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A Simplified Model For General Circulation In Earth's Atmosphere And Ocean

D. B. ADHIKARY

Abstract : In order to understand the behavior of the atmosphere and the ocean, a various types of mathematical models governing the motion and the states of atmosphere and ocean have been already established and studied by different mathematicians. Present paper is devoted to the construction of a simplified linear model that governs the general circulation of atmosphere and ocean.

1. Introduction

Mathematical modeling is what physical applied mathematics is all about. A model is a representation of a process. Usually, a mathematical model takes the form of a set of equations describing a number of variables.

Mathematical modeling begins with the identification of a problem. Once a problem is identified, and a mechanism proposed, then one must formulate it mathematically. Formulation involves equations and boundary conditions. Physical laws governing the motion and states of atmosphere and ocean can be described by the general equations of hydrodynamics and thermodynamics, which are very complicated.

There are essentially two characteristics of both the atmosphere and ocean that are used in simplifying the equations. The first one is that for large-scale geophysical flows, the ratio between the vertical and horizontal scales is very small. Another small parameter is the ratio of the speed (horizontal) of wind to the speed of rotation of the earth around the polar axis. This number, called the Rossby number, is of order of 1/50. The asymptotics corresponding to the small Rossby number leads to the so-called geostrophic and quasi-geostrophic equations.

Reduction is the process whereby a model is simplified, most often by neglecting various small terms. But, what's meant by "small"? For example ; a speed of 1 cm per second is slow for a bullet, but fast for an earthworm. So, the word 'small' is to be used in a relative manner.

Linerization is a type of reduction, which is basic to practically all-analytic models of atmospheric motions. In this process, the flow field is divided into a longitudinally averaged part (zonal mean) and deviations from that average (perturbation). It is then assumed that the perturbations are sufficiently small so that the terms involving products of the perturbations may be neglected compared to linear terms.

It is well known that in order to understand the turbulent behavior of the atmosphere and the ocean, and to predict the weather and the climate, we need to establish some mathematical equations or models governing the motion and the states of the atmosphere and the ocean. We also need to establish and solve the corresponding numerical models (the numerical approximation of the mathematical equations).

The aim of the author is to derive some mathematical models for the coupling of the atmosphere and the ocean to study them from analytical as well as numerical viewpoints. As a first step, the present paper is devoted to the construction of a simplified linear mathematical model that governs the general circulation of atmosphere and ocean. Here in this paper, a very simple global circulation model of the atmosphere and the ocean is obtained, for which the equations of motion for wind and temperature are linear evolution equations similar to the linear Stokes equations.

2. Historical Background

As astronomers discover new planets, the planets are named after them. Similarly, scientists name the equations after the names of those mathematicians, who develop them. There are many equations named after mathematicians such as Laplace's equation, Euler-D 'Alambert's equation, Tricomi's equation, Schrodinger equation, Maxwell's equation and so on.

Euler derived the equations that describe the most fundamental behavior of a fluid in 1755. These are the equations of conservation of momentum and conservation of mass of a fluid that is incompressible and is inviscid. The initial-boundary value problem for the Euler equations is a surprisingly difficult problem. Perhaps, it is one of the most challenging of all problems in partial differential equations that arise directly from physics. Even the basic questions of existence and uniqueness of the solutions in three dimensions remain open.

All real fluids are at least very weakly viscous. Viscosity is necessary to generate flows, and its influence is very complicated. Incorporation of the effects of

viscosity leads to the versions of Euler equations, called Navier–Stokes equations. Since friction is a fact of nature, it could be argued that only the Navier–Stokes equations are physically relevant. But there is much to learn from the Euler equations.

The issue of the stability or instability of a fluid flow became one of the most basic problems in fluid dynamics and was examined experimentally and mathematically by such giants of science as Helmholtz, Kelvin, Rayleigh and Reynolds. An elegant mathematical treatment of non-linear stability is given by Arnold [2], which is applicable to certain two dimensional inviscid fluid motions.

The idea of wide application of mathematical models of rotating fluids to the study of dynamics of atmospheric processes belongs to A. Friedman, who in the beginning of 20th century contributed a series of fundamental works in this direction.

The linearized Navier-Stokes system

$$(2.1)\qquad \begin{cases} \dfrac{\partial \vec{V}}{\partial t} - [\vec{v},\vec{\omega}] - \nu\Delta\vec{v} + \nabla P = \vec{F} \\ \operatorname{div}\vec{V} = 0 \end{cases}$$

describing the motion of a rotating fluid was studied by V.N. Maslennikova [8]. System (2.1) without the Coriolis term $[\vec{v},\vec{\omega}]$ was extensively studied from the mathematical point of view by Oseen and Laray. In particular, Oseen was the first to construct the fundamental solution of system (2.1) without taking rotation into account, the so-called *Oseen tensor*. System (2.1) without viscosity. (i.e., $\upsilon = 0$) was first studied by S.L. Sobolov and is known as *Sobolov system*. Even in a system with viscosity, as well as in a Sovolev system [1], the presence of the terms $[\vec{v},\vec{\omega}]$ causes the solution to have an oscillatory character and leads to the emergence of a vortex, whereas in the absence of these terms a solution of system (2.1) is in a definite sense analogous to a solution of the heat equation.

L. Marchhuk introduced a linearized system [7] of partial differential equations in his book 'Mathematical Models of Circulation in Oceans' in 1980, in which a numerical approach is suggested for its solution.

3. Formulation of the Mathematical Model

Let the whole of the space $\mathbb{R}^3$ be filled with inviscid fluid being at rest at infinity. Let Ox be an inertial frame of reference and Oy be a frame of reference which rotates with respect to Ox with constant angular velocity $\vec{\omega}$. Without loss of generality, we assume

$$\vec{\omega} = (0, 0, \omega)$$

The dynamics of an inviscid fluid with respect to the inertial frame Ox is governed by the Euler equations

$$(3.1)\qquad \begin{cases} \dfrac{\partial \vec{V}}{\partial t} + (\vec{V}, \nabla_x)\vec{V} + \dfrac{1}{\xi}\nabla_x P = \vec{F}(x,t) \\ \dfrac{\partial \xi}{\partial t} + \operatorname{div}_x(\xi \vec{v}) = 0, \end{cases}$$

where

$\vec{V}$ = fluid's velocity field
P = hydrodynamics pressure
ξ = fluid's density
$\vec{F}$ = mass density of external forces

With respect to the rotating frame Oy, the dynamics of inviscid fluid is known to be governed [5] by the equations

$$(3.2)\qquad \begin{cases} \dfrac{\partial \vec{u}}{\partial t} + (\vec{u}, \nabla_y)\vec{u} - 2[\vec{u}, \vec{\omega}] + \dfrac{1}{\xi}\nabla_y Q = \vec{G}(y,t) \\ \dfrac{\partial \xi}{\partial t} + \operatorname{div}_y(\xi \vec{u}) = 0. \end{cases}$$

with $\vec{u}, Q$ and $\vec{G}$ conveying the same physical meaning as $\vec{V}$, P and $\vec{F}$ respectively.

In equations (3.2), the pressure Q involves the centrifugal force :

$$Q = P - \frac{\xi}{2}\,|[\vec{y}, \vec{\omega}]|^2$$

and the velocity $\vec{u} = \vec{v} - [\vec{\omega}, \vec{y}]$. Since the fluid is at rest at infinity with respect to Ox, we have :

$$(3.3)\qquad \lim_{|x| \to \infty} \vec{V}(x,t) = 0.$$

With respect to Oy, the fluid is not at rest at infinity :

$$(3.4)\qquad \lim_{|y| \to \infty} (\vec{u} + [\vec{\omega}, y]) = 0.$$

Note that in (3.1), the pressure $P(x,t)$ is bounded at infinity. But in (3.2), the pressure $Q(y, t)$ at infinity becomes a polynomial in y of the second order. And due to condition (3.4), the velocity field $u\,(y, t)$ in (3.2) becomes at infinity a polynomial in y of the first order.

Technically, the condition (3.4) makes solving any problem for equations (3.2) most complicated, compared to the case of zero condition at infinity. To overcome this technical obstacle, we represent the solution $\vec{u}$ of (3.2) in the form

(3.5) $$\vec{w}(y,t) = \vec{u}(y,t) + [\vec{\omega}, \vec{y}]$$

which reduces equations (3.2) to the following form:

(3.6) $$\begin{cases} \dfrac{\partial \vec{w}}{\partial t} + (\vec{w}, \nabla_y)\vec{w} - 2[\vec{w}, \vec{\omega}] + \dfrac{1}{\xi}\nabla_y\, p = \vec{H}(y,t) \\ \dfrac{\partial \xi}{\partial t} + \operatorname{div}_y\, (\xi\, \vec{w}) = 0, \end{cases}$$

where

$$\vec{w}(y,t) = \vec{u}(y,t) + [\vec{\omega}, \vec{y}] = \vec{v}(S^* y, t)$$

$$p(y,t) = P(S^* y, t)$$

with S being a rotation matrix of the following form

$$S = S(t) = \begin{pmatrix} \mathrm{Cos}\, \omega t & -\mathrm{Sin}\, \omega t & 0 \\ \mathrm{Sin}\, \omega t & \mathrm{Cos}\, \omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Notice that the equation (3.6) could be derived from equation (3.1) by the change of variables

$$y = S(t)\, x,$$

where S is an orthogonal matrix :

$$S^{-1} = S^*$$

In equations (3.6), the pressure $p(y, t)$ does not involve the centrifugal force, hence $p(y, t)$ is bounded at infinity as $P(x, t)$, with the velocity $\vec{w} = \vec{w}(y,t)$ being in fact a velocity of the fluid with respect to Ox, but measured in Oy. And only at this point one can linearize equations (3.6) for sufficiently small velocities :

(3.7) $$\begin{cases} \dfrac{\partial \vec{w}}{\partial t} - 2[\vec{w}, \vec{\omega}] + \dfrac{1}{\xi}\nabla_y\, p = H(y,t) \\ \dfrac{\partial \xi}{\partial t} + \operatorname{div}_y\, (\xi\, \vec{w}) = 0. \end{cases}$$

The equations (3.7) of dynamics are supplemented by the equation of heat transfer for continuous medium :

(3.8) $$\frac{\partial T}{\partial t} + (\vec{w}, \nabla_y\, T) = f(y,t),$$

where

T is the temperature,

f is the heat source density.

The difference in temperature at different points creates a force acting upon the particles of fluid. One of the simplest models of such force [4] is

$$\vec{H}(y,t) = \vec{h}(y,t) - \vec{g}\left(\frac{T-T_0}{T_0}\right), \tag{3.9}$$

where

T_0 is some standard temperature,
$T - T_0$ is the deviation of the temperature T from T_0 ,
$\vec{h}(y,t)$ is the mass density of external forces,
$\vec{g}$ is the vector of free fall acceleration

Equation (3.9) describes the free heat convection in the Earth's atmosphere. We consider the simple case of vectors $\vec{g}$ and $\vec{\omega}$ being collinear.

Now, we restrict ourselves to the case of incompressible fluid, and without loss of generality, we assume $\xi = 1$. So. equations (3.7) – (3.9) take the form :

$$\begin{cases} \dfrac{\partial \vec{w}}{\partial t} - 2\,[\vec{w},\vec{\omega}] + \nabla_y p + \dfrac{T'}{T_0}\,\vec{g} = \vec{h}(y,t) \\ \operatorname{div} \vec{w} = 0 \\ \dfrac{\partial T'}{\partial t} + (\vec{w}, \nabla_y T') = f(y,t), \end{cases} \tag{3.10}$$

where $T' = T - T_0$ Equations (3.10) represent a simplified model for general circulation in Earth's atmosphere and Oceans [6, 7, 10]. Marchhuk and others were the first to introduce and study the equations (3.10). Another version of simplified equations may be found in the recent works of Lion, Temam and Wang [6].

Substituting $\vec{V}, P, T, \vec{\omega}$ and $\vec{F}$ for $\vec{w}, p, T', 2\vec{\omega}$ and $\vec{h}$ respectively, we get

$$\begin{cases} \dfrac{\partial \vec{V}}{\partial t} - [\vec{V},\vec{\omega}] + \nabla p + \vec{g}\left(\dfrac{T}{T_0}\right) = \vec{F} \\ \operatorname{div} \vec{V} = 0 \\ \dfrac{\partial T}{\partial t} + (\vec{V}, \nabla T) = f \end{cases} \tag{3.10$'$}$$

Recall that the vector product

$$[\vec{v},\vec{\omega}] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ V_1 & V_2 & V_3 \\ 0 & 0 & \omega \end{vmatrix} = \omega V_2 \vec{i} - \omega V_1 \vec{j} + 0\vec{k}$$

Let $\vec{V} = (\vec{V}_1, \vec{V}_2, \vec{V}_3)$, $\vec{F} = (F_1, F_2, F_3)$ and $\frac{g}{T_0} = \sigma$, then the dynamics equations in $(3.10')$ can be rewritten as follows :

$$(3.11) \qquad \begin{cases} \frac{\partial V_1}{\partial t} - \vec{\omega} V_2 + \frac{\partial P}{\partial x_1} = F_1 \\ \frac{\partial V_2}{\partial t} + \vec{\omega} V_1 + \frac{\partial P}{\partial x_2} = F_2 \\ \frac{\partial V_3}{\partial t} + \frac{\partial P}{\partial x_3} + \sigma T = F_3 \end{cases}$$

Taking into account only the vertical gradients of temperature, we find

$$\begin{aligned} (\vec{V}, \nabla T) &= V_1 \frac{\partial T}{\partial x_1} + V_2 \frac{\partial T}{\partial x_2} + V_3 \frac{\partial T}{\partial x_3} \\ &= V_3 \frac{\partial T}{\partial x_3}, \text{ taking } \frac{\partial T}{\partial x_1} = \frac{\partial T}{\partial x_2} = 0 \\ &= -r V_3 \ \ (r > 0), \end{aligned}$$

where $r = -\frac{\partial T}{\partial x_3}$ is being treated as a prescribed constant. The heat transfer equation in $(3.10')$ now reduces to the form

$$(3.12) \qquad \frac{\partial T}{\partial t} - r V_3 = f.$$

Thus, we have derived the linear system of partial differential equations

$$(3.13) \qquad \begin{cases} \frac{\partial V_1}{\partial t} - \omega V_2 + \frac{\partial P}{\partial x_1} = F_1 \\ \frac{\partial V_2}{\partial t} + \omega V_1 + \frac{\partial P}{\partial x_2} = F_2 \\ \frac{\partial V_3}{\partial t} + \sigma T + \frac{\partial P}{\partial x_3} = F_3 \\ \frac{\partial T}{\partial t} - r V_3 = f \\ \operatorname{div} \vec{V} = 0. \end{cases}$$

System (3.13) represents a simplified linear model for general circulation of the atmosphere and ocean [1].

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Probability Models For The Number Of Rural Out–Migrants At Micro–Level

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Abstract: Probability models have been proposed to describe the distribution of rural out-migrants from rural areas of Palpa and Rupendehi districts of Mid-western Nepal. The maximum likelihood estimation technique has been used to estimate the parameters involved in the models and asymptotic variances and co-variances of the estimators have also been obtained. The proposed models fit the data sets satisfactorily well.

1. Introduction.

Demographers and social scientists have given their due attention on the formulation of models and their applications due to its usefulness and applicability in social sciences. Several studies in different regions of the developing countries have dealt with the economic aspects of migration. However, majority of them have concentrated with the differentials & determinants of migration focusing mainly on causes and consequences of migration (Afsar, 1995 ; Mehta 1991 ; Mehta and Kohli, 1991 ; Wintle, 1992 ; Yadava, 1987). Apart from social and economic impact, migration of an individual produces a demographic impact as well on his/her household at the place of origin. The physical separation between husband and wife as a result of migration gives the female partner less scope for conception that results

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in low fertility of the household (Sharma, 1992). Therefore, it is important to understand intention of migration, extent of migration and its effect on the growth of urban population for a proper urban planning as well as for furthering rural development. The macro-level migration studies have their own importance since this approach describes aggregate flow or rate of migration and identify the factors influencing out-migration (Benerjee, 1986). On the other hand, micro-level (i.e. at the household level or individual level) studies help to identify the behavioural phenomena of the process.

Usually, two types of households are observed according to the occurrence of migration i.e. (i) male members aged 15 years and above migrate singly leaving their wives and children at home and (ii) male members migrate with their wives, children and other dependent members. It is to be mentioned here that persons migrating from both types of households maintain close tie with remaining (non-migrant) members of the households in their place of origin through regular visit and sending remittances. It is obvious fact that the first category maintains closer tie than the second category of household. Basically mathematical models are being frequently used in social sciences to provide concise representations of extensive data sets. On the view of this, several attempts have been made to study the pattern of rural out-migration through the use of probability models (Iwunor, 1995; Sharma, 1988 ; 1985 ; Yadava, 1993 ; Yadava and Singh, 1983 ; Yadava *et al.* 1991 ; 1994). Singh and Yadava, (1981) proposed negative binomial distribution to describe the trend of rural out-migration for male migrants aged 15 years and above. The idea of cluster was incorporated by Yadava and Singh (1983) and found that Thomas distribution is well suited to describe the number of migrants from a household. Singh and Yadava (1988) extended the idea of cluster and assumed that the occurrence of migration in cluster varies from household to household and the number of migrants to a cluster follows truncated displaced geometric distribution. A probability distribution under such assumptions fitted well the distribution of male migrants aged 15 years and above.

Most of the studies mentioned above have used moment technique or mean-zero frequency method (equating observed and theoretical zero'th cell frequencies and means) to estimate the parameters involved in their proposed models. Under these techniques of estimation, it is observed that about 80 to 85 percent variation in migration is equated through zero'th cell frequencies since all the non-migrant households are counted in this cell. So, only about 15 to 20 percent variations are explained by the estimated parameters when mean-zero frequency method is applied. Further, moment estimates are usually consistent, but they are often less efficient. Considering these limitations into account, the maximum likelihood estimation technique is proposed in this study to estimate the parameters involved in the models. Needles to mention that the maximum likelihood method has the advantage that the standard error of the estimators can also be obtained and this

method measures the total variation of the distribution. Further the estimates obtained by this method have the optimum properties in terms of consistency and efficiency.

This paper is concerned with mathematical modelling to study the pattern of rural out-migration in the developing countries like Nepal. The proposed model is to describe the distribution of male migrants aged 15 years and above. Maximum likelihood estimates technique has been used to estimate the parameter involved in the model. The standard errors of the estimated parameters have also been obtained. The models have been applied to the data collected from rural areas of Palpa and Rupandehi distracts of Mid-western Nepal.

2. Probability Models for Male Migrants Aged 15 Years and Above

Model M_1

A probability model for describing the variation in the number of rural male out-migrants aged 15 years and above from a households has been derived on the basis of following assumptions:

(i) At the survey point, the household is either exposed to the risk of migration or it is not exposed to the migration risk. Let α and $(1-\alpha)$ be the respective probabilities.

(ii) The probability of migrating one male form a household is greater than the probability of migrating two males and probability of two males migrating is greater than that of three males from a household and so on. Thus, the pattern of migration from a household is a decreasing function and follows a logarithmic series distribution with parameter λ.

Let X represent the number of male rural out-migrants aged 15 years and above from a household, then under the assumptions (i) and (ii), the probability function of x is given by

$$(1) \qquad \left.\begin{aligned} P(X=k) &= 1-\alpha, \qquad \text{for } k=0 \\ &= \alpha\left[\frac{-\lambda^k}{k\log(1-\lambda)}\right] \text{ for } k=1,2,3,\ldots;\, 0<\lambda<1;\, 0<\alpha<1 \end{aligned}\right\}$$

The log-series distribution has a long positive tail and the shape of the tail is similar to that of geometric distribution for large values of k. The log-series distribution have the advantage that it has only one parameter instead of two parameters of Negative Binomial Distribution (Chatfield *et al*, 1966).

Estimation of Parameters

Consider a random sample $X_1, X_2, \ldots, X_n$ of n observations on the random variable X with probability function given in expression (1). Then, each X_i counts the number of male rural out-migrants aged 15 years and above from a household. Suppose that n_k (k = 0, 1, 2, . . . , m) represents the number of observations with value k and $\sum_{k=0}^{m} n_k = n$. The likelihood function for the given sample $(X_1, X_2, \ldots, X_n)$ can be expressed as :

$$\text{(2)} \qquad L[\alpha, \lambda \mid (X_1, X_2, \ldots, X_n)] = (1-\alpha)^{n_0} \prod_{k=1}^{m} \left[\alpha \left(\frac{-\lambda^k}{k \log (1-\lambda)} \right) \right]^{n_k}$$

$$= \frac{(1-\alpha)^{n_0} (-\alpha)^{n-n_0} \lambda^{\sum_{k=1}^{m} n_k k}}{\left(\prod_{k=1}^{m} k^{n_k} \right) [\log(1-\lambda)]^{n-n_0}}$$

$$\text{(3)} \qquad = \frac{(1-\alpha)^{n_0} (-\alpha)^{n-n_0} \lambda^{\sum_{i=1}^{n} X_i}}{\left(\prod_{k=1}^{m} k^{n_k} \right) [\log (1-\lambda)]^{n-n_0}}$$

Taking logarithms of (3) and differentiating with respect to α and λ respectively and equating to zero gives the following estimating equations :

$$\text{(4)} \qquad \frac{\partial \log L}{\partial \alpha} = -\frac{n_0}{1-\alpha} + \frac{n-n_0}{\alpha} = 0$$

$$\text{(5)} \qquad \frac{\partial \log L}{\partial \alpha} = \frac{\sum_{i=1}^{n} X_i}{\lambda} + \frac{n-n_0}{(1-\lambda) \log (1-\lambda)} = 0$$

The equation (4) yields the estimator of α as

$$\hat{\alpha} = \frac{n-n_0}{n}$$

The estimating equation for λ is obtained by solving equation (5) as :

$$\text{(6)} \qquad (1-\lambda) \log (1-\lambda) \sum_{i=1}^{n} X_i + (n-n_0) \lambda = 0$$

where $\sum_{i=1}^{n} X_i = \sum_{k=1}^{m} n_k\, k$

This equation can not be solved analytically, but it can be solved numerically and the numerical solution of (6) is the desired maximum likelihood estimate for λ.

The second partial derivatives of log L is as follows :

(7) $$\frac{\partial^2 \log L}{\partial \alpha^2} = -\frac{n_0}{(1-\alpha)^2} - \frac{n-n_0}{\alpha^2}$$

(8) $$\frac{\partial^2 \log L}{\partial \lambda^2} = \frac{\sum_{i=1}^{n} X_i}{\lambda^2} + \frac{(n-n_0)\,[1+\log(1-\lambda)]}{[(1-\lambda)\log(1-\lambda)]^2}$$

(9) $$\frac{\partial^2 \log L}{\partial \alpha \partial \lambda} = 0$$

Using the fact that $E(n_0) = E\,[\sum_{i=1}^{n} 1_{\{X_i=0\}}] = \sum_{i=1}^{n} 1\,p(X_i = 0) = \sum_{i=1}^{n}(1-\alpha) = n(1-\alpha)$

and $E(n-n_0) = n - E(n_0) = n\alpha$

$$E(k) = \frac{-\alpha\lambda}{(1-\lambda)\log(1-\lambda)} \text{ and } E\left(\sum_{k=1}^{m} n_k k\right) = -\frac{n\alpha\lambda}{(1-\lambda)\log(1-\lambda)}$$

The expected values of second partial derivatives is obtained as :

$$-E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) = -\frac{E(n_0)}{(1-\alpha)^2} - \frac{E(n-n_0)}{\alpha^2} = \frac{n}{\alpha(1-\alpha)} = \varphi_{11} \text{ (say)}$$

$$-E\left(\frac{\partial^2 \log L}{\partial \lambda^2}\right) = -\frac{E\left(\sum_{k=1}^{n} n_k k\right)}{\lambda^2} + \frac{[1+\log(1-\lambda)]\,E(n-n_0)}{[(1-\lambda)\log(1-\lambda)]^2}$$

$$= -n\alpha\left[\frac{1}{\lambda(1-\lambda)\log(1-\lambda)} + \frac{1+\log(1-\lambda)}{[(1-\lambda)\log(1-\lambda)]^2}\right] = \varphi_{22} \text{ (say)}$$

The co-ovariance between α and λ is zero since $E\left(\frac{\partial^2 \log L}{\partial \alpha \partial \lambda}\right) = 0$ and hence the variance of α and λ can be obtained as $v(\hat{\alpha}) = \frac{1}{\phi_{11}}$ and $v(\hat{\lambda}) = \frac{1}{\phi_{22}}$.

Model M_2

Sharma (1985) has proposed a probability model for the number of rural male out-migrants aged 15 years and above from a household under following assumptions :

(i) At any point in time, let α be the probability migrating out from a household and $1-\alpha$ be the probability of not migrating from a household.

(ii) If p represents the probability of a single individual migrating from a household, the pattern of migration from each household follows the geometric distribution.

If X represent the number of rural male out-migrant from a household, then X follows the inflated geometric distribution with probability density function :

$$\left.\begin{aligned} P(X=0) &= 1-\alpha+\alpha\, p \\ P(X=k) &= \alpha\, q^k\, p \ \text{ for } \ k=1,2,3,\dots \end{aligned}\right\} \tag{10}$$

where $p+q=1$.

As mentioned above, Sharma (1985) used method of moments to estimate the parameters α and p of model (10) and obtained the asymptotic expressions for variance and covariance of the estimators using multivariate central limit theorem. Iwunor (1995) proposed an alternative estimation technique based on likelihood function and obtained the variance and covariance of the estimators. Though he used the likelihood function using multinomial combination, but finally estimated the parameters by mean-zero frequency method. An alternative estimation techniques based on likelihood function is worked out by using all the observations to estimate the parameters. The expressions for exact variance and covariance of the estimators have also been derived.

Estimation of Parameters

Let $(X_1, X_2, \dots, X_n)$ denote a random sample of size n from the above probability model. Then, each X_i counts the number of male rural out-migrants aged 15 years and above from household. Further, assume that n_k $(k=0,\ 1,2,\dots,m)$ denote the number of observations with value k. The likelihood function for estimating the parameters α and p can be expressed as :

$$\begin{aligned} L &= (1-\alpha+\alpha\, p)^{n_0} \prod_{k=1}^{m} (\alpha\, pq^k)^{n_k} \\ &= (1-\alpha+\alpha\, p)^{n_0}\ \alpha^{n-n_0}\ p^{n-n_0}\ q^S \end{aligned} \tag{11}$$

where, $S = n_1 + 2n_2 + 3n_3 + \dots + mn_m = \sum_{k=1}^{m} n_k k$ and $n_0 + n_1 + n_2 + \dots + n_m = n$

Taking logarithms of the likelihood function (11) and differentiating with respect to α and p respectively and equating to zero gives the following estimating equations :

$$\frac{\partial \log L}{\delta \alpha} = \frac{n_0(p-1)}{(1-\alpha+\alpha p)} + \frac{n-n_0}{\alpha} = 0 \tag{12}$$

$$\frac{\partial \log L}{\delta p} = \frac{n-n_0}{p} - \frac{s}{1-p} + \frac{n_0 \alpha}{(1-\alpha+\alpha p)} = 0 \tag{13}$$

Solution of equation (12) provides the estimator of α as

$$\hat{\alpha} = \frac{n-n_0}{n(1-p)}$$

Substituting the value of α and after rearranging equation (13) yields the estimator of p as :

$$\hat{p} = \frac{n-n_0}{\sum_{k=1}^{m} n_k k}$$

The second partial derivatives of $Log\, L$ can be obtained as

$$\frac{\partial^2 \log L}{\delta \alpha^2} = \frac{-n_0(p-1)^2}{(1-\alpha+\alpha p)^2} - \frac{(n-n_0)}{\alpha^2} \tag{14}$$

$$\frac{\delta^2 \log L}{\delta p^2} = -\frac{n-n_0}{p^2} - \frac{S}{(1-p)^2} - \frac{n_0 \alpha^2}{(1-\alpha+\alpha p)^2} \tag{15}$$

$$\frac{\delta^2 \log L}{\delta \alpha \partial p} = \frac{n_0}{(1-\alpha+\alpha p)^2} \tag{16}$$

Using the fact $E(n_0) = E\left[\sum_{i=1}^{n} 1_{\{X_i=0\}}\right] = \sum_{i=1}^{n} 1 p(X_i = 0) = \sum_{i=1}^{n}(1-\alpha+\alpha p)$ $= n(1-\alpha+\alpha p)$ and $E(n-n_0) = n - E(n_0) = n\alpha(1-p)$, the expected value of second partial derivatives of $Log\, L$ can be obtained as

$$-E\left(\frac{\delta^2 \log L}{\delta \alpha^2}\right) = \frac{n(1-p)}{\alpha(1-\alpha+\alpha p)} = \phi_{11} \text{ (say)}$$

$$-E\left(\frac{\delta^2 \log L}{\delta p^2}\right) = \frac{n\alpha q}{p} + \frac{n\alpha}{pq} + \frac{n\alpha^2}{1-\alpha+\alpha p} = \phi_{22} \text{ (say)}$$

$$-E\left(\frac{\delta^2 \log L}{\delta\alpha\partial p}\right) = -\frac{n}{(1-\alpha+\alpha p)} = \phi_{12} \quad \text{(say)}$$

Therefore, by inverting the information matrix, the expressions for asymptotic variances and covariance of the estimators can be obtained as $V(\hat{\theta}) = \dfrac{\phi_{22}}{\phi_{11}\,\phi_{22} - \phi_{12}^2}$,

$V(\hat{p}) = \dfrac{\phi_{11}}{\phi_{11}\,\phi_{22} - \phi_{12}^2}$ and $\operatorname{cov}(\hat{\theta}, \hat{p}) = \dfrac{\phi_{12}}{\phi_{11}\,\phi_{22} - \phi_{12}^2}$

3. Application

Models M_1 and M_2 are applied to the data collected from rural areas of Rupandehi and Palpa districts of Mid-western Nepal. These data were collected under a sample survey "Demographic Survey on Fertility and Mobility in Rural Nepal (DSFM, 2000) : A Study of Palpa and Rupandehi Districts" during January-June, 2000. The detail about the sample survey is discussed in Aryal (2002).

Table 1 shows the estimated value of the parameters, variances and covariance, observed and expected number of households according to the number of male migrants (aged 15 years and above) by models M_1 and M_2 for rural household of Nepal.

The estimated value of risk parameter α under model M_1 was found 0.2318126. The corresponding values for model M_2 were found to be 0.7004229. The estimated value of λ was found to be 0.5306358 from the model M_1. The estimated value of p was found to be 0.6090391 from the model M_2.

The value of χ^2 for the models M_1 and M_2 were found to be 1.89 and 0.05 respectively, which is insignificant at 5 per cent level of significance. This indicates that the validity of the proposed model was found to be a reasonable approximation to describe the pattern of rural out-migration to the situation at least at the micro-level. However, the χ^2 value suggest that the model M_2 was found to be more suitable for Nepal data then the model M_1.

4. Conclusions

The study indicates that the proposed models M_1 and M_2 are a reasonable approximation to describe the distribution of household for the male migrants aged 15 years and above at least at the micro-level. The exact variances and co-variance of the estimators for both the models have also been computed. For the development of a more effective and equitable rural and urban policies in the developing countries like Nepal, the policy planners and social researchers may get an idea from this study.

Table 1 Distribution Of Observed And Expected Number Of Households According To The Number Of Male Migrants (Aged 15 Years and Above) In Rural, Nepal.

Number of migrants per household	Observed	Expected (Model M_1)	Expected (Model M_2)
0	623	623.00	623.00
1	126	132.01	125.80
2	42	35.03	41.63
3	13	12.39	13.78
4	4	8.57	6.79
5	2		
6	1		
7	0		
Total	811	811.00	811.00
χ^2		1.89	0.05
d.f		2	2
$\hat{\alpha}$		0.2318126	0.7004229
$\hat{\lambda}$		0.5306358	
$\hat{p}$			0.6690391
$v(\hat{\alpha})$		0.0000744	0.0031271
$v(\hat{\lambda})$		0.0004415	
$v(\hat{p})$			0.0001021
$Cov(\hat{\alpha}, \hat{\lambda})$		0.0000001	
$Cov(\hat{\alpha}, \hat{p})$			0.0005707

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Pasch Geometric Spaces As Orbits Of Vector Spaces

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Abstract: Pasch geometric spaces over geometric skewfields generalise the usual vector spaces over skewfields. It is shown in this paper that an abstract Pasch geometric space of appropriate dimension over a geometric skewfield may be represented as orbits of a usual vector space over a skewfield.

1. Introduction

The set of double cosets of a group with respect to a subgroup and the set of orbits of a group with respect to a group of automorphisms inherit certain structures from the group which have been studied as Pasch geometries. In particular, the category of the projective spaces form a subcategory of the category of Pasch geometries (see [5]). The study of projective geometry in the language of Pasch geometry has the advantage of dealing with morphisms and homomorphisms in a natural way with availability of homomorphism theorems similar to other algebraic structures [2]. The points of the projective space $P(V)$ of a vector space V over a (skew)-field F can be taken as the set of orbits V/F^* and the inherited structure provides the geometry of the classical projective space. However, if Γ is a normal subgroup of the multiplicative group F^*, then the geometry of orbits V/Γ becomes a Pasch geometric space over the geometric skewfield F/Γ [3]. Now a natural converse question would be: what are the necessary and sufficient conditions for an abstract geometric space over a geometric skewfield to be realized as an orbit space of a vector space over a skewfield as above ? It is classically known that an abstract projective space of sufficient dimension can be realized as $P(V)$, which is the orbit geometry V/Γ, when $\Gamma = F^*$, the whole multiplicative group of the nonzero elements of the skewfield F. So generalizing the fundamental theorem of projective geometry, it is shown here that an abstract Pasch geometric space of appropriate dimension can be represented as V/Γ over geometric skewfield F/Γ. A similar result has been proved in [4] for an elementary abelian Pasch geometry. We wish to point out that

the prototypes of Pasch geometries are the double cosets and orbits of groups and that the above results is a step to the broader question : what are the necessary and sufficient conditions for a Pasch geometry to be represented as the orbit space or the double cosets of a group ?

2. Preliminaries

In this section we briefly present the basic concepts and preliminary results on Pasch geometries. The details can be found in the references, particularly in [5].

Definition 2.1. By a Pasch geometry is meant a triple (A, e, Δ) where A is a set, $e \in A$, and $\Delta_A = \Delta \subseteq A \times A \times A$ subject to the following axioms :

1. $\forall a \in A, \exists$ a unique $b \in A$ with $(a, b, e) \in \Delta$. Let $b = a^\#$.
2. $e^\# = e$ and $(a^\#)^\# = a \forall a \in A$.
3. $(a, b, c) \in \Delta \Rightarrow (b, c, a) \in \Delta$.
4. $(a_1, a_2, a_3), (a_1, a_4, a_5) \in \Delta \Rightarrow \exists a_6 \in A$ with $(a_6, a_4^\#, a_2), (a_6, a_5, a_3^\#) \in \Delta$

The identity element e and the inverse $a^\#$ are unique. Throughout this paper, geometry will mean Pasch geometry.

A geometry is called *Abelian* if $(a, b, c) \in \Delta \Rightarrow (b, a, c) \in \Delta$. A geometry is called *sharp* if $(a, b, c), (a, b, d) \in \Delta \Rightarrow c = d$. Also, a geometry is called *projective* if $a^\# = a \forall a \in A$ and $(a, a, b) \in \Delta \Rightarrow b = e$ or $b = a$.

For an abelian geometry, we have a useful :

Lemma 2.2. Let A be an abelian geometry. Then $(x_1, s_1, t_1), (x_2, s_2, t_2), (x_1, x_2, x_3) \in \Delta \Rightarrow \exists s_3, t_3 \in A$, such that $(s_1, s_2, s_3), (t_1, t_2, t_3), (s_3, t_3, x_3) \in \Delta$.

Examples 2.3.

1. Let G be a group. Define $(a, b, c) \in \Delta$ if and only if $a.b.c = 1$, the identity of G. Then G becomes a sharp geometry with $e = 1$ and $a^\# = a^{-1}$ Conversely every sharp geometry is a group.

2. Let P be the set of points of a projective space. Let $A = P \cup \{e\}, e \notin P$. On A, let $(a, b, c) \in \Delta$ if and only if : a, b, c are distinct and collinear points of P ; or one of the elements a, b, c of A is e and the other two are equal ; $a = b = c$ if the lines of P have more than three points. It verifies that A is a geometry which is projective. Conversely, every geometry which is projective is a projective space including degenerate ones.

2.1. Subgeometry and Factor geometry. Let A be a geometry and $B \subseteq A$. Then B is called a subgeometry if $e \in B$ and $(b_1, b_2, x) \in \Delta$, $b_1, b_2 \in B \Rightarrow x \in B$. Let

$\Delta_B = \Delta_A \cap (B \times B \times B)$. Then (B, e, Δ_B) is a geometry. The subgeometry B is called normal if $\forall x, y \in A, (b,x,y) \in \Delta$, for some $b \in B \Rightarrow \exists b_1 \in B$ with $(b_1, y, x) \in \Delta$.

Let B be a subgeometry of A. For $a, b \in A$, define $a \sim b$ if $\exists b_1, b_2 \in B$ and $x \in A$ such that $(a, b_1, x^{\#})$, $(x, b^{\#}, b_2) \in \Delta$. This defines an equivalence relation on A. Let $A//B = \{[a] : a \in A\}$ be the set of all equivalence classes. Let ([a], [b], [c]) $\in \Delta_{A//B}$ if $\exists x \in [a]$, $y \in [b]$, $z \in [c]$ with $(x, y, z) \in \Delta_A$. Then $A//B$ is a geometry, the factor geometry.

In particular, if $A = G$ is a group and $B = H$ is a subgroup, then the geometry $G//H$ is the geometry of double cosets.

2.2. Morphisms and Homomorphisms. Let $f : A \to B$ be a map between geometries. It is called a morphism if $f(e_A) = e_B$ and $(x, y, z) \in \Delta_A \Rightarrow (f(x), f(y), f(z)) \in \Delta_B$. If, in addition, $(f(x), f(y), b) \in \Delta_B. \Rightarrow \exists z \in A$ with $f(z) = b$ and $(x, y, z) \in \Delta_A$, then the map f is called a homomorphism. A bijective morphism is an isomorphism, if f^{-1} is also a morphism. However, a bijective homomorphism is an isomorphism.

The natural map $A \to A//B$ is a morphism and is a homomorphism only when B is normal in A.

2.3. Geometry of Orbits. Let A be a geometry. A group Γ is said to act on A if there is a homomorphism from Γ to the geometry automorphisms of A. Thus for $\alpha \in \Gamma$ and $a \in A$, we get $\alpha a \in A$ satisfying obvious properties. In such case, we call A a Γ-geometry. For $a \in A$, let $\langle a \rangle = \{\alpha a : \alpha \in \Gamma\}$ denote the orbit of a and $A/\Gamma = \{\langle a \rangle : a \in A\}$ be the set of orbits. Let $(\langle a \rangle, \langle b \rangle, \langle c \rangle) \in \Delta_{A/\Gamma}$ iff $\exists x \in \langle a \rangle, y \in \langle b \rangle, z \in \langle c \rangle$ with $(x, y, z) \in \Delta_A$. This makes A/Γ a geometry called the geometry of orbits of A by Γ. In particular, if V is a (left) vector space over a skewfield F and Γ is a normal subgroup of F^*, then we get the geoemetry of orbits V/Γ. If $\Gamma = F^*$, then the geometry V/F^* is that of the classical projective space $P(V)$.

2.4. Geometric spaces over geometric skewfields. Let $(A, 0_A, \Delta_A)$ be an abelian geometry. Suppose, in addition, $(A,.)$ is a semigroup with 1 such that $0.a = a.0 = 0$. It is called a geometric ring if $(a, b, c) \in \Delta, x \in A \Rightarrow (ax, bx, cx)$, $(xa, xb, xc) \in \Delta_A$. It is called a geometric sfield if $A^* = A - \{0\}$ is a group. Suppose $(V, 0_V, \Delta_V)$ is an abelian geometry and the geometric sfield A acts on V compatibly as scalars satisfying : $a(bv) = (ab)v$; $0_A . v = a . 0_V = 0$; $1 . v = v$; $(u, v, w) \in \Delta_V \Rightarrow (au, av, aw) \in \Delta_V$; $(a, b, c) \in \Delta_A \Rightarrow (av, bv, cv) \in \Delta_V$; $(av, bv, cv) \in \Delta_V, v \neq 0 \Rightarrow (a, b, c) \in \Delta_A$; $(av, bv, w) \in \Delta_V \Rightarrow w = cv$ for some c ; where $a, b, c \in A$ and $u, v, w \in V$. Then V is said to be a geometric space over the geometric sfield A. In case

V and A have sharp geometries, the geometric space V is a vector space over the usual skewfield A.

Suppose V is a geometric space over a geometric sfield A, $W \subseteq V$. Then W is a subspace of V if W is a subgeometry of V and $a \in A$, $w \in W \Rightarrow aw \in W$. If $X \subseteq V$, then $sp(X)$ denotes the smallest subspace containing X. It verifies, using lemma 2.2, that $sp\,(v) = \{av : a \in A\}$, $sp\,(v_1, v_2) = \{v : (v, av_1, bv_2) \in \Delta,\ a, b \in A\}$ and if W is a subspace, then $sp\,(\{u\} \cup W) = \{v : (v, au, w) \in \Delta,\ a \in A,\ w \in W\}$. This definition of spanning satisfies all the formal axioms of [7] and consequently there exists a basis and a well defined dimension of V over A. If $(v, av_1, bv_2) \in \Delta, v_1, v_2$ are independent, then a, b are unique.

Example 2.4.

Let V be a vector space over a skewfield F and Γ be a normal subgroup of F^*. Then V/Γ and F/Γ are geometries. Note that $(\bar{u}, \bar{v}, \bar{w}) \in \Delta_{V/\Gamma}$ iff $u + \gamma_1 v + \gamma_2 w = 0$ for some $\gamma_1, \gamma_2 \in \Gamma$. For $\bar{a}, \bar{b} \in F/\Gamma$, let $\bar{a}.\bar{b} = \overline{ab}$. It is well defined and makes F/Γ into a geometric sfield and for $\bar{a} \in F/\Gamma$ and $\bar{v} \in V/\Gamma$, $\bar{a}.\bar{v} = \overline{av}$ makes V/Γ into a geometric space over F/Γ.

More generally, if V is a geometric space over a geometric sfield A, and Γ is a normal subgroup of A^*, then V/Γ is a geometric space over A/Γ, Then $v_1, v_2 \ldots v_n$ in V are independent over A iff $\bar{v}_1, \bar{v}_2, \ldots \bar{v}_n$ are independent in V/Γ over A/Γ. So $\dim_A V = \dim_{A/\Gamma} V/\Gamma$. In particular, the geometry of orbits $(V/\Gamma)/(A^*/\Gamma) \cong V/A^*$ is projective and so represents a projective space of the same dimension. Note that $(\bar{v}_1, \bar{v}_2, \bar{v}_3) \in \Delta_{V/\Gamma}$ iff $\exists \gamma_1, \gamma_2 \in \Gamma$ with $(v_1, \gamma_1 v_2, \gamma_2 v_3) \in \Delta_V$. Also $\overline{v^\#} = \bar{v}^\#$.

2.5. Semi-isomorphism. Let V and W be geometric spaces over A and B respectively. A pair of maps $(\sigma, \hat{\sigma}) : (V, A) \to (W, B)$ is called a semi-isomorphism if $\sigma : V \to W$ is an isomorphism of geometries, $\hat{\sigma} : A \to B$ is an isomorphism of geometric sfields. (i.e. isomorphism of geometries, with $\hat{\sigma}(ab) = \hat{\sigma}(a)\,\hat{\sigma}(b)$) and $\sigma(av) = \hat{\sigma}(a)\,\sigma(v)\ \forall v \in V,\ \forall a \in A$. Clearly if $v_1, v_2 \ldots v_r$ are independent in V, then $\sigma(v_1), \sigma(v_2), \ldots, \sigma(v_r)$ will be independent in W. So a semi-isomorphism preserves basis and dimension. In particular, if V and W are sharp and so vector spaces, then a semi-isomorphism is a bijective semi-linear transformation.

3. Generalization Of The Fundamental Theorem Of Projective Geometry

In this section, W is a geometric space over a geometric sfield A, $\dim_A W \geq 3$. The following theorem is a generalization of the fundamental theorem of projective geometry.

Theorem 3.1. *Let V be a vector space over a skewfield F, $\dim_F V \geq 3$ and W a geometric space over a geometric sfield A. Let Γ_1 and Γ_2 be normal subgroups of F^* and A^* respectively. Let*

$$(\sigma, \hat{\sigma}) : (V/\Gamma_1, F/\Gamma_1) \to (W/\Gamma_2, A/\Gamma_2)$$

be a semi-isomorphism between geoemtric spaces. Then there exists a normal subgroup Γ of $F^*, \Gamma \subseteq \Gamma_1$ and a semi-isomorphism

$$(\psi, \hat{\psi}) :: (V/\Gamma, F/\Gamma) \to (W, A)$$

such that it induces the given semi-isomorphism as : $\overline{\psi(\tilde{v})} = \sigma(\bar{v})$ and $\overline{\hat{\psi}(\tilde{a})} = \hat{\sigma}(\bar{a})$.

Proof: For $v \in V, a \in F, x \in W, \alpha \in A$, we will denote by $\bar{v}$, $\bar{a}$, $\bar{x}$, $\bar{\alpha}$ the corresponding elements of $V/\Gamma_1, F/\Gamma_1$, $W/\Gamma_2, A/\Gamma_2$ respectively. The proof follows steps similar to those of theorem (2.1) of [4].

Step 1: Let $v \in V, v \neq 0$ and $x \in W$ such that $\sigma(\bar{v}) = \bar{x}$. For any $u \in V$, u, v independent, $\exists$ unique $y \in W$ such that $\sigma(\bar{u}) = \bar{y}$, $\sigma(v-u) = \bar{t}$ and $(x^{\#}, y, t) \in \Delta$

Proof: We have $(\overline{-v}, \bar{u}, \overline{v-u}) \in \Delta_{V/\Gamma_1}$, so $(\sigma(\overline{-v}), \sigma(\bar{u}), \sigma(\overline{v-u})) = (\bar{x}'', \bar{y}_1, \bar{t}_1)$ $\in \Delta_{W/\Gamma_2}$. So $\exists \gamma_1, \gamma_2 \in \Gamma_2$ with $(x^{\#}, \gamma_1 y_1, \gamma_2 t_1) \in \Delta_W$. Then $\gamma_1 y_1 = y$ and $\gamma_2 t_1 = t$ are as required. Note that y, t are independent. If also $(x^{\#}, y', t') \in \Delta$ with similar properties, then $\exists s \in W$ with $(s, y'^{\#}, y)$, $(s, t', t^{\#}) \in \Delta$. If $s \neq 0$, then as $y' = \gamma\, y$, we get $s = \alpha y$ and similarity $s = \beta t$, contradicting independence. So, $s = 0$ giving uniqueness : $y' = y$, $t = t'$.

Thus given $u, v \in V, x \in W$ as above, we have $\phi_{(v,x)}(u) = y$, a unique element with $\sigma(\bar{u}) = \bar{y}$. It is easily seen that $\phi_{(v,x)}(u) = y \Leftrightarrow \phi_{(u,y)}(v) = x$.

Step 2: Let u, v, w be independent in V. Then $\phi_{(u,x)}(v) = y$, $\phi_{(u,x)}(w) = z \Rightarrow$ $\phi_{(v,y)}(w) = z$.

Proof: Given that $\sigma(\bar{u}) = \bar{x}$, $\sigma(\bar{v}) = \bar{y}$, $\sigma(\bar{w}) = \bar{z}$, $\sigma(\overline{u-v}) = \bar{t}_1$, $\sigma(\overline{u-w}) = \bar{t}_2$, $(x^{\#}, y, t_1)$, $(x^{\#}, z, t_2) \in \Delta$. The last relations imply $\exists\, t_3 \in W$ with $(t_3, y^{\#}, z)$, $(t_3, t_1, t_2^{\#}) \in \Delta$. It suffices to show $\sigma(\overline{v-w}) = \bar{t}_3$. Let $\sigma(\overline{v-w}) = \bar{s}$. We have $(\overline{v-w}, \overline{u-v}, \overline{u-w^{\#}}) \in \Delta$. Taking σ, we get $(\bar{s}, \bar{t}_1, \bar{t}_2^{\#}) \in \Delta$. So $\exists\, \gamma_1, \gamma_2 \in \Gamma_2$ with $(s, \gamma_1 t_1\, \gamma_2 t_2^{\#}) \in \Delta$. Also $(\overline{v-w}, \overline{-v}, \bar{w}) \in \Delta$, so $(\bar{s}, \bar{y}^{\#}, \bar{z}) \in \Delta$ giving

$(s,\delta_1 y^{\#},\delta_2 z) \notin \Delta$, $\delta_1,\delta_2 \in \Gamma_2$. So s, $t_3 \in sp(y,z) \cap sp\,(t_1,t_2)$. Since $v, w, u-v$ are independent, so are $y, z\ t_1$ and so $sp(y,z) \cap sp\,(t_1,t_2)$ has dimension at most one. So $\exists \alpha \in A^*$ with $s = \alpha t_3$. So $(\alpha t_3, \delta_1 y^{\#}, \delta_2 z) \in \Delta$ giving $(t_3, \alpha^{-1}\delta_1 y^{\#}, \alpha^{-1}\delta_2 z) \in \Delta$. Also, $(t_3, y^{\#}, z) \in \Delta$. y, z independent, so by uniqueness of coefficients $\alpha^{-1}\delta_1 = 1 = \alpha^{-1}\delta_2$ giving $\alpha \in \Gamma_2$. So $\bar{s} = \bar{t}_3$ giving $\sigma(\overline{v-w}) = \bar{t}_3$. Hence $\phi_{(v,y)}(w) = z$.

Step 3 : There is a surjective geometry morphism $\phi : V \to W$ such that $\overline{\phi(v)} = \sigma(\bar{v}) \forall v \in V$.

Proof: Choose $e_1, e_2, e_3 \in V$ linearly independent. Let $x_1 \in W$ with $\sigma(\bar{e}_1) = \bar{x}_1$ and $\phi_{(e_1,x_1)}(e_2) = x_2$, $\phi_{(e_1,x_1)}(e_3) = x_3$. Note that x_1, x_2, x_3 are independent in W. We fix these elements

Now define $\phi : V \to W$ as follows : $\phi(0) = 0$. For $0 \neq u \in V$, choose $e_i = e_1$, say, such that u, e_1 are independent and let $\phi(u) = \phi_{(e_1,x_1)}(u) = x$. If also e_1, e_2, u are independent, then by step 2, $\phi_{(e_2,x_2)}(u) = x$. If e_1, e_2, u are dependent then e_1, e_3, u and also e_3, e_2, u are independent, so $\phi_{(e_3,x_3)}(u) = x = \phi_{(e_2,x_2)}(u)$. So ϕ is well defined. Note that $\sigma(\bar{v}) = \bar{x}$, so $\overline{\phi(v)} = \sigma(\bar{v})$.

To show ϕ is a morphism, let $(u_1, u_2, u_3) \in \Delta_V$. Then $u_3 = -(u_1 + u_2)$, so it suffices to show $(\phi(u), \phi(v), \phi(u+v)^{\#}) \in \Delta_W$.

Case (1): Let u, v be independent. Choose e_i such that u, v, e_i are independent, say $e_i = e_1$. Let $\bar{t}_1 = \overline{\phi(e_1-u)}$ and $\bar{t}_2 = \overline{\phi(e_1-v)}$. By definition, $(x_1^{\#}, \phi(u), t_1)$, $(x_1^{\#}, t_2, \phi(v)) \in \Delta$. So $\exists t \in W$ with $(t, t_2^{\#}, \phi(u))$, $(t, t_1^{\#}, \phi(v)) \in \Delta$. But $(\overline{\phi(e_1-u-v)}, \overline{\phi(e_1-v)}^{\#}, \overline{\phi(u)}) = (\overline{\phi(e_1-u-v)}, \bar{t}_2^{\#}, \overline{\phi(u)}) \in \Delta$ and similarly $(\overline{\phi(e_1-u-v)}, \bar{t}_1^{\#}, \overline{\phi(v)}) \in \Delta$, so $\overline{\phi(e_1-u-v)}, \bar{t} \in sp(\bar{t}_2, \overline{\phi(u)}) \cap sp(\bar{t}_1, \overline{\phi(v)})$. But $sp(\bar{t}_2, \overline{\phi(u)}) \neq sp(\bar{t}_1, \overline{\phi(v)})$, so $\overline{\phi(e_1-u-v)} = \bar{\alpha}\,\bar{t}, \bar{\alpha} \in A/\Gamma_2$. So $(\overline{\alpha t}, \bar{t}_2^{\#}, \overline{\phi(u)}) \in \Delta \Rightarrow (\alpha t, \gamma_1 t_2^{\#}, \gamma_2 \phi(u)) \in \Delta$. But $(t, t_2^{\#}, \phi(u)) \in \Delta$, so $\alpha^{-1}\gamma_1 = 1 = \alpha^{-1}\gamma_2$, giving $\alpha \in \Gamma_2$ and $\overline{\phi(e_1-u-v)} = \bar{t}$. Also, $(\overline{u+v}, \bar{u}^{\#}, \bar{v}^{\#}) \in \Delta \Rightarrow (\overline{\phi(u+v)}, \overline{\phi(u)}^{\#}, \overline{\phi(v)}^{\#}) \in \Delta$ and similarly $(\overline{\phi(u+v)}, \overline{\phi(e_1-u-v)}, \overline{\phi(e_1)}^{\#}) = (\overline{\phi(u+v)}, \bar{t}, \bar{x}_1^{\#}) \in \Delta$. But $(t_1^{\#}, \phi(v), t)$, $(t_1^{\#}, \phi(u)^{\#}, x_1) \in \Delta \Rightarrow \exists\, s \in W$ with $(s, \phi(u), \phi(v))$, $(s, x_1, t^{\#}) \in \Delta$. So $\bar{s}$, $\overline{\phi(u+v)} \in sp\,(\overline{\phi(u)}, \overline{\phi(v)}) \cap sp(\bar{t}, \bar{x}_1)$, so $\overline{\phi(u+v)} = \beta\bar{s}$. But $(\bar{\beta}\bar{s}, \bar{t}, \bar{x}_1^{\#})$, $(\bar{s}^{\#}, \bar{t}, \bar{x}_1^{\#}) \in \Delta \Rightarrow \bar{\beta} = \bar{1}^{\#}$ and so

$\phi(\overline{u+v}) = \bar{s}$. Thus $\sigma(\overline{u+v}) = s^{\#}$, $\sigma(\overline{e_1 - u - v}) = \bar{t}^{\#}$, $(x_1^{\#}, s^{\#}, t) \in \Delta \Rightarrow \phi(u+v) = s^{\#}$. Hence $(s, \phi(u), \phi(v)) = (\phi(u+v)^{\#}, \phi(u), \phi(v)) \in \Delta$.

Case (2) : Suppose u, v are dependent. Obvious if $u = 0$, or $v = 0$ or $u + v = 0$. So let $u \neq 0$, $v = au$, $a \neq 0, -1$. Let $w \in V$ with u, w independent and let $e_i = e_1$, u, w be independent. Then by case (1), $(\phi(u), \phi(w), \phi(u+w)^{\#})$, $(\phi(v), \phi(w), \phi(v+w)^{\#})$, $(\phi(u+v), \phi(w), \phi(u+v+w)^{\#})$, $(\phi(u+w), \phi(v), \phi(u+v+w)^{\#}) \in \Delta$. Thus, $(\phi(u+v+w)^{\#}, \phi(v), \phi(u+w))$, $(\phi(w), \phi(u), \phi(u+w)^{\#})$, $(\phi(u+v+w)^{\#}, \phi(w), \phi(u+v)) \in \Delta$, so by lemma 2.2, $\exists s, t \in W$ with $(\phi(u), \phi(v), s)$, $(\phi(u+w), \phi(u+w)^{\#}, t)$, $(\phi(u+v), s, t) \in \Delta$. Since $\phi(v) = \hat{\phi}(a)\phi(u)$ so $s = \alpha\phi(u)$, $\alpha \in A$. Also, $\phi(u+v) = \hat{\phi}(1+a)\,\phi(u)$, so $(\hat{\phi}(1+a)\,\phi(u), \alpha\phi(u), t) \in \Delta \Rightarrow t = \beta\,\phi(u)$. If $\beta \neq 0$, then from second relation $\phi(u)$, $\phi(u+w)$ and so u, $u+w$ would be dependent, contradiction. So $t = 0$ and $s = \phi(u+v)^{\#}$ Hence $(\phi(u+v)^{\#}, \phi(u), \phi(v)) \in \Delta$.

Step 4 : There is a surjective geometry morphism $\hat{\phi} : F \to A$ such that $\hat{\phi}(ab) = \hat{\phi}(a)\hat{\phi}(b)\,\forall a,b \in F$ and inducing $\hat{\sigma} : \overline{\hat{\phi}(a)} = \hat{\sigma}(\bar{a})$.

Proof: Let $0 \neq a \in F$. Then $\hat{\sigma}(\bar{a}) = \bar{\alpha}$ for some $\alpha \in A$. Take $0 \neq v \in V$. Since $\sigma(\overline{av}) = \hat{\sigma}(\bar{a})\,\hat{\sigma}(\bar{v}) = \overline{\alpha\phi(v)}$, we get $\overline{\phi(av)} = \overline{\alpha\,\phi(v)}$, so $\exists \gamma_{(a,v)} \in \Gamma_2$ with $\phi(av) = \gamma_{(a,v)}\,\alpha\phi(v)$. We show $\gamma_{(a,v)}\,\alpha$ is independent of v. Take $0 \neq u \in V$. Let u,v be independent. Then, $(au, av, -a(u+v)) \in \Delta \Rightarrow (\gamma_{(a,u)}\,\alpha\phi(u), \gamma_{(a,v)}\,\alpha\phi(v), \gamma^{\#}_{(a,u+v)}\,\alpha\,\phi(u+v)) \in \Delta$. Also, $(\phi(u), \phi(v), \phi(u+v)^{\#}) \in \Delta$. So comparing unique coefficients gives $\alpha^{-1}\gamma^{-1}_{(a,u)}\,\gamma_{(a,v)}\,\alpha = 1 = \alpha^{-1}\,\gamma^{-1}_{(a,u)}\,\gamma_{(a,u+v)}\,\alpha$ giving $\gamma_{(a,u)}\,\alpha = \gamma_{(a,v)}\,\alpha = \gamma_{(a,u+v)}\,\alpha$. If u, v are dependent, choose w independent of both to give $\gamma_{(a,v)}\,\alpha = \gamma_{(a,w)}\,\alpha = \gamma_{(a,u)}\,\alpha$. Let the element be denoted by $\gamma_a\alpha$. Define $\hat{\phi}(a) = \gamma_a\alpha$ and $\hat{\phi}(0) = 0$. It is well defined map such that $\phi(av) = \hat{\phi}(a)\,\phi(v)\;\forall v \in V$. Note that $\hat{\phi}(a) = \hat{\sigma}(a)\,\forall a \in F$.

Let $(a_1, a_2, a_3) \in \Delta_F$. For $0 \neq v \in V$, we get $(a_1 v, a_2 v, a_3 v) \in \Delta \Rightarrow (\hat{\phi}(a_1)\phi(v), \hat{\phi}(a_2)\phi(v), \hat{\phi}(a_3)\phi(v)) \in \Delta$ and $\phi(v) \neq 0$, so $(\hat{\phi}(a_1), \hat{\phi}(a_2), \hat{\phi}(a_3)) \in \Delta$. So $\hat{\phi}$ is a morphism.

Also, $\phi(abv) = \hat{\phi}(a)\,\phi(bv) = \hat{\phi}(a)\,\hat{\phi}(b)\,\phi(v)$ shows that $\hat{\phi}(ab) = \hat{\phi}(a)\,\hat{\phi}(b)$. For surjectiveness, let $0 \neq \alpha \in V$. Take $0 \neq v \in V$. Then $\exists u \in V$ with $\phi(u) = \alpha\phi(v)$. Hence, u, v are dependent in V also, so $u = av$, $a \in F$. Then $\phi(u) = \hat{\phi}(a)\,\phi(v) = \alpha\phi(v)$ giving $\hat{\phi}(a) = \alpha$.

Step 5: The morphisms $(\phi, \hat{\phi})$ induce a semi-isomorphism $(\psi, \hat{\psi}): (V/\Gamma, F/\Gamma) \to (W, A)$, where Γ is a normal subgroup of F^*.

Proof : From above, $\hat{\phi} : F^* \to A^*$ is a group homomorphism. Let $\Gamma = ker\,(\hat{\phi})$. Let $\hat{\psi} : F^*/\Gamma \to A^*$ be the isomorphism of groups. Extending the map obviously to F/Γ, we get a bijection map $\hat{\psi} : F/\Gamma \to A$. If $(\widetilde{a}, \widetilde{b}, \widetilde{c}) \in \Delta_{F/\Gamma}$, then $(a, \gamma_1 b, \gamma_2 c) \in \Delta_F$, so $(\hat{\phi}(a), \hat{\phi}(\gamma_1 b), \hat{\phi}(\gamma_2 c)) = (\hat{\phi}(a), \hat{\phi}(b), \hat{\phi}(c)) = (\hat{\psi}(\widetilde{a}), \hat{\psi}(\widetilde{b}), \hat{\psi}(\widetilde{c})) \in \Delta$. So $\hat{\psi}$ is a morphism.

Also, we get the geometry V/Γ, which is a geometric space over F/Γ where $\widetilde{a}\widetilde{v} = \widetilde{av}$ for $\widetilde{a} \in F/\Gamma$ and $\widetilde{v} \in V/\Gamma$. Now define $\psi : V/\Gamma \to W$ by $\psi(\widetilde{v}) = \phi(v)$. If $\widetilde{v}_1 = \widetilde{v}_2$, then $v_1 = \gamma\, v_2$, so $\phi(v_1) = \phi(\gamma v_2) = \hat{\phi}(\gamma)\phi(v_2) = \phi(v_2)$, so the map is well defined. If $(\widetilde{u}, \widetilde{v}, \widetilde{w}) \in \Delta$,then $(u, \gamma_1 v, \gamma_2 w) \in \Delta$, so $(\phi(u), \phi(\gamma_1 v), \phi(\gamma_2 w)) = (\psi(\widetilde{u}), \psi(\widetilde{v}), \psi(\widetilde{w})) \in \Delta$. So ψ is a morphism. Thus $(\psi, \hat{\psi}) : (V/\Gamma, F/\Gamma) \to (W, A)$ are bijective morphisms such that $\psi(\widetilde{a}\widetilde{v}) = \hat{\psi}(\widetilde{a})\psi(\widetilde{v})$. For isomorphism, remains to show that the inverses are also morphisms.

Let $(x, y, z) \in \Delta_W$ and $\psi(\widetilde{u}) = x, \psi(\widetilde{v}) = y, \psi(\widetilde{w}) = z$. We show $(\widetilde{u}, \widetilde{v}, \widetilde{w}) \in \Delta_{V/\Gamma}$. Since $(\overline{x}, \overline{y}, \overline{z}) \in \Delta_{W/\Gamma_2}$, and σ is isomorphism, we get $(\overline{u}, \overline{v}, \overline{w}) \in \Delta$. So $\exists \delta_1, \delta_2 \in \Gamma_1$ with $(u, \delta_1 v, \delta_2 w) \in \Delta$. So $(\phi(u), \hat{\phi}(\delta_1)\,\phi(v), \hat{\phi}(\delta_2)\,\phi(w)) = (x, \gamma_1 y, \gamma_2 z) \in \Delta$, $\gamma_i = \hat{\phi}(\delta_i)$.

Case (1) : y, z and so v, w independent. Then by uniqueness, $\gamma_1 = \gamma_2 = 1$ in A, so $\delta_i \in \Gamma$. Hence $(\widetilde{u}, \widetilde{\delta_1 v}, \widetilde{\delta_2 w}) = (\widetilde{u}, \widetilde{v}, \widetilde{w}) \in \Delta$.

Case (2) : y, z dependent. Obvious if $y = 0$, or $z = 0$, or $x = 0$. So let $y = \alpha x$, $z = \beta x$, $\alpha, \beta \in A^*$. If $\psi^{-1}(x) = \widetilde{u}$, then $\psi^{-1}(y) = \widetilde{a}\widetilde{u}$, $\psi^{-1}(z) = \widetilde{b}\widetilde{u}$, where $\hat{\psi}(\widetilde{a}) = \alpha, \hat{\psi}(\widetilde{b}) = \beta$. Choose $t \in W$ with x, t independent. Let $(x, t, s) \in \Delta$, so t, s independent. Now $(x, t, s), (x, y, z) \in \Delta \Rightarrow \exists t_1 \in$ W with $(t_1, y^\#, t), (t_1, z, s^\#) \in \Delta$. By case (1), $(\psi^{-1}(t_1), \psi^{-1}(y)^\#, \psi^{-1}(t)), (\psi^{-1}(t_1), \psi^{-1}(z), \psi^{-1}(s)^\#) \in \Delta \Rightarrow \exists \widetilde{v} \in$ V/Γ with $(\widetilde{v}, \psi^{-1}(z)^\#, \psi^{-1}(y)^\#), (\widetilde{v}, \psi^{-1}(s)^\#, \psi^{-1}(t)^\#) \in \Delta$. Then, $(\widetilde{v}, \widetilde{a}\widetilde{u}, \widetilde{b}\widetilde{u}) \in \Delta \Rightarrow \widetilde{v} = \widetilde{c}\widetilde{u}$, so $(\widetilde{c}\widetilde{u}, \psi^{-1}(s)^\#, \psi^{-1}(y)^\#) \in \Delta \Rightarrow (\hat{\psi}(\widetilde{c})x, s^\#, t^\#) \in \Delta$. But (x, s, t) $\in \Delta$. So $\hat{\psi}(\widetilde{c}) = 1^\#$ giving $\widetilde{v} = \widetilde{u}^\# = \psi^{-1}(x)^\#$. So from above $(\psi^{-1}(x)^\#, \psi^{-1}(y)^\#, \psi^{-1}(z)^\#) \in \Delta$ which gives the required result. Thus ψ: $V/\Gamma \to W$ is an isomorphism

of geometries. Also let $(\alpha, \beta, \gamma) \in \Delta_A$. Take $0 \neq x \in W$. Then $(\alpha x, \beta x, \gamma x) \in \Delta_W \Rightarrow$ $(\hat{\psi}^{-1}(\alpha)^{\#}\ \psi^{-1}(x), \hat{\psi}^{-1}(\beta)\psi^{-1}(x), \hat{\psi}^{-1}(\gamma)\ \psi^{-1}(x)) \in \Delta_{V/\Gamma}$, so $(\hat{\psi}^{-1}(\alpha), \hat{\psi}^{-1}(\beta), \hat{\psi}^{-1}(\gamma)) \in \Delta_{F/\Gamma}$. Hence $\hat{\psi} : F/\Gamma \to A$ is an isomorphism of geometric sfields such that $\psi(\widetilde{a}\widetilde{v}) = \hat{\psi}(\widetilde{a})\psi(\widetilde{v})$.

Note that if $\gamma \in \Gamma$, then $\hat{\phi}(\gamma) = 1_A$, so $\hat{\sigma}(\gamma) = 1_{A/\Gamma_2}$, so $\gamma \in \Gamma_1$. So $\Gamma \subseteq \Gamma_1$. Note also that $\hat{\psi}(\Gamma_1 / \Gamma) = \phi(\Gamma_1) = \Gamma_2$.

Hence $(\psi, \hat{\psi}) : (V/\Gamma, F/\Gamma) \to (W, A)$ is semi-isomorphism of geometric spaces as required.

In case A is sharp and hence a skewfield, we get

Corollary 3.2. Let V be a vector space over a skewfield F_1 $\dim_{F_1} V \geq 3$ and W a vector space over $A = F_2$. Let Γ_1 and Γ_2 be normal subgroups of F_1^* and F_2^* respectively. Let

$$(\sigma, \hat{\sigma}) : (V/\Gamma_1, F_1/\Gamma_1) \to (W/\Gamma_2, F_2/\Gamma_2)$$

be a semi-isomorphism between geometric spaces. Then there exists a semi-linear transformation

$$(\psi, \hat{\psi}) : (V, F_1) \to (W, F_2)$$

such that $\psi(\Gamma_1) = \Gamma_2$ and induces the given semi-isomorphism as : $\psi(\overline{v}) = \sigma(v)$ and $\hat{\psi}(a) = \hat{\sigma}(a)$.

Proof : By the theorem there is a semi-isomorphism

$$(\psi, \hat{\psi}) : (V/\Gamma, F_1/\Gamma) \to (W, F_2)$$

But F_2 is sharp and the isomorphism $\hat{\psi}$: $F_1/\Gamma \to F_2$ implies F_1/Γ is also sharp which is possible iff $\Gamma = \{1\}$. So $\hat{\psi}$ is isomorphism of skewfields and $(\psi, \hat{\psi})$ is a bijective semi-linear transformation as required.

In particular, if $\Gamma_1 = F_1^*$ and $\Gamma_2 = F_2^*$, then F_1 / F_1^* and F_2 / F_2^* are trivial and we get the well known Fundamental Theorem of Projective Geometry.

Corollary 3.3. If $\sigma : V/F_1^* \to W/F_2^*$ is an isomorphism of geometries, then there exists a bijective semi-linear transformation $(\psi, \hat{\psi}) : (V, F_1) \to (W, F_2)$ which induces the given isomorphism.

4. The Representation Theorem

Now the following theorem gives the representation of a geometric space as orbits of a vector space.

orem 4.1. *Suppose W is a geometric space over a geometric sfield A,*

$\dim_A W \geq 4$. *Then there exists a vector space V over a skewfield F and a normal subgroup* Γ *of* F^* *such that there is a semi-isomorphism* $(\psi, \hat{\psi}) : (V/\Gamma, F/\Gamma) \to (W, A)$. *Moreover if also* $(\psi_1, \hat{\psi}_1) : (V_1/\Gamma_1, F_1/\Gamma_1) \to (W, A)$ *is a semi-isomorphism, then there exists a semi-linear transformation* $(\phi, \hat{\phi}) : (V, F) \to ((V_1, F_1)$ *such that* $\hat{\phi}(\Gamma) = \Gamma_1$. *If* $\dim_A W = 3$, *then the above is true if W is a* D-*geometry* [4].

Proof: Since W is a geometric space over a geometric sfield A, the geometry W/A^* is a geometric space over the geometric sfield $A/A^* = \{\overline{0}, \overline{1}\}$. But the geometry of W/A^* is projective and its non-zero elements form the points of a projective space, say $P(W)$. By hypothesis, it is Desarguesian of proper dimension, so by a well known theorem of projective geometry [1], there is a vector space V over a skewfield F such that $P(W)$ is isomorphic to $P(V)$. This isomorphism clearly extends to a geometry isomorphism $\sigma : V/F^* \to W/A^*$. Now $F/F^* = \{\overline{0}, \overline{1}\}$ is sharp iff V/F^* is sharp iff each line of $P(V)$ and hence of also $P(W)$ has three points iff A/A^* is sharp. Otherwise both are non-sharp. Since there are only two non-isomorphic geometric sfields with two elements $\{0,1\}$, sharp or non-sharp, we see in either case that the map $\hat{\sigma}: F/F^* \to A/A^*$ by $\hat{\sigma}(\overline{0}) = \overline{0}, \hat{\sigma}(\overline{1}) = \overline{1}$ is an isomorphism of geometric sfields. Then clearly

$$(\sigma, \hat{\sigma}) : (V/F^*, F/F^*) \to (W/A^*, A/A^*)$$

is a semi-isomorphism. So by theorem 3.1, there is a normal subgroup Γ of F^* and a semi-isomorphism

$$(\psi, \hat{\psi}) : (V/\Gamma, F/\Gamma) \to (W, A)$$

which induces the given semi-isomorphism.
Now suppose also there is another semi-isomorphism

$$(\psi_1, \hat{\psi}_1) : (V_1/\Gamma_1, F_1/\Gamma_1) \to (W, A)$$

Then,

$$(\psi_1^{-1} o \psi, \hat{\psi}_1^{-1} o \hat{\psi}) : (V/\Gamma, F/\Gamma) \to (V_1/\Gamma_1, F_1/\Gamma_1)$$

is a semi-isomorphism. So by corollary 3.2, there is a semi-linear transformation $(\phi, \hat{\phi}) : (V, F) \to (V_1, F_1)$ such that $\hat{\phi}(\Gamma) = \Gamma_1$. Thus the representation of (W,A) is unique upto semi-linear isomorphism of vector spaces.

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Lorentz Equation For Constant Electromagnetic Fields

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Abstract: We show that the Lorentz equation admits exact integration when the external electromagnetic field is constant. Our process uses the eigenvectors of the Faraday tensor and a 2nd order differential equation for the tangent vector to world line, which originates a method more simple comparated with the techniques of Plebañski [1], Synge [2] and Piña [3].

I. Introduction:

The trajectory of a particle in Minkowski space is given by a sequence of events $x_r(s)$:

(1) $$x_1 = x,\ x_2 = y,\ x_3 = z,\ x_4 = it,\ i = \sqrt{-1}$$

where s is the proper time and the light velocity is equal to one. Then the corresponding velocity, acceleration and superacceleration are (we shall employ the quantities and notation of Synge [2,4,5]) ;

(2) $$\lambda_r = \frac{dx_r}{ds},\ \mu_r = \frac{d\lambda_r}{ds},\ \nu_r = \frac{d\mu_r}{ds}$$

$$\lambda_r\lambda_r = -1,\quad \mu_r\lambda_r = 0,\quad \lambda_r\nu_r = -\mu_r\mu_r$$

If the particle has a charge e, then the Lorentz equation [4,6,7] :

(3) $$\mu_r = bF_{rj}\lambda_j,\ \ b = \frac{e}{m}$$

indicate its interaction with an external electromagnetic field characterized by the Faraday tensor [4,8] :

$$(F_{ab}) = \begin{pmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{pmatrix} \tag{4}$$

where $\vec{E}$ and $\vec{B}$ are the electric and magnetic vectors, respectively. In Sec 2 we introduce the dual tensor of (4) which permits to construct the classification of F_{ab} as proposed by Synge [2] and Piña [3], with great relevance in the study of the solutions for (3).

Our problem is to integrate the Lorentz equation when F_{ab} is constant, which it may be realized through the following technique :

a) Algebraical Process

Synge [2,9–11] showed that the curvatures K_r , $r = 1, 2, 3$ of the world line are constants, and from Frenet-Serret formulae [10-12] he obtains a 4th order differential equation for the velocity λ_r :

$$\frac{d^4}{ds^4}\lambda_r + \left(k_2^2 + k_3^2 - k_1^2\right)\frac{d^2}{ds^2}\lambda_r - k_1^2\, k_3^2\, \lambda_r = 0 \tag{5}$$

whose solutions depend of the corresponding Euler's characteristic equation. Therefore,

$$x_r(s) = x_r(0) + \int_0^s \lambda_r(\theta)\, d\theta \tag{6}$$

gives us the path in terms of the initial conditions $x_a(0)$ and $\lambda_a(0)$.

b) Tensorial Method

It is immediate the integration of (3) :

$$\lambda_r(s) = \exp(bsF_{rc})\,\lambda_c(0). \tag{7}$$

thus Plebañski [1] and Piña [3] indicate how to calculate the exponential function of an antisymmetric matrix (or tensor). This determines $\lambda_r(s)$ and then (6) again gives the trajectory.

In Sec. 3 we exhibit a new process (named algebraical-tensorial method) to resolve (3), which it uses the proper values and eigenvectors of Faraday tensor.

2. Algebraic Classification F_{ab}

Using the totally antisymmetric symbol $\in_{armn}$ of Levi-Civita we can construct the dual tensor of Faraday [4, 8] :

(8) $$*F_{ac} = \frac{1}{2} \in_{acmn} F_{mn} ,$$

with matricial expression

(9) $$(*F_{ac}) = \begin{pmatrix} 0 & E_3 & -E_2 & iB_1 \\ -E_3 & 0 & E_1 & iB_2 \\ E_2 & -E_1 & 0 & iB_3 \\ -iB_1 & -iB_2 & -iB_3 & 0 \end{pmatrix}$$

Any antisymmetric tensor and its dual satisfy the identities [1,3,8,13-15]:

(10) $$F_{ra} F_{rc} - *F_{ra} * F_{rc} = \frac{I_1}{2} \delta_{ac} , \quad *F_{ra} F_{rc} = \frac{I_2}{4} \delta_{ac}$$

where :

(11) $$I_1 = F_{ac} F_{ac} = 2\left(E^2 - B^2 \right), \quad I_2 = *F_{ac} F_{ac} = 4 \vec{E} \cdot \vec{B}$$

are the unique invariants (under Lorentz transformations) of F_{ij} , which leads to the Synge classification [2, 8, 10] for the electromagnetic field:

(12)
$$\begin{aligned} &\text{Type A} : I_2 \neq 0 \\ &\text{Type B} : I_1 < 0, \ I_2 = 0 \\ &\text{Type C} : I_1 = 0, \ I_2 = 0 \ \text{Null field} \\ &\text{Type D} : I_1 > 0, \ I_2 = 0 . \end{aligned}$$

The cases indicated in (12) are of interest because the properties of the world line depend of the Type for F_{ac} plus the initial conditions.

We also can write (11) in the form of Piña [3]:

(13) $$I_1 = 2HCos\gamma , \quad I_2 = 2HSin\gamma , \quad H \geq 0, \quad 0 \leq \gamma < 2\pi ,$$

thus (12) is reduced to :

(14)
$$\begin{aligned} &\text{Type A} : \gamma \neq 0, \pi , \qquad \text{Type B} : \gamma = \pi , \\ &\text{Type C} : \text{H} = 0, \quad \text{Type D} : \gamma = 0. \end{aligned}$$

The electromagnetic field is non-null if I_1 or/and I_2 are differents to zero, then in this case there are [4,13, 16-18] two null eigenvectors γ_r and η_r with proper values $\pm\lambda$:

$$F_{rc}\gamma_c = \lambda\gamma_r, \ \ F_{rc}\eta_c = -\lambda\eta_r, \ \ \gamma_r\gamma_r = \eta_r\eta_r = 0,$$

(15)
$$\tilde{\lambda} = \frac{1}{4}\left(I_1^2 + I_2^2\right)^{1/2} = \frac{H}{2} > 0,$$

$$\lambda = \left(\tilde{\lambda} - \frac{I_1}{4}\right)^{1/2} = \sqrt{H}\ Sin\frac{\gamma}{2} > 0,$$

which permits to express F_{ij} and its dual in terms of their null principal directions :

$$F_{rc} = \frac{1}{\gamma_a\eta_a}[\lambda(\gamma_r\eta_c - \gamma_c\eta_r) + i\beta \in_{rcmn}\gamma_m\eta_n],$$

(16)
$${}^*F_{rc} = \frac{1}{\gamma_a\eta_a}[-\beta(\gamma_r\eta_c - \gamma_c\eta_r) + i\lambda \in_{rcmn}\gamma_m\eta_n],$$

$$\beta = \in\left(\tilde{\lambda} + \frac{I_1}{4}\right)^{1/2} = \in\sqrt{H}\ \ Cos\frac{\gamma}{2} \geq 0,\ \ \in = \pm 1.$$

We note that always $\gamma_a\eta_a \neq 0$ because these null eigenvectors have linear independence

3. Algebraical–Tensorial method

This method is not explicitly in the literature, and we consider that it is more simple and elementary than the processes of [1-3] because it only involve relations very known in tensorial algebra.

In fact, if we employ (10) in (3) results a 2nd order differential equation for the acceleration :

(17)
$$\frac{d^2}{ds^2}\mu_r + \frac{b^2}{2}I_1\mu_r = -\frac{b^3}{4}I_2\,{}^*F_{rc}\lambda_c$$

which permits to study easily the types B,C and D because they have $I_2 = 0$, and thus one integration gives us :

(18)
$$\frac{d^2}{ds^2}\lambda_r + \frac{b^2}{2}I_1\lambda_r = \nu_r(0) + \frac{b^2}{2}I_1\lambda_r(0)$$

with more simplicity than (5). The nature of the characteristic roots α of (18) depends of the invariant I_1, therefore :

Type B:

$$\alpha = \pm\, b\lambda\,, \qquad \lambda = \sqrt{-\frac{I_1}{2}} > 0.$$

Then the 4-velocity solution of (18) is :

(19) $$\lambda_r(s) = \lambda_r(0) - \frac{1}{b^2\lambda^2}\, \nu_r(0) + A_r e^{b\lambda S} + B_r e^{-b\lambda S}$$

where the integration's constants A_r and B_r are determined in terms of the initial conditions :

(20) $$\mu_r(0) = bF_{rc}\lambda_c(0),\;\; \nu_r(0) = b^2 F_{rc}F_{cn}\lambda_n(0)$$

If we remember that $e^{\pm b\lambda S} = Cosh\,(b\lambda s) \pm\; Sinh\,(b\lambda s)$, then (19) adopts the form

$$\lambda_r(s) = \lambda_r(0) + \frac{1}{b\lambda}\, \mu_r(0) Sinh(b\lambda s) + \frac{1}{b^2\lambda^2}\, \nu_r(0)\,(Cosh(b\lambda s) - 1),$$

and thus (6) implies the path :

(21) $$x_r = x_r(0) + \left[s\,\delta_{rn} + \frac{1}{b^2\lambda^2}(Cosh(b\lambda s) - 1)\, F_{rn} + \right.$$
$$\left. + \frac{1}{\lambda^2}\left(\frac{1}{b\lambda} Sinh(b\lambda s) - s\right) F_{rc}\, F_{cn}\right] \lambda_n(0)$$

Type C

In this case we have $I_1 = 0$ and it is trivial to resolve the equation (18) :

$$\lambda_r(s) = \lambda_r(0) + \mu_r(0)s + \nu_r(0)\,\frac{s^2}{2}\,,$$

then (6) give us the worldline :

(22) $$x_r(s) = x_r(0) + \left(s\;\delta_{rn} + \frac{bs^2}{2}\, F_{rn} + \frac{b^2 s^3}{6}\, F_{rc}\, F_{cn}\right) \lambda_n(0)$$

Type D

$$\alpha = \pm\; i\beta b\,, \qquad \beta = \sqrt{\frac{I_1}{2}} > 0.$$

Here (18) has the solution

$$\lambda_r(s) = \lambda_r(0) + \frac{1}{b\beta}\, \mu_r(0) Sin(b\beta\, s) - \frac{1}{b^2\beta^2}\, \nu_r(0)\,(Cos(b\beta\, s) - 1).$$

where :

$$x_r(s) = x_r(0) + \Big[s\,\delta_{rn} - \frac{1}{b^2\beta^2}(Cos(b\beta s) - 1)\,F_{rn} - \qquad (23)$$

$$- \frac{1}{\beta^2}\left(\frac{1}{b\beta} Sin(b\beta s) - s\right) F_{rc}\,F_{cn}\Big]\,\lambda_n(0)$$

Now we must consider the type A, thus we shall study the right side of (17). The expressions (16) permit to write the Faraday tensor in function of its dual if into them we eliminate the Levi–Civita symbol, then :

$$*F_{rc} = \frac{\lambda}{\beta} F_{rc} - \frac{1}{\gamma_a \eta_a}\left(\beta + \frac{\lambda^2}{\beta}\right)(\gamma_r \eta_c - \gamma_c \eta_r)$$

which with (3) in (17) implies

$$\frac{d^2}{ds^2}\mu_r + b^2 H^2\, Cos^2\frac{\gamma}{2}\,\mu_r = f_r(s) \qquad (24)$$

where

$$f_r(s) = \frac{b^3 \lambda \widetilde{\lambda}}{\gamma_a \eta_a}(\gamma_r \eta_c - \gamma_c \eta_r)\,\lambda_c. \qquad (25)$$

However, from (3) and (15) it is easy to see that

$$\gamma_c \lambda_c = \lambda_c(0)\,\gamma_c\, e^{-b\lambda s},\ \ \eta_c \lambda_c = \lambda_c(0)\,\eta_c\, e^{b\lambda s}$$

which with (15) and (16) determine $f_r(s)$ via (25) :

$$f_r(s) = M_r\, Cosh(b\lambda s) + N_r\, Sinh(b\lambda s),$$

$$M_r = b^3\left(\lambda^2 F_{rc} - \frac{I_2}{4} *F_{rc}\right)\lambda_c(0), \qquad (26)$$

$$N_r = b^3 \lambda\left[\left(\widetilde{\lambda} + \frac{I_1}{4}\right)\delta_{rc} - F_{rn}\,F_{cn}\right]\lambda_c(0).$$

If we put (26) into (24) we obtain the solution :

$$\mu_r(s) = B_r Cos\left(bHs\,.\,Cos\frac{\gamma}{2}\right) + C_r Sin\left(bHs\,.\,Cos\frac{\gamma}{2}\right) + \frac{1}{2b^2\widetilde{\lambda}} f_r(s),$$

$$B_r = \mu_r(0) - \frac{1}{2b^2\widetilde{\lambda}} M_r, \qquad (27)$$

$$C_r = \frac{1}{bH\,Cos\frac{\gamma}{2}}\left(\nu_r(0) - \frac{\lambda}{2b\tilde{\lambda}}N_r\right),$$

with the corresponding trajectory :

$$x_r(s) = x_r(0) - \frac{1}{bH\,Cos\frac{\gamma}{2}}\left[\left(Cos\left(bHs\,.\,Cos\frac{\gamma}{2}\right) - 1\right)B_r - \right.$$

$$\left. - Sin\left(bHs\,.\,Cos\frac{\gamma}{2}\right)C_r\right] + \frac{1}{2b^4\lambda^2\tilde{\lambda}}\left(f_r(s) - M_r\right) \tag{28}$$

due to the following identity :

$$\lambda_r(0) + \frac{1}{bH\,Cos\frac{\gamma}{2}}C_r - \frac{1}{2b^3\lambda\tilde{\lambda}}N_r = 0 \tag{29}$$

with $\mu_r(0)$ and $\nu_r(0)$ given by (20).

Thus is completely known the motion of a point charge under the various types of constant electromagnetic fields. Our expressions (21), (22), (23) and (28) are equivalent to (4.6), (4.7), (4.8) and (4.9) of Piña [3].

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Any Empty Spacetime Has Not Constant Timelike Vectors

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Abstract: Using a Synge's invariant we show that any vacuum geometry accepts constant timelike vectors.

1. Introduction.

Synge [1] deduced an invariant with the property that, when it is zero at a particular event of the spacetime then the curvature tensor is also zero at these event. This result of Synge is surprising because it means that one zero valued scalar implies nullification of the twenty independent components of the Riemann tensor. Thus we have the :

Theorem, " Let τ_r be any unitary timelike vector and

$$H = \frac{1}{20} H_0 + \frac{23}{60} H_2 + \frac{30}{10} H_4 . \tag{1}$$

with

$$H_0 = \frac{3}{2} R_{abcd} R^{abcd} + 2 R_{ab} R^{ab} + \frac{1}{2} R^2 ,$$

$$H_2 = 2 \left[R_{pqra} (R^{pqrb} + R^{rqpb}) - R^{pq} R^{b}_{paq} + R^{p}_{a} R^{b}_{p} + \frac{R}{2} R^{b}_{a} \right] \tau^a \tau_b , \tag{2}$$

$$H_4 = \left[R^{paqb} R_{pcqd} + \frac{1}{2} R^{ab} R_{cd} \right] \tau_a \tau_b \tau^c \tau^d ,$$

then for each event in R_4 :

(3) $$H = 0 \Rightarrow R_{ijkm} = 0,"$$

We remember than $R_{ab} = R^{c}_{abc}$ is the Ricci tensor and $R = R^{a}_{a}$ is the scalar curvature. It is clear that $H = 0$ in every point of R_4 implies a flat spacetime,

This Synge's theorem can be very useful in general relativity : we give here one application of it showing that a vacuum metric not admits constant timelike vectors.

2. Empty Spacetime.

The ref. [2] motivates us to recognize the importance of the existence of constant vectors :

(4) $$\tau^{r}{}_{,c} = 0.$$

Thus the non-commutative property of covariant derivative leads to :

(5) $$R^{abcd}\ \tau_d = 0$$

In this Section we consider the case $R_{ab} = 0$, and with the use of H and (5) we show that there are no constant timelike vectors in empty spacetime. This can be applied to several vaccum geometries such as those of Taub, Schwarzschild, C. Siklos, Kerr, etc.

In fact we suppose there exists a unitary constant timelike vector τ^{r} (note there is no loss of generality because the norm of a constant vector is also a constant), then from (1), (2) and (5) we obtain :

(6) $$H_0 = \frac{3}{40} R_{abcd}\ R^{abcd}$$

But in [2] it was proved (using a result of Horndeski [3]) that the Lanczos scalar [4] $R_{abcd}\ R^{abcd}$ for an empty spacetime is zero in presence of a non-null constant vector. Therefore $H = 0$ and thus (3) affirms that our R_4 is flat, that is, only the Minkowski spacetime admits constant timelike vectors q.e.d.

The proof here presented is simpler than other ones, for example, in [5] special coordinate systems are required.

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On Algebraic Characterizations in Shop Scheduling Problems

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Abstract: We consider classical *NP*-hard shop scheduling problems where one of the major tasks is to minimize the makespan criterion over the set of all feasible (cycle-free) combinations (*sequences*). The collection of all *machine order matrices* (*row latin rectangles*) and *job order matrices* (*column latin rectangles*) forms the *super sequence group* isomorphic to $S_m^n \times S_n^m$. The function on the maximal order of an element of this group is generalized. The investigated algebraic characterizations yield a mathematical decomposition that corresponds to a practical classification of the considered problems.

Key Words: shop problems, super sequence group, maximal order.

1. Introduction

In an $n \times m$ classical shop scheduling problem, each job i, $i \in I = \{1,2, \ldots, n\}$ has to be processed on each *machine* j, $j \in j = \{1,2, \ldots, m\}$ exactly once without preemption for the positive processing time. Here, assume that each machine can process at most one job and each job can be processed on at most one machine at a time. Let $SIJ = I \times J$ and $P = [p_{ij}]$ be the set of all operations o_{ij} and the matrix of processing times p_{ij} with $i \in I \wedge j \in J$, respectively. We denote the completion time of job i on *machines* by C_i and the matrix of completion times by $C = [c_{ij}]$ so that $C_i = \max_j c_{ij}$ holds. The *job order* on machine j is the order of jobs processed on machine j and the machine order for job i is the order of

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machines which process job i. We have to find a feasible combination of machine orders and job *orders* (*sequence*) which minimizes the maximum completion time. A *schedule* gives the corresponding time table. Here, we consider the strongly *NP-hard* shop scheduling problem denoted by $\alpha|\beta|\gamma$ (cf. GRAHAM et al. [13]) with $\alpha \in \{O, F, J\}$ and $\beta = \phi$. In a job shop problem $\alpha = J$ all machine orders are given in advance, in the flow shop problem $\alpha = F$ machine orders are identical for each job, and in the open shop problem $\alpha = O$ all machine orders are arbitrary.

A set S of sequences is called *potentially optimal* if it contains an optimal sequence with respect to the objective γ independent of processing times. AKERS/FRIEDMAN [1] considered the job shop problem $J|n=2|C_{\max}$ and obtain a necessary and sufficient condition for removing some of the sequences. Some structural properties on the set of possibly optimal sequences for the flow shop problem are proved by CONWAY et al. [10]. Different decompositions of job shop sequences and schedules are presented and further analyzed for their removal conditions by ASHOUR [2]. One of the aims to study irreducible sequences is to investigate a potentially optimal sequence set of minimal cardinality and it is unavoidable. But the existence of a unique unavoidable sequence set is not known in general.

BRÄSEL/KLEINAU [8] and KLEINAU [16] investigated the irreducible sequences introducing a dominance relation $\preceq$ on the set of all sequences with a fixed format $n \times m$. A sequence is called *irreducible* if there exists no better sequence replacing it for arbitrary processing times; otherwise it is a *reducible* one. This extension depends on the trees of *dependence* which is valid for all $\alpha \in \{O, F, J\}$. We further refer to BRÄSEL et al. [6,7] and HARBORTH [15] for the extended investigations on the reducibility and irreducibility for $\alpha \in \{O, J\}$ to minimize the makespan criterion. They present several sufficient conditions for reducibility each of which can be tested in polynomial *time*. Computational results show that the ratio of the number of irreducible sequences to the number of all sequences decreases as the size of the shop problem instance increases. A new decomposition approach of sequences is introduced and some sufficient conditions for the irreducibility of sequences are presented by DHAMALA [12]. Up to today, no polynomial time algorithm is known for the decision whether a sequence is irreducible in the general case and it is still unknown if such an algorithm can exist.

Recall that counting sequences by considering the cardinality of special latin rectangles or the chromatic polynomial of the Hamming graph $K_n \times K_m$ is hard (see HARBORTH [15]). A closed formula for this unsolved counting problem is unknown up to now and only upper and lower bounds are available in general (e.g., BRÄSEL/DHAMALA [3], DHAMALA [12] and HARBORTH [15]). Introducing group operations on the set of all combinations in the open shop problem, BRÄSEL/DHAMALA [3], DHAMALA [12], give a mathematical decomposition which corresponds to a practical classification of shop scheduling problems. Furthermore,

they study the maximal orders of the elements in the corresponding groups of small formats to decompose the whole group as the union of its subgroups. Investigations in the fields of graph theory, algebra, and latin squares and rectangles are combined to obtain results in scheduling theory by DHAMALA [12].

In Section 2, the mathematical models along with some basic notions of graphs and sequences are summarized. Sections 3 and 4 are devoted to the study of algebraic characterizations in shop scheduling problems and the maximal order of the elements in the respective groups, respectively. Some concluding remarks are contained in the final section.

2. Basic Concepts

Unlike the polyhedral approaches and the disjunctive graph model we adopt the block matrices model (cf. BRÄSEL [9]) in shop scheduling problem, where all graph theoretical structures are basically described by means of special latin rectangles also called *sequences*. Given SIJ and a pair of machine orders and job orders in an $n \times m$ shop scheduling problem, we define the *machine order graph* $G_{MO} = (SIJ, E_{MO})$, where the set of arcs contains the precedence constraints of all machine orders and the *job order graph* $G_{JO} = (SIJ, E_{JO})$, where the set of arcs contains the precedence constraints of all job orders. The acyclic graphs G_{MO} and G_{JO} consist of n and m acyclic components, respectively. The *rank* of a vertex in an acyclic directed graph is the number of vertices on a longest path from a source to the vertex itself. Here, $MO = [mo_{ij}]$ and $JO = [jo_{ij}]$ represent the $n \times m$ rank matrices of G_{MO} and G_{JO}, called machine order matrix (*row latin rectangle*) and *job order matrix* (*column latin rectangle*), respectively. Moreover, jo_{ij} is the position of job i in the job order on machine j and mo_{ij} is the position of machine j in the machine order for job i.

For given (MO, JO), we define the shop graph $G_{MO,JO} = (SIJ, E_{MO,JO})$, where the arc set $E_{MO,JO} = E_{MO} \cup E_{JO}$ reflects all machine orders and all job orders. The connected directed graph $G_{MO,JO}$ may or may not be acyclic. If the shop graph is acyclic (cyclic) we call it a *sequence graph* (*non-sequence graph*). Note that the sequence graph is an acyclic orientation of the disjunctive graph (see SUSSMAN [20]). Also, $C_{\max} = \max\{c_{ij} \mid o_{ij} \in SIJ\}$ is given by the weight of a critical path in the sequence graph. The decision problem whether a given connected directed graph is a shop graph is efficiently solved (cf. BRÄSEL/DHAMALA [3] and DHAMALA [12]). Moreover, the recognization of a sequence graph is also a polynomial solvable problem (cf. BRÄSEL et al. [5], HARBORTH [15]).

For each sequence graph $G_{MO,JO}$, we can describe the sequence (MO, JO) by a rank matrix $A = [a_{ij}]$ which contains the rank of each o_{ij} in $G_{MO,JO}$. The rank matrix A is a special latin rectangle with *sequence property* : for each integer $a_{ij} > 1$

there exists $a_{ij} - 1$ in row i or in column j or in both. Recall that a *latin rectangle* $LR\,[n, m, q] = [l_{ij}]$ is a matrix of size $n \times m$ with its entries $l_{ij} \in \{1, 2, \ldots, q\}$ such that each integer of the symbol set occurs at most once in each row and at most once in each column of LR (see DÉNES/ KEEDWELL [11]). If $n = m = q$ holds, then the matrix is a latin square of order n and is denoted by $LS\,[n]$. On the other hand, given any latin rectangle $LR\,[n, m, q] = [l_{ij}]$ satisfying the sequence property, we can define a sequence graph by means of l_{ij} as a level of o_{ij}. Therefore, there exists a one-to-one correspondence between the set of all latin rectangles with sequence property and the set of all sequence graphs for the open shop problem (cf. BRÄSEL [9]). Moreover, the transformation of an individual member can be performed in

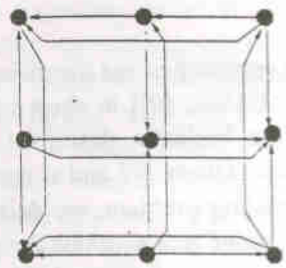

Figure 1: The Sequence Graph G_A Associated to the Sequence A

$$A = \begin{pmatrix} 5 & 4 & 3 \\ 1 & 5 & 6 \\ 6 & 1 & 2 \end{pmatrix}, \text{ where } (MO, JO) = \left(\begin{pmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 2 \\ 1 & 3 & 3 \\ 3 & 1 & 1 \end{pmatrix} \right)$$

linear time $O(nm)$ (cf. BRÄSEL [9]). The *comparability* graph of a sequence B is denoted by $[G_B^{tr}]$, where $[G]$ represents the underline undirected graph of a directed graph G. Note that a sequence contains all information about machine orders and job orders of the corresponding sequence graph.

A sequence $A \in \mathbf{S}^{nm}(\alpha)$ is called *reducible* to $B \in \mathbf{S}^{nm}(\alpha)$ if $C_{\max}(B) \leq C_{\max}(A)$ for all $P \in \mathbf{P}_{nm}$, we write $B \preceq A$. A sequence $A \in \mathbf{S}^{nm}(\alpha)$ is called *strongly reducible* to $B \in \mathbf{S}^{nm}(\alpha)$, denoted by $B \prec A$ if $B \preceq A$ but not $A \preceq B$. Two sequences $A, B \in \mathbf{S}^{nm}(\alpha)$ are called *similar*, denoted by $A \simeq B$, if $B \preceq A$ and $A \preceq B$. A sequence $A \in \mathbf{S}^{nm}(\alpha)$ is called *irreducible* if there exists no other non-similar $B \in \mathbf{S}^{nm}(\alpha)$ to which A can be reduced.

If $C_i(B) \leq C_i(A)$ for all $i \in I$ and for arbitrary processing times, then $\gamma(B) \leq \gamma(A)$ for all regular γ and all $P \in \mathbf{P}_{nm}$. A function which is monotonously nondecreasing with respect to all variables C_i is known as regular objective. The dominance relation $\preceq$ is an equivalence relation on $\mathbf{S}^{nm}(\alpha)$ decomposing the set into disjoint equivalence classes. A sequence A is irreducible if $B \preceq A$ implies

$A = B$. The irreducible elements are the minimal sequences with respect to the partial order $\prec$ and hence are the locally optimal sequences. The set of all pairwise non-similar irreducible sequences is potentially optimal for $O \parallel \gamma$. We may choose the lexicographically minimal one for the class representative for a potentially optimal sequence set $\mathbf{S}_{I^*}^{nm}(\alpha)$ of still smaller cardinality $\frac{1}{2}|\mathbf{S}_I^{nm}(\alpha)|$ for $O\|C_{\max}$. In general, an optimal solution of a shop problem is not unique. However, $B \in \mathbf{S}^{nm}(\alpha)$ is optimal for $\alpha \parallel \gamma$ if $B \preceq A$ for all $A \in \mathbf{S}^{nm}(\alpha)$.

A path w_A with vertex set $V(w_A)$ in the sequence A is called *maximal* if there does not exist another path w_A^* with $V(w_A) \subset V(w_A^*)$. Denoting the set of all maximal paths in A by $\mathbf{W}_A$, one of the paths in $\mathbf{W}_A$ becomes the longest depending on p_{ij}. Note that $B \prec A$ does not necessarily imply $C_{\max}(B) < C_{\max}(A)$ for arbitrary p_{ij}. It remains true if for each critical path w_A in A, there does not exist a maximal path in B covering all vertices in $V(w_A)$. Clearly, $B \preceq A$, if and only if for all $w_B \in \mathbf{W}_B$, there exists $w_A \in \mathbf{W}_A$ such that $V(w_B) \subseteq V(w_A)$. If $B \prec A$, then there exists $w_B \in \mathbf{W}_B$ such that $V(w_B) \subset V(w_A)$ for some $w_A \in \mathbf{W}_A$. For example, $A = \begin{pmatrix} 3 & 4 & 2 & 1 \\ 2 & 1 & 4 & 3 \end{pmatrix} \succ B = \begin{pmatrix} 3 & 4 & 2 & 1 \\ 1 & 2 & 4 & 3 \end{pmatrix}$, since the operations $\{o_{11}, o_{12}, o_{22}, o_{21}\}$ belong to a common path in G_A^{lr} but not in G_B^{lr} and whenever certain operations belong to a common path in B, these operations also belong to a common path in A.

Because of the existence of exponential number of maximal paths in a sequence graph, the decision problem whether a given sequence is reducible, similar or strongly reducible to another given sequence by definition takes exponential time. This problem is solved with time complexity $O(n^2m^2)$ (cf. BRÄSEL et al [7]) using the algorithms on the transitive closures for graphs as it has this complexity to determine the transitive closure of a sequence graph and to check if $[G_B^{lr}]$ is a subgraph of $[G_A^{lr}]$.

Theorem 1. *Let $A, B \in \mathbf{S}^{nm}(O)$. Then A is similar, str-reducible or reducible to B for $O\|C_{\max}$ if and only if $[G_B^{lr}] = [G_A^{lr}]$, $[G_B^{lr}] \subset [G_A^{lr}]$ or $[G_B^{lr}] \subseteq [G_A^{lr}]$, respectively.* □

To look on some algebraic consequences, we denote by M_{-1} the reversed matrix of the rank matrix M obtained by reversing the orientations of all arcs in the associated graph. Furthermore, we denote by π_r, π_c, Φ, and Ψ, respectively, a *row permutation* $\pi_r \in S_n$, a *column permutation* $\pi_c \in S_m$, a transposition $\Phi \in Z_2$, and a *reversion* $\Psi \in Z_2$ of a matrix, where Z_2 is the cyclic group of order two. Two given sequences A and B are called *structure isomorphic*, *graph isomorphic* or *permutation isomorphic*, denoted by $A \cong_s B$, $A \cong_g B$ or $A \cong_p B$, if there exists a mapping such that $(\pi_r, \pi_c, \Phi, \Psi)\, A = B$, $(\pi_r, \pi_c, \Phi)\, A = B$ or $(\pi_r, \pi_c)\, A = B$, respectively (see BRÄSEL et al [6]). For example, because

$((2\ 1\ 3), (2\ 1\ 3), \psi)\ \left(\begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix}\right) = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix}$, the mapping $((2\ 1\ 3), (2\ 1\ 3), \psi)$ is a structure automorphism. Note that for given two sequences A and B of the same format $n \times m$, the decision problem whether these sequences are isomorphic can be solved in $O(\min\{mn^2, nm^2\})$ time. The concepts of isomorphisms of sequences can be extended for the corresponding machine order matrices, too.

Let $G(X)$ be the group of isomorphisms of sequences of a certain type. Then for each sequence $A \in S^{nm}(O)$, the sets $\{f \in G(X) : f(A)=A\}$ and $\{f(A) : f \in G(X)\}$ are the *stabilizer* and the *orbit* of A, respectively, satisfying :

$$|\{f \in G(X) : f(A) = A\}| \cdot |\{f(A) : f \in G(X)\}| = |G(X)|.$$

The elements of the stabilizer of A are the *sequence automorphisms*, whereas the orbits are the isomorphic classes decomposing the set of all sequences into disjoint equivalent classes. Given a system of distinct representatives (SDR) for each isomorphic class, the total number of sequences is given by the class equation:

$$|S^{nm}(O)| = \sum_{A \in SDR} |\{f(A) : f \in G(X)\}| = \sum_{A \in SDR} \frac{|G(X)|}{|\{f \in G(X) : f(A) = A\}|}.$$

Therefore, the sequence automorphisms play very important roles for determining the exact number of sequences for the classical open shop problem. The concepts of isomorphisms of sequences can be extended for the corresponding machine order matrices, too.

Given the number $s(m,k)$ of permutations of order k in the symmetric group S_m, the number of pairwise non-isomorphic machine order matrices with respect to the permutation isomorphism is $\frac{1}{m!}\sum_{k|n} s(m,k) \binom{\frac{m!}{k} + \frac{n}{k} - 1}{\frac{n}{k}}$ (cf. BRÄSEL et al [6] and HARBORTH [15]). These concepts are implemented for an enumeration of all sequences in the classical open shop problem as well. Therefore, the orders of elements in the symmetric groups play important roles for determining the total number of sequences. In this paper, we are investigating the orders of the elements in the generalized groups.

The properties of isomorphisms and irreducibility play very important roles for an enumeration of isomorphisms and irreducible classes in shop scheduling problems. The os-irreducibility with respect to the maximum completion time is invariant within each isomorphic class. Given two isomorphic sequences $A, B \in S^{nm}(O)$, the sequence A is os-irreducible with respect to C_{max} if and only if B is os-irreducible (cf. BRÄSEL et al. [7]). However, if two job shop problems have structure isomorphic machine order matrices, then there is a one-to-one mapping between the corresponding solution sets which preserve irreducibiliy.

3. The Super Sequence Group

The concept of a *row latin square* (*column latin square*) was already developed by D.A. Norton in 1952 (see DÉNES / KEEDWELL [11]). A *nonempty set X* together with an *associative binary operation* ★ forms a group $G(X) = (X, \star)$ if there exists the *identity element* and *inverse elements* for each of its member. The collections of all row latin squares and column latin squares form groups under the composition of permutations (see [11]). We denote these groups by $\mathbf{G}(\mathbf{LS}^{\mathbf{R}})$ and $\mathbf{G}(\mathbf{LS}^{\mathbf{C}})$, respectively.

Moreover, the sets **JO** and **MO** of all $n \times m$ job order matrices and machine order matrices form groups, called *job order group* and *machine order group*, under the composition of permutations columnwise and rowwise, respectively (cf. BRĂSEL/DHAMALA [3] and DHAMALA [12]). We denote these groups by **G(JO)** and **G(MO)** and the corresponding binary operations by $\bullet$ and $\circ$, respectively. The identity elements of these groups are denoted by $\mathbf{E}^* = (e^*, e^*, \dots, e^*)$ and $\mathbf{E} = (e, e, \dots, e)$, where $e^* = (1, 2, \dots, n)$ and $e = (1, 2, \dots, m)$ represent the identity permutations on the sets of n and m elements, respectively.

The product of these two groups is called the *super sequence group*, denoted by $\mathbf{G}(\mathbf{MO} \times \mathbf{JO})$, which is isomorphic to the product group $S_m^n \times S_n^m$, where S_q denotes the symmetric group on q symbols (cf. [3, 12]). This group hints many interesting algebraic characterizations in classical shop scheduling problems.

The super sequence group corresponds to the classical open shop problem. One of its normal subgroups corresponds to the classical flow shop problem. An equivalence class with respect to the flow shop normal subgroup **N(E, JO)** contains all sequences in a classical job problem. Taking the class representative of the k^{th} equivalent class as the elements (MO_k, E^*) the factor group $(\{[(MO_k, E^*)]_{k=1}^{(m!)^n}\}, (\star))$ with respect to the flow shop normal subgroup can be described as follows :

$$[(MO_q, E^*)]\ (\star)\ [(MO_r, E^*)] = [(MO_q, E^*) \star (MO_r, E^*)] = [(MO_q \circ MO_r, E^*)].$$

A subgroup of the super sequence group is called *sequence subgroup* if it contains only sequence elements; otherwise it is called *weak sequence subgroup*. Both normal subgroups **N(E, JO)** and **N(MO, E*)** are sequence subgroups of the super sequence group whose all elements are strongly reducible for the open shop problem with $C_{\max}$ objective. The factor groups with respect to both normal subgroups are isomorphic if $n = m$. The normal subgroup $\mathbf{N(E, JO)} \cap \mathbf{N(MO, E^*)}$ corresponds to an element of the permutation flow shop.

Let π and π_{-1} be any two permutations on the finite set $X_q = \{1, 2, \dots, q\}$ such that

$$\pi = \begin{pmatrix} 1 & 2 & \dots & q \\ u_1 & u_2 & \dots & u_q \end{pmatrix} \text{ and } \pi_{-1} = \begin{pmatrix} 1 & 2 & \dots & q \\ v_1 & v_2 & \dots & v_q \end{pmatrix}$$

We say that π_{-1} is the *permutation reversion* of π if $u_k + v_k = q+1$ holds for all $k \in X_q$. Note that $(MO, JO) \in \mathbf{G}(\mathbf{MO} \times \mathbf{JO})$ is a sequence element if and only if its reversion $(MO, JO)_{-1} \in \mathbf{G}(\mathbf{MO} \times \mathbf{JO})$ is a sequence element. Moreover, independent of the processing times, $C_{\max}((MO, JO), P) = C_{\max}((MO, JO)_{-1}, P)$ holds. If we consider the following sub-families:

$$(\mathbf{MO}, \mathbf{W}^*) := \{(MO, JO) \mid MO \in \mathbf{MO}, \pi_j^* \in \{e^*, e^*_{-1}\}\}$$
$$(\mathbf{W}, \mathbf{JO}) := \{(MO, JO) \mid \pi_i \in \{e, e_{-1}\}, JO \in \mathbf{JO}\}.$$

Lemma 1. *Both* $\mathbf{H}(\mathbf{MO}, \mathbf{W}^*)$ *and* $\mathbf{H}(\mathbf{W}, \mathbf{JO})$ *are weak sequence subgroups of the super sequence group.* □

The lemma above is straightforward. The non-commutative subgroup $\mathbf{H}(\mathbf{W}, \mathbf{JO})$ represents a job shop like problem, where machine orders are already restricted to identity machine orders and/or their permutation reversions and the job orders are arbitrarily allowed. Note that one can easily obtain sequence subgroups on more restricted sub-families of this structure (cf. BRÄSEL/DHAMALA [4] and DHAMALA [12]).

Let X and Y be simultaneously nonempty subsets of the sets X_m and X_n, respectively. A combination (MO, JO) is called *constantly ordered* subject to the pair (Y, X) if $\pi_k = \pi \in S_m \wedge \pi_l^* = \pi^* \in S_n$ for all $k \in Y \wedge l \in X$. A combination (MO, JO) with additional constraints $\pi_k(j) = j \wedge \pi_l^*(i) = i$ for all $(j \in X \wedge k \in X_n) \wedge (i \in Y \wedge l \in X_m)$ is defined to be a *vectorwise stabilizer* of the pair (X, Y) with respect to the super sequence group. The collection of all constantly ordered pairs (MO, JO) with respect to (Y, X) forms a subgroup denoted by $\mathbf{H}_c(Y, X)$ of the super sequence group. Likewise, the collection of all vectorwise stabilizers of (X, Y) subject to the super sequence group is a subgroup denoted by $\mathbf{H}_v(X, Y)$ of the super sequence group.

Theorem 2. *All possible sequence elements of* $\mathbf{H}_v(\phi, \{1\})$ *and* $\mathbf{H}_v(\{1\}, \phi)$ *are strongly reducible for* $O||C_{\max}$. □

Note that both subgroups $\mathbf{H}_c(\phi, X_m)$ and $\mathbf{H}_c(X_n, \phi)$ are sequence subgroups of the super sequence group, since the pair (MO, JO) is a sequence if the condition $MO = (\pi, \pi, \dots, \pi)$ or $JO = (\pi^*, \pi^*, \dots, \pi^*)$ holds. Moreover, the sequence subgroup $\mathbf{H}_c(X_n, \phi)$ enumerates the flow shop, whereas the intersection $\mathbf{N}(\mathbf{E}, \mathbf{JO}) \cap \mathbf{H}_c(\phi, X_m)$ corresponds to the permutation flow shop. Clearly, the sequence

subgroup $\mathbf{H}_c(X_n, X_m)$ contains the classes of all constantly ordered machine order matrices and all constantly ordered job order matrices.

Theorem 3. *All elements of* $\mathbf{H}_c(\phi, X_m)$ *and* $\mathbf{H}_c(X_n, \phi)$ *are strongly open shop reducible with respect to the makespan objective.*

In particular, each $(MO, JO) \in \mathbf{H}_c(X_n, X_m)$ is strongly reducible for the open shop problem with makespan objective. It is remarkable that there exist some sequence subgroups which contain a potentially optimal sequence set in the sequence space of the flow shop problem $F \parallel \gamma$, with γ a regular objective. We can restate the results of CONWAY et al. [10] that a potentially optimal sequence set for the problem $F \parallel \gamma$ is $\{(MO, JO) \mid MO = E, JO = (\pi_1^*, \pi_2^*, \dots, \pi_m^*)\}$ where $\pi_1^* = \pi_2^*$ and γ is any regular function. Moreover, if the objective function is restricted to the makespan criterion, the potentially optimal solution set for the flow shop is $\{(MO, JO) \mid MO = E, JO = (\pi_1^*, \pi_2^*, \dots, \pi_{m-1}^*, \pi_m^*)\}$ where $\pi_1^* = \pi_2^*$ and $\pi_{m-1}^* = \pi_m^*$. Note that both sets form certain sequence subgroups of the super sequence group in the flow shop.

We refer to DHAMALA [12] for many quite interesting algebraic characterizations of shop scheduling solution spaces.

Theorem 4. *To each sequence subgroup* $\mathbf{H}$ *of the super sequence group, there exists an isomorphic group* $(\mathbf{LR}^{\mathbf{P}}, \#)$ *in the set of all sequences of classical open shop problem.*

Proof : Let **H** be any sequence subgroup of the super sequence group. We define a mapping $\Sigma : \mathbf{LR} \to \mathbf{MO} \times \mathbf{JO}$ such that $A \mapsto (MO, JO)$, where **LR** denotes the set of all $n \times m$ sequences in the open shop. Then Σ is a one-to-one mapping of sequence set **LR** into the product space $\mathbf{MO} \times \mathbf{JO}$. Let $LR_1, LR_2 \in \mathbf{LR}$ be any two sequences. Then we define an operation # on the domain of the mapping Σ with

$$\begin{aligned} LR_1 \# LR_2 &= \Sigma^{-1}((MO_1, JO_1)) \# \Sigma^{-1}((MO_2, JO_2)) \\ &= \Sigma^{-1}((MO_1 \circ MO_2, JO_1 \cdot JO_2)) = LR_3 . \end{aligned}$$

The isomorphic group $\Sigma^{-1}(\mathbf{H})$ in the set of all sequences is denoted by $(\mathbf{LR}^{\mathbf{P}}, \#)$. □

4. Maximal Orders

In a finite group $G(X)$, the order of $x \in G(X)$ is the least positive integer, denoted by $o(x)$, such that $x^{o(x)}$ yields the identity element of the group. Various attempts have been made in order to calculate the effective bounds for the maximal order of an element and the largest prime dividing the maximal order in the symmetric group, but up to now no exact formula is obtained. We refer to (e.g. GRANTHAM [14], MILLER [19] and MASSIAS et al. [18]) and the articles cited therein

for the different improved results. Note that more effort has been made for the error estimation in the various approximations (see [18]).

The *Landau function* $g(n)$ on the maximal order of the symmetric group S_n, which is obviously nondecreasing, is based on the formula

$$g(n) = \max \{lcm(n_1, n_2, \ldots, n_k) \mid \sum_i n_i = n\} = \max \{\prod p_i^{\alpha_i} \mid \sum_i p_i^{\alpha_i} \leq n\}$$

where $(n_1, n_2, \ldots, n_k)$ is a partition of n, p_i's are distinct primes, $\alpha_i \geq 1$ for all i and the product $\prod_i p_i^{\alpha_i}$ represents the decomposition of n into prime factors. We say that a function ϕ_1 is *asymptotic* to a function ϕ_2, denoted by $\phi_1(n) \sim \phi_2(n)$, if $\lim_{n\to\infty} \frac{\phi_1(n)}{\phi_2(n)} = 1$. Note that the relation $\sim$ is transitive. Already in 1903, LANDAU proved that $\lim_{n\to\infty} \frac{\log g(n)}{\sqrt{n \log n}} = 1$, where $g(n) = \max\{o(\pi) \mid \pi \in S_n\}$ and log denotes the natural logarithm.

Theorem 5. Let p *be the largest prime such that the sum of all primes less than* $\bar{p}$ *does not exceed n, and let* $f(n)$ *be the product of all primes less than* $\bar{p}$. *Then* $\log g(n) \sim \log f(n)$. *Moreover*, $\log f(n) \sim \sqrt{n \log n}$. □

A proof of Theorem 5 (see MILLER [19]) makes frequent use of the *Prime Number Theorem* which basically states a deep number theoretic result known in the literature that, if $p(x)$ be the number of primes p less than or equal to x then $p(x) \sim \frac{x}{\log x}$.

The maximal order of elements in the group of row (column) latin rectangles is not frequently considered. For $MO = (\pi_1, \pi_2, \ldots, \pi_n) \in MO$, if l_k be the order of π_k $(k = 1, 2, \ldots, n)$ then $o(MO) = lcm\ (l_1, l_2, \ldots, l_n)$ holds in the group $\mathbf{G(MO)}$. Moreover, for $(MO, JO) \in (MO, JO)$, we have $o((MO, JO)) = lcm\ (o(MO), o(JO))$. LAYWINE/ MULLEN [17] sketch a proof, referring to J. DÉNES and H. RETIKIN, on the maximal order of the group of row latin squares. We define the following functions of maximal orders by

$$g_r(n,m) = \max \{o(MO) \mid MO \in \mathbf{G(MO)}\} \ \&\ g_c(n,m) = \max \{o(JO) \mid JO \in \mathbf{G(JO)}\}.$$

Lemma 2. *The maximal order of an element of* $\mathbf{G(LS^R)}$ *is* $g_r(n,n) = \prod_{k=1}^{t(n)} q_k^{s_k}$, *where* q_k *denotes the* k^{th} *prime less than or equal to n and* s_k *is the largest integer such that* $q_k^{s_k} \leq n$ *for all* $k = 1, 2, \ldots, t(n)$. □

For $n = m$ $g_c(n,n) = g_r(n, n)$ as the groups $\mathbf{G}(\mathbf{LS^R})$ and $\mathbf{G}(\mathbf{LS^C})$ are isomorphic. The maximal order of an element of the group of machine (job) order matrices is obtained by DHAMALA [12] and BRÄSEL/DHAMALA [3].

Theorem 6. *Let* $\mathbf{G}(\mathbf{MO})$ *be the group of machine order matrices* $MO\,[n,m,m]$, *and let* $n \geq t(m)$ *where* $t(m)$ *denotes the number of all primes not exceeding m. Then the maximal order of an element of the machine order group is*

$$g_r\,(n,m) = \prod_{k=1}^{t(m)} q_k^{s_k} \tag{1}$$

where q_k *denotes the* k^{th} *prime less than or equal to m, and* s_k *is the largest integer such that* $q_k^{s_k} \leq m$ *for all* $k = 1, 2, \ldots, t(m)$.

Proof: First we note that the stated formula is nothing but a unique prime factorization of the integer $g_r(n,m)$. Here, the factors are pairwise relatively prime to each other and, therefore, their product gives the least common multiple of the factors. Furthermore, we construct a machine order matrix of order $g_r(n,m)$ and show that there exists no machine order matrix of higher order than $g_r(n,m)$.

Without loss of generality, let π_k be the k^{th} row of a machine order matrix consisting only of one cycle structure of length $q_k^{s_k}$ and the remaining part of this row with arbitrary cycles of lengths not exceeding the number $m - q_k^{s_k}$ for $k = 1, 2, \ldots, t(m)$. Moreover, each of the remaining rows (if any) π_k, where $k = t(m) + 1, \ldots, n$, can be replaced by arbitrary permutations on m symbols. Then it is easily seen that the order of such constructed machine order matrix is $g_r(n,m)$. Finally, as the least common multiple is already represented by the unique factorization, we further claim that there is no any other machine order matrix with a larger order than $g_r(n,m)$. □

Corollary 1. *Let* $m \geq s(n)$, *where* $s(n)$ *denotes the number of all prime numbers not exceeding n. Then the maximal order of an element of the job order group* $\mathbf{G}(\mathbf{JO})$ *is*

$$g_c(n,m) = \prod_{k=1}^{s(n)} q_k^{t_k} \tag{2}$$

where q_k *denotes the* k^{th} *prime less than or equal to n, and* t_k *is the largest interger such that* $q_k^{t_k} \leq n$ *for all* $k = 1, 2, \ldots, s(n)$. □

Corollary 2. *Let* $n \geq t(m) = \sum_{p \leq m} 1$, *and* $m \geq s(n) = \sum_{p \leq n} 1$ *where p is a prime number. Then the maximal order of an element of the super sequence group is*

$$g_{rc}(n,m) = \max\,\{g_r(n,m),\ g_c(n,m)\}$$

where $g_r(n,m)$ *and* $g_c(n,m)$ *are given by Formulas* (1) *and* (2), *respectively.* □

The example in Section 2 associated to Figure 1 has the maximal order 6 for the super sequence group with $n = m = 3$. Note that our all formulas on maximal orders depend on the approximation functions of prime number for large values of n and m. On the other hand, if we release the restrictions on the number of rows or/and columns, our results yield only upper bounds on the maximal order of the elements of the corresponding groups.

5. Concluding Remarks

In this paper, some algebraic concepts of row latin squares are generalized. We study the machine order group, job order group and the super sequence group in classical shop scheduling problems which yield a practical classification and some interpretations of the considered shop problems. As a consequence, corresponding to each sequence subgroup of the super sequence group, there is associated an isomorphic group in the set of all sequences in the open shop. We cover the set of all sequences by sequence subgroups for $m = n = 2$ but it is impossible in general. Therefore, to find a minimal set of subgroups containing the minimum number of non-sequences which covers the sequence space from outside is an interesting theoretical problem in this field. From the practical point of view, the sequence subgroups have to be applied in search algorithms for solving the problems. Moreover, we present results on the maximal orders of the elements of the investigated groups. The results in this field are of both theoretical and practical interests.

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Interdependent Machining System with Spares, Controllable Arrival Rates and Additional Repairmen

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Abstract: The provision of spares and repair facility has utmost significance for machining systems of manufacturing / production processes. This paper deals with machine repair system consisting of operating units along with spare units under the care of a repair facility having permanent as well as removable additional repairmen. It is considered that machines are interdependent. The breakdown times and renovation times of units are exponentially distributed. The repair facility controls the breakdown units by rendering the service in FCFS order. The mean queue length is obtained by using recursive method.

Key words: Machine repair, FCFS, Interdependent, Spares, Repairmen, Mean queue length.

1. Introduction

The applicability of machines can be realised in any manufacturing production organization to fulfil the desired requirements of the jobs. To design a better machining system, the system designer must have good technical knowledge of performance prediction apart from sufficient knowledge of various designs issues.

Several queue theorist have studied machining systems in different frameworks. Gaglio and Wagner (1964) gave an approximate solution for three-machine scheduling problem. Wang (1990) studied machine repair problem with single service station subject to breakdowns and analysed the profit of the system. The cost analysis of the machine repair problem with non-reliable service stations was investigated by Wang and Hsu (1995). Two-machine production line with state-dependent rate was considered by Jain (1999). Jain et al. (2002) also studied a flexible manufacturing system to evaluate its performance indices. A little bit literature whatsoever is available in which the skilful research workers conducted their research works incorporating the provision of spares provisioning, is worth mentioning. Some useful works in this direction are also made by many researchers.

Natarajan (1968) analysed a reliability problem with spares and multiple repair facilities. Srinivasan and Gopalan (1973) studied a two-unit system having warm standby spares and estimated availability and reliability of the system. Jain (1997) analysed state-dependent (m.M) machine repair problem with spares.

The facility of additional repairmen may be cost effective to improve the system availability in particular when there is sufficient backlog of failed units in the system M/M/C/K/N machine repair problem with balking, reneging, spares and additional repairman was studied by Jain et al. (2000a). Jain et al. (2000b) also considered machine repair problem with spares, reneging, additional repairman and two modes of failure. Jain and Baghel (2001) considered a multi-component repairmen repairable system with facilitating spare parts and state-dependent rates.

Agnihothri (1989) studied a machining problem with ingenuous repairmen and interrelationship between their performance measures. Gupta (1995) developed a queueing system with interrelationship between controlled arrival and service rates. An M/M/1 interdependent queueing system having controllable arrival rates was analysed by Rao et al. (2000). Recently, M/M/c interdependent queueing model with controllable arrival rates was studied by Begum and Maheswari (2002).

In present investigation, we consider a multi-component system having correlated failure and repair rates. We have incorporated the provision of spares and two additional repairmen in our model. The whole paper is ramified into four sections. In section 2, we describe assumptions and notations to describe the mathematical model. The governing equations and their solutions in explicit form are given in section 3. The average queue length is obtained. The last section 4 includes the scope of future research work and concluding remarks.

2. System Postulates with Notations

We consider an interdependent machining system consisting of M operating machines, R permanent repairmen and S Warm spares. There is provision of two additional repairmen to increase efficiency of the system in case of heavy traffic of failed machines. The following assumptions are made to elaborate the model as :

- There is a correlation between breakdown times and repair times of machines, which are exponentially distributed with mean rate $1/\lambda$ and $1/\mu$ respectively.
- Whenever a machine fails, it is sent to renovate in repair facility in order of their breakdowns.
- After renovation, the machine is as good as at the time of failure and joins the group of standby or functioning machines.
- The spare units are assumed to fail according to exponential distribution with rate α
- When all the spares are used, the operating machines fail with degraded failure rate λ.

- To reduce the queue size, the first additional repairman is provided at threshold level T_0+1 and preceded till queue length drops to Q_0.
- The second additional repairman starts repair at threshold level T_1+1 and continue till queue length drops to Q_1

3. The Equations and Analysis

Denote

$$\Lambda_n = M(\lambda - e) + (S-n)(\alpha - e) \qquad n < S$$

$$\Lambda_n = (M + S - n) + (\lambda' - e) \qquad n \geq S$$

and

$$R_j = R(\mu - e) + (\mu_j - e)$$

The steady-state equations governing the model are given as follows :

(1) $\Lambda_0 P_0(0) = (\lambda - e) P_1(0)$

(2) $[\Lambda_n + n(\mu - e)] P_n(0) = \Lambda_{n-1} P_{n-1}(0) + (n+1)(\mu - e) P_{n+1}(0), \qquad 1 \leq n < R$

(3) $[\Lambda_n + R(\mu - e)]\ P_n(0) = \Lambda_{n-1} P_{n-1}(0) + R(\mu - e) P_{n+1}(0), \qquad 1 \leq n < R$

(4) $[\Lambda'_S + R(\mu - e)] P_S(0) = \Lambda_{S-1} P_{S-1}(0) + R(\mu - e) P_{S+1}(0)$

(5) $[\Lambda'_n + R(\mu - e)] P_n(0) = \Lambda'_{n-1} P_{n-1}(0) + R(\mu - e) P_{n+1}(0), \qquad S < n < Q_0$

(6) $[\Lambda'_{Q_0} + R(\mu - e)] P_{Q_0}(0) = \Lambda'_{Q_0-1} P_{Q_0-1}(0) + R(\mu - e) P_{Q_0+1}(0) + R'_1 P_{Q_0+1}(1)$

(7) $[\Lambda'_n + R(\mu - e)] P_n(0) = \Lambda'_{n-1} P_{n-1}(0) + R(\mu - e) P_{n+1}(0), \qquad Q_0 + 1 \leq n < T_0$

(8) $[\Lambda'_{T_0} + R(\mu - e)] P_{T_0}(0) = \Lambda'_{T_0-1} P_{T_0-1}(0)$

(9) $[\Lambda'_{Q_0+1} + R'_1] P_{Q_0+1}(1) = R'_1 P_{Q_0+2}(1)$

(10) $[\Lambda'_n + R'_n] P_n(1) = \Lambda'_{n-1} P_{n-1}(1) + R'_1 P_{n+1}(1), \qquad Q_0 + 2 < n < T_0 + 1$

(11) $[\Lambda'_{T_0+1} + R'_1] P_{T_0+1}(1) = \Lambda'_{T_0} P_{T_0}(0) + \Lambda'_{T_0} P_{T_0}(1) + R'_1 P_{T_0+2}(1)$

(12) $[\Lambda'_n + R'_1] P_n(1) = \Lambda'_{n-1} P_{n-1}(1) + R'_1 P_{n+1}(1), \qquad T_0 + 2 \leq n < Q_1$

(13) $[\Lambda'_{Q_1} + R'_1] P_{Q_1}(1) = \Lambda'_{Q_1-1} P_{Q_1-1}(1) + R'_1 P_{Q_1+1}(1) + R'_2 P_{Q_1+1}(2)$

(14) $[\Lambda'_n + R'_1] P_n(1) = \Lambda'_{n-1} P_{n-1}(1) + R'_1 P_{n+1}(1), \qquad Q_1 + 1 < n < T_1$

(15) $[\Lambda'_{T_1} + R'_1] P_{T_1}(1) = \Lambda'_{T_1-1} P_{T_1-1}(1)$

(16) $[\Lambda'_{Q_1+1} + R'_2] P_{Q_1+1}(2) = R'_2 P_{Q_1+2}(2)$

(17) $[\Lambda'_n + R'_2] P_n(2) = \Lambda'_{n-1} P_{n-1}(2) + R'_2 P_{n+1}(2), \qquad Q_1 + 2 \leq n < T_1 + 1$

(18) $[\Lambda'_{T_1+1} + R'_2] P_{T_1+1}(2) = \Lambda'_{T_1} P_{T_1}(1) + \Lambda'_{T_1} P_{T_1}(2) + R'_2 P_{T_1+2}(2)$

(19) $[\Lambda'_n + R'_2] P_n(2) = \Lambda'_{n-1} P_{n-1}(2) + R'_2 P_{n+1}(2), \qquad T_1 + 2 \leq n \leq M + S$

(20) $\Lambda'_{M+S-1} P_{M+S-1}(2) = R'_2 P_{M+S}(2)$

On solving equations (1)-(4), we have

$$P_n(0) = \begin{cases} \dfrac{\prod_{i=0}^{n-1}\Lambda_i}{n!(\mu-e)^n} P_0(0), & 1 \le n \le R \\ \dfrac{\prod_{i=0}^{n-1}\Lambda_i}{R!R^{n-R}(\mu-e)^n} P_0(0), & R < n \le S \\ \dfrac{\prod_{i=0}^{S-1}\Lambda_i \prod_{i=0}^{n-1}\Lambda'_i}{R!R^{n-R}(\mu-e)^n} P_n(0), & S < n \le Q_0 \end{cases} \tag{21}$$

Substituting the values of $P_{Q_0-1}(0)$ and $P_{Q_0}(0)$ from equation (21) in equations (5)-(7), we get

$$P_{Q_0+1}(0) = \frac{\prod_{i=0}^{S-1}\Lambda_i \prod_{i=S}^{Q_0}\Lambda'_i}{R!R^{Q_0+1-R}(\mu-e)^{Q_0+1}} P_{Q_0}(0) - \left[1+\frac{(\mu'_1-e)}{R(\mu-e)}\right] P_{Q_0+1}(1) \tag{22}$$

and in general we have

$$P_n(0) = \frac{\prod_{i=0}^{S-1}\Lambda_i \prod_{i=S}^{n-1}\Lambda'_i}{R!R^{n-R}(\mu-e)^n} P_n(0) - \left[1+\sum_{i=Q_0+1}^{n-1}\prod_{j=i}^{n-1}\frac{\Lambda'_j}{R(\mu-e)}\right] \times \left[1+\frac{(\mu_1-e)}{R(\mu-e)}\right] P_{Q_0+1} \;\cdots (1) \qquad Q_0 < n \le T_0 \tag{23}$$

Now substituting the value of $P_{T_0-1}(0)$ and $P_{T_0}(0)$ from equation (23), equation (8) yields

(24) $P_n(0) =$

$$\frac{\prod_{i=0}^{S-1}\Lambda_i \prod_{i=S}^{T_0}\Lambda_i}{R!R^{T_0-R}(\mu-e)^{T_0}\left[\sum_{i=Q_0+1}^{T_0-1}\prod_{j=1}^{T_0-1}\frac{\Lambda'_i}{R(\mu-e)} \sum_{i=Q_0+1}^{T_0-2}\prod_{j-1}^{T_0-2}\frac{\Lambda'_i}{R(\mu-e)}\right]\left[1+\frac{(\mu_1-e)}{R(\mu-e)}\right]} P_0(0)$$

$$= WP_0(0)$$

where

$$W = \frac{\prod_{i=0}^{S-1} \Lambda_i \prod_{i=S}^{T_0} \Lambda_i'}{R!\,R^{T_0-R}(\mu-e)^{T_0}\left[\sum_{i=Q_0+1}^{T_0-1} \prod_{j=1}^{T_0-1} \frac{\Lambda_i'}{R(\mu-e)} - \sum_{i=Q_0+1}^{T_0-2} \prod_{j=1}^{T_0-2} \frac{\Lambda_i'}{R(\mu-e)}\right]\left[1+\frac{(\mu_1-e)}{R(\mu-e)}\right]}$$

Now equation (23) reduces to

$$(25) \quad P_n(0)=\left[\frac{\prod_{i=0}^{S-1} \Lambda_i \prod_{i=S}^{n-1} \Lambda_i'}{R!\,R^{n-R}(\mu-e)^n} - W\left\{\sum_{i=Q_0+1}^{n-1} \prod_{j=i}^{n-1} \frac{\Lambda_j'}{R(\mu-e)}\right\}\left[1+\frac{(\mu_1-e)}{R(\mu-e)}\right]\right] P_n(0)$$

$$Q_0 < n < T_0$$

Using equations (9), (10) and (34), we find

$$(26) \qquad P_n(0) = \left[1+\sum_{i=Q_0+1}^{n-1} \prod_{j=i}^{n-1} \frac{\Lambda_j'}{R_1'}\right] W\, P_0(0) \qquad Q_0+2 \le n < T_0+1$$

Equation (11) provides

$$P_{T_0+2}(1)=\left[\left(1+\frac{\Lambda_{T_0+1}'}{R_1'}\right)\left\{1+\sum_{i=Q_0+1}^{T_0} \prod_{j=i}^{T_0} \frac{\Lambda_j'}{R_1'}\right\} - \frac{\Lambda_{T_0}'}{R_1'}\left\{1+\sum_{i=Q_0+1}^{T_0-1} \prod_{j=i}^{T_0-1} \frac{\Lambda_j'}{R_1'}\right\}\right] W\, P_0(0)$$

$$(27) \qquad -\frac{\Lambda_{T_0}'}{R_1'}\left[\frac{\prod_{i=0}^{S-1} \Lambda_i \prod_{i=S}^{T_0-1} \Lambda_i'}{R!\,R^{T_0-R}(\mu-e)^n} - W.\prod_{i=Q_0+1}^{T_0-1}\left[1+\frac{\Lambda_1'}{R(\mu-e)}\right]\left[1+\frac{(\mu_1-e)}{R(\mu-e)}\right]\right] P_n(0)$$

With the help of equation (12), we have

$$(28) \qquad P_{T_0+3}(1) = \left(1+\frac{\Lambda_{T_0+2}'}{R_1'}\right) P_{T_0+2}(1) - \frac{\Lambda_{T_0+1}'}{R_1'} P_{T_0+1}(1)$$

...

in general equation (12) gives

$$(29) \quad P_n(1)\ \left(1+\sum_{i=T_0+2}^{n-1} \prod_{j=i}^{n-1} \frac{\Lambda_j'}{R_1'}\right) P_{T_0+2}(1) - \frac{\Lambda_{T_0+1}'}{R_1'}\left(1+\sum_{i=T_0+3}^{n-1} \prod_{j=i}^{n-1} \frac{\Lambda_j'}{R_1'}\right) P_{T_0+1}(1)\ .$$

$$T_0+4 \le n < Q_1$$

Equation (13) provides

$$P_{T_1+1}(1) = \left(1 + \frac{\Lambda'_{Q_1}}{R'_1}\right) P_{Q_1}(1) - \frac{\Lambda_{Q_1-1}}{R'_1} P_{Q_1-1}(1) \frac{R'_2}{R'_1} P_{Q_1+1}(2) \tag{30}$$

Also equation (13) yields to

$$P_{Q_1+2}(1) = \left(1 + \frac{\Lambda'_{Q_1+1}}{R'_1}\right) P_{Q_1+1}(1) - \frac{\Lambda'_{Q_1}}{R'_1} P_{Q_1}(1) \tag{31}$$

Similarly in general, we find

$$P_n(1) = \left(1 + \sum_{i=Q_1+1}^{n-1} \prod_{j=i}^{n-1} \frac{\Lambda'_j}{R'_1}\right) P_{Q_1+1}(1) - \frac{\Lambda'_{Q_1}}{R'_1}\left(1 + \sum_{i=Q_1+2}^{n-1} \prod_{j=i}^{n-1} \frac{\Lambda'_j}{R'_1}\right) P_{Q_1}(1) \tag{32}$$

$$Q_1 + 3 \le n < T_1' - 1$$

On solving equation (15), we have

$$P_{T_1}(1) = \frac{\Lambda_{T_1-1}}{\Lambda'_{T_1} R'_1} P_{T_1-1}(1) \tag{33}$$

From equation (16), we find

$$P_{Q_1+2}(2) = \left(1 + \frac{\Lambda'_{Q_1+1}}{R'_2}\right) P_{Q_1+1}(2) \tag{34}$$

Now equation (17) gives

$$P_{Q_1+3}(2) = \left(1 + \frac{\Lambda'_{Q_1+2}}{R'_2}\right) P_{Q_1+2}(2) - \frac{\Lambda'_{Q_1+1}}{R'_2} P_{Q_1+1}(2) \tag{35}$$

and in general, we have

$$P_n(2) = \left(1 + \sum_{i=Q_1+2}^{n-1} \prod_{j=i}^{n-1} \frac{\Lambda'_j}{R'_2}\right) P_{Q_1+2}(2) - \frac{\Lambda'_{Q_1+1}}{R'_2}\left(1 + \sum_{i=Q_1+3}^{n-1} \prod_{j=i}^{n-1} \frac{\Lambda'_j}{R'_2}\right) P_{Q_1+1}(2) \tag{36}$$

$$Q_1 + 4 \le n < T_1 + 1$$

The equation (18) yields

$$P_{T_1+2}(2) = \left(1 + \frac{\Lambda_{T_1+1}}{R'_2}\right) P_{T_1+1}(1) - \frac{\Lambda_{T_1}}{R'_2} P_{T_1}(2) - \frac{\Lambda_{T_1}}{R'_2} P_{T_1}(1) \tag{37}$$

Further equation (19) provides

$$P_{T_1+3}(2) = \left(1 + \frac{\Lambda'_{T_1+2}}{R'_2}\right) P_{T_1+2}(1) - \frac{\Lambda'_{T_1+1}}{R'_2} P_{T_1+1}(2) \tag{38}$$

and in general we find

$$P_n(2) = \left(1+\sum_{i=T_1+2}^{n-1}\prod_{j=i}^{n-1}\frac{\Lambda'_j}{R'_2}\right) P_{T_1+2}(2) - \frac{\Lambda'_{T_1+1}}{R'_2}\left(1+\sum_{i=T_1+3}^{n-1}\prod_{j=i}^{n-1}\frac{\Lambda'_j}{R'_2}\right) P_{T_1+1}(2) \tag{39}$$

$$T_1 + 4 \le n < M+S-1$$

Now equation (20) and (39) give

$$P_{M+S}(2) - \frac{\Lambda'_{T_1+1}}{R'_2}\left(1+\sum_{i=T_1+2}^{M+S-2}\prod_{j=i}^{M+S-2}\frac{\Lambda'_j}{R'_2}\right) P_{T_1+2}(2)$$

$$\frac{\Lambda_{T_1+1}}{R'_2}\left(1+\sum_{i=T_1+3}^{M+S-2}\prod_{j=i}^{M+S-2}\frac{\Lambda'_j}{R'_2}\right) P_{T_1+2}(2) \tag{40}$$

Mean Queue Length

The mean queue length is obtained as

$$L = \sum_{n=0}^{T_0} nP_n(0) + \sum_{n=Q_0+1}^{T_1} nP_n(1) + \sum_{n=Q_1+1}^{M+S} nP_n(2) \tag{41}$$

4. Discussion

In this paper, steady state queue size distribution for repairable machining system with spare provisioning has been determined using recursive method. By using spare parts support and controllable arrival rates in the presence of permanent as well as additional repairmen, we have studied more versatile problem of machining system. For higher productivity and efficiency, the conventional machining system has the provision of spare part support. The additional repairmen may be helpful to reduce the backlog of the system which is essential for economical as well as reliability requirement viewpoints.

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A Generalization Of Unified Common Fixed Point Theorem

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Abstract: The aim of the present paper is to obtain a common fixed point theorem for a set of four mappings, which includes all the known contractive definitions as particular cases and employs a Lipschitz type analogue of known contractive definitions. Also we provide a new type of answer to the open problem posed by Rhoades [21] on the existence of a contractive definition.

Key Words and Phrases : Common fixed point, compatible mappings, contractive conditions.

1. Introduction

The study of common fixed point of mappings satisfying contractive type conditions has attracted a great deal of research activity during the last two decades. The most general of the common fixed point theorems pertain to four mappings, say A, B, S and T of a metric space (X, d), and use either a Banach type contractive condition of the form

(1) $$d(Ax, By) \le h\, m(x, y),\ 0 \le h < 1,$$

where $m(x,y) = \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), [d(Sx, By) + d(Ax, Ty)]/2\}$, or, a Meir-Keeler type (ε, δ)-contractive condition of the form given $\varepsilon > 0$ there exists a $\delta > 0$ such that

(2) $$\varepsilon \le m(x, y) < \varepsilon + \delta \Rightarrow d(Ax, By) < \varepsilon,$$

or, a ϕ-contractive condition of the form

(3) $$d(Ax, By) \le \phi(m(x,y)),$$

involving a contractive gauge function $\phi : R_+ \to R_+$ is such that $\phi(t) < t$ for each $t > 0$.

Clearly, condition (1) is a special case of both conditions (2) and (3). A ϕ-contractive condition (3) does not guarantee the existence of a fixed point unless

some additional condition is assumed. Therefore, to ensure the existence of common fixed point under the contractive condition (3), the following conditions on the function ϕ have been introduced and used by various authors.

(I) ϕ (t) is non decreasing and $t/(t - \phi(t))$ is non increasing ([2]),

(II) ϕ (t) in non decreasing and $\lim_n \phi^n(t) = 0$ for each $t > 0$([4], [9]),

(III) ϕ is upper semi continuous ([1], [4], [8], [14]) or equivalently,

(IV) ϕ is non decreasing and continuous from right ([20]).

It is now known (e.g. [4], [16]) that if any of the conditions (I), (II), (III), or (IV) is assumed on ϕ, then a ϕ-contractive condition (3) implies an analogous (ε,δ)-contractive condition (2) and both the contractive conditions hold simultaneously. Similarly, a Meir-Keeler type (ε,δ)-contractive condition does not ensure the existence of a fixed point. The following example illustrates that an (ε,δ)-contractive condition of type (2) neither ensures the existence of a fixed point nor implies an analogous ϕ-contractive condition (3).

Example 1. ([16]) Let X = [0, 2] and d be the Euclidean metric on X. Define $f: X \to X$ by $fx = (1 + x)/2$ if $x < 1$; $fx = 0$ if $x \geq 1$. Then, it satisfies contractive condition $\varepsilon \leq \max\{d(x, y), d(x, fx), d(y, fy), [d(x, fy) + d(y, fx)]/2\} < \varepsilon + \delta$ $\Rightarrow d(fx, fy) < \varepsilon$; with $\delta(\varepsilon) = 1$ for $\varepsilon \geq 1$ and $\delta(\varepsilon) = 1 - \varepsilon$ for $\varepsilon < 1$ but f does not have a fixed point. Also f does not satisfy the contractive condition

$$d(fx, fy) \leq \phi(\max\{d(x,y), d(x, fx), d(y, fy), [d(x, fy) + d(y, fx)]/2\}),$$

since the desired function $\phi(t)$ cannot be defined at t = 1.

Hence, the two type of contractive conditions (2) and (3) are independent of each others. Thus, to ensure the existence of common fixed point under the contractive condition (2), the following condition on the function δ have been introduced and used by various authors.

(V) δ is non decreasing ([12], [14]),

(VI) δ is lower semi continuous ([6], [7]).

Jachymski [4] has shown that the (ε,δ)-contractive condition (2) with a non decreasing δ implies a ϕ-contractive condition (3). Also, Pant *et al.* [16] have shown that the (ε,δ)-contractive condition (2) with a lower semi continuous δ, implies a ϕ-contractive condition (3). Thus, we see that if additional conditions are assumed on δ then the (ε,δ)-contractive condition (2) implies an analogous ϕ-contractive condition (3) and both the contractive conditions hold simultaneously.

It is thus clear that contractive conditions (2) and (3) hold simultaneously whenever (2) or (3) is assumed with additional condition on δ or ϕ respectively. Besides the contractive condition (2), the ϕ-contractive condition is also assumed simultaneously with or even without imposing any additional restriction either on ϕ or on δ [13,16]. Such theorems not only unify the Meir-Keeler type fixed point

theorem and Boyd-Wong type fixed point but also improve them. A more general approach of generalizing these results consists of assuming Lipschitz type analogue of contractive condition. It follows, therefore, that the known common fixed point theorems can be extended and generalized if instead of assuming one of the contractive condition (2) or (3) with additional conditions on δ and, we assume contractive condition (2) together with a Lipschitz type analogue of condition (3); that is, a condition of the form

$$d(Ax, By) < \max \{d(Sx, Ty), k[d(Ax, Sx) + d(By, Ty)]/2, [d(Sx, By) + d(Ax, Ty)]/2\}, \ 1 \le k < 2.$$

We prove a common fixed point theorem for four mappings using this approach. It may be noted that all the known results have been dealt with the case $k = 1$ and so all such results are obtained as special case of our theorem.

Two self-mappings A and S of a metrix space (X, d) are called *compatible* (see Jungck [6]) if, $\lim_n d(ASx_n, SAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_n Ax_n = \lim_n Sx_n = t$ for some t in X. It is easy to see that compatible maps commute at their coincidence points.

2. Results

We prove the following theorem with the notation M(x, y) defined as

$$M(x, y) = \max \{d(Sx, Ty), d(Ax, Sx), d(By, Ty), [d(Sx, By) + d(Ax, Ty)]/2\}.$$

Theorem 1. *Let* (A, S) *and* (B, T) *be compatible pairs of self mappings of a complete metric space* (X, d) *such that*

i) $AX \subset TX$, $BX \subset SX$,

ii) given $\varepsilon > 0$ there exists $\delta > 0$ such that $\varepsilon \le M(x, y) < \varepsilon + \delta \Rightarrow d(Ax, By) < \varepsilon$, and

iii) $d(Ax, By) < \max \{d(Sx, Ty), k[d(Ax, Sx) + d(By, Ty)]/2$
$[d(Sx, By) + d(Ax, Ty)] / 2\}, \ 1 \le k < 2.$

If one of the mappings A, B, S and T is continuous then A, B, S and T have unique common fixed point.

Proof : Let x_0 be any point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X given by the rule

(1.1) $$y_{2n} = Ax_{2n} = Tx_{2n+1} \ ; \ y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$$

This can be done by virtue of (i). We claim that $\{y_n\}$ is a Cauchy sequence.

Using (ii), we get $d(y_{2n}, y_{2n+1}) = d(Ax_{2n}, Bx_{2n+1}) \le M(x_{2n}, x_{2n+1}) < d(y_{2n-1}, y_{2n})$.

Similarly, we get $d(y_{2n-1}, y_{2n}) < d(y_{2n-2}, y_{2n-1})$ and so on.

Thus, $\{d(y_n, y_{n+1})\}$ is a strictly decreasing sequence of positive numbers and, therefore, tends to a limit $r \geq 0$. If possible, suppose $r > 0$. Then given $\delta > 0$ there exists a positive integer N such that for each $n \geq N$, we have

(1.2) $$r < d(y_{2n}, y_{2n+1}) = d(Ax_{2n}, Bx_{2n+1}) \leq M(x_{2n}, x_{2n+1}) < r + \delta.$$

Selecting δ in (1.2) in accordance with (ii), for each $n \geq N$, we get

$d(y_{2n+2}, y_{2n+1}) = d(Ax_{2n+2}, Bx_{2n+1}) < r$. This, in turn, gives

$d(y_{2n+3}, y_{2n+2}) < d(y_{2n+1}, y_{2n+2}) < r$. This contradicts (1.2). Hence, $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$.

We now show that $\{y_n\}$ is a Cauchy sequence.

Suppose it is not, then there corresponds an $\varepsilon > 0$ and a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $d(y_{n_i}, y_{n_{i+1}}) > 2\varepsilon$. Selecting δ in (ii), so that $0 < \delta < \varepsilon$. Since $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$, so there exists an integer N such that $d(y_n, y_{n+1}) < \delta / 6$ whenever $n \geq N$. Let $n_i \geq N$ then there exists integer m_i satisfying $n_i < m_i < n_{i+1}$ such that $d(y_{n_i}, y_{m_i}) > \varepsilon + (\delta / 3)$. If not, then

$$d(y_{n_i}, y_{n_{i+1}}) \leq d(y_{n_i}, y_{n_{i+1}-1}) + d(y_{n_{i+1}-1}, y_{n_{i+1}}) < \varepsilon + (\delta / 3) + (\delta / 6) < 2\varepsilon,$$

which is a contradiction. Now, without loss of generality, we can assume n_i to be odd.

Let m_i be the smallest even integer such that $d(y_{n_i}, y_{m_i}) > \varepsilon + (\delta / 3)$. Then $d(y_{n_i}, y_{m_i-2}) < \varepsilon + (\delta / 3)$ and

$$\varepsilon + (\delta / 3) < d(y_{n_i}, y_{m_i}) \leq d(y_{n_i}, y_{m_i-2}) + d(y_{m_i-2}, y_{m_i-1}) + d(y_{m_i-1}, y_{m_i})$$
$$< \varepsilon + (\delta / 3) + (\delta / 6) + (\delta / 6) = \varepsilon + (2\delta / 3);$$

that is,

(1.3) $$\varepsilon + (\delta / 3) < d(y_{n_i}, y_{m_i}) \leq \varepsilon + (2\delta / 3).$$

Also, using (ii), we get $d(y_{n_i+1}, y_{m_i+1}) \leq M(x_{n_i+1}, x_{m_i+1}) < \varepsilon + (2\delta / 3) + (\delta / 6) < \varepsilon + \delta$;

that is, $\varepsilon + (\delta / 3) < M(x_{n_i+1}, x_{m_i+1}) < \varepsilon + \delta$. Using (ii), we get $d(y_{n_i+1}, y_{m_i+1}) < \varepsilon$.

But then, $$d(y_{n_i}, y_{m_i}) \leq d(y_{n_i}, y_{n_i+1}) + d(y_{n_i+1}, y_{m_i+1}) + d(y_{m_i+1}, y_{m_i})$$
$$< (\delta / 6) + \varepsilon + (\delta / 6) = \varepsilon + (\delta / 3),$$

which contradicts (1.3). Hence $\{y_n\}$ is a Cauchy sequence in X. But X is complete so there exists a point z in X such that $y_n \to z$. Also, using (1.1), we have

(1.4) $$y_{2n} = Ax_{2n} = Tx_{2n+1} \to z \text{ and } y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \to z.$$

Suppose that S is continuous. Then $SSx_{2n} \to Sz$, $SAx_{2n} \to Sz$ and compatibility of A and S implies that $ASx_{2n} \to Sz$. Also, since $AX \subset TX$, corresponding to each value of n, there exists z_{2n} in X such that $ASx_{2n} = Tz_{2n}$. Thus,

$ASx_{2n} = Tz_{2n} \to Sz$ and $SSx_{2n} \to Sz$. We show that $\lim_n Bz_{2n} = Sz$. If not, then there exists a subsequence $\{Bz_{2m}\}$ of $\{Bz_{2n}\}$, a number $r > 0$ and a positive integer N such that for each $m > N$, we have $d(ASx_{2m}, Bz_{2m}) \geq r$, $d(Sz, Bz_{2m}) \geq r$ and in view of (iii), we get

$$d(ASx_{2m}, Bz_{2m}) < \max\{d(SSx_{2m}, Tz_{2m}),\ k[d(ASx_{2m}, SSx_{2m}) + d(Bz_{2m}, Tz_{2m})]/2,\ [d(SSx_{2m}, Bz_{2m}) + d(ASx_{2m}, Tz_{2m})]/2\},$$

which, on letting $m \to \infty$, yields

$$d(Sz, Bz_{2m}) \leq k[d(Sz, Bz_{2m})]/2 < d(Sz, Bz_{2m}), \text{ a contradiction.}$$

Hence $\lim_{n\to\infty} Bz_{2n} = Sz$. We claim that $Az = Sz$. If $Az \neq Sz$, then by virtue of (iii), for sufficiently large values of n, we get

$$d(Az, Bz_{2n}) < \max\{d(Sz, Tz_{2n}),\ k[d(Az, Sz) + d(Bz_{2n}, Tz_{2n})]/2,\ [d(Sz, Bz_{2n}) + d(Az, Tz_{2n})]/2\},$$
$$= k[d(Az, Bz_{2n})]/2.$$

On letting $n \to \infty$, this yields $d(Az, Sz) < k[d(Az,Sz)]/2 < d(Az, Sz)$, a contradiction. Hence $Az = Sz$. Again, since $AX \subset TX$, there exists a point w in X such that $Az = Tw$. If $Bw \neq Tw$, using (iii) we get

$$d(Az, Bw) < \max\{d(Sz, Tw),\ k(d(Az, Sz) + d(Bw, Tw)]/2,\ [d(Sz, Bw) + d(Az, Tw)]/2\}.$$
$$= k[d(Bw, Tw)]/2 < d(Bw, Tw) = d(Bw, Az), \text{ a contradiction.}$$

Hence $Az = Bw$ and so, $Sz = Az = Tw = Bw$.

Since compatible maps commute at their coincidence points, we get $ASz = SAz$ and $BTw = TBw$. Moreover, $AAz = ASz = SAz = SSz$ and $BBw = BTw = TBw = TTw$.

If $Az \neq AAZ$, using (iii), we find

$$d(Az, AAz) = d(AAz, Bw) < \max\{d(SAz, Tw),\ k[d(AAz, SAz) + d(Bw, Tw)]/2,\ [d(SAz, Bw) + d(AAz, Tw)]/2\}.$$
$$= k[d(Bw, AAz)]/2 < d(AAz, Bw), \text{ a contradiction.}$$

So that $Az = AAz = SAz$ and so Az is a common fixed point of A and S. Similarly, Bw (=Az) is a common fixed point of B and T. Uniqueness of the common fixed point follows from (ii). The proof is similar when T is assumed continuous in place of S. Moreover, since $AX \subset TX$ and $BX \subset SX$, the proof follows on similar lines when A or B is assumed to be continuous. This establishes the theorem.

We now give an example to illustrate the above theorem.

Example 2. Let $X = [2, 20]$ and d be the Euclidean metric on X. Define A, B, S and T: $X \to X$ as follows :

$Ax = 2$ for each x:

$Bx = 2$ if $x < 4$ or ≥ 5, $Bx = 3 + x$ if $4 \leq x < 5$;

$Sx = x$ if $x \leq 8$. $Sx = 8$ if $x > 8$;

$Tx = 2$, if $x < 4$ or ≥ 5, $Tx = 9 + x$ if $4 \leq x < 5$.

Then A, B, S and T satisfy all the conditions of the above theorem and have a unique common fixed point $x = 2$. It can be seen in this example that A, B, S and T satisfy the condition (ii) when $\delta(\varepsilon) = 1$ if $\varepsilon \geq 6$ and $\delta(\varepsilon) = 6-\varepsilon$ if $\varepsilon < 6$. Thus, $\delta(\varepsilon)$ is neither non decreasing nor lower semi continuous. It can also be verified that the mappings A, B, S and T do satisfy the contractive condition (iii) with $k = 1$. However, A, B, S, and T do not satisfy the ϕ-contractive condition (3) since the required function $\phi(t)$ can not be defined at $t = 6$. Hence we see that the present example does not satisfy the conditions of any previously known common fixed point theorem for contractive type mappings, since neither the mappings satisfy a ϕ-contractive condition nor δ is lower semi continuous or non decreasing.

Now, as a corollary of Theorem 1, we obtain the following Theorem 2, which provides a new type or affirmative answer to an open problem (e.g. see Rhoades [21], p. 242) on the existence of a contractive definition, which is strong enough to generate a fixed point but does not force the map to be continuous at their common fixed point. It may be observed in this context that fixed point theorems either explicitly assume continuity of mappings or, as shown by Rhoades [21] and Hicks and Rhodes [3], the contractive definitions themselves imply continuity at the fixed point. This makes, the next theorem an interesting result.

Theorem 2. *Let f be a self mapping of a complete metric space* (X, d) *such that for any* x, y *in* X (i) *given* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\varepsilon \leq \max\{d(x, y), d(x, fx), d(y, fy), [d(x, fy) + d(fx, y)]/2\} < \varepsilon + \delta \Rightarrow d(fx, fy) < \varepsilon,$$

and

(ii) $d(fx, fy) \leq \max\{d(x, y), k[d(x, fx) + d(y, fy)]/2, [d(x, fy) + d(fx, y)]/2\}$,

where $\phi : R_+ \rightarrow R_+$ is such that $\phi(t) < t$ for each $t > 0$. Then f has a unique fixed point.

Proof : Theorem 2 follows from Theorem 1 by taking $S = T = I_X$, an identity mapping on X and $A = B = f$, together with $k = 1$ in Theorem 1.

It may also be noted that f need not be continuous in Theorem 2, as illustrated in the next example.

Example 3. Let $X = [0, 10]$ and d is the usual metric on X. Define $f : X \rightarrow X$ as follows: $fx = (x^2 + 5)/2$ if $x \leq 5$ and $fx = 0$ if $x > 5$. Then f satisfies all the conditions

of the above Theorem 2 and has a unique fixed point $x = 5$. It can be verified that conditions (i) and (ii) of Theorem 2 are satisfied with $\delta(\varepsilon) = 5$ if $\varepsilon \geq 5$ and $\delta(\varepsilon) = 5 - \varepsilon$ if $\varepsilon < 5$ when $k = 1$. It can be seen that f is discontinuous at the fixed point $x = 5$.

Remarks : As various assumptions either on ϕ or on δ have been considered to ensure the existence of common fixed points under contractive conditions, so our Theorem 1 improves the results of Boyd and Wong [1], Carbone et al. [2], Matkowski [9], Pant [11, 12, 15], Pant and Pant [13, 14], Park and Rhoades [20], Singh and Kashhara [23], Jungck [6], Jungck *et al.* [7]. Jachymski [4, 5], Maiti and Pal [8], and Park and Bae [19] for the case when $k = 1$. All such results are obtained as special case of Theorem 1 when $k = 1$. Also our theorem, thus, generalizes the results of Pant [15], Pant and Pant [13, 14], Pant *et al.* [16], Pant and Jha [17, 18] and all other similar results for fixed points and allows k to take values other than 1 by taking a Lipschitz type contractive condition. Moreover Theorem 2 provides a new type of affirmative answer to the open problem since our theorem assumes Lipschitz type analogue of a plane contractive condition instead of a ϕ-contractive condition.

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On Submanifolds Immersed In A HSU-Quaternion Manifold

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Abstract: Integrability conditions of an almost quaternion manifold were studied by Yano and Ako [4]. Quaternion submanifolds of co-dimension r have been defined and studied by Prof. A. Hamoui [1] and others. In this paper, we have defined a Hsu-quaternion manifold and showed that a submanifold of codimension r of the Hsu quaternion manifold admits Hsu-$(F, \underset{x}{U}, u^x, \eta^y_x)$-structure.

1. Preliminaries

A Hsu-quaternion manifold is the manifold M^{4n} admitting a set of tensor fields $\overset{*}{F}, \overset{*}{G}, \overset{*}{H}$ of type (1,1) satisfying following relations [2].

$$(1.1)\qquad \overset{*}{F}{}^2 = a^r I_n, \quad \overset{*}{G}{}^2 = b^r I_n$$

$$\overset{*}{H}{}^2 = c^r I_n ; \quad 0 \le r \le n \text{ and } c^r = a^r b^r$$

I_n being identity operator ; a, b, c complex numbers and r an integer such that

$$(1.2)\qquad \begin{aligned} &(a) \quad b^r \overset{*}{F} = \overset{*}{G}\overset{*}{H} = \overset{*}{H}\overset{*}{G} \\ &(b) \quad a^r \overset{*}{G} = \overset{*}{H}\overset{*}{F} = \overset{*}{F}\overset{*}{H} \\ &(c) \quad \overset{*}{H} = \overset{*}{F}\overset{*}{G} = \overset{*}{G}\overset{*}{F} \end{aligned}$$

2. Structure in the submanifold M^{4n-r}

Let M^{4n-r} be a submanifold of co-dimension r of the Hsu-quaternion manifold M^{4n}. Let τ denotes the immersion $M^{4n-r} \to M^{4n}$. If $B = d\tau$, then a vector field X tangent to M^{4n-r} corresponds to a vector field BX tangent to M^{4n}.

Let $\underset{x}{N}$, $x = 1, 2, \ldots, r$ be r mutually orthogonal unit normals to M^{4n-r}. The transformation for $\overset{*}{F}BX$ and $\overset{*}{F}\underset{x}{N}$ can be expressed in the form

$$\overset{*}{F}BX = BFX + \sum_{x=1}^{r} \overset{x}{u}(x)\underset{x}{N} \tag{2.1}$$

F is a tensor field of type (1,1) and $\overset{x}{u}$ are $r(C^{\infty})$ 1- forms on submanifold M^{4n-r}.

Also,

$$\overset{*}{F}\underset{x}{N} = -B\underset{x}{U} + \sum_{y=1}^{r} \eta_x^y \underset{y}{N} \tag{2.2}$$

$\underset{x}{U}$, $x = 1, 2, \ldots, r$ are $r(C^{\infty})$ vector fields on the submanifold M^{4n-r}

Similarly for the tensor fields $\overset{*}{G}$ and $\overset{*}{H}$ we have the following set of transformations

$$\text{(2.3)} \quad \text{(a)} \quad \overset{*}{G}BX = BGX + \sum_{x=1}^{r} \overset{x}{v}(X)\underset{x}{N}$$

$$\text{(b)} \quad \overset{*}{G}\underset{x}{N} = -B\underset{x}{V} + \sum_{y=1}^{r} \eta_x^y \underset{y}{N}$$

$\overset{x}{v}$ and $\underset{x}{V}$ $r(C^{\infty})$ 1-forms and vector fields respectively. G is the tensor field of type (1,1) on the submanifold M^{4n-r}.

$$\text{(2.4)} \quad \text{(a)} \quad \overset{*}{H}BX = BHX + \sum_{x=1}^{r} \overset{x}{w}(X)\underset{x}{N}$$

$$\text{(b)} \quad \overset{*}{H}\underset{x}{N} = -B\underset{x}{W} + \sum_{y=1}^{r} \eta_x^y \underset{y}{N}$$

$\overset{x}{w}$, $\underset{x}{W}$ and H have their usual meanings as in (2.2) and (2.3).

A manifold Vm is said to possess Hsu-$(F, \underset{x}{U}, u^x, \eta_x^y)$- structure if there exists tensor F of type (1,1), $r(C^\infty)$ vector fields $\underset{x}{U}$, $r(C^\infty)$ 1-forms $\overset{x}{u}$, $x = 1,2,\ldots, r$ and scalar functions η_x^y satisfying

$$\text{(a)} \qquad F^2 = a^r I_n + \sum_{x=1}^{r} \overset{x}{u} \otimes \underset{x}{U}$$

$$\text{(b)} \qquad \overset{y}{u}\, oF = \sum_{x=1}^{r} \eta_x^y\, \overset{x}{u} = 0$$

$$\text{(c)} \qquad F\underset{x}{U} + \sum_{y=1}^{r} \eta_x^y\, \underset{y}{U} = 0$$

$$\text{(d)} \qquad -\overset{z}{u}(\underset{x}{U}) + \sum_{y=1}^{r} \eta_y^z\, \eta_x^y = a^r I_n$$

A manifold Vm will be to possess Hsu-$(F, G, H, \underset{x}{U}, \underset{x}{V}, \underset{x}{W}, \overset{x}{u}, \overset{x}{v}, \overset{x}{w}, \eta_x^y)$- structure if there exists tensor fields F, G, H each of type (1,1), $r(C^\infty)$ vector fields $\underset{x}{U}, \underset{x}{V}, \underset{x}{W}$, and $r(C^\infty)$ 1-forms $\overset{x}{u}, \overset{x}{v}, \overset{x}{w}$ and scalar functions η_x^y satisfies

$$\text{(a)} \qquad GH = b^r F + \sum_{x=1}^{r} \overset{x}{w} \otimes \underset{x}{V}$$

$$\text{(b)} \qquad \overset{y}{v}\, oH = b^r \overset{y}{u} - \sum_{x=1}^{r} \eta_x^y\, \overset{x}{w}$$

$$\text{(c)} \qquad G\underset{x}{W} + \sum_{y=1}^{r} \eta_x^y\, \underset{y}{V} = b^r \underset{x}{U}$$

$$\text{(d)} \qquad -\overset{z}{v}(\underset{x}{W}) + \sum_{y=1}^{r} \eta_x^y\, \eta_y^z = \eta_x^z$$

$$, x, y, z = 1, 2, \ldots,$$

3. Some results

In this section, we shall prove some theorems on the submanifolds M^{4n-r}

Theorem 3.1. *A submanifold* M^{4n-r} *of co-dimension* r *of the Hsu-quaternion manifold* M^{4n} *admits* $(F, \underset{x}{U}, \overset{x}{u}, \eta_x^y)$*-structure.*

Proof: Applying $\overset{*}{F}$ to (2.1) and making use of equations (2.2) and (1.1) we get

$$a^r BX = BF^2X + \overset{y}{u}(FX)\underset{y}{N} + \sum_{x=1}^{r} \overset{x}{u}(X)\left\{-B\underset{x}{U} + \sum_{y=1}^{r} \eta_x^y \underset{y}{N}\right\}$$

Comparison of tangent and normal vector fields gives

(3.1)

(a) $$F^2 = a^r I_n + \sum_{x=1}^{r} \overset{x}{u} \otimes \underset{x}{U} \text{ and}$$

(b) $$\overset{y}{u}\, oF = \sum_{x=1}^{r} \eta_x^y \, \overset{x}{u} = 0$$

Again applying $\overset{*}{F}$ on (2.2) and use of (1.1), (2.1) and (2.2) we get

$$a^r \underset{x}{N} = -\left\{BF\underset{x}{U} + \sum_{z=1}^{r} \overset{z}{U}(\underset{x}{U})\underset{z}{N}\right\} + \sum_{y=1}^{r} \eta_x^y \left\{-B\underset{y}{U} + \sum_{z=1}^{r} \eta_x^z \underset{z}{N}\right\}$$

Comparing of tangential and normal vector fields we get

(3.2)

(a) $$F\underset{x}{U} + \sum_{y=1}^{r} \eta_x^y \underset{y}{U} = 0$$

(b) $$-\overset{z}{u}\left(\underset{x}{U}\right) + \sum_{y=1}^{r} \eta_y^z \, \eta_x^y = a^r I_n$$

In view of the equations (3.1 (a), (b)) and (3.2(a), (b)) we evidently observe that the submanifold M^{4n-r} of co-dimension r of M^{4n} admits $(F, \underset{x}{U}, \overset{x}{u}, \eta_x^y)$-structure.

Corollary (3.1)

The submanifold M^{4n-r} of co-dimension r of the Hsu-quaternion manifold M^{4n} also admits the structure $(G, \underset{x}{V}, \overset{x}{v}, \eta_x^y)$ and $(F, \underset{x}{W}, \overset{x}{w}, \eta_x^y)$ relative to tensor fields $\overset{*}{G}$ and $\overset{*}{H}$ respectively.

Theorem 3.2. *An orientable submanifold of co-dimension r of the Hsu-quaternion manifold* M^{4n} *admits,* F, G, H 3*–structures expressed* $(F, G, H, \underset{x}{U}, \underset{x}{V}, \underset{x}{W}, \overset{x}{u}, \overset{x}{v}, \overset{x}{w}, \eta_x^y)$.

Proof: From (1.2(a)) we have

$$\overset{*}{G}\,\overset{*}{H}\,BX = b^r\ \overset{*}{F}\,BX$$

which in view of (2.1) and (3.1(a)) yields

$$\text{(3.3)}\qquad BGHX + \sum_{y=1}^{r} \overset{y}{v}(HX)\underset{y}{N} + \sum_{x=1}^{r} \overset{x}{w}(X)\left\{-B\underset{x}{V} + \sum_{y=1}^{r} \eta_x^y\,\underset{y}{N}\right\} = b^r\left\{BFX + \sum_{y=1}^{r} \overset{y}{u}(X)\underset{y}{N}\right\}$$

Comparison of tangential and normal vector fields yields

$$\text{(3.4)}\qquad \begin{aligned} &\text{(a)}\qquad GHX = b^r FX + \sum_{x=1}^{r} \overset{x}{w}(X)\underset{x}{V} \\ &\text{(b)}\qquad b^r\,\overset{y}{u}(X) = \overset{y}{v}(HX) + \sum_{x=1}^{r} \eta_x^y\,\overset{x}{w}(X) \end{aligned}$$

$$x, y = 1,2,\ldots, r.$$

We also have from the same equation (2.2)

$$\overset{*}{G}\,\overset{*}{H}\,\underset{x}{N} = b^r\,\overset{*}{F}\,\underset{x}{N}$$

$$-\left\{BG\underset{x}{W} + \sum_{z=1}^{r} \overset{z}{v}(\underset{x}{W})\ \underset{z}{N}\right\} + \sum_{y=1}^{r} \eta_x^y\left\{-B\underset{y}{V} + \sum_{z=1}^{r} \eta_y^z\ \underset{y}{N}\right\} = b^r\left\{-B\underset{x}{U} + \sum_{z=1}^{r} \eta_x^z\ \underset{z}{N}\right\}$$

Equating tangential and normal vector fields we get

$$\text{(3.5)}\qquad \begin{aligned} &\text{(a)}\qquad G\underset{x}{W} + \sum_{y=1}^{r} \eta_x^y\ \underset{y}{V} = b^r\ \underset{x}{U} \\ &\text{(b)}\qquad -\overset{z}{v}\left(\underset{x}{W}\right) + \sum_{y=1}^{r} \eta_x^y\ \eta_y^z = \eta_x^z \end{aligned}$$

Similarly we obtain

(3.6)

$$\text{(a)} \qquad HF = a^r G + \sum_{x=1}^{r} \overset{x}{u} \otimes \underset{x}{W} \quad \text{etc.}$$

$$\text{(b)} \qquad FG = H + \sum_{x=1}^{r} \overset{x}{v} \otimes \underset{x}{U} \quad \text{etc.}$$

Further in view of the relations (1.2(a)) it follows that

$$\overset{*}{G}\overset{*}{H} BX = \overset{*}{H}\overset{*}{G} BX$$

which in view of equations (2.3(a)) and (2.4(a)) becomes

$$BGHX + \sum_{y=1}^{r} \overset{y}{v}(HX) \underset{y}{N} + \sum_{x=1}^{r} \overset{x}{w}(X) \left\{ -B\underset{x}{V} + + \sum_{y=1}^{r} \eta_x^y \underset{y}{N} \right\}$$

$$= BHGX + \sum_{y=1}^{r} \overset{y}{w}(GX) \underset{y}{N} + \sum_{x=1}^{r} \overset{x}{v}(X) \left\{ -B\underset{x}{W} + \sum_{y=1}^{r} \eta_x^y \underset{y}{N} \right\}$$

Equating tangential and normal vector fields we obtain

(3.7)

$$\text{(a)} \qquad (GH - HG)X = -\sum_{x=1}^{r} \left\{ \overset{x}{v}(X) \underset{x}{W} + \overset{x}{w}(X) \underset{x}{V} \right\}$$

$$\text{(b)} \qquad \overset{x}{v}(HX) - \overset{y}{w}(GX) = \sum_{x=1}^{r} \eta_x^y \left\{ \overset{x}{v}(X) - \overset{x}{w}(X) \right\}$$

Again

$$\overset{*}{G}\overset{*}{H}\underset{x}{N} = \overset{*}{H}\overset{*}{G}\underset{x}{N}, \qquad x = 1, 2, \ldots, r$$

$$BG\underset{x}{W} + \sum_{z=1}^{r} \overset{z}{v}(W) \underset{z}{N} + \sum_{y=1}^{r} \eta_x^y \left\{ -B\underset{y}{V} + \sum_{z=1}^{r} \eta_y^z \underset{z}{N} \right\}$$

$$= BH\underset{x}{V} + \sum_{z=1}^{r} \overset{z}{w}(\underset{x}{V}) \underset{z}{N} + \sum_{y=1}^{r} \eta_x^y \left\{ -B\underset{y}{W} + \sum_{z=1}^{r} \eta_y^z \underset{z}{N} \right\}$$

Equating tangential and normal vector fields we get

$$(3.8)\qquad \begin{aligned} &(a) \quad G\underset{x}{W} - H\underset{x}{V} = -\sum_{y=1}^{r} \eta_x^y \left(\underset{y}{W} - \underset{y}{V}\right) \\ &(b) \quad \overset{z}{v}(\underset{x}{W}) - \overset{z}{w}(\underset{x}{v}) = 0 \end{aligned}$$

for $x, y, z = 1, 2, \ldots, r$.

In a similar manner, we can prove the following of relations

$$(3.9)\qquad \begin{aligned} &(a) \quad (HF-FH)X = -\sum_{x=1}^{r} \left\{ \overset{x}{v}(X)\underset{x}{U} + \overset{x}{u}(X)\underset{x}{V} \right\} \\ &(b) \quad \overset{y}{w}(FX) - \overset{y}{u}(HX) = \sum_{y=1}^{r} \eta_x^y \left\{ \overset{y}{w}(X) - \overset{y}{u}(X) \right\} \\ &(c) \quad H\underset{x}{U} - F\underset{x}{W} = \sum_{y=1}^{r} \eta_x^y \left(\underset{y}{U} - \underset{y}{W}\right) \\ &(d) \quad \overset{z}{w}(\underset{x}{U}) - \overset{z}{w}(\underset{x}{V}) = 0 \quad \text{etc.} \end{aligned}$$

The theorem follows by virtue of equation (3.5 to (3.9).

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Flow Past Streamline Shaped Bodies

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Abstract : In this paper, the fluid motions in which the vortex lines coincides with the stream lines past stream line shaped bodies has been discussed and their aerodynamical application as well as solutions in particular cases are given.

Key words : Stream lines, Vortex lines, Gradual damping.

Introduction :

The paper "Gradual Damping of Solitary Waves" by Garbis H. Kenlegan, [6], suggests that fluid motions of which the vortex lines coincides with steam lines, and which according to Durand [1] represent the state of flow past an aeroplane wing should be capable of representing the graduate damping of wave motion when a ship moves through water. Ramballabh had come across these motions in their study of 'superposable fluid motions' published in the proceedings of Banaras mathematical society vol. II (1940).

One of the important characteristics of these motions is that Bernoullis equation

$$\frac{P}{\rho} + \frac{1}{2} q^2 + V = \text{Constant},$$

where p = fluid pressure, ρ = fluid density

q = fluid velocity, & V = force potential,

is applicable to them although they may represent an unsteady state of flow of a viscous fluid. Two such fluid motions are capable of combining in a natural order.

The results represent exact solutions of the equations of viscous motion and are capable of aerodynamic application as well.

Derivation

The equation of motion [2] of a viscous homogeneous in compressible fluid can be written as

$$\frac{\partial u}{\partial t}-(v\zeta - w\eta) = -\frac{\partial \chi}{\partial x} + \nu\, \nabla^2 u,$$

$$\frac{\partial v}{\partial t}-(w\xi - u\zeta) = -\frac{\partial \chi}{\partial y} + \nu\, \nabla^2 v,$$

and

$$\frac{\partial w}{\partial t}-(u\eta - v\xi) = -\frac{\partial \chi}{\partial z} + \nu\, \nabla^2 w,$$

where ξ, η, ζ are the velocity components, ν the kinematic viscosity,

$$\chi = \frac{P}{\rho} + \frac{1}{2} q^2 + V$$

and

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

From these it is apparent that if $\xi = \lambda u,\ \eta = \lambda v\ \xi = \lambda w$ i.e., if the vortex lines of the motion coincide with the stream lines, the equations of motion reduces to the simple form

(1) $$\frac{\partial u}{\partial t} - \nu\, \nabla^2 u = -\frac{\partial \chi}{\partial x}$$

(2) $$\frac{\partial v}{\partial t} - \nu\, \nabla^2 v = -\frac{\partial \chi}{\partial y}$$

and

(3) $$\frac{\partial w}{\partial t} - \nu\, \nabla^2 w = -\frac{\partial \chi}{\partial t}$$

The necessary and sufficient conditions that these be integrable are

$$\frac{\partial^2 \chi}{\partial z\, \partial y} = \frac{\partial^2 \chi}{\partial y\, \partial z},\ \frac{\partial^2 \chi}{\partial x\, \partial z} = \frac{\partial^2 \chi}{\partial z\, \partial x},\ \frac{\partial^2 \chi}{\partial y\, \partial x} = \frac{\partial^2 \chi}{\partial x\, \partial y}$$

From (1), (2) and (3) we then obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)=\nu\,\nabla^2\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right).$$

(4) $$\frac{\partial \xi}{\partial t}=\nu\,\nabla^2\,\xi$$

with two similar equations, namely

(5) $$\frac{\partial \eta}{\partial t}=\nu\,\nabla^2\,\eta$$

and

(6) $$\frac{\partial \zeta}{\partial t}=\nu\,\nabla^2\,\zeta$$

which shows that the vorticity components for a motion of the type under contemplation obey the law of heat conduction through an isotropic medium.

Now, substituting

$$\xi=\lambda u,$$

$$\eta=\lambda v,$$

$$\zeta=\lambda w,$$

in (4), (5) and (6) and assuming λ to be independent of the space variables x, y and z, we get

(7) $$\frac{\partial}{\partial t}(\lambda u)=\nu\,\lambda\nabla^2\,u$$

(8) $$\frac{\partial}{\partial t}(\lambda v)=\nu\,\lambda\nabla^2\,v$$

and

(9) $$\frac{\partial}{\partial t}(\lambda w)=\nu\,\lambda\nabla^2\,w$$

But $$\xi=\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}$$

So that $$\lambda u=\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}$$

$$=\frac{\partial}{\partial y}\left(\frac{\zeta}{\lambda}\right)-\frac{\partial}{\partial z}\left(\frac{\eta}{\lambda}\right)$$

i.e.,
$$\lambda^2 u = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2}$$

(10)
$$= -\nabla^2 u$$

because from the equation of continuity $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

we have $\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x}$

It can similarly be seen that

(11)
$$\lambda^2 v = -\nabla^2 v$$

and

(12)
$$\lambda^2 w = -\nabla^2 w$$

substituting for $\nabla^2(u, v, w)$ in (7), (8), (9) we have

$$\frac{\partial}{\partial t}(\lambda u) = -\nu \lambda^3 u$$

$$\frac{\partial}{\partial t}(\lambda v) = -\nu \lambda^3 v$$

and
$$\frac{\partial}{\partial t}(\lambda w) = -\nu \lambda^3 w$$

Integrating,

(13)
$$\left.\begin{aligned} \lambda u &= F e^{-\int \nu\lambda^2 dt} \\ \lambda v &= G e^{-\int \nu\lambda^2 dt} \\ \lambda w &= H e^{-\int \nu\lambda^2 dt} \end{aligned}\right]$$

where, F, G, H are functions of x, y, z only, satisfying the equation of continuity

$$\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}+\frac{\partial H}{\partial z}=0$$

and the equations

(14)
$$\frac{\partial H}{\partial y}-\frac{\partial G}{\partial z}=\lambda F$$
$$\frac{\partial F}{\partial z}-\frac{\partial H}{\partial x}=\lambda G$$
$$\frac{\partial G}{\partial x}-\frac{\partial F}{.\partial y}=\lambda H$$

obtained from $\xi'=\lambda u$ etc.

But λ must be an absolute constant. Solution (13) therefore simplifies to

(15)
$$\lambda u = Fe^{-\nu\lambda^2 t}$$
$$\lambda v = Ge^{-\nu\lambda^2 t}$$
$$\lambda w = He^{-\nu\lambda^2 t}$$

on substituting these values of u, v and w in the equations of motion (1), (2) and (3), we get

$$\frac{\partial\chi}{\partial x}=\frac{\partial\chi}{\partial y}=\frac{\partial\chi}{\partial t}=0.$$

i.e. χ is a constant, the value of which is the same for all points of the fluid. $\chi=\frac{P}{\rho}+\frac{1}{2}q^2+V$, we conclude that Bernoulli's theorem for steady irrotational flow of a non-viscous homogeneous incompressible fluid is applicable to unsteady viscous flows of the type (15) which possess the following properties

(i) The vortex-lines coincide with stream lines

(ii) The velocity decays exponentially with time, the decay being rapid for fluids of high kinematic viscosity and slow for fluids of low kinematic viscosity. In the case of glycerine [3] (ν = 6.9 at 20° C) or cylinder oil [3] (ν = 10.4 at 20° C) the decay would be rapid. In the case of water [3] (ν = . 01 at 20° C), the decay would be gradual unless the vortcity bears a high ratio to velocity.

Aerodynamical Application

Attention to flows of the above type has been drawn by Durand [4] in the study of flow past an aerofoil where he says 'steady motion is possible only when the vertex lines coincide with the line of flow'.

If the viscosity effect be neglected and the motion be steady, we again have from (1), (2) and (3)

$$\frac{P}{\rho} + \frac{1}{2}q^2 + V = \text{Constant}$$

at all points of the fluid and the value of the constant will be the same for all stream lines, although the flow will be rotational.

In aerodynamical problems the variations in V are small so that Bernoullis equation can be written as

$$\frac{P}{\rho} + \frac{1}{2}q^2 = \frac{p_0}{\rho}$$

where p_0 is the rest pressure.

A Particular Case

To study a particular case, let us suppose $w = 0$

Then from (14),

$$\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0,$$

$$-\frac{\partial G}{\partial z} = \lambda F,$$

$$\frac{\partial F}{\partial z} = \lambda G,$$

$$\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} = 0,$$

the complete solution of which is

$$F = \chi_x \, Cos\ \lambda z + \chi_y \, Sin\ \lambda z$$

$$G = \chi_y \, Cos\ \lambda z - \chi_x \, Sin\ \lambda z\ ,$$

where χ is a function of x and y having continuous derivatives and satisfying the equation

$$\chi_{xx} + \chi_{yy} = 0$$

so that from (15),

$$\lambda u = e^{-\nu\lambda^2 t}\ (\chi_x \, Cos\ \lambda z + \chi_y \, Sin\ \lambda z)$$

$$\lambda v = e^{-\nu\lambda^2 t}\ (\chi_y \, Cos\ \lambda z - \chi_x \, Sin\ \lambda z)$$

$$w = 0,$$

this represents a damped periodic motion arising out of the movement of a streamline shaped body through a viscous fluid. The damping in the case of water will be gradual unless the velocity is large compared with velocity as remarked earlier.

A Special Feature

Fluid motions of which the vertex lines coincide with streamlines have another characteristic property. If u_r, v_r, $r = 1, 2, 3$ be the velocity components of two such motions in the same fluid, and p_1, p_2 the corresponding pressures, we have from (1), (2) and (3)

$$\frac{\partial u_r}{\partial t} - \nu \nabla^2 u_r = -\frac{\partial \chi_1}{\partial x_r}$$

and

$$\frac{\partial v_r}{\partial t} - \nu \nabla^2 v_r = -\frac{\partial \chi_2}{\partial x_r},$$

where χ_1, χ_2 are the values of χ for the two motions and x_1, x_2, x_3 stand for x, y, z representively.

Adding,

$$\frac{\partial}{\partial t}(u_r + v_r) - \nu \nabla^2 (u_r + v_r) = -\frac{\partial}{\partial x_r}(\chi_1 + \chi_2)$$

so that it is possible to super impose one such motion upon another in a way that the velocity of the resulting flow is the vector sum of the velocities of separate flows.

The pressure P for the combined flow will be given by

$$\frac{p}{\zeta} + \frac{1}{2}\sum_{r=1}^{3}(u_r + v_r)^2 + V$$

$$= \frac{p_1}{\zeta} + \frac{1}{2}\sum_{r=1}^{3} u_r^2 + V + \frac{p_2}{\zeta} + \frac{1}{2}\sum_{r=1}^{3} v_r^2 + V,$$

i.e.,

$$p = p_1 + p_2 + \rho\left\{V - \sum_{r=1}^{3} u_r v_r\right\}$$

The pressure head for the combined flow will be in excess of the sum of the pressure heads for separate motions if the force potential $V > \sum_{r=1}^{3} u_r v_r$,which condition may be assumed to be true for slow motion under gravity.

Since the fluid motions are capable of combining in such a simple way, it is expected that an increase in velocity at any point in the fluid will not create any discontinuity in fluid flow.

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On Complete Lifts of (1,1) Tensor Field F Satisfying Structure $F^{\upsilon+1} - \lambda^2 F^{\upsilon-1} = 0$

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Abstract: The complete lifts from a differentiable manifold M^n of class C^∞ to its cotangent bundle $T^*(M^n)$ have been studied by professor Yano and Patterson [4, 5]. Yano and Ishihara [6] studied lifts of an f-structure in the tangent and cotangent bundles. F-structures manifolds of degree $\upsilon \geq 3$ have been studied by Kim J.B. [2]. The present paper deals with some problems on complete lifts of structures mentioned above both in tangent and cotangent bundles and the prolongation in the third Tangent space $T_3(M^n)$.

1. Preliminaries

Let M^n be a differentiable manifold of class C^∞ and dimension n. Let $T^*(M^n)$ be the cotangent bundle of M^n, then $T^*(M^n)$ is also a differentiable manifold of class C^∞ and of dimension 2n. Throughout this paper we make use of the following notations and conventions.

(i) $\pi : T^*(M^n) \to (M^n)$ is the projection map of $T^*(M^n)$ onto M^n

(ii) $\Im^r_s(M^n)$ denotes the set of tensor field of class C^∞ and type (r, s) in M^n and $\Im^r_s(T^*(M^n))$ denotes the corresponding set of tensor fields in $T^*(M^n)$

(iii) Vector fields in M^n denoted by X, Y, Z . . . . . lie derivatives with respect to X is denoted by 0_X

If A is a point in M^n then $\pi^{-1}(A)$ is the fibre over A. Any point $P \in \pi^{-1}(A)$ is an ordered pair(A, P_A), where P is a 1-form in M^n and P_A is its value at A. Let U be a co-ordinate neighbourhood $\pi^{-1}(U)$ in $T^*(M^n)$, then we have

[1.1] $$(X+Y)^c = X^c + Y^c \qquad \text{and}$$

[1.2] $$(F^c)(Z)^c = (F^z)^c + (0_z\ F)^v$$

let M^n be an n-dimensional connected differentiable manifold of class C^∞. Let there be given in M^n, a (1,1) tensor field F of class C^∞ satisfying

[1.3] $$F^{\upsilon+1} - \lambda^2 F^{\upsilon-1} = 0$$

where λ is non zero complex number. Also

$$\text{rank}(F) = \tfrac{1}{2}(\text{rank } F^{\upsilon-1} + \dim M^n)$$
$$= r \text{ (a constant every where on } M^n)$$

Let the operators l^* and m^* be defined as

[1.4] $$l^*\ \underline{\text{def}}\ (F/\lambda)^{\upsilon-1},\ m^* = I - (F/\lambda)^{\upsilon-1},$$

where I denotes the identity operator on M^n. Then the operators I^* and m^* applied to the tangent space at a point of the manifold be complementary projection operators.

Agreement [1.1]

In what follows we make use of the following results [6] for any $X, Y \in \Im_0^1(M^n)$ we have

[1.5] (a) $[X^c, Y^c] = [X, Y]^c$ and

(b) $F^c X^c = [FX]^c$

F-Structure manifold of degree υ (≥ 3)

Let F be a non zero tensor field of type (1,1) and of class C^∞ on an n-dimensional M^n such that [2]

[1.6] $$F^\upsilon + (-1)^{\upsilon-1} F = 0 \qquad \text{and}$$
$$F^u + (-1)^{u-1} F \neq 0 \qquad \text{for } 1 < u < \upsilon$$

Where υ is fixed positive integer greater than 2. Such a structure on M^n is called an F-structure of rank r and degree υ. If the rank of F is constant and equal to r, then M^n is called F-structure manifold of degree υ (≥ 3). The case when υ is odd has been considered here.

Let the operators on M^n be defined as follows [2]

[1.7] $$l = -(-1)^{\upsilon+1}\,\frac{F^{\upsilon+1}}{\lambda^2} \quad \text{and}$$

$$m = I + (-1)^{\upsilon+1}\,\frac{F^{\upsilon+1}}{\lambda^2},$$

where I denotes the identity operator on M^n
From the operators defined by [1.7] we have

[1.8] $$l + m = I,\ l^2 = l \text{ and } m^2 = m$$

For F satisfying [1.6] there exists complementary distribution L and M corresponding to the projection operators l and m respectively

If rank (F) be r, constant on M^n then
dim L = r and
dim M = n −r

We have the following results

[1.9] (i) $Fl = lF = F$ and
$Fm = mF = 0$

(ii) $F^{\upsilon-1} = -\lambda^2\, l$ and
$F^{\upsilon-1}m = 0$

2. The Complete lift of F in the tangent Bundle T(M^n)

The complete lifts F^c of an element of $\Im^1_1(M^n)$ with local component F^h_i has components of the form [6]

[21] $$F^c = \begin{pmatrix} F^h_i & 0 \\ 2F^h_i & F^i_h \end{pmatrix}$$

Now we prove some theorems on the complete lifts of F((υ +1), λ^2(υ −1))– Structure satisfying [1.3].

Theorem 2.1. *The complete lift of a* F((υ +1), λ^2(υ −1))- *Structure also has* F((υ +1), λ^2(υ −1))- *Structure in the tangent bundle.*

Proof : Let F, G ∈ $\Im^1_1(M^n)$ then we have

[2.2] $$(FG)^c = F^c G^c$$

Putting F = G we obtain

[2.2] $$(F^2)^c = (F^c)^2$$

Putting $G = F^2$ in [2.2] and making use of [2.3] we get

$$(F^3)^c = (F^c)^3$$

Continuing the above process of replacing G in equation [2.2] by some higher degree of F we obtain

[2.4] $$(F^{\upsilon})^c = (F^c)^{\upsilon}$$

where υ is any positive integer. Taking complete lift on both side of equation [1.3] we get

$$(F^{\upsilon+1})^c - (\lambda^2 F^{\upsilon-1})^c = 0$$

which in view of the equation [2.4] gives

$$(F^c)^{\upsilon+1} - \lambda^2 (F^c)^{\upsilon-1} = 0$$

Thus the complete lift of F also has $F((\upsilon+1),\ \lambda^2(\upsilon-1))^{\upsilon-1}$-Structure in $T(M^n)$

The complete lift l^{*c} and m^{*c} of l^* and m^* are complementary projection tensor in $T(M^n)$. Thus there exist in $T(M^n)$ two complementary distributions L^{*c} and M^{*c} determined by l^{*c} and m^{*c} respectively.

Theorem 2.2. *The complete lift* M^{*c} *of the distribution* M^* *in* $T(M^n)$ *is integrable, iff* M^* *is integrable in* M^n.

Proof: It is well known that the distribution M^* is integrable in M^n iff

[2.6] $$l^*[m^*X, m^*Y] = 0$$

taking complete lifts on both sides we get

[2.7] $$l^{*c}[m^{*c}X^c, m^{*c}Y^c] = 0$$

Where

$$l^{*c} = (I - m)^{*c} = I - m^{*c} \text{ as } I^c = I$$

In consequence of equation [2.7] M^{*c} is integrable in $T(M^n)$

In the same way we can proof the theorem

Theorem 2.3. *The complete lift* L^{*c} *of* l^* *in* $T^*(M^n)$ *is integrable iff* L^* *integrable in* M^n.

Theorem 2.4. *The structure* F^c *is partially integrable iff* F *is partially integrable in* M^n.

Proof: We know that F is partially integrable iff

[2.8] $$N(l^*X, l^*Y) = 0$$

Taking complete lifts on both sides we obtain

[2.9] $$N^c(l^{*c}\, X^c,\ l^{*c}\ Y^c) = 0$$

Hence F^c is partially integrable iff F is partially integralbe in M^n.

Theorem 2.5. *For any* X, Y $\in \Im_0^1(M^n)$, *let* F *be integrable in* M^n, *Thus* F^c *is integrable in* $T^*(M^n)$ *iff*

$$N^c(X^c, Y^c] = 0.$$

Proof : We know that F is integrable iff

[2.10] $$N(X,Y) = 0$$

where N(X,Y), the Nijenhuis tensor of F satisfying [1.3] and it is given by [6]

[2.11] $$N_{F,F}(X,Y) = [FX, FY] - F[FX, Y] - F[X, FY] + F^2[X,Y]$$

Taking complete lift of both sides we have

[2.12] $$N^c(X^c, Y^c) = [F^cX^c, F^cY^c] - F^c[F^cX^c, Y^c] - F^c[X^c, F^cY^c] + (F^2)^c[X^c, Y^c]$$

also taking complete lift of [2.10] we get

$$N^c(X^c, Y^c] = 0.$$

Which is view of Equation [2.11] and [2.12] and the fact F is integrable in M^n shows that F^c is integrable in $T(M^n)$

3. The compete lift of $F((\upsilon+1), \lambda^2(\upsilon-1)$-Structure in Cotangent Bundle

In this section we prove some theorems on complete lift of F satisfying $F((\upsilon+1), \lambda^2(\upsilon-1)$-Structure.

Theorem 3.1. *The Nijenhuis tensor of the complete lift of* $F^{\upsilon+1}$ *vanishes if the lie derivative of the tensor field* $F^{\upsilon+1}$ *with respect to* X *and* Y *are both zero and* F *is an almost* π *- structure on* M^n.

Proof : In consequence of (2-11) the Nijenhuis tensor of $F^{\upsilon+1}$ is given by

$$[3.1] \qquad N_{(F^{\upsilon+1})^c (F^{\upsilon+1})^c}(X^c, Y^c) = [(F^{\upsilon+1})^c X^c, (F^{\upsilon+1})^c X^c] - (F^{\upsilon+1})^c [(F^{\upsilon+1})^c X^c, Y^c] - (F^{\upsilon+1})^c [X^c, (F^{\upsilon+1})^c Y^c] + (F^{\upsilon+1})^c (F^{\upsilon+1})^c [X^c, Y^c]$$

which in view of [1.3] takes the form

$$[3.2] \qquad N_{(F^{\upsilon+1})^c (F^{\upsilon+1})^c}(X^c, Y^c) = \lambda^4 [(F^{\upsilon-1})^c X^c, (F^{\upsilon-1})^c Y^c] - \lambda^4 (F^{\upsilon-1})^c [(F^{\upsilon-1})^c X^c, Y^c] - \lambda^4 (F^{\upsilon-1})^c [X^c, (F^{\upsilon-1})^c Y^c] + \lambda^4 (F^{\upsilon-1})^c (F^{\upsilon-1})^c [X^c, Y^c].$$

In Consequence of [1.2] we have

$$[3.3] \qquad (F^{\upsilon-1})^c X^c = (F^{\upsilon-1}X)^c X^c + (\mathcal{L}_x F^{\upsilon-1})^v$$

Hence we get

$$[3.4] \qquad N_{(F^{\upsilon+1})^c (F^{\upsilon+1})^c}(X^c, Y^c) = [(F^{\upsilon-1}X)^c, (F^{\upsilon-1}Y)^c] + [(\mathcal{L}_x F^{\upsilon-1})^v (F^{\upsilon-1}, Y^c] + [(F^{\upsilon-1}X)^c, (\mathcal{L}_y F^{\upsilon-1})^v] + [(\mathcal{L}_y F^{\upsilon-1})^v, (\mathcal{L}_y F^{\upsilon-1})^v] - [(F^{\upsilon-1})^c, [(F^{\upsilon-1}X)^c, Y^c] - (F^{\upsilon-1})^c, [(\mathcal{L}_y F^{\upsilon-1}X)^v, Y^c] - (F^{\upsilon-1})^c, [X^c, (F^{\upsilon-1}, Y)^c] - (F^{\upsilon-1})^c, [X^c, (\mathcal{L}_y F^{\upsilon-1})^v] + (F^{\upsilon-1})^c, (F^{\upsilon-1})^c, [X^c, Y^c]$$

If the lie derivatives of the tensor field $F^{\upsilon-1}$ with respect to X and Y are both zero we have

$$\mathcal{L}_x F^{\upsilon-1} = 0 \quad \text{and} \quad \mathcal{L}_y F^{\upsilon-1} = 0$$

Therefore equation [3.4] takes the form

[3.5] $$N_{(F^{\upsilon+1})^c(F^{\upsilon+1})^c}(X^c, Y^c) = [(F^{\upsilon-1}X)^c, (F^{\upsilon-1}Y)^c] - (F^{\upsilon-1})^c[(F^{\upsilon-1}X)^c, Y^c] - (F^{\upsilon-1})^c[X^c, (F^{\upsilon-1}Y)^c] + (F^{\upsilon-1})^c(F^{\upsilon-1})^c[X^c, Y^c]$$

In view of equation [1.5] the equation [3.5] reduces to

[3.6] $$N_{(F^{\upsilon+1})^c(F^{\upsilon+1})^c}(X^c, Y^c) = +[F^{\upsilon-1}X, F^{\upsilon-1}Y]^c - (F^{\upsilon-1})^c[F^{\upsilon-1}X, Y]^c - (F^{\upsilon-1})^c[X, F^{\upsilon-1}Y]^c + (F^{\upsilon-1})^c(F^{\upsilon-1})^c[X, Y]^c$$

Let F be an almost π-structure on M^n then $F^2 = \lambda^2 I$, where I is unit tensor field. Hence $F^{\upsilon-1} = \lambda^2 I$ or I and therefore [3.6] takes the form

$$N_{(F^{\upsilon+1})^c(F^{\upsilon+1})^c}(X^c, Y^c) = [X, Y]^c - [X, Y]^c - [X, Y]^c + [X, Y]^c = 0$$

Theorem 3.2. *The Nijenhuis tensor of the complete lift of* $F^{\upsilon+1}$ *is equal to* λ^{4} *multiplied by the complete lift of Nijenhuis tensor of* $F^{\upsilon+1}$, *if*

(3.7) (i) $\mathcal{L}_X F^{\upsilon-1} = 0$, $\mathcal{L}_Y F^{\upsilon-1} = 0$ and

(ii) $[X, Y]^c = 0$, $\tilde{\upsilon} = 0$

Where

$$\tilde{\upsilon} \ \underline{\underline{\text{dif}}} \ \mathcal{L}_{[F^{\upsilon-1}X, Y]}F^{\upsilon-1} + \mathcal{L}_{[X, F^{\upsilon-1}Y]}F^{\upsilon-1} - \mathcal{L}_{[X, Y]}F^{2\upsilon-2}$$

Proof : In view of equation [1.1] and [2.11] we have

[3.8] $$N_{F^{\upsilon-1}F^{\upsilon-1}}(X, Y)^c = [F^{\upsilon-1}X, F^{\upsilon-1}Y]^c - (F^{\upsilon-1})[F^{\upsilon-1}X, Y]^c - (F^{\upsilon-1}[X, F^{\upsilon-1}Y])^c + (F^{\upsilon-1}F^{\upsilon-1}[X, Y])^c$$

Which on account of [3.3] yield

$$N_{F^{\upsilon-1}F^{\upsilon-1}}(X, Y)^c = [F^{\upsilon-1}X, F^{\upsilon-1}Y]^c - (F^{\upsilon-1})^c[F^{\upsilon-1}X, Y]^c - (\mathcal{L}_{[F^{\upsilon-1}X, Y]}F^{\upsilon-1})^v - (F^{\upsilon-1})^c[X, F^{\upsilon-1}Y]^c - (\mathcal{L}_{[X, F^{\upsilon-1}Y]}F^{\upsilon-1})^v + (F^{2\upsilon-2})^c[X, Y]^c - (\mathcal{L}_{[X, Y]}F^{2\upsilon-2})^v$$

But we have [6]

$$[3.9] \qquad (F^{\upsilon-1})^c (F^{\upsilon-1})^c = (F^{2\upsilon-2})^c + [N_{F^{\upsilon-1} F^{\upsilon-1}}]^v$$

Hence in view of [3.9], the equation [3.8] becomes

$$[3.10] \qquad N_{F^{\upsilon-1} F^{\upsilon-1}} (X,Y)^c = [F^{\upsilon-1} X, F^{\upsilon-1} Y]^c - (F^{\upsilon-1})^c [F^{\upsilon-1} X, Y]^c$$
$$- (F^{\upsilon-1})^c [X, F^{\upsilon-1} Y]^c + (F^{2\upsilon-2})^c [X,Y])^c$$
$$- (\mathcal{K}_{[F^{\upsilon-1}X,Y]} F^{\upsilon-1})^v + \mathcal{K}_{[X,F^{\upsilon-1}Y]} F^{\upsilon-1})^v$$
$$- (\mathcal{K}_{[X,Y]} F^{2\upsilon-2})^v$$

Now from [3.9] we have

$$(F^{2\upsilon-2})^c = (F^{\upsilon-1})^c (F^{\upsilon-1})^c - (N_{[F^{\upsilon-1}, F^{\upsilon-1}]})^v$$

Thus

$$[3.11] \qquad N_{F^{\upsilon-1} F^{\upsilon-1}} (X,Y)^c = [(F^{\upsilon-1} X, F^{\upsilon-1} Y]^c - (F^{\upsilon-1})^c [F^{\upsilon-1} X, Y]^c$$
$$- (F^{\upsilon-1})^c [X, F^{\upsilon-1} Y]^c + (F^{\upsilon-1})^c (F^{\upsilon-1})^c [X,Y]^c$$
$$- (N_{[F^{\upsilon-1}, F^{\upsilon-1}]})^v (X, Y^c) + (\mathcal{K}_{[F^{\upsilon-1}X,Y]} F^{\upsilon-1})^v$$
$$+ (\mathcal{K}_{[X,F^{\upsilon-1}Y]} F^{\upsilon-1})^v - (\mathcal{K}_{(X,Y)} F^{2\upsilon-2})^v$$

In view of equation [3.11] the equation [3.5] takes the form

$$N_{(F^{\upsilon+1})^c (F^{\upsilon+1})^c} (X^c, Y^c) = N_{F^{\upsilon-1} F^{\upsilon-1}} (X,Y)^c + (N_{[F^{\upsilon-1}, F^{\upsilon-1}]})^v [X,Y]^c$$
$$- \{ \mathcal{K}_{[F^{\upsilon-1}X,Y]} F^{\upsilon-1} - \mathcal{K}_{[X,F^{\upsilon-1}Y]} F^{\upsilon-1}$$
$$+ \mathcal{K}_{(X,Y)} F^{2\upsilon-2} \}^v$$

In consequence of [3.7] we have

$$[3.12] \qquad (N_{(F^{\upsilon+1})^c (F^{\upsilon+1})^c} (X^c, Y^c) = \lambda^4 (N_{[F^{\upsilon-1}, F^{\upsilon-1}]} (X,Y)^c) + \lambda^4 (N_{[F^{\upsilon-1}, F^{\upsilon-1}]})^v$$
$$[X,Y]^c - \tilde{\upsilon}^v$$

Let $[X,Y]^c = 0$ and $\tilde{\upsilon}^v = 0$ then [3.12] reduces to

$$(N_{(F^{\upsilon+1})^c (F^{\upsilon+1})^c} (X^c, Y^c) = \lambda^4 (N_{[F^{\upsilon-1}, F^{\upsilon-1}]} (X,Y)^c)$$

Theorem 3.3. *The Nijenhuis tensor of the complete lift of* $F^{\upsilon-1}$ *is equal to the complete lift of the Nijenhuis tensor of* $F^{\upsilon-1}$ *if*

(i) $\mathcal{L}_X F^{\upsilon-1} = 0, \quad \mathcal{L}_Y F^{\upsilon-1} = 0$

and

(ii) $\mathcal{L}_X Y = 0 \qquad \tilde{\upsilon}^v = 0$

Proof : Since $[X, Y]^c = 0$ implies that

$$[X, Y] = 0 \quad \text{if} \quad \mathcal{L}_X Y = 0,$$

Therefore from [3.2] the result follows.

Theorem 3.4. *The process of computing the Nijenhuis tensor of* $F^{\upsilon-1}$ *and taking complete lift are commutative.*

Proof: Theorem follows easily with the help of [3.1] and above theorem.

4. Prolongation of a $F((\upsilon+1), -\lambda^2(\upsilon-1))$-Structure in third tangent space $T_3(M^n)$

Let us denote $T_3(M^n)$ the third order tangent bundle over M^n and let F^{III} be the third lift on F in $T_3(M^n)$ then we have

For any F, $G \in \Im^1_1 M^n$ the following holds

[4.1] $$(G^{III} F^{III}) X^{III} = G^{III}(F^{III} X^{III})$$

$$G^{III}(FX)^{III}$$

$$(G(FX))^{III}$$

$$(GF)^{III} X^{III}$$

For every $X \in \Im^1_0 (M^n)$, therefore we have

$$G^{III} F^{III} = (GF)^{III}$$

If P(t) denotes a polynomial of variable then we have

[4.2] $$(P(F))^{III} = P(F)^{III}$$

where $$F \in \Im^1_1 (M^n)$$

Theorem 4.1. *The third lift* F^{III} *defines* a $F((\upsilon+1), -\lambda^2(\upsilon-1))$*-structure in* $T_3(M^n)$ *iff* F *defines a* $F((\upsilon+1), -\lambda^2(\upsilon-1))$*-structure in* M^n.

Proof : Let F satisfy [1.3] then F defines a $F((\upsilon+1), -\lambda^2(\upsilon-1))$- structure in M^n satisfying $F^{\upsilon+1} - \lambda^2 F^{\upsilon-1} = 0$.
which in view of [4.2] takes the form

[4.3] $$(F^{III})^{\upsilon+1} - \lambda^2 (F^{III})^{\upsilon-1} = 0$$

Therefore F^{III} defines $F((\upsilon+1), -\lambda^2(\upsilon-1))$ structure in $T_3(M^n)$

Theorem 4.2. *The third lift* F^{III} *is integrable in* $T_3(M^n)$ *iff F is integrable in* M^n

Proof: Let us denote N^{III} and N Nijenhuis tensors of F^{III} and F, respectively. Then we have [6]

[4.4] $$N^{III}(X,Y) = (N(X,Y))^{III}$$

We know that $F((\upsilon+1), -\lambda^2(\upsilon-1))$ structure is integrable in M^n iff $N(X,Y) = 0$

Which in view of [4.4] is equivalent to

[4.5] $$N^{III}(X,Y) = 0$$

Thus F^{III} is integrable iff F is integrable in M^n

Theorem 4.3. *The third lift* F^{III} *of F is partially integrable in* $T_3(M^n)$ *iff F is partially integrable in* M^n.

Proof: We know that For F to be partially integrable in M^n the following holds

$N(l^* X, l^* Y) = 0$ and $N(m^* X, m^* Y) = 0$

Which in view of Equation [4.4] takes the form

[4.6] $$N^{III}(l^{*III} X^{III}, l^{*III} Y^{III}) = 0 \qquad \text{and}$$
$$N^{III}(m^{*III} X^{III}, m^{*III} Y^{III}) = 0$$

Where l^{*III}, m^{*III} are operators in $T_3(M^n)$ which defines the distributions L^{*III} and M^{*III} respectively. Thus the equation [4.6] gives the condition for F^{III} to be partially integrable.

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Lower Radical And Condition Q of Hemirings

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Abstract: In this paper, we geneeralize a few results of [7] for lower radical classes of rings for radical classes of hemirings, by using the Lee construction for lower radical classes of hemirings.

1. Introduction and Preliminaries :

D. M. Olson and T.L. Jenksins [6] discussed general Radical Theory of Hemirings. The theory was further enriched by many authors (see [3, 4, 9]). The certain condition (condition Q) was investigate by [7] for radical classes of rings. Here we are interesting to generalize a several results of [7] in the frame work of hemiring which is quite different from ring theoretical approach discussed in [7].

A semiring (R, +, .) is called a hemiring if (i) '+' is commutative (ii) there exists an element $0 \in R$ such that 0 is the identity of (R, +) and the zero element of (R, .). i.e. $0r = r0 = 0,\ \forall\, r \in R$.

Lower radical classes for hemirings can be constructed similar to the construction of lower radicals for rings (see [8, 9]).

If R is a hemiring then $HR, D_1(R)$ denote the set of all homomorphic images of R and the set of all semi-ideals of R respectively. Observe that $\mathcal{M}$ is a homomorphically closed if and only if $HR \subseteq \mathcal{M},\ \forall\, R \in \mathcal{M}$ and ideally closed or hereditary if and only if $D_1(R) \subseteq \mathcal{M},\ \forall\, R \in \mathcal{M}$. If I is an semi-ideals of R, then we denote $I \leq R$.

First we include necessary preliminaries, let ω be the universal class of all hemirings and $\mathcal{M}$ be a sub-class of ω and let $\mathcal{M}_0$ be the homomorphic closure of $\mathcal{M}$ in ω. For each $A \in \omega$, let $D_1(R)$ be the set of all semi-ideals of R. Inductively we define.

$$D_{n+1}(R) = \{I : I \text{ is an semi-ideal of some hemiring in } D_n(R)\}.$$

Let $D(R) = \bigcup_{n \in N} D_n(R)$, $n = 1, 2, 3, \ldots$ By using ring theoretical approach discussed in [7], (see also [5, 10]), we have

$\pounds\mathcal{M} = \{R \in \omega : D(R/I) \cap \mathcal{M}_0 \neq 0$, for each proper semi-ideal I of R$\}$, is the Lee construction for lower radical determined by $\mathcal{M}$, and $\mathcal{M} \subseteq \pounds\mathcal{M}$, (see also [6, 9, 10]). For undefined terms of hemirings we may refer (see [1, 2, 6]).

2. Condition Q:

We extend the result of [7] by using the above Lee construction of lower radical for hemiring which is indeed provides an excellent and different approach to handle the many results of [7] in the frame work of hemiring.

If ρ is a radical class then its semisimple class is denoted by $S\rho$. The following definition is closely inspired by [7].

Definition 2.1 : Let $\mathcal{M}$ be an arbitrary class of hemirings. A non simple hemiring R is called **H_M - hemiring** if

i) $R / I \in \mathcal{M}$, for every $(I \neq 0) \leq R$

ii) Every minimal semi-ideal J or R belongs to $\mathcal{M}$.

A radical class ρ satisfies the **condition Q** for $\mathcal{M}$ if $H_M \subseteq \rho \cup S\rho$. We assume that $0 \in H_M$. Observe that a non-zero simple hemiring $R \in \mathcal{M}$ if and only if $R \in H_M$.

If R is a simple hemiring, $R \in \mathcal{M}$, $R / R = 0 \in \mathcal{M}$, i.e. (i) is satisfied. Moreover R is minimal semi-ideal of R, i.e. $R \in \mathcal{M}$, thus (ii) is satisfied and hence $R \in H_M$. Conversely assume that $R \in H_M$. Since (i), (ii) are satisfied. Also R is minimal semi-ideal of R, therefore by (ii) $R \in \mathcal{M}$.

The following theorem was proved by H. J. le Roux and G.A.P Heyman [7] for rings. Here we generalize it for hemiring, which can be obtained on the lines of ring theoretical approach.

Theorem 2.2. *Let $\mathcal{M}$ be a homomorphically, ideally closed class of hemirings, which is further closed under finite direct sum. Let ρ be a radical class such that $\rho \cap H_M \neq 0$. Then $H_M \subseteq \rho \cup S\rho$ if and only if $\mathcal{M} \subseteq \rho$.*

Theorem 2.3 : *Let $\mathcal{M}$ be a hereditary class which is homomorphically closed and closed under the finite direct sums respectively. A radical class $\rho = \pounds\mathcal{M}$ if and only if*

i) ρ satisfies the condition Q i.e. $H_M \subseteq \rho \cup S\rho$

ii) $\rho \cap H_M \neq 0$

iii) for any radical class ς satisfies (i), (ii) implies that $\rho \subseteq \zeta$..

Definition 2.4: *A hemiring R is subdirectly irreducible if and only if the intersection of any collection of non-zero k-ideals of A is again a non-zero k-ideal.*

Indeed intersection of all non-zero k-ideal of subdirectly irreducible hemiring is non-zero k-ideal which is uniquely determined and is called heart of R and is denoted by H.

Theorem 2.5. *If* ρ *is hereditary radical class and A is subdirectly irreducible with heart H then*

i) $H \in \rho \cup S\rho$

ii) $H \in S\rho$ if and only if $A \in S\rho$.

iii) Then H is either simple hemiring or a zero-hemiring.

Proof:

i) By [6, Theorem 3], $\rho(H)$ is k-ideal of H. By definition of heart, we have $\rho(H) = H$ or $\rho(H) = 0$. This implies that $H \in \rho$ or $H \in S\rho$ and hence $H \in \rho \cup S\rho$.

ii) Let $A \in S\rho$. As $H \leq A$, by hereditary of $S\rho$ [9, Theorem 1] we have $H \in S\rho$.

Conversely let $H \in S\rho$. This implies that $\rho(H) = 0$. Since ρ is hereditary radical class. So by [6, Lemma 8] we have $\rho(H) = H \cap \rho(A)$. Thus $\rho(H) = 0$ and hence $H \cap \rho(A) = 0$. Now, $H \cap \rho(A) = 0$. We claim that $\rho(A) = 0$. If $\rho(A) \neq 0$. As $H \neq 0$ then $\{H, \rho(A)\}$ is family of non zero semi-ideal of A such that $H \cap \rho(A) = 0$. This implies that A is not subdirectly irreducible, which is a contradiction. Hence $\rho(A) = 0$. This implies that $A \in S\rho$.

iii) As H is minimal semi-ideal, $H^2 \leq H$, therefore, we have $H^2 = \{0\}$ or $H^2 = H$. If $H^2 = \{0\}$ then H is zero hemiring. If $H^2 = H$, let $(I \neq 0)$ be non-zero semi-ideal of H. Assume that

$$I^* = I + AI + IA + AIA$$

then $(0 \neq I^*)$ is k-ideal of A, generated by I. Moreover $(I^*)^3 \subseteq I$ (see [6], Lemma 9, p. 28). By $I \subseteq H$ and $H \leq A$, we have $AI \subseteq AH \subseteq H$, $IA \subseteq HA \subseteq H$, $AIA \subseteq AHA \subseteq H$. This implies that $I^* = I + AI + IA + AIA \subseteq H$ i.e. $I^* \subseteq H$. By definition of H, we have $I^* = H$ or $I^* = 0$. As $I \neq 0$, therefore $I^* \neq 0$ ($\because I \subseteq I^*$) and hence $I^* = H$. This implies that $(I^*)^3 = H^3$. As $(I^*)^3 \subseteq I$, this implies that $H^3 = I$ or $H^2 H \subseteq I$ and hence $H^2 \subseteq I$ ($\because H^2 = H$). Consequently we have $H \subseteq I$. This implies that $H = I$ and hence H is a simple hemiring.

First we note that Z = class of all zero hemirings and β_s = upper radical determined by the class of all prime simple hemirings.

The following theorem provides a necessary and sufficient condition for given radical class ρ of hemiring to satisfy the condition Q associated to certain class of hemirings, which is indeed an extension of Theorem 2 of [7].

Theorem 2.6. *Let $\mathcal{M}$ be a hereditary class which is homomorphically closed and also closed under the finite direct sum of hemirings and $Z \subseteq \mathcal{M}$. A hereditary radical class $0 \neq \rho \subseteq \beta_s$ satisfies condition* Q *if and only if $\mathcal{M} \subseteq \rho$.*

Proof: If $\mathcal{M} \subseteq \rho$. *then $M \cap \rho = M \neq 0$, and hence $H_M \cap \rho \neq 0$. By Theorem 2.2,* ρ *satisfies condition* (Q).

Conversely assume that ρ satisfies condition (Q). Let us suppose $\mathcal{M} \not\subset \rho$. Now there are two cases :

CASE I : Let $\mathcal{M} \cap \rho \neq 0$, as $\mathcal{M} \subseteq H_M$. This implies that $H_M \cap \rho \neq 0$. As $\mathcal{M} \not\subset \rho$, by Theorem 2.2, ρ does not satisfy (Q), a contradiction to the assumption.
Hence $\mathcal{M} \subseteq \rho$.

CASE II : Let $\mathcal{M} \cap \rho = 0$, we will show that, this is impossible.
Let $(0 \neq R) \in \rho$. As R can be decomposed into sub-direct sum of subdirectly irreducible hemirings R_i, s.

Let R_i be an arbitrary component of R, and let H be heart of R, then $H^2 = H$ or H= 0. If $H^2 = H$, then H is a simple prime hemiring. Since β_s= upper radical determined by all simple prime hemirings, therefore $H \in S\beta_s$. Since $\rho \subseteq \beta_s$, therefore $S\beta_s \subseteq S\rho$ (see [6]) and hence $H \in S\rho$. By Theorem 2.5, $H \in S\rho$ if and only if $R_i \in S\rho$.

As R is the direct sum of R_i, therefore there exists is a projection mapping $\pi_i : R \rightarrow R_i$. If ker $\pi_i = I_i$ then $R / I_i \cong R_i$. Since $R \in \rho$, and ρ is a radical class, therefore $R_i \in \rho$, $\forall$ i. Since $R_i \in S\rho$, therefore $R_i \in S\rho \cap \rho$ (See [10]). This means $R_i = 0$. As R_i is an arbitrary component of R such that $R_i = 0$, therefore R = 0, a contradiction. Hence $H^2 \neq H$, the minimality of H implies $H^2 = 0$. i.e. H is zero-hemiring i.e. $H \in Z \subseteq \mathcal{M}$. This implies that $H \in \mathcal{M}$. Since $R \in \rho$, and $H \leq R$ and ρ is hereditary. This implies that $H \in \rho$, this implies that $H \in \rho \cap \mathcal{M} = 0$. Consequently, we have H = 0, which contradicts the definition of heart H. Hence $\rho \cap \mathcal{M} = 0$ is impossible i.e. Thus $\rho \cap \mathcal{M} \neq 0$, so result follows by case (I).

Theorem 2.7: *A hereditary radical class* ρ *under the hypothesis of the Theorem 2.6 coincides with £$\mathcal{M}$ if and only if*

i) ρ satisfies condition Q
ii) for only hereditary class ς satisfying condition Q then $\rho \subseteq \varsigma$.

Proof : Let £$\mathcal{M} = \rho$. By Theorem 2.3, ρ satisfies condition Q and $\rho \cap H_M \neq 0$. Let ς be a radical class such that ς satisfies condition Q, then by Theorem 2.6, $\mathcal{M} \subseteq \varsigma$. This implies that $\mathcal{M} \cap \varsigma \neq 0$. Thus $H_M \cap \varsigma \neq 0$. Thus ς satisfies (i), (ii), of Theorem 2.3, therefore by (iii) of Theorem 2.3, $\rho \subseteq \varsigma$. Conversely assume that (i), (ii) of the

statement are valid, ρ satisfies condition Q, therefore by Theorem 2.6, $\mathcal{M} \subseteq \rho$. This implies that $£\mathcal{M} \subseteq \rho$. Now $\mathcal{M} \subseteq £\mathcal{M}$. This implies that $£\mathcal{M}$ satisfies condition Q (by Theorem 2.6). By (ii) of the statement $\rho \subseteq £\mathcal{M}$. This implies that $£\mathcal{M} = \rho$.

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