}$$

where I denotes the unit tensor field. Then it is easy to show

$$\lambda^2 = \lambda,\ m^2 = m,\ \lambda + m = I,\ \lambda m = m\lambda = 0. \tag{2.3}$$

Thus the operators λ and m when applied to the tangent space of M at a point are complementary projection operators. Hence there exist complementary distributions L^* and M^* corresponding to the projection operators λ and m respectively. If the rank of 'f' is constant everywhere and equal to r, the dimensions of L^* and M^* are r and $(n-r)$ respectively. Let us call such a structure as Hsu-(4,2) structure of rank r.

Let f_i^h be the component of f at A in the coordinate neighbourhood U of M. Then the complete lift f^C of f is also a tensor field of type (1,1) in C_{TM} whose components $\tilde{f}_B^A$ in $\pi^{-1}(U)$ are given by

$$\tilde{f}_i^h = f_i^h \tag{2.4.1}$$

$$\tilde{f}_{\bar{i}}^h = 0 \tag{2.4.2}$$

$$\tilde{f}_i^{\bar{h}} = p_a\,[\partial f_h^a/\partial x^i - \partial f_i^a/\partial x^h] \tag{2.4.3}$$

and

$$\tilde{f}_{\bar{i}}^{\bar{h}} = f_h^i \tag{2.4.4}$$

where $(x^1, x^2, x^3, \ldots, x^n)$ are coordianates of A in U and p_A has components $(p_1, p_2, p_3, \ldots, p_n)$. Thus we can write

(2.5) $$f^C = (\tilde{f}^A_B) = \begin{bmatrix} f^h_i & 0 \\ p_a(\partial_i f^a_h - \partial_h f^a_i) & f^i_h \end{bmatrix}$$

where $\partial_i = \partial/\partial x^i$.

If we put

(2.6) $$\partial_i f^a_h - \partial_h f^u_i = 2\partial\,[if^a_h].$$

then the equation (2.5) can be written as

(2.7) $$f^C = (\tilde{f}^A_B) = \begin{bmatrix} f^h_i & 0 \\ 2p_a\partial[if^a_h] & f^i_h \end{bmatrix}$$

$$\Rightarrow \qquad (f^C)^2 = \begin{bmatrix} f^h_i & 0 \\ 2p_a\partial[if^a_h] & f^i_h \end{bmatrix}\begin{bmatrix} f^i_j & 0 \\ 2p_t\partial[jf^t_i] & f^j_i \end{bmatrix}$$

$$= \begin{bmatrix} f^h_i f^i_j & 0 \\ 2p_a f^i_j\,\partial[if^a_h] + 2p_t f^i_h\,\partial[jf^t_i] & f^i_h f^j_i \end{bmatrix}$$

If we substitute

(2.8) $$2p_a f^i_j\,\partial[if^a_h] + 2p_t f^i_h\,\partial[jf^t_i] = L_{hj}$$

then we can write

(2.9) $$(f^C)^2 = \begin{bmatrix} f^h_i f^i_j & 0 \\ L_{hj} & f^j_i f^i_h \end{bmatrix}$$

Squaring (2.9) again we get

(2.10) $$(f^C)^4 = \begin{bmatrix} f^h_i f^i_j & 0 \\ L_{hj} & f^j_i f^i_h \end{bmatrix}\begin{bmatrix} f^j_k f^k_\lambda & 0 \\ L_{j\lambda} & f^\lambda_k f^k_j \end{bmatrix}$$

$$= \begin{bmatrix} f^h_i f^i_j f^j_k f^k_\lambda & 0 \\ f^j_k f^k_\lambda L_{hj} + f^j_i f^i_h L_{j\lambda} & f^\lambda_k f^k_j f^j_i f^i_h \end{bmatrix}$$

Putting

(2.11) $$f^j_k f^k_\lambda L_{hj} + f^j_i f^i_h L_{j\lambda} = \lambda^r L_{hj}$$

Then in view of equation (2.11) and (2.1) the equation (2.10) take the form

$$(f^{C})^{4} = \begin{bmatrix} \lambda^{r} f_t^h f_\lambda^t & 0 \\ \lambda^{r} L_{h\lambda} & \lambda^{r} f_t^\lambda f_h^t \end{bmatrix} = \lambda^{r} \begin{bmatrix} f_t^h f_\lambda^t & 0 \\ L_{h\lambda} & f_t^\lambda f_h^t \end{bmatrix}$$

$$\Rightarrow \quad (f^{C})^{4} = \lambda^{r} (f^{C})^{2}$$

$$\Rightarrow \quad (f^{C})^{4} - \lambda^{r} (f^{C})^{2} = 0$$

Thus the complete lift f^{C} of f also has Hsu–(4,2) structure in the cotangent bundle C_{TM}.

Thus we have

Theorem 2.1. *In order that the complete lift f^{C} of a* (1,1) *tensor field f admitting* Hsu-(4,2) *structure in M may have the similar structure in the cotangent bundle C_{TM} it is necessary and sufficient that*

$$f_k^j f_\lambda^k L_{hj} + f_i^j f_h^i L_{j\lambda} = \lambda^{r} L_{h\lambda}.$$

3. Nijenhuis Tensor of Complete Lift of f^4

Nijenhuis tensor of a (1,1) tensor field f on M is given by

$$N_{f,f}(X,Y) = [fX, fY] - f[fX,Y] - f[X,fY] + f^{2}[X,Y] \tag{3.1}$$

Also for the complete lift of f^{4}, the Nijenhuis tensor is given by

$$\begin{aligned} N_{(f^4)^C,(f^4)^C}(X^C,Y^C) = {} & [(f^4)^C X^C, (f^4)^C Y^C] \\ & - (f^4)^C [(f^4)^C X^C, Y^C] \\ & - (f^4)^C [X^C, (f^4)^C Y^C] \\ & + (f^4)^C (f^4)^C [X^C, Y^C], \end{aligned} \tag{3.2}$$

In view of the equation (2.1) the above equation (3.2) takes the form

$$\begin{aligned} N_{(f^4)^C,(f^4)^C}(X^C,Y^C) = {} & [(\lambda^r f^2)^C X^C, (\lambda^r f^2)^C Y^C] \\ & - (\lambda^r f^2)^C [(\lambda^r f^2)^C X^C, Y^C] \\ & - (\lambda^r f^2)^C [X^C, (\lambda^r f^2)^C Y^C] \\ & + (\lambda^r f^2)^C (\lambda^r f^2)^C [X^C, Y^C] \end{aligned}$$

or

$$(3.3)\quad N_{(f^4)^C,(f^4)^C}(X^C,Y^C) = \lambda^{2r}\{[(f^2)^C X^C,(f^2)^C Y^C] - (f^2)^C[(f^2)^C X^C,Y^C] - (f^2)^C[X^C,(f^2)^C Y^C] + (f^2)^C(f^2)^C[X^C,Y^C]\}.$$

We also know that ([4] page 243)

$$(3.4)\quad (f^2)^C X^C = (f^2 X)^C + \nu(\mathcal{L}_X f^2),$$

where νf has components

$$(3.5)\quad \nu f = \begin{pmatrix} 0 \\ p_a f_i^a \end{pmatrix}$$

In view of (3.4), the equation (3.3) takes the form

$$(3.6)\quad N_{(f^4)^C,(f^4)^C}(X^C,Y^C) = \lambda^{2r}\{[(f^2X)^C,(f^2Y)^C] + [\nu(\mathcal{L}_X f^2),\nu(f^2Y)] + [(f^2X)^C,\nu(\mathcal{L}_Y f)^2] + [\nu(\mathcal{L}_X f^2),\nu(\mathcal{L}_Y f^2)] - (f^2)^C[(f^2X)^C,Y^2] - (f^2)^C[\nu(\mathcal{L}_X f^2),Y^2] - (f^2)^C[X^C,(f^2Y)^C] - (f^2)^C[X^2,\nu(\mathcal{L}_Y f^2)] + (f^2)^C(f^2)^C[X^2,Y^2]\}.$$

Let us now suppose that

$$(3.7)\quad \mathcal{L}_X f^2 - \mathcal{L}_Y f^2 = 0$$

then the equation (3.6) takes the form

$$(3.8)\quad N_{(f^4)^C,(f^4)^C}(X^C,Y^C) = \lambda^{2r}\{[(f^2X)^C,(f^2Y)^C] - (f^2)^C[(f^2X)^C,Y^2] - (f^2)^C[X^C,(f^2Y)^C] + (f^2)^C(f^2)^C[X^C,Y^C]\}.$$

Let us now suppose that f acts as Hsu-structure on M[1]. Then

$$(3.9)\quad f^2 = \lambda^r I.$$

Thus the equation (3.8) becomes.

$$N_{(f^4)^C,(f^4)^C}(X^C,Y^C) = \lambda^{4r}\{[X^C,Y^C] - [X^C,Y^C] - [X^C,Y^C] + [X^C,Y^C]\} = 0.$$

Hence we have.

Theorem 3.1. *The Nijenhuis tensor of the complete lift of f^4 vanishes if the Lie derivative of the tensor field f^2 with respect to X and Y are both zero and f acts as Hsu-structure operator on M.*

4. Horizontal Lift of *HSU*-(4,2) Structure

Let f and g be two tensor fields of type (1,1) on the manifold M. If f^{H} denotes the horizontal lift of f, we have [5].

(4.1) $$f^{\mathrm{H}}g^{\mathrm{H}}+g^{\mathrm{H}}f^{\mathrm{H}}=(fg+gf)^{\mathrm{H}}$$

Taking f and g identical, we get

(4.2) $$(f^{\mathrm{H}})^2=(f^2)^{\mathrm{H}}.$$

Squaring the above equation both sides and making use of the equation (4.1) we get

(4.3) $$(f^{\mathrm{H}})^4=(f^4)^{\mathrm{H}}$$

Since f gives Hsu-(4,2) structure on M, we have

$$f^4-\lambda^r f^2=0.$$

Taking horizontal lift in the above equation we get

(4.4) $$(f^4)^{\mathrm{H}}-\lambda^r(f^2)^{\mathrm{H}}=0.$$

In view of the equation (4.2) and (4.3) the above equation (4.4) takes the form

$$(f^{\mathrm{H}})^4-\lambda^r(f^{\mathrm{H}})^2=0.$$

Thus the horizontal lift f^{H} of f also admits Hsu-(4,2) structure in the cotangent bundle C_{TM}. Hence we have

Theorem 4.1. *Let f be a tensor field of type* (1,1) *satisfying Hsu-*(4,2) *structure on the manifold M. Then the horizontal lift f^{H} of f also admits the same structure in the cotangent bundle C_{TM}.*

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Applications of the Class *SDCP* with Plane Harmonic Mappings

CHINTA MANI POKHREL

Abstract : In [2] we introduced and fully characterized the class *SDCP* (Strongly Direction Convexity Preserving). In this paper we shall discuss its applications to harmonic mappings in plane.

1. Introduction to the class *SDCP*

Let A denote the set of analytic functions in $\mathbb{D}$. Let

$$f(z)=\sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z)=\sum_{n=0}^{\infty} b_n z^n$$

be two members of A. Then the Hadamard product or convolution between f and g, denoted by $f * g$, is defined by

$$(f * g)(z)=\sum_{n=0}^{\infty} a_n b_n z^n .$$

A domain $D \subset \mathbb{C}$ is said to be *convex in the direction* $e^{i\varphi}$, $\varphi \in \mathbb{R}$, if and only if for every $a \in \mathbb{C}$ the set

$$D \cap \{\, a+te^{i\varphi} : t \in \mathbb{R}\,\}$$

is either connected or empty. Accordingly we define the classes $K(\varphi) \subset A$, $\varphi \in \mathbb{R}$, of the functions *convex in the direction* $e^{i\varphi}$ as

$$K(\varphi):=\{\, f \in A : f \text{ univalent and } f(\mathbb{D}) \text{ convex in the direction } e^{i\varphi}\,\}.$$

Finally, a function $g \in A$ is called *Direction-Convexity-Preserving* ($g \in DCP$) if and only if

$$g * f \in K(\varphi) \text{ for all } f \in K(\varphi) \text{ and all } \varphi \in \mathbb{R}$$

Functions in *DCP* have many intriguing convolution-type properties, for instance the preservation of convex harmonic functions in $\mathbb{D}$, and of Jordan curves in the plane with convex interior domain we refer to [13],[14] for more details. There one also finds a complete description of the members of *DCP*, namely

$$g \in DCP \Leftrightarrow g(z) + itzg'(z) \in k\left(\tfrac{\pi}{2}\right) \text{ for all } t \in \mathbb{R}.$$

Further it is known, that *DCP* functions are convex univalent. The class *DCP* is not rotation invariant. That is, $f \in DCP$ does not always imply that $e^{i\varphi} f$ is in *DCP*, $\forall \alpha \in \mathbb{R}$. However, $f \in DCP \Rightarrow Af + B \in DCP$ for all $A : \mathbb{R} \setminus \{0\}$ and $B \in \mathbb{C}$. Hence we can normalize the class *DCP* by the conditions $f(0) = 0$ and $|f'(0)| = 1$. Motivated by the non-existent rotation invariance property of the class *DCP*, we are interested in studying the subclass of those functions in *DCP* which are rotation invariant.

Definition 1.1. A function $f \in A$ is said to be in the class *SDCP* (Strongly Direction Convexity Preserving) if and only if $e^{i\alpha} f \in DCP$ for all $\alpha \in \mathbb{R}$

Every $f \in SDCP$ is univalent and hence fulfills $f'(0) \neq 0$. At the same time, it is clear that

$$\frac{f(z) - f(0)}{f'(0)} \in SDCP.$$

Hence we can normalize the class *SDCP* by the conditions $f(0) = 0$ and $f'(0) = 1$. From now on, by the class *SDCP*, we always mean the normalized class defined as follows :

$$SDCP := \{ f \in S : e^{i\alpha} f \in DCP, \forall \alpha \in \mathbb{R} \}.$$

We refer [2] for detail about the class *SDCP*.

2. Introduction to Harmonic Mappings in the Plane

A complex valued function $f(x,y)$ is harmonic in a domain D in the plane if it satisfies the Laplace's equation $\Delta f = 0$, where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

The complex notation $z = x + iy$ for points in the plane leads to the differential operators

$$(1) \qquad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

and

(2) $$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

A simple calculation gives

$$\Delta f = 4\frac{\partial^2 f}{\partial z \partial \bar{z}} = 4\frac{\partial^2 f}{\partial \bar{z} \partial z},$$

so the Laplace's equation can be written

(3) $$f_{\bar{z}z} = \frac{\partial^2 f}{\partial z \partial \bar{z}} = 0.$$

For a complex-valued function with continuous first order partial derivatives, the equation $f_{\bar{z}} = 0$ is equivalent to Cauchy-Riemann equations. Thus a function f is analytic if and only if $f_{\bar{z}} = 0$. From (3) it follows that every analytic function is harmonic, and that the z-derivative of every harmonic function is analytic.

The word harmonic mapping will be reserved for univalent (one-to-one) function. A complex–valued harmonic function f is a harmonic mapping of a domain $D \subset \mathbb{C}$ if it maps D unvalently on to some planer domain $\Omega = f(D)$. Although harmonic mappings are natural generalization of conformal mappings, they were studied originally by differential geometers because of their natural role in parametrizing minimal surfaces. Only in the mid 1980s did harmonic mappings begin to attract wide-spread interest among complex analysists. Particularly after the landmark paper [4] by J. Clunie and T. Sheil-Small which pointed out that many of the classical result for conformal mappings have clear analogues for harmonic mapping. Since that time the study of harmonic mapping developed rapidly, although a number of basic problems remain unsolved. In this paper we will only discuss applications of the class *DCP* with harmonic mappings in a plane.

If we write f as a sum of two real valued function $u(x,y)$ and $v(x,y)$ say, $f = u + i\,v$, the Jacobian of f is

$$J_f = \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}.$$

In the case of an analytic function we have the Cauchy-Riemann equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

so that $J_f = |f_x|^2 = |f_y|^2 = |f'|^2$. In the more general case of harmonic function, a simple calculation, with the help of (1) and (2), we can express the Jacobian J_f of f as

(4) $$J_f = |f_z|^2 = |f_{\bar{z}}|^2.$$

A sufficient condition for a differentiable function to be locally univalent (one-to-one) near a point $z_0 = x_0 + iy_0$ is that $J_f(z_0) \neq 0$. For analytic function it is also a necessary condition for locally univalence at z_0 as $J_f(z_0) = |f'(z_0)|^2 \neq 0$. Lewy [10] has shown that this property generalizes to harmonic function.

Lewy's Theorem : If f is a complex valued harmonic function which is locally univalent in a domain $D \subset \mathbb{C}$, then its Jacobian J_f is different from zero for all $z \in \mathbb{C}$.

Thus a harmonic mapping is a domain D is either sense preserving with $J_f > 0$ or sense reversing with $J_f < 0$. More precisely a harmonic mapping is sense preserving if $|f_z(z)| > |f_{\bar{z}}(z)|$ for all $z \in D$ and in sense reversing if $|f_{\bar{z}}(z)| > |f_z(z)|$ for all $z \in D$. In particular if a harmonic mapping is sense preserving, then $f_z(z) \neq 0$ for all $z \in D$. A mapping is sense reserving if and only if its complex conjugate is sense preserving. Conformal mapping are sense-preserving.

If f is a complex valued harmonic function on a simply connected domain $D \subseteq \mathbb{C}$, then it has the representation.

$$f = \overline{g} + h, \tag{5}$$

where g and h are analytic functions unique up to an additive constant in D, to justify this simple fact, first recall that since f is harmonic so $f_{z\bar{z}} = 0$. From this we see that f_z is analytic. Then let $h' = f_z$ and choose an antiderivative h. Let $g = \overline{g} - \overline{h}$ and observe that $g_{\bar{z}} = \overline{f_z} - \overline{h_z} = 0$, so that g is analytic. If 0 is in the domain D, we shall choose h so that $h(0) = f(0)$ and refer to $f = h + \overline{g}$ as the canonical representation of f. We call h the analytic and g the co-analytic part of f.

3. Application of the Class *SDCP* with Harmonic Mapping in the Plane

Let S_H denote the class of functions f of the form (5) that are univalent harmonic and sense preserving in the unit disk $\mathbf{D}$ and normalized by $f(0) = f_z(0) - 1 = 0$, and let S_H^0 denote the subclass of S_H for which $f_{\bar{z}}(0) = 0$. The class S_H obviously reduces to the familiar class S of normalized univalent functions in the unit disk D if the co-analytic part of f is zero.

In this section we shall look at various subclasses of S_H^0. We note that functions in S_H^0 have the form

$$f = \overline{g} + h, \text{ where } h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \;\; g(z) = \sum_{n=2}^{\infty} b_n z^n, \tag{6}$$

Let S_H^{*0} and C_H^0 be the subclasses of S_H^0 consisting of functions f that map $\mathbb{D}$ onto a starlike domain and convex domain, respectively, and let, as usual, C be the subclass of S consisting of those functions which map $\mathbb{D}$ onto a convex domain. In [7], the following theorem was proved.

Theorem 1. *If a function f of the form* (6) *satisfies* $\sum_{n=2}^{\infty} n\,(|a_n|+|b_n|) \le 1$, *then* $f \in S_H^{*0}$.

It has been proved in [3] that for $f \in S_H^{*0}$, the function $\int_0^z \left(\frac{f(\varsigma)}{\varsigma}\right) d\varsigma\ C_H^0$. Hence we have the following corollary :

Corollary 1. *If f of the form* (6) *satisfies* $\sum_{n=2}^{\infty} n^2\,(|a_n|+|b_n|) \le 1$, *then* $f \in S_H^0$..

Theorem 2. *If a function f of the form* (6) *satisfies* $\sum_{n=2}^{\infty} n\,(|a_n|+|b_n|) \le 1$, *then*

$$\varphi \,\tilde{*}\, f = \overline{\varphi * g} + \varphi * h \in S_H^{*0}, \quad \forall\, \varphi \in C.$$

Proof : Let $\varphi(z) = z + \sum_{n=2}^{\infty} c_n z^n \in C$. Then the coefficients of φ satisfy the condition $|c_n| \le 1$. Therefore the coefficients of the function $\varphi \,\tilde{*}\, f = \overline{\varphi * g} + \varphi * h$, whose analytic and co-analytic part have the form

$$\varphi * h = z + \sum_{n=2}^{\infty} c_n a_n z^n \ \text{ and } \ \varphi * g = \sum_{n=2}^{\infty} c_n b_n z^n.$$

respectively, satisfy the condition

$$\sum_{n=2}^{\infty} n(|c_n a_n| + |c_n b_n|) = \sum_{n=2}^{\infty} n|c_n|\,(|a_n|+|b_n|) \le \sum_{n=2}^{\infty} n(|a_n|+|b_n|) \le 1.$$

Hence by Theorem 5.4.1, the function $\varphi \,\tilde{*}\, f \in S_H^{*0}$

Corollary 2. *If f of the form* (6) *satisfies* $\sum_{n=2}^{\infty} n^2\,(|a_n|+|b_n|) \le 1$, *then*

$$\varphi \,\tilde{*}\, f = \overline{\varphi * g} + \varphi * h \in C_H^{*0}, \quad \forall\, \varphi \in C.$$

Proof: Let the function φ be as in the previous theorem. Then the coefficients of the function

$$\varphi \,\tilde{*}\, f = \overline{\varphi * g} + \varphi * h$$

satisfy the condition

$$\sum_{n=2}^{\infty} n^2\,(|c_n a_n| + |c_n b_n|) = \sum_{n=2}^{\infty} n^2|c_n|(|a_n|+|b_n|) \le \sum_{n=2}^{\infty} n^2\,(|a_n|+|b_n|) \le 1.$$

Hence $\varphi \,\tilde{*}\, f \in S_H^0$, by Corollary 5.4.1.

Theorem 3. *If f of the form (6) satisfies $\sum_{n=2}^{\infty}(|a_n|+|b_n|)\le 1$, then*

$$\varphi \tilde{*} f = \overline{\varphi * g} + \varphi * h \in S_H^{*0}, \quad \forall\ \varphi \in SDCP.$$

Proof : Let $\varphi(z) = z + \sum_{n=2}^{\infty} c_n z^n$. Since $\varphi \in SDCP$, the coefficients of φ satisfy the condition $|c_n| \le \frac{1}{n}$. Therefore the coefficients of the function $\varphi \tilde{*} f = \overline{\varphi * g} + \varphi * h$, whose analytic and co-analytic part have the form

$$\varphi * h = z + \sum_{n=2}^{\infty} c_n a_n z^n, \quad \varphi * g = \sum_{n=2}^{\infty} c_n b_n z^n,$$

respectively, satisfy the condition

$$\sum_{n=2}^{\infty} n(|c_n a_n| + |c_n b_n|) = \sum_{n=2}^{\infty} n|c_n|\,(|a_n| + |b_n|) \le \sum_{n=2}^{\infty} (|a_n| + |b_n|) \le 1.$$

Hence by Theorem 5.4.1, the function $\varphi \tilde{*} f \in S_H^{*0}$.

We need the following result of J. Clunie and T.Shell-Small [4] for our next theorem

Lemma 1. Let $f = \overline{g} + h$ *be locally univalent and harmonic in* $\mathbb{D}$. *Then* f *is a univalent mapping of* $\mathbb{D}$ *onto a domain convex in the direction of the real axis if, and only if,* $h-g$ *is a conformal univalent mapping of* $\mathbb{D}$ *onto a domain convex in the direction of the real axis.*

Theorem 4. *Let* $C_H(\theta)$ *consist of functions* $f = \overline{g} + h$ *in* S_H *which are convex in the direction* $e^{i\theta}$, *Then*

$$\varphi \tilde{*} f = \overline{\varphi * g} + \varphi * h \in C_H(\theta), \ \forall f \in C_H(\theta) \ \text{ and } \ \varphi \in DCP.$$

provided the function $\varphi \tilde{*} f = \overline{\varphi * g} + \varphi * h$ *is locally univalent.*

Proof : Let $f = \overline{g} + h \in C_H(\theta)$. Then the function

$$e^{-i\theta} f = \overline{e^{i\theta} g} + e^{-i\theta} h$$

is convex in the direction of the real axis. Hence, by the above lemma, the function

$$e^{-i\theta} h - e^{i\theta} g$$

is a conformal map which is convex in the direction of the real axis. Since $\varphi \in DCP$ the function

$$\varphi * (e^{-i\theta} h - e^{i\theta} g) = e^{-i\theta}(\varphi * h) - e^{i\theta}(\varphi * h) \tag{7}$$

is also convex in the direction of the real axis.

Now, by hypothesis, the function

$$\varphi \,\widetilde{*}\, f = \overline{\varphi * g} + \varphi * h$$

is locally univalent. Hence so is the function

$$e^{-i\theta}(\varphi \,\widetilde{*}\, f) = \overline{e^{i\theta}(\varphi * g)} + e^{-i\theta}(\varphi * h).$$

If we take into account that (7) is a conformal map convex in the direction of the real axis, we can apply the above lemma to conclude that

$$e^{-i\theta}(\varphi \,\widetilde{*}\, f) = \overline{e^{i\theta}(\varphi * g)} + e^{-i\theta}(\varphi * h)$$

is convex in the direction of the real axis. Hence the function

$$\varphi \,\widetilde{*}\, f = \overline{\varphi * g} + \varphi * h \in C_H(\theta).$$

This completes the proof of the theorem.

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A Note on Stokes Drag on Axi-symmetric Body : Oblique : Angle of Attack

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Abstract : In the paper [1], author have proposed a simple formulae for evaluating the axial and transverse Stokes drag on axi-symmetric bodies. Continuing the efforts in this regard, the expression for drag has been given when the axially-symmetric body is placed in the slow uniform incompressible viscous flow with oblique angle of attack 'ζ'. The drags in such flow have been calculated for sphere, spheroids (prolate and oblate), deformed sphere, cycloidal body, egg-shaped body, cassini body and hypocycloidal body.

Key words : Stokes drag, axially symmetric body, oblique angle, cycloidal body, egg-shaped body, cassini body, hypocycloidal body.

AMS Subject Classification : 76 D

1. Introduction

In the recent paper [Datta and Srivastava, 1999,1], authors have proposed a simple formulae, based on the integral [p. 122,2] used to evaluate drag on a sphere, for finding the axial and transverse Stokes drag on axi-symmetric bodies.

The Axial Flow

The drag on body, when it is placed in axi-symmetric Stokes flow with uniform stream U along x-axis is given as [1].

(1.1) $$F_x = \frac{1}{2}\frac{\lambda(y_{max})^2}{h},$$

where

(1.2) $$\lambda = 6\pi\mu U$$

and

$$h = \left(\frac{3}{8}\right) \int_{\alpha=0}^{\pi} R \, \mathrm{Sin}^3\alpha \, d\alpha \tag{1.3}$$

Here, R is the intercepting length between the point on the meridional curve and axis of symmetry (x-axis) of the body and 'α' is the slope of normal. In Cartesian coordinates, 'h' can be expressed as

$$h = \left(-\frac{3}{4}\right) \int_{0}^{a} \frac{yy''}{(1+y'^2)^2} \, dx, \tag{1.4}$$

where, x = a is maximum axial length and dashes represents derivative with respect to x.

The Transverse Flow

Let us consider an axially-symmetric body placed in a uniform stream 'U' along transverse axis (y-axis). The Stokes drag on this body is given to be [1].

$$F_y = \left(\frac{1}{2}\right) \frac{\lambda (y_{max})^2}{h_y}, \tag{1.5}$$

where

$$\lambda = 6\pi\mu U$$

and

$$h_y = \left(\frac{3}{16}\right) \int_{\alpha=0}^{\pi} (2R \, \mathrm{Sin}\,\alpha - R \, \mathrm{Sin}^3\alpha) \, d\alpha, \tag{1.6}$$

In Cartesian coordinates, h_y can be expressed as

$$= \left(\frac{3}{8}\right) \int_{0}^{a} \left[\frac{yy''}{(1+y'^2)} - \frac{yy''}{(1+y'^2)^2}\right] dx. \tag{1.7}$$

For the details, the reader is referred to the paper [1]. Now, in the next section, the method for Stokes drag on axi-symmetric body placed in the uniform stream attacking at an angle 'ζ' with the axis of symmetry, is proposed.

2. The Method

Let us consider an axially symmetric body placed in uniform stream 'U' attacking an angle of attack 'ζ' with the axis of symmetry (x-axis) [see, Fig. 1]

Let us consider e_x, e_y as unit vectors representing x and y directions and F_x, F_y are axial and transverse drags, then force vector F can be written as

$$F = F_x e_x + F_y e_y, \tag{2.1}$$

where F_x and F_y (axial and transverse Stokes drags) : are defined in (1.1) and (1.5). Since the uniform stream U makes an angle 'ζ' with the x-axis (Fig.1) then its components U Cos ζ and U Sin ζ will be in x and y direction and in general we can have the axial and transverse drag as

$$F_{x_1} = F_x \cos\zeta, \tag{2.2}$$

and

$$F_{y_1} = F_y \cos\zeta, \tag{2.3}$$

Then force vector **F** can be written as

$$\begin{aligned} \mathbf{F} &= F_{x_1} e_x + F_{y_1} e_y \\ &= F_x \cos\zeta\, e_x + F_y \sin\zeta\, e_y, \end{aligned} \tag{2.4}$$

and its magnitude will be given by $F = |\mathbf{F}|$ which reduces to axial and transverse drag as $\zeta = 0$ and $\zeta = \pi/2$.

Now, in the next section, we use the result (2.4) to obtain force vector with magnitude for the axi-symmetric bodies.

3. Flow Past Spheroid Prolate Spheroid

Let us consider the prolate spheroid, generated by the rotation of ellipse

$$x = a \,\text{Cost}\, t,\ \ y = b \sin t,\ \ 0 \le t \le \pi, \tag{3.1}$$

about the x-axis.

By using (2.4), together with (1.1) and (1.5), the expression for drag will be

$$F = 16\pi\mu U a e^3 \left[\frac{\cos^2\varsigma}{\{-2e + (1+e^2)L\}^2} + \frac{4\sin^2\varsigma}{\{2e + (3e^2-1)L\}^2} \right]^{\frac{1}{2}}, \tag{3.2}$$

where, $L = \ln \dfrac{(1+e)}{(1-e)}$.

Oblate Spheroid

Let us consider the oblate spheroid, generated by the rotation of ellipse

(3.3) $x = a \operatorname{Cos} t, \quad y = b \operatorname{Sin} t, \quad 0 \le t \le \pi$
about x-axis

By using (2.4), together with (1.1) and (1.5), the expression for drag will be

$$F = 8\pi\mu U a e^3 \left[\frac{\operatorname{Cos}^2 \varsigma}{\left\{e\sqrt{(1-e^2)} - (1-2e^2)\operatorname{Sin}^{-1} e\right\}^2} + \right.$$

$$\left. + \frac{4\operatorname{Sin}^2 \varsigma}{\left\{-e\sqrt{(1-e^2)} + (1+2e^2)\operatorname{Sin}^{-1} e\right\}^2} \right]^{\frac{1}{2}} \tag{3.4}$$

4. Flow Past a Deformed Sphere

Consider the axially symmetric body defined by

$$r = a\left[1 + \varepsilon\left\{d_0 + d_2 P_2(\mu) + \sum_{k=0}^{\infty} d_{2k+1} P_{2k+1}(\mu)\right\}\right], \quad \mu = \operatorname{Cos}\theta, \tag{4.1}$$

where (r, θ) are spherical polar coordinates and $P_k(\mu)$ is Legenedre function of first kind. For small parameter ε, this represents a deformed sphere. By using (2.4), together with (1.1) and (1.5), the expression for drag will be

$$F = \lambda a\left[1 + 2\varepsilon\left\{d_0 + \frac{d_2}{10}(1 - 3\operatorname{Sin}^2\varsigma)\right\} + 0(\varepsilon^2)\right]^{\frac{1}{2}} \tag{4.2}$$

5. Flow Past Cycloidal Body of Revolution

A. Let us take the inverted cycloid

(5.1) $x = a(t + \operatorname{Sin} t), \quad y = a(1 + \operatorname{Cos} t), \quad -\pi \le t \le \pi:,$

with vertex at (0, 2a), and revolve it about x-axis, the base, to generate the cycloidal body of revolution. By using (2.4), together with (1.1) and (1.5), the expression for drag is given to be

$$F = \frac{128}{15}\mu U a\ [36 - 11\operatorname{Cos}^2\varsigma]^{\frac{1}{2}}. \tag{5.2}$$

B. Let us consider the body generated by the rotation about x-axis of the curve composed of arcs of two cycloidal parts represented parametrically by

$$\left.\begin{array}{l} x = a(1+\text{Cos } t),\ y = a(t+\text{Sin } t),\ 0 \le t \le \pi; \\ x = -a(1+\text{Cos } t),\ y = a(t+\text{Sin } t),\ 0 \le t \le \pi; \end{array}\right\} \tag{5.3}$$

by using (2.4), together with (1.1) and (1.5), the required drag will be

$$F = 96\pi^3 \mu Ua \left[\frac{\text{Cos}^2\varsigma}{(3\pi^2+16)^2} + \frac{4(1-\text{Cos}^2\varsigma)}{(9\pi^2+32)^2} \right]^{\frac{1}{2}} \tag{5.4}$$

6. Flow Past an Egg-Shaped Body

Let us consider an egg-shaped body in which right portion is in the shape of a half prolate spheroid given parametrically by

$$\left.\begin{array}{c} x = a\text{ Cos } t,\ y = b\text{ Sin } t,\ 0 \le t \le \pi/2, \\ \text{and the left half portion is a hemisphere given by} \\ x = b\text{ Cos } t,\ y = b\text{ Sin } t,\ \pi/2 \le t \le \pi. \end{array}\right\} \tag{6.1}$$

By using (2.4) together with (1.1) and (1.5), the drag is given by

$$F = 8\pi\mu Ua\sqrt{1-e^2}\left[\frac{\text{Cos}^2\varsigma}{\left\{\frac{2}{3} + \frac{\sqrt{1-e^2}(-2e+(1+e^2)L)}{4e^3}\right\}^2} \times \frac{4(1-\text{Sin}^2\varsigma)}{\left\{\frac{4}{3} + \frac{\sqrt{1-e^2}(2e+(3e^2-1)L)}{4e^3}\right\}^2} + L \right]^{\frac{1}{2}} \tag{6.2}$$

7. Flow Past Cassini Body of Revolution

Let us consider the cassini body obtained by revolving the curve

$$y^2 = \frac{2}{3}(1+3x^2)^{\frac{1}{2}} - x^2 - \frac{1}{3}, \quad 0 \le x \le 1, \tag{7.1}$$

about the axis of symmetry (x-axis).

By using (2.4) together with (1.1) and (1.5), the expression for drag will be

$$\left.\begin{aligned} F &\approx [(0.8)^2 \operatorname{Cos}^2 \varsigma + (0.82)^2 \operatorname{Sin}^2 \varsigma]^{\frac{1}{2}} \lambda, \lambda = 6\pi \mu U \\ &\approx [0.6724 - 0.0324 \operatorname{Cos}^2 \varsigma]^{\frac{1}{2}} \lambda. \end{aligned}\right\}, \tag{7.1}$$

8. Flow Past Hypocycloidal Body of Revolution

Let us consider the body generated by rotating the curve

$$y^2 = -3x^2 + (1 + 8x^4)^{\frac{1}{2}}, \quad 0 \le x \le 1, \tag{8.1}$$

about axis of symmetry (x-axis).

Now, the expression for drag can be written by the use of (2.4), together with (1.1) and (1.5)

$$\left.\begin{aligned} F &\approx [(1.044)^2 \operatorname{Cos}^2 \varsigma + (1.32)^2 (1 - \operatorname{Cos}^2 \varsigma)]^{\frac{1}{2}} \lambda, \lambda = 6\pi \mu U \\ &\approx [1.7424 - 0.6525 \operatorname{Cos}^2 \varsigma]^{\frac{1}{2}} \lambda. \end{aligned}\right\}, \tag{8.2}$$

All the above results are found to be new for the axially symmetric bodies lies under the restricted class [1] and never existed in the literature

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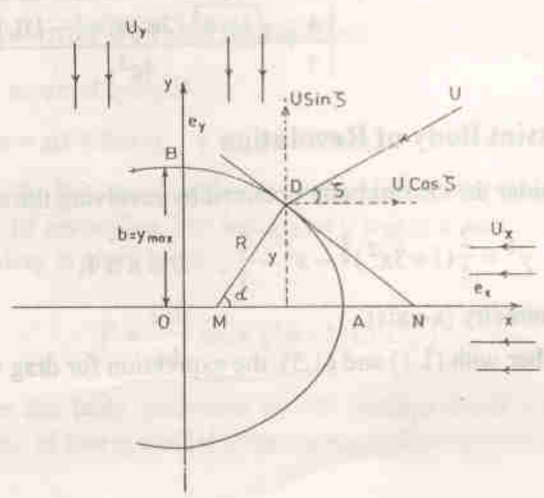

Fig. 1- Geometry of axially-symmetric body ; oblique angle of attack

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Infinitesimal Variation of Hypersurfaces of an Almost *r*-Contact Hyperbolic Structure Manifold

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Summary: The Infinitesimal variation of the structure tensors of an almost contact metric structure induced on the hyper surface of a Kahlerian manifold under various conditions has been studied by Yano. In this paper we have studied the infinitesimal variation of the structure tensors of an almost r-contact hyperbolic structure induced on the hyper surface of a differentiable manifold equipped with an almost r-contact hyperbolic structure.

1. Introduction :

Let M^{n+r} be an $(n+r)$ dimensional differentiable manifold of differentiability class C^∞. Let there exist on M^{n+r} a C^∞ vector valued linear function F, rC^∞ linearly independent and non zero contravariant vector fields $T^1, T^2, \ldots T^r$ such that

$$F^2 X = X + \sum_{l=1}^{r} A_l(x) T^l \tag{1.1}$$

for arbitrary vector field X on M^{n+r}. Also

$$F(X) = \overline{X} \tag{1.2}$$

In view of (1.1), let M^{n+r} be endowed with the Riemannian metric G such that it satisfies the following condition.

$$G(\overline{X}, \overline{Y}) + G(X, Y) + \sum_{l=1}^{r} A_l(X) A_l(Y) = 0 \tag{1.3}$$

Thus M^{n+r} satisfying the conditions (1.1) and (1.3) will be called an almost r-contact hyperbolic structure manifold [2].
In M^{n+r} the following results hold

$$\text{(1.4)} \qquad \begin{aligned} &\text{(a)} \quad \overline{T}^{l} = 0, \\ &\text{(b)} \quad A_l(\overline{X}) = 0, \text{ for arbitrary vector field } X. \\ &\text{(C)} \quad A_l(T^m) + \delta_l^m = 0, \end{aligned}$$

Where δ_l^m is Kronecker delta and ℓ, m take the values $1, 2, \ldots, r$.

Let us imbed a hypersurface M^{n+r-1} into M^{n+r} by the isometric immersion $b : M^{n+r-1} \rightarrow M^{n+r}$. Corresponding to this we have the Jacobian b^* of b denoted by B which carries $T_q(M^{n+r-1})$ into $T_b(q)M^{n+r}$ injectively. Since the immersion is isometric, we have

$$\text{(1.5)} \qquad G(BX, BY) \, o \, b = g(X,Y).$$

g being the metric induced on the hyper surface and X, Y denote arbitrary vector fields, We have

$$\text{(1.6)} \qquad G(BX,N) = 0,$$

$$\text{(1.7)} \qquad G(N,N) = 1.$$

The transformation equations are

$$\text{(1.8)} \qquad FBX = BfX + \alpha(X)N,$$

$$\text{(1.9)} \qquad FN = BP + \eta N,$$

where f is a tensor field of type (1.1) and α is a 1-form on M^{n+r-1}. From equation (1.8) and the relations

$$\text{(1.10)} \qquad \begin{aligned} &\text{(a)} \quad T^l = Bt_l + P_l N, \\ &\text{(b)} \quad A_l(BX) \, o \, b = a_l(X), \\ &\text{(c)} \quad \alpha(X)P = 0. \end{aligned}$$

we get

$$\text{(1.11)} \qquad f^2 X = X + \sum_{l=1}^{r} a_l(X) t^l$$

The metric g in (1.5) is found to satisfy

$$\text{(1.12)} \qquad g(fX, fY) + g(X,Y) + \sum_{l=1}^{r} a_l(X) a_l(Y) = 0.$$

Consequently an almost r-contact hyperbolic structure gets induced on M^{n+r-1}.

Let D be the Riemannina connecxion induced on M^{n+r-1}. Then we have the Gauss and Weingartan equations [1].

$$E_{BX}\, BY = BD_X Y + H(X,Y)N, \tag{1.13}$$

$$E_{BX}\, N = -B'HX, \tag{1.14}$$

Where H is the 2[nd] fundamental form of M^{n+r-1} and $'H$ is a tensor field of type (1,1) associated with H. Let K and $\widetilde{K}$ stand for the curvature tensors of the hyper surface and the enveloping manifold. Then we have Gauss and Codazzi equations.

$$\widetilde{K}(BX, BY, BZ, BU) = {}'K(X,Y,Z,U) - H(Y,Z)\, H(X,U) + H(X,Z)\, H(Y,U) \tag{1.15}$$

and

$${}'\widetilde{K}(BX, BY, BZ, N) = (D_X H)(Y,Z) - (D_Y H)(X,Z), \tag{1.16}$$

Where $'K$ and $'\widetilde{K}$ are the associate covariant curvature tensors of M^{n+r-1} and M^{n+r}. Now let us differentiate equation (1.8) along the hyper surface and use $E_{\widetilde{X}} F = 0$ hence

$$E_{BX}\, B f Y = F(E_{BX}\, BY) - \{(D_X A)Y + A(D_X Y)\}N - A(Y) E_{BX} N.$$

In view of (1.9), (1.13) and (1.14) we get

$$(D_X f)Y = H(X,Y)\, P + \alpha(Y)\, 'HX, \tag{1.17}$$

$$(D_X \alpha)Y = H(X,Y)\eta - H(X, fY). \tag{1.18}$$

Covariant differentiation of (1.9) along M^{n+r-1} yields

$$D_X P = \eta\ 'HX - 'HfX. \tag{1.19}$$

Definition 1.1 *An almost r-contact hyperbolic structure is said to be normal if*

$$S(X,Y) = N(X,Y) + \sum_{l=1}^{r} \{(D_X \alpha)Y - (D_Y \alpha)X\}\, t^l = 0, \tag{1.20}$$

Where

$$N(X,Y) = (D_{fX} f)Y - (D_{fY} f)X + f(D_Y f)X - f(D_X f)Y + \sum_{l=1}^{r} a_l [X,Y]\, t^l,$$

So that the normality condition (1.20) takes the form

$$S(X,Y) = (D_{fX} f)Y - (D_{fY} f)X + f(D_Y f)X - f(D_X f)Y + \sum_{l=1}^{r} a_l [X,Y] t^l + \sum_{l=1}^{r} \{(D_X \alpha)Y - (D_Y \alpha)X\} t^l = 0.$$

If almost r-contact hyperbolic structure induces on M^{n+r} be normal, from the last equation and from (1.17) and (1.18) we obtain

$$\alpha(X)\{'Hf - f\,'H\}Y - \alpha(Y)\{'Hf - f\,'H\}X = 0$$

(1.21) $$'Hf = f\,'H$$

Therefore it follows that [1]

(1.22) $$H(P,P) = \,'HP$$

Showing that $H(P,P)$ is an eigen value of $'H$ and the corresponding eigen vector is P. Let us denote $H(P,P)$ by τ.

Definition 1.2. *An almost r-contact hyperbolic structure is called r-hyperbolic Sasakian if*

(1.23) $$\sum_{l=1}^{r} \{(D_X \alpha_l)Y - (D_Y \alpha_l)X\} = r'f(X,Y).$$

We have, $$'f(X,Y) = g(fX,Y).$$

More generally in a normal r-contact hyperbolic structure hyper surface of M^{n+r} we assume that [3]

(1.24) $$\sum_{l=1}^{r} \{(D_X \alpha_l)Y - (D_Y \alpha_l)X\} = r\beta\,'f(X,Y).$$

Applying (1.18) to the above equation we have

(1.25 $$'H\eta = \,'Hf = -r'\beta f.$$

Thus we obtain

(1.26) $$'HX = -r'\beta x + (\tau + r'\beta)\,\alpha(X)P,$$

Equation (1.17), (1.18), (1.19) then transform as

(1.27) $$(D_X f)Y = -r'\beta\,\{g(X,Y)P + \alpha(Y)X\} + 2(\tau + r'\beta)\,\alpha(X)\,\alpha(Y),$$

(1.28) $$(D_X \alpha)Y = r'\beta f(X,Y),$$

(1.29) $$D_X P = -(\eta - f)\, r'\beta X.$$

Let β be a constant so that from (1.27) and (1.29) we obtain

$$K(X, Y, P) = -r'^2 \beta^2\ \eta(\alpha(Y))X - \alpha(X)Y),$$

which shows that for a normal r-contact hyperbolic structure hypersurface satisfying (1.24) and involving constant $r'\beta$, the sectional curvature with respect to a plane section containing P is $r'^2 \beta^2$.

Let us call such a structure a normal r-contact hyperbolic structure with f sectional curvature $r'^2 \beta^2$.

2. Infinitesimal Variation of a Hypersurface of an Almost *r*-contact Hyperbolic Structure Manifold

Let us take the restriction of an almost decomposable killing vector field U on the enveloping manifold of the hypersurface. According the variation of the differential of imbedding is given by [4].

(2.1) $$(\delta B)X = \in \mathrm{E}_{BX}\mathrm{U}$$

where $\in$ is infinitesimally small number. Splitting U into its tangential and normal parts as

(2.2) $$\mathrm{U} = BV + \lambda N$$

and from (1.13), (1.14) we express (2.1) as

(2.3) $$(\delta B)(X) = \in \{B(D_X V - \lambda\, 'HX) + (X\lambda + H(X,V))N\}.$$

Infinitesimal *V*ariation of N is given by [5]

(2.4) $$\delta N = \in L_{\mathrm{U}} N = \in BW$$

The Lie derivative of N (i.e., $L_{\mathrm{U}} N$) being orthogonal to N. Infinitesimal variation of equation (1.6) yields

$$G(BD_X V + H(X,V)N + (X\lambda)N - \lambda B\,'HX, N) = -\,G(BX, BW)$$

which implies that $W = -('HV + \Lambda)$

Where Λ stands for the vector field associate to the gradient of λ. Thus we have

$$\delta N = -\in B('HV + \Lambda)$$

Now varying equation (1.8) infinitesimally, we get

$$(\delta B)(fX) + B(\delta f)X = F((\delta B)X) - (\delta N)\,\alpha(X) - \delta\alpha(X)N.$$

Making use of (1.8), (2.3) and (2.4) in it we find

$$B(\delta f)X + (\delta\alpha)(X)N = \in[\{Bf(D_X V - \lambda\,'HX)N + \alpha(D_X V - \lambda\,'HX)N + (X\lambda + H(XV)(BP + \eta N) + \alpha(X)B('HV + \Lambda)\} - \{B(D_{fX}V - \lambda\,'HfX) + (fX\lambda + H(fXV)N\}].$$

Comparing the tangential and normal components, we have

(2.5) $$(\delta f)X = \in\{f(D_X V - \lambda\,'HX) + (H(X,V) + X\lambda)P + \alpha(X)('HV + \Lambda) - D_{fX}V + \lambda\,'HfX\}.$$

and

(2.6) $$(\delta\alpha)X = \in\{f(D_X V - \lambda\,'HX) + \eta(H\lambda + H(X,V) - X\lambda - H(fX,V)\}$$

Since the derivative of f along V is given by

$$(L_V f)X = L_V(fX) - f(L_V X) = D_V(fX) - D_{fX}V - f(D_V X - D_X V).$$

Therefore equation (2.5) assumes the following form

(2.7) $$(\delta f)X = \in\{(L_V f)X + \lambda('Hf - f'H)X + X\lambda P + \alpha(X)\Lambda + 2H(X,V)P\}$$

Applying equation (1.18) and the definition

$$(L_V\alpha)X \overset{\text{def}}{=} (D_V\alpha)X + (D_X V)$$

(2.8) $$(\delta\alpha)X = \in[\{(L_V\alpha)X - \alpha\lambda\,'HX - (fX)\lambda\} + 2H(X,V)\eta + 2h(V,fX)]$$

Next varying equation (1.9) infinitesimally, we get

$$-\in FB('HV + \Lambda) = [B(\delta P) + \in\{B(D_P V - \lambda\,'HP) + P\lambda + H(P,V)N\} - \in\eta B('HV + \Lambda)].$$

Which by virtue of (1.8) and (2.3) yields

$$B\,\delta P + \in\{B(D_P V - \lambda\,'HP) + (P\lambda + H(P,V)N\} - \in\eta B('HV + \Lambda) = -\in[Bf('HV + \Lambda) + \alpha('HV + \Lambda)N],$$

whose tangential part reduces in virtue of (1.19) to the form

(2.9) $$\delta P = \in [\,\lambda' HP + L_U P + \Lambda(\eta - f)].$$

Again varying equation (1.5) infinitesimally, we get

(2.10) $$(\delta g)(X,Y) = G((\delta B)X, BY) + G(BX, (\delta B)Y),$$

which in virtue of (2.3) reduces to

(2.11) $$(\delta g)(X,Y) + \in \{(L_V\, g)(X,Y) - 2\lambda\, H(X,Y)\}.$$

Thus we establish the following theorem.

Theorem 2.1. *When a hyper surface of an almost r-contact hyperbolic structure manifold varied infinitesimally by means of a vector field* $U = BV + \lambda N$ *the structure tensors of almost r-contact hyperbolic structure hypersurface vary according to equations* (2.7), (2.8), (2.9) *and* (2.10).

Corollary 2.1. *When a hypersurface of an almost r-contact hyperbolic structure manifold is given infinitesimally tangential variation by means of BV, the variation of the induced almost r-constant hyperbolic structure tensors on the hypersurface are given by their Lie-derivatives along V.*

Corollary 2.2. *When a hypersurface of an almost r-contact hyperbolic structure manifold is given infinitesimal normal variation by means of λN, the variation of the induced almost r-contact hyperbolic structure tensors on the hyper surface are given by*

(a) $(\delta f)X = \in [\,\lambda('Hf - f'H)X + X\lambda P + \alpha(X)\Lambda + 2H(X,V)P\,],$

(b) $(\delta\alpha)X = \in [-\alpha\lambda\, 'HX - fX\lambda + 2H(X,V)\,\eta + 2H(V, fX)],$

(2.11) (c) $(\delta P) = \in [\,\lambda\, 'HP + \Lambda(\eta - f)].$

(d) $(\delta g)(X,Y) = -2 \in \lambda\, H(X,Y).$

The infinitesimal variation is said to be parallel when BX and $B\overline{X}$ are both parallel equivalently and when $(\delta B)X$ is tangential to the original hyper surface. Since

$$(\delta B)X = \in [\,B(D_X V - \lambda\, 'HX) + (X\lambda + H(X,V)N\,].$$

Therefore for an infinitesimal parallel variation it is necessary and sufficient that

(2.12) $$X\lambda + H(X,V) = 0.$$

Corollary 2.3. *When a hyper surface of an almost r-contact hyperbolic structure manifold is given infinitesimal parallel variation the hypersurface variation the hypersurface*

$$
\begin{aligned}
&\text{(a)}\quad (\delta f)X = \in[\lambda(f\,'H - f'H)X + \alpha(X)\Lambda],\\
&\text{(b)}\quad (\delta\alpha)X = \in[-\alpha\lambda\,'HX],\\
(2.13)\quad &\text{(c)}\quad (\delta P) = \in\lambda\,'HP,\\
&\text{(d)}\quad (\delta g)(X,Y) = -2\in\lambda H(X,Y).
\end{aligned}
$$

Corollary 2.4. *Let the structure induced on a hypersurface of an almost r-contact hyperbolic structure manifold be a normal r-contact hyperbolic structure with f-sectional curvature-$\gamma'^2\beta^2$ then the infinitesimal normal parallel variation of the hypersurface makes the structure tensor vary as*

$$
(2.14)\qquad
\begin{aligned}
&(\delta f)X = \alpha(X)\Lambda,\\
&(\delta\alpha)X = -\lambda\tau P,\\
&\delta P = \in\lambda\tau P,\\
&(\delta g)(X,Y) = -2\in\lambda\{-r'\beta\, g(X,Y) + (\tau + r'\beta)\,\alpha(X)\,\alpha(Y)\}.
\end{aligned}
$$

3. Variation of r–Hyperbolic Sasakian Hypersurface with f– Sectional Curvature $r'^2\beta^2$

We now assume that an almost r-contact hyperbolic structure induced on the hyper surface is a *r*-hyperbolic Sasakian structure with f-sectional curvature $r'\beta$, we have [1]

$$(3.1)\qquad H(X,\,'HY) = r'^2\beta^2 g(X,Y) + (\tau^2 + r'^2\beta^2)\,\alpha(X)\,\alpha(Y)$$

and

$$(3.2)\qquad H(X,Y) = -r'\beta g(X,Y) - r'\beta(\delta g)(X,Y) + \delta(\tau + r'\beta)\,\alpha(X)\,\alpha(Y).$$

The variation in the connexions and the second fundamental form are given by [1].

$$(3.3)\qquad (\delta D)(X,Y) = \in\{(L_U D)(X,Y) - (D_Y\lambda\,'H)X - (D_X\,\lambda\,'H)Y + H(X,Y) + \lambda H^*(X,Y)\}$$

where

$$g\,H^*(X,Y)Z = (D_Z\,H)(X,Y)$$

and

$$(3.4)\qquad (\delta H)(X,Y) = \in\{(L_V\,H)(X,Y) - \lambda H(X,\,'HY) + XY\lambda - (D_X Y)\,\lambda + \lambda\,'K(N, BX, BY, N)\}$$

If the infinitesimal variation of the hyper surface are normal the variation of D would be given by [1].

(3.5) $(\delta D)(X,Y) = \in [X Y\lambda - (D_X Y)\lambda + {}'K(N, BX, BY, N) - \lambda H(X\ {}'HY)]$.

Varying equation (3.2) infinitesimally, we have

$$(3.6)\qquad (\delta H)(X,Y) = -(\delta\, r'\beta)\, g(X,Y) - r'\beta(\delta g)(X,Y) + \delta(\tau + r'\beta)\,\alpha(X)\,\alpha(Y) + (\tau + r'\beta)\{(\delta\alpha)(X)\,\alpha(Y) + \alpha(X)(\delta\alpha)(Y)\}.$$

which with the help of equations (2.8), (2.9), (2.10), (3.5) and

$$(3.7)\qquad (L_V H)(X,Y) = -r'\beta(L_V g)(X,Y) + \{(L_V H)(P,P) + 2H(L_V P,P)\}\alpha(X)\,\alpha(Y) + (\tau + r'\beta)\{(L_V\alpha)(X)\alpha(Y) + \alpha(X)(L_V\alpha)(Y)\}.$$

becomes

$$\begin{aligned}
&\in \{X, Y\lambda - (D_X Y)\lambda + \lambda\, {}'K(N, BX, BY, N) - \lambda H(X\ {}'HY)\}\\
&= -2\, r\beta \in \lambda H(X,Y) + \in (PP\lambda - (D_P P)\lambda\\
(3.8)\qquad &- \lambda H(P,HP) - 2H(P, \lambda HP - \Lambda(\eta - f)\} + \delta r'\beta/\in\}(X)\,\alpha(Y)\\
&+ \in (\tau + r'\beta)(-\alpha\lambda\, {}'HX - f X\lambda + 2H(X,V)\,\eta\\
&+ 2H(V, fX\}\,\alpha(Y) + (-\alpha\lambda\, {}'HY + f Y\lambda\\
&+ 2H(Y,V\} + 2H(V, fY\}\,\alpha(X),
\end{aligned}$$

Conversely if λ satisfies the differential equation (3.8) then by retreating the steps we get (3.3).

Hence we have the following theorem

Theorem 3.1. *In order that for an infinitesimal variation (2.1) may have the a r-hyperbolic Sasakian hypersurface with f-sectional curvature $-r'^2\beta^2$ in a r-hyperbolic Sasakian with f-sectional curvature $-r'^2\beta^2 - \delta r'^2\beta^2$, it is necessary and sufficient that the function λ satisfies the relation*

$$\begin{aligned}
&\in \{XY\lambda - (D_X Y)\lambda + \lambda\{{}'K(N, BX, BY, N)\} + \gamma'^2\beta^2\,(g(X,Y) - \alpha(X) - \alpha(Y))\\
&+ (PP\lambda - D_P P)\lambda)\,\alpha(X)\,\alpha(Y) + (\tau + r'\beta)\{-fX\lambda\,\alpha(Y)\\
&- f Y\lambda\,\alpha(X)\}\, f\{2H(X,V)\,\tau + 2HV, fX)\}\,\alpha(Y)\\
&+ \{2H(Y,V) + 2H(V, fY)\}\,\alpha(X)]\\
&= \delta r'\beta(\alpha(X)\,\alpha(Y) - g(X,Y)).
\end{aligned}$$

Corollary 3.1. *The infinitesimal normal parallel variation carries a normal r-hyperbolic Sasakian hypersurface with f-sectional curvature* $-r'^2 \beta^2$ *to a normal r-hyperbolic Sasakian hypersurface with f-sectional curvature* $-r'^2 \beta^2 - \delta r'^2 \beta^2$ *if and only if*

$$\lambda \in \{ {}'K(N, BX, BY, N)\} + r'^2 \beta^2 \, (g(X,Y) - \alpha(X)\, X(Y)\} = \{\alpha(X)\, \alpha(Y) - g(X,Y)\, \delta r' \beta \tag{3.9}$$

Corollary 3.2. *If the enveloping manifold of corollary* (3.1) *be flat the condition reduced to* $\delta r' \beta = -\lambda \in r'^2 \beta^2$.

Hence the proof is obvious.

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CONTENTS

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