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Study of Some Property of Density Topology

M.P. CHAUDHARY

Abstract: In this paper we prove that in the plane density topology exists without power property.

Key words: Discrete topology, density topology, power property.

Introduction

Let X be a topological space. Any continuous transformation, on a topological space X will be called mapping. We say that $f : A \to X$ is light on $A \subset X$ if f is non-constant on any non-degenerate continuous mapping in A.

We say that any topology on X has the power property if for any open set $U \subset X$, any open and light mapping $f : U \to X$ and any point $x \in U$ there exist a number $n \in N$ and a $nbd\ V \subset U$ of x such that for every $y \in [(f(V)\backslash (f\{x\})]$ the set $f^{-1}(\{y\})$ has cordiality n. In order way the power property of a topology means that each open and light mapping on any open set is locally n to one.

Results

1. Family Lemma: For any $t \in (0,1)$, we define the family, $Ht = \{\frac{1}{2^k} : k \geq 0\}$. Given prison set $M \subset (0,1)$ of the lebesque measure $\lambda(m) = \frac{1}{2^n} \exists\ t \in (0,1)$ with a free family $Ft = Ht / M$ of cardiality at least n.

Proof : The assertion is obviously fulfilled with $M = \left(0, \frac{1}{2^n}\right)$. We can try to avoid the free family of cordiality at least $(n+1)$ with a prison of measure $\frac{1}{2^n}$. For each $t \in (0.5,1)$ there must be at most n members of the family $Ht = \{t / 2^k : K > 0\}$ free. The most efficient way is to build the prison $m = \left(0, \frac{1}{2^n}\right)$
Hence the family lemma proved.

2. Theorem : *The density topology does not have the power property in the plane.*

Proof : Let $D = \{r\,(\mathrm{Cos}\ \pi t + i \sin \pi t) \in C : 0 \le r < 1,\ 0 < t < 2\}$ and define a corkscrew type mapping $f : D \to C$ by $f(r(\cos \pi t + i \sin \pi t) = r\{\mathrm{Cos}\ 2\pi\phi(t) + i \sin 2\pi\phi(t)\}$. When,

$$\phi(t) = \begin{cases} t-1, \text{for } t \in (1,2] \\ \phi(2^{n+1} t), \text{for } t \in \left(\frac{1}{2^{n+1}} \cdot \frac{1}{2^n}\right], n \ge 0 \end{cases}$$ with t decreasing from 2 to 1 the lower half of D is mapped onto D Anti-clockwise, then the speed of rotation increases in such a way that,

$$f[\{r(\cos \pi t + i \sin \pi t) \in C : 0 \le r \ \ 1, t \in \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right)\}] = D \text{ for } n \ge 0.$$

We can consider the mapping $f : D \to C$ with the density topology on both D and C. We have

A) D is a density open set (the missing segment has the lebesgues measure zero).

B) f is density continuous at $D \setminus \{0\}$.

C) f is density continuous at 0.

Proof of (C) : For any density open set V containing 0, the density of a set $f^{-1}(V)$ at 0 can be calculated using the radical segments.

$[\{r(\mathrm{Cos}\ \pi t + i \sin \pi t) \in C : \mathrm{O} \le r < 1,\ t \in \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right)\}]$ of D defined by the segments $t \in \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right)$, the density of V gives the density of $f^{-1}(V)$ at 0.

D) f is density open at $D \setminus \{0\}$.

E) f is density open at $\{0\}$

Proof of (E) : The density of U at o gives the estimate of the lebesgue measure of

$$U \cap \{r\,(\mathrm{Cos}\ \pi t + i \sin \pi t) \in C : \mathrm{O} \le r < 1,\ t \in (1,2)\}$$

And we obtain the estimate of the density of $f(U)$ at 0.

F) f is a light and open mapping on an open set D on a topological space C with the density topology.

G) For any density open set $V \subset D$ containing 0 and $n \in N$ there exists $y \in f(V)$ such that the set $V \cap f^{-1}(y)$ has cardinality at least n.

Proof of (G) : There is a density open set $U \subset V$, containing 0 and a open set G (Euclidean open) containing the density closed set $C \setminus V$, such that G and U are disjoint (See–[2]), the Lusin–Menchoff property of the density topology. When U reaches the density $\left(1 - \frac{1}{2^n}\right)$ at 0 for some $R \in (0, 1)$, i.e., $\lambda\,[\{r\,(\mathrm{Cos}\ \pi t + i \sin \pi t) \in C : 0 \le r < r,\ t \in (0, 2)\}] > \left(1 - \frac{1}{2^n}\right).\ \pi R^2$. We can using the polar coordinates obtain $r \in (0, R)$ such that the set,

$$M = \{t \in (0,2) : r(\cos \pi t + i \sin \pi t) \in G\}$$

is Euclidean open in (0,2) with $\lambda(M) \leq \frac{1}{2^n}$.

By family lemma with M as the prison set we conclude that there exists $t \in (0,1)$ with the free family set $Ft = \{t_1, \dots t_n\} \subset (0,1)$ disjoint with M, being of cardinality at least n.

Then $\quad y \quad = f\{r(\cos \pi t_i + i \sin \pi t_i)\}$

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$= \dots\dots\dots\dots\dots\dots\dots\dots\dots$

$= f\{r(\text{Cos}\ \pi t_n + i \sin \pi t_n) \in f(V)$ due to the definitions of Ft and f, and consequently $f^{-1}(y)$ has cardinality at least n.

The mapping $f : D \to C$ shows that the density topology does not have the power property. Hence the theorem

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A History of Fixed Point Theorems

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Abstract: In this paper, we present a brief historical account of the development of Fixed Point Theorems (*fpths*), There are about seven thousand results on *fpths* and this paper includes almost all initial generalizations and extensions of major and interesting results on *fpths*, through different types of mappings, like Contraction; Non-expansive: Multifunctions; Family of mappings, Set valued mappings, etc.

Keywords and Phrases: Fixed points, contraction mapping, non-expansive mapping, multifunctions, commuting mappings.

1. Introduction:

Let T be a self mapping on a set X. An element u in X is said to be a fixed point of the mapping T if $Tu = u$. The *fpth* is a statement which asserts that under certain conditions (on the mapping T and on the space X), a mapping T of X into itself admits one or more fixed points. History is a meaningful record of man's achievement & historical research is the application of scientific method to the description and analysis of past events. In this paper, we have tried briefly to present a history of *fpths*. There are plenty of results on different cases of *fpths* and this paper is basically a survey work which deals with almost all earlier settings of *fpths*, with suitable examples. This paper considers some short abbreviations like *fpths*, *Bfth*, *Sfpth*, & **CMTH** for fixed point theorems, **Brouwer's** fixed point theorem, Schauder's fixed point theorem & Contraction Mapping Theorem respectively.

Historically, the most important result in the *fpth* is the famous theorem of **L.E.J. Brouwer** which says that every continuous self-mapping of the closed unit ball in $\mathbb{R}^n$, the n-dimensional Euclidean space, possesses a fixed point. This result, published by **Brouwer** (1910), was previously known to **H. Pointcaré** in an equivalent form. In 1986, **Pointcaré** proved the following result: If $f: E_n \to E_n$ is any continuous function with the property that, for some $r > 0$ and any $\alpha > 0$, $f(x) + \alpha\, x \neq 0$, $\| x \| = r$ then there exists a point x_0, $||x_0|| \leq r$ such that $f(x_0) = x_0$. Now it is known that this assertion is equivalent to the *Bfpth*. Another interesting fact is that the **Poincaré** theorem was also rediscovered by **P.Bohl** (1904). Also, **A.L. Cauchy** (1844) was the first mathematician to give a proof for the existence and uniqueness of the solution of the differential equations $\frac{dy}{dx} = f(x,y)$;

$y(x_0) = y_0$, when f is a continuous differentiable function. **R. Lipschitz** (1877) simplified **Cauchy's** proof using which is known today as the 'Lipschitz Condition'. Latter **G.Peano** (1890) established a deeper result, supposing only the continuity of **F. Eeano's** approach is more related to modern ***fpth***, which is used to obtain existence theorem.

Also, **E.Sperner** (1928) proved the combinatorial geometric lemma on the decomposition of a triangle, which plays an important role in the theory of fixed points. Due to its wide applications, there are plenty of results and still more results to come on ***fpths***. These are the most important tools for proving the existence and uniqueness of solutions to various mathematical models (differential, integral, ordinary and partial differential equations, variational inequalities). Other fields are Steady-state temperature distribution, Chemical reactions, Neutron transport theory, Economic theory, game theory, Epidemics, Flow of fluids, Optimal control theory, Fractals, etc.

2. Brouwer's Schauder's and Tychonoff's Epths:

Brouwer proved his famous theorem in 1912. As such theorems, where the spaces are subsets of are $\mathbb{R}^n$ are not of much use in Functional analysis where one is generally concerned with infinite dimensional subset of some function spaces. The first infinite dimensional ***fpth*** was investigated by **C.D. Birkhoff & O.D. Kellogg** (1922). There exist many proofs of the original ***Bfpth***. **Birkhoff & Kellogg** gave one proof of ***Bfpth*** with the assumption about convexity and compactness. **P.J. Schauder** in 1927 extended the **Birkhoff-Kellogg** theorem to metric linear space and in 1930, **Schauder** extended ***Rfpth*** to the result that every compact convex set in a Banach space has the fixed point property for continuous mapping, as well as that every weakly compact convex set in a separable Banach space has the fixed point property for weakly continuous mapping. An improvement of the last assertion was obtained by **M. Krein & V. Smulian** (1940).

Among the several proofs of ***Bfpth*** namely topological, analytic and degree theoretic, the proof of ***Bfpth*** depending on various definitions of the degree of a mapping (i.e. rotation of a vector field) were given by **Brouwer** (1910, 1912); **J.W. Alexander** (1922), **S. Lefschetz** (1926); **H Hopf** (1929); **J. Leray & J.Schauder** (1934); **E.Rothe** (1937); **S. Kakutani** (1943); **J. Leray** (1950); **M. Nagumo** (1951); **J.Dugundji** (1951); **V.L.Klee** (1960); **A. Granas** (1962); **P.Whittlesey** (1963); **J Cronin** (1964); **E. Fadell** (1970); **F.E. Browder & J.A.B. Potter** (1972); **Alexander** (1922), under the impression about ***Bfpth*** was proved for homeomorphism only, gave a new proof. The first continuos theorems applicable to non-linear problems were due to **Leray & Schaunder** (1934), known as "the Leray-Schaunder theorem", using the linearisation trick. But this theorem cannot be stated or applied without a knowledge of degree theory. Various attempts have been made

to replace Leray-Schauder theorem by theorems in which the degree is not used. These theorems use conditions which are less general but more easily established in applications. The most useful result is that of **Schaefer** (1955) and **Browder** (1966). In 1926, **Lefcshetz** gave an extension of ***Bfpth*** to orientable n-manifolds without boundary, using what is called now the Lefschetz number $\wedge(f)$, and also extended this to the case of n-manifolds with boundary. This result was further extended to finite polyhedra by **Hopf** (1929). **Lefschetz** (1930,42) gave ***fpth*** for compact contractible sets. **E.Spanier** (1966) gave a modern proof of **Lefschetz's** result in 1942. **S.Kinoshita** (1953). **E.F. Whittlesey** (1963); **R.H. Bing** (1969), **E. Fadell** (1970) have given ***fpths*** on contractible sets with many interesting examples and references.

Another proof of ***Bfpth*** depending on classical method (Calculus & determinants) were given by **Birkhoff & Kellog** (1922) the most proof of ***Bfpth*** is by simplicial subdivision of an n-simplex due to **B Knaster, C. Kuratwoski & S. Mazurkiewicz** (1929); **C. Kuratwoski** (1933) and **L.M. Graves** (1946). **Hirsch's** (1963). **Hirsch's** proof plays an important role in the algorithms for fixed point in *Bfpth*. Unlike **CMT**, **Bfpth** .does not give any computational scheme for obtaining a *fpth*. However, in 1967, **H. Scarf** gave some sort of algorithm for computing a fixed point of a mapping with some additional conditions. This gives a new proof of *Bfpth*. We find many other algorithms in a book edited by **S.Karamardian** (1977). **H. Robbins** (1967) gave compliments of ***Bfpth***. The most interesting generalization of ***Bfpth*** is the so called **K. Borsuk- Ulam theorem** and Borsuk's theorem about antipodal points the proof of **Borsuk-Ulam theorem** is that of **M.D.Meyerson & A.L Wright** (1979).

The condition of compactness in ***Sfpth*** was a very strong condition. As many problems in analysis do not have compact setting, it was natural to modify this theorem by relaxing the condition of compactness. **A.N. Tyechonoff** (1935) proved a generalizations of ***Sfpth*** for the case of compact operators on locally convex linear spaces, and **M. Hukuhara** in 1950. **Tychonoff** need simplicial subdivision method to prove his ***fpth***. An interesting extension was obtained by **Browder** (1959), under some deep conditions for the iterations of the mappings. **H.H. Schaeffer** (1955) gave a slight but very useful variation of ***Sfpth*** for compact mapping on Banach space. Browder extends ***Sfpth*** for the compact sets. **Rothe** extended it in 1937 and it was latter proved by **Potter** (1972) to the more general case of convexity, adopting the argument of **Browder Branas** (1962) considered a general region for the same theorem and need the method of **Potter**.

The generalization of ***Sfpth*** concerning set valued mappings was proved by **Ky Fan & I. Glicksbert** (1952). A proof of **Tychonoff's** theorem using the fixed point property for the Hilbert cube is given by **N. Dunford & J.T. Schwartz** (1958). An interesting generalization of both ***Sfpth*** and Tychonoff's theorem was obtained by **Ky Fan** (1961). The extension given by **Ky Fan** depends upon a lemma which is

essentially the infinite dimensional version of the **Knaster-Kuratowski-Mazurkiewicz** theorem (1929). Also, a new and important step in extending the **Sfpth** to more general class of mappings was made by **G.Darbo** (1955). In 1967, **V.N.Sdovski**, using a new measure of non compactness, proved a generalization of the ***Sfpth*** for mappings which are known as condensing or densifying. **M Volato** (1953) extended ***Sfpth*** to mappings without strong relation with compactness. **G.Jones** (1953) extended ***Sfpth*** which was generalized by **V. Istrătescu** (1978). Also, **M.A. Krasnoselskii** (1953). **M.Altman** (1957) and **W.V. Petryshin** (1967) proposed some conditions for the mapping for the computation of fixed point. **Altmann** proved his ***fpth*** using ***Sfpth***. A proof of altmann's theorem using the concept of degree theory was found by **M.S. Berger & M. Berger** (1968). Latter on, **M. Edelstein** (1966) has generalized the theorem of **Krasnoselskii**. **Browder** (1970) used the homeomorphism as initial condition. **Sadvskii** (1972) developed the generalized degree theorem who extended the concept of degree to the class of limit compact operators. **B.V. Singbal** has shown that the ***Sfpths*** is true for locally convex spaces in its full generality, using a technique due to **Nagumo** (1951). A paper of **K.L.Stepaneek** (1957) includes different kind of generalizations of ***Sfpth***. **M.G. Krein & M.A.Rutman** gave results related to the transition from nonlinear to linear problem. Also **N. Aronszajn** (1942) gives general regular condition on T sufficient to establish that the set of its fixed points is a homeomorphic image of the intersection of decreasing sequence of absolute retracts.

3. Banach Contraction Principle:

There are ***fpths*** that can be approached without any combinatorial topology as background. One result applies to contraction that is, distance diminishing mappings of a complete metric space into itself. The concept of Banach space was introduced by **Stefen Banach** and obtained a ***fpth*** for contraction mappings in 1922, famous as Banach Contraction Principle **(BCP)** or **CMTH**. Recently there have been numerous generalization of **BCP** by weakening its hypothesis while retaining the convergence property of the successive iterates to the unique fixed point of the mapping. One result is due to **R.Caccioppoli** in 1930. **BCP** is very useful in the existence and uniqueness theories. **S. C. Chu & J.B. Diaz** in 1964, 65 gave one generalization of **BCP**. The result due to **Chu & Diaz** has been further extended by **V. M. Sehegal** (1969). **Krasnoseiskii** generalized **CMT** in 1964. Also **E. Rakotch** (1962), **D.Boyd & J.S.W. Wong** (1969) and **Browder** (1968) have attempted to generalize **BCP** by replacing the Lipschitz constant by some real valued function whose values are less than 1. It is noted that the class of **Boyd & Wong** is strictly larger than the class of **Rakotch, A.Meir & E.Keeler** (1969) has generalized **BCP** for the case of weakly uniformly strict contraction.

Some significant generalizations of **Boyd & Wong** theorem are those due to **S.Park & B.E.Rhoades** (1981); **S.L. Singh & S. Kasahara** (1982); **S.A. Hussain**

& V.M. Sehegal (1975); **S.P. Singh & B.A. Meade** (1977); **J. Jachymski** (1994) and **R.P. Pant** (1996). Similarly, some of the well known generalizations of the **Meir & Keeler** theorem are those due to **S.Park & J. S. Bae** (1981); **Park & Rhoades** (1981); **I. H. N. Rao** (1985); **Pant** (1986); **G.Jungck** (1986); **Jungck, K.B. Moon, Park & Fhoades** (1993). **Edelstein** in 1962 has shown that compactness of metric space X will guarantee a unique fixed point for a contractive mapping on **S. D. F. Bailey** (1966) extended this result for contraction mappings. **Edelstein** (1961) gave a local version of the **CMTH**. The generalization of the **BCP** to a class of mappings on C-chainable space is due to **Edelstein (1966)** and **Sehgal** (1969). **S.P. Singh & Zorzitto** (1971) have obtained more general results by replacing the metric by some real valued function with continuity condition. **Wong** in 1972 proved the result for lower semi continuous mapping **on a compact** Hausdorff space.

R. Kannan (1968); **Hussain & Sehgal** (1975) and **J.V. Caristi** (1975) have considered several generalizations of contraction mappings. **Rhoades** (1977) have given more results on contractive mappings and its generalizations. **G.E.hardy & T.D. Regers** proved a ***fpth*** concerning Kannan-Reich type mapping in 1972. **Hussain & Sehgal** (1975) proved a ***fpth*** which generalizes the Kannan-Reich's and Ciric type of generalized **CMTHS**. An extension of **Hussain & Sehgal's** result was obtained by **Singh & Meade** in 1977. Both results deal with common ***fpth*** of a pair of mappings. **Singh & Meade** proved ***fpth*** under the assumption that Φ is upper semi continuous. We have some generalizations due to **S.Reich** (1971), **Maia** (1968), **Singh** (1970) and **Hardy & Rogers** (1973) for two mappings on a complete metric space. **Kannan** (1969) proved a theorem in which the completeness of the space is not required. **Kannan's** results have been generalized by **Singh** in 1969. We have fixed point results from **L.P. Belluce & W.A. Kirk** (1969) and **Fukushima** (1970) on diminishing orbital diameters. **S.B. Nadler** (1969) represented the extension of the **BCP** to the case of set-valued contraction mapping.

Browder (1965) gave a result which did not assume compactness. The result remains true for the case of a uniformly convex Banach space. We have some more results on ***fpth*** due to **Browder** (1965), **Kirk** (1965) or **K. Goebel** (1969). Converses of **CMTH** have been discussed by **P.R.Meyers** (1967), **L.Janos** (1967) and **Edelstein** (1969). In 1970, **L.F. Guseman Jr**. gave a ***fpth*** that was first proved by **Sehgal** in 1972. In 1976, **Caristi** rediscovered independently a **fpth** which turned out to be an abstraction of a Lemma of **E.Bishop & P.R. Phelps** (1963). Its applications were discussed by **Kirk & Caristi** (1974); **Kirk** (1975). **D. downing & Kirk** (1977). **Caristi's** proof involves transfinite induction. The proof of **Caristi's** theorem is given by **Kirk** (1976) and implicit in a paper by **A. BrΦnsted** (1974). The localization of contractive mapping condition was given by **R.D. Holmes** (1976). **David Hilbert** (1895) introduced a metric which is interesting in its own right but also applications to analysis, as was proved by **Birkhoff** (1957) and by

U.Urabe (1956) . It is possible to show that, after a suitable change of the metric, the mapping is actually a contraction mapping. The first result of this type seems to be that of **C Bessage** in 1959. Generalizations of **Bessage's** result as well as of related results were obtained by **Meyers** (1970). **S.Leader** (1977), **I.Rosenholtz** (1976), etc. Various applications of the **CMTH** have been given in **Copson** [pp 111-136]; **Heuser** [pp 17-23]; **Martin** [pp 114-117]; **Pitts** [pp 88-89]; **Simmons** [pp 339-340], **Singh** [pp 10-11], **Smart** [pp 41-52].

4. FPTHS For Non- Expansive Mappings:

As the fundamental properties of contraction mapping do not extend to non-expansive mappings, so it is of great importance in applications to find out if non-expansive mappings have fixed points. The study of non-expansive mappings has been one of the main features in recent development of fixed point properties. Contractive mappings, isometries and orthogonal projections are all non-expansive mappings. The problem of the existence of an extension for non-expansive mappings on $\mathbb{R}^n$ was first considered by **M.D Kirszbraun** (1934) .**M. Markov & Kakutani** in 1938 referred to simultaneous fixed points of suitable families of continuous mappings of compact convex subset of a topological vector space into itself. The mappings in the Markov-Kakutani theorem must satisfy a condition close to line .linearity .the first important result in the theory of fixed points for non-expansive mapping was obtained by **R. de Marr** in 1963 who has proved an interesting extensive of the famous result of Markov-Kakutani. This result greatly influenced to the development of fixed point theory . **R.de Marr** gave various ***fpths*** concerning families of mappings which need not to be affine, using downward induction argument

Kirk in 1963 proved a ***fpth*** using a characterization of reflexive due to **V.Smulian** and a concept (normal structure) of **M.S. Brodski & D.P. Milman** in 1948 to prove the ***fpth*** for mapping which do not increase distances. **Brodski** and **Milman** gave conditions under which a convex set in a Banach space has a point invariant under all isometric self mappings. **E.W. Cheney & Goldstein** in 1959 have given the results for a non-expansive mapping in a metric space. **Kirk** in 1965 proved ***fpth*** for a non-expansive self mapping of a bounded, closed, convex subset of a reflexive Banach space. An immediate consequence of Kirk's theorem was proved independently by **Browder** (1965); **D.Gohde & Kirk** (1965). They proved that a non-expansive self mapping of a bounded closed convex subset of a uniformly convex Banach space has a fixed point. Latter on, **Kirk** in 1970 proved the same result under slightly weaker assumptions that the space is reflexive and a bounded closed convex subset has normal structure.

Theorems for approximating fixed point concerning the convergence of some sequence defined using iteration techniques for general non-expansive

mappings are given by **Browder & Petryshin** (1966, 67); **Edelstein** (1966), **Diaz & F.T. Metcalf** (1967); A. **Pazy** (1971); **S. Kaniel** (1971); **H.F. Senter & W.G. Dotson Jr.** (1974); **Reich** (1976); **Browder** (1976). In recent works include papers of **Browder** (1966) dealing with the relationship of non-expansive mappings to the theory of monotone operators in Hilbert space and, in more general setting to the theory of J-monotone operators and accretive operators. **Browder & Petryshin** (1966) proved ***fpth*** for non-expansive and asymptotically regular mappings in Banach space. Petryshin (1966) proved that the class of demi compact operators is more general than the compact operators. In 1967, **Browder** proved a well known result in strictly convex Banach space. **Browder**'s result is false in the most general case of Banach space and a beautiful example for it is due to **de Marr**. The weak convergence of successive approximations for a non-expansive mappings is dealt by **Z. Opial** in 1967.

Edelstein in 1966 proved some interesting results in uniformly convex Banach spaces. In 1972, he gave the original notion of asymptotic centre and proved some of its properties and used it to prove a ***fpth*** for a class of mappings which includes non-expansive mappings. A semi contraction is a generalization obtained by intertwining of non-expansive mappings with strongly continuous mappings. **Browder**'s result is false in the most general case of Banach space and a beautiful example for it is due to de **Marr**. The weak convergence of successive approximations for a non-expansive mappings is dealt by **Z. Opial** in 1967.

Edelstein in 1966 proved some interesting results in uniformly convex Banach spaces. In 1972, he gave the original notion of asymptotic centre and proved some of its properties and used it to prove a ***fpth*** for a class of mappings which includes non-expansive mappings. A semi contraction is a generalization obtained by intertwining of non-expansive mappings with strongly continuous mappings. **Browder** in 1966 was the first to develop ***fpth*** for semi contraction mappings. Subsequently, **T.C. Lim** (1980), **K. Yanagi** (1980) and others dealt with **Edelstein's** result extensively. Also further contributions on semi contraction mapping were made by **Browder** (1968); **Petryshin** (1968); **Nassbaum** (1969); **Kirk** (1969) and **Webb** (1970). The concept which **Dotson Jr.** (1972) has labelled quasi-non-expansive, was essentially introduced, along with some other related ideas, by **Diaz & Metcalf** in 1962. It is clear that a non-expansive mapping with at least one fixed point is quasi-non expansive mapping **Sadovski** in 1972 gave an example showing that non-expansive mapping may fail to have fixed points in general Banach space. **Dotson** in 1972 gave an example which is continuous quasi-non expansive but not a non-expansive. **R. K. Bose & R.N. Mukherjee** in 1975 showed that **Dotson**'s theorem remains valid if there is an associated family of compact subset of a complete metric space. **Petryshin & Williamson** (1972), 73) proved a theorem concerning the quasi-non-expansive mapping. In a uniformly convex Banach space, **Senter & Dotson Jr**, have given conditions under which certain types of iterates

(Mann type) of a quasi-non-expansive mappings converge to a fixed points of the mapping. **Bose & Mukherjee** in 1981 considered approximation of fixed points of generalized non-expansive mapping.

Reich in 1976 considered the iteration scheme for non-expansive mappings in uniformly convex space with a Fréchet differentiable norm. For a commuting family of non-expansive mappings, we have results from **Belluce & Kirk** in 1966 and **P.K.F. Kuhfitty** in 1980. **Ng** in 1970 gave a result for non-expansive mapping on cluster set. **Caristi** (1975,76) ; **Lim** (1980); **Downing & Kirk** (1977) and **Yanagi** (1980) considered inward mappings and proved ***fpth*** for such mappings (both single valued and multivalued). **B.Halpern** in 1965 first considered inward mapping in his Ph.D. thesis. **Goebel & Kirk** in 1972 proved **fpth** for asymptotically non-expansive mappings. **S.C. Bose** in 1978 proved a ***fpth*** as an extension of **Opial's** convergence theorem in 1967 for non-expansive mappings to the class of asymptotically non-expansive mappings. In 1982. **G.B. Passty** extended **Bose's** result. **J. Lindenstraus** in 1975 has constructed an example of a non-expansive mapping defined no a closed, convex and bounded set of a **Banach** space such that the sequence does not converge. An interesting class of non-expansive mappings for which the **Cauchy-Picard** sequence of iterations converges, was discovered by **J.J. Moreau** in 1978. His result refers to the **Hilbert** space nonlinear mappings. The extension of **Moreau's** result to the case of uniformly convex **Banach** spaces was obtained by **B.Beauzamy** in 1978. **D. de Figueiredo** and **Petryshin** in 1967 computed fixed point for class of non-expansive mappings.

5. Fpths For Many Values Mappings :

If each point x of a set m is mapped onto a set U(x) then U is called a many valued mappings (multi functions). The study of fixed point problem of multifunctions was initiated by **Kakuntani** in 1941, when U(x) is compact & convex in finite dimensional spaces. This is the extension of ***Bfpth*** to the point compact convex set-valued mappings on a compact convex set in Euclidean space. It was extended to infinite dimensional Banach spaces by **Bohnenblust & Karlin** in 1950 by a method similar to **Schauder**'s proof in 1930 and to locally convex spaces by **Ky Fan** in 1952 and by **Glicksberg** in 1952. **S. Eilenberg & D.Montgomery** in 1946 allowed U(x) to be the acyclic (homologically trivial); so did **de Begle** (1950) and **Górniewiez & Granas** (1970). **Ky Fan** in 1961 merely requires U(x) to be compact but here U(x) must depend continuously on x. **R.E. Smithson** in 1965 considers cases where U(x) is finite-valued. Among two significant sets of methods in the fixed points of multivalued mappings, the first homological method started in 1946 by **Eilnberge & Montgomery** whereas the second method started in 1935 by **J. Von Neumann.**

Fan's result also generalizes **Schauder-Tychonoff's theorem. J.P. Dauer** in 1972 have considered ***fpth*** for multifunctions providing natural settings for many

problems in Control theory involving differential equations. The developments of geometric ***fpth*** for multifunctions was initiated by **Nadler Jr.** In 1969 and subsequently persued by **J.T.Markin** (1973); **Browder** (1968); **N.A. Assad & Kirk** (1972); **Goebel, E. Lami-Dogo** (1973) and others. **S.C.J. Himmelbert** in 1972 generalized **Fan**'s result and **Sehgal & E.A. Morrison** has further generalized **Himmelberg**'s work in 1973. **R.L. Plluket** (1956) and **L.E. Ward Jr.** (1961) have shown the spaces which have fixed point property for multivalued contraction mappings. These theorems do not place servere restrictions on the images of points and, in general, the space is required to be complete metric space. **Fleischman** (1970) and **Smithson** (1972) have given various recent related contributions.

Assad & Kirk and **Markin** worked on multivalued contraction, while **Smithson** worked on contractive multifunctions in 1971, which extends **Edeltein's** ***fpth*** for contractive single valued mappings to multifunctions. **Reich** (1971) and **Bose & Mukherjee** (1977) have extended the work of **Nadler Jr.** and obtained ***fpths*** for generalized multivalued contraction mappings. In 1980, **Bose & Mukherjee** proved ***fpth*** wihich's a generalization of a theorem of **Iseki**'s result for single valued mappings. They also gave a generalization of a theorem of **Wong** to multi-valued functions. **S.Itoh & W.Takahashi** (1977) and **Yanagi** (1980) proved ***fpth*** of multivalued non-expansive mappings on non-convex domain, more precisely on star-shaped domains. **Downing & Kirk** (1977) proved a ***fpth*** in conjunction with an elegant approach of **Goebel**. For single valued mappings, it is known that a non-expansive mapping is a <u>pseudo-contractive</u>. **Dowing & W.O.Ray** (1981) showed that the same is not true in the set valued case. **J.P. Aubin & J.Siegel** in 1980 proved a ***fpth*** that has relevance in <u>Control theory</u>. For the case of pseudo contractive mappings. **Browder & Petryshin** in 1967 gave methods to compute fixed points. We have some more results for ***fpth*** of pseude-contractive mappings due to **N.G. Crandall & Pazy** (1969); **T.Kato** (1970); **J. Reinermann & Schoneberg** (1976), **Kirk & R.Schoneberg** (1977) and **Kirk & Ray** (1979).

6. Special Cases on FPTHS:

The first theorem regarding to the continuity of fixed points of contraction mappings was proved by **F.F. Bonsall** in 1962. Subsequently, **Nadler Jr.** In 1968 obtained results concerning sequences of contraction mapping and also gave an application suggested by **Dorroh**. The first result about the convergence of sequence for the case s =1/2 in (0.1) was obtained by **Krasnoselski** (1955). This result was extended by **Schaefer** (1957) by proving the convergence for any fixed s in (0,1) and then weakening the assumptions about the mapping. The second result of **Schaefer** was extended by **Edlstein** (1966) to the case of <u>rotund spaces</u>.

The first result about fixed points for family of mappings was proved by **Markov** in 1936, depending on **Tychonoffs** theorem (1935). **Kakutani** in 1938 found a direct proof of **Markov**'s result and also proved a ***fpth*** for groups of affine

equicontinuous mappings. Thus, for affine mappings, fixed points have a natural geometric significance and we have famous Markov-Kakutani theorem. Latter **Edward** in 1965 gave a result which includes both Markov-Kakutani theorem and the case of a solvable group of affine mappings. **M.M.Day**'s theorem in 1961 is still more general. **Day** gave a connection between amenability of a group of mappings of semi groups and fixed points. An important extension of **Markov-Kakutani** results was given by **C.Ryll-Nardzewski** in 1967. It is interesting to note that the proof given by **Nardzewski** was probabilities in nature. Latter he found a proof which do not use probabilistic ideas, and **Asplund & Namioka** in 1967 have found a simplified proof. We find some more fixed point results on families from **N.W. Riekert** (1967); **F.F. Greenleaf** (1969); **R.E.Huff** (1970) and **T.Mitchell** (1970).

The first extension of topological fixed point theory of continuous mappings to the case of set-valued mappings was made by **John Von Newman** in 1937 in connection with the proof of the fundamental theorem of Game theory. The behaviour of the fixed points of set valued mappings has been considered by **Nadler Jr.** (1969) and **Markin** (1973). Both established conditions implying the strong convergence of the fixed points of a sequence of set valued contractions. These results were extended further by **Nadler & Fraser** in 1969 and **H. Covitz & Nadler**. Using **Urysohn**'s lemma and ***Bfpth***, we obtain ***fpth*** for a class of set valued mappings which generalizes and extends the results of **Kakutani**, **Bohnenblust**, **Karlin**, **Blicksberg** and **Fan**. Also, using the Liapunov function, **Sehgal & Smithson** were able to extend many results from the single valued mappings to set-valued mappings. In 1966, **Ky Fan** gave an analytic formulation of his ***fpth***, using so called quasi-concave functions. The weak convergence of fixed points of set-valued non-expansive mappings in a Banach space was obtained by **Markin** in 1978 who used it to obtain a stability result for generalized differential equations. We have more results related to this mapping due to Ng (1968); **M. Furi & A. Vignoli** (1969); **Singh & Russel** (1969); **Singh** (1970); **Reich** (1971); **G.W. Collins** (1973); **Dube & Singh** (1973).

Eldon Dyer (1954), **Allon Schields** (1955) and **Lester Dubins** asked the following question that whether any two continuous commuting self functions defined on [0,1] have a common fixed point or not ? An interesting problem related to fixed points for families of commuting mappings on [0,1] was noted by **Isbel** in 1957. This was a main source of inspiration for a decade and several mathematicians tried to solve this problem. However, in 1967, **W.M. Boyee & Huneke** independently disproved the conjecture. **Kakutani** produced an example that this theorem does not hold for infinite dimensional spaces. **Ryll-Nardzewski** in 1966 proved a more general form of common fixed points in which norm topology is replaced by any locally convex topology. **Folkmann** in 1966 gave results for common ***fpth***. In 1975, **Hussain & Sehgal** proved a common ***fpth*** and later on, it was improved upon by **Singh & B.A. Meade** in 1977 in a slightly different form. An

iteration scheme which converges strongly in one case and weakly in another case to a common fixed point of a finite family of non-expansive mapping were obtained by **Kuhfitting** in 1981. Another iteration scheme converging weakly in more general setting then Kuhfitting's were proved by **R.K.Bose & D.Sahani** in 1984. The study of common fixed points of non-commuting generalized contraction mappings was initiated by **S.Sessa** in 1982, introducing the notion of weakly commuting mapping.

In ordered Banach space, we find fixed point results due to **Krasnoselskii** (1964); **J.A. Gatica & H.L. Smith** (1977) ; **Gustufson & Schmitt** (1976); **R.W. Leggett & L.R. Williams** (1980); **H. Amann** (1976) ; **Turner** (1975). **Amann** has shown that using the asymptotic behaviour of a mapping, the existence of fixed points of the mapping can be derived. **Williams & Leggett** obtained some multiple ***fpths*** which they applied to problems in Chemical reactor theory. The theory of measure of non-compactness and densifying operators have applications in general topology, geometry of Banach spaces and the theory of differential equations. The most widely used measure of non-compactness on metric spaces are the α-measure introduced by **Kuratowskii** (1958) and used by **Darbo** (1955); **Furi & Vignoli** (1969); **Nussbaum** (1970); **Petryshin** (1971) and others. In 1972, **Sadovskii** introduced the concept of a condensing operator for mappings defined on subsets of a Banach space ad there by obtained a generalization of ***sfpth***. **Petryshym** (1971) points out that α and the Banach space are different although they have a good deal in common. We find some further results involving densifying mapping by **Diaz & Metcalf** (1969); **Kirk** (1971); **Singh & Guerra** (1971); **Singh & Riggio** (1972); **Singh** (1972); **Yadav** (1972); **Singh & Yadav** (1973). **J. Halle** in 1974 obtained a result concerning the continuous dependence of fixed points for densifying mappings.

In many problems of analysis, one encounters operators which may be expressed in the form $T=A+B$, where A is a contraction mapping and B is compact and T itself neither of these properties ? Thus neither the **BCP** nor the ***Sfpth*** applies directly and it becomes desirable to develop fixed point for such situations. **Krasnoselskii** first introduced well known theorem of this kind in 1955. In 1967, **Zabreiko & Krasnoselskii** proved the stronger variation of **Kransnoselskii**'s theorem. **Nashed & Wong** in 1969 gave extensions of **Kransnoselskii's** theorem. Also we have related results due to **Sadovskii** (1967); **Nussbaum** (1969); **Furi & Vignoli** (1970); **Srinivasa Chargulu** (1971); **Singh** (1973). **Browder** (1965) gave example which illustrates that a non-expansive mapping under perturbation by a compact mapping loses it fixed point. Then the question arises, does a non-expansive mapping have a fixed point under any perturbation ? This question has answered affirmatively by **Edmunds** (1967). **Zabreiko, Krasnoselskii & Kachurovskii** (1967) and **Reinermann** (1971).

A comprehensive account of history, properties and applications of convex functions upto 1946 has been given by **Beckenback** (1948). **J.W. Green** (1954)

published an elegant article on convex function. He remarked that convex functions have been consistently of value of Analysis, Geometry and other branches of mathematics, notably Mathematical Economics. **Belluce & Kirk** (1969) **and Singh & Veitch** have given fixed point results for convex functions. **Poljak** (1966) gave a useful fixed point result for strongly quasi convexity that is natural assumption for minimization problems. The concept of associating a distribution function with the point pairs was first introduced by **K. Menger** (1942) under the name "Statistical metric spaces". Shortly, it is named as "Probabilistic Metric Spaces" or **PM-Spaces.** The important paper of **B.Schweizer** and **A.Sklar** in 1960 has given a new impulse to the theory of **PM-Spaces**, introducing the notion of convergence in **PM-Spaces.** The notion of a contraction mapping defined on a **PM-Space** was first defined by **Sehgal** who has also proved that every contraction mapping in a complete **Nebger** Space has a unique fixed points. The notion of C-chainabililty for **PM-Spaces** is first defined by **Sehgal** and **Barucha-Reid**. The notion of **Kuratowski** probabilistic measure of non-compactness was introduced by **Bocsan & Constantin** in 1873, under the name of Kuratowski functions. **V. Istrătescu** in 1974 has given a detailed discussion of **PM-Spaces.**

Finally the existence theorems for ordinary differential equations are due to **Cauchy-Lipschitz**, **Nirenberg** (1953) ; **Peano**, **Picard & Bass** (1958); **Stokes** (1960); **Cronin** (1964); **Edwards** (1965); **Jones** (1965); **Güssefeldt**(1970); **Browder** (1973). Similarly, the existence theorems for partial differential equations are due to **Caccioppoli** (1930) and **Nemyckii** (1936). A fixed point method for finding periodic solutions of dynamical problems was used by Poincare in 1912. In 1920, **Lawson** gave a formulation as an implicit function theorem in abasract spaces and this result is universal, in the sense that the convergence is proved for any arbitrary initial value. A local version of Lawson's theorem was obtained by **Hildebrandt & Grave in** 1929.

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On the Cauchy Problem For a Sobolev Type System in Hydrodynamics of Rotating Fluid with Heat Transfer

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Abstract: The solution and properties of various equations or systems of equations in mathematical physics are generally applied in different fields of natural sciences, such as in the science of oceans, prognosis of weather, theory of hydro-nuclear reactors, etc. One of such systems of equations is the Sobolev system studied by eminent Russian mathematician S.L. Sobolev. Here the solution of Sobolev type homogeneous system of partial differential equations with initial conditions, where the heat transfer is also taken into account, has been constructed in explicit form. For the construction, basically, Fourier-Laplace transformations have been applied. Then, using Duhamel's principle, the solution of the corresponding non-homogeneous system has been found. In the process of investigation, the uniqueness of the obtained solution has been proved and the estimation of the solution in Sobolev space is established by using Mareinkiewicz theorem on multiplicators. The central part is the study of asymptotic behavior of the solution for large time. Remarkable results have been obtained by investigating the improper multiple integral depending on parameters. For this, mainly, the method of stationary phase is used.

1. Introduction

We know that the construction and investigation of mathematical models of physical phenomena constitute the subject of mathematical physics. The solutions and properties of various equations or systems of equations in mathematical physics are generally applied in different fields of natural sciences, such as in the science of Oceans, prognosis of weather, theory of hydro-nuclear reactors, etc. Our everyday life is full of examples of fluid motion, for instance, stirring a cup of tea, flows in rivers, ocean waves, hurricanes and so on. The equations that describe the most fundamental behavior of an inviseid fluid were derived by Euler two and half centuries ago in 1755. Incorporation of the effects of viscosity leads to versions of Euler equations, called Navier-Stokes equations.The idea of wide application of mathematical models of rotating fluid to the study of atmospheric processes belongs to Russian mathematician A. A. Friedman. In the beginning of 20th century. Friedman contributed a series of fundamental works in dynamics of atmospheric processes. Later on, different types of Cauchy problems and initial boundary value problems for hydrodynamics system were studied by various mathematicians, especially, S. L. Sobolev, V. P. Mikhailov, O. A. Ladyzhenskaya, V.N. Maslennikova, M.E. Bogovskii. Eminent Russian mathematicians S. L. Sobolev

initiated the study of a system of partial differential equations during the World War II, when it became necessary to study the stability of trajectory for rotating projectile filled with fluid. Now a days, this system is known as Sobolev system and has the following form [10].

$$\text{(1)} \qquad \begin{cases} \dfrac{\partial \vec{v}}{\partial t} - [\vec{v}, \vec{\omega}] + \nabla P = \vec{F}(x,t), & x \in \Omega\ \mathbb{R}^2,\ t \geq 0 \\ \operatorname{div} \vec{v} = 0 : \end{cases}$$

where $\vec{v}$ – velocity fields

$\vec{\omega}$ – angular velocity of rotation fluid

p – pressure

t – time

$\vec{F}$ – mass density of external forces

V. N. Maslennikova, one of the former research students of S. L. Sobolev, studied the asymptotic behavior of solutions of different linearized systems of hydrodynamics of rotating fluids with and without the consideration of compressibility and viscosity [6, 7, 8, 9]. M. E. Bogovskii, a student of Maslennikova, also studied and is still continuing the study of various types of boundary value problems in hydrodynamics.The study of a Sobolev type system with heat transfer has even more practical applications than the Sovolev system itself, M. L. Marchhuk introduced such a system [5] in his book "Mathematical Models of Circulation in Oceans" in 1980, in which a numerical approach is suggested for solution. Here, a Sobolev type system in hydrodynamics with account of heat transfer is taken under consideration. The solution of a Cauchy problem for the system is constructed in explicit form. In the process of investigating the solution, uniqueness theorem is established for the Cauchy problem solution and the solution is estimated in Sobolev spaces. The most important part of the work is the study of asymptotic behavior of the solution for large time. The system undertaken for study is the following :

$$\text{(2)} \qquad \begin{cases} \dfrac{\partial v_1}{\partial t} - \omega v_2 + \dfrac{\partial P}{\partial x_1} = f_1 \\ \dfrac{\partial v_2}{\partial t} - \omega v_1 + \dfrac{\partial P}{\partial x_2} = f_2 \\ \dfrac{\partial v_3}{\partial t} - \sigma T + \dfrac{\partial P}{\partial x_3} = f_3 \\ \dfrac{\partial T}{\partial t} - \gamma v_3 = f \\ \dfrac{\partial v_1}{\partial x_1} - \dfrac{\partial v_2}{\partial x_2} + \dfrac{\partial v_3}{\partial x_3} = 0 \end{cases}$$

where

v_1, v_2, v_3 – components of velocity $\vec{v}$

p –pressure

T–temperature (deviation of temperature from some standard value T_0 corresponding to the plane $x_3 = 0$)

σ–free convection coefficient (positive constant)

γ–mean gradient of density (positive constant)

ω–constant vector of angular velocity

f_1, f_2, f_3 – components of mass density $\vec{F}$ of external forces

f– heat source density

Without any loss of generality, we can take $\vec{\omega} = (0, 0, \omega)$, The homogeneous system corresponding to (2) is as follows:

$$(2')\qquad \begin{cases} \dfrac{\partial v_1}{\partial t} - \omega v_2 + \dfrac{\partial P}{\partial x_1} = 0 \\ \dfrac{\partial v_2}{\partial t} + \omega v_1 + \dfrac{\partial P}{\partial x_2} - 0 \\ \dfrac{\partial v_3}{\partial t} + \sigma T + \dfrac{\partial P}{\partial x_3} = 0 \\ \dfrac{\partial T}{\partial t} - \gamma v_3 = 0 \\ \dfrac{\partial v_1}{\partial x_1} + \dfrac{\partial v_2}{\partial x_2} + \dfrac{\partial v_3}{\partial x_3} = 0 \end{cases}$$

The solution of (2) is considered in the domain $\Omega = \{(x, t) : x \in \mathbb{R}^3,\ t \geq 0\}$ with the following initial conditions :

$$(3)\qquad \begin{cases} \vec{v}(x,t)\big|_{t=0} = \vec{v}^0(x) \\ T(x,t)\big|_{t=0} = T^0(x) \\ \operatorname{div}\ \vec{v}^0 = (x) \end{cases}$$

The main results of the work are given in the form of four theorems. Theorem 1 is on the explicit solution and other theorems are on the properties of the solution.

2. Construction of Solution

The solution of the Cauchy problem (2′), (3) for homogeneous system is first constructed in explicit form [1]. For this, basically, Fourier-Laplace

transformations have been applied. Then, using Duhamel's principle, the solution of the corresponding non-homogeneous system is found.

In the process of construction of solution different special cases are considered. For the most general case, the following kernels are found.

$$(4)\qquad \begin{cases} \mathcal{K}_1(x,t) = \frac{1}{2\pi^2 r} \int\limits_0^{\pi/2} \cos(t\, g(\psi))\, d\psi \\ \mathcal{K}_2(x,t) = \frac{1}{2\pi^2 r} \int\limits_0^{\pi/2} \frac{\sin(t\, g(\psi))}{g(\psi)}\, d\psi \\ \mathcal{K}_3(x,t) = \frac{1}{2\pi^2 r} \int\limits_0^{\pi/2} \frac{[1-\cos(t\, g(\psi))]}{[g(\psi)]^2}\, d\psi. \end{cases}$$

where

$$g(\psi) = \sqrt{(\omega^2 - \sigma\gamma)\left(\tfrac{\rho}{r}\right)^2 \sin^2\psi + \sigma\gamma}\,,$$
$$\rho = |x'|\,.\ x' = (x_1, x_2).\ r = |x|\,.\ x = (x_1, x_2, x_3).$$

By the help of these kernels, the solution of (2′), (3) is written. It is given by

$$(5a)\quad v_1(x,t) = \int\limits_{\mathbf{R}^3} -\Delta v_1^0(y)\mathcal{K}_1(x-y,t)dy + \int\limits_{\mathbf{R}^3}\left\{\omega\frac{\partial^2 v_2^0(y)}{\partial y_3^2} - \omega\frac{\partial^2 v_2^0(y)}{\partial y_2 \partial y_3} + \sigma\frac{\partial^2 T^0(y)}{\partial y_1\partial y_3}\right\}\times \mathcal{K}_2\ (x - y, t)\ dy + \int\limits_{\mathbf{R}^3}\left\{\sigma\gamma\frac{\partial^2 v_1^0(y)}{\partial y_3^2} - \sigma\gamma\frac{\partial^2 v_2^0(y)}{\partial y_1\partial y_2} + \sigma\omega\frac{\partial^2 T^0(y)}{\partial y_2\partial y_3}\right\}\times\mathcal{K}_3\ (x-y, t)\ dy$$

$$(5b)\quad v_2(x,t) = \int\limits_{\mathbf{R}^3} -\Delta v_2^0(y)\mathcal{K}_1(x-y,t)dy + \int\limits_{\mathbf{R}^3}\left\{-\omega\frac{\partial^2 v_1^0(y)}{\partial y_3^2} - \omega\frac{\partial^2 v_3^0(y)}{\partial y_1\partial y_3} + \sigma\frac{\partial^2 T^0(y)}{\partial y_2\partial y_3}\right\}\times\mathcal{K}_2\ (x-y, t)\ dy + \int\limits_{\mathbf{R}^3}\left\{\sigma\gamma\frac{\partial^2 v_1^0(y)}{\partial y_1\partial y_2} + \sigma\gamma\frac{\partial^2 v_2^0(y)}{\partial y_1^2} - \sigma\omega\frac{\partial^2 T^0(y)}{\partial y_1\partial y_3}\right\}\times\mathcal{K}_3\ (x-y, t)\ dy$$

$$(5c)\quad v_3(x,t) = \int\limits_{\mathbf{R}^3} -\Delta v_3^0(y)\mathcal{K}_1(x-y,t)dy + \int\limits_{\mathbf{R}^3}\left\{\omega\frac{\partial^2 v_1^0(y)}{\partial y_2\partial y_3} - \omega\frac{\partial^2 v_2^0(y)}{\partial y_1\partial y_3} - \sigma\left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}\right)T^0(y)\right\}\mathcal{K}_2(x-y, t)dy$$

$$(5d)\quad P(x,T) = \int\limits_{\mathbf{R}^3}\left\{\omega\frac{\partial^2 v_1^0(y)}{\partial y_2} - \omega\frac{\partial v_2^0(y)}{\partial y_1} + \sigma\frac{\partial T^0(y)}{\partial y_3}\right\}\mathcal{K}_1\ (x-y, t)\ dy$$

$$-\int_{\mathbb{R}^3}\left\{\sigma\gamma\frac{\partial v_1^0(y)}{\partial y_1}+\sigma\gamma\frac{v_2^0(y)}{\partial y_2}+\omega^2\frac{\partial v_3^0(y)}{\partial y_3}\right\}\mathcal{K}_2(x-y,t)\,dy$$

$$+\,\omega\int_{\mathbb{R}^3}\left\{\sigma\gamma\frac{v_1^0(y)}{\partial y_2}-\sigma\gamma\frac{\partial v_2^0(y)}{\partial y_1}+\sigma\omega\frac{\partial T^0(y)}{\partial y_3}\right\}\mathcal{K}_3(x-y,t)\,dy$$

(5e) $$T(x,t)=\int_{\mathbb{R}^3}-\Delta T^0(y)\mathcal{K}_1(x-y,t)dy-\gamma\int_{\mathbb{R}^3}-\Delta v_3^0(y)\mathcal{K}_2(x-y,t)dy$$

$$+\,\omega\int_{\mathbb{R}^3}\left\{\gamma\frac{\partial^2 v_1^0(y)}{\partial y_2\partial y_3}-\gamma\frac{\partial^2 v_2^0(y)}{\partial y_1\partial y_3}+\omega\frac{\partial^2 T^0(y)}{\partial v_3^2}\right\}\mathcal{K}_3(x-y,t)\,dy.$$

Now, the solution of the non-homogeneous system (2) with the same initial conditions, i.e., the solution of the Cauchy problem (2), (3) is found from the solutions (5) of the corresponding homogeneous system by using the Duhamel's principle.

In this connection, for the external force $\vec{F}=(f_1,f_2,f_3,)\in L_2(R^3)$, we assume, without loss of generality that

$$\operatorname{div}\vec{F}=0$$

The solution of the Cauchy problem (2), (3) has the following form:

$$v_1^*(x,t)=v_1(x,t)+\tilde{v}_1(x,t)$$

where

(6) $$\tilde{v}_1(x,t)=\int_0^t\int_{\mathbb{R}^3}-\Delta f_1(y,\tau)\mathcal{K}_1(x-y,t-\tau)\,dy\,d\tau$$

$$+\int_0^t\int_{\mathbb{R}^3}\left\{\omega\frac{\partial^2 f_2(y,\tau)}{\partial y_3^2}-\omega\frac{\partial^2 f_3(y,\tau)}{\partial y_2\partial y_3}+\sigma\frac{\partial^2 f(y,\tau)}{\partial y_1\partial y_3}\right\}\mathcal{K}_2(x-y,t-\tau)\,dy\,d\tau$$

$$+\int_0^t\int_{\mathbb{R}^3}\left\{\sigma\gamma\frac{\partial^2 f_1(y,\tau)}{\partial y_2^2}-\sigma\gamma\frac{\partial^2 f_2(y,\tau)}{\partial y_1\partial y_2}+\sigma\omega\frac{\partial^2 f(y,\tau)}{\partial y_2\partial y_3}\right\}\mathcal{K}_3(x-y,t-\tau)dyd\tau$$

Expressions for $v_2^*(x,t)$, $v_3^*(x,t)$, $P^*(x,t)$ and $T^*(x,t)$ are found in the same way as that in finding $v_1^*(x,t)$.

So, we have the following result.

Theorem 1. *Let the initial data $\vec{v}^0(x)$ and $T^0(x)$ in* (3) *be sufficiently smooth and decrease as $|x|\to\infty$. Then the solution of* (2′), (3) *for $\omega^2\geq\sigma\gamma$ is given by* (5a) – (5c). *The solution of* (2). (3) *with additional condition div $\vec{F}=0$, where $\vec{F}=(f_1,f_2,f_3)$ is as follows*

$$v_1^*(x,t) = v_1(x,t) + \tilde{v}_1(x,t),$$

where $\tilde{v}_1(x,t)$ is given by (6). Other components $v_2^*(x,t)$, $v_3^*(x,t)$, $P^*(x,t)$ have similar forms

2. Uniqueness and Estimation of Solution

In the process of investigation of the obtained solution, uniqueness theorem is proved and then the estimates of solution are established in Sobolev spaces by using Marcinkiewez theorem on multiplicators [4].

For Cauchy problem (2′), (3), the following uniqueness theorem holds [2].

Theorem 2 *The solutions $\vec{v}(x,t)$ and $T(x,t)$ of the Cauchy problem* (2′), (3) *are given in L_2, while the solution $P(x,t)$ is determined up to a function of t. In addition, ∇P is again unique in L_2.*

The following theorem on the estimation of the solutions holds true [2].

Theorem 3. *If the initial data $\vec{v}^0(x)$, $T^0(x) \in W_p^\ell(\mathbb{R}^3)$, then the following opriori estimates for the solutions of the Cauchy problem* (2′), (3) *take place:*

$$\|\vec{v}\|_{W_{p,t,x}^{k+1,\ell}(\mathbb{R}_H^4)} + \|T\|_{W_{p,t,x}^{k+1,\ell}(\mathbb{R}_H^4)} + \|\nabla_x P\|_{W_{p,t,x}^{k,\ell}(\mathbb{R}_H^4)}$$
$$\le C_H(\ell,p,k)\,[\|\vec{v}^0\|_{W_p^\ell(\mathbb{R}^3)}\ \|T^0\|_{W_p^\ell(\mathbb{R}^3)}],$$

where $\mathbb{R}_H^4 = \{(x,t) : x \in \mathbb{R}^3, 0 \le t \le H\}$.

In addition, if $\vec{F} \in W_{p,t,x}^{k+1,\ell}(\mathbb{R}_H^4)$, then for the solutions of (2), (3), we will have;

$$\|\vec{v}\|_{W_{p,t,x}^{k+1,\ell}(\mathbb{R}_H^4)} + \|T\|_{W_{p,t,x}^{k+1,\ell}(\mathbb{R}_H^4)} + \|\nabla_x P\|_{W_{p,t,x}^{k,\ell}(\mathbb{R}_H^4)}$$
$$\le C_H(\ell,p,k)\,[\|\vec{v}^0\|_{W_p^\ell(\mathbb{R}^3)} + \|T^0\|_{W_p^\ell(\mathbb{R}^3)} + \|\vec{F}\|_{W_{p,t,x}^{k+\ell}(\mathbb{R}_H^4)}]$$

4. Asymptotic Behavior of Solution

When we consider a Cauchy problem, that is, large volumes of rotating fluids, there arise the problems of determining the behavior of solution as time $t \to \infty$. It is very important, even in numerical methods, to study the behavior of the solution for large time. In classical problems of mathematical physics (for example, the heat conduction equation), the question of whether a solution of the Cauchy problem for a homogeneous equation with smooth finite data tends to zero as $t \to \infty$, and at what rate, is usually simply solved. In the case of the system (2′), this question is not so simple because of the fact that the system contains a constantly operating

Coriolis term and therefore the equation of the damping of solution as $t \to \infty$ for the homogeneous system (2′) even when the initial data are sufficiently smooth and well decreasing as $|x| \to \infty$, requires further investigation. To obtain the asymptotic expansion of the solution, the kernels (4) are expressed in terms of Bessel's functions as follows:

$$
(7)\quad \begin{cases}
K_1(x,t) = \frac{1}{4\pi r}\left[J_0\left(\alpha\left(\frac{\rho}{r}\right)t\right) - \beta t \int\limits_0^t J_0\left(\alpha\left(\frac{\rho}{r}\right)\eta\right) \frac{J_1(\beta\sqrt{t^2-\eta^2})}{\sqrt{t^2-\eta^2}}\, d\eta \right] \\
K_2(x,t) = \frac{1}{4\pi r} \int\limits_0^t J_0\left(\alpha\left(\frac{\rho}{r}\right)\eta\right) J_0(\sqrt{t^2-\eta^2})\, d\eta \\
K_3(x,t) = \frac{1}{4\pi^2 r} \int\limits_0^{\pi/2} \frac{1}{\alpha^2\left(\frac{\rho}{r}\right)^2 \sin^2\psi + \beta^2} \left[1 - \cos\left\{\alpha\left(\frac{\rho}{r}\right)\sin\psi t\right\} \right. \\
\qquad \left. + \beta t \int\limits_0^t \cos\left[\alpha\left(\frac{\rho}{r}\right)\eta\sin\psi\right] \frac{J_1(\beta\sqrt{t^2-\eta^2})}{\sqrt{t^2-\eta^2}}\, d\eta \right] d\psi.
\end{cases}
$$

Then, in investigating the convolutions in the solution's representation, change of variables, integration by parts, Chebyshev polynomials and their properties are used repeatedly. The integrals are approximated, mainly, by the method of stationary phase [3] and a Watson type lemma proved by Fedoryk is also used.

Before stating the theorem on asymptotics, we need to introduce the following condition:

Condition *A*. The initial data $\vec{v}^0(x)$ is said to satisfy condition *A*, if $\exists$ a positive constant C_3 such that $\forall$ multi-index β, $2 \le |\beta| \le 2\ell + 4$.

$$\int\limits_{\mathbb{R}^3} (1+|x|)^{|\beta|-2}\, |\mathcal{D}_x^\beta \vec{v}^0(x)|\, dx \le C_3$$

where ℓ is some given positive integer.

Theorem 4. *If $\omega^2 = \sigma\gamma$, solution of (2′), (3) is periodic in t. In case $\omega^2 = \sigma\gamma$, for initial data $\vec{v}^0(x)$ and $T^0(x)$ from $C^{2\ell+4}(\mathbb{R}^3) \cap W_p^\ell(\mathbb{R}^3)$, satisfying condition A, the solution of (2′), (3) satisfies the following properties*:

1. The components v_1, v_2, P and T stabilise at a rate $(\ln t)t^{-1/2}$ as $t \to \infty$ to some functions $v_1^*(x)$. $v_2^*(x)$. $P^*(x)$ and $T^*(x)$, respectively, which are determined completely by the prescribed data.
2. The components v_3 vanishes at a rate $t^{-1/2}$, as $t \to \infty$ at an arbitrary compact $K \subset \mathbb{R}^3$.

Here the functions $v_1^*(x)$. $v_2^*(x)$. $P^*(x)$ and $T^*(x)$ and $T^*(x)$ are given by

$$v_1^*(x) = \frac{\sqrt{\sigma\gamma}}{4\pi} \int\limits_{\mathbf{R}^3} \left\{ \frac{\partial^2 v_1^0(x-y)}{\partial x_2^2} - \frac{\partial^2 v_2^0(x-y)}{\partial x_1 \partial x_2} + \frac{\omega}{\gamma} \frac{\partial^2 v_1^0(x-y)}{\partial x_1 \partial x_3} \right\} \frac{dy}{\sqrt{\omega^2 \rho^2 + \sigma\gamma\, y_3^2}}$$

$$v_1^*(x) = \frac{\sqrt{\sigma\gamma}}{4\pi} \int\limits_{\mathbf{R}^3} \left\{ \frac{\partial^2 v_2^0(x-y)}{\partial x_1^2} - \frac{\partial^2 v_1^0(x-y)}{\partial x_1 \partial x_2} + \frac{\omega}{\gamma} \frac{\partial^2 T^0(x-y)}{\partial x_1 \partial x_3} \right\} \frac{dy}{\sqrt{\omega^2 \rho^2 + \sigma\gamma\, y_3^2}}$$

$$P^*(x) = \frac{\omega}{4\pi\sqrt{\sigma\gamma}} \int\limits_{\mathbf{R}^3} \left\{ \frac{\partial^2 v_1^0(x-y)}{\partial x_2} - \frac{\partial^2 v_2^0(x-y)}{\partial x_1} + \frac{\omega}{\gamma} \frac{\partial^2 T^0(x-y)}{\partial x_2} \right\} \frac{dy}{\sqrt{\omega^2 \rho^2 + \sigma\gamma\, y_3^2}}$$

$$v_1^*(x) = \frac{\sqrt{\sigma\gamma}}{4\pi\sigma} \int\limits_{\mathbf{R}^3} \left\{ \frac{\partial^2 v_1^0(x-y)}{\partial x_2^2 \partial x_3} - \frac{\partial^2 v_2^0(x-y)}{\partial x_1 \partial x_3} + \frac{\omega}{\gamma} \frac{\partial^2 T^0(x-y)}{\partial x_3^2} \right\} \frac{dy}{\sqrt{\omega^2 \rho^2 + \sigma\gamma\, y_3^2}}$$

5. Concluding Remarks

For any physical problem to be well-posed, that is meaningful, the existence, uniqueness and stability of its solution are required. Theorems 1 to 4 show that our problems is well-posed. The results obtained for such a problem can be applied in various fields, such as, the science of atmosphere and oceans, the weather forecasting, theory of hydro-nuclear reactors, etc.

Due to the advent of powerful computers and advanced numerical methods, many problems now can be solved numerically. But at the same time, analytical methods are not to be underestimated as the analytical and numerical solutions have to complements each other. Explicit solution and its asymptotics obtained in this work give the possibility to determine the initial data effectively in each step of calculation, which minimizes the time required to solve the problem numerically.

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A Note On Stokes Drag On Axi-symmetric bodies : A New Approach

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Abstract: In the recent paper [1], author have proposed a simple formulae for evaluating the axial and transverse Stokes drag on axially symmetric bodies. Continuing the efforts in this regard, the axial and transverse drag forces have been evaluated for the cassini body, hypo-cycloidal body and cylindrical capsule with circular cross-section of radius 'b' and semispeherical caps on both ends. Further, the moments on these bodies have also been calculated.

Keywords: Stokes drag, axially symmetric body, cassini body, hypocycloidal body, cylindrical capsule.

Introduction:

In the recent paper [Datta and Srivastava, 1999, 1], authors have proposed a simple formulae, based on the integral [p.122,2] used to evaluate drag on a sphere, for finding the axial and transverse Stokes drag on axi-symmetric bodies.

The axial flow

The drag on body, when it is situated in axi-symmetric Stokes flow with uniform stream U_x along x-axis is given as [1].

(1.1) $$F_x = \frac{1}{2}\frac{\lambda(y_{\max.})^2}{h},$$

where

(1.2) $$\lambda = 6\pi\mu U_x$$

and

(1.3) $$h = \left(\frac{3}{8}\right)\int_0^{\pi} R\sin^3\alpha \, d\alpha$$

Here, R is the intercepting length between the point on the meridional curve and axis of symmetry (x-axis) of the body and α is the slope of normal [See figure 1]. In cartesian coordinates, h can be expressed as

(1.4) $$h = \left(-\frac{3}{4}\right)\int_0^{a} \frac{yy''}{(1+y'^2)^2}\, dx,$$

where, $x = a$ is the maximum axial length and dashes represents derivatives with respect to x.

The transverse flow

Let us consider an axially symmetric body [see figure 1] placed in a uniform stream U_y along transverse axis (y-axis). The Stokes drag on this body is given to be [1]

$$F_y = \frac{1}{2} \cdot \frac{\lambda (y_{\max} \cdot)^2}{h_y}, \tag{1.5}$$

where

$$\lambda = 6\pi\mu U_y$$

and

$$h_y = \frac{3}{16} \int_0^{\pi} (2R \sin\alpha - \sin^3\alpha)\, d\alpha, \tag{1.6}$$

in cartesian coordinates, it can be expressed as

$$h_y = \left(\frac{3}{8}\right) \int_0^{a} \left[\frac{yy''}{(1+y'^2)^2} - \frac{yy''}{(1+y'^2)^2}\right] dx. \tag{1.7}$$

The Moment

The moment on the axially symmetric body rotating slowly with uniform angular velocity Ω about axis of symmetry is given as [1]

$$M_x = \left(\frac{2}{3}\right) \left[\frac{(y_{\max.})^2 \Omega}{U}\right] F_x, \tag{1.8}$$

where, F_x is the axial drag on body. The result (1.1, 1.5, 1.8) have already been used for the bodies (viz; sphere, spheroid, deformed sphere, cycloidal body of revolution and egg-shaped body) and are given paper [1]. Now, in this continuation, these results have been used for cassini body of revolution, hyp-ocycloidal body of revolution and cylindrical capsule having semi-spherical caps with same radius. It has been found here that the results of Stokes drag and moments are new and never existed in the literature.

2. Flow past cassini body of revolution

Let us consider the cassini body (figure 2) obtained by revolving the curve

$$y^2 = \left(\frac{2}{3}\right) (1+3x^2)^{\frac{1}{2}} - x^2 - \frac{1}{3}, \quad 0 \le x \le 1, \tag{2.1}$$

about x-axis (axis of symmetry).

By using the (1.1) together with (1.4), the axial drag on cassini body will be, with $y_{\max.} = 0.577$

(2.2) $$F_x \approx 0.8\lambda, \quad \lambda = 6\pi\mu U_x$$

and the transverse Stokes drag on cassini body can be easily obtained by using (1.5) together with (1.6), $y_{\max.} = 0.577$

(2.3) $$F_y \approx 0.82\,\lambda, \quad \lambda = 6\pi\mu U_y$$

Also, by using (1.8), the moment on cassini body rotating with angular velocity Ω about axis of symmetry is given to be, $y_{\max.} = 0.577$

(2.4) $$M_x \approx 1.066\ \pi\mu\,\Omega$$

3. Flow past hypocycloidal body of revolution

Let us consider the hypocycloidal body (figure 3) obtained by the curve

(3.1) $$y^2 = -3x^2 + (1 + 8x^4)^{\frac{1}{2}}, \qquad 0 \le x \le 1,$$

about axis of symmetry (x-axis).

By using (1.1) together with (1.4), $y_{\max.} = 1.0$, the axial Stokes drag on hypocycloidal body will be given to be

(3.2) $$F_x \approx 1.044\,\lambda, \quad \lambda = 6\pi\mu U_x,$$

and the transverse Stokes drag, with same y, on this body can be obtained by using (1.5) and (1.6)

(3.3) $$F_y \approx 1.32\,\lambda, \quad \lambda = 6\pi\mu U_y$$

Also, the moment on hypocycloidal body rotating with angular velocity Ω about the axis of symmetry is given to be, with the help of (1.8)

(3.4) $$M_x \approx 4.176\,\pi\mu\,\Omega.$$

4. Flow past cylindrical capsule

Let us consider the cylindrical capsule (figure 4) with semi-spherical caps on both ends having same radius 'b', obtained by revolving the curves (PA, the circular segment, AA', the line segment, $A'P'$, again circular segment)

(4.1) $$\left.\begin{array}{llll} PA, & x = b\cos t, & y = b\sin t, & 0 \le t \le \pi/2 \\ AA', & y = b, & \theta = \pi/2 & \\ A'P', & x = b\cos t, & y = b\sin t, & \pi/2 \le t \le \pi \end{array}\right\}$$

about the axis of symmetry, x-axis.

By using (1.1) and $y_{\max.} = b$, the axial drag on the capsule is comes out to be

(4.2) $$F_x = 6\pi\,\mu U_x b$$

and the transverse drag on the body is given by using (1.5), $y_{\max.} = b$,

(4.3) $$F_y = 6\pi\,\mu U_y b.$$

Also, the results for moment on cylindrical capsule rotating with small angular velocity Ω about the axis of symmetry with $y_{max.} = b$, by using (1.8) comes out to be

$$M_x = 4\pi\,\mu b^3\Omega.$$

These results (4.2, 4.3, 4.4) are same as to that of sphere, it may be happen due to the fact that straight line segment (AA') occurred in the meridional curve (4.1) which, does not contribute in the drag force.

It should be kept in mind, while using these results, that the numerical value of integral (1.4) involve the error of $o(h^3)$, due to the Simpson's one third rule, where 'h' is the step length. Therefore, all the results of drags and moments for various proposed axially symmetric bodies are in approximation, but to a valid limits.

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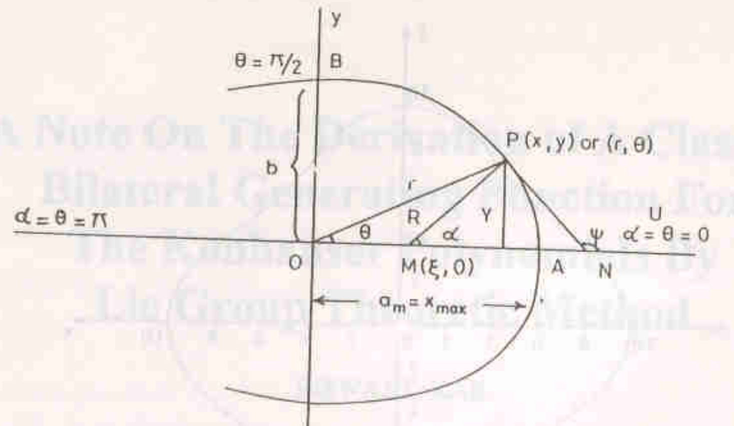

(PN, PM are tangent and normal)

Fig. 1 Geometry of axially symmetric body.

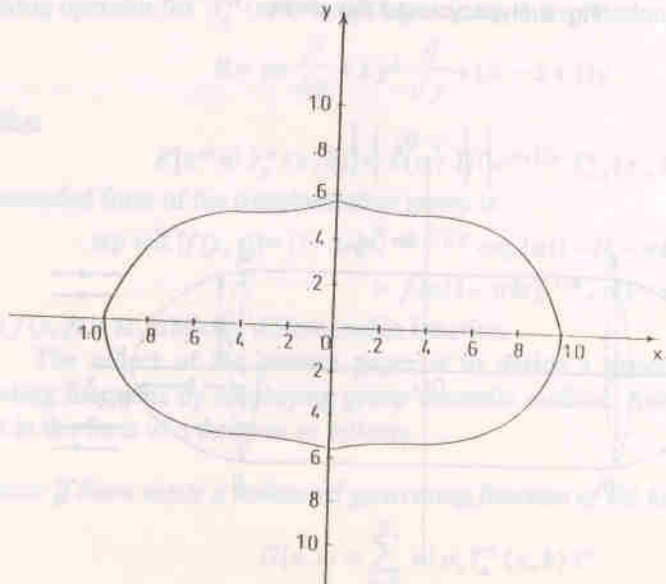

Fig. 2 Cassini curve (cassini body of revolution).

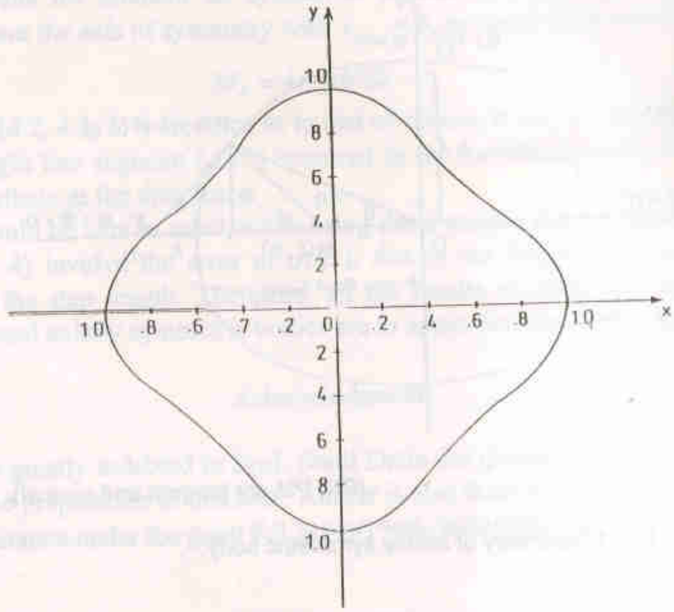

Fig. 3 Hypocycloidal like profile.

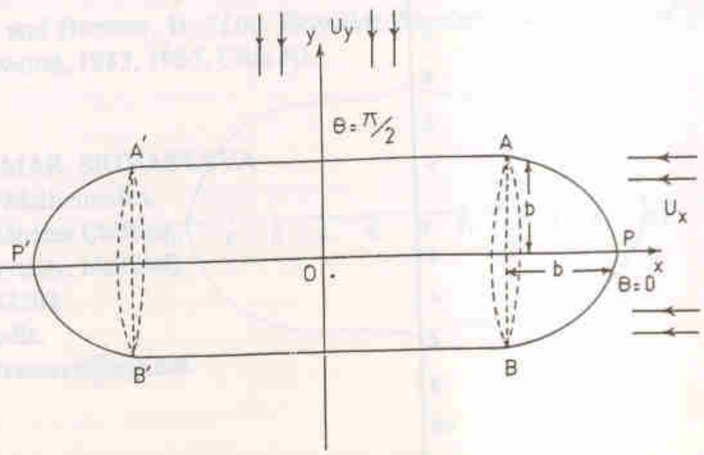

Fig. 4 Cylindrical body with semispherical caps on the ends.

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A Note On The Derivation of A Class of Bilateral Generating Function For The Konhauser Polynomials By Lie Group Theoretic Method

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Abstract: A new class of bilateral generating function for the Konhauser polynomials $Y_n^{\alpha}(x,k)$ [3] is obtained.

Keywords: Special Function, generating functions, Lie group

1. introduction:

Applying L. Weisner's group theoretic method [2] we obtained the following operator for $Y_n^{\alpha}(x, k)$ by giving suitable interpretation to n.

$$R = yx\frac{\partial}{\partial x} + ky^2\frac{\partial}{\partial y} + (\alpha - x + 1)y \tag{1.1}$$

such that

$$R[e^{ny}n!\,Y_n^{\alpha}(x;k)] = k(n+1)!\,e^{(n+1)y}\,Y_{n+1}^{\alpha}(x;k) \tag{1.2}$$

The extended form of the transformation group is

$$\exp wR\,[f(x,t)] = (1-wkt)^{-(\alpha+1)/K}\exp[x\{1-(1-wkt)^{-1/k}\}] \times f(x(1-wkt)^{-1/K}, t(1-wkt)^{-1}) \tag{1.3}$$

where $f(x, t)$ is an arbitrary differentiable function.

The object of the present paper is to derive a general class of bilateral generating functions by employing group theoretic method. Actually our results can be put in the form of a theorem as follows

Theorem: *If there exists a unilateral generating function of the form*

$$G(x,t) = \sum_{n=0}^{\infty} n!\,a_n Y_n^{\alpha}(x,k)\,t^n \tag{1.4}$$

then the following class of bilateral generating functions will hold

$$(1.5) \qquad (1-kt)^{-(\alpha+1)/K}\exp[x\{1-(1-kt)^{-1/k}\}]\,G[x(1-kt)^{-1/k}\},\,ty(1-kt)^{-1}]$$
$$=\sum_{p=0}^{\infty}\sum_{n=0}^{p} a_n\,\frac{k^{(p-n)}p!}{(p-n)!}\,t^p y^n Y_p^{\alpha}\,(x,k)$$

Importance of the result (1.5) is that whenever one knows a generating function of the type (1.4) for a particular value of a_n then the corresponding bilateral generating functions can at once be written from (1.5).

Thus, one can derive a large number of bilateral generating functions by setting different values to a_n.

2. Proof of the theorem:

Let

$$(2.1) \qquad G(x,t)=\sum_{n=0}^{\infty} n!\,a_n Y_n^{\alpha}(x,k)\,t^n$$

Replacing t by ty, we have

$$G(x,t)=\sum_{n=0}^{\infty} n!\,a_n Y_n^{\alpha}(x,k)\,t^n y^n$$

we operate both sides by exp wR and hence

$$(2.2) \qquad \exp wR\,[G(x,ty)]=\exp wR\sum_{n=0}^{\infty} n!\,a_n\,Y_n^{\alpha}(x,k)\,t^n y^n$$

Left hand side of (2.2) becomes

$$(1-wkt)^{-(\alpha+1)/K}\exp[x\{1-(1-wkt)^{-1/k}\}]\,f\,(x\{1-(1-wkt)^{-1/k},\,ty(1-wkt)^{-1})$$

On the other hand, right side of (2.2) reduces to

$$\sum_{p=0}^{\infty}\frac{w^p R^p}{p!}\sum_{n=0}^{\infty} n!\,a^n Y_p^{\alpha}(x,k)\,t^n y^n$$
$$=\sum_{p=0}^{\infty}\sum_{n=0}^{p} a_n\,\frac{w^p}{p!}\,k^p(n+p)!\,t^{(n+p)}\;y^n\;Y_{n+p}^{\alpha}(x,k)$$
$$=\sum_{p=0}^{\infty}\sum_{n=0}^{p} a_n\,\frac{w^{(p-n)}}{(p-n)!}\,p!\,k^{(p-n)}\;t^p\;y^n\;Y_p^{\alpha}\;Y_p^{\alpha}\;(x,k)$$

equating and substituting $w=1$, we obtain (1.5)

Application : Setting $k=1$, we have the following bilateral generating function for $L_n^{\alpha}(x)$.

If $\;G(x,t)=\sum_{n=0}^{\infty} a_n n!\,L_n^{\alpha}(x)\,t^n$

then we have

$$(1-t)^{-(\alpha+1)} \exp\left[\frac{xt}{1-t}\right] G[x(1-t)^{-1}, ty(1-t)^{-1}]$$

$$= \sum_{p=0}^{\infty}\sum_{n=0}^{p} a_n \frac{p!}{(p-n)!} t^p y^n L_p^{\alpha}(x)$$

This was derived by *Al Salam* [1]

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Structure of Ultradistribution

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Abstract: Structure of ultradistribution belonging to $\left(H^{b_q,B}_{\mu,a_k,A}\right)$ has been investigated

1. Introduction:

The classical Hankel transformation is defined by

$$(1.1) \qquad (h_\mu\phi)(x)=\int_0^\infty \phi(y)\sqrt{xyJ_\mu(xy)dy}, \quad \left(\mu=-\frac{1}{2}\right)$$

where J_μ is the Bessel function of the first kind and order μ. Zemanian [5] introduced the spaces H'_μ and its dual H'_μ to extend the above transformation to the space of distributions belonging to H'_μ. Following the technique of Gel'fand and Shilov [1], Lee [2] defined spaces $H_{\mu,\alpha,A}, H^{\beta,B}_\mu$ and $H^{\beta,B}_{\mu,\alpha,A}$, Pathak and Pandey [3] introduced spaces $H_{\mu,a_k,A}$, $H^{b_q,B}_\mu$ and $\left(H^{b_q,B}_{\mu,a_k,A}\right)$ generalizing the aforesaid Lee spaces. The dual spaces of $H_{\mu,a_k,A}$, $H^{\beta_q,B}_\mu$ and $\left(H^{b_q,B}_{\mu,a_k,A}\right)$ are $(H_{\mu,a_k,A})'$, $(H^{b_q,B}_\mu)'$ and $\left(H^{b_q,B}_{\mu,a_k,A}\right)$. The elements of the dual spaces are called ultradistributions. In the present work, we study the structure of ultradistribution.

Throughout the paper, I denote the open interval $(0, \infty)$ and all the testing functions herein are defined on I. We recall here the spaces $H_{\mu,a_k,A}$, $H^{b_q,B}_\mu$ and $H^{b_q,B}_{\mu,a_k,A}$ defined by Pathak and Pandey.

Let $\{a_k\}_{k\in IN}$ and $\{b_q\}_{q\in IN}$ be arbitrary sequence of positive numbers which satisfy the following conditions:

(i) Logarithmic Convexity

$$(1.2) \qquad a_k^2 \le a_{k-1}\, a_{k+1}, \qquad k\ge 1$$

$$(1.3) \qquad a_q^2 \le b_{q-1}\, a_{q+1}, \qquad q\ge 1$$

Immediate consequences of these inequalities are

(1.4) $\quad a_p a_k \le a_0 a_{p+k}; \qquad p, k = 0,1,2$

(1.5) $\quad b_p b_q \le b_0 a_{p+q}; \qquad p, q = 0,1,2$

(ii) Stability under multiplication by x

There are constants c, h, c_1 and h_1 such that $k \ge 0,\ q \ge 0$

(1.6) $\quad a_{k+1} \le ch^k\, a_k$

(1.7) $\quad b_{q+1} \le c_1 h_1^q\, b_q$

(iii) Non-quasi analyticity

(1.8)
$$\sum_{q=1}^{\infty} \frac{b_{q-1}}{b_q} < \infty$$

(iv) Stability under Hankel transformations

Conditions (1.6) and (1.7) are replaced by the following stronger conditions:

(1.9) $\quad a_{r+k} \le LR^{r+k}\, a_r a_k \qquad$ for all $r, k \ge 0$

(1.10) $\quad b_{r+q} \le R_1^{r+q}\, b_r b_q \qquad$ for all $r, q \ge 0$

where L, R, L_1 and R_1 are positive constants.

Pathak and Pandey have introduced spaces $H_{\mu,a_k,A}$, $H_{\mu}^{b_q,B}$ and $H_{\mu,a_k,A}^{b_q,B}$. These spaces are defines as follows :

Let ϕ be an infinitely differentiable function on $I = (0, \infty)$. Then

(a) $\phi \in H_{\mu,a_k,A}$ if and only if

$$\gamma_{k,q}^{\mu}(\phi) = \sup_{x \in I} \left| x^k (x^{-1}D^q (x^{-\mu-1/2})\phi(x) \right| \le C_q^{\mu} (A+\delta)^k a_k; \qquad k, q \in IN_0$$

where the constants A and C_q^{μ} depend on ϕ and $\delta > 0$ is an arbitrary constant

(b) $\phi \in H_{\mu}^{b_q,B}$ if and only if

$$\gamma_{k,q}^{\mu}(\phi) = \sup_{x \in I} \left| x^k (x^{-1}D^q (x^{-\mu-1/2})\phi(x) \right| \le C_k^{\mu} (B+\rho)^q bq, \qquad k, q \in IN_0$$

where the constant C_q^{μ} depend on ϕ and $\rho > 0$ is arbitrary constant.

(c) $\phi \in H_{\mu,a_k,A}^{b_q,B}$ if and only if

$$\begin{aligned} \gamma_{k,q}^{\mu}(\phi) &= \sup_{x \in I} \left| x^k (x^{-1}D)^q (x^{-\mu-1/2})\phi(x) \right| \\ &\le C^{\mu} (A+\delta)^k (B+\rho)^q a_k b_q, \qquad k, q \in IN_0 \end{aligned}$$

where δ and ρ are as above and c^{μ}, A and B are certain positive constants depending on ϕ.

2. Structure of ultradistribution

In this section we shall investigate the structure of ultradistribution belonging to $\left(H_{\mu,a_k,A}^{b_q,B}\right)'$.

Theorem 2.1 *A linear functional u defined on* $H_{\mu,a_k,A}^{b_q,B}$ *belongs to* $\left(H_{\mu,a_k,A}^{b_q,B}\right)'$ *if and only if there exist* $r \in IN$ *and function* $f_p \in L^{\infty}(I)$ $(0 \le p \le r)$ *such that*

$$u = \sum_{p=0}^{r} x^{-p-1/2}\left(-D\frac{I}{x}\right)^{q} x^k f_p.$$

Proof: Let $u \in \left(H_{\mu,a_k,A}^{b_q,B}\right)'$. Then there exists a positive constant c and a non-negative integer r such that for every $\phi \in H_{\mu,a_k,A}^{b_q,B}$.

$$(2.1) \qquad |<u,\phi>| = c \max_{\substack{0\le k\le r\\ 0\le k\le r}} \sup_{x\in I} \left| x^k (x^{-1}D)^q x^{-\mu-1/2}\phi(x)\right|$$

Let Γ denote the direct sum of $(r+1)$ copies of $L'(I)$ normed with

$$\left|(f_j)_{0\le j\le r}\right|_1 = \max_{0\le j\le r} \|f_j\|_1$$

and β be the denote the direct sum of $(r+1)$ copies of $L^{\infty}(I)$ normed with

$$\left|(f_j)_{0\le j\le r}\right|_{\infty} = \sum_{j=0}^{r} \|f_j\|_{\infty}$$

Now consider the mapping

$$F : H_{\mu,a_k,A}^{b_q,B} \to \Gamma$$

$$\phi \to F(\phi) = \left(x^k (x^{-1}D)^q \; x^{-\mu-1/2} \; \phi(x)\right)_{0\le q\le r}$$

Clearly the mapping is one-one.

Define the functional L on $F\left(H_{\mu,a_k,A}^{b_q,B}\right) \subset \Gamma$ by

$$<L, F(\phi)> = <u,\phi> \text{ for } \phi \in H_{\mu,a_k,A}^{b_q,B}$$

L is continuous from F $\left(H_{\mu,a_k,A}^{b_q,B}\right)$ into C by virtue of (1), when $F\left(H_{\mu,a_k,A}^{b_q,B}\right)$ is equipped with the topology induce by Γ. By Hahn- Banach's theorem, we can extend continuously upto Γ without increasing the norm. This extension is also denoted by L. Since $L \in \Gamma'$, therefore for $(g_p)_{p=0}^{r} \in \Gamma$ and for certain $f_p \in L^{\infty}$, Reisz's representation theorem gives

$$< L, (g_p)_{p=0}^{r} > = \sum_{p=0}^{r} \int_0^{\infty} f_p(x) g_p(x) dx$$

Now,

$$< u, \phi > = < L, F(\phi) >$$

$$= \sum_{p=0}^{r} \int_{0}^{\infty} f_p(x)\, x^k (x^{-1}D)^q\, x^{-\mu-1/2}\, \phi(x) dx$$

$$= < \sum_{p=0}^{r} x^{-\mu-1/2} \left(-D\frac{1}{x}\right)^q x^k f_p, \phi >$$

$$\therefore u = \sum_{p=0}^{r} x^{-\mu-1/2} \left(-D\frac{1}{x}\right)^q x^k f_p,$$

Conversely, let

$$< u, \phi > = \sum_{p=0}^{r} \int_{0}^{\infty} f_p(x) x^k (x^{-1}D)^q x^{-\mu-1/2}\, \phi(x) dx, \left(\phi(x) \in H^{b_q, B}_{\mu, a_k, A}\right)$$

Then

$$|< u, \phi >| = \sum_{p=0}^{r} \int_{0}^{\infty} \left| f_p(x) x^k (x^{-1}D)^q x^{-\mu-1/2} \phi(x) \right| dx$$

$$\leq \sum_{p=0}^{r} \|f_p\| \int_{0}^{\infty} \left| x^k (x^{-1}D)^q\, x^{-\mu-1/2}\, \phi(x) \right| dx$$

Now,

$$\int_{0}^{\infty} \left| x^k (x^{-1}D)^q\, x^{-\mu-1/2}\, \phi(x) \right| dx$$

$$= \int_{0}^{-1} \left| x^k (x^{-1}D)^q\, x^{-\mu-1/2} \phi(x) \right| dx + \int_{0}^{\infty} \left| x^k (x^{-1}D)^q x^{-\mu-1/2} \phi(x) \right| dx$$

$$\leq \sup_x \int_{0}^{\infty} \left| x^k (x^{-1}D)^q x^{-\mu-1/2}\, \phi(x) \right| \sup_x \int_{0}^{1} \left| x^{k+2} (x^{-1}D)^q x^{-\mu-1/2}\, \phi(x) \right| \int_{1}^{\infty} \frac{1}{x^2} dx$$

$$\leq c^\mu (A+\delta)^k (B+\rho)^q a_k b_q + c^\mu (A+\delta)^{k+2} (B+\rho)^q a_{k+2} b_q$$

$$\leq c^\mu (A+\delta)^k (B+\rho)^q a_k b_q + c^\mu (A+\delta)^{k+2} (B+\rho)^q b_q LR^{k+2} a_k a_2$$

$$= c^\mu (A+\delta)^k (B+\rho)^q a_k b_q [1 + (A+\delta)^2 a_2 LR^{k+2}]$$

$$= c_1^\mu (A+\delta)^k (B+\rho)^q a_k b_q$$

So that

$$|\langle u, \phi \rangle| = c_1^\mu (A+\delta)^k (B+\rho)^q a_k b_q \sum_{p=0}^{r} \| f_p \|_\infty$$

Hence $u \in \left(H^{b_q, B}_{\mu, a_k, A}\right)'$.

This completes the proof of the theorem.

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Some Operation–Transform Formulas For S_μ–Transform

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1. Introduction:

In an earlier Paper [1] S_μ-Transform has been defined by

$$S_\mu[f(t)] = F(s) = \int_0^\infty f(t)\frac{e^{-\mu(s/t)}}{s+t}\,dt \qquad (\mu \geq 0). \tag{1.1}$$

where $f(t)$ is a suitably restricted conventional function defined on the +ve real line $0 < t < \infty$ & $0 < \text{Res} < \infty$. It has been generalised in the case of generalised functions as

$$S_\mu[f(t)] = F(s) = \langle f(t), \frac{e^{-\mu(s/t)}}{s+t} \rangle \qquad (\mu \geq 0). \tag{1.2}$$

Its invention formula has been also derived. Here it is proposed to discuss some operation transform formulae of the transform given by (1.1)

2. Operation –Transform Formulae for S_μ-Transform

Differentiation : If $\phi \in S_\mu$, where B_μ is the space of all complex valued smooth functions $\phi(t)$ such that for each $\phi(t) \in B_\mu$, we have

$$\rho_n(\phi) = \sup_{0<t<\infty} |D^n\phi(t)| \qquad (n = 0, 1, 2 \ldots..)$$

bounded.
We shall prove that

$$\rho_n[-D\phi] = \rho_{n+1}[\phi] \tag{2.1}$$

Since,

$$\rho_n[-D\phi] = \sup_{0<t<\infty} |D^n(-D\phi)|$$

$$= \sup_{0<t<\infty} |D^{n+1}(-D\phi)|$$

$$= \rho_{n+1}[\phi].$$

Therefore, we get

$$\rho_n[-D\phi] = \rho_{n+1}[\phi].$$

From (2.1) it follows that $\phi \to -D\phi$ is a continuous and linear mapping of β_μ onto itself. Therefore, from Theorem 1.10–1 due to Zemanian [2, p.29], the adjoint mapping $f \to Df$ is also a continuous and linear mapping of β'_μ on to itself where β'_μ is the dual of β_μ and we get

(2.2) $$\langle Df(t), \phi(t) \rangle = \langle f(t), -D\phi(t) \rangle.$$

Now, we prove that the following operation–transform formula

(2.3) $$S_\mu[D^n f] \le K.S_\mu[|f(t)|].$$

Proof: Using, the generalised definition of S_μ–transform and the relation (2.2), we get

$$S_\mu[D^n f] = \langle D^n f(t) \frac{e^{-\mu s/t}}{s+t} \rangle$$

$$= \langle f(t), (-D)^n f \frac{e^{-\mu s/t}}{s+t} \rangle$$

$$= \langle f(t), \sum_{v=0}^{n} n_{c_v} (-D)^{n-v} e^{-\mu} s/t (-D)^v \frac{1}{s+t} \rangle$$

Therefore, we get

(2.4) $$S_\mu[D^n f(t)] = \langle f(t), \frac{e^{-\mu s/t}}{s+t} \cdot \frac{P_n(t)}{Q_n(t)} \rangle,$$

where $P_n(t)$ and $Q_n(t)$ are the polynomials in t such that order of $Q_n(f) \ge$ order of $P_n(t)$. Let us suppose that f is a regular generalised function of β'_μ. Therefore, for $\phi \in \beta_\mu$, we have

$$\langle f, \phi \rangle = \int_0^\infty f(t)\phi(t)\,dt$$

and

$$|\langle f, \phi \rangle| \le \int_0^\infty |f(t)| \; |\phi(t)|\,dt$$

Consequently, we get

(2.5) $$|\langle f, \phi \rangle| \le \langle |f|, |\phi| \rangle$$

An appeal to (2.4) and (2.5) given

$$S_\mu[D^n f] \le \langle |f(t)|, \left| \frac{e^{-\mu s/t}}{s+t} \right| \left| \frac{P_t(t)}{Q_n(t)} \right| \rangle$$

$$\le \langle |f(t)|, \left| \frac{e^{-\mu s/t}}{s+t} \right| \rangle, K \rangle,$$

where $\left|\dfrac{P_n(t)}{Q_n(t)}\right| \le K$ (cont.) ; $0 < t < \infty$; $0 < s < \infty$ & $\mu \ge 0$.

Therefore, we get

$$S_\mu|D^n f| \le K\left\langle |f(t),\left|\frac{e^{-\mu s/t}}{s+t}\right|\right\rangle$$

$$\le K \,.\, S_\mu[f(t)|].$$

This completes the proof.

Multiplication by an Exponential Functions

Let μ be a real number such that $\mu \ge 0$. Now we prove that $\phi(t) \to e^{-\mu t}\,\phi(t)$ is a continuous and linear mapping from B_μ on to itself.

Proof : Let $\phi \in B_\mu$, We have

$$D^n[e^{-\mu t}\phi(t)] = \sum_{v=0}^{n} n_{c_v} D^{nv}\; e^{-\mu t}\; D^v\; \phi(t)$$

$$= \sum_{v=0}^{n} n_{c_v}\;(-\mu)^{n-v}\; e^{-\mu t}\; D^v\; \phi(t)$$

Therefore, we get

$$|D^n[e^{-\mu t}\phi(t)]| \le \sum_{v=0}^{n} K\,|D^v\;\phi(t)|,$$

where $|n_{c_v}(-\mu)^{n-v}\; e^{-\mu t}| \le K$ for $\mu \ge 0$ & $0 < t < \infty$.

Thus we get

(2.6) $$\rho_n[e^{-\mu t}\;\phi(t)] \le K\sum_{v=0}^{n}|\rho_n|\,\phi(t)| \quad (n = 0, 1, 2, \ldots;\; v = 0, 1, 2, \ldots).$$

From (2.6), it follow that $\phi(t) \to e^{-\mu t}$ is a continuous and linear mapping of B_μ on to itself. Therefore, from Theo. 1.10–1 due to Zemanian [2, p.29] the adjoint mapping $f \to e^{-\mu t} f$ is also a continuous and linear mapping of B_μ on to itself and we get.

(2.7) $$\langle e^{-\mu t} f(t), \phi(t)\rangle = \langle f(t), e^{-\mu t}\;\phi(t)\rangle.$$

An appeal to (2.7) & the generalised definition of S_μ–transform.

We get

$$S_\mu[e^{-\mu t} f(t)] = \left\langle e^{-\mu t} f(t), \frac{e^{-\mu s/t}}{s+t}\right\rangle$$

$$= \left\langle f(t), e^{-\mu t}\frac{e^{-\mu s/t}}{s+t}\right\rangle$$

Therefore,

$$|S_\mu[e^{-\mu t} f(t)]| \le \left\langle |f(t)|, |e^{-\mu t}|\left|\frac{e^{-\mu s/t}}{s+t}\right|\right\rangle$$

by (2.5) if f is a regular generalised function

$$\leq M \langle |f(t)|, \frac{e^{-\mu s/t}}{s+t} \rangle$$

$$\leq M S_\mu[|f(t)|],$$

where $|e^{-\mu t}| \leq M$

Thus we get an operation-transform formula

(2.8) $$S_\mu |[e^{-\mu t} f(t)]| \leq M S_\mu[|f(t)]$$

Multiplication by $(S+t)^{-\lambda}$ where $\lambda \to 0$; $0 < t < \infty$ & $0 < s < \infty$.

We prove that $\phi(t) \to (s+t)^{-\lambda} \phi(t)$ is an a continuous and linear mapping of B_μ on itself, where $\lambda > 0$; $0 < t < \infty$ & $0 < s < \infty$.

Proof: Let $\phi \in B_\mu$, We have

$$D_n[(s+t)^{-\lambda} \phi(t)] = \sum_{v=0}^{n} n_{c_v} D^{n-v} (s+t)^{-\lambda} D^v \phi(t)$$

$$= \sum_{v=0}^{n} n_{c_v} (-\lambda)(-\lambda-1)\ldots(-\lambda-(n-v-1)(s+t)^{-\lambda-v+v} D^v \phi(t)$$

Therefore, we get

$$|D^n (s+1)^{-\lambda} \phi(t)| \leq M \sum_{v=0}^{n} |D^v \phi(t)|,$$

where $\quad |n_{c_v} (--\lambda)(-\lambda-1)\ldots(-\lambda-n+v+1)(s+t)^{-\lambda-n+v}| \leq M.$

$(n = 0, 1, 2, \ldots\ldots;\ v = 0, 1, 2, \ldots.)$

Thus we get

(2.9) $$\rho_n[(s+t)^{\mu} \phi(t)] \leq M \sum_{v=0}^{n} \rho_v [\phi(t)]$$

From (2.9) it follows that $\phi(t) \to (S+t)^{-\lambda} \phi(t)$ is a continuous & linear mapping of B_μ on to itself. Therefore, from Theo. 1.10–1 due to Zemainan [2.p.29], the adjoint mapping $f \to (s+t)^{-\lambda} f$ of $\phi \to (s+t)^{-\lambda} \phi$ is also a continuous and linear mapping of β'_μ on itself and we get

(2.10) $$\langle (s+t)^{-\lambda} f(t)\, \phi(t) \quad f(t)\, (s+t)^{-\lambda} \phi(t) \rangle$$

An appeal to (2.10) & the generalised definition of S_μ–transform gives

$$S_\mu[(s+t)^{-\mu} f(t)] = \langle (s+t)^{-\lambda} f(t),\ e^{-\mu s/t} \rangle$$

$$= \langle f(t), (s+t)^{-\lambda} \frac{e^{-\mu s/t}}{s+t} \rangle$$

If f be a regular generalised function then by using (2.5), we get

$$|S_\mu[(s+t)^{-\lambda} f(t)]| \le \langle |f(t)|, |(s+t)^{-\lambda}| \left|\frac{e^{-\mu s/t}}{s+t}\right| \rangle$$
$$\le N\langle |f(t)|, \frac{e^{-\mu s/t}}{(s+t)} \rangle \le N\, S_\mu[|f(t)|],$$

where $$|(s+t)^{-\lambda}| \le N$$

Thus we get an operation-transform formula

(2.11) $$|S_\mu[(s+t)^{-\lambda} f(t)]| \le NS_\mu \langle |f(t)|]$$

Sifting: Let T be a fixed real number such that $0 < t + T < \infty$ & $0 < t < \infty$. Let $\phi(t) \in \beta_\mu$. Now we will prove that $(t+T)$ is a continuous & linear mapping of β_μ on to itself.

Proof: Let us consider

$$D^n[\phi(t+T)] = (d/dt)^n\, |\phi(t+T)|$$
$$= \left[\frac{d}{d(t+T)}\frac{d(t+T)}{dt}\right]^n |\phi(t+T)|$$
$$= \left(\frac{d}{d(t+T)}\right)^n |\phi(t+T)|$$
$$= D^n_{t+T}\,\phi(t+T)$$
$$= D^n_{t_1}[\phi\, t_1], [t_1 = t+T],$$

where $0 < t+T < \infty$ and $0 < t < \infty$.
Therefore, we get

(2.12) $$D^n_t[\phi(t+T) = D^n_t[\phi(t)], |t_1 = t|$$

i.e. $$\rho_n |[\phi(t+T)]| = \rho_1 |\phi(t)|$$

Thus from (2.12), it follows that $\phi(t) \to \phi(t+T)$ is a continuous and linear mapping of β_μ on to itself. Its inners mapping $\phi(t) \to \phi(t+T)$ is also a continuous and linear mapping of β_μ on to itself. Therefore, $\phi(t) \to \phi(t+T)$ is an isomorphism of β_μ onto itself. The adjoint mapping of $\phi(t) \to \phi(t+T)$ is f(t) $\to f(t+T)$ which is also a continuations and linear mapping of β'_μ onto itself due to Theorem 1.10–1 of Zemanian [2.p.29] and we get

(2.13) $$\langle f(t+T),\ \phi(t) \rangle = \langle f(t),\ \phi(t+T) \rangle$$

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Application in Bio-Mathematics of Hypergeometrid Type

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Introduction

In recent research work, Ronghe [4] has determined the equations of Atmospheric pressure and half life period, making use of Fox's H-functions. We shall now here establish certain results involving generalized H-functions which lead in Bio-mathematics. This paper concludes with some interesting special cases for G-functions of the main results established herein. Meijer's G-functions of two variables introduced by Agarwal (I) was extended by the introduction of H-function of two variables by Mittal and Gupta (3). They have given the following notations to define the H-function of two variables as:

$$(1) \qquad H^{0,n:(m_1,n_1);(m_2,n_2)}_{p,q,:(r,s);(k,1)}\left[\begin{array}{c|c} x & \begin{array}{c}(a_1;A_j,\alpha_j)_{1p} \\ (c_j,C_j)_{1r}(e_j,E_j)_{1k}\end{array} \\ y & \begin{array}{c}(b_j;B_j,\beta_j)_{1q} \\ (d_1,D_j)_{1s}(f_j,F_j)_{11}\end{array}\end{array}\right]$$

$$= -\frac{1}{4\pi}\int_{L_1}\int_{L_2}\theta(\xi,\eta)g(\xi)h(\eta)x^{\xi}y^{\eta}\,d\xi d\eta,$$

where,

$$g(\xi,\eta) = \frac{\prod_{j=1}^{n}\Gamma(1-a_j+A_j\xi+\alpha_j\eta)}{\prod_{j=m}^{n}\Gamma(1-b_j+B_j\xi+\beta_j\eta)\prod_{j=n+1}^{n}\Gamma(a_j-A_j\xi-\alpha_j\eta)}$$

$$g(\xi) = \frac{\prod_{j=1}^{m_1}\Gamma(d_j-D_j\xi)\prod_{j=1}^{n_1}\Gamma(1-e_j+C_j\xi)}{\prod_{j=m_1+1}^{s}\Gamma(1-d_j+D_j\xi)\prod_{j=n_1+1}^{k}\Gamma(e_j-C_j\xi)},$$

$$h(\eta)=\frac{\prod_{j=1}^{m_2}\Gamma(f_j-F_j\eta)\prod_{j=1}^{n_2}(1-e_j+E_j\eta)}{\prod_{j=m_2+1}^{m_1}\Gamma(1-f_j+F_j\eta)\prod_{j=n_2+1}^{k}\Gamma(e_j-E_j\eta)}\text{ valid for}$$

(i) $/x/</, /y/</, 0\le n\le p, 0\le m_1\le g, 0\le n_1\le s, 0\le m_2\le k, 0\le n_2\le 1.$

(ii) All $A'S, B'S, C'S, D'S, E'S, F'S$, α's and $\beta' S$ are positive quantities.

(iii) The H-function 1.1(1) converges provided

(a) $U=-\sum_{j=n+1}^{p}A_j-\sum_{j=1}^{q}B_j+\sum_{j=1}^{m_1}D_j-\sum_{j=m_1+1}^{S}D_j+\sum_{j=1}^{n_1}C_j-\sum_{j=n_1+1}^{r}C_j>0$

(b) $V=-\sum_{j=n+1}^{p}d_j-\sum_{j=1}^{q}\beta_j+\sum_{j=1}^{m_2}F_j-\sum_{j=m_2+1}^{1}F_j+\sum_{j=1}^{n_2}E_j-\sum_{j=n_2+1}^{k}E_j>0$

(c) $/\ argx\,/<\frac{1}{2}\pi U,$

(d) $/\ argy\,/<\frac{1}{2}\pi V$

1.2. In this section we shall determine the Equation of Population Growth involving H-function of two variables.

Suppose that $P(t)$ is the Population size at time t and δP the growth in the Population corresponding to time δt, then we have

(1) $\delta P\ \alpha\ P\delta t$, which gives the differential equation

(2) $DP/dt=\lambda P$,

where λ is proportional constant.

Integration of 1.2(2) yields.

$\int\frac{dp}{dt}=\lambda p+k_1$, where k_1 is a constant of integration and therefore

(3) $$\int\frac{\Gamma(P)}{\Gamma(P+1)}dP=\lambda\frac{\Gamma(t+1)}{\Gamma(t)}+k_1$$

We can now determine the value of constant k_1 at the initial conditions $t=0$. $P=P_0$. When time increases, the population also increases. If we make $P\to P+P_1\xi$ and $t\to t+t_1\xi$, then 1.2(3) gives

(4) $$\int\frac{\Gamma(P+P_1\xi)}{\Gamma(1+P+P_1\xi)}dP=\lambda\frac{\Gamma(1+t+t_1\xi)}{\Gamma(t+t_1\xi)}+k_1$$

Similarly, we can deduce.

(5) $$\int\frac{\Gamma(Q+Q_1\eta)}{\Gamma(1+Q+Q_1\eta)}dQ=\lambda_1\frac{\Gamma(1+t^1+t_1^1\eta)}{\Gamma(t^1+t_1^1\eta)}+k_2$$

Combine 1.2(4) and 1.2(5) and multiply both sides of it by $\frac{1}{(2\pi i)^2}\Phi(\xi,\eta)g(\xi)h(\eta)$, further, integrate with respect to ξ and η along the direction of double contours L_1 and L_2 and use 1.1(1) to obtain the following equation of population growth involving H-function of two variables :

$$(6)\ \iint H^{0,n:(m_1+1;n_1+1);(m_2+1,n_2+1)}_{p,q:(r+1,s+1);(k+1,l+1)}\left[\begin{array}{c|c} x & (a_p;A_p,\alpha_p) \\ & (1-P,P_1);(c_r,C_r);(1-Q,Q_1),(c_k,E_r) \\ y & (b_q;B_q,\beta_q) \\ & (d_s,D_s),(-P,P_1;(f_1,F_1),(-Q,Q_{1_r}) \end{array}\right] dPdQ$$

$$= \mu_1 H^{0,n:(m_1+1,n_1+1);(m_2+1,n_2+1)}_{p,q:(r+1,s+1);(k+1,1+1)}\left[\begin{array}{c|c} x & (a_p;A_p,\alpha_p) \\ & (-t,t_1);(c_r,C_r);(-t^1,t_1^1),(e_k,E_k) \\ y & (b_q;B_q,\beta_q) \\ & (d_s,D_s),(1-t_1);(f_1,f_1),(1-t^1,t_1^1) \end{array}\right]$$

$$+ \mu_2 H^{0,n:(m_1+1,n_1+1);(m_2,n_2)}_{p,q:(r+1,s+1);(k,1)}\left[\begin{array}{c|c} x & (a_p;A_p,\alpha_p) \\ & (-t,t_1);(c_r,C_r);(e_k,E_k) \\ y & (b_q;B_q,\beta_q) \\ & (d_s,D_s),(1-t,t_1);(f_1,F_1) \end{array}\right]$$

$$+ \mu_3 H^{0,n:(m_1,n_1);(m_2+1,n_2+1)}_{p,q:(r,s);(k+1,1+1)}\left[\begin{array}{c|c} x & (a_p;A_p,\alpha_p) \\ & (c_r,C_r);(-t^1,t_1),(e_k,E_k) \\ y & (b_q;B_q,\beta_q) \\ & (b_q,D_s);(f_1,F_1),(1-t^1,t_1^1) \end{array}\right]$$

$$+ \mu_4 H^{0,n:(m_1,n_1);(m_2+n_2)}_{p,q:(r,s);(k+1)}\left[\begin{array}{c|c} x & (a_p;A_p,\alpha_p) \\ & (c_r,C_r);(e_k,E_k) \\ y & (b_q;B_q,\beta_q) \\ & (d_{s_2}D_s);(f_1,F_1) \end{array}\right]$$

where

(i) $\mu_1(-\lambda\lambda_1)$, $\mu_2(=\lambda k_2)$, $\mu_3(=\lambda_1 k_1)$ and $\mu_4(=k_1k_2)$ are all constants

(ii) $P_1 > 0$, $Q_1 > 0$, $t_1 > 0$ and $t_1^1 > 0$.

1.3. Special Cases:

If we put $A_j = \alpha_j = 1, (j=1,2, \ldots, p)$, $B_j = \beta_j = 1 (j=1,2, \ldots, q)$, $C_j = 1\ (j=1,2, \ldots, r)$, $D_j = 1\ (j=1,2, \ldots, s)$, $E_j = 1\ (j=1,2, \ldots, k)$,

$F_j = 1$ (j =1,2, ... , I), $P_1 = 1$, $Q_1 = 1$, $t_1 = 1$ and $t_1^1 = 1$. in 1.2(6), then we have

$$(1)\quad \iint G^{\,n,n_1+1,n_2+1,m_1+1,m_2+1}_{\,p\,(r+1,k+1)\,q,(s+1\,;1+1)}\left[\begin{array}{c|c} & (a_p) \\ x & (1-P),(c_r);(1-Q),(e_k) \\ y & (b_q) \\ & (d_s),(-P)\,;(f_1),(-Q) \end{array}\right] dPdQ$$

$$= \mu_1 G^{\,n,n_1+1,n_2+1,m_1+1,m_2+1}_{\,p\,(r+1;k+1),\,q,(s+1\,;1+1)}\left[\begin{array}{c|c} & (a_p) \\ x & (-t),(c_r);(-t^1),(e_k) \\ y & (b_q) \\ & (d_s),(1-t)\,;(f_1),(1-t^1) \end{array}\right]$$

$$+\mu_2 G^{\,n,n_1+1,n_2,m_1+1,m_2}_{\,p,(r+1;k),\,q,(s+1\,;1)}\left[\begin{array}{c|c} & (a_p) \\ x & (-t),(c_r);(e_k) \\ y & (b_q) \\ & (d_s),(1-t)\,;(f_1), \end{array}\right]$$

$$+\mu_3\, G^{\,n,n_1,n_2+1,m_1,m_2+1}_{\,p,(r;k+1),\,q,(s:1+1)}\left[\begin{array}{c|c} & (a_p) \\ x & (c_r);(-t^1),(e_k) \\ y & (b_q) \\ & (d_s),(f_1),(1-t^1) \end{array}\right]$$

$$+\mu_4\, G^{\,n,n_1,n_2,m_1,m_2}_{\,p,(r;k),\,q,(s:1)}\left[\begin{array}{c|c} & (a_p) \\ x & (c_r)\,;(e_k) \\ y & (b_q) \\ & (d_s)\,;(f^1) \end{array}\right]$$

where μ_1, μ_2, μ_3 and μ_4 are all constants and the G-functions of two variables in 1.3(1) are valid for
$/x < 1$, $/y/ < 1$,
$0 \le n \le p$, $0 \le n_1 \le r$, $0 \le n_2 \le k$, $0 \le m_1 \le s$, $0 \le m_2 \le 1$, $0 \le$ q. and for convergence, we have

$$p+q+s+r < 2\,(m_1+n_1+n)$$
$$p+q+1+k < 2\,(m_2+n_2+n)$$
$$/argx/ < \pi\ [m_1+n_1+n-(p+q+s+r)/2]$$
$$/argx/ < \pi\ [m_2+n_2+n-(p+q+1+k)/2]$$

In particular, if we take $p = 0$ and $q = 0$ in 1.3(1) we have.

$$(2)\quad \iint G^{\,m_1+1,n_1+1}_{\,r+1,s+1}\left[x\,\middle|\,\begin{matrix}(1-p),(C_r)\\(d_s),(-P)\end{matrix}\right]\times G^{\,m_2+1,n_2+1}_{\,k+1,1,+1}\left[y\,\middle|\,\begin{matrix}(1-Q),(e_k)\\(f_1),(-Q)\end{matrix}\right]dPdQ$$

$$= \mu_1\, G^{\,m_1+1,n_2+1}_{\,r+1,s+1}\left[x\,\middle|\,\begin{matrix}(-t),(e_r)\\(d_s),(1-t)\end{matrix}\right]\times G^{\,m_2+1,n_2+1}_{\,k+1,1,+1}\left[v\,\middle|\,\begin{matrix}(-t^1),(e_k)\\(f_1),(1-t^1)\end{matrix}\right]$$

$$+ \mu_2\, G^{\,m_1+1,n_1+1}_{\,r+1,s+1}\left[x\,\middle|\,\begin{matrix}(-t),(e_r)\\(d_s),(1-t)\end{matrix}\right]\times G^{\,m_2,n_2}_{\,k,1}\left[y\,\middle|\,\begin{matrix}(e_k)\\(f_1)\end{matrix}\right]$$

$$+ \mu_3\, G^{\,m_1,n_1}_{\,r,s}\left[x\,\middle|\,\begin{matrix}(e_r)\\(d_s)\end{matrix}\right]\times G^{\,m_2+1,n_2+1}_{\,k+1,1,+1}\left[y\,\middle|\,\begin{matrix}(-t^1),(e_k)\\(f_1),(1-t^1)\end{matrix}\right]$$

$$+ \mu_3\, G^{\,m_1,n_1}_{\,r,s}\left[x\,\middle|\,\begin{matrix}(e_r)\\(d_s)\end{matrix}\right]\times G^{\,m_2,n_2}_{\,k,1}\left[y\,\middle|\,\begin{matrix}(e_k)\\(f_1)\end{matrix}\right].$$

where μ_1, μ_2, μ_3 and μ_4 are all constants and the G-functions in 1.3 (2) are valid for $/x/ < 1$, $/y/ < 1$,

$$\begin{aligned}
&0 \le m_1 \le s,\ 0 \le m_1 \le r,\ 0 \le m_2 \le 1,\ 0 \le n_2 \le k\\
&r + s < 2\ (m_1 + n_1)\\
&k + 1 < 2\ (m_2 + n_2)\\
&/argx/ < \pi\ [m_1 + n_1 - (r+s)/2],\\
&/argy/ < \pi\ [m_2 + n_2 - (k+1)/2]
\end{aligned}$$

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On CR-Submanifolds Of A Trans Para Sasakian Manifolds

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1. Introduction

A Bejancu [1] introduced the notion of CR-submanifolds of a Kachlerian manifold. C.R–submanifolds of a Sasakian manifold have been studied by Kobayashi [4]. C.R–submanifolds of a Kenmotsu manifold have been studied by Papaghuic [2]. Oubina [5] introduced a new class of almost contact Riemannian manifold known as trans-Sasakian manifold.

The purpose of the present paper is to define and study CR-submanifolds of a trans para Sasakian manifolds.

2. Preliminaries

Let $\overline{M}$ be an n-dimensional almost para contact metric manifold with structure tensors (F, U, u, g) where F is a (1,1) tensor field, a vector field U, a 1-form u and g is an associated Riemannian metric on $\overline{M}$ which satisfy the following conditions [3]

(2.1) $\quad F^2 = I - u \otimes U, \quad u(U) = 1, \quad F(U) = 0, \quad uoF = 0,$

(2.2) $\quad g(FX, FY) = g(X,Y) - u(X)\, u(y),$

(2.3) $\quad g(FX, Y) = g(X, FY) = 0, \qquad u(X) = g(X, U), \quad \text{for all } X,Y \in \overline{M}$

Definition : An almost para contact metric structure (F,U,u,g) on $\overline{M}$ is called trans para Sasakian if

(2.4) $\quad (\overline{\nabla}_X F)(Y) = \alpha\,(g(X, Y)U - u(Y)X) + \beta\,(g(FX, Y)\, U - u(Y)\, FX)$

for α, β non zero constant and we say that trans para Sasakian structure is of type (α, β).

From the above formula, we get

(2.5) $\quad \overline{\nabla}_X F = \alpha FX + \beta (X - u(X)\, U).$

Definition : A submanifold M of $\overline{M}$ is called a CR–subminifold if U is tangent to M and there exist on M a differentiable distribution D: $x \to D_x \subset T_x M$ satisfying the following conditions:

(i) D_x is invariant under F that is $FD_x \subset D_x$ for each $x \in M$,

(ii) the complimentary orthogonal distribution $D^{\perp}: x \to D_x^{\perp} \subset T_x M$ is totally real under F, that is $FD_x^{\perp}: x \subset T_x^{\perp} M$ for each $x \in M$, where $T_x M$ and $T_x^{\perp} M$ are tangent and normal space of M at x respectively.

M is an invariant (resp. anti-invariant) submanifold of $\overline{M}$ when $\dim D^{\perp}=0$ (resp. $\mathrm{Dim} D = 0$), where D (resp. $D^{\perp}$) is the horizontal (resp. vertical) distribution. The pair $(D, D^{\perp})$ is called U-horizontal (U-vertical) if $U_x \in D_x$ (resp. $U_x \in D_x^{\perp}$) for each $x \in M$.

For a vector field X tangent to M, we put

(2.6) $$X = PX + QX,$$

where PX and QX belong to the distribution D and $D^{\perp}$ respectively. Also for a vector field N normal to M, we put

(2.7) $$FN = BN + CN,$$

where BN (resp. CN)denotes the tangential (resp. Normal) component of FN.

The Gauss and Weingarten formulas are given by

(2.8) $$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y); \quad \overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N, \quad X, Y \in TM, N \in T^{\perp}M,$$

where $\nabla^{\perp}$ is the normal connection, h (resp. A) is the second fundamental form (resp.tensor) of M in M satisfying.

(2.9) $$g(A_N X, Y) = g(h(X, Y), N).$$

If we denote the orthogonal component of $FD^{\perp}$ in $TM^{\perp}$ by μ, then we have $T^{\perp}M = FD^{\perp} \oplus \mu$, it is obvious that $F\mu = \mu$.

3. Some Basic Lemmas

Lemma 3.1. Let M be a CR-submanifold of a trans para Sasakian manifold $\overline{M}$. Then we have

(3.1) $$P\nabla_X FPY - PA_{FQY}X = FP\nabla_X Y + \alpha g(X,Y)PU + \beta g(FPX,Y)PU - \alpha u(Y)PX - \beta u(Y)FPX,$$

(3.2) $$Q\nabla_X FPY - QA_{FQY}X = Bh(X, Y) + (\alpha g(X,Y) + \beta g(FQX,Y)QU - \alpha u(Y)QX.$$

(3.3) $$h(X, FPY) + \nabla_X^{\perp} FQY = FQ\nabla_X Y + Ch(X,Y) - \beta u(Y) FQX,$$

for any $X, Y \in TM$

Proof: From equations (2.4), (2.6), (2.7) and (2.8), we have

$$\overline{\nabla}_X FX - \overline{\nabla}_X Y = \alpha(g(X,Y)U + u(Y)X) + \beta(g(FX,Y)U - u(Y)FX)$$

or

$$\nabla_X FPY + h(X,FPY) + \nabla_X^{\perp} FQY - A_{FQY}X - F\nabla_X Y - Fh(X,Y)$$
$$= \alpha g(X,Y)PU + \alpha g(X,Y)QU - \alpha u(Y)PX - \alpha u(Y)QX + \beta g(FPX+FQX, Y)U - \beta u(U)FPX - \beta u(y)FQX.$$

or $\quad P\nabla_X FPY + Q\nabla_X FPY + h(X, FPY) + \nabla_X^{\perp} FQY - PA_{FQY}X - QA_{FQY}X$

$$= FP\nabla_X Y + FQ\nabla_X Y + Bh(X,Y) + Ch(X,Y) + \alpha g(X,Y)PU + \alpha g(X,Y)QU$$
$$-\alpha u(Y)PX - \alpha u(Y)QX + \beta g(FPX,Y)PU + \beta g(FQX,Y)QU - \beta u(Y)FPX - \beta u(Y)FQX.$$

Now equating the horizontal, vertical and normal components we get the results

Definition: The horizontal distribution D is said to be parallel with respect to the connection ∇ on M if $\nabla_X Y \in D$ for all vector fields $X, Y \in D$.

Proposition 3.1. Let M be a U-horizontal CR-submanifold of a trans para Sasakian manifold $\overline{M}$. Then the distribution D is parallel if and only if

(3.4) $\qquad h(X, FY) = h(FX, Y) = Fh(X, Y)$, for all $X, Y \in D$.

Proof: parallel distribution is involutive, that is

(3.5) $\qquad h(X, FY) = h(FX, Y)$, for all $X, Y \in D$.

From (3.3) and (3.5), we have

(3.6) $\qquad h(X, FY) = Ch(X, Y).$

Also $\nabla_X FY \in D$, $\nabla_Y FX \in D, \forall X, Y \in D$, so from (3.2) and using D-parallelness, we get

$$Bh(X,Y) = 0, \ \forall X, Y \in D.$$

From (2.7), we get

$$Fh(X,Y) = Bh(X,Y) + Ch(X,Y).$$

From (3.6), $Bh(X,Y) = 0$ and the above equation, we get

$$Fh(X,Y) = Ch(X,Y) = h(X, FY), \ \forall X, Y \in D.$$

Which proves (3.4).

Definition : A CR-submanifold M of a trans para Sasakian manifold $\overline{M}$ is said to be mixed totally geodesic if $h(X,Y) = 0$, for $X \in D$ and $Y \in D^{\perp}$

A CR-submanifold is mixed totally geodesic if and only if $A_N X \in D$ for each $X \in D$.

Definition : A normal vector field $N \neq 0$ is D-parallel normal section if $\nabla_x^{\perp} N = 0$, for all $X \in D$.

Proposition 3.2. Let M be a mixed totally geodesic U-vertical CR-submanifold of a trans para Sasakian manifold $\overline{M}$. Then the normal section $N \in FD^{\perp}$ is a D-parallel if and only if $\nabla_X FN \in D$, for all $X \in D$.

Proof: Let $N \in FD^{\perp}$ and as M be a mixed totally geodesic, we have

$$\nabla_X(FN) = \overline{\nabla}_X(FN)$$
$$\nabla_X(FN) = (\overline{\nabla}_X F)N + F\nabla_X N$$

(3.7) $$\nabla_X (FN) = F\nabla_x^{\perp} N - A_N NX.$$

Let normal section be D-parallel means $\nabla_x^{\perp} N = 0$. Let we have $A_N X \in D$ and $\nabla_x^{\perp} N = 0$ then from equation (3.7), we get $\nabla_X FN \in D$, for all $X \in D$. Conversely, we have $A_N X \in D$ and $\nabla_X FN \in D$, then from (3.7), we get $\nabla_x^{\perp} N = 0$, for all $X \in D$. This implies that normal section N is D-parallel.

This proves our assertion.

4. Integrability of Distributions of CR-submanifold:

Lemma 4.1. Let M be a CR-submanifold of a trans para Sasakian manifold $\overline{M}$. Then we have

(4.1) $$A_{FY}Z - A_{FZ}Y + \alpha(u(Z)Y - u(Y)Z) = FP[Y,Z], \text{ for any } Y,Z \in D^{\perp}.$$

Proof: We have

$$\overline{\nabla}_{FY} FZ = (\overline{\nabla}_Y F)(Z) + F + \overline{\nabla}_Y Z.$$

Using (2.4) in the above equation, we get

$$\overline{\nabla}_Y FZ = \alpha(g(Y,Z)U - u(Z)Y) + \beta(g(FY,Z)U - u(Z)FY) + F\nabla_Y Z + Fh(Y,Z)$$
$$= \alpha(g(Y,Z)U - u(Z)Y) + FP\nabla_Y Z + Bh((Y,Z) + Ch(Y,Z) - \beta u(Z)FQY.$$

In view of (2.8) and the above equation, we have

(4.2) $$-A_{FZ}Y + \nabla_Y^{\perp} FZ = \alpha(g(Y,Z)U - u(Z)Y) - \beta u(Z)FQY + FP\nabla_Y Z + + FQ\nabla_Y Z + Bh(Y,Z) + Ch(Y,Z), \text{ for all } Y,Z \in D^{\perp}.$$

From (3.3), for all Y, $Z \in D^{\perp}$, we have

(4.3) $$\nabla_Y^{\perp} FZ = FQ\nabla_Y Z + Ch(Y,Z) + \beta u(Z)FQY.$$

Now from (4.2) and (4.3), we have

$$FP\nabla_Y Z = -A_{FZ}Y - \alpha g(Y,Z)U + \alpha u(Z)Y - Bh(Y,Z).$$

Similarly we have

$$FP\nabla_Z Y = -A_{FZ}Z - \alpha g(Y,Z)U + \alpha u(Y)Z - Bh(Y,Z).$$

Thus from the above two equations, we get

$$FP[Y,Z] = A_{FY}Z - A_{FZ}Y + \alpha(u(Y)Z)Y - u(Y)Z, \text{ for all } Y,Z \in D^{\perp}.$$

Theorem 4.1. *Let M be a CR-submanifold of a trans para Sasakian manifold $\overline{M}$. The distribution $D^{\perp}$ is integrable if and only if*

(4.4) $$A_{FY}Z - A_{FZ}Y = \alpha(u(Y)Z - u(Z)Y), \text{ for all } Y,Z \in D^{\perp}.$$

Proof: Suppose the distribution $D^{\perp}$ is integrable, then $[Y,Z] \in D^{\perp}$ for any $Y,Z \in D^{\perp}$. This gives $P[Y,Z] = 0$ and from (4.1) we get (4.4).

Conversely suppose (4.4) holds. Then by (4.1) we have $FP[Y, Z] = 0$ for any $Y, Z \in D^{\perp}$. From this we have $P[Y, Z] = 0$, which is equivalent to $[Y, Z] \in D^{\perp}$ for all $Y, Z \in D^{\perp}$. This implies that $D^{\perp}$ is integrable.

Theorem 4.2. *Let M be a U-horizontal CR-submanifold of a trans para Sasakian manifold $\overline{M}$. The distribution D is integrable if and only if*

$$h(X,FY) = h(Y,FX), \text{ for all } X, Y \in D.$$

Proof: From (3.3) for all $X, Y \in D$, we have

(4.5) $$h(X,FY) = FQ\,\nabla_X Y + Ch(X,Y).$$

Similarly, we have

(4.6) $$h(Y, FX) = FQ\,\nabla_Y X + Ch(X,Y).$$

From (4.5) and (4.6), we get

(4.7) $$h(X, FY) = h(Y, FX) = FQ\,[X,Y].$$

As the distribution D is integrable, that is, $Q\,[X,Y] = 0$.
Using this in equation (4.7), we get the result.
Conversely we have

$$h(X, FY) = h(Y, FX)$$

From equation (4.7) and above, we get

$$FQ\,(X,Y) = 0 \Rightarrow Q\,(X,Y) = 0$$
$$\Rightarrow D \text{ is integrable}$$

This proves our assertion.
Now from (2.5), we have

(4.8) $$\nabla_X U + h(X,U) = \alpha\,FPX + \alpha\,FQX + \beta\,(X - u\,(X)U).$$

From (4.8), we get the following two equations

(4.9) $$\nabla_X U = \alpha\,FPX + \beta\,(X - u\,(X)U),$$

(4.10) $$h(X,U) = \alpha\,FQX.$$

Now from (4.9) and (4.10), we get the following two relations:

(4.11) $$\nabla_X \mathrm{U} = \beta\,(X - u\,(X)U), \text{ for } X \in D^{\perp},$$

(4.12) $$h(X,U) = 0, \text{ for } X \in D.$$

Definition: A CR-submanifold of a trans para Sasakian manifold $\overline{M}$ is called D-umbilic (resp. $D^{\perp}$ umbilic) if $h(X,Y) = g(X,Y)H$ holds for all $X, Y \in D$ (resp. $X, Y \in D^{\perp}$), where H is a mean curvature vector field.

Proposition 4.1. Let M be a D-umbilic U-horizontal CR-submanifold of trans para Sasakian manifold $\overline{M}$, then M is D-totally geodesic.

Proof: Let M be a D-umbilic U-horizontal CR-submanifold, that is $h(X,Y) = g(X,Y)H$, for all $X, Y \in D$.

Putting $Y = u$, we have

$$h(X,U) = g(X,U)\,H$$

Using (4.12) in the above equation, we get

$$H = 0 \Rightarrow h(X,Y) = 0.$$

This shows that M is D-totally geodesic.

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On a structure defined by a tensor field $f(\neq 0)$ of type (1,1) satisfying $(f^{2v+3}+\lambda^2 f)\,(f^{2v+3}+\mu^2 f)=0$

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Abstract: Androu [1] has studied the structure defined by a tensor field f ($\neq 0$) of type (1,1) satisfying $f^5 + f = 0$. In the present paper, we have defined and studied $f_{\lambda\mu}$(2v+3,1)–structure. We have also obtained a positive definite Riemanian metric with respect to which the complementary distributions are orthogonal.

1. Introduction

$f_{\lambda,\mu}(2v+3,1)$–Structure

Let M^n be an n-dimensional differentiable manifold of class C^∞. Suppose there exists on M^n a non-null tensor field f of type (1.1) of class C^∞ and of rank r satisfying.

(1.1) $$(f^{2v+3}+\lambda^2 f)\,(f^{2v+3}+\mu^2 f)=0, \quad \lambda\neq\mu$$

Let us defined on such M^n tensor fields 'ℓ' and 'm' of type (1.1) as follows

(1.2) $$\ell=\frac{f^{2v+2}+\lambda^2}{\lambda^2-\mu^2} \quad m=\frac{f^{2v+2}+\lambda^2}{\mu^2-\lambda^2}$$

Then it can be easily shown that

(1.3) $$\ell^2=\ell, \quad m^2=m, \quad \ell m=\ell m=0, \quad \ell+m=1$$

Thus the operators ℓ and m when applied to tangent space of M^n at a point all complementary projection operators. Thus there exist complementary distributions L and M corresponding to projection operators ℓ and m respectively. Let us call such structure as $f_{\lambda,\mu}(2v+3,1)$–structure.

For the manifold M^n equipped with $f_{\lambda,\mu}(2v+3,1)$–structure, the following result can be proved easily.

(i) $$f\ell=\frac{f^{2v+2}+f\lambda^2}{\lambda^2-\mu^2}, \quad fm=\frac{f^{2v+2}+f\mu^2}{\mu^2-\lambda^2}$$

(1.4) (ii) $$f^2\ell = -\mu^2\ell \;, \quad f^2 m = -\lambda^2 m$$
and

(iii) $$m - \ell = \frac{2f^{2\nu+2} + \mu^2 + \lambda^2}{\mu^2 - \lambda^2}$$

2. $f_{\lambda,\mu}(2\nu+3,1)$–structure in local Coordinates:

We now introduce in the manifold M^n a local coordinate system and represented by f_i^h, ℓ_i^h and m_i^h the local components of f, ℓ and m respectively. We also introduce in M^n, a positive definite Riemannian metric by taking r mutually orthogonal unit vectors u_a^h in $L(a, b, c \ldots\ldots = 1,2, \ldots\ldots r)$ and $n-r$ mutually orthogonal unit vectors u_B^h in $L(A, B, C, \ldots = 1,2, \ldots\ldots n-r)$ in M. Thus we have

(2.1) $$\ell_i^h, u_b^i = u_b^h, \quad \ell_i^h, u_B^i = 0 \;;$$
$$m_i^h, u_b^i = 0, \qquad m_i^h, u_B^i = u_B^h,$$

Let $(\mathrm{v}_i^a, \mathrm{v}_i^A)$ be the matrix inverse to (u_b^h, u_B^h). Then v_i^a and v_i^A are components of linearly independent covariant vectors satisfying.

(2.2) $$\mathrm{v}_i^a, u_b^i = \delta_b^a, \quad \mathrm{v}_i^a \; u_B^i = 0 \;;$$
$$\mathrm{v}_i^A, u_b^i = 0, \qquad \mathrm{v}_i^A, u_B^i = \delta_B^A,$$

δ_j^i being Kronecker delta. Also

(2.3) $$\mathrm{v}_i^a, u_a^h + \mathrm{v}_i^A \; u_A^h = \delta_i^h$$

In view of equations (2.1) and (2.2), we have

(2.4) $$(\ell_i^h, \mathrm{v}_h^a)\, u_b^i = \delta_b^a, \quad (\ell_i^h, \mathrm{v}_h^a)\, u_B^i = 0$$
$$(m_i^h, \mathrm{v}_h^A)\, u_b^i = 0, \qquad (m_i^h, \mathrm{v}_h^A)\, u_B^i = \delta_B^A$$

Thus we have

(2.5) $$\ell_i^h, \mathrm{v}_h^a = \mathrm{v}_i^a, \quad \ell_i^h, \mathrm{v}_h^A = 0 \;;$$
$$m_i^h, \mathrm{v}_h^a = 0, \qquad m_i^h, \mathrm{v}_h^A = \mathrm{v}_i^A,$$

Since $fm = 0$, we have $f_i^h \; m_h^i = 0$. Hence contracting with v_j^A and making use of (2.5), we obtain

(2.6) $$f_i^h, \mathrm{v}_h^A = 0$$

further, since

$$\ell_j^h, u_a^j = u_a^h,$$

we have $$\ell_j^h, u_a^j \, \mathrm{v}_i^a = \mathrm{v}_i^a, u_a^h$$

or $$\ell_i^h, (\delta_i^j - \mathrm{v}_i^A \; u_A^j) = \mathrm{v}_i^a, u_a^h$$

(2.7) $$\ell_i^h = v_h^a \, u_b^i$$

by virtue of (2.1) and (2.3). Similarly

(2.8) $$m_i^h = v_i^A \, u_A^h$$

Let us now put

(2.9) $$g_{ji} = v_j^a \, v_i^a + v_j^A \, v_i^A$$

Then g_{ji} is globally defined positive definite Riemannian metric with respect to which (u_b^h, u_B^h) form an orthogonal frame and such that

(2.10) $$v_j^a = g_{ji} \, u_a^i, \qquad v_j^A = g_{ji} \, u_A^i$$

If we further put

(2.11) $$\ell_{ji} = \ell_j^t \, g_{ti} \text{ and } m_{ji} = m_j^i \, g_{ti},$$

We have in view of (2.7) and (2.8)

(2.12) $$\ell_{ji} = v_j^a \, v_i^a, \qquad m_{ji} = v_j^A \, v_i^A$$

and consequently

(2.13) $$\ell_{ji} + m_{ji} = g_{ji}$$

The following equations can be proved easily

(i) $\ell_j^t \, \ell_i^s \, g_{ts} = \ell_{ji}$,

(2.14) (ii) $\ell_j^t \, m_i^s \, g_{ts} = 0$

and (iii) $m_j^t \, m_i^s \, g_{ts} = m_{ji}$

For any two vectors x, y with components x^i, y^i, let us put

(2.15) $$m^*(x,y) = m_{st} x^s y^t, \qquad g(x,y) = g_{st} \, x^s y^t$$

(2.16) $$G(x,y) = \frac{1}{2(v+1)} \{g(x,y) + g(fx, fy) + + g(f^2x, f^2y) + \ldots + g(f^{2v+1}x, f^{2v+1}y) + m^*(x,y)\}.$$

Thus we have

$$m^*(u_A, u_a) = g(u_A, u_a) = g(fu_A, fu_a) = = g(f^2u_A, f^2u_a) = \ldots\ldots = g(f^{2v+1} u_A, f^{2v+1}u_a) = 0$$

$$G(u_A, u_a) = \frac{1}{2(v+1)} \{g(u_A, u_a) + g(fu_A, fu_a) + + g(f^2u_A, f^2u_a) + \ldots + g(f^{2v+1}u_A, f^{2v+1}u_a) + m^*(u_A, u_a) = 0.$$

By virtue of the fact that the distributions L and M are orthogonal with respect to Riemannian metric g. Thus L and M are orthogonal with respect to G also. Consequently, we have the following theorem.

Theorem 2.1. *Let M^n be an n-dimensional differentiable manifold equipped with $f_{\lambda,\mu}(2v+3,1)$-Structure of rank r. Then there exist complementary distributions L and M and a positive definite Riemannian metric G with respect to which the distributions are orthogonal.*

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Certain Sequence Spaces and Matrix Transformations From $Sl_\infty(p)$ to c and C.

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Abstract: Necessary and sufficient conditions have been established for an infinite matrix $A = (a_{nk})$ to transform $S\ell_\infty(p)$ into c and C_s.

1. Introduction

The definition and basic properties of paranormed sequence spaces are given in ([5], [6] and [7]). A paranormed sequence space whose topology is normable is called normed sequence space.

The following sequence spaces will be important in our discussion:

$$\ell_\infty = \{x = \{x_k\} - : \sup_k |x_k| < \infty\}$$
$$c = \{x = \{x_k\} : |x_k| \to 1(k \to \infty), \text{ for some } 1 \in C\}$$
$$c_\infty = \{x = \{x_k\} : |x_k| \to 0(k \to \infty)\}$$
$$C_s = \left\{x = \{x_k\} : \left(\sum_{i=i}^{n} xi\right) \text{ is convergent}\right\} (7,[10])$$

If $\{p_k\}$ is a bounded sequence of strictly positive real numbers, then

$$l(p) = \left\{x = \{x_k\} : \sum_{i=i}^{\infty} |x_k|^{p_k} < \infty\right\}$$
$$l_\infty(p) = \{x = \{x_k\} : \sup_k |x_k|^{pk} < \infty\}$$
$$c(p) = \{x = \{x_k\} : |x_k|^{pk} \to \ell(k \to \infty), \text{ for some } 1 \in C\}$$
$$c_0(p) = \{x = \{x_k\} : |x_k|^{pk} \to 0\ (k \to \infty)\}$$

$l(p)$ and $c_o(p)$ are linear metric spaces respectively paranormed by

$$\|x\| = \left[\sum_{k=1}^{\infty} \|x_k\|^{pk}\right]^{1/m} \text{ and } \|x\| = \sup_k |x_k|^{p_k/m}$$

where $M = \max(1, \sup_k p_k)$ $l_\infty(p)$ and $c(p)$ are paranormed by $\|x\| = \sup_k |x_k|^{p_k/m}$ if and only if $p_k > 0$.

For detailed discussions on these spaces we refer ([5], [6], [7] and [9]).
We now define some sequence spaces
Given any $x = \{x_k\}$ we shall write $\Delta x = (x_k - x_{k-1})$, where $x_o = 0$. We define

$$Sl_\infty(p) = \{x = \{x_k\} : \Delta x \in l_\infty(p)\}$$
$$Sp(p) = \{x = \{x_k\} : \Delta x \in c(p)\}$$
$$Sc_o(p) = \{x = \{x_k\} : \Delta x \in c_o(p)\}$$

We write $e = (1,1,1,1,1, \ldots)$, then $S\,l\infty\,(e) = Sl_\infty$, $Sc(e) = Sc$ and $Sc_o(e) = Sc_o$

2. Dual Space

If X is a sequence space, we define

$$X^\beta = \left\{ a = \{a_k\} \sum_{k=1}^{\infty} a_k x_k \text{ is convergent foreach } x \in X \right\}$$

We call X^β is the β-(or, generalised *kö* the – Toeplitz) dual space of X.

Theorem 1(i). *Let $p_k > 0$, for every k. Then*

$$[Sl_\infty(p)]^\beta = \bigcap_{m=1}^{\infty} \{a = \{a_k\} : \left[\sum_{k=1}^{\infty} N^{1/p_m}\right] \text{ converges and } \sum_{k=1}^{\infty} N^{1/p_k} |R_k| < \infty, N > 1 \}$$

where $R_k = \sum_{v=k}^{\infty} a_v$ $\left(\text{We assume that } \sum_{m=1}^{k} z_m = 0 \ (k > 1) \right)$

Proof: Suppose that $x \in Sl_\infty(p)$. We choose $N > 1$, so that $\sup_k |\Delta x_k|^{p_k} < N$
We write

$$\sum_{k=1}^{\infty} a_k x_k = \sum_{k=1}^{\infty} R_k \Delta x_k - R_{m+1} \sum_{k=1}^{\infty} \Delta x_k \quad (m = 1,2,3, \ldots) \tag{1}$$

Since $\sum_{k=1}^{\infty} |R_k||\Delta x_k| \le \sum_{k=l}^{\infty} |R_k|\, N^{1/p_k} < \infty$, it follows that

$\sum_{k=1}^{\infty} R_k \Delta x_k$ is absolutely convergent. By corollary 2 in [2], the convergence of

$\sum_{k=1}^{\infty} a_k \left(\sum_{m=1}^{k} N^{1/p_m} \right)$ implies that $\lim_{m \to \infty} R_{m+1} \sum_{m=1}^{k} N^{1/p_m} = 0.$
Hence, it follows from (1) that

$$\sum_{k=1}^{\infty} a_k x_k \text{ is convergent for each } x \in Sl_\infty(p)$$

This yields $a \in (Sl_\infty(p))^\beta$

Conversely, suppose that $a \in (Sl_\infty(p))^\beta$, then by definition, $\sum_{k=1}^{\infty} a_k x_k$ is convergent for each $x \in Sl_\infty(p)$.

Since $e = (1,1,1,\ldots\ldots..) \in Sl_\infty(p)$ and $x = \left[\sum_{m=1}^{\infty} N^{1/p_m}\right] \in Sl_\infty(p)$, then $\sum_{v=1}^{\infty} a_v$ and $\sum_{v=1}^{\infty} a_v \left[\sum_{m=1}^{\infty} N^{1/p_m}\right]$ are respectively convergent. By using corollary 2 in [3], we find that

$$\lim_{m\to\infty} R_{m+1} \sum_{m=1}^{v} N^{1/p_m} = 0$$

Thus, we get from (1) that the series $\sum_{k=1}^{\infty} R_k \Delta x_k$ converges for each $x \in Sl_\infty(p)$.

Since $x \in Sl_\infty(p)$ if and only if $\Delta x \in Sl_\infty(p)$. This implies that $R = \{R_k\} \in (l_\infty(p))^\beta$. It now follows from a theorem 2 in [4] that

$\sum_{k=i}^{\infty} |R_k| N^{1/p_k}$ convergence for all $N > 1$.

Theorem 1(ii). *Let $P_k > 0$, for every k. Then $Sc_0(p)^\beta = SM_0(p)$*, where

$$SM_0(p) = \bigcup_{N>1} \left\{ a = \{a_k\} : \sum_{k=1}^{\infty} a_k \left[\sum_{m=1}^{k} N^{-1/p_m}\right] \text{ converges and } \sum_{k=1}^{\infty} |R_k| N^{-1/p_k} < \infty, N > 1 \right\}$$

Proof: Let $a \in SM_0(p)$ and $x \in Sc_0(p)$. We choose an integer N such that $|\Delta x_k|^{p_k} < N^{-1}$.

We have

$$\sum_{k=1}^{\infty} a_k x_k = \sum_{k=1}^{\infty} R_k \Delta x_k - R_{m+1} \sum_{k=1}^{\infty} \Delta x_k : m = 1, 2, 3, 4 \ldots$$

Since

$$\sum_{k=1}^{\infty} |R_k \Delta x_k| \le \sum_{k=i}^{\infty} |R_k||\Delta x_k| \le \sum_{k=i}^{\infty} |R_k| N^{1/p_k} < \infty, \text{ it follows that,}$$

$\sum_{k=1}^{\infty} R_k \Delta x_k$ is absolutely convergent. The convergence of

$\sum_{k=1}^{m} a_k \left[\sum_{m=1}^{k} N^{-1/p_m}\right]$ implies that $\lim_{m\to\infty} R_{m+1} \sum_{i=1}^{m} N^{-1/p_i} = 0(1)\ m \to \infty$. Hence $\sum_{k=1}^{\infty} a_k x_k$ converges for each $x \in Sc_0(p)$. That is, a $a \in Sc_0(p)^\beta$

Conversely, let $a \in Sc_0(p)^\beta$. Then, for any $x \in Sc_0(p)$, $\sum_{k=1}^{\infty} a_k x_k$ converges.

Since $x = \left\{ \sum_{m=1}^{k} N^{-1/p_m} \right\}$ by choosing $\in > 1/N$, $N = 2, 3, \ldots \in Sc_0(p)$ it follows

that $\sum_{k=1}^{\infty} a_k \left(\sum_{m=1}^{k} N^{-1/p_m} \right)$ converges. To show that $\sum_{k=i}^{\infty} |R_k| N^{-1/p_k} < \infty$, $N > 1$,

we suppose that, $\sum_{k=i}^{\infty} |R_k| N^{-1/p_k} = \infty$, $N > 1$. Then from Theorem 6 in [8], it follows that $R \notin M_0(p) = [c_0(p)]\beta$. Then there is a sequence $x = \{1/k\}$,

$k \geq 1 \in c_0(p)$ such that $\sum_{k=1}^{\infty} R_k 1/k$ does not converge. Although, if we define

$y = \{y_k\}$ by $y_k = \sum_{n=1}^{k} 1/n$, then $y \in Sc_0(p)$

But

$$\sum_{k=1}^{\infty} a_k y_k = \sum_{k=1}^{\infty} a_k \left\{ \sum_{n=1}^{k} 1/n \right\} = \sum_{k=1}^{\infty} R_k 1/k.$$

Hence, $\sum_{k=1}^{\infty} a_k y_k$ does not converge for $y \in Sc_0(p)$, a contradiction to the fact that $a \in Sc_0(p)^\beta$. So,

$$\sum_{k=i}^{\infty} |R_k| N^{-1/p_k} < \infty, \quad N > 1.$$

This completes the proof.

Matrix Maps

Let X and Y be any two sequence spaces. Let $A = (a_{n,k})$ be an infinite matrix of scalar entries. If $x = \{x_k\} \in X$, then $A_x = (A_n(x))_{n=1}^{\infty} \in Y$

where $A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$ converges for each $n = 1,2,3, \ldots$. We say that A defines a matrix map from X into Y and we write $A \in (X, Y)$. By (X,Y) we mean the class of matrices A such that $A \in (X, Y)$.

The main aim of this section is to characterize the $(Sl_\infty(p), c)$ and $(Sl_\infty(p), Cs)$ We shall first establish the following simple characterization. In order to characterize, we need the following lemma.

Lemma 1. Let X and Y be sequence spaces, and let

$\Delta Y = (y = \{y_k\} : \Delta y = \{y_k - y_{k-1}\}) \in Y, y_0 = 0\}$. Then $A \in (X, Y)$ if and only if $\Delta A = (a_{n,k} - a_{n-1,k}) = (b_{n,k}) = B \in (X, \Delta Y)$.

With lemma 1 and Theorem 1(i,ii) in [4] or Theorem 3 in [4] or Theorem 5b(i) and Theorem 7 in [6], a characterization of the class. $(1(p) ; Sl_\infty)$ or $(l_\infty(p) ; Sl_\infty)$ or $(1(p) ; Sl_\infty(q))$ is immediately follows $q \in l_\infty$.

Theorem 2. *Let* $p_k > 0$, *for every k. Then* $A \in (Sl_\infty(p), c)$ *if and only if*

(i) $R \in (l_\infty(p), c)$

(ii) $A_n\left[\sum_{i=1}^{k} N^{-1/p_i}\right] \in c$ ($n,k = 1, 2, 3, \ldots$), for all integers, $N > 1$,

(iii) $\lim_{n\to\infty} a_{n,k} = \alpha_k$ ($k = 1, 2, 3, \ldots\ldots$)

Where,

$$R = (r_{n,k}) = \left[\sum_{v=k}^{\infty} a_{n,v}\right] \quad (n,k = 1, 2, 3, \ldots\ldots)$$

Proof: Let us first prove the sufficiency. Consider any $x \in Sl_\infty(p)$. We choose $N > 1$, so that

$$\sup_k |\Delta x_k|^{p_k} < N.$$

We, write,

$$\text{(2)} \qquad \sum_{k=1}^{\infty} a_{n,k}\, x_k = \sum_{k=1}^{m} r_{n,k}\, \Delta x_k - r_{n+1,m} \sum_{k=1}^{m} \Delta x_k \;\; (m = 1, 2, 3, 4 \ldots\ldots)$$

By condition (ii), $\sum_{k=1}^{\infty} a_{n,k}\left[\sum_{i=1}^{k} N^{-1/p_r}\right]$ is convergent for each $n = 1, 2, 3, \ldots$.

Hence, by corollary 2 in [3] it follows that

$$\lim_{m\to\infty} \quad r_{n+1},\ m\sum_{i=1}^{m} N^{1/p_i} = 0$$

By condition (i) $R \in (l_\infty(p), c)$, and since $x \in Sl_\infty(p)$ if and only if $\Delta x \in Sl_\infty(p)$.

Hence, by corollary 2 in [3] it follows that $\sum_{k=1}^{\infty} |x_{n,k}|\, N^{1/p_k}$ is uniformly convergent in n and $\lim_{n\to\infty} r_{n,k}$ exists for each $k = 1, 2, 3, \ldots$

Since,

$$\sum_{k=1}^{\infty} |r_{n,k}||\Delta x_k| \le \sum_{k=1}^{\infty} |r_{n,k}|\, N^{1/p_k}$$

Thus, from (2) we find that $\sum_{k=1}^{\infty} a_{n,k} x_k$ is absolutely and uniformly convergent in n.

Finally, we have

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k}\, x_k = \sum_{k=1}^{\infty} a_{n,k}\, x_k .$$

This proves the sufficiency.

The necessities of (iii) and (ii) are respectively obtained by taking

$$x = e = (1,1,1,\ldots\ldots) \in Sl_\infty(p) \text{ and } x = \left[\sum_{i=1}^{k} N^{-1/p_i}\right] \quad (k = 1,2,3,\ldots.) \in Sl_\infty(p).$$

Now consider the necessity of (I). If it is not true, then there exists $x = \{x_v\} \in l_\infty(p)$ with $\sup_v |x_k|^{p_v} = 1$ such that $\left[\sum r_{n,v} x_v\right]_{n=1}^{\infty} \notin c$. Although if we define a sequence $y = \{y_k\}$ by

$$y_v = \sum_{i=1}^{v} x_i \quad (v = 1,2,3,\ldots\ldots),$$

then $y \in Sl_\infty(p)$ but that,

$$\left[\sum_{v=1}^{\infty} a_{n,v} y_v = \sum_{v=1}^{\infty} r_{n,v} r_v\right] \notin c.$$

This contradicts the fact that $A \in (Sl_\infty(p), c)$ and therefore (I) must hold.

Before characterizing the class $(Sl_\infty(p)$, Cs) we add one more notation. For any $n > 1$, we write

$$t_n(AX) = \sum_{i=1}^{N} A_i(x) = \sum_{k=1}^{\infty} b_{n,k}\, x_k [x \in Sl_\infty(p)],$$

where
$$B = (b_{n,k}) = \left[\sum_{i=1}^{k} a_{i,k}\right] (n, k = 1,2,3,\ldots.)$$

Theorem 3. *Let $p_k > 0$, for every k. Then $A \in (Sl_\infty(p), Cs)$ if and only if*

(i) $C \in (l_\infty(p), \text{Cs})$,

(ii) $B_n \left[\sum_{i=1}^{k} N^{\frac{1}{p_i}}\right] \in Cs \;\; (n, k = 1,2,3,\ldots.)\; N > 1$

(iii) $\lim_{n\to\infty} b_{n,k} = \lim_{n\to\infty} \sum_{i=1}^{n} a_{i,k} = \beta_k \qquad (k = 1,2,3,\ldots.)$

where,

$$C = (C_{n,k}) = \left\{\sum_{i=1}^{n}\left[\sum_{v=k}^{\infty} a_{iv}\right]\right\} \qquad (n, k = 1,2,3,\ldots.)$$

This theorem follows immediately from Theorem 2 and Theorem 2 in [8].

We remark that in proving theorem 3, we use $\sum_{k=1}^{m} b_{n,k} x_k = \sum_{k=1}^{m} C_{n,k} \Delta x_k - C_{n,m+1} \sum_{k=1}^{m} \Delta x_k$

$(m = 1,2,3, \ldots\ldots)$ and the convergence of

$$\sum_{k=1}^{\infty} b_{n,k} \left[\sum_{i=1}^{m} N^{\frac{1}{p_i}} \right] \text{ implies that } \lim_{m\to\infty} C_{n,m+1} \sum_{i=1}^{m} N^{\frac{1}{p_i}} = 0$$

Characterization of $(1(p), Sc_0(q)), q \in l_\infty$ follows from Theorem 5(ii) in ([6]) with Lemma 1.

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Origin of Certain Generating Functions of The Charlier Polynomial $C_m(a;x)$ from The View Point of Lie–Algebra

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Abstract: In this paper generating functions for the Charlier polynomial $Cm\ (a;\ x)$ are obtained with the help of the representation of a Lie-group $G(0,1)$ [2]

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1. Introduction

Let, $G(0,1)$ be a complex 4-dimensional Lie-group. This abstract group $G(0,1)$ consists of abl 4 x 4 matrices of the form

$$(1.1)\qquad g=\begin{bmatrix}1 & ce^{\tau} & d & \tau\\ 0 & e^{\tau} & b & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\quad d,b,c,\tau\in C$$

where the group operation is matrix multiplication. We can introduce co-ordinates for the elements g in $G(0,1)$ by setting $g\equiv(d,\,b,\,c,\,\tau)$. The co-ordinates are valid over the entire group. The usual topology of G induces a topology in $G(0,1)$ and is simply connected (Pontrij- agin, ch, 8). $L[G(0,1)]$ can be identified with the space of 4x4 matrices of the form:

$$\alpha=\begin{bmatrix}0 & x_2 & x_4 & x_3\\ 0 & x_3 & x_1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},\ x_1,x_2,x_3,x_4\in C$$

with Lie product $[\alpha,\beta]=\alpha\,\beta-\beta d$; $\alpha,\beta\in L\,[G(0,1)]$. The matrices

$$g^{*}=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},\ g^{-}=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$g^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \varepsilon = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

form a basis of Lie-algebra $L[G(0,1)] = g\,(0,1)$ with the commutation relations.

$$[g^3, g^+] = g^+, [g^3, g^-] = -g^-, [g^+, g^-] = -\varepsilon, [\varepsilon, g^+] = [\varepsilon, g^-] = [\varepsilon, g^3] = 0$$

where 0 is the 4 x 4 zero matrix.

The mapping $\alpha \to \exp \alpha$ is an analytic diffeomorphism of a nbd. of $\theta \in L(G)$ on to a nbd. of e in G(here θ is the additive identity of $L[G(0,1)]$ and e is the identity element of $G(0,1)$. So, the mapping defines a local one to one coordinate transformation in C^4.

Here,

$$\exp \tau g^3 = \begin{bmatrix} 0 & 0 & 0 & \tau \\ 0 & e^{\tau} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \exp b g^+ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\exp C g^- = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \exp d\varepsilon = \begin{bmatrix} 1 & 0 & d & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(\tau, b, c, d \in C)$$

Let, ρ be the representation of g(0,1) on the complex vector space V and let

let $\quad J^+ = \rho(g^+), J^- = \rho(g^+), E = \rho(\varepsilon), J^3 = \rho(g^3)$

These operator obey the commutation relations

$$[J^+, J^-] = -E, [J^3, J^+] = J^+, [J^3, J^-] = -J^-$$

$[J^+, E] = [J^-, E] = [J^3, E] = 0$ where $[A, B] = AB - BA$ for the linear operators A and B on V.

We define the Casimir operator $C_{0,1}$ on V by $C_{0,1} = J^+, J^- - EJ^3$. Let, ρ be the representation of g(0,1) satisfying the conditions : (i)ρ is irreducible (ii) each eigen value of J^3 has multiplicity equal to one. There is a countable basis for V consisting of eigen vectors of J^3. Such a representation ρ of g(0,1), for which $E \neq 0$, is isomorphic to the representation $R(w, m_o, \mu)$ defined for all $w, m_o, \mu \in C$ such that $\mu \neq 0$, $0 \leq \operatorname{Re} m_o < 1$ and $w + m_o$ is not an integer. The spectrum of this representation is the set $S = \{m_o + n\,;\, n \text{ is an integer}\}$ and the representation space V has a basis $\{f_m\}, m \in S$, so that

(A) $$\begin{cases} J^+ f_m = \mu f_{m+1},\ J^3 f_m = m f_m,\ J^- f_m = (m+w) f_{m-1} \\ E f_m = \mu f_m\ ,\ C_{0,1}\ f_m = \mu w f_m\ ; \end{cases}$$

where the differential operators J^-, J^3 are given by

(1.2) $$\begin{aligned} J^- &= e^{-y}\ \left(x\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \\ J^+ &= e^{y}\ \left(-\frac{\partial}{\partial x} + (1 - a/x) \right) \\ J^3 &= \frac{\partial}{\partial y} \end{aligned}$$

such that

(B) $$\begin{cases} J^-\left[C_m(a\,;x)\,e^{my}\right] = m C_{m-1}\,(a\,;x)\,e^{(m-1)y} \\ J^+\left[C_m(a\,;x)\,e^{my}\right] = C_{m+1}\,(a\,;x)\,e^{(m+1)y} \\ J^3\left[C_m(a\,;x)\,e^{my}\right] = m C_m\,(a\,;x)\,e^{my} \end{cases}$$

From the relations (A) and (B), we have $\mu = 1$, $w = 0$. Thus the realization by differential operators yields a multiplier representation T of $G(0,1)$ whose Lie-algebra is $g(0,1)$.

2. Derivation of the generating functions:

It follows that the functions $f_m(x,y) = C_m(a;x)e^{my}$ form a basis for a realisation of the representation R(0, m_0 ,1) of $g(0,1)$. This representation of $g(0,1)$ by Lie-derivatives can be extended to a local multiplier representation of $G(0,1)$. If we denote '*cl*' the space of all entire analytic functions of x and y, the operators will uniquely define a multiplier representation T of $G(0,1)$ on '*cl*'. We compute the multiplier representation and obtain

(2.1) $$\left[T(\exp bJ^+)\,f_m\right](x,t) = e^{bt}(1 - bt/x)^a f_m(x - bt, t) \quad \text{where } b = e^y$$

i.e. $t^m = e^{my}$

(2.2) $$\left[T(\exp cJ^-)\,f_m\right](x,t) = f_m[x(t+c)/t\ ,\ t+c]$$

and

(2.3) $$\left[T(\exp \tau J^3)\,f_m\right](x,t) = f_m(x, te^{\tau})$$

Also,

(2.4) $$\left[T(\exp dE)\,f_m\right](x,t) = e^{df_m}(x,t)$$

Now, we have

(2.5) $$\begin{aligned} &T\left[(\exp bJ^+)(\exp bJ^-)(\exp \tau J^3)(\exp dE)\,f_m\right]\quad (x,\,t) \\ &= e^{d+bt}\,(1 - bt/x)^a f_m\left[(x - bt)(1 + c/t)\,,\ (t+c)e^{\tau}\right] \end{aligned}$$

where

(2.6) $$g = (\exp bJ^+)(\exp cJ^-)(\exp \tau J^3)(\exp dE) \in G(0,1).$$

The matrix elements of this local representation with respect to the basic f_m are uniquely determined by $(R(0, m_o, 1)$ and we obtain the relation.

(2.7) $$[T(g) f_{m_0+k}](x,t) = \sum_{\ell=-\infty}^{\infty} A_{\ell k}(g)\, f_{m_{0}+\ell}(x,t),\ k = 0, \pm 1; \pm 2, \ldots.$$

Thus we have

(2.8) $$e^{d+bt}(1 - bt/x)^a (t+c)^{m_0+k} e^{(m_0+k)\tau}\, C_{m_0+k}[a; (x-bt)(1+c/t)].$$

$$= \sum_{\ell=-\infty}^{\infty} A_\ell, m - m_0 \quad (g) t^{m_0+\ell} \quad C_{m_0+\ell} \quad (a; x) \text{ where } k = m - m_0$$

The matrix elements $A_{\ell k}(g)$ are given by

(2.9) $$A_{\ell k}(g) = \frac{\exp[d + (m_o + k)\tau]\,\Gamma(m_o + k + 1)\, c^{k-\ell}\, {}_1F_1}{(k-\ell)!\,\Gamma(m_o + \ell + 1)} (-m_0 - \ell; k - \ell + 1; -bc)$$

for $k \geq \ell$.

and

(2.10) $$A_{\ell k}(g) = \frac{\exp[d + (m_o + k)\tau](b)^{\ell - k}\, {}_1F_1}{(\ell - k)!} (-m_0 - k; \ell - k + 1; -bc) \text{ for } \ell \geq k$$

Thus our result becomes

(2.11) $$e^{bt}(1 - bt/x)^a (t + c/t)^m\, Cm[a; (x-bt)(1+c/t)]$$

$$= \sum_{n=0}^{\infty} \binom{m}{n} c^m\, {}_1F_1 \quad (-m_0 + n; n+1; -bc) \quad C_{m-n}(a; x)\, t^{-n}$$

where $k - \ell = n$ with $|bt/x| < 1$, $|c/t| < 1$

This gives a new class of generating functions.

Also, we obtain

(2.12) $$e^{bt}(1 - bt/x)^a (t + c/t)^m\, c_m[a; (x-bt)(1+c/t)].$$

$$= \sum_{n=0}^{\infty} \frac{b^n}{n!}\, {}_1F_1 \quad (-m; n+1; -bc)\, c_{m+n}(a; x) \quad t^m \quad \text{where } \ell - k = n$$

with $|b\,t/x| < 1$, $|c/t| < 1$

which gives another class of new generating functions.

3. Applications

(i) If we take $b = 0$, $c = -1$ in (2.11) we have

$$(1 - 1/t)^m\, C_m\left[a; x\left(1 - \frac{1}{t}\right)\right] = \sum_{n=0}^{\infty} (-1)^n \binom{m}{n} \quad C_{m-n} \quad (a; x)\, t^{-n}$$

Replacing $1/t$ by t

$$+\frac{N^n N!}{R!}\sum_{n=rT+1}^{S+K}\frac{\rho^n}{(N+S-n)R^{n-(1-D)R-DT}\left(\frac{n!}{R!}\right)^D}\prod_{k=1}^{r-1}\left\{\frac{(R+k)^{DT}}{\left(1+\frac{\mu_k}{R_\mu}\right)^T}\right\}\frac{(R+r)^{(n-rT)D}}{\left(1+\frac{\mu_r}{R_\mu}\right)^{(n-rT)}}$$

The average number of failed units in the system is obtained by

$$E(N)=\sum_{n=0}^{S+K} n\,P_n \tag{17}$$

$$=\sum_{n=0}^{R}\frac{N^n\rho^n}{(n-1)}P_n+\frac{1}{R!}\sum_{n=R+1}^{S}\frac{N^n\rho^n n}{R^{(1-D)(n-R)}\left(\frac{n!}{R!}\right)}P_0+$$

$$+\frac{N^S N!}{R!}\sum_{n=S+1}^{T}\frac{n\rho^n}{(N+S-n)R^{(1-D)(n-R)}\left(\frac{n!}{R!}\right)^D}P_0$$

$$+\frac{N^S N!}{R!}\sum_{n=mT+1}^{(m+1)T}\frac{n\rho^n}{(N+S-n)R^{n-(1-D)R-DT}\left(\frac{n!}{R!}\right)^D}\prod_{k=1}^{m-1}\left\{\frac{(R+k)^{DT}}{\left(1+\frac{\mu_k}{R\mu}\right)}\right\}\frac{(R+m)^{(n-mT)D}}{\left(1+\frac{\mu_m}{R\mu}\right)^{(n-mT)}}P_0$$

$$+\frac{N^n N!}{R!}\sum_{n=mT+1}^{S+K}\frac{n\rho^n}{(N+S-n)!\,R^{n-(1-D)R-DT}\left(\frac{n!}{R!}\right)^D}\prod_{k=1}^{r-1}\left\{\frac{(R+k)^{DT}}{\left(1+\frac{\mu_r}{R\mu}\right)^T}\right\}\frac{(R+r)^{(n-rT)D}}{\left(1+\frac{\mu_r}{R\mu}\right)^{(n-rT)}}P_0$$

Case II: $S<R$

The failure rate and the repair rate for the model in this case are:

$$\lambda_n=\begin{cases} N\lambda & 0<n<S\\ (N+S-n)\lambda & S\le n<R\\ (N+S-n)\left(\frac{R}{n+1}\right)^b\lambda, & R\le n<T\\ (N+S-n)\left(\frac{R+m}{n+1}\right)^b\lambda, & mT\le n<(m+1)T,\ 1\le m<r\\ (N+S-n)\left(\frac{R+r}{n+1}\right)^b\lambda, & rT\le n<S+K \end{cases} \tag{18}$$

and

$$\mu_n=\begin{cases} n\mu, & 0<n\le R\\ \left(\frac{n}{R}\right)^a R\mu, & R<n\le T\\ \left(\frac{n}{R+m}\right)^b(R\mu+\mu_m) & mT<n\le(m+1)T,\ 1\le m<r\\ \left(\frac{n}{R+r}\right)^b(R\mu+\mu_r) & rT<n\le S+K \end{cases} \tag{19}$$

Solving equations (3)-(12), the probabilities for different states are given by:

$$(13)\quad P_n = \begin{cases} \dfrac{N^n \rho^n}{n!} P_0 & 0 < n \leq R \\[2ex] \dfrac{N^n \rho^n}{R!\, R^{(1-D)(n-R)}\left(\dfrac{n!}{R!}\right)} P_0, & R < n \leq S \\[2ex] \dfrac{N^S N!\, \rho^n}{R!(N+S-n)!\, R^{(1-D)(n-R)}\left(\dfrac{n!}{R!}\right)} P_0, & S < n \leq T \\[2ex] \dfrac{N^S N!\, \rho^n}{R!(N+S-n)!\, R^{n-(1-D)R-DT}\left(\dfrac{n!}{R!}\right)^D} \prod_{k=1}^{m-1} \left\{ \dfrac{(R+k)^{DT}}{1+\left(1+\frac{\mu_k}{R_\mu}\right)^T} \right\} \dfrac{(R+k)^{(n-rT)D}}{\left(1+\frac{\mu_m}{R_\mu}\right)^{(n-mT)}} P_0, & mT < n \leq (m+1)T, \quad 1 \leq m < r \\[2ex] \dfrac{N^S N!\, \rho^n}{R!(N+S-n)!\, R^{n-(1-D)R-DT}\left(\dfrac{n!}{R!}\right)^D} \prod_{k=1}^{r-1} \left\{ \dfrac{(R+k)^{DT}}{1+\left(1+\frac{\mu_k}{R_\mu}\right)^T} \right\} \dfrac{(R+r)^{(n-rT)D}}{\left(1+\frac{\mu_k}{R_\mu}\right)^{(n-rT)}} P_0, & rT < n \leq S+K \end{cases}$$

where

$$(14)\quad \rho \frac{\lambda}{\mu}\; D = a + b\,.$$

In order to determine the value of P_0, we use the normalizing condition.

$$(15)\quad \sum_{n=0}^{S+K} P_n = 1$$

Then we obtain

$$(16)\quad P_0^{-1} = \sum_{n=0}^{R} \frac{N^n \rho^n}{n!} + \frac{1}{R!} \sum_{n=R+1}^{S} \frac{N^n \rho^n}{R^{(1-D)(n-R)}\left(\frac{n!}{R!}\right)} +$$

$$+ \frac{N^n N!}{R!} \sum_{n=S+1}^{T} \frac{\rho^n}{(N+S-n) R^{(1-D)(n-R)}\left(\frac{n!}{R!}\right)^D} +$$

$$+ \frac{N^n N!}{R!} \sum_{m=mT+1}^{(m+1)T} \frac{\rho^n}{(N+S-n) R^{n-(1-D)R-DT}\left(\frac{n!}{R!}\right)^D} \prod_{k=1}^{m-1} \left\{ \frac{(R+k)^{DT}}{\left(1+\frac{\mu_k}{R_\mu}\right)} \right\} \frac{(R+r)^{(n-mT)D}}{\left(1+\frac{\mu_r}{R_\mu}\right)^{(n-mT)}}$$

Here a and b are the parameters that indicate the degree to which repair rate and failure rate respectively affected by state of system.

The Chapman-Kolmogorov steady-state equations developed by the model are given as:

$$(3) \quad -N\lambda P_0 + \mu P_1 = 0,$$

$$(4) \quad -[N\lambda + n\mu]\, P_n + N\lambda\, P_{n-1} + (n+1)\,\mu\, P_{n+1} = 0, \qquad 1 \le n < R$$

$$(5) \quad -\left[N\left(\frac{R}{R+1}\right)^b \lambda + R\mu \right] P_R + N\lambda\, P_{R-1} + \left(\frac{R+1}{R}\right) R\mu\, P_{R-1} = 0$$

$$(6) \quad -\left[N\left(\frac{R}{n+1}\right)^b \lambda + \left(\frac{n}{R}\right)^a R\mu \right] P_n + N\left(\frac{R}{n}\right)^b \lambda\, P_{n-1} + \left(\frac{n+1}{R}\right)^a R\mu\, P_{n+1} = 0, \qquad R < n < S$$

$$(7) \quad -\left[(N+S-n)\left(\frac{R}{n+1}\right)^b \lambda + \left(\frac{n}{R}\right)^a R\mu \right] P_n + (N+S-n+1)\left(\frac{R}{n}\right)^b \lambda\, P_{n-1}.$$
$$+ \left(\frac{n+1}{R}\right)^a R\mu\, P_{n+1} = 0 \qquad S \le n < T$$

$$(8) \quad -\left[(N+S-T)\left(\frac{R+1}{T+1}\right)^b \lambda + \left(\frac{T}{R}\right)^a R\mu \right] P_T + (N+S-T+1)\left(\frac{R}{T}\right)^b \lambda\, P_{T-1}$$
$$+ \left(\frac{T+1}{R+1}\right)^a (R\mu + \mu_1) P_{T+1} = 0,$$

$$(9) \quad -\left[(N+S-n)\left(\frac{R+m}{n+1}\right)^b \lambda + \left(\frac{n}{R+m}\right)^a (R\mu + \mu_m) \right] P_n +$$
$$+ (N+S-n+1)\left(\frac{R+m}{n}\right)^b \lambda\, P_{n-1} + \left(\frac{n+1}{R+m}\right)^a (R\mu + \mu_m)] P_{n+1} = 0$$
$$mT < n < (m+1)T, \quad 1 \le m < r$$

$$(10) \quad -\left[(N+S-n)\left(\frac{R+r}{n+1}\right)^b \lambda + \left(\frac{n}{R+r-1}\right)^a (R\mu + \mu_{r-1}) \right] P_n +$$
$$+ (N+S-n+1)\left(\frac{R+r-1}{n}\right)^b \lambda\, P_{n-1} + \left(\frac{n+1}{R+r}\right)^a (R\mu + \mu_r) P_{n+1} = 0, \qquad n = rT$$

$$(11) \quad -\left[(N+S-n)\left(\frac{R+r}{n+1}\right)^b \lambda + \left(\frac{n}{R+r}\right)^a (R\mu + \mu_r) \right] P_n +$$
$$+ (N+S-n+1)\left(\frac{R+r}{n}\right)^b \lambda\, P_{n-1} + \left(\frac{n+1}{R+r}\right)^a (R\mu + \mu_r) P_{n+1} = 0$$
$$rT < n < S+K$$

$$(12) \quad -\left(\frac{S+K}{R+r}\right)^b (R\mu + \mu_r)]\, P_{S-K} + (N-K+1)\left(\frac{R+r}{S+K}\right)^h \lambda\, P_{S+K-1} = 0$$

production in the system. For model developing purpose, we have made in the following assumptions:

* The unit alternates in both states i.e. in operating state and repair (or failed) state.
* The standby spare units replace the failed operating units.
* The failure rate of the unit and repair rate of the permanent (additional) repairmen is λ and $\mu(\mu_i)$ $(i = 1,2, \ldots, r)$ respectively.
* The repair facility provides repair according to FCFS manner.
* The switch over times from failure to repair, from repair to standby and from standby to operating states are assumed to be negligible.
* When there are $n < T$ failed units, only R permanent repairmen are available to repair them.
* If there are $mT < n \le (m+1)T$ failed units, there will be m special additional apart from R permanent repairmen in the system $(m = 1,2, \ldots, r-1)$.
* When $rT < n \le S + K$ failed units, all the permanent and additional repairmen will be busy to provide service in the system.
* On completion of repair, the unit joins the set of standby units and treated as good as new ones.
* P_n represents the steady-state probability of n^{th} state $(n = 1,2, \ldots, S + K)$ while P_0 is the steady -state probability of empty.

3. Steady-state Equations and their Analysis

We consider two cases for analysis purpose, which are given as below

Case I: $R \le S$. The birth death rates for the model are given by

$$(1) \qquad \mu_n = \begin{cases} N\lambda & 0 < n < R \\ N\left(\dfrac{R}{n+1}\right)^h \lambda, & R \le n < S \\ (N+S-n)\left(\dfrac{R}{n+1}\right)^b \lambda, & S \le n < T \\ (N+S-n)\left(\dfrac{R+m}{n+1}\right)^b \lambda, & mT \le n < (m+1)T, 1 \le m < r \\ (N+S-n)\left(\dfrac{R+r}{n+1}\right)^b \lambda, & rT \le n < S+K \end{cases}$$

and

$$(2) \qquad \mu_n = \begin{cases} N\mu & 0 < n < R \\ \left(\dfrac{n}{R}\right)^a R\mu, & R < n < T \\ \left(\dfrac{n}{R+m}\right)^a (R\mu + \mu_m) & mT < n \le (m+1)T, 1 \le m < r \\ \left(\dfrac{n}{R+r}\right)^a (R\mu + \mu_r) & rT < n \le S+K \end{cases}$$

machine repairmen problems were described by Either [2]. A cost function for the machine interference problem was developed by Moshe [17]. Wang and Hsu [19] carried out the cost analysis of machine repair problem with R non-reliable service stations. The M/G/I machine interference model with spares was studied by Gupta and Rao [5]. Jain [7] introduced diffusion approximation for (m.M) machine repair problem with spares and state dependent rates. M/M/R machine repair problem with spares and additional repairmen was also considered by Jain [8]. N-policy queueing system with finite source and warm spares was discussed by Gupta [3]. Jain and Dhyani [9] considered the transient analysis of M/M/C machine repair problem with spares. Jain considered the transient analysis of M/M/C machine repair problem with spares. Jain and Ghimire [10] investigated machine repair queueing system with non-reliable service stations and heterogeneous service discipline. Optimal repair/replacement policy for a general repair model was given by Jiang et al, [13]. They provided the repair cost-limit and the optimal average cost. Yakasai [20] developed a cost-off replacement policy for a component demanded by two parallel units by taking a minimum total cost function.

In the long queue of failed units, there may be some inconvenience due to which the failed units may be discouraged to join the queue. It means either they may balk or renege without being served. Some efforts are also made to analyze machine repair problem with balking and/or reneging, (m.M) machine repair problems with spares and reneging was investigated by Jain and Singh [12]. Ke and Wang [16] suggested cost analysis of the M/M/R machine repair problem with balking. reneging and server breakdowns. Recently Jain et al. [14] developed M/M/C/K/N machine repair problem with balking, reneging, spares and additional repairman. The two modes of failure machine repair problem with spares, reneging and additional repairman was also tackled by Jain et al. [15].

In the present paper we study a multi-component repairable system with spares and state-dependent rates by using birth-death techniques. The rest of the paper is organized in the following manner: The terminology of the model and notations used are given in section 2. In section 3, the balance equations in steady-state and their product obtain the optimal number of repairmen and spares, a heuristic approach is suggested in section 5. The conclusion and ideas for further development of the work are outlined in the last section 6.

2. The Model

We consider multi-component repairable system with N operating and S spare units. There is a provision of a repair facility consisting of R permanent and r additional repairmen. The system can accommodate only S+K units. The lifetime and repair time are state-dependent and are assumed to follow negative exponential distributions. There is provision of *r* special additional repairmen, which turn on one by one with the additional load of *T* failed units in order to maintain the regular

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A multi-Component Repairable System With Spares and State-sependent Rates

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Abstract: The queue size distribution at equilibrium for a machining system having M operating units along with S spares is obtained. There are R permanent repairmen available which provide the repair to the units failed while operating according to *FCFS*-discipline. The life times and repair times of all the units are assumed to be negative exponentially distributed with rate depending on the existing workload. Since the reliability of the system depends upon the system configuration. Therefore the provision of r special additional repairmen, which turn on according to a threshold rule depending upon the failed units in the system, is also made. This will help in reducing the backlog in case of long queue. Expression for the number of failed units in system is derived by using steady-state queue size distribution. Some other system characteristics are also reported a heuristic approach is facilitated to obtain the optima number of repairmen and spares simultaneously by minimizing the cost function

Key words: Markov, Standby, Machine Repair, Reliability, State-dependent Rates, Cost Function, Queue, Additional Repairmen.

1. Introduction

In the recent years, the advanced technology has been developed such that the system designer may meet out the desired demand of production by improving the reliability and availability of the system. Maintenance has a major impact over a long run. As soon as the system becomes more sophisticated and its units become more interdependent, the impact will increase. In view of such a design, the standby units play an important role so that the system may keep working to provide the desired grade of service all the time. The standby unit may replace the failed unit whenever the operating unit breaks down. In many applications, the behaviour of the failed unit to join the queue may depend upon the number of failed units. When all the spares are used and all permanent repairmen are busy and a unit breaks down, the production will be effected. Therefore, for maintaining continuous magnitude of the production, it is recommended that the special repair facility may be provided.

Many authors have reported research works in the field of machining system by using queue-theoretic approach. A cost model for cold standby was described by Hilliard [6]. M/M/C/m/m model with spares was studied by Gross and Harris [4]. A semi-numerical iterative method for solving a machine interference problem was given by Jain and Sharma [11]. Cherian et al. [1] studied the reliability of standby system with repair. Wang [19] provided the profit analysis of the machine repair problem with a single service station subject to breakdown. Optimal policies for

$$(1-t)^m \, C_m \left[a \, ; x \, (1-t) \right] = \sum_{n=0}^{\infty} \frac{(-m)_n}{n!} \; C_{m-n} \; (a \, ; x) \, t^n$$

which can be well compared with a result of Mcbride by Truesdell method.

(ii) Again, setting $b = 1$, and $c = 0$ in (2.12) we obtain

$$e^t (1 - t/x)^a \, C_m \left[a \, ; x - t \right] = \sum_{n=0}^{\infty} \frac{1}{n!} C_{m+n} \; (a \, ; x) \, t^n$$

This was also obtained by Mcbride by Truesdell method.

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The Chapman-Kolmogorov balance equations governing the model are given by:

(20) $\quad - N\lambda P_0 + \mu P_1 = 0$

(21) $\quad - [N\lambda + n\mu] P_n + N\lambda P_{n-1} + (n+1)\mu P_{n+1} = 0 \qquad 1 \le n < S$

(22) $\quad - [(N+S-n)\lambda + n\mu] P_n + [(N+S-n+1)\lambda] P_{n-1} + (n+1)\mu P_{n+1} = 0, \quad S \le n < R$

$$(23) \quad - \left[(N+S-R)\left(\frac{R}{n+1}\right)^b \lambda + R\mu \right] P_R + (N+S-R+1)\lambda P_{n-1} +$$

$$+ \left(\frac{R+1}{R}\right) R \mu P_{R+1} = 0,$$

$$(24) \quad - \left[(N+S-n)\left(\frac{R}{n+1}\right)^b \lambda + \left(\frac{n}{R}\right)^a R\mu \right] P_n + (N+S-n+1)\left(\frac{R}{n}\right)^b \lambda P_{n-1} +$$

$$+ \left(\frac{n+1}{R}\right) R\mu P_{n+1} = 0 \qquad R \le n < T$$

$$(25) \quad - \left[(N+S-T)\left(\frac{R+1}{T+1}\right)^b \lambda + \left(\frac{T}{R}\right)^a R\mu \right] P_T + (N+S-T+1)\left(\frac{R}{T}\right)^b \lambda P_{T-1} +$$

$$+ \left(\frac{R+1}{R+1}\right)^a (R\mu + \mu_1) P_{T+1} = 0,$$

$$(26) \quad - \left[(N+S-n)\left(\frac{R+m}{n+1}\right)^b \lambda + \left(\frac{n}{R+m}\right)^a (R\mu + \mu_m) \right] P_n + (N+S-n+1)$$

$$\left(\frac{r+m}{n}\right)^b \lambda P_{n-1} + \left(\frac{n+1}{R+m}\right)^a (R\mu + \mu_m) P_{n+1} \Bigg] P_{n+1} = 0.$$

$$MT < n < (m+1)T, \quad 1 \le m < r$$

$$(27) \quad - \left[(N+S-n)\left(\frac{R+r}{n+1}\right)^b \lambda + \left(\frac{n}{R+r-1}\right)^a (R\mu + \mu_{r-1}) \right] P_n +$$

$$+ (N+S-n+1)\left(\frac{R+r-1}{n}\right)^b \lambda P_{n-1} + \left(\frac{R+1}{R+r}\right) (R\mu + \mu_1 \Big] P_{n+1} = 0. \qquad N = rT$$

$$(28) \quad - (N+S-n)\left(\frac{R+r}{n+1}\right)^b \lambda + \left(\frac{n}{R+r}\right)^a (R\mu + \mu_1) \Big] P_n +$$

$$+ (N+S-n+1)\left(\frac{R+r}{n}\right)^b \lambda P_{n-1} + \left(\frac{n+1}{R+r}\right)^a (R\mu + \mu_1)] P_{n+1} = 0, \qquad rT < n < S+K$$

$$(29) \quad -\left(\frac{S+K}{R+r}\right)^a (R\mu + \mu_r)] P_{S+K} + (N-K+1)\left(\frac{R+r}{S+K}\right)^a \lambda P_{S+K-1} = 0,$$

We find the probabilities for different states by solving the equations (20)-(29) as :

$$(30)\quad P_n=\begin{cases} \dfrac{N^n\rho^n}{n!}P_0 & 0<n\le S\\ \dfrac{N^nN!\,\rho^n}{n!(N+S-n)!}P_0, & S<n\le R\\ \dfrac{N^SN!\,\rho^n}{R!(N+S-n)!\,R^{(1-D)(n-R)}\left(\frac{n!}{R!}\right)^D}P_0, & R<n\le T\\ \dfrac{N^SN!\,\rho^n}{R!(N+S-n)!\,R^{n-(1-D)R-DT}\left(\frac{n!}{R!}\right)^D}\prod_{k=1}^{m-1}\left\{\dfrac{(R+k)^{DT}}{\left(1+\frac{\mu_k}{R_\mu}\right)^T}\right\}\dfrac{(R+m)^{(n-mT)D}}{\left(1+\frac{\mu_m}{R\mu}\right)^{(n-mT)}}P_0, & mT<n\le(m+1)T,\quad 1\le m<r\\ \dfrac{N^nN!\,\rho^n}{R!(N+S-n)!\,R^{n-(1-D)R-DT}\left(\frac{n!}{R!}\right)^D}\prod_{k=1}^{r-1}\left\{\dfrac{(R+k)^{DT}}{1+\left(1+\frac{\mu_k}{R\mu}\right)^T}\right\}\dfrac{(R+k)^{(n-rT)D}}{\left(1+\frac{\mu_k}{R\mu}\right)^{(n-rT)}}P_0, & rT<n\le S+K \end{cases}$$

The normalizing condition (15) results in

$$(31)\quad P_0^{-1}=\sum_{n=0}^{S}\frac{N^n\rho^n}{n!}+\sum_{n=S+1}^{R}\frac{N^SN!\,\rho^n}{n!(N+S-n)}+\frac{N^SN!}{R!}\sum_{n=R-1}^{T}\frac{\rho^n}{(N+S-n)R^{(1-D)(n-R)}\left(\frac{n!}{R!}\right)^D}+$$

$$+\frac{N^nN!}{R!}\sum_{n=mT+1}^{(m+1)T}\frac{\rho^n}{(N+S-n)R^{n-(1-D)R-DT}\left(\frac{n!}{R!}\right)^D}\prod_{k=1}^{m-1}\left\{\frac{(R+k)^{DT}}{\left(1+\frac{\mu_k}{R\mu}\right)^T}\right\}\frac{(R+m)^{(n-mT)D}}{\left(1+\frac{\mu_r}{R\mu}\right)^{(n-DT)}}$$

$$+\frac{N^nN!}{R!}\sum_{n=rT+1}^{S+K}\frac{\rho^n}{(N+S-n)!\,R^{n-(1-D)R-DT}\left(\frac{n!}{R!}\right)^D}\prod_{k=1}^{r-1}\left\{\frac{(R+k)^{DT}}{\left(1+\frac{\mu_k}{R\mu}\right)^T}\right\}\frac{(R+r)^{(n-rT)D}}{\left(1+\frac{\mu_r}{R\mu}\right)^{(n-rT)}}$$

The average number of failed units in the system

$$(32)\quad E(N)=\sum_{n=0}^{S+K}n\,P_n$$

$$=\sum_{n=0}^{S}\frac{N^n\rho^n}{(n-1)!}P_0+\sum_{n=S+1}^{R}\frac{N^n\rho^n n}{n!(N+S-n)!}P_0+$$

$$+\frac{N^SN!}{R!}\sum_{n=R+1}^{r}\frac{n\rho^n}{(N+S-n)R^{(1-D)(n-R)}\left(\frac{n!}{R!}\right)^D}P_0$$

$$+\frac{N^n N!}{R!}\sum_{n=mT+1}^{(m+1)T}\frac{n\rho^n}{(N+S-n)R^{n-(1-D)r-DT}\left(\frac{n!}{R!}\right)^D}\prod_{k=1}^{m-1}\left\{\frac{(R+k)^{DT}}{\left(1+\frac{\mu_k}{R\mu}\right)^T}\right\}\frac{(R+m)^{(n-mT)D}}{\left(1+\frac{\mu_m}{R\mu}\right)^{(n-mT)}}P_0$$

$$+\frac{N^n N!}{R!}\sum_{n=rT+1}^{S+K}\frac{n\rho^n}{(N+S-n)!\,R^{n-(1-D)R-DT}\left(\frac{n!}{R!}\right)^D}\prod_{k=1}^{r-1}\left\{\frac{(R+k)^{DT}}{\left(1+\frac{\mu_k}{R\mu}\right)^T}\right\}\frac{(R+r)^{(n-rT)D}}{\left(1+\frac{\mu_r}{R\mu}\right)^{(n-rT)}}P_0$$

4. Some System Characteristics

Using steady-state queue size distribution given equations (13) and (30) for case I and II, we also obtain some more performance measures as follows:

- The average number of spare units in the system is obtained as

(33) $$E(S)=\sum_{n=0}^{S}(S-n)P_n$$

- The average number of operating units in the system is given by

(34) $$E(0)=N-\sum_{n=Y+1}^{S+K}(n-S)P_n$$

- The average number of permanent idle repairmen is given by

(35) $$E(I)=\sum_{n=0}^{R-1}(R-n)P_n$$

The average number of permanent busy serves in the system is

(36) $$E(B)=R-E(I)$$

- The rate of production per unit is given as

(37) $$\text{P.R.}=1-\frac{E(N)}{N+S}$$

- The operating utilization is denoted as

(38) $$\text{O.U.}=\frac{E(B)}{R+r}$$

5. Cost Functions

Our main aim in this section is to provide a cost function, which is to be minimized to determine the optimal number of repairmen and spares by considering different costs. The total average cost is given by :

(39) $$E(C) = C_E \sum_{n=S+}^{S+K} (n-S) P_n + C_S E(S) + C_I E(I) + C_B E(B) + C_A \sum_{n=T+1}^{S+K} (n-T) P_n$$

where,

C_E Cost per unit time when all spares are employed.
C_S Cost per unit time for providing one spare unit.
C_B Cost per unit time when one permanent repairman is in busy state.
C_I Cost per unit time when one permanent repairman is in idle state.
C_A Cost per unit time of providing one additional repairman.

We can illustrate minimum cost as follows:

$$\text{Min } (Z^*) = F(R^*.S^*).$$

subject to constraints $A_v = \sum_{n=0}^{S} P_n \geq A$.

Here Av denotes the availability of the system while A shows the minimum fraction of time. Since analytical solution for evaluating the optimal number of repairmen and spares is very difficult, a direct research technique may be employed to determine the optimum value of the cost.

6. Discussion

In this paper we have tackled the reliability issues of a multi-component system consisting of S warm standby spares along with R permanent repairmen and r additional repairmen so that the system may provide regular magnitude of production up to a desired grade of demand. The steady-state probability distribution of the average number of breakdown units and some system characteristics are developed. In the last section of the paper, a cost function is also facilitated that may be very useful to practitioners and other system designers for a stream of tasks in various manufacturing / production processes.

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Unbiased Estimation of The Population Variance Using Midzuno-Sen Type Sampling Scheme

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Abstract: We have employed Midzuno-Sen type sampling scheme to propose two unbiased strategies for estimating the population variance. These strategies have been compared with certain known ones, and necessary and sufficient conditions have obtained for their superior performance as compared to the known ones. An unbiased variance estimator of the estimator the population variance has also been worked out. Real-life data are shown to yield substantial gains via these strategies.

Key words: Unbiased estimation of the population variance ; Midzuno-Sen type sampling scheme.

1. Introduction

By and large, the estimators of the population variance (based on auxiliary information) that have been proposed in the literature were not mooted from the point of view of statistical property of unbiasedness. Although unbiasedness should be an obsessive property, yet it desirable to seek unbiasedness of estimators whenever it is feasible. For the purpose of obtaining an unbiased estimator of the population variance, we, in this paper, take to Midzuno-Sen type sampling scheme.

2. Some Unbiased Estimators of The Population Variance Under Midzuno-Sen Type Sampling Scheme.

Consider a finite population of N units in which y_i and x_i are the measurements in respect of the study variable y and the ausiliary variable x taken on the ith unit $(i = 1,2, \ \ldots\ldots, N)$ of the population from which a sample s of size n is drawn according to a certain sampling design. Let $\overline{Y}$ and $\overline{y}$ be the population and the sample means respectively of the study variable y and let $\overline{X}$ and $\overline{y}$ be the population and sample means respectively of the auxiliary variable x. We now define the following population and sample quantities:

$$\mu_{r,s} = \frac{1}{N}\sum_{i=1}^{N}(x_i - \overline{X})^r (y_i - \overline{Y})^s,$$

$$m_{r,s} = \frac{1}{N}\sum_{i=1}^{N} x_i^r \, x_i^s$$

(for any specified r and s)

$$\beta_2(y) = \frac{\mu_{04}}{\mu_{02}^2}, \quad \theta = \frac{\mu_{22}}{\mu_{02}\,\mu_{20}}$$

$$S_y^2 = \frac{1}{N}\sum_{i=1}^{N}(y_i - \overline{Y})^2$$

$$s_y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \overline{y})^2$$

We similarly define the quantities $\beta_2(x)$, S_x^2 and S_x^2 for the variable x which being based on the auxiliary information is supposed to be known. Further, later in this paper, we would, to terms of $O\left(\frac{1}{n}\right)$, use the following well-known results :

$$V\left(s_y^2\right) = \frac{\lambda}{n} S_y^4 \left(\beta_2(y) - 1\right)$$

$$V\left(s_x^2\right) = \frac{\lambda}{n} S_x^4 \left(\beta_2(x) - 1\right)$$

$$Cov\left(s_y^2, s_x^2\right) = \frac{\lambda}{n} S_y^4 S_x^2 \left(\theta - 1\right)$$

where $\lambda = \dfrac{n-1}{N}$

An unbiased estimator of the population variance, under the simple random sampling without replacement design, say p_o, when non auxiliary variable is used, is given by

(2.1) $$t = s_y^2$$

Isaki [2] proposed the ratio-type estimator of the population variance

(2.2) $$t_o = \frac{s_y^2}{s_x^2} S_x^2$$

which is based under the sampling design p_o. Although Agrawal an Sthapit [1] alluded to the sampling designs which render t_o unbiased, but, to terms of $O\left(\frac{1}{n}\right)$, the variance of t_o under these design remains equal to the one under the design p_o. Hence, we would continue to discuss t_o under the design p_o.

It is known that, under the Midzuno-Sen sampling scheme, the probability of selecting a specified sample s is given by

$$p(s) = \frac{1}{\binom{N-1}{n-1}} \sum_{i \varepsilon s} p_i$$

where p_i is the initial probability of selecting the ith unit. If we consider p_i in accordance with either of the following schemes for a suitably chosen r ($r = 1, 2, \ldots$)

(a) $p_i \propto x_i^r$

and (b) $p_i \propto (x_i - \overline{X})^r$.

then we obtain

$$p_1(s) = \frac{1}{\binom{N}{n}} \frac{\hat{m}_{r,o}}{m_{r,o}}$$

for scheme (a) and

$$p_2(s) = \frac{1}{\binom{N}{n}} \frac{\hat{\mu}_{r,o}}{\mu_{r,o}}$$

for scheme (b) when $\hat{m}_{r,o}$ and $\hat{\mu}_{r,o}$ are the sample-based quantities corresponding to $m_{r,o}$ and $\mu_{r,o}$. Now, we propose the estimators of the population variance under scheme (a) as

$$t_1 = s_y^2 \frac{m_{r,o}}{\hat{m}_{r,o}} \tag{2.3}$$

and under scheme (b) as

$$t_2 = s_y^2 \frac{\mu_{r,o}}{\hat{\mu}_{r,o}} \tag{2.4}$$

Both the estimator t_1 and t_2 can be verified as being unbiased. For this purpose, we can note that

$$\begin{aligned} E_{p_1}(t_1) &= \sum_{s \varepsilon S} p_1(s)\, t_1(s) \\ &= \frac{1}{\binom{N}{n}} \sum_{s \varepsilon S} s_y^2 \\ &= E_{p_o}\left(s_y^2\right) \\ &= S_y^2 \end{aligned}$$

Similarly, the estimator t_2 can be shown to be unbiased.

Denoting the strategies $(p_o, t), (p_o, t_o), (p_1, t_1)$, and (p_2, t_2), by D_1, D_o, D_1 and D_2 respectively, we compare them in the next section.

3. A Comparison of The Competing Strategies

The variance, to terms of $O\left(\frac{1}{n}\right)$, for the strategy D, when no auxiliary information is used is

$$V_{p_o}(t) = \frac{\lambda}{n} S_y^4 \, [\beta_2(y) - 1] \tag{3.1}$$

The mean square error (MSE), to terms of $O\left(\frac{1}{n}\right)$, for the strategy D is given by

$$MSE_{p_o}(t_o) = \frac{\lambda}{n} S_y^4 \, [\beta_2(y) - 1 + \beta_2(x) - 1 - 2(\theta - 1)] \tag{3.2}$$

Now, we proceed to obtain the variances of the proposed estimators t_1 and t_2 defined by (2.3) and (2.4) under the designs $p_1(s)$ and $p_2(s)$ respectively. For the strategy D_1, we can write.

$$V_{p_1}(t_1) = E_{p_1}(t_1^2) - S_y^4$$

$$m_{r,o} = \frac{1}{\binom{N}{n}} \sum_{s \varepsilon S} (s_y^2 / \hat{m}_{r,o}) - S_y^4$$

$$= m_{r,o} \, E_{P_0}(s_y^4 / \hat{m}_{r,o}) - S_y^4$$

which, after some algebra, is obtainable, to terms of $O\left(\frac{1}{n}\right)$, as

$$V_{p_1}(t_1) = \frac{\lambda}{n} S_y^4 \left[\beta_2(y) - 1 + \left(\frac{\overset{*}{\mu}_{2r,0}}{\overset{*}{\mu}_{r,0}} - 1\right) - 2\left(\frac{\overset{*}{\mu}_{2r,0}}{\overset{*}{\mu}_{0,2} \, \overset{*}{\mu}_{r,0}} - 1\right)\right] \tag{3.3}$$

where $\mu_{r,s}^* = \frac{1}{N} \sum_{i=1}^{N} x_i^r (y_i - \overline{Y})^S$.

In manner similar to the above, we can work out the variance for the strategy D_2, as

$$V_{p_2}(t_2) = \frac{\lambda}{n} S_y^4 \left[\beta_2(y) - 1 + \left(\frac{\mu_{2r,0}}{\mu_{2r,0}} - 1\right) - 2\left(\frac{\mu_{r,2}}{\mu_{0,2} \, \mu_{r,0}} - 1\right)\right]. \tag{3.4}$$

Now, by setting $(y_i - \overline{Y})^2 = w_i, (x_i - \overline{X})^2 = u_i, x_i^r = v_i^*$ and $u^{r/2} = v_i$, the various variance expressions given by (3.1), (3.2), (3.3) and (3.4) can alternatively be expressed as

$$V_{p_o}(t) = \frac{\lambda}{n} \overline{W}^2 \, C_1^2, \tag{3.5}$$

$$V_{p_0}(t) = \frac{\lambda}{n} \overline{W}^2 \, (C_0^2 + C_1^2 - 2 \, \rho_0 \, C_0 \, C_1), \tag{3.6}$$

$$V_{p_1}(t_1) = \frac{\lambda}{n} \overline{W}^2 \, (C_0^2 + C_2^2 - 2 \, \rho_1 \, C_0 \, C_1), \tag{3.7}$$

$$V_{p_2}(t_2) = \frac{\lambda}{n} \overline{W}^2 \, (C_0^2 + C_3^2 - 2 \, \rho_2 \, C_0 \, C_1), \tag{3.8}$$

where C_0, C_1, C_2 and C_3 are the coefficients of variation of w, u, v^* and v respectively and ρ_0, ρ_1 and ρ_2 are the coefficients of correlation between w and u, w and v, and w and v^* respectively.

Needless to say for employing the strategies D_1 and D_2, a proper choice of r has to be made. Regarding the relative performance of the competing strategies D, D_0, D_1 and D_2, we can based on the relevant variances given by (3.5), (3.6), (3.7) and (3.8), arrive at the following conclusions.

(i) The strategy D_0 scores over the strategy D if and only if

$$\rho_o \geq \frac{1}{2}\frac{C_0}{C_1}$$

(ii) The strategy D_1 performs better than D_2 if and only if

$$\frac{1}{2}\frac{C_3}{C_0}\left(\frac{C_2^2}{C_3^2}-1\right)-\left(\rho_o\frac{C_2}{C_3}-\rho_2\right)\leq 0,$$

(iii) The strategy D_1 will outperform the strategy D_0 if and only if

$$\frac{1}{2}\frac{C_1}{C_0}\left(\frac{C_2^2}{C_1^2}-1\right)-\left(\rho_1\frac{C_2}{C_1}-\rho_0\right)\leq 0;$$

while the strategy D_2 performs better then the strategy D_0 if and only if

$$\frac{1}{2}\frac{C_1}{C_0}\left(\frac{C_3^2}{C_1^2}-1\right)-\left(\rho_2\frac{C_3}{C_1}-\rho_0\right)\leq 0;\ and$$

(iv) The strategy D_1 fares better than the strategy D if and only if

$$\rho_1 \geq \frac{1}{2}\frac{C_2}{C_0};$$

While the strategy D_2 scores over the strategy D if and only if

$$\rho_2 \geq \frac{1}{2}\frac{C_3}{C_0};$$

4. Unbiased Variance Estimation

To obtain an unbiased estimator, under the design p_1, of the variance of t_1, we write

$$V_{p_1}(t_1) = E_{p_1}(t_1^2) - S_y^4$$

which yields

(4.1) $$\hat{V}_{p_1}(t_1) = t_1^2 - \hat{S}_y^4$$

Now, with a view to estimating S_y^4, we first express it as

(4.2) $$S_y^4 = \frac{1}{(N-1)^2}\left[\sum_{i=1}^{N} y_i^4 - 2N\bar{Y}^2\sum_{i=1}^{N} y_i^2 + N^2\bar{Y}^4 + \sum_{i\neq j}^{N} y_i^2 y_j^2\right]$$

$$= \frac{1}{N^2(N-1)^2}\left[(N-1)^2\sum_{i=1}^{N} y_i^4 - 4(N-1)\sum_{i\neq j}^{N} y_i^3 y_j + (N^2-2N+3)\sum_{i\neq j}^{N} y_i^2 y_j^2 \right.$$
$$\left. -2(N-3)\sum_{i\neq j\neq k}^{N} y_i^2 y_j y_k + \sum_{i\neq j\neq k\neq 1}^{N} y_i y_j y_k y_1 \right]$$

Since, under the design p_1, we have

$$E_{p_1}\left[\frac{N}{n}\sum_{i=1}^{n}\frac{m_{r,o}}{\hat{m}_{r,o}} y_i^4\right] = \sum_{i=1}^{n} y_i^4,$$

$$E_{p_1}\left[\frac{N(N-1)}{n(n-1)}\sum_{i\neq j}^{n} y_i^2 y_j^2 \frac{m_{r,o}}{\hat{m}_{r,o}}\right] = \sum_{i\neq 1}^{N} y_i^2 y_j^2,$$

and so on, can, thus, replace all the terms on the right hand side of (4.2) by the respective unbiased estimating quantities and then, after some algebra, we obtain an unbiased estimator of S_y^4 as

$$\text{(4.3)} \qquad \hat{S}_y^4 = \frac{1}{AN(N-1)}\frac{m_{r,o}}{\hat{m}_{r,o}}\left[C\sum_{i=1}^{n}\left(y_i^2 - \sum_{i=1}^{n} y_i^2/n\right)^2 \right.$$
$$\left. + 4C\left\{\left(\sum_{i=1}^{n} y_i^2\right)^2/n - \bar{y}\sum_{i=1}^{n} y_i^3\right\} + Bs_y^4\right]$$

where $A = (n-1)(n-2)(n-3)$
$B = n(n-1)^2 (N-2)(N-3)$
$C = (N-n)(N+n+1-Nn)$

which, to terms of $O\left(\frac{1}{n}\right)$, can be expressed as

$$\text{(4.4)} \quad \hat{S}_y^4 = \frac{m_{r,o}}{\hat{m}_{r,o}}\left(1+\frac{4\lambda}{n}\right)s_y^4 - \frac{\lambda}{n^2}\sum_{i=1}^{n}\left(y_i^2 - \sum_{i=1}^{n} y_i^2/n\right)^2 - \frac{4\lambda}{n^2}\left\{\left(\sum_{i=1}^{n} y_i^2\right)^2/n - y\sum_{i=1}^{n} y_i^3\right\}\Bigg]$$

and the same is then inserted in (4.1) to obtain the requisite variance estimator of t_1. In a manner similar to the above, obtain, under the sampling design p_2 unbiased S_y^4 estimator if we replace $m_{r,o}$ *and* $\hat{m}_{r,o}$ *by* $\mu_{r,o}$ *and* $\hat{\mu}_{r,o}$ respectively, and hence the variance estimator of t_2.

5. Empirical Investigation

To illustrate the potential gain that might accrue from the use of the proposed strategies D_1 and D_2 over the known ones, viz., D and D_0, we consider the following data sets:

Data-Set 1 : We consider first fifty four (1–54) observations from Murthy (1967, p. 178) and the following quantities are obtained therefrom:

$$N = 54,\ n = 18,\ \beta_2(y) = 3.799,\ \beta_2(x) = 2.012,\ \theta = 1.627,\ \frac{MSE\ (t_o)}{\frac{\lambda}{n} S_y^4} = 2.557$$

$$\frac{V(t)}{\frac{\lambda}{n} S_y^4} = 2.799,\ \frac{V(t_1)}{\frac{\lambda}{n} S_y^4} = 2.209\ (for\ r = 4)\ and\ \frac{V(t_2)}{\frac{\lambda}{n} S_y^4} = 2.557\ (for\ r = 2)$$

Data-Set 2: We refer to the data available in Kish ([3], p. 213, Ex.6.6). However, treating the given data as unclustered, we compute the following quantities therefrom

$$N = 17,\ n = 7,\ \beta_2(y) = 10.078.,\ \beta_2(x) = 3.079,\ \theta = 5.687,\ \frac{MSE\ (t_o)}{\frac{\lambda}{n} S_y^4} = 2.683$$

$$\frac{V(t)}{\frac{\lambda}{n} S_y^4} = 9.078,\quad \frac{V(t_1)}{\frac{\lambda}{n} S_y^4} = 0.370\ (for\ r = 7)\ and\ \frac{V(t_2)}{\frac{\lambda}{n} S_y^4} = 0.382\ (for\ r = 4).$$

Data-Set 3: We refer to the data available in 'Singh and Chaudhary ([5]..p 141) and have computed the following quantities.

$$N = 22,\ n = 9,\ \beta_2(y) = 13.257,\ \beta_2(x) = 5.579,\ \theta = 7.713,\ \frac{MSE\ (t_o)}{\frac{\lambda}{n} S_y^4} = 3.410$$

$$\frac{V(t)}{\frac{\lambda}{n} S_y^4} = 12.257,\ \frac{V(t_1)}{\frac{\lambda}{n} S_y^4} = 0.524\ (for\ r = 7)\ and\ \frac{V(t_2)}{\frac{\lambda}{n} S_y^4} = 0.528\ \ (for\ r = 6).$$

In respect of the above data-sets, we compute the following percent gains.

$$G_1 = \left[\frac{V(t)}{V(t_1)} - 1\right] \times 100$$

$$G_1' = \left[\frac{MSE(t_0)}{V(t_1)} - 1\right] \times 100$$

$$G_2 = \left[\frac{V(t)}{V(t_2)} - 1\right] \times 100$$

$$G_2' = \left[\frac{MSE(t_0)}{V(t_2)} - 1\right] \times 100$$

and present them in the following table

Table 1: Percent gains of t_1 and t_2 relative to t and t_0

Data-set	G_1	G_1	G_2	G_1
1	26.71 (4*)	15.75 (4*)	9.46 (2*)	0 (2*)
2	2353.51 (7*)	625.14 (7*)	2276.44 (4*)	602.36 (4*)
3	2239.12 (7*)	550.76 (7*)	2221.40 (6*)	545.83 (6*)

(*indicates choice of r)

Table 1 bears it out that, for the estimating the population variance, the newly proposed strategies, D_1 and D_2 that make use of Midzuno-Sen type sampling schemes are, apart from being unbiased, capable of yielding substantial, gains in precision as compared to the known strategies D and D_0. However, between D_1 and D_2 the former is slightly better than the latter.

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Some Models Reflcting and Projecting Nepal's Fertility Scene*

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Abstract: In this paper Double exponential, Gamma, Hadwiger, Gompertz models are fitted to the fertility data of Nepal. Further they have been analyzed for their accuracy. While studying the various models which reflect the fertility scene of Nepal, the results obtained from these models are also compared.

Key words: Age specific fertility rate; Gamma, Hadwiger, Double Exponential ; Gompertz..

Introduction

Fertility is of interest to demographers and in some cases more interesting than mortality. The reason for this preference is that mortality as an event occurs only once in a life time. Fertility on the other hand occurs more frequently.

The word fecundity is the biological capacity to reproduce [3], which varies from woman to woman, and from age to age. It can be studied either as a function of marriage or independently. The stigma associated with illegitimacy varies from place to place and from time to time, so the pressure on a couple to marry in order to have children or to legitimize a conception out of wedlock will vary correspondingly. In Nepal, for example, the social customs and traditions prevent a couple to have children outside a wedlock.

In many developing countries a history of accurately recorded fertility data is strikingly missing. Reliable responses about illegitimate infants cannot be expected. In addition to that, when asked about the number of children, the information about deceased children, especially girls, are not always remembered.

Numerical Methods

In an effort to find a suitable model, an extensive use of numerous computer software packages was made. For extensive numerical interpolation and

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extrapolation, EXCEL worksheets were very useful. In addition to that, EXCEL, was intensively used for data mining and therefore, in the correction of faulty data. A rigorous use of EXCEL in the production of tables as well as in the construction of graphs was made. MATHEMATICA the computer algebra system has been used to find the fast solution of Symbolic mathematical problems (Such as pseudo inverse, higher order determinants, initial solution of non linear equations). Excel has not been programmed for Symbolic Mathematics. Mathematica has given quick and efficient solutions of such problems.

NAG FORTRAN library system and SPSS advanced statistical system were implemented in liner and non-linear optimization problems and modeling. In some cases they have complemented each other.

2. The Double Exponential Model

Multiexponential models are single formulations, which can represent a wide variety of vital events (such as mortality, fertility, marriage). The age specific fertility rate ASFR, which is a unimodal bell shaped curve, can be represented by a four parameter (a, α, μ, β) double exponential curve.

$$m(x) = a \,.\, e^{-\alpha\,.(x-\mu)-e^{-\lambda\,.(x-\mu)}}$$

where a is the scaling parameter. μ locates the schedule along x-axis and an increase in its value, while holding all the other parameters constant, moves the schedule to the right α and λ are the slope parameters.

ASFR

African countries show a high fertility rate in all age groups as compared to Latin America, Caribbean, Asia, Oceania [5]. This difference in the age specific fertility rates is prominent particularly in age groups 25–29 and 30–34, where the difference between the average for Africa and Latin America were 64 and 58 births per 1000 women, respectively. African ASFR are higher than that of Asian in all the age groups. When the strength of the family planning program and the level of development were considered, the countries with high development (group I) had lower fertility at all the age groups as compared to the countries with low development (groups IV). The age specific rates for the groups II and III fell in between. Nepal, Pakistan, Bangladesh fall in group IV where as Sri Lanka is in group II.

ASFR with respect to the strength of the family planning program reflected the same trend. Countries with a strong family planning program had a lower fertility at all ages then those with a very weak or no program. Nepal, Bangladesh and Pakistan fall into the category of countries with weak efforts, whereas Sri Lanka falls into the category of moderate efforts.

2.1. The Method of Fitting

$$m(x) = a \cdot e^{-\alpha \cdot (x-\mu) - e^{-\lambda \cdot (x-\mu)}}$$
$$= h(a, \alpha, \mu, \lambda, x)$$

Here x is the independent variable and it represents age. a, α, μ, λ are the parameters to be estimated.

(1) $$Q(a, \alpha, \mu, \lambda, x_i) = m(x) - h(a, \alpha, \mu, \lambda, x)$$

The Equation (1) represents the error e_i.

The objective is to choose $(a, \alpha, \mu, \lambda)$ from $\Re^4$ such that $\sum_i e_i^2$ would be minimal.

1. Simple Least Squares : The objective is to minimize $\sum_i e_i^2$. Here initial values of a, α, μ, λ is choosen. Through iterative scheme $\sum_i e_i^2$ is minimized. Levenberg-Marquardt method of non-linear optimization is used. That value of $(a, \alpha, \mu, \lambda, x)$ is chosen such that $\sum_i e_i^2$ is minimum.
2. Weighted Least Squares

   The objective is to minimize $\sum_i W_i e_i^2$, or minimize

   (2) $$\sum_i W_i e_i^2 = \sum_i W_i e_i^2 (m(x_i) - h(a, \alpha, \mu, \lambda))^2$$

   where W_i is the relative size of married women in the i-th age group.

   $$W_i = \frac{P_i}{P}$$

   where P_i is the number of married women in the i-th age group. P is the total number of married women.

   So it is reduced to minimize $\sum_i \frac{P_i}{P}(m(x_i) - h(a, \alpha, \mu, \lambda))^2$

   or, $$\sum_i \left(\left(\frac{P_i}{P}\right)^{\frac{1}{2}} . m(x_i) - \left(\frac{P_i}{P}\right)^{\frac{1}{2}} . h(a, \alpha, \mu, \lambda) \right)^2$$

   Since social and religious customs hinder the births to be outside the wedlock, instead of taking women in reproductive age-group, married women in reproductive age span were taken. The Levenberg-Marquardt method of non-linear optimization was used

2.2. Interpolation

As seen from the table 1, the age distribution for the year 1976 and 1986 was interpolated using Polynomial Interpolation (Lagranges interpolation formula), which states that suppose population in 1971 is P_a and population in 1981 is P_b and population in 1991 is P_c, then we use three point interpolation to estimate the population of 1976. *Population in* $1976 = f(1976)$.

$$= P_a \cdot \frac{(1976-1981).(1976-1991)}{(1971-1981).(1971-1991)} + P_b \cdot \frac{(1976-1971).(1976-1991)}{(1981-1971).(1981-1991)}$$

$$+ P_c \cdot \frac{(1976-1971).(1976-1981)}{(1991-1971).(1991-1981)}$$

The Table 1 was obtained from the table percentage of currently married females [2]. This age-distribution table 1, obtained by the above mentioned method was used as weights in the weighted least square method.

3. The Gamma Density

While testing the efficacy of various models, which best reflect Nepal's fertility, a step was taken towards fitting a Gamma density. The result was surprisingly good, better than the results obtained by other models. Hoem [1] has fitted Gamma density among several functions such as Coale-Trussell, Hadwiger, Beta to Danish fertility data, and his observation was the same. The performance of Gamma density was as good as Coale-Trussell, followed Hadwiger, Beta density did not show very promising results.

Table 1: The age distribution married female

Age	1971	1981	1991	1976 *Interpolated*	1986 *Interpolated*
00-04	843600.66	1121107.959	1334897.712	990319.0028	1235967.529
05-09	855078.22	1069815.438	1371978.204	951518.6355	1209968.628
10-14	596833.12	791370.324	1121684.883	677129.5526	939555.4341
15-19	499273.86	630165.258	917742.177	545133.8689	754368.0274
20-24	505012.64	696112.785	862121.439	603699.1489	782253.5484
25-29	476318.74	593527.743	723069.594	533381.6355	656757.0625
30-34	424669.72	505597.707	602557.995	463129.6759	552073.8134
35-39	355804.36	432322.677	509856.765	393936.5471	470962.7496
40-44	309894.12	373702.653	435695.781	342025.3121	404926.1426
45-49	218073.64	285772.617	361534.797	250915.2281	322645.8066
50-54	195118.52	249135.102	287373.813	224099.0449	270226.6914
55-59	126253.16	161205.066	213212.829	141597.1309	185076.9654
60-64	154947.06	175860.072	213212.829	163348.5979	192481.4824
65+	183640.96	227152.593	315184.182	199831.782	265603.393
Total	5744518.78	7312847.994	9270123		

3.1. The Model

$$h(x;b,c,d) = \frac{1}{\Gamma b, c^b}(x-d)^{(b-1)} e^{\frac{-(x-d)}{c}} \tag{3}$$

Here d is the lowest childbearing age.

$$\Gamma b = (b-1)!$$

Let μ denotes the mean of the distribution.
Let m denotes the mode of the distribution
σ^2 denotes the Variance.

3.2. The Fitting

The model was fitted to the ASFR (Age specific fertility rate) data of Nepal. Let $h(.\,;\theta_1,\theta_2,\dots,\theta_r)$ be the probability density function on a real line $\theta_1,\theta_2,\dots,\theta_r$ are the parameters whose optimum value has to be estimated. In this case $h(.;\theta_1,\theta_2,\dots,\theta_r) = h\,(x\,;\,b,\,c,\,d)$, as three parameters in the probability density function has to be estimated. $0 < h(x\,;\,b,\,c,\,d) < 1$ where $x \in \Re^+$.

Here x denotes the age component.

Now,

(4) $$\overset{*}{g}(x; R,\theta_1,\theta_2,\dots,\theta_r) = R,\, h(x\,;\theta_1,\theta_2,\dots,\theta_r)$$

R is the scaling parameter, $R = TFR$ (Total fertility rate)

The ASFR data were fitted to (4).

The objective is to choose that value of $(R,\theta_1,\theta_2,\dots,\theta_r)$ where

$$(R,\theta_1,\theta_2,\dots,\theta_r) \in \Re^{r+1}$$

such that

$$\sum_i (g(x_i\,;R\,\theta_1,\theta_2,\dots,\theta_r) - R,h(x;\theta_1,\theta_2,\dots,\theta_r))^2$$

was minimized.

The rates are available for age groups, rather than for single ages. Let m_j be the ASFR for the age group $(a_j, a_{j+1}]$, $j = 1, 2, \dots, k-1$. Then

$$TFR = R = \sum_{j=1}^{k-1}(a_j - a_{j+1})\,.\,m_j$$

$$M = Mean = \frac{1}{R}\sum_{j=1}^{k-1} x_j\,.\,m_j\,,$$

where r_j is the mid point of the age group $(a_j, a_{j+1}]$

$$S^2 = \frac{1}{R}\sum_{j=1}^{k-1}(a_j^2\,.\,m_j - M^2$$

(5) $$f(x) = \left(\frac{x}{b}\right)^{(c-1)} .\, e^{\frac{-x}{b}}\ \ \frac{1}{b.\Gamma c}$$

Where Γc is the gamma function of the parameter c.

$$\Gamma c = \int_0^\infty e^{-u}.\,u^{c-1}$$

Then $Mean = \mu = bc$

$Variance = \sigma^2 = b^2 c$

$Mode = b(c-1)$ Comparing (4) and (5)

$$h(x) = \frac{1}{\Gamma b . c^b}(x-d)^{(b-1)} e^{\frac{-(x-d)}{c}}$$

let $x - d = y$

$dx = dy$

$h(y) = \frac{1}{\Gamma b . c^b}(y)^{(b-1)} . e^{-\frac{y}{c}}$ for $y > 0$

$$= \frac{1}{\Gamma b . c}\left(\frac{y}{c}\right)^{(b-1)} . c^{-\frac{y}{c}}$$

where $y > 0$. Then

(6) $$Mean = \mu = bc$$

(7) $$Variance = \sigma^2 = c^2 b$$

(8) $$Mode = m = c\,(b-1)$$

From (6), (7) and (8) we get; $c = \mu = mb = \frac{\mu - d}{c} = \frac{\sigma^2}{c^2}$. Substituting the sample estimates of μ, m, σ^2 the initial values of the parameters b, c, d can be estimated.

4. Hadwiger function

Hoem [1] fitted the Hadwiger density to the Danish fertility data. The performance of Hadwiger density was second best, right next to Gamma and Coale-Trussell. Hadwiger density showed promising results, when fitted to the fertility data of Nepal

4.1. The Fitting

The model was fitted to the ASFR (Age specific fertility rate) data of Nepal Let $h(.;\theta_1, \theta_2, \ldots, \theta_r)$ be the probability density function on a real line. $\theta_1, \theta_2, \ldots, \theta_r$ are the parameters whose optimum value has to be estimated. In this case $h(.;\theta_1, \theta_2, \ldots, \theta_r) = h\,(x\,;\,H,T,D)$, as three parameters in the probability density function has to be estimated . $0 < h\,(x, H,T,D) < 1$ where $x \in \Re^+$.

Here x denotes the age component.

Now,

(9) $$g(x; R, \theta_1, \theta_2, \ldots, \theta_r) = R, h(x\,; \theta_1, \theta_2, \ldots, \theta_r)$$

R is the scaling parameter. R=TFR(Total fertility rate)

The ASFR data are fitted to (9).

The objective is to choose that value of $(R\,; \theta_1, \theta_2, \ldots, \theta_r)$ where

$$(R\,; \theta_1, \theta_2, \ldots, \theta_r) \in \Re^{r+1}.$$

such that

$$\sum_i (g(x_i\,; R, \theta_1, \theta_2, \ldots, \theta_r) - R, h(x; \theta_1, \theta_2, \ldots, \theta_r))^2$$

was minimized.

The rates are available for age groups, rather than for single ages. let m_j be the ASFR for the age group $(a_j, a_{j+1}]$, $j = 1,2, \ldots k-1$. Then

$$TFR = R = \sum_{j=1}^{k-1} (a_j - a_{j+1})\,.\,m_j$$

4.2. The Model

$$h(x) = \frac{H}{T\sqrt{\Pi}} \left(\frac{T}{x-D}\right)^{\frac{3}{2}} e^{-H^2\left(\frac{T}{x-D}+\frac{x-D}{T}-2\right)} \qquad (10)$$

where $x > D$[4]. The parameters $\theta_1 = R$,

$$\theta_2 = D + T\frac{\left(1+\frac{16.H^4}{9}\right)^{\frac{1}{2}}-1}{\frac{4}{3}\cdot H^2}$$

$$\theta_3 = T + D$$

$$\theta_4 = \frac{1}{4}\cdot\frac{T^2}{H^2}\,,$$

where $\theta_1, \theta_2, \theta_4$ are the mode, mean and variance respectively.

5. Gompertz Model

In earlier days, in order to smooth the mortality data, Gompertz formula was in frequent use. It may be written as,

$$\mu_x = Be^{kx}$$

where μ_x is the force of mortality of age at age x ;
B and k are constants and
k lies in the vicinity of .09

This law had been used to describe the mortality experience of earlier times. It takes neither infant nor childhood mortality nor excess accident mortality in early adult life into consideration. Pollard [4] showed that using Gompertz law many short cuts and approximations (resulting from this law), help work out numerical problems faster. In addition to that, these formula are robust. Gompertz function has been quiet successful in describing fertility experience.

Cumulative Fertility

Cumulative fertility (age specific) is the total number of children ever born to a woman belonging to a specific age group..

The women aged 40-49 in countries of high development and in countries with strong family planning program had about 6.1 children, whereas those in countries in a low development group and in countries with a very weak program had 5 children more at the end of the fertility span.

5.1 The Model

Here an attempt was made to fit a Gompertz function $Y = K.A^{B^t}$ to the cummulative fertility rates of Nepal. In addition to that, the mathematical meaning of the parameters of Gompertz function was analyzed.

$$Y = K.A^{B^t}$$

the origin is changed to t_0, so

(11) $$Y(t) = K.A^{B^{t-t_0}}$$

Let $t_1 - t_0 = p$ $t_2 - t_0 = t_2 - t_1 + t_1 - t_0 = 2p$

the age points t_0, t_1, t_2 are equidistant and $t_1 - t_0 = t_2 - t_1 = p$

Substituting $t = t_0$, $t = t_1$, $t = t_2$ in (11) respectively, we get:

(12) $$y(t_0) = y(0) = K.A^{B^{t-t_0}} = K.A^{B^0} = K.A$$

(13) $$y(t_1) = y(1) = K.A^{B^{t_1-t_0}}$$

(14) $$y(t_2) = y(2) = K.A^{B^{t_2-t_0}}$$

Applying logarithmic transformations on both sides of equation (12).(13) and (14) we get,

(15) $$ln\,(y_0) = ln\,K + ln\,A$$

(16) $$ln\,(y_1) = lnK + B^P.\,ln\,A.$$

(17) $$ln\,(y_2) = lnK + B^{2P}.\,ln\,A$$

From equation (15)

$$lnK = lny_0 = lnA$$

$$K = e^{lny_0 - lnA}$$

Subtracting (15) from (16)

(18) $$lny_1 - lny_0 = lnA(B^p - 1)$$

or,

$$\frac{lny_1 - lny_0}{B^p - 1} = lnA$$

or,

$$A = e^{\frac{\ln y_1 - \ln y_0}{B^p - 1}}.$$

Substracting (16) from (17)

$$lny_2 - lny_1 = lnA.(B^{2p} - B^p)$$

(19) $$lny_2 - lny_1 = B^p ln.A.(B^p - 1)$$

Dividing (19) by (18)

$$\frac{lny_2 - lny_1}{lny_1 - lny_0} = B^p \frac{ln.A.(B^p - 1)}{ln.A.(B^p - 1)} = B^p$$

or,

$$B^p = \left(\frac{lny_2 - lny_1}{lny_1 - lny_0}\right)^{\frac{1}{p}}$$

So the starting values of the parameters A, B, K can be found by the following formula.

$$B = \left(\frac{lny_2 - lny_1}{lny_1 - lny_0}\right)^{\frac{1}{p}}$$

$$A = e^{\frac{\ln y_1 - \ln y_0}{B^p - 1}}$$

$$K = e^{lny_0 - lnA}.$$

The curve (11) is fitted to the cumulative fertility rate.

6. Interpretation

While modeling the fertility scene of Nepal, Gamma density showed very good results, in comparison to Hadwiger. Gompertz and Double exponential have also performed well. Low M.S.E and R^2 close to 1 reflect this fact in the Table 4-6. Figures 1-4 show the performance of all the four models. Whereas Figure 5 shows the comparative performance. The parameters have been projected using linear extrapolation. The projected ASFR values can be seen in Table 2 and Table 3. Figure 6 shows the overall behaviour of ASFR over time. It can be seen that the modal age of fertility is gradually increasing.

Table 2: Projected ASFR 2001-2015; Nepal

Age	2001	2005	2011	2015
17	0.0431	0.0376	0.0294	0.0246
22	0.0974	0.0914	0.07845	0.0695
27	0.1824	0.1767	0.1616	0.1499
32	0.2569	0.2528	0.2440	0.2362
37	0.2288	0.2405	0.2521	0.2572
42	0.0906	0.1290	0.1629	0.1806
47	0.0070	0.0298	0.0583	0.0750

Table 3: Projected Cumulative ASFR 2001-2015; Nepal

Age	2001	2005	2011	2015
17	0.1186	0.1244	0.1334	0.1396
22	0.3792	0.3857	0.3946	0.3999
27	0.6626	0.6612	0.6572	0.6534
32	0.8662	0.8547	0.8355	0.8216
37	0.9852	0.9658	0.9353	0.9142
42	1.0480	1.0237	0.9864	0.9609
47	1.0795.	1.0525	1.0113	0.9836

Table 4: Result of Gamma fitting to the A.S.F.R. Nepal

Age	1991	1986	1081	1971
R	5.6	6	6.3	6.3
M	28.699374	28.633499	31.453748	29.924603
S^2	55.6090	53.9652	68.3818	56.4665
S	7.4571	7.3461	8.2693	7.5144
mode	22	22	27	27
Initial				
d	14.5	14.5	14.5	14.5
r	5.6	6	6.3	6.3
c	6.699376	6.633499	4.453748	2.924603
b	2.119507	2.130625	3.806625	5.274084
Final				
R	6.511	6.683748	8.15934	7.08345353
B	2.644093	2.081419	2.4153388.	3.812741
C	5.812354	6.932186	8.442468	4.8009388
D	14.772587	16.005439	15.085	12.9716545
R^2	0.98875	0.97253	0.9878	0.99012
MSE	2.2394E-04	6.9013E-04	1.4554E-04	2.1840E-04

Table 5: Result of Hadwiger fitting to the A.S.F.R. Nepal

Age	1991	1986	1076
Initial			
R	5.6	6	6.4
H	21.9561	21.9085	22.5313
T^2	38.3635	38.2803	39.3686
D	38.1515	38.0714	39.1536
Final			
R	5.92762	6.495693	6.7473002
H	1.5265	1.276313.	6.520562
T	21.58	19.749947	79.520829
D	8.759098	11.151191	-51.581816
R^2	0.98554	0.96718	0.9849
MSE	2.8802E-04	8.2457E-04	3.5360E-04

Table 6: Result of Gompertz fitting to Cummulative A.S.F.R. Nepal

Age	1991	1986	1081	1976	1971
Initial					
B	0.827458706	0.810509347	0.843590829	0.860293202	0.842008546
A	0.103400063	0.100036289	0.072797666	0.125221248	0.070725666
K	0.91876153	0.939659009	0.906622478	1.15795045	1.046296258
Initial					
B	0.868032	0.867425	0.895601	0.872698	0.87423
A	0.094775	0.094931	0.063997	0.1061	0.065043
K	1.162837	1.252469	1.37882	1.332043	1.328708
R^2	0.99918	0.99802	0.99926	0.99901	0.99956
MSE	.0001877	.000529	.0002224	.0002208	.0001386

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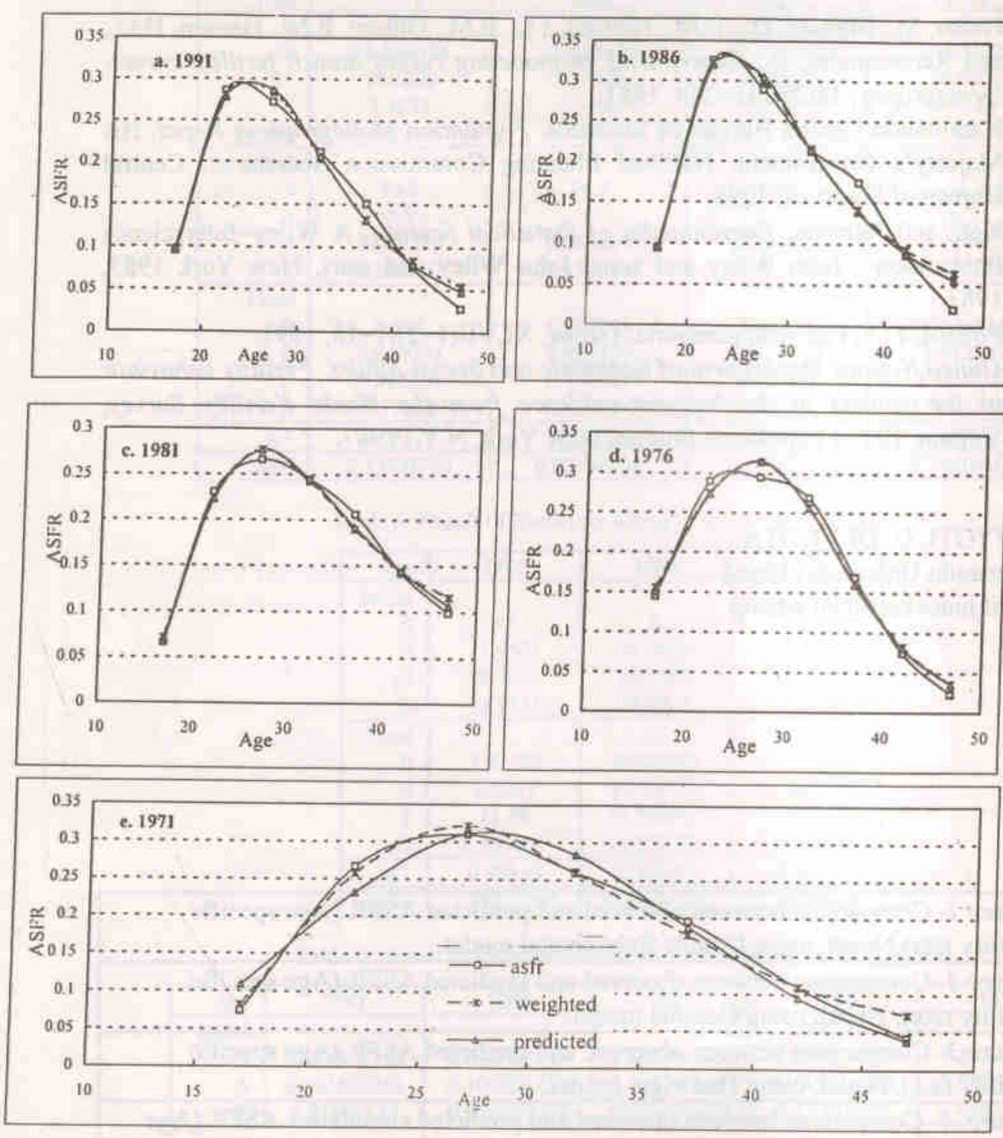
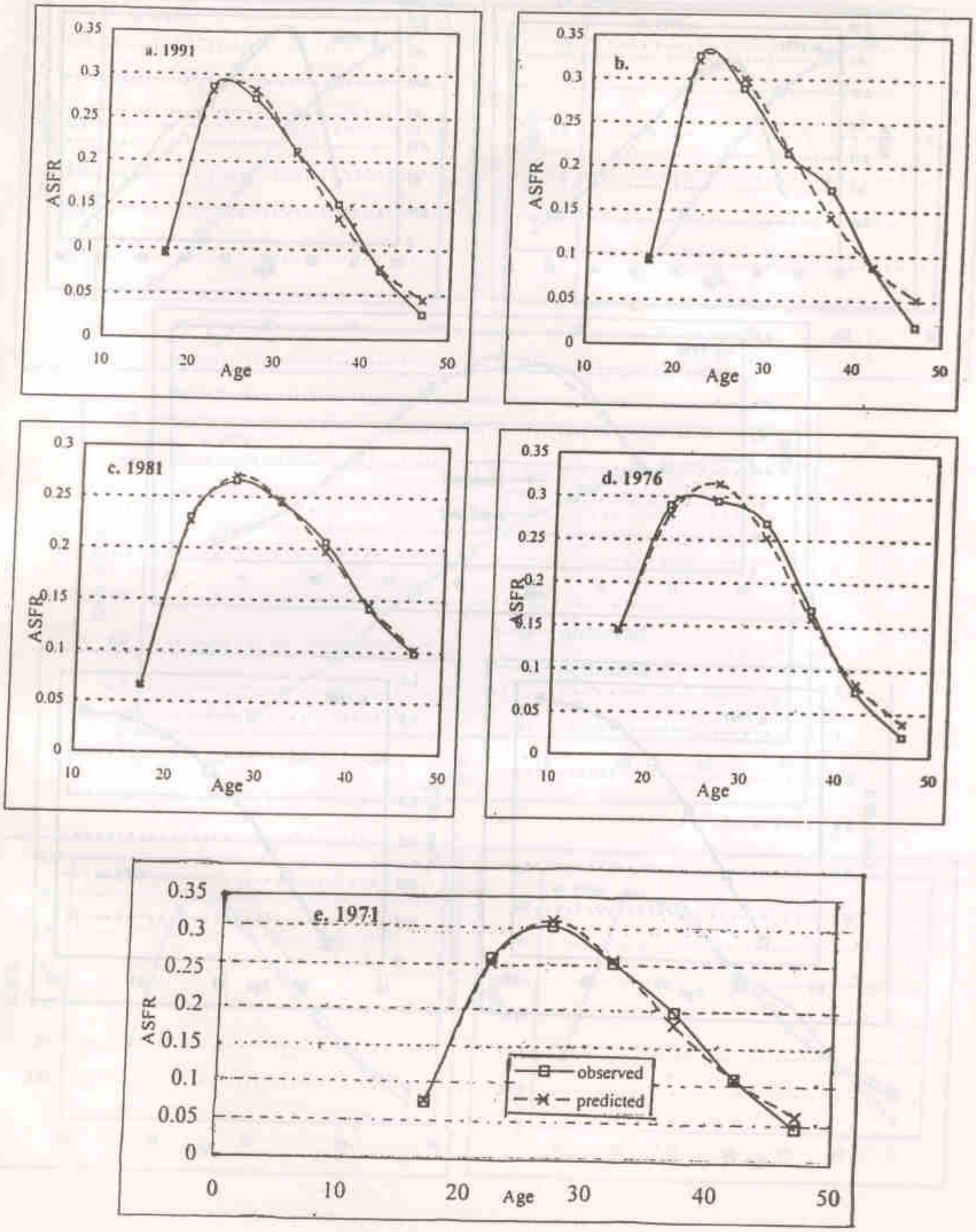
Figure 1- Comparison between observed and predicted ASFR (Age specific fertility rate),Nepal, using Double Exponential model.
Figure 2- Comparison between observed and predicted ASFR (Age specific fertility rate), Nepal, using Gamma model.
Figure 3- Comparison between observed and predicted ASFR (Age specific fertility rate), Nepal, using Hadwiger model.
Figure 4- Comparison between observed and predicted cumulative ASFR (Age specific fertility rate), Nepal, using Gompertz model.
Figure 5- Comparison between Gamma, Hadwiger, Weighted double exponential and Double exponential model, Nepal.
Figure 6- Movement of ASFR (Age specific fertility rate), Nepal, with respect to time.


a. 1991
b. 1986
c. 1981
d. 1976
e. 1971
ASFR
Age
asfr
weighted
predicted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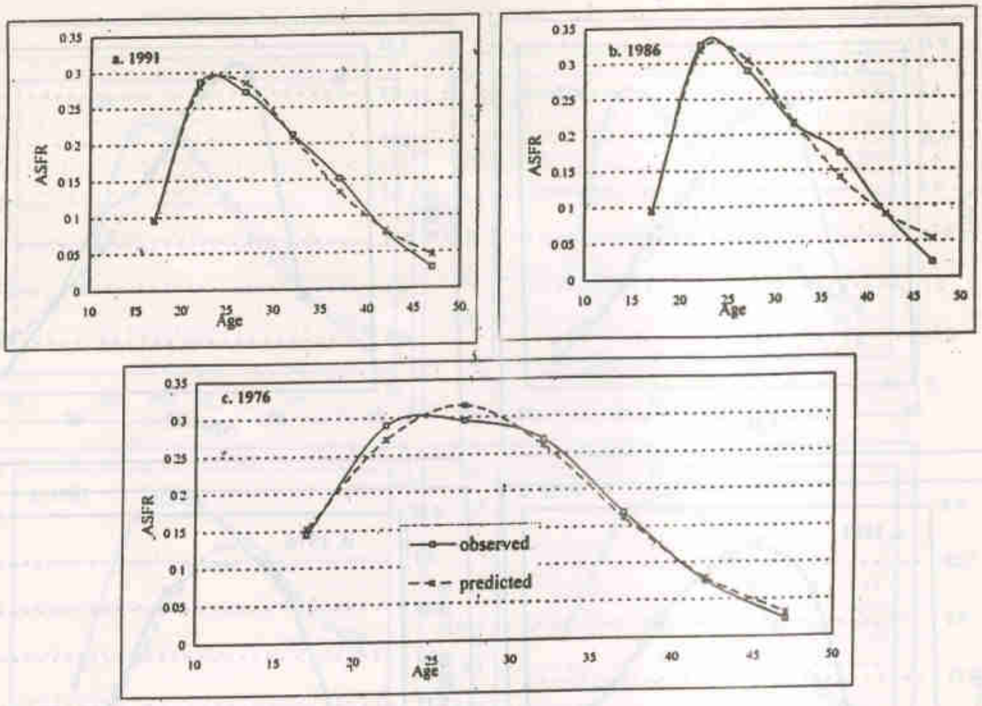
c. 1981
d. 1976
e. 1971
ASFR
Age
observed
predicted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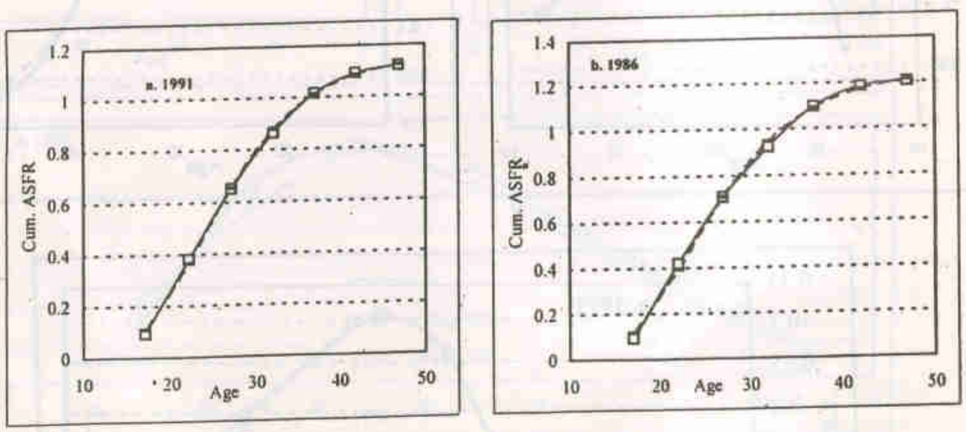
ASFR
Age
observed
predicted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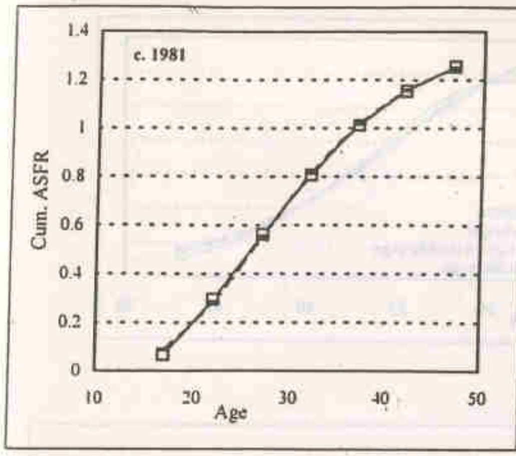
b. 1986
Cum. ASFR
Age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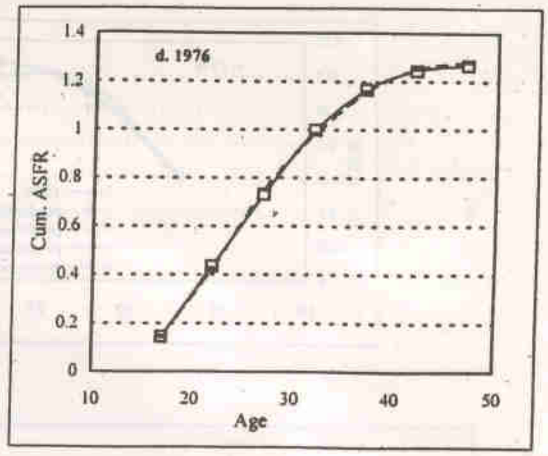
Cum. ASFR
Age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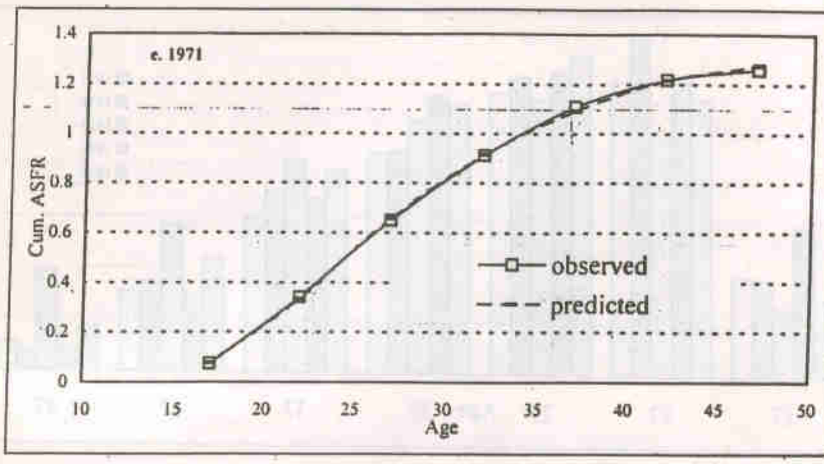
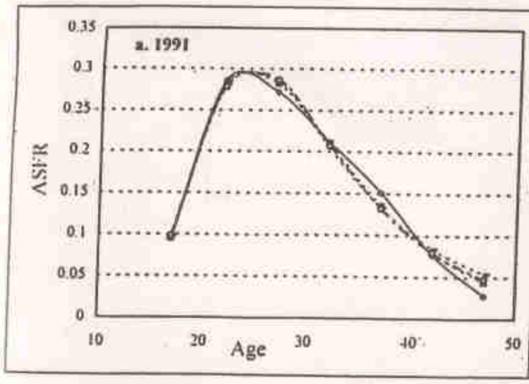
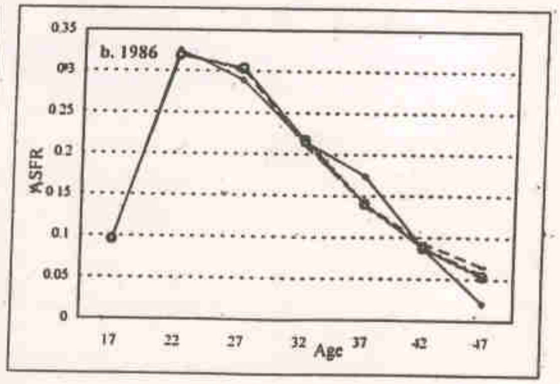
Cum. ASFR
Age


e. 1971
Cum. ASFR
observed
predicted
Age


a. 1991
ASFR
Age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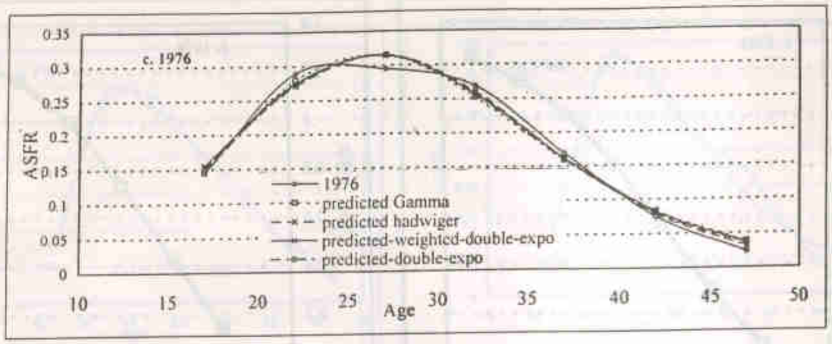
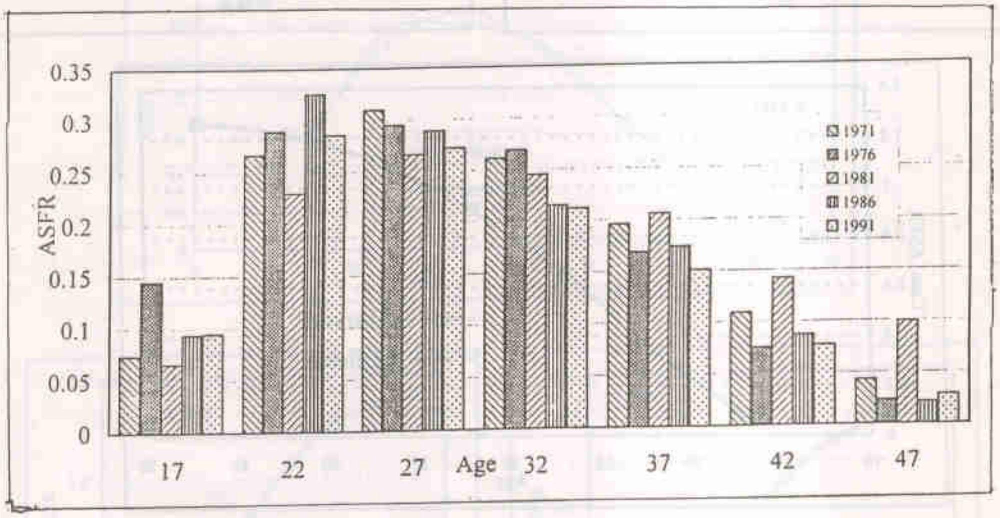
ASFR
Age


c. 1976
ASFR
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
10
15
20
25
30
35
40
45
50
Age
1976
predicted Gamma
predicted hadwiger
predicted-weighted-double-expo
predicted-double-expo


ASFR
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
17
22
27
Age
32
37
42
47
1971
1976
1981
1986
1991

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