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TRIBHUVAN UNIVERSITY
KATHMANDU, NEPAL

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Multiplier Theorem Revisited

PARTHA PRATIM DEY

Abstract: An abelian (v, k, λ)-difference set in an abelian group G is a set D consisting of k group elements with the property that the list of the "differences" xy^{-1} with $x,y \in D$ contains every non-identity element of G exactly λ times. We investigate these sets from the viewpoint of the group algebra KG. Using the idempotents in KG we give a new proof of the Hall Multiplier Theorem.

Introduction

For every known (v, k, λ)-difference set, a prime p is a multiplier if p divides $n = k-\lambda$ and $(p, v) = 1$. This fact known as multiplier theorem was first proved by Hall in 1951 in his paper on Cyclic Incidence Matrices [1]. In this paper, we use idempotents in KG to provide a new proof of the multiplier theorem. Throughout, G will denote an abelian group of exponent μ and K will be a field containing a primitive μ th root of unity. Notice that this necessarily requires that the characteristic of K does not divide μ.

Preliminary Results

A character of G is a homomorphism ϕ from G to a multiplicative group of K. For example, the trivial character (or principal character), denoted by ϕ_0, is the map such that $\phi_0(g) = 1$, $\forall_g \in G$. It is not difficult to determine all the characters of G. To see this, let us decompose G as a product of cyclic groups,

$$G = G_1 \times \ldots \times G_n$$

where G_i is generated by g_i. A character ϕ must carry each g_i to a $|G_i|$ th root of unity, and conversely ϕ is completely determined by knowing to which root of unity each g_i is carried. Hence there are precisely $|G| = |G_1| \times \ldots \times |G_n|$ characters of G. In fact the characters form a group isomorphic to G under the rule $(\phi\psi)(g) = \phi(g)\psi(g)$.

Lemma (2.1) [2]. Let G be an abelian group and $ch(G)$ be the group of characters of G with values in K.

(i) For $\forall_g \in G$

$$\sum_{g \in ch(G)} \phi(G) = \begin{cases} |G| & \text{if } g \text{ is an identity element} \\ 0 & \text{otherwise} \end{cases}$$

(ii) For $\forall \phi \in ch(G)$,

$$\sum_{g \in G} \phi(G) = \begin{cases} |G| & \text{if } \phi = \phi_0 \\ 0 & \text{otherwise} \end{cases}$$

Proof: (i) Let $S(g)$ denote the sum in question. If g is the identity element then $S(g)$ certainly equals $|G|$. So let g be a non-identity element. Choose a character ψ such that $\psi(g) \neq 1$. Then

$$S(g) \sum_{\phi \in ch(G)} \phi(G) = \sum_{\phi \in ch(G)} (\phi\psi)(g) = \psi(g)\Big(\sum_{\phi \in ch(G)} \phi(G)\Big) = \psi(g)S(g).$$

Hence $S(g) = 0$.

(ii) If ϕ is the principal character ϕ_0, then clearly

$$\sum_{g \in G} \phi(g) = |G|.$$

Let ϕ be nonprincipal and let $S(\phi)$ be the sum in question. Choose $h \in G$ such that $\phi(h) \neq 1$. Then

$$S(\phi) = \sum_{g \in G} \phi(G) = \sum_{g \in G} \phi(gh) = \sum_{g \in G} \phi(g)\,\phi(h) = \phi(h)\Big(\sum_{g \in G} \phi(g)\Big) = \phi(h)S(\phi).$$

Hence $S(\phi) = 0$. *QED.*

Another concept that we will use in our proof of the Multiplier theorem is the group-ring KG which consists of all formal sums

$$\sum_{g \in G} a_g g$$

with $a_g \in K$. Addition is defined in the usual manner. Multiplication is defined by using the distributive law and the group operation of G:

$$\Big(\sum_{g \in G} a_g g\Big)\Big(\sum_{h \in G} b_h h\Big) = \sum_{g \in G} \sum_{h \in G} (a_g\, h_n) gh.$$

Clearly KG is a ring with identity and is commutative as G is abelian. The characters of G can easily be extended to be maps from the group-ring KG onto K and it is convenient to do so. If ϕ is a character of G, simply let

$$\phi\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} a_g \phi(g)$$

We should also note that any subset A of G can be identified with an element in KG by setting

$$A = \sum_{g\in A} g.$$

Under this definition, a (v, k, λ)-difference set D may be written as

$$D = \sum_{d\in D} d.$$

Also if $A, B \subseteq G$ and ϕ is a character of G, then $\phi(AB) = \phi(A)\phi(B)$. For any integer t and subset A of G we define:

$$A^t = \sum_{g\in G} g^t.$$

Lemma (2.2) [2]. Let D be a (v, k, λ)-difference set in an abelian group G. Then

$$DD^{-1} = (k-\lambda)\, e + \lambda G = ne + \lambda G.$$

where $k-\lambda = n$ and e is the identity element of G.

Proof: Note that

$$DD^{-t} = \sum_{x\in y} xy^{-1} + \sum_{x\neq y} xy^{-1}$$

where $x, y \in D$. Since $|D| = k$, we have

$$\sum_{x=y} xy^{-1} = ke$$

and the definition of difference set yields

$$\sum_{x\neq y} xy^{-1} = \lambda(G - e).$$

Hence

$$DD^{-1} = ke + \lambda(G - e) = (k-\lambda)\, e + \lambda G = ne + \lambda G. QED.$$

Lemma (2.3) [2]. Let D be a (v, k, λ)-difference set in an abelian group G and let ϕ be a nonprincipal character of G. Then

$$\phi(D)\phi(D^{-1}) = n.$$

Proof: We apply ϕ to both sides of $DD^{-1} = ne + \lambda G$. Then

$$\phi(D)\phi(D^{-1}) = n\phi(e) + \lambda\, \phi(G) = n$$

as $\phi(G) = 0$ by lemma (2.1). *QED.*

Lemma (2.4) [2]. Let G be an abelian group of exponent μ and let K be a field containing a μ th root of unity. Given ϕ, a character of G, we define e_ϕ as follows :

$$e_\phi = \frac{1}{|G|}\sum_{g\in G}\phi(g^{-1})g$$

Then (i) If e is the identity of G, we have

$$\sum_{\phi\in ch(G)} e_\phi = e$$

(ii) Moreover for any A,

$$A = \sum_{g\in G} e_g g$$

in KG, we obtain

$$A = \sum_{\phi\in ch(G)} \phi(A)e_\phi$$

Proof. (i) Note that

$$\sum_{\phi\in ch(G)} e_\phi$$

$$= \sum_{\phi\in ch(G)} \left(\frac{1}{|G|}\sum_{g=G}\phi(g^{-1}g)\right)$$

$$= \frac{1}{|G|}\sum_{\phi\in ch(G)}\left(\sum_{g=G}\phi(g^{-1})g\right)$$

$$= \frac{1}{|G|}\sum_{g\in G} g\left(\sum_{\phi\in ch(G)}\phi(g^{-1}g)\right)$$

$$= \frac{1}{|G|}e\sum_{\phi\in ch(G)}\phi(e^{-1})$$

$$= \frac{1}{|G|}e\,|G|$$

$$= e$$

Note that for $g \neq e$.

$$\sum_{\phi\in ch(G)}\phi(g^{-1}) = 0$$

by lemma (2.1).

(ii) Let $h \in G$. Then

$$he_\phi$$

$$= h\left(\frac{1}{|G|}\sum_{g\in G}\phi(g^{-1})g\right)$$

$$= \frac{1}{|G|}\sum_{g\in G}\phi(g^{-1})hg$$

$$= \frac{1}{|G|}\sum_{g\in G}\phi(g^{-1})\phi(h^{-1})\ \phi(h)hg$$

$$= \frac{1}{|G|}\phi(h)\sum_{g\in G}\phi(g^{-1})\phi(h^{-1})hg$$

$$= \frac{1}{|G|}\phi(h)\sum_{g\in G}\phi(g^{-1}h^{-1})hg$$

$$= \frac{1}{|G|}\phi(h)\sum_{g\in G}\phi(hg)^{-1})hg$$

$$= \phi(h)\left(\frac{1}{|G|}\sum_{g\in G}\phi(hg)^{-1})hg\right)$$

$$= \phi(h)e_\phi$$

Then

$$Ae_\phi$$

$$= \left(\sum_{g\in G} a_g g\right)e_\phi$$

$$= \sum_{g\in G} a_g g\, e_\phi$$

$$= \sum_{g\in G} a_g \phi(g)\, e_\phi$$

$$= \left(\sum_{g\in G} a_g \phi(g)\right)e_\phi$$

$$= \phi(A)e_\phi$$

Hence

$$A$$

$$= Ae$$

$$= A\left(\sum_{\phi\in ch(G)} e_\phi\right)$$

$$= \sum_{\phi \in ch(G)} Ae_\phi$$

$$= \sum_{\phi \in ch(G)} \phi(A) e_\phi \ .QED.$$

3. The Multiplier Theorem

Finally we are ready to prove the Multiplier Theorem by Hall.

Theorem 3.1. (*Multiplier Theorem*). *Let D be an abelian* (v, k, λ)-difference set of *an abelian group G. Suppose that p is a prime such that* $(p,v) = 1$, $p > \lambda$, $p|n = k-\lambda$. *Then p is a multiplier of D.*

Proof. Let K be a field of characteristic p such that K contains a primitive μ^{th} root of unity, where μ is the exponent of G. Then $D^p = D^{(p)}$ in Z_pG, which implies $\phi(D)^p = \phi(D)^p = \phi(D^{(p)})$ for any character ϕ of G. We now compute $D^{(p)}Dg^{(-1)}$.

$$D^{(p)}Dg^{(-1)}.$$

$$= \sum_{\phi \in ch(G)} \phi(D^{(p)}) Dg^{(-1)} e_\phi$$

$$= \sum_{\phi \in ch(G)} \phi(D^{(p)})\, \phi(Dg^{(-1)})\, e_\phi$$

$$= \sum_{\phi \in ch(G)} \phi(D^{(p)})\, \phi(D)\, \phi(g^{(-1)})\, e_\phi$$

$$= \sum_{\phi \in ch(G)} \phi(D)^{p-1} \phi(D)\, \phi(D^{(-1)})\, \phi(g^{-1})\, e_\phi .$$

If ϕ is nonprincipal, then $\phi(D)\phi(D^{(-1)}) = n = 0$ by Lemma (2.3) and the fact that p divides n. Hence

$$D^{(p)}\, Dg^{(-1)}$$

$$= \phi_0(D)^{p-1} \phi_0(D)\, \phi_0(D^{(-1)})\, \phi_0(g^{(-1)})\, e_{\phi 0}$$

$$= k^{p+1} \frac{1}{|G|} \sum_{g \in G} g$$

$$= \frac{k^{p+1}}{v} \sum_{g \in G} g$$

Because $|D^{(p)} \cap Dg|$ is the coefficient of e in $D^{(p)}\, Dg^{(-1)}$, we have $|D^{(p)} \cap Dg| = \frac{k^{p+1}}{v}$. Further $k(k-1) = \lambda(v-1)$ implies $k^2 - n = \lambda v$. If p divides k, then p divides λv and since $p > \lambda$, p divides v, a contradiction. Hence p does not divide k. Finally $k(k-1) = \lambda(v-1)$ implies $k = v\ (modp)$. Thus

$$\frac{k^{p+1}}{v} = \frac{\lambda^{p+1}}{\lambda} = \lambda^p = \lambda (modp)$$

Hence $|D^{(p)} \cap Dg| = \lambda + tp$. As $p > \lambda$, the integer t has to be nonnegative which shows $|D^{(p)} \cap Dg| \geq \lambda$. As $D^{(p)}$ intersects every block Dg in at least λ points, $D^{(p)}$ is a block of the development of D. Thus p is a multiplier. *QED.*

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Blood Flow Through Diseased Artery

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Abstract: Localized narrowing in any artery, commonly referred to as a stenosis, is a frequent result of arterial disease. It is caused mainly due to the intravascular atherosclerotic plaques, which develop at the arterial wall and protrude into the lumen of the vessels. Such contractions disturb normal blood flow through artery and there is considerable evidence that hydrodynamic factors such as wall shear stress, pressure distribution, separation etc. can play a significant role in the development and progression of this disease. Hence considerable attention has been given to the theoretical and experimental studies of stenotic region flow under different conditions. [Young (1968), Caro et al 1971), Deschpande et al (1977), Nerm (1974), Fox & Hugo (1966), Roadbard (1970). Rodkiewiez (1974), Steinman (2000), Gijsen F.I.H (1999).

Many authors have investigated analytically the flow characterisation of blood in artery with mild and non-critical stenosis.

In the above mentioned studies only the effect of single stenosis was considered and the tube has been taken to be of uniform cross-section. However, it may be noted that many of the blood vessels either converge or diverge slowly along their lengths. Also it is possible to have multiple stenosis in a series along the length of the tube.

In view of this, in this paper, the flow through tubes of non-uniform cross-section and with multiple stenosis is investigated. Solution for very mild stenosis have been obtained and the effects of heights of stenosis, number of stenosis etc. on the resistance to the flow λ and wall shear distribution γ have been studied.

Analysis

Let us consider the flow of blood through an artery having mild multiple stenosis in its lumen. We assume that the artery has two stenosis in series along the length and is such that the first stenosis is in a portion of uniform cross-section and the other one in a portion where the tube radius varies axially i.e. R=R*(z). Thus assuming that the two stenosis have developed symmetrically and are very mild, the geometry of the tube can be defined as:

$$(1)\quad R(z)=\begin{cases} R_0 : 0 \le z \le d_1 \ \text{ and } \ d_1 + L_1 \le z \le B_1 \\ \{R(z) : B_1 \le z \le d_2 \ \text{ and } \ d_2 + L_2 \le z \le B \\ \{R_0 - (\delta_1/2)\,\{1+\mathrm{Cos}\,(2\pi/L_1)\,(z-d_1-L_1/2)\} : d_1 \le z \le d_1 + L_1 \\ \{R(Z) - (\delta_2/2)\,\{1+\mathrm{Cos}\,(2\pi/L_2)\,(z-d_2-L_2/2)\} : d_2 \le z \le d_2 + L_2 \end{cases}$$

Where $R(z)$ is the tube radius at any cross-section, R_0 is the constant radius of the first portion, $R^*(z)$ is the tube radius in the second portion, δ_1 and δ_2 are the amplitude of the two stenoses and L_1 and L_2 are their lengths such that :

(2) $$\delta_1 \leq \min (R_0, R \text{ out}) \leq L_1$$

The basic equation governing the flow in a tube with mild constriction is,

(3) $$\therefore \quad dp/dz = (\mu/r)\, \partial/\partial r\, (\partial w/\partial r)$$

Where w is the axial velocity, p is the pressure, μ is the coeff. of viscosity. The boundary conditions are provided by the no–slip condition at the boundary of the tube and by the axial symmetry of the tube i.e.,

(4) $$\delta w/\partial r = 0 \quad \text{at } r = 0$$
$$w = 0 \quad \text{at } r = R(z)$$

The axial velocity w on solving eqn. (3) along with the conditions (4) is obtained as

(5) $$\omega = \{(-1/4\mu)\, \delta p/\delta z\}\, (R^2(z) - r^2)$$

The volumetric flow rate is defined as :

$$Q = \int_0^{R(z)} 2\pi r\ w\ dr$$

Which on using eqn. (5) is given by following expression :

(6) $$dp/dz = -8\mu Q/\pi R(z)$$

Integrating (6) alongwith the condition

$$\therefore p = p_1 \text{ at } z = 0$$

and $$p = p_0 \text{ at } z = B$$

We get

(7) $$p_1 - p_0 = (8\mu Q/\pi) \int_0^B dz/R^4(z)$$

Which gives the resistance to the flow λ as,

(8) $$\lambda = (p_1 - p_0)/Q = (8\mu/\pi) \int_0^B dz/R^4(z)$$

The shearing stress, τ_w, on the wall of the tube is given by,

(9) $$\tau = \{(1 - d\,R/dz)^2 / (1 + d\,R/dz)^2\}\, [4Q\mu/\pi R^3(Z)]$$

Let λ_n and τ_n be the resistance to the flow and the shearing stress at the wall for the flow of blood in a tube of uniform cross-section of radius R_0 and with no stenosis.

Then

(10) $$\lambda_n = 8\,\mu\, B/\pi R_0^4,$$

(11) $$\tau_n = 4\,\mu\, Q/\pi R_0^3$$

Thus from equations (8) -(10) and (9)-(11) we have,

(12) $$\bar{\lambda} = \lambda/\lambda_n = R_0^4 / B \int_0^B dz/R^4(Z)$$

(13) $$\bar{\tau} = \tau/\tau_n = \{R_0/R(z)\}^3[(1-dR/dz)^2/(1+dR/dz)^2]$$

Discussion

The total resistance to the flow $\bar{\lambda}$ and the shear stress acting on the wall $\bar{\tau}$ are given by equations (12) and (13). To see explicitly the effects of the various parameters on the resistance to the flow and the wall shear the following function has been assumed for the tube radius in the second portion,

$$R^*(z) = [e^{K(Z-B_1)}]^2$$

Where K is the wall exponent parameter.

Numerical calculations have been done using the following values of the parameters:

$\bar{B}_1 = 0.4$
$\bar{B}_2 = 1$ $\qquad K = -0.1, 0.0, +0.1$

$\bar{L}_1 = 0.04$
$\bar{L}_2 = 0.06$

$\bar{\delta}_1 = 0.0–0.2$
$\bar{\delta}_2 = 0.0–0.2$

The resistances to the flow and the wall shear stresses are plotted in figures and the following effects have been observed :

(1) $\bar{\lambda}$ increases with the heights and the numbers of the stenosis.

(2) $\bar{\lambda}$ is more in a convergent tube compared to its value for a straight tube whereas far a divergent tube it decreases.

From the figures it can be seen that the sheer stress acting on the wall of the second region, τ_2 increases with the heights of the stenosis. For a convergent tube τ_2 is more compared to its value for a straight tube. However τ_2 decrease for a divergent tube compared to a straight tube.

Same behaviour is observed for the shear stress acting on the wall of the first region τ_1 These observations have been found to be in conformity with the experimental findings of Talukdar etc. al (Ref. 12).

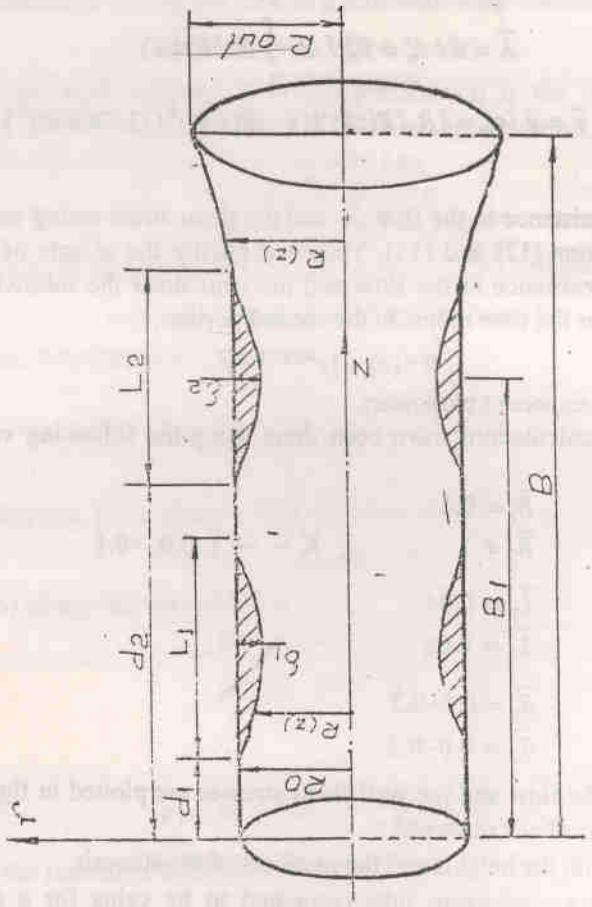

Fig 1.1 Geometry of the tube with multiple stenosis.

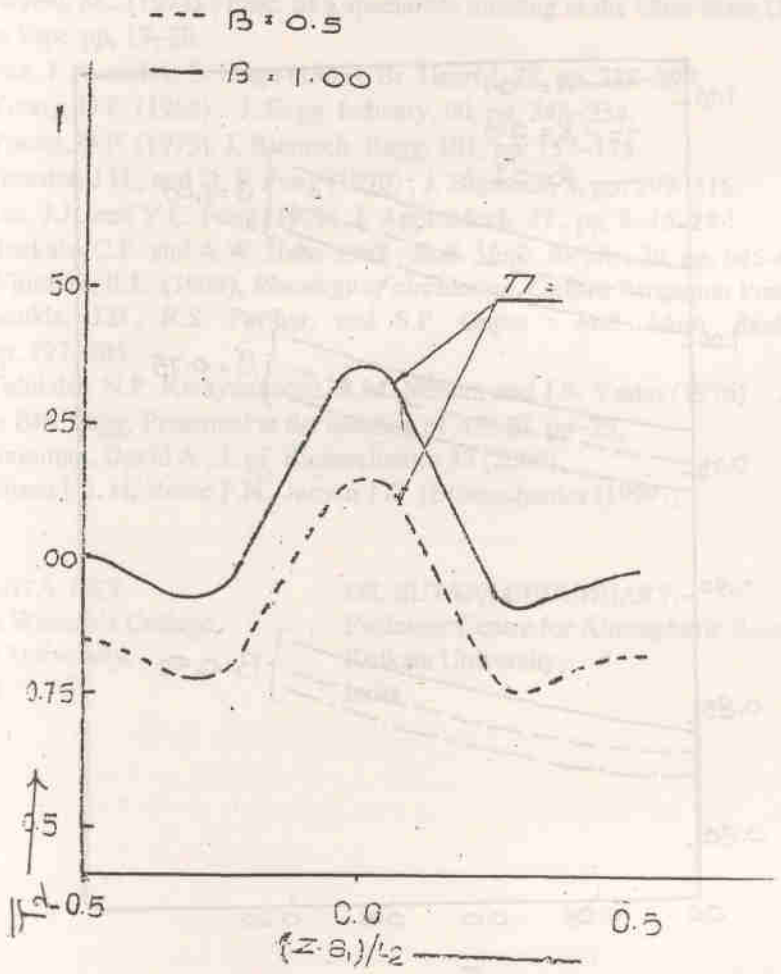

Fig. 1.2. Effect of 'B' and 'n' on $\overline{T}_2$ ($\bar{\delta}_2 = 0.1$; k = 0)

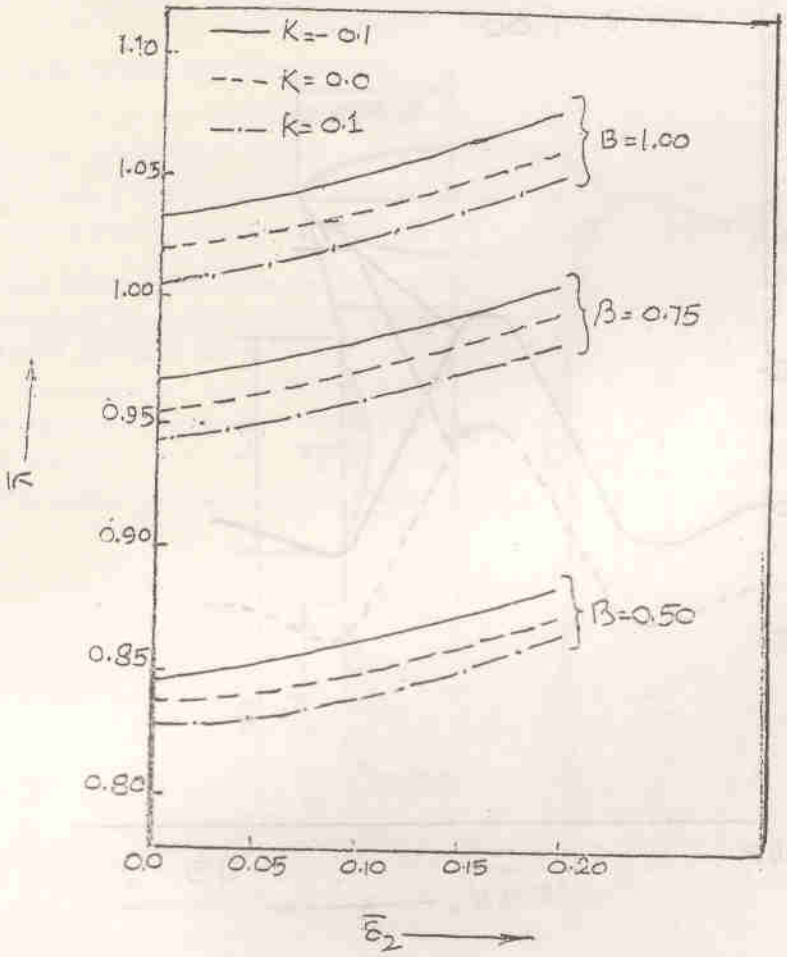

Fig. 1.3 Effect of $\bar{\delta}_2$ on $\bar{\lambda}$ with variation in B and K ($n = 0.15$ $\bar{T}_2$ ($n = 0.15$ $\bar{\delta}_1 = 0.1$).

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Recognition and Isomorphism Algorithms of Shop Graphs

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Abstract: Shop graphs are the graphic representations of classical shop scheduling problems. We present an efficient algorithm to decide whether a given directed graph is a shop graph. This extends a result on the recognition algorithms of sequence graph (acyclic orientation of the Hamming graph $K_n \times K_m$). Orientations of the $K_n \times K_m$ play important roles in practical shop problems. Moreover, we give a short review on the concept of sequence (acyclic shop graph) isomorphisms.

Key Words: shop problems, recognition algorithm, isomorphic sequences.

1. Introduction

In an $n \times m$ *shop scheduling problem*, each job i with $i \in I = \{1, \ldots, n\}$ has to be processed on each *machine* j with $j \in J = \{1, \ldots, m\}$ exactly once without preemption for the processing time $p_{ij} > 0$. We assume that, at a time, each machine can process at most one job and each job can be processed on at most one machine. Let $SIJ = I \times J$ be the set of all operations o_{ij} with $i \in I \wedge j \in J$. The matrix of processing times is denoted by $P = [p_{ij}]_{n \times m}$. We denote the completion time of job i on machines by C_i and the matrix of completion times by $C = [c_{ij}]_{n \times m}$ so that $C_i = \max_j (c_{ij})$ holds, where c_{ij} is the completion time of operation o_{ij}. A scheduling problem is denoted by a triple $\alpha \,|\beta|\, \gamma$, where α describes the machine environment, β gives the job characteristics and γ represents the objective function (cf. [11]). The machine order for job i is the order of machines which process job i, whereas the job order on machine j is the order of jobs processed on machine j. We have to find a feasible combination of machine orders and job orders

* The author would like to extend sincere thanks to DAAD, and to Prof. Heidemarie Bĭasel

(a sequence) which minimised the makespan $C_{max} = \min_i \{C_i\}$. A *schedule* give the corresponding time table.

A computational problem is a function $\prod : Z \to Y$, where Z is the set of all problem instances I and Y is the set of solutions both reasonably encoded as strings of symbols in predefined alphabet. A problem $\prod$ is called decision problem if $Y = \{yes, no\}$. Each optimisation problem has its decision counterpart which is associated by defining an additional threshold value y for the corresponding objective function γ. For example, given an additional threshold value y for the objective function γ we ask : does there exist a feasible solution $x \in X$ such that $\gamma(x) \leq y$? The significant meanings of each of the space and time complexities from the computational point of view are systematically analysed in [9]. Informally, a decision problem is said to be in class $\mathcal{P}$ if there is a polynomial time algorithm solving it. A decision problem belongs to the class $N\mathcal{P}$ if a positive answer can be verified in polynomial time. One of the major open problems of modern mathematics is whether $\mathcal{P} = N\mathcal{P}$. A decision problem in $N\mathcal{P}$ is called *$N\mathcal{P}$-complete* if it can be solved polynomially only if $\mathcal{P} = N\mathcal{P}$.

In Section 2, we summarise the block-matrices model and give some basic notions of graphs applicable in the considered shop scheduling problems. In the block-matrices model, all graph theoretical structures in shop problems are basically described by means of special latin rectangles. For a detailed description of these structures we refer to [5] and to **http://fma2.math.uni-magdeburg.de/ ~ lisa.**

In section 3, given a connected digraph we mainly deal the problem of deciding whether it is a shop graph in a shop scheduling problem. We give an efficient recognition algorithm with linear time and space complexities. One of the objectives to consider such a problem is to investigate some interesting properties of Hamming graph in shop scheduling problems.

The problem of efficiently recognising whether a given graph is a Hamming graph is often treated in the literature (cf. [1,13,14]). A first non-algorithmic proof is : up to isomorphism, all finite connected graphs have unique prime factorisation (cf. [21]). Recall that none of the algorithms which belongs to the class of general decomposition algorithms of graphs with respect to the Cartesian product is linear. Several algorithms for recognising Hamming graphs have already been proposed in the literature. The running time complexity of the first algorithm is bounded by $O(|V|^5)$ (cf. [23]). The fastest known approach for recognising whether a given graph is a Hamming graph, by solving prime factorisation algorithms with respect to the graph Cartesian product, is presented in [1]. Given an undirected connected graph $G = (V,E)$ in its adjacency list data structure, a (unique) prime factorisation of G is obtained with respect to the Cartesian product in $O(|E| \log (|V|))$ time (cf. [1]). An algorithm with linear time complexity with respect to the edges of the graph is

presented in [14] and [13]. Note that the space complexity is also linear in [13] with adjacency list representations. In this way, the considered Hamming graph recognition problem is solved in linear time. We refer to a couple of algorithms in the cited articles for detailed descriptions and the references therein for many striking characterisations of Hamming graphs and several other classes of graphs closely related to Hamming graphs.

In section 4, we consider the decision version of a sequence graph isomorphism problem arising from shop scheduling problems. The decision version of the graph isomorphism problem is to decide whether given two graphs G_1 and G_2 are isomorphic. The Graph automorphism problem is : given a graph G, decide whether its automorphism group contains non-trivial automorphism.

Much effort has been made to find efficient algorithms for the graph isomorphism problem but no polynomial algorithm for this problem has been developed and it is unknown if such an algorithm can exist. The graph isomorphism problem belongs to the class NP but it is still unknown whether it belongs to the class $\mathcal{P}$ or $\mathcal{NP}$-complete. However, for certain special subclasses of graphs, the isomorphism problem is efficiently solvable. For example, planar graphs solved by J. E. HOPCROFT and J.W. WONG in 1974, undirected graphs generated from latin squares solved by G.L. MILLER in 1978, graphs of bounded valence (cf. [17]), cyclic tournaments (cf. [19]) and graphs of bounded average genus (cf. [6]). Because all undirected edges can be replaced by two anti-parallel arcs, the isomorphism problem for undirected graphs is polynomial time reducible to a corresponding isomorphism problem for the directed graphs. On the other hand, the directed graph isomorphism problem is polynomial time reducible to the problem of undirected graph isomorphism (cf. [18]); the problem of directed graph isomorphism and undirected graph isomorphism are polynomially equivalent.

Some concluding remarks are contained in the final section of the paper

2. Basic Concepts

Given a combination of machine orders and job orders in a shop problem of n jobs and m machines, we define the following pair of acyclic digraphs with vertex set SIJ:

- *Machine order graph* $G_{MO} = (SIJ, E_{MO})$, where the set of arcs contains the precedence constraints of all machine orders.
- *Job order graph* $G_{JO} = (SIJ, E_{JO})$, where the set of arcs contains the precedence constraints of all job orders.

The graphs G_{MO} and G_{JO} consist of n and m acyclic components, respectively. We represent by $MO = [mo_{ij}]$ and $JO = [jo_{ij}]$ the $n \times m$ rank matrices of given graphs G_{MO} and G_{JO}, called machine order matrix and job order matrix, respectively. The rank of a vertex in an acyclic directed graph is the number of vertices on a longest

path from a source to the vertex itself. Moreover, jo_{ij} is the position of job i in the job order on machine j and mo_{ij} is the position of machine j in the machine order for job i. The collections of all job order matrices and machine order matrices are, respectively, denoted by JO and MO.

- For given MO and JO, and the set SIJ of operations we define a digraph $G_{MO,JO} = (SIJ, E_{MO,JO})$, where the arc set $E_{MO,JO} = E_{MO} \cup E_{JO}$ reflects all machine orders and all orders.

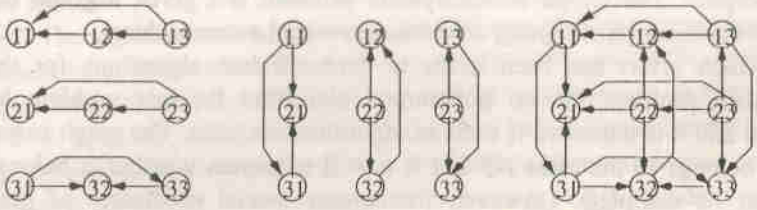

Figure 1: The Transitive Graphs G_{MO}, G_{JO} and $G_{MO,JO}$

Note that , the graph $G_{MO,JO}$ is connected and it may or may not be acyclic. We use [G] for the underlying undirected graph of a digraph G.

Definition 1. *For any pair* (MO, JO), *the graph* $G_{MO,JO}$ *is called a transitive shop graph. If the graph is acyclic* (*cyclic*) *we call it a transitive sequence graph* (*non-sequence graph*).

If we consider only direct precedence constraints in the foregoing graphs, the attribute "transitive" is dropped. The pair (MO, JO) is a sequence (non-sequence) if the shop graph is acyclic (cyclic), respectively. Here, transitive shop graphs and shop graphs are denoted by the same notations for the easiness. The graphs in Figure 1 illustrate the transitive graphs of machine orders, job orders and non-sequence graph with $m = n = 3$.

For each sequence graph $G_{MO,JO}$ we can describe the sequence (MO, JO) by a rank matrix, too. The corresponding matrix $A = [a_{ij}]$ contains the rank of the vertex o_{ij} for each operation o_{ij} in the sequence graph $G_{MO,JO}$. Note that the rank matrix A is a special latin rectangle with *sequence property* : for each integer $lr_{ij} > 1$ there exists the integer $lr_{ij} - 1$ in row i or in column j or in both. Recall that a latin rectangle $LR[n, m, q] = [lr_{ij}]$ is a matrix of size $n \times m$ with its entries $l_{ij} \in \{1, 2, \ldots, q\}$ such that each integer of the symbol set occurs at most once in each row and at most once in each column of LR (cf. [7]). If $n = m = q$ holds, then the matrix is a latin square of order n and is denoted by $LS[n]$.

On the other hand, given any latin rectangle $LR\ [n, m, q] = [l_{ij}]$, we can define a sequence graph by means of its entries l_{ij} as a level of the vertex o_{ij}. Therefore, in particular, any latin rectangle satisfying the sequence property obviously produces an acyclic digraph for a shop scheduling problem with n jobs and m machines. Therefore, there exists a one-to-one correspondence between the set of all latin rectangles with sequence property and the set of all sequence graphs for the open shop scheduling problem (cf.[5]).

The sets of all sequences and sequence graphs of format $n \times m$ in machine environment α are denoted by $\mathcal{LR}(\alpha\ ;\ n, m)$ and by $\mathcal{G}_{\mathcal{LR}}(\alpha\ ;\ n, m)$, respectively. If the context is clear from the considerations, then we may denote these sets simply by $\mathcal{LR}$ and $\mathcal{G}_{\mathcal{LR}}$, respectively. If $m = n = q$, then the set of all latin squares $\mathcal{LS}[n]$ concentrated in machine environment α is denoted by $\mathcal{LS}(\alpha\ ;\ n)$. Clearly, all latin squares satisfy the sequence property. Each element of $\mathcal{LR}(\alpha\ ;\ n, m)$ is also called sequence. A sequence contains all information about machine orders and job orders of the corresponding sequence graph. The terms like source, sink, operation, path, etc., are interchangeable in the sequence graph and the sequence accordingly. By the one-to-one correspondence, we see that the determination of the cardinality of $\mathcal{G}_{\mathcal{LR}}(\alpha\ ;\ n, m)$, which in general is an unsolved counting problem, can also be described as the problem of determination of cardinality $\mathcal{LR}\ (\alpha\ ;\ n, m)$ in the open shop scheduling problem, but the latter problem is also quite hard. One possibility to handle the former problem is the chromatic polynomial of the Hamming graph $K_n \times K_m$, however, the calculation is hard (see [12]).

Note that an infinite set of schedules can be assigned to each sequence. On the other hand, each schedule contains all information about its unique sequence. More formally, we can define an equivalence relation $\mathcal{R}$ on the set of all schedules as:

$S_1 \mathcal{R} S_2 \Leftrightarrow$ both schedules S_1 and S_2 base on the same sequence LR.

It is clear that each equivalence class contains an infinite number of schedules but the number of classes is finite; a trivial upper bound is $(n!)^m (m!)^n$; we refer to [8, 12] for some improvements. In order to find a set of distinct representatives, we may use the semiactive schedules under unit processing times, i.e., all sequences. A schedule is called semiactive if each operation $o_{ij} \in SIJ$ is started as early as possible with respect to the given machine orders and job orders. However, to every sequence of a shop scheduling problem, we can associate a unique semiactive schedule $C = (A, P)$ in linear time $O(nm)$ (cf. [5]). The same time complexity holds for the calculation of the weights of a longest path through the operation o_{ij} for each operation $o_{ij} \in$ SIJ (cf. [5]). Given a sequence of certain

format, determining the associated semiactive schedule is a polynomially solvable problem. Therefore, the main difficulty on the complexity of shop scheduling problems lies also to the construction of appropriate sequences.

The transitive sequence graph is an acyclic oriented digraph of the disjunctive graph (see [22]), where all vertices $o_{ij} \in SIJ$ have unit processing time weights. Disjunctive graphs are widely used to represent certain schedules for general shop scheduling problems. On the other hand, if we assign in a graph $\mathcal{G}_{\mathcal{LR}}(SIJ, E_{MO\cdot JO})$ a vertex cost p_{ij} to each $o_{ij} \in SIJ$, we can consider different objective functions on the set $\mathcal{G}_{\mathcal{LR}}(\alpha\,; n, m)$ of sequences graphs and hence on the set $\mathcal{LR}(\alpha\,; n, m)$ of sequences, too. In particular, if $p_{ij} = 1$ for all i and j, then we have the equation $C = \mathcal{LR}[n, m, r]$. Moreover, $C_{\max} = \max\{c_{ij} | o_{ij} \in \text{SIJ}\}$ is given by the weight of a critical path (the length of a longest path from a source to a vertex itself) in G_{LR} in the case of operations set $SIJ = I \times J$. Then the problem in this case is to determine a sequence with minimal cardinality of the insertion set, for instance.

The graph $[G_{LR}]$ corresponds to the linegraph of bipartite graph $G = (I \cup J, E)$ with edge $(i, j) \in$ E if and only if job i with $i \in \{1, 2, \ldots, n\}$ is processed machine j with $j \in \{1, 2, \ldots, m\}$. For example, the underlying graph $[G_{LR}]$ for $O2\,|n = 2|\,\gamma$ is isomorphic to the 4–cycle Z_4. The linegraph $L(G) = (V_L, E_L)$ of a graph $G = (V, E)$ is a graph with $V_L = E$ and $\{ab, xy) \in E_L$ if and only if $\{a,b\}$ and $\{x, y\}$ belonging to the edge set E are adjacent in the graph G. A graph $G = (V, E)$ is a bipartite if there exist disjoint subsets U and W of V with $V = U \cup W$ such that $\{u, w\} \in E$ implies either $(u \in U \wedge w \in W)$ or $(w \in U \wedge u \in W)$. The graph $[G_{LR}^{tr}]$ of the transitive closure G_{LR}^{tr} of a sequence graph G_{LR} is known as a comparability graph. The transitive closure of a digraph $G = (V,E)$ is the digraph denoted by $G^{tr} = (V, E^{tr})$ such that for each are $(x, y) \in E^{tr}$ there is a path $w_G = (v_0, v_1, \ldots, v_k)$ in acyclic digraph $G = (V, E)$ with $x = v_0$ and $y = v_k$. An undirected graph $G = (V,E)$ is a comparability graph (cf. [10]) if there exists an acyclic orientation E^* of edge set E such that the corresponding digraph $G^* = (V, E^*)$ is transitive closure.

Two graphs $G_i = (V_i, E_i)$ $(i = 1,2)$ are said to be isomorphic denoted by $G_1 \cong^{\psi} G_2$, if there exists a bijection $\psi\colon V_1 \to V_2$ such that for all $v,w \in V_1$ we have $\{v, w\} \in E_1$ if and only if $\{\psi(v), \psi(w)\} \in E_2$; the bijection ψ is called graph isomorphism. A permutation on the set of vertices of a graph is called an automorphism of a graph $G = (V, E)$ if it is adjacency preserving.

The Cartesian product $G := G_1 \times G_2$ of two simple finite graphs G_1 and G_2 is the graph with vertex set $V := V_1 \times V_2$ and edge $\{ux, vy\} \in E_1 \times E_2 =: E$ whenever $\{u,v\} \in E_1$ and $x = y$, or $\{x, y\} \in E_2$ and $u = v$. It is easy to see that the graph Cartesian product is commutative and associative and has the one-vertex simple graph K_1 as a unit such that $K_1 \times G = G = G \times K_1$ for any graph G. Because of the

associativity property, one may write $G = G_1 \times G_2 \times \ldots \times G_r$ for a product G of graphs and the vertex set of such a product the set of all r-tuples $u_1 u_2 \ldots u_r$ where u_k is the vertex in the k-th component. Moreover, the Cartesian product of two graphs is connected if and only if both factors are connected. A graph G is called prime if $G = G_1 \times G_2$ implies $G_1 = K_1$ or $G_2 = K_1$. A set $\{G_1, G_2, \ldots, G_r\}$ of r-graphs is called a prime factorisation of G if $G = G_1 \times G_2 \times \ldots \times G_r$ and $G_k \neq K_1$ for all $k = 1,2, \ldots r$, where G_i is a prime graph.

A Hamming graph $H(G)$ is the Cartesian product of complete graphs, and the Hamming distance, which is the shortest path distance in graph $H(G)$, between two r-tuples $u = u_1 u_2 \ldots u_r$ and $v = v_1 v_2 \ldots v_r$ is the number of positions in which the entries in u and v differ. More formally, Hamming distance is defined as the following discrete metric function:

$$\text{for all } u, v \in H(G),\ d_H(u,v) := |\{k : u_k \neq v_k\}|,$$

so that the Hamming graph $H(G)$ is a discrete metric space. In terms of Hamming distance one may characterise two adjacent vertices $u = u_1 u_2 \ldots u_r$ and $v = v_1 v_2 \ldots v_r$ in $H(G)$ if and only if $d_H(u,v) = 1$. For example, the Cartesian product of r copies of the complete graph K_2 is a special Hamming graph, called hypercube Q_r(r-cube for short) or binary Hamming graph. This class of graphs is a well known object in the field of graph theory as well as computer science.

The definition of Hamming labelling can be described as follows. For all $k = 1, 2, \ldots, r$, let $t_k \geq 2$ be given integers. Furthermore, we consider the vertices of a r–dimensional Hamming graph as all $\prod_{k=1}^{r} t_k$ such r–tuples $u := u_1 u_2 \ldots u_r$ with $1 \leq u_k \leq t_k$ for all $k = 1, 2, \ldots, r$, where two vertices are adjacent if and only if they differ in exactly one place. Then, this type of labelling of the vertices of Hamming graph is called a Hamming labelling. Note that the product graph $H(G) = K_{t_1} \times K_{t_2} \times \ldots \times K_{t_r}$ is an $(\sum_{k=1}^{r} t_k - r)$ –regular graph on $\prod_{k=1}^{r} t_k$ vertices, and hence it has $\frac{1}{2} \prod_{k=1}^{r} t_k \ (\sum_{k=1}^{r} t_k - r)$ edges. Moreover, the neighbourhood of a vertex v in this graph $H(G)$ induces a disjoint union of complete graphs.

Let $G = (V, E)$ be a connected digraph. A topological sorting of the vertices of V is a mapping $\sigma: V \to \{1, 2, \ldots, |V|\}$ such that $\sigma(v) < \sigma(w)$ for all $(v, w) \in E$. For example, $\sigma(v) = 1$ for all sources $v \in$ V. It is well known that a digraph G is acyclic if and only if there exists a topological sorting of G. The Topological Sorting Algorithm is implemented in $O(|E| + |V|)$ time (cf. [15, 16]) using an appropriate data structure.

3. The Recognition Algorithm

In this section we present an efficient algorithm for the following decision problem:

Given : A connected digraph $G = (V, E)$.

Question : Is G a transitive shop graph ?

Moreover, if the answer of this question is "yes", the algorithm yields the matrices *MO* and *JO* for the (transitive) shop graph $G_{MO,JO}$.

We use two well-known algorithms, namely, the Hamming Graph Algorithm (cf. [13 14]) and the Topological Sorting Algorithm (cf. [15,16]), for solving this problem. Here for our purpose, the prescribed labelling algorithm is applicable to the Cartesian product $K_n \times K_m$ of two complete graphs starting from the label o_{11} for the first chosen vertex (cf. [2,8]).

A clique in a graph is its subgraph which is isomorphic to a complete graph and a tournament is an oriented complete graph. We call E^{tr} a transitive tournament if for all vertices $(x,y) \in E^{tr}$ and $(y, z) \in E^{tr}$ implies $(x,z) \in E^{tr}$. Clearly, a tournament is transitive if and only if it is acyclic. Each 2-dimensional Hamming graph $K_n \times K_m$, which is of order *nm*, contains m disjoint subgraphs isomorphic to K_m (so-called n-cliques), and n disjoint subgraphs isomorphic to K_m (so-called m-cliques). On the other hand, a transitive shop graph contains m disjoint subgraphs isomorphic to transitive tournaments of order *n* (so-called column tournaments) and n disjoint subgraphs isomorphic to transitive tournaments of order *m* (so-called row tournaments).

Now, we are able to formulate the following algorithm and verify its validity.

Algorithm 1. Transitive Shop Graph Recognition

Input : Connected digraph $G = (V, E)$.

Output : *MO* and *JO*, if *G* is a transitive shop graph.

Step 1. *Call the Hamming Graph Algorithm for the graph* $[G]$ *of* $= (V, E)$.
If $[G]$ *is not a Hamming Graph,* ***then goto Step 5;***
else *output of the Hamming graph Algorithm:*
n,m, for all $v \in V$ *: label* $(v) = o_{ij}$ *with* $1 \le i \le n$, $1 \le j \le m$;

Step 2. *Identify each vertex* v *with the associated label* $(v) = o_{ij}$;
the cliques K_i^r *and* K_j^c *are the complete subgraphs of* $G = (V, E)$ *induced by the vertex sets* $\bigcup_{j=1}^{m}\{o_{ij}\}$, $i = 1, 2, \ldots, n$, *and* $\bigcup_{i=1}^{n}\{o_{ij}\}$, $j = 1, 2, \ldots, m$, *respectively* ;

Step 3. *For all* $i \in \{1, 2, \ldots, n\}$ ***do***
begin
Call the Topological Sorting Algorithm for the clique K_i^r *;*
if K_i^r *contains a cycle then goto Step 5;*
else output of this algorithm : $rk(o_{ij})$ *for all* $o_{ij} \in V(K_i^r)$ *;*
insert the values $rk(o_{ij})$ *into row i of matrix MO;*
end;

Step 4. ***For all*** $j \in \{1, 2, \ldots, m\}$ ***do***
begin
Call the Topological Sorting Algorithm for the clique K_j^c *;*
if K_j^c *contains a cycle* ***then goto*** *Step 5;*
else output of this algorithm : $rk(o_{ij})$ for all $o_{ij} \in V(K_j^c)$ *;*
insert the values $rk(o_{ij})$ *into column j of matrix JO;*
end;

Output : *G is a transitive shop graph ; MO, JO ;* ***stop.***

Step 5. *G is not a transitive shop graph ;* ***stop***

Because a tournament is transitive if and only if it is acyclic, it provides a linear time complexity for recognising transitive (equivalently, acyclic) tournaments (cf. [10]). The matter is first to calculate the in-degree (or, out-degree) of each vertex, and then to verify that there are no repetitions among the in-degrees (or, out-degrees). It is well known (cf. [20]) that each tournament contains an oriented Hamiltonian path. Moreover, such an acyclic (equivalently, transitive) tournament contains exactly one Hamiltonian path. Afterwards, each tournament which is a subgraph of a transitive shop graph is spanned by a unique path.

Theorem 1. Let $G = (V, E)$ *be a connected digraph. Then the problem of deciding whether G is a transitive shop graph is solvable in* $O(\max\{mn^2, m^2n\})$ *time.*

Proof: Given a connected undirected graph $G = (V, E)$, the problem of deciding whether $[G]$ is a Hamming graph is solvable by the Labelling Algorithm and the Hamming Graph Algorithm in $O(|E|)$ time complexity (cf. [13, 14]). Here, given the connected digraph $G = (V, E)$ we consider the underlying undirected graph $[G]$. Afterwards, we call the Hamming Graph Algorithm [13, 14] and decide if it is a Hamming graph of this kind.

Total time complexity for the Steps 3 and 4 is approximately the same as that of Topological Sorting Algorithm, namely, $O(|V|+|E|)$ and this bound can be reduced to $O(|E|)$ in this case, because $|V| < |E|$ in our shop model. Because each acyclic orientation of a complete graph contains exactly one Hamiltonian path (cf.

[20]), by calling the Topological Sorting Algorithm [16] for all cliques in the Hamming graph we determine the ranks of vertices which reflect the precedence constraints. Thus, since the Steps 2 to 4 do not need more time than Step 1 in the algorithm, the overall time complexity of the algorithm is not worse than the complexity of the Hamming Graph Algorithm.

Moreover, since the total number of edges in a Hamming graph associated to any classical shop scheduling problems of our interest in $n\binom{m}{2} + m\binom{n}{2}$, the time complexity reaches $O(\max\{mn^2, m^2n\})$. ■

Clearly, the presented recognition algorithm is linear in time with respect to the arcs of the input digraph.

A transitive sequence graph is an acyclic orientation of the Hamming graph $K_n \times K_m$ (cf. [3, 12]). Therefore, if we consider the transitive sequence graph, then the number of arcs of such a digraph G_{MO_JO} is $n\binom{m}{2} + m\binom{n}{2}$, and it is reduced to $n(m-1) + m(n-1)$ in the case of a sequence graph without transitive arcs. Therefore,

Theorem 2 *For a given digraph $G = (V, E)$, the problem of deciding if it is a transitive sequence graph is solvable in $O(\max\{mn^2, m^2n\})$ time. Moreover, the computational complexity of the sequence according to a given $n \times m$ transitive sequence graph is $O(\max\{mn^2, m^2n\})$.*

Proof: The prescribed algorithms in the cited references [3, 12] are the Hamming Labelling Algorithm and the Hamming Graph Algorithm [14], and the Topological Sorting Algorithm [16]. The focus point of the proof of stated statement is to consider the problem into two different algorithmic parts. In the first part, one considers the former two algorithms in order to check and label the vertices if it is a Hamming graph. The final part is to calculate the rank if it is acyclic which can be performed by topological sorting. Precisely, the rank of the vertex labeling o_{ij} in the sequence graph is associated with the value of the element o_{ij} in the sequence. In this sorting algorithm, we have to mark all sources instead of a single source at a time and delete the marked sources and the adjacency arcs. The time complexity is clear because of the number of arcs in this specific sequence graph. Moreover, the space complexities remain $O(\max\{mn^2, m^2n\})$ by using the advantages of the algorithms in [13], where the adjacency lists data structures are implemented. ■

However, note that the complexity $O(mn)$ is correct in the case of given $n \times m$ sequence graphs without any transitive arcs. Furthermore, it is also clear that if the number of jobs is completely dominated by the number of machines ($n \ll m$), then the computational time complexities $O(\max\{mn^2, m^2n\})$ and $O(mn)$ reduce to $O(m^2)$ and $O(m)$, respectively.

4. Isomorphism of Sequences

In this section we review an efficient algorithm (cf. [3, 12, 8]) for the decision problem :

Given : Two sequences of the same format.

Question : Does there exists a sequence isomorphism ?

Moreover, if the answer of this question is "*yes*" the algorithm yields an isomorphism. We denote by π_r, π_c, Φ, and ψ, respectively, a row permutation $\pi_r \in S_n$, a column permutation $\pi_c \in S_m$, a transposition $\Phi \in Z_2$, and $\psi \in Z_2$, of a matrix, where S_t is the usual symmetric group on $\{1,2, \ldots, t\}$ and Z_2 is the cyclic group of order two. The transposition of a matrix means reflecting in the main left-to-right diagonal. Note that Φ maps sequence A to the transposed sequence A^T. For matrix reversion of sequence A, we have to replace each arc $(o_{ij}, o_{kl}) \in E_{MO,JO}$ by an oppositely oriented arc (o_{kl}, o_{ij}) in the sequence graph $G_A = (SIJ, E_{MO,JO})$. In order to classify the main structural differences and to get deeper study of equivalent sequence properties between shop scheduling problems, the following definition is more useful (see [4]).

Definition 2 *Two sequences A and B are called structure isomorphic, graph isomorphic or permutation isomorphic, denoted by $A \cong_s B$, $A \cong_g B$, or $A \cong_p B$, if there exists a mapping such that $(\pi_r, \pi_c, \Phi, \Psi)A = B$, $(\pi_r, \pi_c, \Phi)A = B$ or $(\pi_r, \pi_c)A = B$, respectively.*

For the sake of simplicity, we simply say that two sequences A and B are isomorphic if they are isomorphic under any one of the above isomorphism relations. The notion $A \cong B$ is used to represent such an arbitrary isomorphism. Note that the collection of all isomorphisms of the same type under the same formats forms a group. Therefore, there are three groups, namely, $S_n \times S_m$, $S_n \times S_m \times Z_2$ and $S_n \times S_m \times Z_2 \times Z_2$ according to each isomorphism type mentioned above. The order of the group of permutation isomorphisms is always $n!m!$, whereas, for $m \neq n$, this number is $n!m!$, and $2n!m!$ if we consider the group of graph isomorphisms, and group of structure isomorphisms, respectively. Furthermore, if $m = n$, then the latter two numbers are exactly doubled, because transposition of a sequence is also applicable in this case.

Clearly, each of the relations $A \cong_s B$, *or* $A \cong_g B$ and $A \cong_p B$ defined above yields an equivalence relation on the set $\mathcal{LR}$ decomposing the set of all sequences into disjoint isomorphism classes. We denote by $\mathcal{PI}$, $\mathcal{GI}$ and $\mathcal{SI}$, respectively, the collections of all permutation isomorphism classes, graph isomorphism classes and structure isomorphism classes. Obviously, $|\mathcal{PI}| \geq |\mathcal{GI}| \geq |\mathcal{SI}|$, since the relations $A \cong_p B \Rightarrow A \cong_g B \Rightarrow A \cong_s B$ hold. The

sequences of an isomorphism class in $\mathcal{PI}$ are equivalent with respect to an arbitrary reindexing of the machines and of the jobs. Also, the sequences of an isomorphism class in $\mathcal{GI}$ are equivalent with respect to an arbitrary renumbering of the completely interchangeable machine set and job set. Finally, any two sequences belonging to the same isomorphism class in $\mathcal{SI}$ are equivalent in the sense that the orientations of each operations are also allowed in addition to an arbitrary reindexing of the completely interchangeable machine set and job set.

If an isomorphism $\cong$. applied to the sequence A results to the same sequence A, it is called a sequence automorphism. Naturally, there corresponds to three types of sequence automorphisms, namely, permutation automorphism, graph automorphism and structure automorphism. The collection of all automorphisms in each class forms a subgroup of the group of isomorphisms with respect to their corresponding classes. Because, ((2, 1, 3), (2, 1, 3), Φ) $A = A$ holds if we have the sequence $A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$, the mapping ((2, 1, 3), (2, 1, 3), Φ) is a graph automorphism.

Definition 3 A sequence A = $[a_{ij}]_{n\times m}$ is called in normal form if $a_{11} = 1$, $a_{1j} < a_{1l}$ $(2 \le j < l \le m)$ and $a_{i1} < a_{k1}$ $(2 \le i < k \le n)$.

Two sequences A and B are graph isomorphic if and only if their associated (transitive) sequence graphs G_A and G_B are isomorphic in terms of graph theory. In [3] a polynomial time Sequence Isomorphism Algorithm (*SIA*) is presented in order to decide whether two sequences are graph isomorphic. The main idea of this algorithm is to put one of the given sequences in normal form and to make appropriate permutations of the other so as to put it as the first. If there is no success, one concludes that the given sequences are not graph isomorphic. In this way, for given $n \times m$ sequences A and B, the permutation isomorphism of A and B is decidable in $O(\max\{mn^2, m^2n\})$ time. With the same time complexity, the graph isomorphism of two sequences A and B as well as the sequence graph isomorphism between the transitive sequence graphs G_A and G_B are decidable.

The validity of the following theorem follows from the summarised *SIA* (cf. [8]). The summarised *SIA* is a compact form of the *SIA* in [3] and the concept of structure isomorphism of sequences in [4]. Moreover, the time and space complexities of compact *SIA* remain similar to that of *SIA*. Therefore,

Theorem 3 *Let A and B be $n \times m$ sequences. Then the isomorphism of A and B is decidable in $O(\min\{mn^2, m^2n\})$ time.* ■

Algorithm 2 The Sequence Isomorphism (*SIA*)

Input : Two arbitrary $n \times m$ sequences $A = [a_{ij}]$ and $B = [b_{ij}]$.

Step 1 : *Find* $[\pi_{r_A}, \pi_{c_A}]$ *such that* $[\pi_{r_A}, \pi_{c_A}]\, A = A^n$, *where sequence* A^n *is normal*;

Step 2 : *For all entries* o_{ij} with $b_{ij} = 1$ *in* **B** *do*:

1. Find $[\pi_{r_B}, \pi_{c_B}]$ *such that* $[\pi_{r_B}, \pi_{c_B}]\, B = B^n$, *where* $b^n_{11} = b_{ij}$ *and* B^n *is normal*;
2. If $B^n = A^n$, return the isomorphism $(\pi_{r_A}\, \pi^{-1}_{r_B}, \pi_{c_A}\, \pi^{-1}_{c_B})$, ***stop***;
3. *For* $m = n$, *if* $B^{n^T} = A^n$, ***return*** *the isomorphism* $(\pi_{r_A}\, \pi^{-1}_{r_B}, \pi_{c_A}\, \pi^{-1}_{c_B}, \Phi)$, ***stop*** ;
4. *If* $B^n = A^n_{-1}$, ***return*** *the isomorphism* $(\pi_{r_A}\, \pi^{-1}_{r_B}, \pi_{c_A}\, \pi^{-1}_{c_B}, \Psi)$, ***stop*** ;
5. *For* $m=n$, *if* $B^{n^T} = A^n_{-1}$, ***return*** *the isomorphism* $(\pi_{r_A}\, \pi^{-1}_{r_B}, \pi_{c_A}\, \pi^{-1}_{c_B}, \Phi, \Psi)$, ***stop*** ;

Output : *An isomorphism* $(\pi_r, \pi_c, \Phi, \Psi)$ *if it exists.*

$\begin{pmatrix}1&2\\2&1\end{pmatrix}$	$\begin{pmatrix}2&1\\1&2\end{pmatrix}$		
$\begin{pmatrix}1&2\\2&3\end{pmatrix}$	$\begin{pmatrix}2&1\\3&2\end{pmatrix}$	$\begin{pmatrix}2&3\\1&2\end{pmatrix}$	$\begin{pmatrix}3&2\\2&1\end{pmatrix}$
$\begin{pmatrix}1&2\\4&3\end{pmatrix}$	$\begin{pmatrix}2&1\\3&4\end{pmatrix}$	$\begin{pmatrix}3&4\\2&1\end{pmatrix}$	$\begin{pmatrix}4&3\\1&2\end{pmatrix}$
$\begin{pmatrix}1&4\\2&3\end{pmatrix}$	$\begin{pmatrix}2&3\\1&4\end{pmatrix}$	$\begin{pmatrix}3&2\\4&1\end{pmatrix}$	$\begin{pmatrix}4&1\\3&2\end{pmatrix}$

Table 1: Class Distribution of 2×2 Sequences

Table 1 Illustrates rowwise distribution of all sequences for $O2 \mid n = 2 \mid \gamma$ into different sequence isomorphism classes. Namely, four different rows represent the four permutation isomorphic classes whereas the last two rows together give one class of graph isomorphism as well as structure isomorphism; here, four permutation classes and three graph/structure isomorphism classes. Clearly, the class representatives form a system of distinct representatives for the isomorphism classes of sequences. The number of nonisomorphic sequences can be calculated by applying Algorithm 2 on the set of all sequences, but this procedure is not sufficient from computational point of view. We refer to [4, 12] for its implementation with small formats. However, a set of sequences generated only on the class representatives (for example, the lexicographically minimal) of their equivalence classes plays an important role for further investigation . In [4, 12] , properties of sequence isomorphisms are applied to investigate a set of sequences which contains at least one optimal solution independent of processing times.

5. Concluding Remarks

In this paper, we have considered recognition algorithms for special class of graphs, so-called transitive shop graphs arising from shop scheduling problems. Investigations of such efficient algorithms allow us to study interesting properties of the 2-dimensional Hamming graph $K_n \times K_m$ in shop problems. Namely, both cyclic as well as acyclic orientations of $K_n \times K_m$ provide useful hints for further structural analysis of shop problems. A study of isomorphic properties of sequences not only provides a decomposition of sequences but also gives some useful hints for further structural analysis of sequences from algebraic point of view. On the other hand, a study of isomorphic properties of cyclic orientations of $K_n \times K_m$ could be one of the interesting subjects from theoretical point of views. Results included in this field are again of both theoretical and practical interests.

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Cox's Regression Model for the Growth of a Tumour Followed by Death from that Tumour

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Abstract: In this paper we introduce Cox's Regression Model to study the growth process of a tumour and death from that tumour. The hazard rate is taken to be a function of the explanatory variables and unknown regression coefficients multiplied by an arbitrary and unknown function of time. Formulas for the density function, the distribution function and the expectation of lifetime are outlined. An equivalent reformulation of the model is also given in terms of the intensities of counting processes. Estimation procedures in the model are discussed and large sample properties of the estimators are outlined.

1. Introduction

A tumour is a swelling of any kind in any par of the body. Tumours occur when the cells of a tissue or organ multiply in an uncontrolled fashion unrelated to the biological requirements of the body and not to meet the needs of repair or of normal replacement. The growth of tumour in man is supposed to depend upon an interplay between factors in the environment and the genetic component of body cells. When an individual is exposed to one or more environmental factors (or carcinogenic agents) such as different occupations, lifestyles, exposure to injurious chemical agents, drugs, ultraviolet light, certain tumour inducing viruses etc., the body's metabolic and other biological reactions render most of the absorbed molecules inactive and are excreted from the body system, but a few metabolites remain and become carcinogenic which are responsible for the growth of tumour in man. The potential carcinogenic metabolites bind with DNA in cells to cause mutation of DNA and result in tumour. The formation of tumour is a manifestation of this continuous biological process that depends on the amount of toxic material retained in the body.

The appearance of tumour cannot always be detected, either because the tumour has not yet reached the detectable size, or because it is hidden in the body. The retained amount of toxic material in the body becomes carcinogenic only after a

specified period of time which may be months or years. The shortest period of time is known as the Latent Period. Suppose this period is denoted by $[0, t_0]$, where t_0 is the length of this time.

For every $t > t_0$, the amount of toxic material absorbed during the interval $[0, t-t_0]$, will be carcinogenic at time t and the amount absorbed during the interval $[t-t_0, t]$ will be potentially carcinogenic.

The growth process of tumour can be explained by using Cox's Regression Model. Cox'x Regression model, based on the method of 'Partial Likelihood', for analysing censored survival data allowing for covariates, is beautifully adapted to the kind of data obtained in clinical cancer trials. By incorporating time-varying random covariates, it becomes a highly flexible tool for model building. The hazard rate at any point of time does not depend on time only but also on a host of explanatory variables or covariates, some of which may not be expressed in quantitative form.

2. Development of the Model

Suppose $T_i, i = 1,2,\ldots,n$, are independent continuously distributed positive random variables representing the times of death of n individuals suffering from tumour, each of whom is observed for a fixed time interval $[0, c_i]$ for certain censoring times $c_i, i = 1,2,\ldots,n$. Suppose that individual i has hazard rate.

$$\lambda_i(t) = \lim_{dt \to 0} \frac{1}{dt} P\left[T_i \le t + dt \mid T_i \ge t\right], \tag{2.1}$$

which can be written as

$$\lambda_i(t) = \lambda_0(t)\, e^{\beta' z_i(t)} \tag{2.2}$$

where

$\lambda_i(t) =$ hazard rate for the i^{th} individual at time t which is a function of overall hazard rate $\lambda_0(t)$, a function of time t only irrespective of other covariates, and hazard rate $e^{\beta' z_i(t)}$ with respect to p factors having corresponding time independent intensities $(\beta_1, \beta_2, \ldots, \beta_p)$ with which the i^{th} individual will be affected at time t. β' is the transpose of the column vector β of p unknown coefficients and $\underset{\sim}{z}_i$ is a column vector of p time-dependent covariates.

Let

$$R(t) = \{i : T_i \ge t, c_i \ge t\}$$

denote the risk set at time t, i.e., the set of individuals i under observation at time t. Given $R(t)$ and that at time t one individual in $R(t)$ is observed to die, the probability

that it is individual i or the probability of death of the i^{th} individual at time t when his censoring time is c_i can be calculated as:

$$\frac{\lambda_0(t)e^{\beta' z(t)}}{\sum_{i\in R(t)} \lambda_0(t)e^{\beta' z(t)}}$$

Thus, the likelihood function is given by

$$L(\beta) = \prod_{T_i \le c_i} \prod_{i=1}^{n} \frac{\exp[\beta' z_i(t)]}{\sum_{j\in R(t)} \exp[\beta' z_i(t)]}$$

This likelihood function, according to Cox (1975) was an example of Partial Likelihood

3. Distribution of Time to Death

Since T_i is a random variable representing the lifetime of the i^{th} individual, who is observed for a fixed time interval $[0, c_i]$, therefore, for every t, $0 \le t \le c_i$, the hazard rate at time t is given by (2.2) , i.e.,

$$\lambda_i(t) = \lambda_0(t)e^{\beta' z_i(t)}$$

The cumulative hazard function is given by

$$\int_0^t \lambda_i(x)\,dx = \int_0^t \lambda_0(x)\, e^{\beta z_i(x)} dx \; ; 0 \le t \le c_i, \tag{3.1}$$

The survival function is given by

$$S(t) = P \text{ (of survival of } i^{th} \text{ individual atleast time } t)$$

$$= \exp\left\{-\int_0^t \lambda_i(x)\,dx\right\} \; ; \; 0 \le t \le c_i \tag{3.2}$$

$$\Rightarrow \; S(t) = \left\{-\int_0^t \lambda_0(x)\, e^{\beta' z_i(x)}\, dx\right\} \; ; \; 0 \le t \le c_i \tag{3.3}$$

The corresponding distribution function and the density function of the lifetime T_i of the i^{th} individual can be obtained from the following expressions:

$$F(t) = 1 - S(t)$$

$$\Rightarrow \; F(t) = 1 - \exp\left\{-\int_0^t \lambda_0(x)\, e^{\beta z_i(x)}\, dx\right\} \; ; \; 0 \le t \le c_i \tag{3.5}$$

and

$$(3.4) \qquad f(t) = -\frac{d}{dt}\left[\exp\left(-\int_0^t \lambda_0(x)\, e^{\beta z_i(x)}\, dx\right)\right] \; ; \; 0 \le t \le c_i$$

The expectation of lifetime of i^{th} individual, T_i for $T_i \le c_i$, can be obtained by using the following expression

$$(3.6) \qquad E[T_i \mid T_i \le c_i] = \left(\int_0^{c_i} t\, f(t)\, dt\right)[F(c_i)]^{-1}$$

4. Formulation of Cox-Regression Model Based on Counting Process Theory

Cox's Regression Model can be reformulated as a model for the random intensity of a multivariate counting process. The original hazard rate definition of Cox's model can be interpreted as specifying the stochastic intensity of multivariate counting process (counting occurrences of the event "death" for each of the individuals under observation).

The observation of the i^{th} individual may be considered as the observation of a counting process.

$$N_i = \{N_i(t) \; ; t \ge 0\}$$

where N_i counts 1 if death is observed in the i^{th} individual, otherwise zero, i.e.,

$$(4.1) \qquad N_i = \begin{cases} 1, & \text{if } T_i \le t,\ T_i \le c_i \\ 0, & \text{otherwise} \end{cases}$$

The counting process N_i has a random intensity process.

$$\Delta_i = \{\Delta_i(t) \; ; t \ge 0\}$$

defined by

$\Delta_i(t)dt = P\{$death is observed in the i^{th} individual in a time interval of length dt around time $t \mid F_t.\}$

$$(4.2) \qquad \Delta_i(t) = \lim_{dt \to 0} \frac{1}{dt}\, P\{N_i(t+dt) - N_i(dt) = 1 \mid F_t\}$$

where

F_t denotes the past upto the beginning of the small interval dt, i.e., everything that has happened until just before time t.

Now, given what has happened before the time interval dt, if the i^{th} individual dies at the observed time $T_i \le t$ and $T_i \le c_i$, or if the i^{th} individual was censored at time $c_i < t$ then

P(of death of the i^{th} individual in the interval dt) = 0

But, if the i^{th} individual is still alive and uncensored, $T_i \in dt$ or $T_i \geq t$, then by (2.1)

$$P(\text{of death of the } i^{th} \text{ individual in the interval } dt) = \lambda_i(t)dt$$

Define

$$Y_i(t) = \begin{cases} 1, & \text{if individual } i \text{ is under observation just before time } t \\ 0, & \text{otherwise} \end{cases} \tag{4.3}$$

From (2.2) and (4.2) we get

$$\Delta_i(t)dt = Y_i(t)\,\lambda_i(t)\,dt$$

$$\Rightarrow \quad \Delta_i(t)\,dt = Y_i(t)\,\lambda_0(t)\,e^{\beta' z_i(t)}\,dt \tag{4.4}$$

Given the past upto (but not including) time t, $Y_i(t)$ & $\Delta_i(t)$ are fixed or non random. Thus Y_i and Δ_i are predictable.

Now, N_i is a simple multivariate counting process, each component of which jumps utmost once, with intensity process Δ_i, satisfying

$$\Delta_i(t)\,dt = Y_i(t)\,\lambda_0(t)\,e^{\beta' z_i(t)}\,dt \tag{4.5}$$

where

The fixed covariate $z_i(t)$ is replaced by a random covariate $Z_i(t)$. N_i, Y_i and Z_i are processes that can be observed and Y_i and Z_i are predictible since $Y_i(t)$ and $Z_i(t)$ are fixed given what has happened before time t. Thus, the Cox's Model is reformulated by

$$L(\beta) = \prod_{t\geq 0}\prod_{i=1}^{n}\left(\frac{Y_i(t)\exp(\beta' Z_i(t))}{\sum_{j=1}^{n} Y_j(t)\exp\{\beta' Z_j(t)\}}\right)^{dN_i(t)} \tag{4.6}$$

where $dN_i(t)$ is the increment of N_i over a small interval dt around the time t and the product over t is a product over disjoint intervals. So (4.6) reduces to a finite product over all i and t for which N_i jumps at time t, i.e., death is observed in the ith individual at time $t(dN_i(t) = 1)$; elsewhere $dN_i(t) = 0$. Now, (4.6) is the original likelihood function for β_0 which is the true value β. Let $\hat{\beta}$ be the value of β maximising $L(\beta)$. Define $L(\beta,u)$ as the likelihood function for β based on the observations on the time interval $[0,u]$, in which the product over $t \geq 0$ in (4.6) is replaced by a product $0 \leq t \leq u$. From (4.6) we have

$$L(\beta_0, u) = L(\beta_0) = \prod_{0\leq t\leq u}\prod_{i=1}^{n}\left(\frac{Y_i(t)\exp(\beta_0' Z_i(t))}{\sum_{j=1}^{n} Y_j(t)\exp\{\beta_0' Z_j'(t)\}}\right)^{dN_i(t)} \tag{4.7}$$

5. Large-Sample Properties of $\hat{\beta}$

Taking logarithm of the Cox's likelihood (4.7) we get

$$\log L(\beta_0, u) = \sum_{i=1}^{n} \sum_{t \le u} dN_i(t) \log \left(\frac{Y_i(t) \exp(\beta_0' Z_i(t))}{\sum_{j=1}^{n} Y_j(t) \exp\{\beta_0' Z_j(t)\}} \right) \tag{5.1}$$

$$\Rightarrow \log L(\beta_0, u) = \sum_{i=1}^{n} \sum_{t \le u} dN_i(t) \left[\log(Y_i(t) \exp(\beta_0' Z_j(t))) - \log \left(\sum_{j=1}^{n} Y_j(t) \exp\{\beta_0' Z_j'(t)\} \right) \right]$$

Now,

$$D \log L(\beta_0, u) = \sum_{i=1}^{n} \sum_{t \le u} dN_i(t) \left[\frac{Y_i(t) Z_i(t) e^{\beta_0' Z_i(t)}}{Y_i(t) e^{\beta_0' Z_i(t)}} - \frac{\sum_{j=1}^{n} Y_j(t) Z_j(t) \exp\{\beta_0' Z_j(t)\}}{\sum_{j=1}^{n} Y_j(t) \exp\{\beta_0' Z_j(t)\}} \right] \tag{5.2}$$

where Dlog $L(\beta)$ denotes the vector of partial derivatives $\frac{\partial}{\partial \beta} \log L(\beta)$ evaluated at β.

$$n^{-1/2} D \log L(\beta_0) = n^{-1/2} \sum_{i=1}^{n} \sum_{t \le u} dN_i(t) \left[Z_i(t) - \frac{\sum_{j=1}^{n} Y_j(t) Z_j(t) e^{\beta_0 Z_j(t)}}{\sum_{j=1}^{n} Y_j(t) e^{\beta_0 Z_i(t)}} \right]$$

$$= n^{-1/2} \sum_{i=1}^{n} \sum_{t \le u} [Z_i(t) - E_0(t)] \, dN_i(t)$$

$$\Rightarrow n^{-1/2} D \log L(\beta_0) = n^{-1/2} \sum_{i=1}^{n} \int_{t=0}^{u} [Z_i(t) - E_0(t)] \, dN_i(t) \tag{5.3}$$

where,

$$E_0(t) = \frac{\sum_{j=1}^{n} Y_j(t) Z_j(t) \exp[\beta_0' Z_j(t)]}{\sum_{j=1}^{n} Y_j(t) \exp[\beta_0' Z_j(t)]} \tag{5.4}$$

$$\Rightarrow \quad n^{-1/2}\, D\log L(\beta_0, u) = \sum_{i=1}^{n} \int_{t=0}^{u} n^{-1/2} [Z_i(t) - E_0(t)]\, dN_i(t)$$

Now, since $N_i(t)$ is a counting process with the corresponding intensities $\Delta_i(t)$, therefore, $M_i(t)$ defined as

$$M_i(t) = N_i(t) - \int_0^1 \Delta_i(\tau)\, d\tau \tag{5.6}$$

is a Martingale.

$$\Rightarrow \quad n^{-1/2}\, D\log L(\beta_0, u) = \sum_{i=1}^{n} \int_{t=0}^{u} n^{-1/2} [Z_i(t) - E_0(t)]\, dM_i(t) \tag{5.7}$$

since

$$dM_i(t) = dN_i(t) - \Delta_i(t)\, dt$$

and

$$\begin{aligned}
&\sum_{i=1}^{n} [Z_i(t) - E_0(t)]\, \Delta_i(t) \\
&= \sum_{i=1}^{n} Z_i(t)\, Y_i(t) \lambda_0(t) \exp(\beta_0' Z_i(t)) - E_0(t) \sum_{i=1}^{n} Y_i(t) \lambda_0(t) \exp(\beta_0' Z_i(t)) \\
&= \sum_{i=1}^{n} Z_i(t)\, Y_i(t) \lambda_0(t) \exp(\beta_0' Z_i(t)) - \sum_{j=1}^{n} Z_j(t) Y_j(t) \lambda_0(t) \exp(\beta_0' Z_j(t)) \\
&= 0
\end{aligned}$$

Now, $n^{-1/2}[Z_i(t) - E_0(t)]$ is a vector of predictable processes and it only depends on the fixed parameter β_0 and the predictable processes Y_j and Z_j, $j = 1, 2, \ldots, n$. Therefore, by Martingale Transform Theorem, $M^{(n)}(t) = n^{-1/2}\, D\log L(\beta_0, t)$, considered as a stochastic process in t, is the sum of n (vector) martingales, hence also a martingale.

Now, expanding Dlog $L(\beta, u)$ around β_0, using Taylor's expansion, we get

$$D\log L(\beta, u) = D\log L(\beta_0, u) - I(\beta^*, u)\ (\beta - \beta_0) \tag{5.8}$$

where β^* is on the line segment between β and β_0, and the positive semidefinite matrix $I(\beta, u)$ is minus the second derivative of $\log L(\beta, u)$ with respect to β i.e.,

$$I(\beta, u) = \frac{-\partial^2 \log L(\beta, u)}{\partial \beta^2} = \frac{-\partial}{\partial \beta} D \log L(\beta, u)$$

Then, $I(\beta, u) = \int_0^u \times$

$$\times \frac{\sum_{j=1}^{n} Y_j(t) e^{\beta' Z_j(t)} \sum_{j=1}^{n} Y_j(t) Z_j(t)^{\otimes 2} e^{\beta' Z_j(t)} - \sum_{j=1}^{n} Y_j(t) Z_j(t) e^{\beta' Z_j(t)} \sum_{j=1}^{n} Y_j(t) Z_j(t) e^{\beta' Z_j(t)}}{\left(\sum_{j=1}^{n} Y_j(t)\, e^{\beta' Z_i(t)} \right)^{\otimes 2}}$$

$$\times d\bar{N}(t)$$

where for a column vector 'a' the matrix aa' is denoted by $a^{\otimes 2}$ or $a \otimes a$ where $a = (a_1, a_2, \ldots, a_n)'$ is a p-vector and $\bar{N} = \sum_{i=1}^{n} N_i$

$$(5.9) \Rightarrow I(\beta, u) = \int_0^u \left[\frac{\sum_{j=1}^{n} Y_j(t) Z_j(t)^{\otimes 2}\, e^{\beta' Z_j(t)}}{\sum_{j=1}^{n} Y_j(t)\, e^{\beta' Z_j(t)}} - \left(\frac{\sum_{j=1}^{n} Y_j(t) Z_j(t)\, e^{\beta' Z_j(t)}}{\sum_{j=1}^{n} Y_j(t)\, e^{\beta' Z_j(t)}} \right)^{\otimes 2} \right] d\bar{N}(t)$$

Now, from (5.8)

$$D \log L(\beta, u) - D \log L(\beta_0, u) = - I(\beta^*, u)\ (\hat{\beta} - \beta_0).$$

Inserting $\hat{\beta}$ we get

(5.10) $$D \log L(\hat{\beta}, u) - D \log L(\beta_0, u) = - I(\beta^*, u)\ L(\hat{\beta}, \beta_0)$$

But $$D \log L(\hat{\beta}, u) = 0$$

since $\hat{\beta}$ is the solution of the likelihood equation $\frac{\partial}{\partial \beta} \log L(B, u) = 0$

$$\Rightarrow \quad D \log L(\beta_0, u) = I(\beta^*, u)\ (\hat{\beta} - \beta_0)$$

$$\Rightarrow \quad n^{-1/2} D \log L(\beta_0, u) = \{n^{-1} I(\beta^*, u)\} n^{1/2} (\hat{\beta} - \beta_0).$$

Now, to prove the asymptotic normality of $n^{1/2}(\hat{\beta}_0, \beta_0)$ we have to prove that the martingale $n^{-1/2}\ D \log L(\beta_0, u)$ weakly converges to a Gaussian Process and $n^{-1} I(\beta^*, u)$ converges in probability to a non-singular matrix. This can be derived by using the Martingale Central Limit Theorem (Rebolledo, 1980) and the counting process approach of Andersen and Gill (1982) with some regularity conditions.

Thus, $\hat{\beta}$ is consistent and asymptotically normally distributed as $n \to \infty$ under mild regularity conditions on the covariate processes and the variance of $\hat{\beta}$ can be estimated consistently from the second derivative $L(\beta)$ evaluate at $\hat{\beta}$.

Consistency and asymptotic normality of the maximum partial likelihood estimates of the regression parameters can also be established by using the weak convergence results as suggested by Tsiatis (1981).

6. Conclusion

The purpose of this paper is to develop a model for the study of growth of a tumour followed by death from that tumour based on partial likelihood method of Cox. It has been shown that how this model can be developed further by using counting process theory. Distribution of time to death has also been outlined. The model also permits the estimation of parameters affecting the growth process of tumour. Consistency and asymptotic normality for the maximum partial likelihood estimate of the regression parameter in Cox's Regression Model can be established by using Martingales techniques suggested by Andersen and Gill (1982) or by using weak convergence results suggested by Tsiatis (1981).

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A Cylindrically Symmetric Cosmological Model with Cosmic Cloud Strings in Bimetric Relativity

V. MAHURPAWAR AND S. D. DEO

Abstract: In this paper a cylindrically symmetric cosmological model is studied in the context of bimetric relativity taking the source as cosmic cloud strings and is found that the model does not exist in bimetric relativity

Key Words: Cylindrically symmetric cosmological model, Cosmic cloud strings, Bimetric relativity.

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1. Introduction:

Rosen[3] proposed a bimetric theory of relativity where there exist two metric tensors at each point of space-time–g_{ij} , which describes gravitation and background metric $-\gamma_{ij}$, which enters into the field equations directly with mater.

Accordingly, at each space-time point one has two line elements

$$ds^2 = g_{ij}\, dx^i\, dx^j$$

and

$$d\sigma^2 = \gamma_{ij}\, dx^i\, dx^j$$

This theory is based on a simple form of Lagrangian and has a simpler mathematical structure than that of the general relativity.

Deo[1] studied this model with the source perfect fluid distribution and found that the model does not exist in this theory.

Here a cylindrically symmetric non-static Einstein-Rosen cosmological model is studied with the source cosmic cloud strings and obtained the vacuum solutions.

2. Field Equations and Model

Field equations of bimetric relativity formulated by Rosen[3] are

$$K_i^j = N_i^j - \tfrac{1}{2} N g_i^j = -8\pi k T_i^j \tag{2.1}$$

where

$$N_i^j = \tfrac{1}{2}\gamma^{\alpha\beta}\left(g^{hj} g_{hi|\alpha}\right)_{|\beta} \tag{2.2}$$

$$N = N_\alpha^\alpha, \; k = (g/\gamma)^{1/2} \tag{2.3}$$

$$g = \det(g_{ij}) \text{ and } \gamma = \det(\gamma_{ij})$$

and a vertical bar (|) denotes the covariant differentiation with respect to γ_{ij}.

The energy-momentum tensor T_i^j for cosmic cloud strings is given by

$$T_i^j = T_{i\,\text{strings}}^j$$

$$T_{i\,\text{strings}}^j = \rho v_i v^j - \lambda x_i x^j \tag{2.4}$$

Here ρ is the rest energy density for a cloud of strings with particle attached along the extension.

Thus

$$\rho = \rho_p + \lambda,$$

where ρ_p is the particle energy density, λ is the tension density of the cloud strings, v^i is the four-vector representing the velocity of the cloud of particle and x^i is the four-vector representing the direction of anisotropy i.e. Z-axis.

So that

$$v_4 v^4 = -1, x_3 x^3 = 1 \text{ and } v_i x^i = 0$$

Now consider the cylindrically symmetric non-static Einstein-Rosen cosmological model[2] given by

$$ds^2 = e^{2(\alpha-\beta)}(dT^2 - (dR^3) - R^2 e^{-2\beta} d\phi^2 - e^{2\beta} dZ^2 \tag{2.5}$$

where α and β are functions of R and T and the convention is

$$x^1 = R, x^2 = \phi, x^3 = Z \text{ and } x^4 = T$$

The flat metric corresponding to (2.5) is

$$d\sigma^2 = dT^2 - dR^2 - R^2 d\phi^2 - dZ^2 \tag{2.6}$$

For this model non-vanishing Christoffel symbols are

$$\Gamma_{12}^2 = \Gamma_{21}^2 = R^{-1}, \Gamma_{22}^1 = -R \tag{2.7}$$

Using equations (2.1) to (2.7) the field equations are

$$R^{-2} \sin h(2\alpha) = 0 \tag{2.8}$$

$$\alpha'' + R^{-1}\alpha' - \alpha^{\cdot\cdot} - R^{-2} \sin h(2\alpha) = 0 \tag{2.9}$$

$$\alpha'' + R^{-1}\alpha' - \alpha'' - 2\beta'' - 2R^{-1}\beta' + 2\beta^{\cdot\cdot} = 8\pi k\lambda \tag{2.10}$$

$$0 = 8\pi k\rho, \tag{2.11}$$

where $\alpha' = d\alpha / dR,\ \alpha'' = d^2\alpha / dR^2$ etc.

and $\alpha' = d\alpha / dT,\ \alpha'' = d^2\alpha / dT^2$ etc.

By using equation (2.11)

$$\rho = 0$$

i.e. there is no contribution from energy density for a cloud of strings to the cylindrically symmetric non-static Einstein-Rosen cosmological model.

Thus the vacuum field equations are

(2.12) $$R^{-2} \sin h(2\alpha) = 0$$

(2.13) $$\alpha'' + R^{-1}\alpha' - \alpha'' - R^{-2} \operatorname{Sin} h(2\alpha) = 0$$

(2.14) $$\alpha'' + R^{-1}\alpha' - \alpha'' - 2\beta - 2R^{-1}\beta' + 2\beta'' = 0$$

From equation (2.12) we get

(2.15) $$\alpha = 0$$

Equation (2.14) and (2.15) gives

(2.16) $$\beta'' + R^{-1}\beta' - \beta'' = 0$$

For solving equation (2.16)

Case I: Let us consider $\beta = H(r, t) + G(t)$

Then (2.16) gives the Bessel equation

(2.17) $$H'' + (1/R)\, H - H'' = 0$$

representing cylindrical waves provided

(2.18) $$G'' = 0$$

The Bessel equation (2.17) results

(2.19) $$H = a\, J_0(hR) \operatorname{Cos}(ht + c_1) + b Y_0(hR) \operatorname{Sin}(ht + c_2)$$

where J_0 (hR) and Y_0 (hR) are Bessel functions of first and second kind of order zero respectively, h is the frequency and a, b, c_1, c_2 are arbitrary constants

On integration , (2.18) gives

(2.20) $$G = c_3 t + c_4$$

where c_3, c_4 are arbitrary constants of integration

Then

(2.21) $$\beta = a\, J_0(hR) \operatorname{Cos}(ht + c_1) + b Y_0(hR) \operatorname{Sin}(ht + c_2) + c_3 t + c_4$$

Case II: Let us consider

$$\beta = H(r, t) + g(r)$$

In this case again we obtain equation (2.17) provided

(2.22) $$g'' + R^{-1} g' = 0$$

It gives us

(2.23) $$g = c_5 \log R$$

where c_5 is arbitrary constant of integration

Now the solution turns out to be

(2.24) $$\beta = a J_0(hR) \operatorname{Cos}(ht + c_1) + b Y_0(hR) \operatorname{Sin}(ht + c_2) + c_5 \log R,$$

where constants could be found using initial as well as boundary conditions from the above equations.

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On Induced Structures and Curvature Tensors In the Tangent Bundle

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Abstract: In this paper, we have studied structures induced In the tangent bundle T(M) if the base space M admits almost GF-contact structure. Certain results on curvature tensors on T(M) are also studied assuming that M admits the Riemannian connection.

1. Preliminaries :

Let M be an n-dimensional differentiable manifold and $T(M)$denotes tangent bundle of M. Then $T(M)$ is also a differentiable nanifold of dimension $2n$ [3]

Suppose over the basespace M, there exists a tensorfield F of type (1,1), a vectorifield ξ and a 1-form η satisfying.

$$\text{(1.1)} \qquad \begin{aligned} &\text{(i)} \quad F^2 = a^2 I_n + \eta \otimes \xi \\ &\text{(ii)} \quad F\xi = 0 \qquad \text{and} \\ &\text{(iii)} \quad \eta(\xi) = -a^2 \end{aligned}$$

'a' any complex number not zero. Then we say that the basepace M admits almost GF-contact structure.

(1.2) If $$F^2 = a^2 I_n ,$$

we say that M admits GF- Structure [2].

It is well known that F^C, F^V, F^H etc are complete, vertical and horizontal lifts in $T(M)$ of (1.1) tensorfield F or M if we define

$$\text{(1.3)} \qquad P = F^C + \frac{1}{a}\eta^V \otimes \xi^V + \frac{1}{a}\eta^C \otimes \xi^C$$

and

$$\text{(1.4)} \qquad Q = F^C + \frac{1}{a}\eta^V \otimes \xi^V + \frac{1}{a}\eta^H \otimes \xi^H$$

then it is easy to show that P and Q define almost GF-structure on T(M) [1]. If G be Riemannian metric on M then G^C given by

(1.5) $$G^C(X^C, Y^C) = G(X, Y)^C$$

for each $X, Y \in \Im_0^1(M)$ defines the Riemannian metric on T(M)

2. Induced Structures in T(M)

In this section we shall prove the following theorems.

Theorem 2.1. If *F gives an almost GF-contact structure on the basespace M then* (1,1) *tensorfield K given by*

$$K = F^C + \left(\frac{\beta^{2+1}}{a\gamma}\right)\eta^V \otimes \xi^C + \frac{\beta}{a}\eta^C \otimes \xi^H - \frac{\beta}{a}\eta^H \otimes \xi^V + \frac{\gamma}{a}\eta^C \otimes \xi^C$$

defines GF-structure an T(M), $\beta, \gamma \in R, \gamma \neq 0$.

Proof: Proof follows easily by virtue of equation (1.1) of previous section and equation on (3.26) on page 20 in [3].

Theorem 2.2. *For* (1,1) *tensorfield F admitting almost GF-contact structure on M, the* (1,1) *tensorfield L given by*

$$L = F^C + \left(\frac{\beta^{2+1}}{a\gamma}\right)\eta^V \otimes \xi^V + \frac{\beta}{a}\eta^V \otimes \xi^H - \frac{\beta}{a}\eta^H \otimes \xi^V + \frac{\gamma}{a}\eta^H \otimes \xi^H$$

gives on almost GF-structure on T(M).

Proof: Proof follows easily in view of equation (1,1) of previous section and equation on page 119 [3].

Theorem 2.3. *The* (1,1) *tensofield J defined as*

(2.3) $$JX^V = aX^H, \; JX^H = aX^V$$

for each $X \in \Im_0^1(M)$ gives almost GF-structure on T(M)

Proof: We have in view of equation (2.3)

$$J^2X^V = aJX^H = a^2X^V$$

and $$J^2X^H = aJX^V = a^2X^H$$

Hence, $J^2 = a^2 I_{2n}$ on T(M) which proves the proposition.

3. Curvature Identities

Suppose the basespace M admits the Riemannian metric G and the Riemannian connection ∇. It is well known that G^C given by

(3.1) $$G^C(X^C, Y^C) = G(X, Y))^C$$

defines the Riemannian metric and ∇^C given by

$$G^C_{X^C} Y^C = (\nabla_X Y)^C \tag{3.2}$$

defines the Riemannian connection in T(M). If R(X,Y) Z be the curvature tensor of M with respect to connection ∇.

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{3.3}$$

It is well known that

$$R^C(X^C, Y^C)\, Z^C = \nabla^C_{X^C} \nabla^C_{Y^C} Z^C - \nabla^C_{Y^C} \nabla^C_{X^C} Z^C - \nabla^C_{[X^C, Y^C]} Z^C \tag{3.4}$$

is curvature tensor in T(M)

Theorem 3.1. *If* R(X,Y)Z *be the Riemannian curvature tensor for the basespace M then.*

$$R^C(X^C, Y^C)Z^C + R^C(Y^C, Z^C)X^C + R^C(Z^C, X^C)Y^C = 0 \text{ in } T(M)$$

Proof: We have

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

Taking complete lift, we have

$$R^C(X^C, Y^C)\, Z^C = \nabla^C_{X^C} \nabla^C_{Y^C} Z^C - \nabla^C_{Y^C} \nabla^C_{X^C} Z^C - \nabla^C_{[X^C, Y^C]} Z^C$$

Interchanging X^C, Y^C, Z^C cyclically and adding all the three equations we get

$$\begin{aligned} R^C(X^C, Y^C)\, Z^C + R^C(Y^C, Z^C)\, X^C + R^C(Z^C, X^C)Y^C \\ = \nabla^C_{X^C} \{\nabla^C_{Y^C} Z^C - \nabla^C_{Z^C} Y^C\} \\ + \nabla^C_{Y^C} \{\nabla^C_{Z^C} X^C - \nabla^C_{X^C} Z^C\} \\ + \nabla^C_{Z^C} \{\nabla^C_{X^C} Y^C - \nabla^C_{Y^C} X^C\} \\ - \nabla^C_{[Y^C, Z^C]} X^C - \nabla^C_{[Z^C, X^C]} Y^C - \nabla^C_{[X^C, Y^C]} Z^C \end{aligned}$$

Since $\nabla^C_{X^C} Y^C - \nabla^C_{Y^C} X^C = [X^C, Y^C]$ as ∇^C is Riemannian connection in T(M) hence

$$\begin{aligned} & R^C(X^C, Y^C)Z^C + R^C(Y^C, Z^C)X^C + R^C(Z^C, X^C)Y^C \\ & = \{\nabla^C_{X^C}[Y^C, Z^C] - \nabla^C_{[Y^C, Z^C]} X^C\} + \{\nabla^C_{Y^C}[Z^C, X^C] - \nabla^C_{[Z^C, X^C]} Y^C\} \\ & \quad + \{\nabla^C_{Z^C}[X^C, Y^C] - \nabla^C_{[X^C, Y^C]}\, Z^C\} \\ & = [X^C, [Y^C, Z^C]] + [Y^C, [Z^C, X^C]] + [Z^C, [X^C, Y^C]] \\ & = 0 \end{aligned}$$

by Jacob Identity.

Hence the proposition.

Theorem 3.2. *If in the basespace* M *for each* $X, Y \in \mathfrak{I}_0^1(M)$

$$K(K,Y) = \frac{g(X, R(X,Y)Y)}{g(X,X)\, g(Y,Y) - \{g(X,Y)\}^2}$$

Then in T(M), for each real numbers s, t

$$K^C(X^C, Y^C) = K^C(sX^C, tY^C).$$

Proof: As given

$$K(K,Y) = \frac{g(X, R(X,Y)Y)}{g(X,X)\, g(Y,Y) - \{g(X,Y)\}^2}$$

Taking complete lift we obtain

$$K^C(X^C, Y^C) = \frac{g^C(X^C, R^C(X^C, Y^C)Y^C}{g^C(X^C, X^C)\, g^C(Y^C, Y^C) - \{g^C(X^C, Y^C)\}^2}$$

It is easy to show that

$$K^C(sX^C, tY^C) = K^C(X^C, Y^C).$$

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On A Class of Bilateral Generating Function For Hermite Polynomials of Two Variables* $H_n(x,y)$

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Abstract: A new class of bilateral generating functions (3) for Hermite polynomials of two variables $H_n(x,y)$ is obtained. Applications of our result are pointed out.

Key words & phrases : Special functions & Bilateral Generating functions.

1. Introduction

Noticing the existence of the following type of generating function of Hermite polynomials of two variables $H_n(x,y)$ (4)

$$(1.1)\qquad e^{2xt-yt^2}\left(1-\frac{yt}{x}\right)^{2\alpha} H_\alpha\left[x\left(1-\frac{yt}{x}\right)^{-1}, y\left(1-\frac{yt}{x}\right)^{-4}\right] = \sum_{n=0}^{\infty}\frac{H_{\alpha+n}(x,y)\,t^n}{n!},$$

where α is a non negative integer.

We are led to investigate a more general class of generating function by Lie group-theoretic method.

The main result of our investigation is the following theorem :

Theorem 1. *If there exists a linear generating function of the form*

$$(1.2)\qquad G(x,y,t)=\sum_{n=0}^{\infty} a_n\, H_{\alpha+n}(x,y)\,t^n$$

then the following new bilateral generating function (3) will exist.

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* A. M. S. Subject classification : 33C 45

$$(1.3)\quad \left(1-\frac{yz}{x}\right)^{2\alpha} \exp(2xz-yz^2)\, G\left[x\left(1-\frac{yz}{x}\right)^{-1},\ y\left(1-\frac{yz}{x}\right)^{-4},\ tz\left(1-\frac{yz}{x}\right)^{2}\right]$$
$$= \sum_{K=0}^{\infty}\sum_{n=0}^{k} \frac{a_n}{(k-n)!}\ H_{\alpha+k}(x,y)\, t^n z^k \quad \text{with } \left|\frac{yz}{x}\right| < 1$$

2. Group Theoretic Discussion

For $H_n(x,y)$ we have the following differential operator

$$(2.1)\quad B = Z\left(y\frac{\partial}{\partial x}+\frac{4y^2}{x}\frac{\partial}{\partial y}-\frac{2yz}{x}\frac{\partial}{\partial z}+2x\right) \quad (4)$$

such that (3)

$$(2.2)\quad B\left[Hn(x,y)\, z^n\right] = H_{n+1}(x,y)\, z^{n+1}$$

The extended form of the transformation group generated by B is (3)

$$(2.3)\quad [\exp bB]\ f(x,y,z)$$
$$= \exp\left\{2bxz\left(1-\frac{byz}{2x}\right)\right\}\cdot f\left[x\left(1-\frac{byz}{x}\right)^{-1},\ y\left(1-\frac{byz}{x}\right)^{-4},\ z\left(1-\frac{byz}{x}\right)^{2}\right]$$

3. Derivation of the generating functions

Let us consider the generating functic ı

$$(3.1)\quad G(x,y,t) = \sum_{n=0}^{\infty} a_n\ H_{\alpha+n}(x,y)\, t^n$$

Multiplying both sides by t^α, we get

$$(3.2)\quad t^\alpha G(x,y,t) = \sum_{n=0}^{\infty} a_n\ H_{\alpha+n}(x,y)\, t^{\alpha+n}$$

Replacing t by tz, we get

$$(3.3)\quad t^\alpha z^\alpha G(x,y,tz) = \sum_{n=0}^{\infty} a_n\ H_{\alpha+n}(x,y)\, z^{\alpha+n}\ t^{\alpha+n}$$

Operating both member of (3.3) by $\exp(bB)$

$$\{\exp(bB)\}\ [t^\alpha z^\alpha G(x,y,tz)] = \{\exp(bB)\}\sum_{n=0}^{\infty} a_n\ H_{\alpha+n}(x,y)\, z^{\alpha+n}\ t^{\alpha+n}$$

The left hand side of the last equation becomes

$$t^{\alpha}\, z^{\alpha}\left(1-\frac{byz}{x}\right)^{2\alpha} \exp\left\{2bxz\left(1-\frac{byz}{2x}\right)\right\} G\left[x\left(1-\frac{byz}{x}\right)^{-1}, y\left(1-\frac{byz}{x}\right)^{-4}, tz\left(1-\frac{byz}{x}\right)^{2}\right]$$

On the other hand using (2.2) the right hand side is reduced to

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} a_n \frac{b^k}{k!}\, H_{\alpha+n+k}(x,y)\, z^{\alpha+n+k}\, t^{\alpha+n}$$

Hence we get

$$(3.4)\qquad \left(1-\frac{byz}{x}\right)^{2\alpha} \exp\left\{2bxz\left(1-\frac{byz}{2x}\right)\right\} G\left[x\left(1-\frac{byz}{x}\right)^{-1}, y\left(1-\frac{byz}{x}\right)^{-4}, tz\left(1-\frac{byz}{x}\right)^{2}\right]$$

$$= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} a_n \frac{b^k}{k!}\, H_{\alpha+n+k}(x,y)\, z^{n+k}\, t^{n}$$

Now putting $b = 1$ we get (1.3) using Lemma 10 of Rainville (Page 56, 57)

4. Application

If we consider $a_n = \frac{1}{n!}$ then $G(x,y,t) = \sum_{n=0}^{\infty} \frac{H_{\alpha+n}(x,y)\, t^n}{n!}$

Also from (4) we have

$$(4.1)\qquad G(x,y,t) = e^{2xt-yt^2}\left(1-\frac{yt}{x}\right)^{2\alpha} H_{\alpha}\left[x\left(1-\frac{yt}{x}\right)^{-1}, y\left(1-\frac{yt}{x}\right)^{-4}\right]$$

Again we obtain from (4)

$$(4.2)\qquad G\left[x\left(1-\frac{yz}{x}\right)^{-1}, y\left(1-\frac{yz}{x}\right)^{-4}, tz\left(1-\frac{yz}{x}\right)^{2}\right]$$

$$= e^{2xzt-2yz^2-yt^2z^2}\left(\frac{x-yz-ytz}{x-yz}\right)^{2\alpha} \cdot H_{\alpha}\left[x\left\{1-\frac{yz(1+t)}{x}\right\}^{-1}, y\left\{1-\frac{yz(1+t)}{x}\right\}^{-4}\right]$$

From (1.3) & (4.2) we obtain

$$(4.3)\qquad \exp\left[2xz(1+t)-yz^2(1+t)^2\right]\left\{1-\frac{yz(1+t)}{x}\right\}^{2\alpha}$$

$$H_\alpha\left[x\left\{1-\frac{yz(1+t)}{x}\right\}^{-1}, y\left\{1-\frac{yz(1+t)}{x}\right\}^{-4}\right]$$

$$=\sum_{k=0}^{\infty}\sum_{n=0}^{k}\frac{1}{n!(k-n)!}\ H_{\alpha+k}(x,y)\,t^n\,z^k$$

Now applying the relation $H_n(x,y)=y^{n/2}\ H_n(x/\sqrt{y})$ we get on putting $y=1$.

(4.4) $$\exp\left[2xz(1+t)-z^2(1+t)^2\right]\ H_\alpha\left[x-z(1+t)\right]$$

$$=\sum_{k=0}^{\infty}\sum_{n=0}^{k}\frac{1}{n!(k-n)!}\ H_{\alpha+k}(x)\,t^n\,z^k,$$

which is a new bilateral generating function for Hermite polynomial of Single variable.

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Applications of Fixed Points To Approximation Theory

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Abstract: The theory of fixed point is a very extensive field, which has various applications. Also, there is an interesting relation between fixed point theory and best approximation. This paper is the survey work on some applications of fixed point theorems in approximation theory.

Key Words and Phrases : fixed points, best approximation, non-expansive mappings, invariant point, star-shaped set.

1. Introduction

Fixed point theory is an important area of analysis. If T is self mapping of a metric space (E, d) then a point x in E is said to be fixed point of T if $Tx = x$; that is, a point which remains invariant under a self mapping is called a fixed point. Theory of fixed point is a very extensive field that has various applications. Fixed point theory has played a central role in problems of non-linear functional analysis and it provided power tools in demonstrating the existence of solutions to a large variety of problems in applied mathematics. Also, approximation theory is concerned with the approximation of function of a certain kind (for instance, continuous functions on some interval) by other (probably simpler) functions (for example, polynomial). A natural setting for the problem of approximation is as follows :

Let E be a normed linear space and C is a non-empty fixed subset of E and $x \in E$. An element y in C is called an element of best approximation of x (by the elements of the set C) if we have $\|x-y\| = d(x, C) = \inf \{\|x - z)\| \; ; z \in C\}$; that is, if y is 'nearest' to x among the elements of C. We see that a best approximation y is an element of minimum distance from the given x. Such a y in C may or may not exist. This raises the problem of existence. Also, if $T : C \to E$ is a function such that, for $x \in C$,

$$(*) \qquad \| Tx - x\| = d(Tx, x) = \inf \{\| Tx - y\| : y \in C\}.$$

Then y is called a solution of (*) if and only if y is a fixed point of $P_C \, oT$, where the set valued mapping P_C is the metric projection on C. The set of best approximation

to x is given by $P_C(x) = \{z \in C : \|x - z\| = d(z, C)\}$. There is very interesting relation between fixed point theory and best approximation. S. Park [15] established that if T satisfies a suitable boundary condition, for example, $Tx \in C$ for all $x \in C$ that is $T : C \to C$ then the set of solutions of (*) coincides with the fixed point set of T.

Fixed point theorems have been used in many instances in approximation theory. To prove existence of best approximates, we have results from Brandt [1], Brosowski [2], Ky Fan [12], Hicks and Humphries [8], Reich [16], Sahney, Singh and Whitfield [18], Singh ([19], [20], [21]), Singh and Watson [22] and Subrahmanyam [24]. For different types of applications of fixed point theorems (mainly Schauder's fixed point theorem), we find results from Brosowski [3], Klee [9] and Subrahmanyam [24]. Also, application of the fixed point theorem to simultaneous best approximation is given by Sahney and Singh [17]. For further references we refer to Brosowski [2], Cheney [4] and Dhage [5]. Between the fixed point theorems and their ultimate application in approximation theory, there is often another level of abstract theorems, usually stating that under suitable condition, a non-linear mapping has a root. Some examples of theorems on this intermediate level have been given by Cheney [4]. In this paper, we briefly present some applications of fixed points to approximation theory.

2. Basic Definitions

We have following basic definitions and well-known results on best approximations.

Definition 2.1. If $P_C(x)$ is non empty for every $x \in E$, then C is called proximinal. If $P_C(x)$ contains at most one element for every $x \in E$ then C is called a chebyshev set.

Definition 2.2. A subset D of E is called boundedly (weakly) compact if, for every $x \in E$ and every $r > 0$, the set $K[x, r] \cap D$ is (weakly) compact, where $K[x, r] = \{y \in E : \|x - y\| \leq r\}$.

Definition 2.3. Let X, Y be Hausdorff spaces and let $A(Y)$ be the set of all non-empty and closed subset of Y. A mapping $A : X \to A(Y)$ is called lower (or upper) semi-continuous on X if, for every x in X, the set $\{x \in X : A(x) \cap U \neq \phi\}$ is open (or closed) whenever U is open (or closed) subset of Y.

Definition 2.4. A normed linear space E is called uniformly convex if, given $\varepsilon > 0$, there exists $\delta(\varepsilon)$ such that $\|x - y\| \geq \varepsilon$ for $\|x\| \leq 1$ implies $\|(x + y)/2\| \leq 1 - \delta(\varepsilon)$. It is noted that every inner product space is uniformly convex. However, the converse is not true. For example, l^p spaces are uniformly convex for $1 < p < \infty$, as they are not inner product spaces for $p \neq 2$.

Definition 2.5. A normed linear space E is called strictly convex if, $\|\lambda x + (1-\lambda)y\| < 1$ for all λ, $0 < \lambda < 1$ and all x, y in E such that $\|x\| = \|y\| = 1$. It is noted that uniformly convex normed linear spaces are strictly convex. However, the converse is not true [25, p. 111].

Definition 2.6. A Banach space is said to have Oshman property if, the metric projection on every closed convex subset is upper semicontinuous.

Definition 2.7. A subset K of E is called star-shaped, provided there is a point $p \in K$ such that for each $x \in K$, the sequence joining x to p is contained in K. It is noted that if K is convex, it is star-shaped. However, the converse is not true [19].

Definition 2.8. A mapping T of a metric space E into itself is said to be locally contractive [7] if, for every $x \in E$ there exists $\varepsilon > 0$ and λ, $0 \leq \lambda < 1$, which may be defined on x such that $p, q \in S(x, \varepsilon) = \{y \in E : d(x, y) < \varepsilon\}$ implies $d(Tp, Tq) \leq \lambda\, d(p,q)$. Also, T is called (ε,λ)-uniformly locally contractive if it is locally contractive and neither ε nor λ depends on x.

Definition 2.9. A metric space E is called η-chainable if, for every a, b in E there exists an η-chain; that is, there exists a finite set of points $a = x_0 < x_1 < x_2 < \ldots < x_n = b$, n may depend on both a and b, such that $d(x_{i-1}, x_i) < \eta$ for $i = 1,2,3, \ldots, n$.

Definition 2.10. Let E be a locally convex Hausdorff topological linear space and let $\mathcal{P}$ be a (fixed) family of continuous seminorms which generates the topology of E. Let C be a nonempty subset of E and let p be a continuous seminorm. For $x \in E$, we define $d_p(x,C) = \inf\{p(x-y) : y \in C\}$ and $P_C(x) = \{y \in C : p(x-y) = d_p(x,C)\}$. Then, the set C is said to be proximinal with respect to p if, for all x in E, $P_C(x)$ is nonempty [16]. It is called approximately compact with respect to p (approximately p-compact) if for each $y \in E$, every net $\{x_\alpha : \alpha \in \wedge\} \subset C$ such that $p(y - x_\alpha) \to d_p(y - C)$ has a subnet that converges to an element of C.

Definition 2.11. Let C be a nonempty subset of E, and let $F = \{f_\alpha\}_{\alpha \in C}$ be a family of functions from [0,1] into C having the property that for each $\alpha \in C$, we have $f_\alpha(1) = \alpha$. Such a family F is said to be contractive, provided there exists a function $\phi : (0,1) \to (0,1)$ such that for α and β in C and for all t in [0,1] and p in $\mathcal{P}$, we have $p(f_\alpha(t) - f_\beta(t)) \leq \phi(t)\, p(\alpha - \beta)$. The function F is said to be jointly continuous, provided that if $t \to t_0$ in [0,1] and $\alpha \to \alpha_0$ in C, then $f_\alpha(t) \to f_{\alpha_0}(t)$ in C.

Definition 2.12. A mapping $T : C \to C$ is said to be p-contractive if for each p in $\mathcal{P}$ there is a k_p, with $0 \leq k_p < 1$ such that $p(Tx - Ty) \leq k_p p(x - y)$ for all x, y in C. If $p(Tx - Ty) \leq p(x - y)$, for each p in $\mathcal{P}$, then T is called p-nonexpansive.

Definition 2.13. Let $\{x_n\}$ be a sequence in E. Then x_n is Cauchy if and only if for each p in Q, $p(x_n - x_m) \to 0$ as $n, m \to \infty$. Also, E is sequentially complete if every Cauchy sequence in E converges to some element in E. Moreover, E is quasi complete if every bounded, closed subset of E is complete. Clearly, every complete space is quasicomplete and every quasicomplete space is sequentially complete. However, the converse is not true [9].

Definition 2.14. A mapping $T: C \to C$ is said to be demicompact if, each bounded net $\{x_\alpha\}$ in C such that $(I-T)(x_\alpha)$ converges, has a convergent subnet. T is called compact if T is continuous and maps bounded subsets of C into relatively compact subsets of C. It is noted that every compact mapping is demicompact. However, the converse is not true [19].

Definition 2.15. Let C be a subset of E and $T: C \to E$ be a mapping. Then mapping T is said to be demiclosed if, for any net x_α in C such that $x_\alpha \to x$ weakly and $Tx_\alpha \to y$, it follows that $y = Tx$. In other words, the mapping T is demiclosed if its graph in $C \times E$ is closed in Cartesian product topology induced in $C \times E$ by the weak topology in C and the strong topology in E.

Definition 2.16. Let $x \in C[a, b]$ and $y \in Y$, where Y is a subspace of the real space $C[a,b]$. A set of points $t_0, t_1, t_2, \ldots, t_k$ in $[a,b]$, where $t_0 < t_1 < t_2 < \ldots < t_k$, is called an alternating set for $x-y$ if $x(t_j) - y(t_j)$ has alternatively the values $+ \| x-y\|$ and $- \| x-y\|$ at consecutive points t_j.

Definition 2.17. A finite dimensional subspace Y of the real space $C[a,b]$ is said to satisfy the Haar condition if, every $y \in Y, y \neq 0$, has at most $n-1$ zeros in $[a,b]$, where $n = \dim Y$.

3. Existence of Best Approximations

We have the following well-known results on the existence of best approximations.

Theorem 3.1. *If C is a finite dimensional subspace of a strictly convex normed space E then there exists at most one best approximation to an $x \in E$ out of C.*

Theorem 3.2. *If C is a finite dimensional subspace of the real space $E = C[a,b]$ then the best approximation out of C is unique for every $x \in E$ if and only if C satisfies the Haar condition.*

Theorem 3.3. *Let C be a subspace of the real space $E = C[a,b]$ satisfying the Haar condition. Given $x \in E$, let $y \in C$ be such that for $x - y$, there exists an alternating*

set of n +1 points, where n = dim C. Then y is the best uniform approximation to x out of C.

Theorem 3.4. *If C is a closed linear subspace or only a closed non-empty convex subset of the reflexive Banach space E, then for every $x \in E$, the equation $\|x-y\| = d(x, C)$, for $y \in C$ is solvable in C.*

Example I. Consider $\Re^2$ with l_2^2 norm. Setting $x = (1,1)$ and $x_1 = (1,0)$, we have $\|x - \alpha x_1\| = \|1-\alpha, 1\|$ which is minimum iff $\alpha = 1$. Thus x_1 is the unique best approximation to x in the closed linear subspace spanned by x_1, denoted by $[x_1]$.

Example II. Consider $\Re^2$ with l_2^∞ norm, with $x = (1,1)$ and $x_1 = (1,0)$ we have $\|x - \alpha x_1\| = \max(|1-\alpha|, 1)$ which is minimum iff $|1-\alpha| \leq 1$ or equivalently, $0 \leq \alpha \leq 2$. Thus there exists infinitely many best approximations of x in $[x_1]$ of the form $\{(\alpha,0) : 0 \leq \alpha \leq 2\}$.

Here, the example II shows that the element y_0, for which $\|x - y_0\|$ is minimized, may not be unique.

Theorem 3.5. Let *C be a non-empty closed convex subset of the uniformly convex Banach space E. Then, for every $x \in E$, the equation $\|x-y\| = d(x,C)$, where $y \in C$, is uniquely solvable. That is, there exists exactly one $y_0 \in C$ such that $\|x - y_0\| \leq \|x-y\|$ for all $y \in C$.*

Theorem 3.6. *Let E be a strictly convex normed linear space and C a weakly compact convex subset of E. Then for every $x \in E$, the problem $\|x-y\| = d(x,C)$ for $y \in C$ is uniformly solvable. That is, there exists exactly one $y_0 \in C$ such that $\|x - y_0\| \leq \|x-y\|$ for all $y \in C$.*

Theorem 3.7. *Let C be a closed convex subset of a Banach space E with the Oshman property. If $T : C \to E$ is continuous and $T(C)$ is relatively compact, then there exists a point $y \in C$ such that $d(Ty, C) = \|Ty - y\|$.*

Theorem 3.8. *A convex boundedly compact subset of a normed linear space is proximinal* [3]. *Also,* a *W*- closed subset C of a dual space E* is Proximinal.*

Theorem 3.9. *Let C be a non-empty subset of E, and let x be an element of E. The mapping $A_x : C \to A(C)$ has a fixed point iff x has a best approximation by means of elements of C.* If *y_0 is a fixed point of the mapping A_x, then every element in $A_x(y_0)$ is a best approximation of x by the elements of C, where $A_x(y_0) = C \cap K\{x, [\|x - y_0\| + P_C(x)]/2\}$, a non-empty and closed set (in C) for every y in C.*

A similar theorem was proved by Brandt [1].

Theorem 3.10. *If C is a boundedly compact subset of E, then the mapping $A_x : C \to A(C)$ is upper semicontinuous on C for every $x \in E$. Also, if C is a closed convex subset of E, then mappings $A_x : C \to A(C)$ is lower semicontinuous on C for $x \in E$.*

4. Invariance of Best Approximation

In the subject of best approximation, one often wishes to know whether some useful property of the function being approximated, is inherited by the approximating function. Meinardus [13] seems to have been the first to observe the general principle that could be applied and the first to employ a fixed point theorem to establish it. We have the following well-known results of Meinardus.

Theorem 4.1 [13] *Let C be an approximately p-compact convex subset of a locally convex Hausdorff topological linear space E, and let $T: C \to E$ be continuous. If either C is compact or $T(C)$ is relatively compact, then for each continuous seminorm p in E, there is a point y in C such that $d_p(Ty, C) = p(Ty-y)$.*

Theorem 4.2. [13] *Let C be a non-empty weakly compact and star-shaped subset of a Hilbert space E. Suppose that T is a nonexpansive mapping of C into itself. Then T has a fixed point.*

Theorem 4.3.[13] *Let $T : B \to B$ be continuous where B is a compact metric space. If $C[B]$ is the space of all continuous real or complex functions on B with the supremum norm. Let $A : C[B] \to C[B]$ be of Lipschitz class with Lipschitz constant I. Suppose further that $Af(T(x)) = f(x)$, $Ah(T(x)) \in V$,* where $h(x) \in V$, *where V is a finite dimensional subspace of $C[B]$. Then there is a best approximation g of f with respect to V such that $Ag(T(x)) = g(x)$.*

This Theorem 4.3 was extended and simplified by Brosowski [2].

Theorem 4.4. [2] *Let T be a nonexpansive mapping (contractive linear operator) on a normed linear space E. Let C be a T-invariant subset of E and y a T-invariant point. If the set of C-approximates $P_c(y)$ to y is non-empty, compact and convex, then it contains a T-invariant point.*

For generalised rational approximation in a space $C(T)$, we have the following analogue result due to Brosowski.

Theorem 4.5.[2] *Let U and V be finite dimensional subspace of $C(T)$ and Y be the set of function u/v, where $u \in U$, $v \in V$, and $v \geq \varepsilon$, ε being fixed and positive. Let A be a contractive linear operator on $C(T)$. If Y contains an A-invariant and if x is an A-invariant, then Y contains an A-invariant best approximation to x.*

Subrahmanyam [24] and Singh [21] have given further extensions of Meinardus results.

Theorem 4.6. [24] *Let E be a normed linear space, C be a finite dimensional subspace, and $T : E \to E$ having a fixed point f be such that $\|x - y\| \leq d_f(C)$ implies $\|Tx - Ty\| \leq \|x - y\|$, where $d_f(C)$ denotes the distance of f from C. If T maps C into itself, then f has a best approximation in C, which is another fixed point of T.*

Theorem 4.7. [24] If *$T : E \to E$ be a nonexpansive operator with a fixed point f and leaving a finite dimensional subspace C of E invariant, then f has a best approximation in C, which is a fixed point of T.*

Theorem 4.8. [17] *Let* E be a locally convex Hausdorff topological vector space and let *$T : E \to E$ be a p-nonexpansive mapping. Let C be a T-invariant set and y a T-invariant point. Assume that for every $p \in \mathcal{P}$, $P_c(y)$ is non-empty, weakly compact and star-shaped. If I–T is demiclosed, then T has a fixed point, which is a best approximation to y in E.*

As condition on $P_c(y)$ is nonempty and compact, and it may be difficult to verify in some instances. So, this leads to the consideration of special cases when it is possible to replace compactness by weak compactness as in the above Theorem 4.2.

Theorem 4.9. [18] *Let E be a locally convex Hausdorff topological linear space and let $T : E \to E$ be a p-nonexpansive mapping. Let C be a T-invariant set and y a T-invariant point. Assume that for every $p \in \mathcal{P}$, $P_c(y)$ is non-empty, sequentially complete, bounded and star-shaped. Further, assume that at least one of the following holds :* (i) *(I–T) $P_c(y)$ is closed* (ii) *T is demicompact and* (iii) *T is compact. Then, T has a fixed point, which is a best approximation to y in $P_c(y)$.*

Theorem 4.10 [19] *Let E be a locally convex Hausdorff topological vector space and let T: $E \to E$ be a p-nonexpansive mapping. Let C be a T-invariant set and y a T-invariant point. Suppose that for every $p \in \mathcal{P}$, $P_c(y)$ is non-empty and compact, and there is a contractive, jointly continuous family of functions associated with star-shaped $P_c(y)$. Then T has a fixed point, which is a best approximation to y in $P_c(y)$.*

Dotson, Jr. [6] has noted that star-shaped subsets have the property of contractivenss and joint continuity. Thus, the immediate consequence of the above result is the following theorem in [20].

Theorem 4.11. [20] *Let E be a locally convex Hausdorff topological vector space $(T_2 - lts)$ and let $T : E \to E$ be a p-nonexpansive mapping. Let C be a T-invariant subset of E and y a T-invariant point. If the set of best C-approximants to y is non-empty, compact and star-shaped, then it contains a T-invariant point.*

Theorem 4.12 [21]. *Let E be a normed linear space and $T : E \to E$ be a mapping. Let C be subset of E such that C is a T-invariant and x be a T-invariant point in E. If the set of best C-approximants to x, $P_c(x)$ is non-empty, compact and starshaped and T is : (i) continuous on $P_c(x)$ and* (ii) $\| z - y\| \leq d(x, C) \Rightarrow \| Tz - Ty\| \leq \| z - y\|$ for z, y in $P_c(x) \cup \{x\}$; *where d(x,C) denotes the distance of x from C ; then it contains a T-invariant point, which is a best approximation to x in C.*

Theorem 4.13 [21] *Let T be a contractive operator on a normed linear space E. Let C be a T-invariant subset of E and y a T-invariant point in X. If the set of best C-approximants $P_c(y)$ to y is non-empty, compact and starshaped, then it contains a T-invariant point.*

Theorem 4.14 [21] *Let E be a normed linear space and $T : E \to E$ be a non-expansive mapping. Let T have a fixed point, say y, and leaving a finite dimensional subspace C of E invariant. Then T has a fixed point, which is a best approximation to y in C.*

Clearly, Theorem 4.3 follows from Theorems 4.7 and 4.13 by defining a mapping $F : C[B] \to C[B]$ by $F(g(x)) = A(g(T(x)))$, as it satisfies all the hypotheses of Theorem 4.3. A theorem similar to Theorem 4.11. was proved by Narang [14] in case of metric linear space as follows

Theorem 4.15. [14] *Let T be a contractive map on a locally convex linear metric space (E, d), C a T-invariant subset of E and y a T-invariant point. If the set of best C-approximants to y is non-void, convex and compact then it contains a T-invariant point.*

Also, the paper of Narang [14] deals with the following applications of fixed points to approximation theory when the underlying spaces are metric linear space.

Theorem 4.16. *Let (E,d) be a metric linear space with strictly monotone metric d(i.e, $d(tx, 0) < d(x, 0)$ for every $x \neq 0$, $0 \leq t < 1$) and let $T : E \to E$ be a p-nonexpansive mapping. Also, set C be a subset of E, $T : \partial C \to C$ and y a T-invariant point. If the set $P_c(y)$ is non-void, compact and star-shaped then it contains a T-invariant point.*

Theorem 4.17. [14] *Let E be a strictly convex metric linear space (i.e., $d(x,0) \leq r$, $d(y,0) \leq r$ imply $d((x + y)\backslash 2,0) < r$ unless $x = y$; $x, y \in E$), y a* T-invariant point. If the set $P_c(y)$ is non-void and star-shaped, $T : \partial C \to C$ *and T satisfies $d(Ty,Tx) \leq d(y, x)$ for all x in $P_c(y)$ then $P_c(y) = \{y_0\}$ with $Ty_0 = y_0$.*

Theorem 4.18. *Let C be a non-empty compact convex subset of a strongly locally convex metric linear spaces (E,d) (i.e., each open sphere in E is convex set). Then*

for any continuous mapping $f : C \to E$, *there exists* y_0 *in* C *such that* $d(y_0, f(y_0)) = \min_{y \in C} d(y, f(y_0))$. *In particular, if* $f(C) \subset C$ *then* y_0 *is a fixed point of* f.

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Baro-diffusion and Thermal–diffusion in a Binary Mixture Near A Stagnation Point–A Numerical Study

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Abstract: The effects of pressure gradient and temperature gradient on separation of a binary mixture of incompressible viscous fluids have been discussed when one of the components of the fluid mixture is present in a small quantity. The flow has been discussed when a stream of such a mixture impinges on an impervious wall at right angles and flows along this wall in all radial directions. Equations of motion and energy together with the equation for species conservation have been solved numerically by Runga-Kutta shooting technique. It has been found that there is no separation effect when pressure gradient and temperature gradient are ignored. The effects of the pressure gradient as well as the temperature gradient are to separate the two components of the mixture in such a manner that the heavier and more abundant component gets deposited near the wall.

1. Introduction

Consider a mixture of two components of fluids the composition of one of which is described by the concentration c_1 defined as the ratio of mass of that component to the total mass of the fluid in a given volume element. In the flow of such a mixture the diffusion of individual species takes place by three mechanisms namely concentration gradient, pressure gradient and temperature gradient. The diffusion flux $\vec{i}$ is given by Landau and Lifshitz [5] as :

$$\vec{i} = -\rho D[Vc_1 + k_p V_p + k_T VT] \tag{1}$$

where ρ is the density of the binary mixture, D is the diffusion coefficient, k_p is baro-diffusion ratio, $k_p D$ is the baro-diffusion coefficient, p is the pressure, k_T is thermal diffusion ratio, $k_T D$ is the thermal-diffusion coefficient and T is the temperature. The first term in the right hand side of equation (1) represents the ordinary diffusion whose contribution to the mass flux depends in a complicated way on the concentration gradients of the substances present in the binary mixture. The second one representing the pressure diffusion term indicates that there may be

a net movement of the i^{th} species in the mixture if there is a pressure gradient imposed on the system. The last one representing the thermal diffusion term describes the tendency for species to diffuse under the influence of a temperature gradient. The effects of the last two terms are quite small, but devices can be arranged to produce very steep pressure gradients and temperature gradients so that separations of mixtures may be effected.

Much interest has been attached to the separation processes wherein one of the components is present in extremely small proportion in normal occurrence of the mixture. Separation of isotopes from their naturally occurring mixture is one such example. It is well known that because of their small relative mass difference the isotopes of heavier molecules offer greatest practical challenge to isolate the rarer component. Sharma [7] has discussed the problem of baro-diffusion in a binary mixture of viscous incompressible fluids when an infinite disk rotates with a constant angular velocity and there is a suction of the mixture at the disk. Srivastava [10] has discussed the baro-diffusion in a binary mixture confined between two disks when one of the disks is rotating and the other is at rest. Sharma and Gogoi [9] have discussed the effect of curvature of a curved annulus on separation of a binary mixture. Shrivastava [11] has discussed the baro-diffusion in a binary mixture in an axi-symmetric stagnation-point flow also. All these problems have been discussed under isothermal conditions. To investigate the effect of the temperature gradient in addition to the pressure gradient on separation of species of a binary mixture of thermally conducting incompressible fluids we have considered in this paper the flow discussed by Srivastava [11] when a stream of such a mixture impinges on a stationary impervious wall perpendicular to the stream and flows away in all radial directions. The equations of motion, energy and also the equation for conservation of species have been reduced to ordinary differential equations and numerical solutions of these ordinary non-linear differential equations have been obtained for various values of Schmidt number, baro-diffusion number, thermal-diffusion number and Prandtl number. Thus in this paper we have discussed the diffusion of the rarer component of the binary mixture under all the three effects namely the concentration gradient, the pressure gradient and the temperature gradient. Thus, the analysis provides a more accurate picture of the separation of the binary mixture of incompressible viscous fluids in the problem than the usual analysis under isothermal conditions.

2. Mass Transfer Equation

We consider here the case when one of the components of the binary mixture of incompressible fluids is present in a small quantity, hence the density and the viscosity of the mixture are independent of the distribution of the components. The flow problem of the binary mixture is identical to that of a single fluid but the

velocity is to be understood as the mass average velocity $\vec{v} = (\rho_1 \vec{v}_1 + \rho_2 \vec{v}_2)/\rho$ and the density $\rho = \rho_1 + \rho_2$ where the subscripts 1 and 2 denote the rarer and more abundant components respectively. The equations of motion and the equation of continuity in the steady case are :

(2) $$\rho(\vec{v}.\nabla)\vec{v} = -V_p + \mu \nabla^2 \vec{v},$$

(3) $$\nabla.\vec{v} = 0,$$

where μ is the coefficient of viscosity of the binary mixture. The equation governing the temperature is given by

(4) $$\rho c_p(\vec{v}.\nabla)T = k\nabla^2 T + \phi_d,$$

where c_p is the specific heat at constant pressure, k is the thermal conductivity of the fluid and ϕ_d is the viscous dissipation function. The additional equation for the species concentration is given by

(5) $$\rho(\vec{v}.\nabla)c_1 = -\nabla.\vec{i}$$

Substituting $\vec{i}$ from equation (1) in (5); we get the following equation for c_1

(6) $$\rho(\vec{v}.\nabla)c_1 = \rho D[\nabla^2 c_1 + \nabla.(k_p \nabla_p) + \nabla.(k_T \nabla T)].$$

The explicit expression for the baro-diffusion ratio has been given by Landau and Lifshitz (5) as:

(7) $$k_p = (m_2 - m_1)[(c_1/m_1) + (c_2/m_2)c_1c_2/p_\infty,$$

where p_∞ denotes the pressure in the working medium, m_1 and m_2 are the masses of two kinds of particles and c_2 is the concentration of the second component of the binary mixture given by

(8) $$c_1 + c_2 = 1$$

since $c_1 = \rho_1/\rho$ and $c_2 = \rho_2/\rho$. Assuming c_1 to be small so that its square is negligible, we get the following expression for k_p :

(9) $$k_p = (m_2 - m_1)c_1/(m_2 p_\infty).$$

The expression for k_T has been suggested by Hurle and Jakeman [4] as :

(10) $$k_T = c_1 s_T(1 - c_1),$$

where s_T is Soret coefficient Neglecting the square of c_1 in this case also, we can write the expression for the thermal diffusion ratio k_T as:

(11) $$k_T = c_1 s_T.$$

3. Boundary Conditions on c_1

The boundary conditions on c_1 are different in different cases. At the solid surface of a body, insoluble in the fluid, the mass flux of the rarer component of the

mixture normal to the solid surface is zero. This boundary condition can be written mathematically as :

(12) $\rho v_n c_1 - D[(\partial c_1/\partial n) + k_p(\partial p/\partial n) + k_T(\partial T/\partial n)] = 0$ at the surface,

where v_n is the fluid velocity normal to the surface and $\partial/\partial n$ denotes derivative normal to the surface. The first part represents the convective flux and the second part in the parenthesis denotes the diffusion flux. If, however, there is diffusion from a body that dissolves in the fluid, the equilibrium is rapidly established at the surface of the body and that the concentration in the fluid adjoining the body is, therefore, the saturation constant c_0. The boundary condition at such a surface is, therefore.

(13) $$c_1 = c_0.$$

4. Formulation of the Problem

In this section we discuss the flow, heat transfer and diffusion of a binary mixture of incompressible viscous fluids when a stream of such a mixture impinges on an impervious insulated wall $z = 0$ and flows away in all radial directions. We consider the temperature of the rarer component of the binary mixture, far away from the surface of the wall, to be constant. We take here cylindrical polar co-ordinates with stagnation point as the origin and the flow direction as negative z-axis. We denote the radial and axial component of the velocity in the frictionless flow region by U and W respectively whereas those in viscous flow region are denoted by $u = u(r,z)$ and $w = w(r, z)$. The boundary conditions on velocity and temperature fields are

(14) $u = 0\,,\ w = 0\,,\ (\partial T / \partial z) = 0$ at $z = 0$ and $u \to \mathrm{U},\ T \to T_\infty$ as $z \to \infty$.

For frictionless case we have [See Schlichting and Gerston [8],

(15) $$\mathrm{U} = ar,\ W = -2az,\ \ P_0 = P + (\rho a^2/2)(r^2 + 4z^2),$$

where a is a constant and P_0 is the total pressure at the stagnation point. We take the following form for u, w and p in the viscous region :

(16) $$u = (av)^{1/2}\,\xi\,\phi'(\eta),\ w = -2(av)^{1/2}\,\phi(\eta),$$

(17) $$p - p_0 = \rho a v P(\xi, \eta),$$

(18) $$T - T_\infty = (\mu c_p/k)\theta(\xi, \eta),$$

where $\eta = (a/v)^{1/2} z$, $\xi = (a/v)^{1/2} r$, $v = \mu/\rho$ and T_∞ is the temperature at a large distance from the surface of the wall.

Substituting expressions for u and w from (16), p from (17), T from (18), k_p from (9) and k_T from (11), the equation of motion (2) becomes

(19) $$\phi''' + 2\phi\phi'' - \phi'^2 + 1 = 0,$$

and also the expression for $P(\xi, \eta)$ becomes

$$P(\xi,\eta) = -\frac{\xi^2}{2} - 2(\phi^2 + \phi'), \tag{20}$$

the equation of energy (4) reduces to

$$\Pr\left(\xi\phi'\frac{\partial\theta}{\partial\xi} - 2\phi\frac{\partial\theta}{\partial\eta}\right) = \left(\frac{\partial}{\xi\partial\xi}\left(\xi\frac{\partial\theta}{\partial\xi}\right) + \frac{\partial^2\theta}{\partial\eta^2}\right) + \Pr(12\phi'^2 + \xi^2\phi''^2) \tag{21}$$

and the diffusion equation (6) becomes

$$Sm\left(\xi\phi'\frac{\partial c_1}{\partial\xi}\right) - 2Sm\left(\phi\frac{\partial c_1}{\partial\eta}\right) = \frac{\partial}{\xi\partial\xi}\left(\xi\frac{\partial c_1}{\partial\xi}\right) + \tag{22}$$

$$+ \frac{\partial^2 c_1}{\partial\eta^2} + \beta\left(\frac{\partial}{\xi\partial\xi}\left(\xi c_1\frac{\partial P}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(c_1\frac{\partial P}{\partial\eta}\right)\right) +$$

$$+ \tau\left(\frac{\partial}{\xi\partial\xi}\left(\xi c_1\frac{\partial\theta}{\partial\xi}\right)\right) + \frac{\partial}{\partial\eta}\left(c_1\frac{\partial\theta}{\partial\xi}\right),$$

where $\Pr = \mu c_p/k$ is the Prandtl number, $Sm = (\nu/D)$ is the Schmidt, β is the baro-diffusion number given by

$$\beta = \frac{m_2 - m_1}{m_2 p_\infty}\cdot\frac{\rho a^2}{2}$$

and τ is the thermal diffusion number given by

$$\tau = \frac{\mu c_p}{k}\cdot S_T$$

The boundary conditions (14) on velocity and temperature become

$$\phi = 0,\ \phi' = 0,\ \theta' = 0 \text{ at } \eta = 0; \qquad \phi' \to 1,\ \theta \to 1 \text{ as } \eta \to \infty \tag{23}$$

and the boundary conditions (12) and (13) on c_1 become

$$\frac{\partial c_1}{\partial\eta} + \beta c_1\frac{\partial P}{\partial\eta} + \tau c_1\frac{\partial\theta}{\partial\eta} = 0 \text{ at } \eta = 0; \quad \text{and} \quad c_1 \to c_0 \quad \text{as } \eta \to \infty \tag{24}$$

5. Solution of Equations

To get solution of partial differential equations (21) and (22) we, at first, convert them to ordinary differential equations. For this we assume θ and c_1 in the forms:

$$\theta(\xi',\eta) = \theta_0 + \xi^2\theta_2 \tag{26}$$

and

$$c_1(\xi,\eta) = c_0 f(\xi,\eta) = c_0\left(f_0(\eta) + \xi^2 f_2(\eta)\right). \tag{20}$$

Putting these values of θ and c_1 in equations (21) and (22) and equating the coefficient of ξ^0 and ξ^2 separately from both sides, we get.

(27) $$\theta_0'' + 4\theta_2 + 2\Pr\left(\phi\theta_0' + 6\phi'^2\right) = 0,$$

(28) $$\theta_2'' - \Pr\left(2\phi'\theta_2 - 2\phi\theta_2' - \phi''^2\right) = 0,$$

(29) $$Sm\left(-2\phi f_0'\right) = f_0'' + 4f_2 - 2\beta\left(2f_0 + 2\phi'^2 f_0 + 2\phi\phi'' f_0 + 2\phi\phi' f_0' + \phi''' f_0 + \phi'' f_1'\right) + \tau\left(4f_0\theta_2 + f_0\theta_2'' + f_0'\theta_2'\right)$$

(30) $$2Sm\left(\phi f_2' - \phi' f_2\right) = 2\beta\left(2f_2 + 2\phi'^2 f_2 + 2\phi\phi'' f_2 + 2\phi\phi' f_2' + \phi''' f_2 + \phi'' f_2'\right) + \tau\left(8f_2\theta_2 + f_0\theta_2'' + f_0'\theta_2' + f_2\theta_0'' + f_2'\theta_2'\right)$$

The boundary conditions (23) and (24) in terms of θ_0, θ_2, f_0 and become

(31) $$\left.\begin{aligned}\theta_0' = 0, \theta_0(\infty) = 0\\ \theta_2' = 0, \theta_2(\infty) = 0\end{aligned}\right\}$$

and

(32) $$f_0' - 2\beta\phi'' f_0 = 0 \text{ at } \eta = 0 \text{ ; and } f_0 \to 1 \text{ as } \eta \to \infty,$$

(33) $$f_2' - 2\beta\phi'' f_2 = 0 \text{ at } \eta = 0 \text{ ; and } f_2 \to 0 \text{ as } \eta \to \infty$$

It is not possible to get analytical expressions for ϕ and hence for θ_0, θ_2, f_0 and f_2. Equation (19) was first solved numerically by Homann [3], and later by Frossling [2] under the boundary conditions (23). Here we have solved equations (19), (27)–(30) under boundary conditions (23), (31)–(33) numerically by using Runga-Kutta shooting technique [See Conte and Boor [1] Robert and Shlpman [6].

6. Discussion

For $\beta = 0$ and $\tau = 0$ the function $f(\xi, \eta)$ becomes 1 throughout the fluid, which shows that there is no separation effect in the binary mixture when pressure gradient and temperature gradient are ignored. This confirms the results of Sarma [7], Srivastava [11] and Sharma and Gogol [9]. The value of $f_2(\eta)$ is found to be zero for all values of η hence from (26) we can conclude that the concentration of the rarer component is independent of the distance from z axis. Taking Schmidt number Sm = 1, Prandti number Pr = 1 and Thermal diffusion number $\tau = 0$, the function $f(\eta)$ has been plotted for baro-diffusion number $\beta = 0.025, 0.050, 0.075$ and 0.100 in Fig. 1. The graph reveals that the value of f is always less than 1 near the upper edge of the boundary layer and its values at the wall are 0.777254, 0.603558, 0.467569 and 0.360866 respectively for the above mentioned values of β. This means that the concentration of the rarer component is much less near the wall than that maintained at a large distance from it. But, $c_1 + c_2 = 1$ which shows that the heavier component gets deposited more near the wall. This again confirms the results of Srivastava [11].

By taking $Sm = Pr = 1$ and $\beta = 0.025$, $f(\eta)$ has been plotted in Fig. 2 for τ = 0.000, 0.025 and 0.050. In this figure we see that the curves for above-

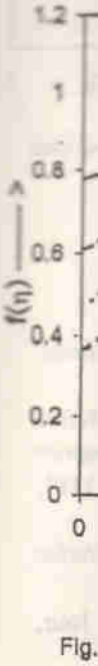

Fig.

mentioned values of τ intersect at $\eta = 0.2$. This indicates that the effect of the thermal diffusion number is to decrease the concentration of the rarer component from the upper edge of the boundary layer to the points corresponding to $\eta = 0.2$ and to decrease it beyond these points to the surface of the wall. The points corresponding to $\eta = 0.2$ are some special points, since concentration at these points remain unaffected by the thermal diffusion number.

In Fig. 3 we have plotted $f(\eta)$ for $Pr = 0.7$, 1.0 and 2.0 by taking $Sm = 1$ and $\beta = \tau = 0.025$. Fig. 4 represents the graph of $f(\eta)$ for Sm = 0.50, 0.75, 1.00, 1.25 and for fix values of $\beta = \tau = 0.025$ and $Pr = 1$. These graphs show that by reducing the values of Pr and/or Sm the rate of change in concentration of the rarer component of the binary mixture can be enhanced. Hence, we can conclude, from above analysis, that the effects of the pressure gradient and the temperature gradient are to collect the heavier component of the binary fluid mixture near the surface of the wall, i.e., to effect the separation of the species in the binary mixture. The rate of separation can be enhanced by reducing the values of Prandtl number and Schmidt number.

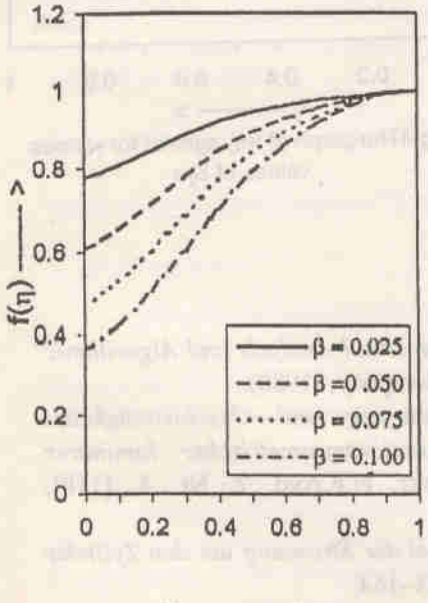

Fig.1The graph of f(η) against for various values of β

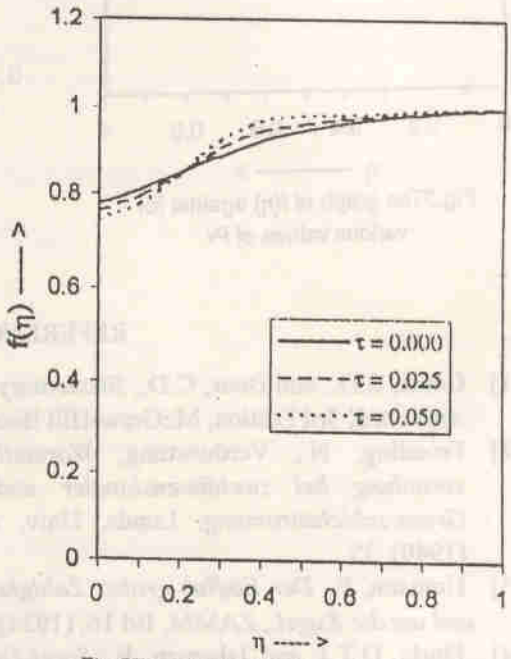

Fig.2The graph of f(η) against for various values of τ

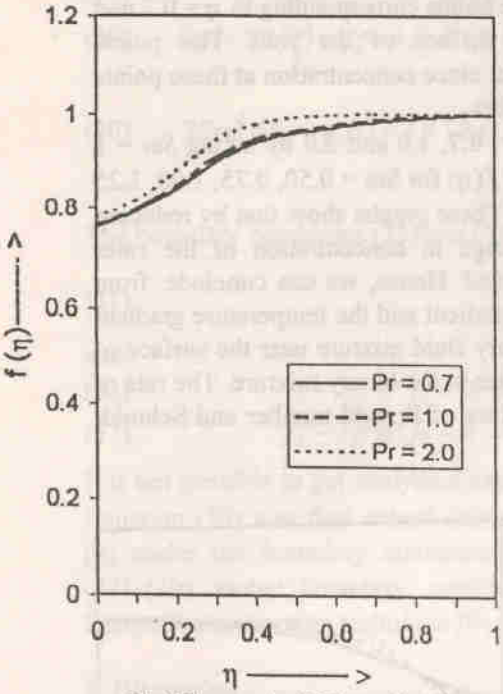

Fig.3The graph of f(η) against for various values of Pr

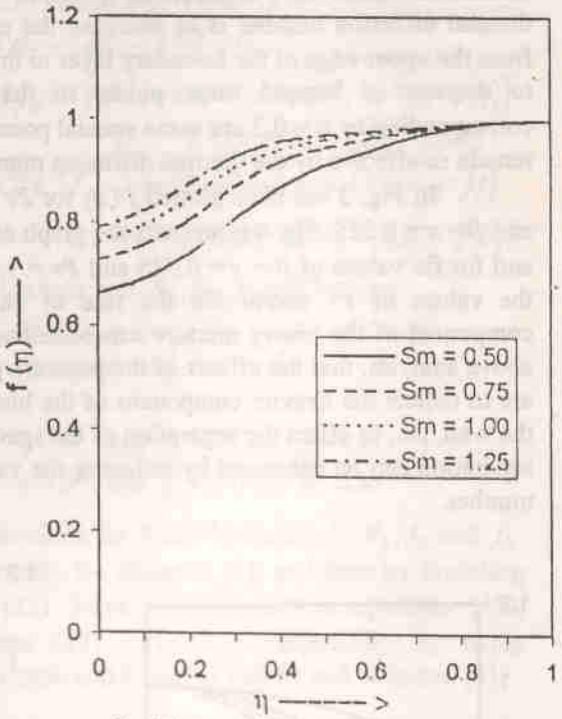

Fig.4The graph of f(η) against for various values of Sm

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Modeling Incompressible Navier–Stokes Flows by Least Squares Approximation

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Abstract: In this paper the least squares particle method (LSQ) is used to model incompressible flows. We present a constrained least squares approximation for the reconstruction of the flow and show reliable and accurate simulations of the viscous Navier-Stokes equations. The corresponding incompressible limit can be implemented and its accuracy tested against numerical and analytical solutions. The incompressible Poiseuille and viscous multi-vortex flows are studied and compared with analytical solutions. Furthermore results for cavity flows at different Reynolds numbers are presented and discussed.

Keywords: Navier-Stokes equations, Particle method, Least squares approximation

AMS subject classification: 76D05, 76M28

1. Introduction

The study of incompressible fluid flows is one of the main field of computational fluid dynamics. This subject becomes even more appealing when one can see the incompressible flow as limit of the compressible, viscous Navier-Stokes equations. The aim of this paper is to show how the compressible, viscous Navier-

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Stokes equations can be simulated by using the least squares method in the framework of particle method and the incompressible flow can be reproduced as limit of small Mach numbers.

The particle method has certain advantages over other methods. This method is a meshfree and a fully Lagrangian method and numerical solutions are computed over a moving grid defined by particles treating the geometry and the corresponding moving boundaries in a natural way. However the classical particle method, smoothed particle hydrodynamics (SPH) [11], is based on an integral interpolant with sufficiently smooth symmetric kernel function and has two fundamental problems: the approximation of derivatives of order larger than one and the implementation of the boundary conditions.

In the classical particle approach the approximation of derivatives near the boundary is not accurate. Alternatively, the approximation of derivatives in a grid free structure can be obtained by moving least squares methods [1, 3, 10]. In [10] it is shown that the moving least squares method gives a good approximation of the function and derivative near the boundary. Both of the approaches are similar to the finite difference discretization but show well known problems of instability and artificial viscosity should be introduced in order to stabilize the scheme. In [11] viscosity is introduced in the momentum and energy equations and in [10] an artificial viscous term is proposed for all the equations of the system. Both approaches do not give good approximations of the second order spatial derivative and therefore, they cannot compute efficiently the Navier-Stokes equations. In this paper, we approximate the first and second order derivatives by a constrained weighted least squares method and the natural Navier-Stokes viscous term is used. In this approach the solutions of the compressible Euler system can be obtained from the Navier-Stokes equations by letting the viscosity and heat conductivity tend to zero. In [15] the scheme for the $1D$ case is shown to be stable and numerical solutions converge to the Euler solutions when the number of particles tend to infinity and the viscosity and heat conductivity tend to zero.

The particle method should reproduce the results obtained by other well known methods with comparable accuracy. Many treatments of the boundary conditions are "ad hoc" implementations and cannot be reproduced easily. In this paper we propose to reconstruct the field over a fix grid and impose the boundary conditions over such a field. The fix grid can be used over the entire domain or only over part of the domain containing the boundary. We show that our method is consistent and it is a solid starting point for a moving particle method. In this paper we are discussing only the method over domains with fix boundaries leaving to further works the discussion of the rules necessary for consistent moving or adaptive particle grids. For further applications of the particle hydrodynamics to moving boundary problems we refer to [10].

The particle scheme is used to find the velocity and pressure fields and a new constrained least squares method is used to reconstruct the function and the derivatives. The use of the constrained least squares approximation allows us to compute with accuracy the second order space derivatives and the solution of the viscous Navier-Stokes equations. Since most of the numerical and analytical works are in the field of incompressible flows over bounded domains we apply this new method to incompressible flows. We solve basically the compressible Navier-Stokes equations in the low Mach number limit.

The paper is organized as follows. In §2 we introduce the compressible model and the incompressible limit. In §3 the constrained least squares method is described and the discrete set of equations are written. In §4 we present some numerical tests.

2. Model

Let Ω be an open bounded domain in $\mathbb{R}^s$ ($s = 1, 2, 3$) with boundary Γ. Let $\hat{\rho}, \hat{v}$ and $\hat{p}$ be the density, velocity and pressure fields representing the state variables. The compressible Navier-Stokes system in the Lagarangian form can be written as [6]

$$\frac{D\hat{\rho}}{Dt} = -\hat{\rho}\,\nabla.\hat{v} \tag{2.1}$$

$$\hat{\rho}\frac{D\hat{v}}{Dt} = -\nabla\hat{p} + \mu\,\vec{\nabla}\,.\,\tilde{\sigma}(\hat{v}), \tag{2.2}$$

where μ is the dynamic viscosity. By D/Dt we denote the Lagrangian derivative and by $\tilde{\sigma}$ the stress tensor $\tilde{\sigma}_{ij} = \frac{\partial\hat{v}_i}{\partial x_j} + \frac{\partial\hat{v}_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\,\nabla.\hat{v}$. The system (2.1–2.2) is closed by the state equation $\hat{p} = \hat{p}(\hat{\rho})$. In this paper we assume a simple linear state law $\hat{p} = A\hat{\rho} + B$ with $A = c^2$ and B constant where c is the characteristic sound speed. For an ideal compressible fluid a power law $\hat{p} = A\hat{\rho}^{\gamma} + B$ can also be used when A/γ is set to be equal to c^2. The Lagrangian form of the equations in (2.1–2.2) can be easily implemented over fix or free boundary domains. However, before solving problems in complex or moving geometries we would like to show that the method is accurate and it can be used to simulate incompressible flows.

The Navier-Stokes system for incompressible fluid flow is a different set of equations which can be written as

$$\nabla\,.\,\hat{v} = 0 \tag{2.3}$$

$$\hat{\rho}\frac{D\hat{v}}{Dt} = -\nabla\hat{p} + \mu\,\vec{\nabla}\,.\,\tilde{\sigma}. \tag{2.4}$$

Under appropriate conditions over some data one expects that the system (2.1–2.2) converges to (2.3–2.4). We denote the nature of (2.1) is deeply different from (2.3) and we should expect the limit to be singular.

Numerical simulations of compressible flows at low Mach numbers is not easy due to the multiple time and length scales involved in the computation. In order to rewrite the system (2.1–2.2) in a more suitable form we use an asymptotic expansion which also gives a good insight into the solution behaviour of the compressible equations in the limit of vanishing Mach number. To cover small scale flow as well as the long-wave phenomena a single time scale is performed. Similar results can be found in literature in many forms and flavors. For details one can consult for example [8, 9, 13, 7].

Let ρ_{ref} , p_{ref} , V_{ref} be the reference state. We consider the state variables (ρ, $\vec{v}$, p) in the non-dimensional form, namely $\rho = \hat{\rho} / \rho_{ref}$, $\vec{v} = \hat{v} / V_{ref}$ and $p = \hat{p} / p_{ref}$. By taking a typical speed V_0 of the incompressible flow and the length scale L of the flow we set the time scale $\theta = L / V_{ref}$. We note that V_{ref} should be in some way related, but not necessarily equal, to V_0 or c. This leads to write the non-dimensional Navier-Stokes system, for the state variables ($\rho, p, \vec{v}$), as

$$\frac{D\rho}{Dt} = -\rho \nabla.\vec{v} \tag{2.5}$$

$$\rho \frac{D\vec{v}}{Dt} = -\frac{p_{ref}}{\rho_{ref} V_{ref}^2} \nabla p + \frac{\mu}{L V_{ref} \rho_{ref}} \nabla.\vec{\sigma}. \tag{2.6}$$

Since we use the linear state law, we write $\hat{p} = c^2 (\hat{\rho} - \rho_{ref}) + V_0^2 \rho_{ref}$. In the limit of $\hat{\rho} \to \rho_{ref}$ we have $\hat{p} = V_0^2 \rho_{ref}$, which is normally used as reference pressure in the incompressible flow. Now by solving (2.5–2.6) through an explicit method we should take into account the fact that we have quasi-incompressible flows and not fully divergent free flow. Let $\hat{\rho}_{hm}$ be the approximate average value for $\hat{\rho}$ then we can define $\Delta\hat{\rho}_h = \hat{\rho}_{hm} - \rho_{ref}$. In this numerical limit the reference pressure is defined by $\rho_{ref} = c^2 \Delta\rho_h + V_0^2 \rho_{ref}$ which agrees with the incompressible reference pressure when $\Delta\hat{\rho}_h$ tends to zero. From the definition of p_{ref} we have $p_{ref} / \rho_{ref} = c^2 \Delta\hat{\rho}_h + V_0^2$. The state equation in the non-dimensional variable becomes $p = ((\rho - 1) + M_a^2) / (\Delta\rho_h + M_a^2)$, with the Mach number defined by $M_a^2 = V_0 / c^2$. The Mach number as a global parameter characterizing the non-dimensional limit is defined with respect to V_{ref} by $M = V_{ref} / (dp/d\rho)^{1/2} = \sqrt{\Delta\rho_h + M_a^2}$, which is equal to M_a in the limit of $\Delta\rho_h$ tending to zero. We note that $M \to 0$ implies both $M_a \to 0$ and $\Delta\rho_h \to 0$. If we set $V_{ref}^2 = p_{ref} / \rho_{ref}$ the system (2.5–2.6) becomes

(2.7) $$\frac{D\rho}{Dt} = -\rho \nabla . \vec{v}$$

(2.8) $$\rho \frac{D\vec{v}}{Dt} = -\nabla p + \frac{\mu}{LV_{ref}\ \rho_{ref}} \nabla . \tilde{\sigma}$$

If M_a is small then V_{ref} is equal to $c\sqrt{\Delta\rho_h}$ and the time is scaled with a characteristic time for sound wave propagation. We have $M = \sqrt{\Delta\rho_h}$ and the non-dimensional velocity cannot be of order 0(1) in the limit of vanishing M_a.

If $\Delta\rho_h$ is small then V_{ref} is equal to V_0 and $M = M_a$. The time is scaled with a characteristic time for the incompressible flow propagation and the pressure term is singular (proportional to $1/M^2$). If one expands the variable p as $p = p_0 + M_a^2 p_2$ (see for examples [8,9,13,7] and references therein) there is no longer one single pressure term to influence the leading order velocity in the limit of low Mach number but a clean separation of different physical effects associated with these pressure terms. The leading term p_0 tends in the limit to be spatially homogeneous and acts as a thermodynamics variable satisfying the state equations. The second order term p_2 represents a balance between the inertial and viscous force and also guarantees the free divergence motion. In the vanishing limit this term should be decoupled completely from the total pressure and therefore from the state equation.

In spite of the fact that $M_a \to 0$ and $\Delta\rho_h \to 0$ are both singular limits the limit $M \to 0$ is nice if the ratio $\Delta\rho_h/M^2$ is approximately constant. The limit in $\Delta\rho_h$ is the most difficult to be imposed and in general a projection over a free divergence velocity field should be used [2, 12]. In this paper we use an explicit method to solve the Navier-Stockes system and therefore an asymptotic form of the (2.3–2.4) and the pressure state equation is appropriate. In (2.8) we take the limit $\Delta\rho_h \to 0$ and in the state equation the limit $M_a \to 0$. The system (2.3–2.4) and the pressure state equation become

(2.9) $$\frac{D\rho}{Dt} = -\rho \nabla . \vec{v}$$

(2.10) $$\frac{D\vec{v}}{Dt} = -\nabla p + \frac{1}{\mathrm{Re}} \nabla . \tilde{\sigma}.$$

(2.11) $$p = \frac{(\rho - 1)}{\delta}$$

where $\mathrm{Re} = L\rho_{ref} V_0/\mu$ is the Reynolds number and δ a positive small real number. The (2.10) is clearly the limit equation for M tending to zero but the (2.11) should be

understood in the limit of small δ. We note that the (2.9–2.10) has the same form of the compressible system in (2.1–2.2) and therefore suitable for a Lagrangian particle method.

We would like to remark that (2.11) is not anymore a state equation but an equation for the incompressible pressure which is completely decoupled from the compressible state equation. The constant δ determines the pressure in agreement with the variable ρ which is not anymore representing the real density but a sort of error in the divergence field.

3. LSQ–particle Discretization

3.1. Least Squares Approximation of the derivatives

The least squares method has been used to approximate the first order space derivatives and solve the compressible Euler equations in fix and moving geometrics (see for example [4, 5]). The aim of this paper is to approximate the full Navier - Stokes equations and therefore both first and second order space derivatives. The main advantage of the least squares methods is that it is very general and can be applied to very irregular moving geometries. The idea is to substitute the smoothing functions used to interpolate particle solutions with a very general least squares interpolant which can cope with a large class of mesh configurations.

Let $f(t,\vec{x})$ be a scalar function and $f_i(t)$ its values at $\vec{x}_i$ for $i = 1, 2, \ldots, N$ and time t. Consider the problem to approximate the function and the spatial derivatives of the function $f(t,\vec{x})$ at $\vec{x}$ in terms of the values of a set of neighboring points. In order to limit the number of points we associate a weight function $w = w(\vec{x}_i - \vec{x}\,; h))$ with small compact support, where h determines the size of the support. In the classical smoothed particle hydrodynamics method, h is known as smoothing length. The weight function can be quite arbitrary but in our computations, we consider a Gaussian weight function in the following form

$$w = w(\vec{x}_i - \vec{x}\,; h)) = \begin{cases} \exp\left(-\alpha \frac{\|\vec{x}_i - \vec{x}\|^2}{h^2}\right), & \text{if } \frac{\|\vec{x}_i - \vec{x}\|^2}{h} \le 1 \\ 0, & \text{else,} \end{cases}$$

with α a positive constant. The smoothing length defines a set of neighboring particles around $\vec{x}$. Let $P(\vec{x}) = \{\vec{x}_i : i = 1, 2, \ldots, n\}$ be the set of n neighboring points of $\vec{x}$. The distribution of neighboring points needs not to be uniform and it can be quite arbitrary. For consistency reasons some obvious restrictions are required, namely for example the particles should not be on the same line.

We approximate the function $f(t,\vec{x})$ by $f_h(t,\vec{x})$ as $f_h(t,\vec{x}) = \sum_{i=1}^{N} f_i(t)\, \phi_h(\vec{x}_i, x)$, where the shape function $\phi_h(\vec{x}_i, \vec{x})$ is computed at each point $\vec{x}$ by the least squares method over its own compact support. It is important to stress that this expression

is consistent only if the function ϕ_h is 1 at $\vec{x}_i$, namely $\phi_h(\vec{x}_i,\vec{x}_j)=\delta_{ij}$ for all $i, j = 1, 2, \ldots, N$.

The approximation of the first and second order derivatives can be computed directly from $f_h(t,\vec{x})$ or directly by using the least squares method. The first method is known in literature as moving least squares method [3, 10]. Usually the function $f_h(t,\vec{x})$ and its derivatives $f_{kh}(t,\vec{x})$ are not smooth enough to be differentiable and therefore the second order derivatives cannot properly computed.

In this paper we approximate the derivatives $\partial f(t,\vec{x})/\partial x_k$ by $f_{kh}(t,\vec{x}) = \sum_{i=1}^{N} f_i(t)\,\eta_{kh}(\vec{x}_i,\vec{x})$ for $k = 1,2,3$, where $\eta_{kh}(\vec{x}_i,\vec{x})$ is directly computed by the least squares interpolation. In a similar manner we define the approximation for the second order derivatives $\partial^2 f(t,\vec{x})/\partial x_l\,\partial x_k$ by $f_{klh}(t,\vec{x}) = \sum_{i=1}^{N} f_i(t)\,\psi_{kh}(\vec{x}_i,\vec{x})$ for $k = 1, 2, 3$. The determination of the functions $f_h(t,\vec{x})$, $f_{kh}(t,\vec{x})$, and $f_{klh}(t,\vec{x})$ $(=f_{lkh}(t,\vec{x}))$ for $k, l = 1, 2, 3$ can be computed easily and accurately by using the Taylor series expansion and the least squares approximation. We write a Taylor's expansion around the point $\vec{x}$ with unknown coefficients and then compute these coefficients by minimizing a weighted error over the neighboring points. The optimization is constrained to satisfy $\phi_h(\vec{x}_1,\vec{x}_1)=1$ where $\vec{x}_1$ is the closest point, namely the approximation must interpolate the closest point.

In order to approximate the function and its derivatives at $\vec{x}$ by using a quadratic approximation through the n neighboring points sorted with respect to its distance from $\vec{x}$ we let

$$f(t,\vec{x}_i) = f_h(t,\vec{x}) + \sum_{k=1}^{3} f_{kh}(t,\vec{x})\,(x_{ki}-x_k) + \frac{1}{2}\sum_{k,l=1}^{3} f_{lkh}(t,\vec{x})\,(x_{ki}-x_k)\,(x_{li}-x_k) + e_i,$$

where e_i is the error in the Taylor's expansion at the point $\vec{x}_i$. The unknowns f_h, f_{kh} and f_{klh} for $k, l = 1,2,3$ are computed by minimizing the error e_i for $i = 2,3, \ldots, n$ and setting the constraint $e_1 = 0$. Our method to solve this constrained least squares problem is straightforward. By subtracting the first equation with $e_1 = 0$ to all the other equations the system can be written as $\vec{e} = M\vec{a} - \vec{b}$, where

$$M = \begin{pmatrix} \Delta x1_2 & \Delta x2_2 & \Delta x3_2 & \Delta x11_2 & \Delta x12_2 & \Delta x13_2 & \Delta x22_2 & \Delta x23_2 & \Delta x33_2 \\ \Delta x1_3 & \Delta x2_3 & \Delta x3_3 & \Delta x11_3 & \Delta x12_3 & \Delta x13_3 & \Delta x22_3 & \Delta x23_3 & \Delta x33_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta x1_n & \Delta x2_n & \Delta x3_n & \Delta x11_n & \Delta x12_n & \Delta x13_n & \Delta x22_n & \Delta x23_n & \Delta x33_n \end{pmatrix},$$

where $\vec{a} = [f_{1h}, f_{2h}, f_{3h}, f_{11h}, f_{12h}, f_{13h}, f_{22h}, f_{23h}, f_{33h}]^T$, $\vec{b} = [f_2 - f_1, (f_3 - f_1), \ldots, f_n - f_1]^T$, $\vec{e} = [e_2, e_3, \ldots, e_n]^T$. The symbol Δxk_i denotes $x_{ki} - x_{k1}$, Δxkl_i denotes $(x_{ki} - x_k)(x_{li} - x_l)$ and Δxkk_i the quantity $(x_{ki} - x_k)(x_{ki} - x_l)/2$ for k, $l = 1,2,3$ and $i = 2,3, \ldots, n$.

For $n > 9$, this system is over-determined for the nine unknowns f_{kh} and f_{klh} for $k, l = 1,2,3$.

The unknowns $\vec{a}$ are obtained from a weighted least squares method by minimizing the quadratic form $J = \sum_{i=1}^{n} w_i e_i^2$. The above equations can be expressed in the form $J = (M\vec{a} - \vec{b})^T W(M\vec{a} - \vec{b})$ where $W = \delta_{ij} w_i$. The minimization of J formally yields $\vec{a} = (M^T WM)^{-1}(M^T W)\vec{b}$. Now from the equation for the closest point x_1 we can compute the value of $f_h(t, \vec{x})$ at $\vec{x}$ as

$$f_h(t,\vec{x}) = f(t,\vec{x}_1) - \sum_{k=1}^{3} f_{kh}(t,\vec{x})(x_{1i} - x_k) - \frac{1}{2}\sum_{k,l=1}^{3} f_{lkh}(t,\vec{x})(x_{k1} - x_k)(x_{l1} - x_k)$$

since f_{kh} and f_{klh} for $k, l = 1,2,3$ are now known..

The solution of the constrained least squares problem is straightforward and more sophisticated techniques can be used. For example minimization or singular decomposition techniques can be very helpful to determine efficiently the unknowns.

We note that if the approximation is computed at $\vec{x}_i$ we have $f_h(t, \vec{x}_i) = f_i(t)$ which implies $\phi(\vec{x}_i, \vec{x}_j) = \delta_{ij}$ for all $i, j = 1,2, \ldots, N$. Also we note that if the weight function is chosen in a suitable form then the constraint $\phi(\vec{x}_i, \vec{x}_j) = \delta_{ij}$ can be approximated very closely performing the unconstrained least squares minimization over all the n equations.

3.2. LSQ-particle discretization

The idea behind the least squares particle method is to approximate a space-time function by mans of an expansion which is represented by scaled and displaced approximate delta functions at the particle position.

Let $f(t, \vec{x})$ be a scalar function and $f_i(t)$ be the set of its values at the particle points $\vec{x}_i$ for $i = 1, 2, \ldots, N$ and time t. We approximate the function $f(t,\vec{x}) = \prod f_h(t,\vec{x}) = \sum_{i=1}^{N} f_i(t)\,\phi(\vec{x}_i, \vec{x})$, its derivatives f_k and f_{kl} as $f_{kh}(t,\vec{x};\vec{z}) = \Pi_k f_h(t,\vec{x}) = \sum_{i=1}^{N} f_i(t)\,\eta_k(\vec{x}_i, \vec{x})$ and $f_{klh}(t,\vec{x};\vec{z}) = \Pi_{lk} f_h(t,\vec{x}) = \sum_{i=1}^{N} f_i(t)\psi_{kl}(\vec{x}_i, \vec{x})$ respectively. The functions ϕ, η_k and ψ_{kl} are computed at each point through the constrained least squares approximation described in the previous section by using the neighboring points over their compact support. The operators Π, Π_k and Π_{lk} for

$k, l = 1, 2, 3$ are well defined and give the values of the function and its derivatives as a linear combination of the neighboring points.

We can summarize some abstract properties for Π which allow us to write the discrete particle approximation. The operator Π satisfies the following properties:

1) the operator Π is linear and the approximation depends linearly from the particle point values ;
2) the approximation obtained by applying the least square method is consistent and the evaluation at the particle points gives the interpolating value. Therefore $\Pi f(t,\vec{x}_i) = f_i(t) = f(t,\vec{x})$ for all $i = 1, 2, \ldots, N$.
3) From the above formalism we have $\Pi f_k\ (t,\vec{x}) = \Pi_k f(t,\vec{x})$ and $\Pi f_{kl}(t,\vec{x}) = \Pi_{kl} f(t,\vec{x})$ for $k,l = 1, 2, 3$.

Consider the system in (2.9–2.11), by introducing the least square approximation and by using the properties in (2–3) we have

$$\frac{D\rho_i(t)}{Dt} = -\rho_i(t)\Pi\nabla \cdot \vec{v}(t,\vec{x}_i) \tag{3.12}$$

$$\rho_i(t)\frac{D\vec{v}_i(t)}{Dt} = -\Pi\nabla p(t,\vec{x}_i) + \frac{1}{\mathrm{Re}}\Pi\nabla \cdot \tilde{\sigma}(\vec{v}(t,\vec{x}_i)) \tag{3.13}$$

$$p_i(t) = \frac{(\rho_i(t)-1)}{\delta} \tag{3.14}$$

at $\vec{x}_i$ for all $i = 1, 2, \ldots, N$. Since the partial derivatives on the *rhs* are approximated by the operators Π, Π_k, Π_{lk} for $k,l = 1, 2, 3$ the system of partial differential equations reduces to a time dependent system of ordinary differential equations.

In addition to the Navier-Stokes system the equations that determine the particle positions should be included as

$$\frac{d\vec{x}_i}{dt} = \beta\vec{v}_i, \text{ for } \quad i = 1, \ldots, N, \tag{3.15}$$

with $0 \le \beta \le 1$. For the case $\beta = 1$ each particle moves with its own velocity along the streamlines ; if $\beta < 1$ the particle moves with reduced velocity field and if $\beta = 0$ the motion is considered with respect to a fix particle grid. It is clear that if β is not equal to 1 the fields must be reconstructed over the points of interest. This approach allows a great variety of possibilities: the particles can be traced along their streamlines or traced at fix positions. For problems over fix domains the use of moving grids could be not effective since the grid may deform and the solution may loose accuracy. Furthermore the implementation of the boundary conditions which is easy in the Euler formulation cannot be done efficiently in the Lagrangian

formulation. Our approach is to apply the boundary conditions always over a fix grid of particles. This grid must cover fix boundaries but can also be extended to cover all the domain that does not have free surfaces.

After the approximation of the spatial derivatives in (3.12–3.15) these equations reduce to a system of ordinary differential equations. This system can be solved by a simple integrations scheme. One can use the explicit Euler scheme, but it requires very small time step. The simple explicit forward Euler scheme is in some cases insufficient to give satisfactory results and, if possible, higher order methods should be used. Here a two Runge-Kutta time steps is proposed which is sufficient for many of the tests proposed in the next section [14].

Let $\mathbf{y}_i = [\vec{x}_i, \rho_i, \vec{v}_i]$ and

$$\mathbf{F}_i(t, \mathbf{y}, \nabla\mathbf{y}, \Delta\mathbf{y}) = \begin{bmatrix} \vec{v}_i \\ -\rho_i \Pi\nabla . \vec{v}(t, \vec{x}_i) \\ -\Pi\nabla . p(t, \vec{x}_i) + \frac{1}{\mathrm{Re}} \Pi\tilde{\sigma}(\vec{v}(t, \vec{x}_i)) \end{bmatrix}$$

for $i = 1, 2, \ldots, N$. In agreement with the notation introduced above the discrete system can be rewritten in a compact form as

$$\frac{d\mathbf{y}_i}{dt} = \mathbf{F}_i(t, \mathbf{y}, \nabla\mathbf{y}, \Delta\mathbf{y}), \tag{3.16}$$

for $i = 1, 2, \ldots, N$, where $\mathbf{F}_i$ denote the discrete approximation of the right hand sides in (3.12-3.13) and (3.15).

4. Numerical Tests

In this section we present some numerical tests. We always consider discretizations over bounded domains with N particles at $\vec{x}_i$ for $i = 1, 2, \ldots, N$ and constant time step Δt.

In this section we denote by f_i^m the values $f(t, \vec{x}_i)$ with $t = m\Delta t$ for $m = 0.1, \ldots$. The discrete form of the Navier-Stokes equations are the discrete version of the asymptotic equations in (3.12-3.15) for Runge-Kutta two time steps, namely we solve

$$\rho_i^{m+\frac{1}{2}} = \rho_i^m - \frac{\Delta t}{2} \rho_i^m \Pi\nabla . \vec{v}_i^m \tag{4.17}$$

$$\vec{v}_i^{m+\frac{1}{2}} = \vec{v}_i^m - \frac{\Delta t}{2\rho_i^m} \left(\Pi\nabla p_i^m + \frac{1}{\mathrm{Re}} \Pi\nabla . \tilde{\sigma}(\vec{v}_i^m) \right) \tag{4.18}$$

$$p_i^{m+\frac{1}{2}} = \frac{(\rho_i^{m+\frac{1}{2}} - 1)}{\delta} \tag{4.19}$$

$$\vec{x}_i^{m+\frac{1}{2}} = \vec{x}_i^m + \frac{\Delta t}{2}\vec{v}_i^m \tag{4.20}$$

$$\rho_i^{m+\frac{1}{2}} = \rho_i^m - \Delta t\ \rho_i^{m+\frac{1}{2}}\,\Pi\nabla\cdot\vec{v}_i^{m+\frac{1}{2}} \tag{4.21}$$

$$\vec{v}_i^{m+\frac{1}{2}} = \vec{v}_i^m - \frac{\Delta t}{\rho_i^{m+\frac{1}{2}}}\left(\Pi\nabla p_i^{m+\frac{1}{2}} + \frac{1}{\mathrm{Re}}\Pi\nabla\cdot\tilde{\sigma}(\vec{v}_i^{m+\frac{1}{2}})\right) \tag{4.22}$$

$$p_i^{m+1} = \frac{(\rho_i^{m+1}-1)}{\delta} \tag{4.23}$$

$$x_i^{m+1} = \vec{x}_i^m + \Delta t\vec{v}_i^{m+\frac{1}{2}} \tag{4.24}$$

for $i = 1, 2, \ldots, N$ and $m = 0, 1, 2, \ldots$, where the initial conditions are given by $(\rho_i^0, \vec{v}_i^0)$ for all $i = 1, 2, \ldots, N$ and δ small positive number.

Many strategies can be adopted. One possibility is to perform two time steps along streamlines and then interpolate the solution over a more regular grid of points. This is convenient in order to control the grid points and keep them regular. In this case the boundary conditions can be imposed simply fixing the velocity at the boundary. Another possibility is to move continuously the particles over the streamlines with its own or reduced velocity. The boundary conditions must be imposed through special boundary particles which are sitting over a boundary fix grid.

First we propose Poiseuille flow in order to test the constrained least squares method where we compute the viscous and body forces. In particular this flow tests the approximation of the second derivatives in space. However in these two tests the pressure does not play any role and the incompressibility constraint acts in a straightforward manner. Then we test the solution for moderate distribution of pressure against analytical solutions. Finally we test the driven cavity flow against the corresponding finite element approximation where the distribution of pressure is not trivial as in the previous cases.

4.1 Poiseuille Flow

The first test case is a stationary forced flow through a channel between two infinite parallel plates. The solution $\vec{v} = (u, v)$ of this simple flow can be written in series form as

$$u(\vec{x}, t) = \frac{F}{2v}y(y-L) + \tag{4.25}$$

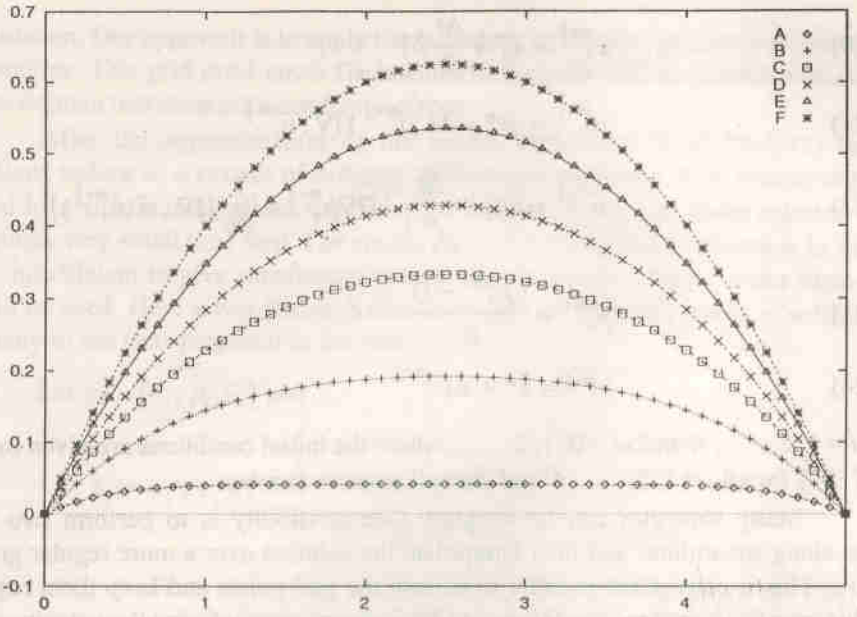


Figure 1: Test 1. Exact flow and solution for Poiseuille flow

$$+\frac{4FL^2}{\upsilon\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\sin\left(\frac{\pi y}{L}(2n+1)\right)\exp\left(-\upsilon\frac{(2n+1)^2\pi^2}{L^2}t\right),$$

$$(\vec{x},t)=0,$$

where L is the width of the channel and F the force. We note that, we have to add the body force F in the momentum equation.

The test was performed with $L = 5$, $F = 1$ and $Re = 1$. Fig. 1 shows the exact and computed solution. We note that the solution obtained by the particle method approximates closely the flow. The solution has been computed both reconstructing the solution over a fix point grid at each time step or letting the particle flow along streamlines. In both cases the matching is excellent.

In Fig. 1 we plot the exact and particle solutions obtained by reconstructing the velocity field over the initial regular particle distribution at each time step. The solutions are plotted at $t = 0.041$ (A), $t = 0.201$ (B), $t = 0.401$ (C), $t = 0.601$ (D) $t = 1.001$ (E), and $t = \infty$ (F) respectively. The limit in δ does not present particular problems and the fact that the pressure is constant allows the viscous term to dominate the pressure term even for very small values of δ.

4.2. Flow in a Square

The Poiseuille flow is one-dimensional flow and do not produce variations in dynamics pressure. In those cases the incompressible limit defined by δ tending to zero cannot be tested in a proper way. In this subsection we present a test where the pressure is changed in agreement with a smooth quadratic distribution. We test the solution against the analytical solution for a flow driven by a given force in the square (0,1) × (0,1) with homogeneous Dirichlet boundary conditions. Let $\vec{v}_d = (u_d, v_d)$ be the desired velocity defined by

$$u_d = \frac{d\phi(x,y)}{dy} \qquad v_d = -\frac{d\phi(x,y)}{dx} \tag{4.26}$$

where $\phi(x, y)$ is $\phi(x, y) = \Phi(x)\,\Phi(y)$ and $\Phi(z)$ is

$$\Phi(z) = (1-\cos(4\pi z))\,(1-z^2).$$

The pressure is given by $p_d = 10^5\ (x^2 - .25x)$. For given $(\vec{v}_d, p_d)$, the corresponding body force is

$$\vec{F} = (\vec{v}_d \cdot \vec{\nabla})\vec{v}_d + \frac{\nabla p_d}{\rho} - \frac{1}{\mathrm{Re}}\nabla\vec{\sigma}(\vec{v}_0) \tag{4.27}$$

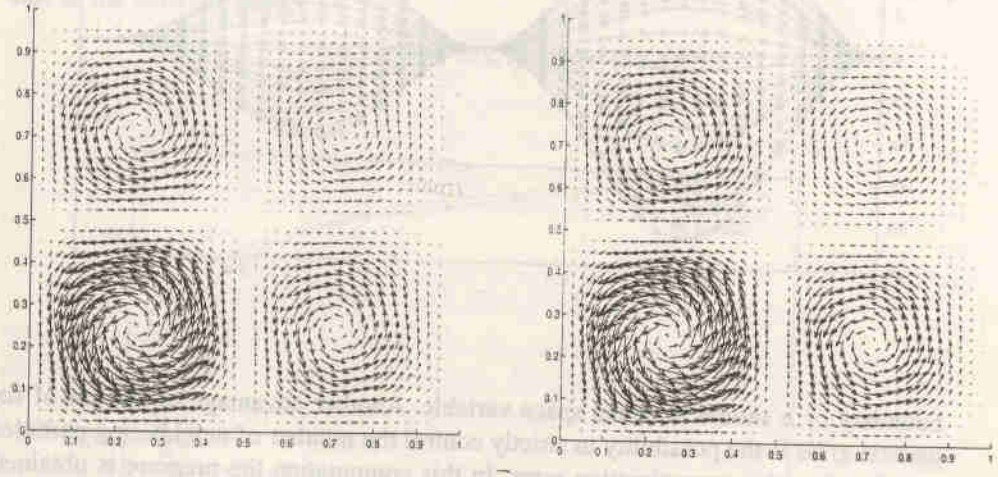
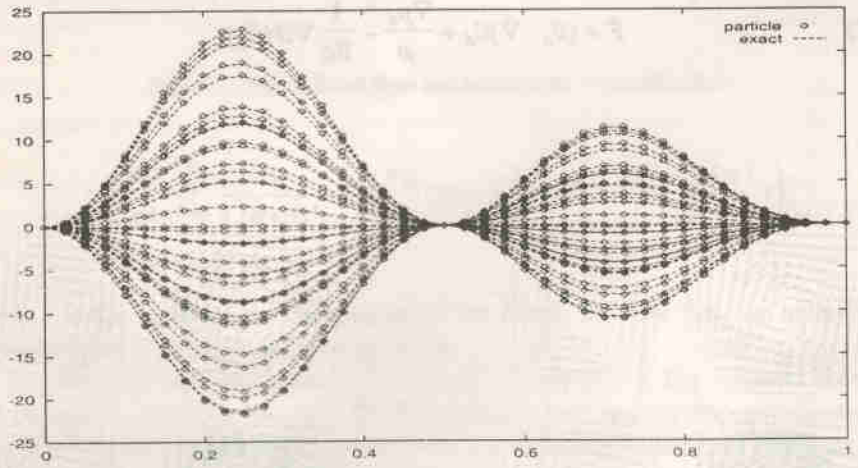


Figure 2: Test 2. Exact flow (right) and solution (left)

The flow consists of four vortices rotating in different parts of the domain. In order to obtain this solution we use a fix particle grid generated at the initial time. The boundary conditions are imposed simply by setting zero velocity at the boundary and reconstructing the fields at each time step. Since the flow is extremely complex the grid has been improved to 41×41 particles.

In Figs. 2–3 we have the solution obtained by the particle method against the exact expression. We can see that the solutions match perfectly: the u component is shown in Fig.3. In these figures all the 41 sections are plotted along the x-axis showing a high degree of symmetry and accuracy. The pressure, which is assumed to be parabolic is matched almost perfectly. Again the smoothing length h is 2.5 times the characteristics mesh length Δx. The ideal value for h over a regular grid should be evaluated on the basis of the number of neighboring particles necessary to compute the unknowns quantities. Usually $h = 1.5\ \Delta x$ is sufficient in the interior of the domain but not on the boundary, where the number of neighboring particles available is reduced. Therefore the values of h is dictated by the topology of the



boundary or h should be taken space variable. Another advantage of the use of fix particle grids is the possibility to strictly control the number of neighboring particles and therefore the approximation error. In this computation the pressure is obtained by $\delta = 10^9$ and density error less than 0.1%.

4.3. Driven cavity flow

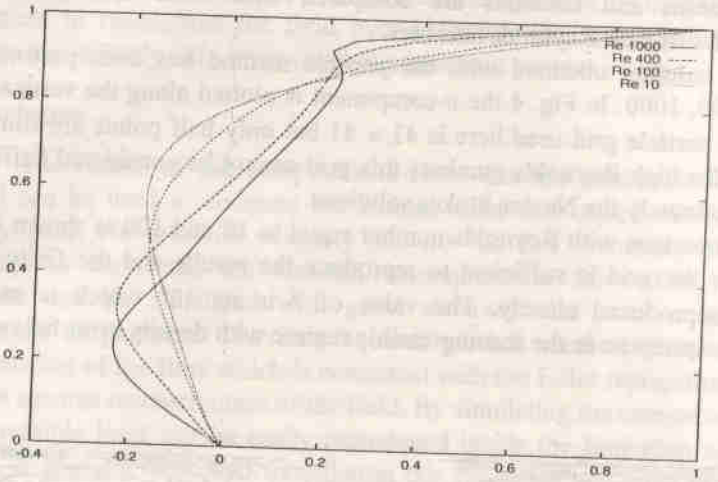


Figure 4: Test 3. U-component along the y-axis at $x = 0.5$.

The flow in a cavity driven by the velocity on the top has become a popular example for testing and comparing numerical methods. The velocity $\vec{u}$ on the top side of the cavity is chosen as

$$v(x,1) = 0 \qquad u(x,1) = 16x^2(1-x)^2,$$

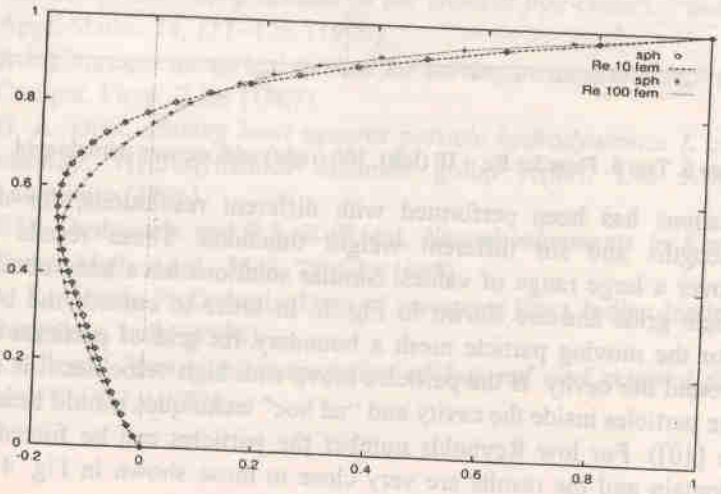


Figure 5: Test 3. U-component along the y-axis at $x = 0.5$.

with homogeneous Dirichlet boundary conditions over the rest of the boundary. Computations have been performed with the iterative method described in the previous sections and solutions are compared with finite element numerical solutions with different Reynolds numbers.

The solution obtained with the particle method has been performed for Re 10,100,400, 1000. In Fig. 4 the u-component is plotted along the vertical line at $x = 0.5$. The particle grid used here is 41×41 but only half points are shown. We remark that for high Reynolds numbers this grid cannot be considered sufficient to reproduce accurately the Navier-Stokes solutions.

Comparison with Reynolds number equal to 10 and 100 is shown in Fig.5. In this range the grid is sufficient to reproduce the results and the finite element solution is reproduced closely. The value of δ is set 10^4 which is enough to reproduce the pressure in the incompressible regime with density error below 0.1%.

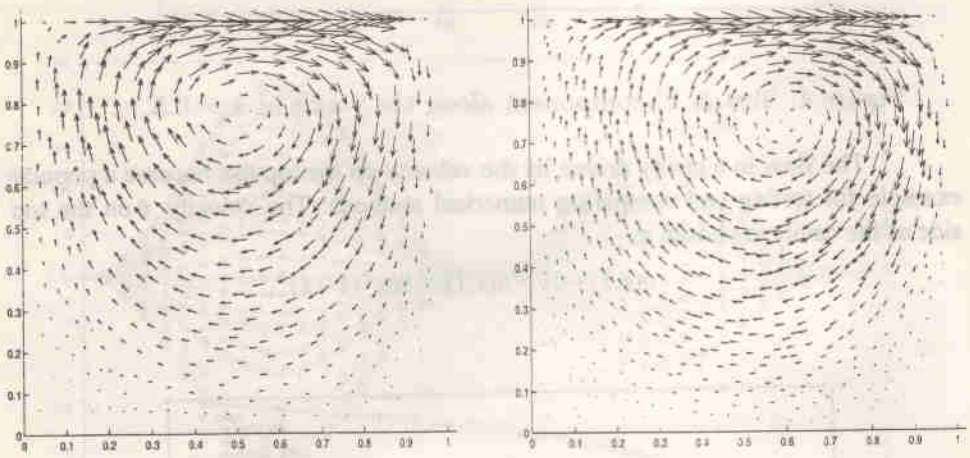


Figure 6. Test 3. Flow for Re = 10 (left), 100 (right) with moving particle grid.

The computations has been performed with different resolutions, for different smoothing lengths and for different weight functions. These results can be reproduced over a large range of values. Similar solutions have been obtained for moving particle grids and are shown in Fig. 6. In order to enforce the boundary conditions for the moving particle mesh a boundary fix grid of particles must be generated around the cavity. If the particles move with high velocities it is not easy to contain the particles inside the cavity and "ad hoc" techniques should be used (see for example [10]). For low Reynolds number the particles can be forced to stay inside the domain and the results are very close to those shown in Fig. 4 and the

velocity profiles in Fig. 5 are matched by the solutions shown in Fig. 6. However for higher Reynolds a robust "ad hod" strategy is necessary to keep the particles inside the boundary. For these reasons, if flows are investigated in fix domains, it is always convenient to reconstruct the field over a "controlled" particle grid and let the particles move freely only if moving boundaries are present.

5. Conclusion

The results of the computations show that the constrained least squares method can be used to compute the second order derivatives and reconstruct the velocity field. No artificial viscosity and no "ad hoc" boundary conditions are necessary to reproduce the viscous, incompressible flow. The Navier-Stokes equations can be simulated with good accuracy and over a quite arbitrary distribution of particles. The least squares particle method gives a Lagrangian representation of the flow which is consistent with the Euler representation given by the least squares reconstruction of the field. By simulating the compressible flow the incompressible limit can be easily reproduced inside the limitation of the explicit scheme. Within the proposed formulation this limitation can only be improved by the development of an implicit or projection scheme. An improvement in this direction can be reached by enforcing the incompressibility constraint directly into the constrained least square approximation of the field. Also these scheme can be easily extended to problems with non-isothermal flow, complex geoemtries and moving boundaries.

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OBITUARY

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Prof. Dr Rameswor Prakash Manandhar was born in a middle class business family at Pakanajol, Kathmandu in 1933 AD. His father Bir Bahadur Manandhar used to trade commercial items with the village food stuffs brought by the villagers. R.P. Manandhar used to help his father in his early days. Later, he developed passion for learning and started his education in his teens, on his own effort. He passed M.Sc. degree in Mathematics in 1963 and under Colombo Plan he was awarded a fellowship for Ph.D. degree in India (1965-1968). From Ranchi University, under the gudance of Prof. K.M. Saksena he was awarded Ph.D.degree in Integral Transform in 1968.

After returning from India, he joined Tri-Chandra College as a lecturer and then he joined the Central Department of Mathematics as a permanent staff of Tribhuvan University in 1970. He became professor in 1986 and then took charge as the head of the department in 1988 and continued to be in the chair till 1998. In the same year he got retirement from the University service but he helped the department as a contract professor for the last four years. Because of his deteriorated health he was unable to attend the department since February, 2003. He passed away after a short illness on 7th June 2003.

Prof. Manandhar has produced two Ph.Ds in Generalized Integral Transform and Distribution Theory. He has published nearly a dozen of research papers in the Mathematical Journals in Nepal, India and abroad. Dr. Manandhar has contributed a lot in the development of Central department of Mathematics and his contribution for the upliftment of the research journal Nepal Mathematical Sciences Report, is highly appreciable.

As a person he was a determined scholar who affected the lives and thoughts of his collegues and students equally.

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