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Reliability of Repairable System with Removable Multi–Repairmen

MADHU JAIN AND R.P. GHIMIRE

Abstract: In this investigation, the explicit expressions for reliability function and mean time to failure (MTTF) of repairable system with provision of spares and removable multi-repair facility have been established. In removable repairmen strategy, the repairmen turn on when there are N or more than N units fail and turn-off when system is empty. The failure times of operating/spare units and repair time of failed units are exponentially distributed.

1. Introduction

The provision of spare part support in repairable system is important in order to increase the reliability of the system. Some interesting results on the reliability of the standby redundant system can be found in Gopalan [1975], Kumar and Agarwal [1980] and Gupta et al. [1990]. Goel and Srivastava [1992] provided the transient analysis of multi-unit redundant system. Jain and Sharma [1986] and Jain [1993] established diffusion equation for multi-repairmen general machine repair system with spares. Sivalion and Wong [1989] gave cost analysis for M/M/R system with warm standby spare machines. Cost analysis of a two cold standby system with three modes of failure was discussed by Singh and Singh [1994]. Agrafiotis and Singh [1995] investigated the stochastic behaviour of a two units standby redundant system with two spare units from cost analysis view point. Agnihotri et al. [1995] considered two units redundant system with n-failure modes and developed fault-detection method. Gopalan and Bhanu [1995] gave cost analysis of a two units repairable system subject to on-line preventive maintenance.

In many real life problems, the provision of full-time repairmen costs the system which will inherently not important from the reasonable cost analysis view point. Yadin and Naor [1963] first introduced the concept of removable single-server system with exponential inter-arrival and service time distributions. Bell [1971] studied the M/G/I queueing system with provision of removable server. Recently Hsieh and Wang [1995] tackled removable single–repairman system with arbitrary spare units. In some situations, there may be multi-repairmen facility in which repairmen turn on only when a threshold number of units are in failed condition. This

motivates us to develop the reliability characteristics of a repairable system in which there are removable multi-repairmen and spare units. Such type of problems can be encountered in machining systems in the production processes.

2. The Model and Mathematical Analysis

We consider a multi-repairmen service facility with the following assumptions :

i) There are M operating and S spare units in the system.
ii) The facility have C repairmen.
iii) System fails if there are less than K operating units present in the system.
iv) The operating (spare) units have exponentially distributed failure time with mean rate $\lambda(\alpha)$. The repair times of failed units are exponentially distributed with rate μ.
v) Repaired units are assumed to be as good as new one.
vi) The repairmen turn on when there are N or more than N units fail and turn off when system is again empty.

Our main objective is to derive the explicit expressions for reliability function $R(t)$ and mean time to failure ($MTTF$) by using Laplace transform technique.

We denote

$P_{0,n}(t)$ = Probability that the repairmen are turned off and there are n failed units ($n = 0, \ldots, N-1$) at time t.

$P_{1,n}(t)$ = Probability that the repairmen are turned on and there are n failed units ($n = 1, 2, \ldots, L$) at time t.

$\overline{P}_{i,n}(S)$ = Laplace transform of $P_{i,n}(t)$, $i = 0, 1$.

The main failure rate and repair rate are respectively given by

$$\lambda_n = \begin{cases} M\lambda + (s-n)\alpha & \text{if } n = 0, 1, \ldots, s \\ (M+s-n)\lambda & \text{if } n = s+1, \ldots, L-1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mu_n = \begin{cases} 0, & i = 0, 0 \le n \le N-1 \\ n\mu, & i = 1 \text{ and } 1 \le n \le c-1 \\ c\mu, & i = 1 \text{ and } c \le n \le L \end{cases}$$

Using birth death process, we obtain the governing equations (in terms of Laplace of probabilities) for our model as follows :

(1) $(s+\lambda_0)\,\overline{p}_{0,0}(s) - \mu\, p_{1,1}(s) = p_{0,0}(0)$

(1.2) $(s+\lambda_n)\,\overline{p}_{0,n}(s) - \lambda_{n-1}\overline{p}_{0,n-1}(s) = p_{0,n}(0)\,, \qquad 1 \le n \le N-1$

(1.3) $(s+\lambda_1+\mu)\,\overline{p}_{1,1}(s) - \lambda_0\overline{p}_{1,0}(s) - 2\mu\,\overline{p}_{1,2}(s) = \overline{p}_{1,1}(0)$

(1.4) $(s+\lambda_{n-1}+(n-1)\mu)\,\overline{p}_{1,n-1}(s) - \lambda_{n-2}\,\overline{p}_{1,n-2}(s) - n\mu\,\overline{p}_{1,n}(s) = p_{1,n-1}(0)$

$$2 \le n \le C-1$$

(1.5) $(s+\lambda_n+c\mu)\,\overline{p}_{1,n}(s) - \lambda_{n-1}\,\overline{p}_{1,n-1}(s) - c\mu\,\overline{p}_{1,n+1}(s) = p_{1,n}(0). \quad C \le n \le N-1$

(1.6) $(s+\lambda_n+c\mu)\overline{p}_{1,N}(s) - \lambda_{N-1}\,\overline{p}_{1,N-1}(s) - \lambda_{N-1}\overline{p}_{0,N-1} - c\mu\,\overline{p}_{1,N+1}(s) = p_{1,N}(0)$

(1.7) $(s+\lambda_n+c\mu)\overline{p}_{1,N}(s) - \lambda_{n-1}\,\overline{p}_{1,n-1}(s) - c\mu\,\overline{p}_{1,n+1}(s) = p_{1,n}(0).$

$$N+1 \le n \le L-2$$

(1.8) $(s+\lambda_{L-1}+c\mu)\overline{p}_{1,L-1}(s) - c\mu\,\overline{p}_{1,L} - \lambda_{1,-2}(s)\,\overline{p}_{1,L-2}(s) = p_{1,L-1}(0)$

(1.9) $s\overline{p}_{1-L}(s) - \lambda_{L-1}(s)\,\overline{p}_{1,L-1}(s) = p_{1,L}(0).$

Also we note that

$$p_{0,0}(0) = 1\,;\, p_{0,n}(0) = 0 \text{ for } n = 1, 2, \ldots, N-1$$

$$p_{1,n}(0) = 0 \qquad \text{for } n = 1, 2, \ldots, L.$$

The above set of equations (1.1–1.9) can be written in the matrix form as

(2) $$A(s)\overline{P}(s) = P(0),$$

where A is the square matrix of order $(N+L) \times (N+L)$ and $\overline{p}(s)$ and P(0) are column vectors of order $(N+L) \times 1$. The matrix A is shown in figure 1, the matrices $\overline{P}(s)$ and $\overline{P}(0)$ are given respectively as

$$\overline{P}(s) = [p_{0,0}(s) \quad p_{0,1}(s) \ldots p_{0,N-1} \quad p_{1,1}(s) \ldots p_{1,N+1}(s) \quad p_{1,N+2}(s) \ldots p_{1,L}]^T$$

and

$$\overline{P}(0) = [\overline{p}_{0,0}(0) \quad p_{0,1}(0) \ldots p_{0,N-1} \quad p_{1,1}(0) \ldots p_{1,N+1}(0) \quad p_{1,N+2}(0) \ldots p_{1,L}]^T$$

$$
A = \begin{bmatrix}
\begin{pmatrix}8+\\ \lambda_0\end{pmatrix} & 0 & 0 \;\cdots\; 0 & 0 & 0 & 0 & 0 \;\cdots & 0 & 0 & 0 & 0 \;\cdots\; 0 & 0 & 0 & 0 \;\cdots\; 0 & 0 & 0 \\
-\lambda_0 & 8+\lambda & 0 \;\cdots\; 0 & 0 & 0 & 0 & 0 \;\cdots & 0 & 0 & 0 & 0 \;\cdots\; 0 & 0 & 0 & 0 \;\cdots\; 0 & 0 & 0 \\
0 & -\lambda_1 & 8+\lambda_1 \;\cdots\; 0 & 0 & 0 & 0 & 0 \;\cdots & 0 & 0 & 0 & 0 \;\cdots\; 0 & 0 & 0 & 0 \;\cdots\; 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 \;\cdots\; -\lambda_{N2} & 8+\lambda_{N1} & 0 & 0 & 0 \;\cdots & 0 & 0 & 0 & 0 \;\cdots\; 0 & 0 & 0 & 0 \;\cdots\; 0 & 0 & 0 \\
0 & 0 & 0 \;\cdots\; 0 & 0 & 8+\lambda_1+\mu & 0 & 0 \;\cdots & 0 & 0 & 0 & 0 \;\cdots\; 0 & 0 & 0 & 0 \;\cdots\; 0 & 0 & 0 \\
0 & 0 & 0 \;\cdots\; 0 & 0 & -\lambda_1 & \begin{pmatrix}8+\lambda_2\\ 8+2\mu\end{pmatrix} & 3\mu \;\cdots & 0 & 0 & 0 & 0 \;\cdots\; 0 & 0 & 0 & 0 \;\cdots\; 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 \;\cdots\; 0 & 0 & 0 & 0 & 0 \;\cdots\; -\lambda_{L1} & 8+\lambda_N+\mu c & -\mu c & 0 & 0 & 0 & 0 & 0 \;\cdots\; 0 & 0 & 0 \\
0 & 0 & 0 \;\cdots\; 0 & 0 & 0 & 0 & 0 \;\cdots\; 0 & -\lambda_l & 8+\lambda_{l+1}+\mu c-\mu c & 0 \;\cdots\; 0 & 0 & 0 & 0 \;\cdots\; 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 \;\cdots\; 0 & 0 & 0 & 0 & 0 \;\cdots\; 0 & 0 & 0 & 0 & -\lambda_{N1} & 8+\lambda_N+\mu c & -\mu c & 0 \;\cdots\; 0 & 0 & 0 \\
0 & 0 & 0 \;\cdots\; 0 & 0 & 0 & 0 & 0 \;\cdots\; 0 & 0 & 0 & 0 & 0 & -\lambda_N & \begin{pmatrix}8+\lambda_{N+1}\\ +\mu c\end{pmatrix} & -\mu c \;\cdots\; 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 \;\cdots\; 0 & 0 & 0 & 0 & 0 \;\cdots\; 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \;\cdots\; -\lambda_{L2} & \begin{pmatrix}8+\lambda_{L1}+\\ \mu c\end{pmatrix} & -\mu c \\
0 & 0 & 0 \;\cdots\; 0 & 0 & 0 & 0 & 0 \;\cdots\; 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \;\cdots\; 0 & -\lambda_{L1} & 0
\end{bmatrix}
$$

Fig. 1: Matrix A

Equation (2) can be solved by Cramer's rule for $\bar{p}_{1,L}(s)$ as

(3) $$\bar{p}_{1,L}(s) = \frac{|A_{N+L}(s)|}{|A(s)|}.$$

Before giving the solution for $|A_{N+L}(s)|$ it is better to construct the sequence of tridiagonal matrices

$$A_1 = (a_{11})$$

$$A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$A_3 = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}$$

$$A_{m-1} = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \cdots & \ddots & \ddots & a_{m-2,m-2} & a_{m-2,m-1} \\ 0 & \cdots & \cdots & \cdots & a_{m-1,m-2} & a_{m-1,m-1} \end{bmatrix},$$

where

$$a_{kp} = \begin{cases} -\mu_{k+1} & \text{for} \quad p = k+1 \\ s + \lambda_k + \mu_k & \text{for} \quad p = k \\ -\lambda_{k-1} & \text{for} \quad p = k-1 \\ 0 & \text{for} \quad |p-k| \geq 2. \end{cases}$$

Also let

$$\Delta_1(s) = |A_1|$$

$$\Delta_2(s) = |A_2|$$

$$\vdots \qquad \vdots$$

$$\vdots \qquad \vdots$$

$$\Delta_{m-1}(s) = |A_{m-1}|.$$

For $\Delta_0(s) =$ and using recursive formula, we obtain

(4) $$\Delta_k(s) = a_{kk}\ \Delta_{k-1}(s) - \lambda_{k-1}\ \mu_k\ \Delta_{k-2}(s),$$

where $k = 1, 2, \ldots, m-1 (m \geq 2)$

The complete solution for $|A_{N+L}(s)|$ is given by

$$|A_{N+L}(s)| = \left[\prod_{k=0}^{L-1} \lambda_k(\lambda_k)\right] \Delta_{N-1}(s)$$

where

$$\Delta_{n-1}(S) = \begin{bmatrix} s+\lambda_1+\mu & -2\mu & 0 & - & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 \\ -\lambda_2 & s+\lambda_2+\mu & -3\mu & - & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 \\ 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & 10 & 1 \\ 0 & 0 & 0 & - & -\lambda_{0-1} & s+\lambda_{0-1}+(c-1)\mu & -\mu c & 0 & 0 & - & 0 & 0 \\ 0 & 0 & 0 & - & 0 & -\lambda_{0-1} & s+\lambda_0+\mu c & -\mu c & 0 & - & 0 & 0 \\ 0 & 0 & 0 & - & 0 & 0 & -\lambda_{0-1} & s+\lambda_{0-1}+\mu c & -\mu c & - & 0 & 0 \\ 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & 1 & 1 \\ 0 & 0 & 0 & - & 0 & 0 & 0 & 0 & 0 & - & s+\lambda_{N0-2}+\mu c & -\mu c \\ 0 & 0 & 0 & - & 0 & 0 & 0 & 0 & 0 & - & -\lambda_{N-1} & s+\lambda_{N-1}+\mu c \end{bmatrix}$$

Now we solve numerator $|A(s)|$. It is clear that $s = 0$ is a root of $|A(s)| = 0$ which is trivial solution. For non-trivial, let $s = -\gamma$, then we have

(6) $$A(-\gamma) = A(0) - \gamma I,$$

where γ is an eigen value, I being identity matrix. from (2) and (6), we have

(7) $$A(-\gamma)\,\bar{p}(s) = (A - \gamma I)\,\bar{p}(s) = p(0).$$

Let $\gamma_1, \gamma_2, \ldots, \gamma_i$ be i distinct non-zero real eigen values and $(\gamma_{i+1}, \bar{\gamma}_{i+1})$ $(\gamma_{i+2}, \bar{\gamma}_{i+2}), \ldots, (\gamma_{i+j}, \bar{\gamma}_{i+j})$ be j pairs distinct conjugate complex eigen values where $i + 2j = N + L - 1$. Hence $|A(s)|$ is given by

(8) $$|A(s)| = s\left\{\prod_{k=1}^{i}(s+\gamma_k)\right\}\left(\prod_{k=1}^{j}[s^2 + (\gamma_{i+k} + \bar{\gamma}_{i+k})s + \gamma_{i+k}, \bar{\gamma}_{i+k}]\right).$$

Using equations (5) and (8), equation (4) gives

$$\bar{p}_{1,L}(s) = \frac{\left[\prod_{k=0}^{L-1}\lambda_k\right]\Delta_{N-1}(s)}{s\left[\prod_{k=1}^{i}(s+\gamma_k)\right]\left[\prod_{k=1}^{j}(s^2 + (\gamma_{i+k} + \bar{\gamma}_{i+k})s + \gamma_{i+k}\,\bar{\gamma}_{i+k})\right]}$$

(9) $$= \frac{b_0}{s} + \frac{b_1}{s+\gamma_1} + \ldots + \frac{b_i}{s+\gamma_i} + \frac{c_1 s + d_1}{s^2 + (s_{i+1} + \bar{\gamma}_{i+1})s + \gamma_{i+1}\,\bar{\gamma}_{i+1}}$$

$$+\ldots+\frac{c_j s+d_j}{s^2+(s_{i+j}+\bar{\gamma}_{i+j})\,s+\gamma_{i+j}\,\bar{\gamma}_{i+j}}$$

where

(10) $$b_0=\frac{\left[\prod_{k=0}^{L-1}\lambda_k\right]\Delta_{N-1}(0)}{\left[\prod_{k=0}^{i}\gamma_k\right]\left[\prod_{k=1}^{j}\gamma_{i+k},\bar{\gamma}_{i+k}\right]}$$

and

(11) $$b_r=\frac{\left[\prod_{k=0}^{L-1}\lambda_k\right]\Delta_{N-1}(-\gamma_r)}{(-\gamma_r)\left[\prod_{k=1}^{i}(\gamma_k-\gamma_r)\right]\left[\prod_{k=1}^{j}(\gamma_r^2+(\gamma_{i+k}+\bar{\gamma}_{i+k})(-\gamma_r)+\gamma_{i+k}\,\bar{\gamma}_{i+k})\right]},\ 1\le r\le i$$

Similarly from equation (9), we have

$c_r(-\gamma_{i+r})+d_r=$

(12) $$\frac{\left[\prod_{k=0}^{L-1}\lambda_k\right]\Delta_{N-1}(-\gamma_{i+r})}{(-\gamma_{i+r})\left[\prod_{k=1}^{i}(\gamma_k-\gamma_{i+r})\right]\left[\prod_{\substack{k=1\\k\neq r}}^{j}(-\gamma_{i+r})^2+(\gamma_{i+k}+\bar{\gamma}_{i+k})(-\gamma_{i+r})+(\gamma_{i+k}\bar{\gamma}_{i+k})\right]}$$

Inverting Laplace transform in equation (9), we get

(13) $$\rho_{1,L}(t)=b_0+\sum_{r=1}^{i}b_r\,e^{-\gamma_r t}+\sum_{r=1}^{j}\left[b_r e^{u_r t}\cos(v_r t)+\frac{d_r-c_r u_r e^{-u_r t}\sin(v_r t)}{v_r}\right],$$

where u_r and v_r represent the real and imaginary parts of complex number γ_{i+r} and b_0, b_r, c_r, and d_r, all are real numbers.

Since the system has failed during the infinite period of time, therefore $\lim_{t\to\infty}p_{1,L}(t)=a_0=1$. Hence the reliability function is given by

(14) $$R(t)=1-p_{1,L}(t).$$

The mean time to system failure (MTTF) is given by

$$MTTF=\int_0^{\infty}R(t)dt=\lim_{s\to 0}\bar{R}(s)$$

(15) $$=-\sum_{k=1}^{i}\frac{b_k}{\gamma_k}-\sum_{k=1}^{j}\frac{d_k}{\gamma_{i+k},\bar{\gamma}_{i+k}}$$

where b_k and d_k can be determined by equations (11) and (12) respectively.

3. Discussions

Repairable units and removable repairmen policy employed in this investigation may be of great importance in various complex problems of machining system in the production processes of new technology world to minimize the expected cost function. We have developed reliability characteristics which may be helpful for system designer to determine the optimal number of repairmen and spare units.

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Simulation of Adaptive Optical Systems

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Abstract: A problem of simulation for an adaptive optical system realized as a discrete mirror consisted of a finite number of identical elements is solved. Discrete mirror element is considered as a nonsymmetrique rigid body, the position of which is defined by six generalized coordinates. The control is realized by three parallel forces applied in fixed points of element or by forces, acting along directions given in the immobile space. The purpose of control is a displacement of mirrors elements in accordance with variation of wave front set. The method of construction of the discrete mirror element dynamics equations as a mechanical system with programmed constraints and corresponding discretization schemes henceforth is recommended.

1. Simulation of Adaptive Optical Systems

It was shown in [1–4] that Adaptive Optical Systems (AOS), designed as a telescope with variable form of reflecting glace, suppose essentially reduce the perturbations of wave front, evoking by nonhomogeneous atmosphere. AOS, consisting of a finite number of mirrors, making motions of translations along parallels directions was considered [1].

Here the solving of discrete AOS simulation problem had been proposed. Discrete mirror element is considered as a nonsysmmetrique rigid body, the position of which is defined [4] by six generalized coordinates $q_1 = x_c$, $q_2 = y_c$, $q_3 = z_c$, $q_4 = \psi$, $q_5 = \vartheta$, $q_5 = \varphi$. The displacements of element are limited by three holonomic constraints, keeping its three points in corresponding directions. The control is realized by three parallel forces, applied in fixed points of element or by forces, acting along fixed directions in immobile space.

The use of well-known Lagrange equations for elements dynamics description does not guarantee asymptotic stability of the integral manifold corresponding to the constraint-equations and purpose of control [5–7]. Naturally, by numerical integration, the integral manifold proves to be unstable. For the stabilization of the constraints, it is necessary to consider simultaneously changes of the deviations from the constraint-equations and their respective derivatives. The deviations of the constraints and purpose of control are accounted by the equations of programmed constraints $f(q, t) = \alpha(t)$, $f = (f_1, \ldots, f_m)$, $m \leq 6$.

The changes of the constraints are described by the constraint perturbation differential equations $\ddot{\alpha} = \alpha(\dot{\alpha}, \alpha, \dot{q}, q, t)$, $\alpha(0,0,\dot{q},q,t) = 0$

The dynamic equations of elements programmed motion is constructed in general form

$$\frac{d}{dt}\frac{\partial T}{\partial v} - \frac{\partial T}{\partial q} = Q(v,q,t) + G(v,q,t)R,$$

where $v = \dot{q}$, $q = (x_c, y_c, z_c, \psi, \vartheta, \varphi)$, $2T = m\mathcal{v}_c^2 + A\omega_x^2 + B\omega_y^2 + C\omega_z^2$, $\mathcal{v}_c^2 = \dot{x}_c^2 + \dot{y}_c^2 + \dot{z}_c^2$, $A \neq B \neq C \neq A$, Q is the vector of external forces, G is a matrix and R is a control forces vector. The programmed motion is given by equation of constraints and by equations of reflecting rays displacements. In consequence of expression complexity the symbolic computation was used. The general solution of a linear-algebraic equation $\mathcal{F}\delta q = \delta\alpha$ with rectangular matrix $\mathcal{F}$, $\mathcal{F} = (f_{ij})$, $f_{ij} = \dfrac{\partial f_i}{\partial q_j}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, $m \leq n$, is constructed in order to determinate a set of all possible displacements δq of the system. The substitution of the obtained expression of the vector of possible displacements of the system into the expression of appropriate mechanical principle makes possible to obtain corresponding equations of dynamics.

Discretization schemes of solving equations of constraints and dynamics equations are defined by the choice of a numerical method and by the right hand side expressions in the constraint perturbation equations. Introduction of programmed constraints allows to construct the stable discretization schemes. The possibility of construction of numerical methods of solving equations connected in no way the evaluation of the inverse matrix is demonstrated. The corresponding conditions of exponential stability of the integral manifold are defined.

Software, which makes possible to define the controlled forces variations, transient complying with the given index of quality and reactions of constraints, is developed.

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On the Existence of Desarguesian Planes of Order 8 Satisfying a Condition of Suetake

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Abstract: The affine plane of order 8 does not admit a transitive collineation group G that partitioned l_∞ into sets Δ and $l_\infty \backslash \Delta$ with $|\Delta| = 2$ and $|G(l_\infty, l_\infty)| = 8$. $|G(P, l_\infty)| = 2$ for $P \in l_\infty \backslash \Delta$, and $|G(P, l_\infty)| = 1 \ \forall \ P \in \Delta$.

1. Introduction:

Let π be an affine plane of order 2^m, where $m \geq 1$. Let $G(l_\infty, l_\infty)$ denote the subgroup of translations which are in a group G of collineations of π. Let G act transitively on the points of π and partition the line l_∞ at infinity into two subsets Δ, $l_\infty \backslash \Delta$ with $|\Delta| = 2$. Then according to Suetake [5] one of the following holds:

(i) π is a translation plane and all the translations of π are in G (ii) $|G(l_\infty, l_\infty)| = 2^m$, $|G(P, l_\infty)| = 1 (P \in \Delta)$, $|G(P, l_\infty)| = 2 \ (P \in l_\infty \backslash \Delta)$. The non-existence of Desarguesion plane satisfying condition (ii) of suetake is studied in [4]. The purpose of this article is to show the following.

Theorem: *An affine plane of order 8 has no transitive collineation group satisfying Suetake's condition.*

We accomplish the proof by classifying the collineations in terms of the length of their orbits on the line at infinity using matrices and then deriving conclusion using order of collineations and coordinatizing P using GF (8) and set $l_\infty = <(1,0,0)><(0,1,0)>$ and $\Delta = \{<(1,0,0)>, <(0, 1, 0)>\}$.

2. Preliminary:

Let $F = GF(q)$ be Galois field of order $q = 2^m$ for some $m \geq 1$. Let $V(3, q)$ be a 3-dimensional vector space over the field F. The projective plane $P(V)$ is denoted by $PG(2,q)$ which is Desarguesian of order q (see 2). Points of $PG(2,q)$ are 1-dimensional subspace of $V(?,q)$ and lines are 2-dimensional subspaces of $V(3,q)$.

Definition:

A permutation λ of V (3,q) called a semilinear automorphism of V(3,q) with respect to the automorphism θ of the field F if

$$(x+y)\,\lambda = (x)\,\lambda + (y)\,\lambda \;\; \forall\, x \in V(3,q), y \in V(3,q)$$
$$(ax)\,\lambda = (a)\,\theta(x)\,\lambda\,(x \in V, a \in F).$$

In particular, if θ is an identity automorphism of F, then λ is linear

Remark:

Let $\{\alpha_1, \alpha_2, \alpha_3\}$ be a basis of V (3,q). If $\beta_1, \beta_2, \beta_3$ are arbitrary vectors in $V(3, q)$ (thought of as images of the basis α_i) and if $x = \Sigma_i\, x_i\, \alpha_i$ is an arbitrary vector of V(3,q), then

$$(x)\lambda = \sum_{i=1}^{3} (x_i)\,\theta\beta_i$$

is a semilinear automorphism of V(3,q) if and only if β_i's form a basis of V(3,q).

The group of all semilinear automorphism of V is denoted by $\Gamma L(V)$. The $\Gamma L(V)$ contains the general linear group $GL(V)$ as subgroup. Since the underlying field F is Galois, there are m field automorphisms and the index $[\Gamma L(V): GL(V)] = m$.

Let ϕ be a collineations of the projective plane P = PG (2, q). Then ϕ maps one dimensional subspaces of $V(3, q)$ into one dimensional subspaces of $V(3, q)$ and if $U_1 \subset U_2$, then $(U_1)\,\phi \subset (U_2)\,\phi$.

Suppose $\lambda : V(3,q) \to V(3, q)$ is a semilinear automorphism of V(3,q) and we define $(U)\,\phi \subset (U)\,\lambda$.

We say ϕ is induced by λ. The group of all induced collineations ϕ is denoted by $P\Gamma L(V)$ provided dim $V \geq 3$. In other words, every collineations is induced by a semilinear automorphism.

Let π be an affine plane and P = π U l_∞ be its projective extension. Let us choose $l_\infty = <(1, 0, 0)> <(0, 1, 0)>$ then Aut (P) = $P\Gamma L$ (3, q). Let cls [0, g] be the element of $P\Gamma L$ (3, q) induced by [0, g] ; $(x, y, z) \to (x^\theta, y^\theta, z^\theta)\, g$, where $\theta \in$ Aut (F) and $g \in GL(V(3, q))$. In particular, set cls g = cls [id, g], set $P_1 = <(1,0,0)>$, $P_2 = <(0, 1, 0)>$, $\Delta = \{P_1, P_2\}$and H = Aut (π).

Hypothesis:

(i) $G \leq H$ (G is a subgroup of H = Auto (π).

(ii) G acts transitively on affine points of π.

(iii) G has two orbits Δ and $l_\infty \backslash \Delta$ on l_∞.

(iv) $|G\,(l_\infty, l_\infty)| = q = 2^m$, $|G(P_1, l_\infty)|$
$= |G(P_2\,;\, l_\infty)| = 1$

and $|G(P_1, l_\infty)| = 2$ for all $P \in l_\infty \backslash \Delta$

Result 1. $G(l_\infty, l_\infty)$ is an elementary abelian 2−group and $|G(l_\infty, l_\infty)| \geq 2$.

Proof : See [6, 5], lemma 4.2 page 275.

Result 2.

(i) $$H = \left\{ \text{cls}\left[\theta, \begin{pmatrix} a_1 & a_4 & 0 \\ a_2 & a_5 & 0 \\ a_3 & a_6 & 1 \end{pmatrix}\right] \middle| \theta \in \text{Aut}(F) \right\}$$

$F = GF(q), a_1, a_2, a_3, a_4, a_5, a_6 \in GF(q), a_1\, a_5 + a_2\, a_4 \neq 0\}$

(ii) $$H(l_\infty, l_\infty) = \left\{ \text{cls}\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{pmatrix}\right] \middle| a,b \in GF(q) \right\}$$

(iii) $$H(<(1,a,0)>, l_\infty) = \left\{ \text{cls}\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & ax & 1 \end{pmatrix}\right] \middle| x \in GF(q) \right\}$$

for all $a \in GF(q)$.

Proof : See [(6), Lemma 1. Page 1].

Result 3.

Let $S = \{\phi \in H \mid \Delta^\phi = \Delta\}$

(i) $$S = \left\{ \text{cls}\left[\theta, \begin{pmatrix} 0 & a & 0 \\ b & 0 & 0 \\ c & d & 1 \end{pmatrix}\right], \text{cls}\left[\theta, \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ c & d & 1 \end{pmatrix}\right] \right.$$

$a, b, c, d \in GF(q), ab \neq 0, \theta \in \text{Aut}\, GF(8)\}$

(ii) $$N_s(G(l_\infty, l_\infty)) \setminus \left\{ \text{cls}\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{pmatrix}\right] \right\} |a,b \in GF(q),$$

$ab \neq 0, (a, b) \neq (0,0)\} \supset G.$

3. Affine plane of order 8:

Lemma 1. Let G be a collineation group of an affine plane π of order 8 which is transitive on the affine points of π and let $\mathbb{P} = \pi \cup l_\infty$ be its projective extension. If G has two orbits Δ and Γ on l_∞ such that $|\Delta| = 2|\Gamma| = 7$, which satisfies the following:

(i) $|G(l_\infty, l_\infty)| = 8$.
(ii) $|G(P, l_\infty)| = 1 \ \forall\, P \in \Delta$
(iii) $|G(P, l_\infty)| = 2. \ \forall\, P \in \Gamma$.

Then the collineations in G have the following orbits on Γ:

1. The collineations fixing each of the points of Δ.

(a) Collineations of type

$$\text{cls}\left[\begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ a & d & 1 \end{pmatrix}\right],$$

where $x, y \in GF(8)\backslash\{0\} = GF(8)^*$, $x \neq y$ have Γ as an orbit and fix each of the points of Δ.

(b) Collineations of type

$$\text{cls}\left[\begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ c & d & 1 \end{pmatrix}\right],$$

where $x \in GF(8)^*$ and fix all the points of l_∞

(c) Collineations of type

$$\text{cls}\left[\theta, \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ c & d & 1 \end{pmatrix}\right],$$

where $x, y \in (8)^*$, and θ is an automorphism of $GF(8)$. And fix $< (1,1,0) > \in \Gamma$ and two cycles of length three of the points of $\Gamma \backslash \{< (1,1,0) >\}$.

(d) Collineations of type

$$\text{cls}\left[\theta, \begin{pmatrix} x & 0 \cdot 0 \\ 0 & y & 0 \\ c & d & 1 \end{pmatrix}\right],$$

where $x \in GF(8)^*$, $x \neq y$ fix the points $<(1, y^{-1}x, 0)>$ of Γ and have two cycles of length three of the remaining points of Γ.

II. The collineations interchanging the points of Δ.

(a) Collineations of type

$$\text{cls}\left[\begin{pmatrix} 0 & x & 0 \\ x & 0 & 0 \\ c & d & 1 \end{pmatrix}\right],$$

where $x \in GF(8)^*$ and map the point $<(1, P, 0)>$ to point $<(1, P^{-1}, 0)>$. In particular, they fix $<(1, 1, 0)> \in \Gamma$.

(b) Collineations of type

$$\mathrm{cls}\left[\begin{pmatrix} 0 & 1 & 0 \\ x & 0 & 0 \\ c & d & 1 \end{pmatrix}\right],$$

where $x \in GF(8)^*$ and map the point $<(1, P, 0)>$ to point $<(1, x^{-1}P^{-1}, 0)>$ and fix the point $<(1, F, 0)>$ where $F^2 = x^{-1}$.

(c) Collineations of type

$$\mathrm{cls}\left[\begin{pmatrix} 0 & x & 0 \\ 1 & 0 & 0 \\ c & d & 1 \end{pmatrix}\right],$$

where $x \in GF(8)^*$ and map the point $<(1, P, 0)>$ to point $<(1, xP^{-1}, 0)>$ and fix the point $<(1, F, 0)>$ where $F^2 = x$.

(d) Collineations of type

$$\mathrm{cls}\left[\begin{pmatrix} 0 & x & 0 \\ y & 0 & 0 \\ c & d & 1 \end{pmatrix}\right],$$

where $x, y \in GF(8)^*$, $x \neq y$ and map the point $<(1, P, 0)>$ to point $<(1, xy^{-1}) P^{-1}, 0)>$ and fix the point $<(1, F, 0)>$ where $F^2 = xy^{-1}$.

(e) Collineations of type

$$\mathrm{cls}\left[\theta, \begin{pmatrix} 0 & x & 0 \\ x & 0 & 0 \\ c & d & 1 \end{pmatrix}\right],$$

where $x \in GF(8)^* \theta$ is a nontrivial automorphism of $GF(8)$ and have cycle of length six of points of Γ and fix $<(1, 1, 0)>$.

(f) Collineations of type

$$\mathrm{cls}\left[\theta, \begin{pmatrix} 0 & x & 0 \\ y & 0 & 0 \\ c & d & 1 \end{pmatrix}\right],$$

where $x, y \in GF(8)^*$, $x \neq y$ fix the point $<(1, F, 0)>$ and have a cycle of length six of the points of Γ where $F^3 = xy^{-1}$.

Proof:

I(a) : Let $<(1, P, 0)> \in \Gamma$, then $P \neq 0$, and $<(1, P, 0)>\text{cls}\left[\begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ c & d & 1 \end{pmatrix}\right] = <(x, yP, 0)>$

$= <(1, x^{-1} yP, 0)> \neq <(1, P, 0)>$ because $x \neq y$.

Also $<(1, P, 0)> \left(\text{cls}\left[\begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ c & d & 1 \end{pmatrix}\right]\right)^2$

$= <(x, yP, 0)\ \text{cls}\left[\begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ c & d & 1 \end{pmatrix}\right] = <(x^2, y^2P, 0)> \neq <(1, P, 0)>$

Otherwise we get $1 = x^{-2} y^2 = (x^{-1} y)^2$ which implies that $x = y$, for $< GF(8)^*, . >$ is a group of odd order. Since for any element x, y of $GF(8)^*$ where $x \neq y$, the equation $1 = (x^{-1} y)^k = x^{-k} y^k$ implies that $k = 7$. We conclude that $\text{cls}\left[\begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ c & d & 1 \end{pmatrix}\right]$ has Γ as an orbit.

I(b) : Let $<(1, P, 0)> \in \Gamma$. Then

$$<(1, P, 0)>\ \text{cls}\left[\begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ c & d & 1 \end{pmatrix}\right] = <(x, xP, 0)> = <(1, P, 0)> \text{ even when } P = 0.$$

I(c) : Let $<(1, P, 0)> \in \Gamma$. Then $<(1, P, 0)> \left(\text{cls}\left[\theta, \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ c & d & 1 \end{pmatrix}\right]\right)^k = <(1, P^{\theta^k}, 0)>$

Since θ is an automorphism of $GF(8)$ and is of order 3, the point $<(1, 1, 0)>$ is fixed by this collineation and the other points belong to orbits of length three.

I(d) : Let $<(1, P, 0)> \in \Gamma$. Then

$$<(1, P, 0)>\ \text{cls}\left[\theta, \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ c & d & 1 \end{pmatrix}\right] = <(x, yP^{\theta}, 0)>$$

$$= <(x^{-1} x, x^{-1} yP, \theta)> = <(1, x^{-1} yP^{\theta}, 0)>,$$

$$<(1, P, 0)> \left(\text{cls}\left[\theta, \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ c & d & 1 \end{pmatrix}\right]\right)^2 = <(xx^{\theta}, yy^{\theta}P^{\theta}, 0)>$$

$$<(1, P, 0)> \left(\mathrm{cls}\left[\theta, \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ c & d & 1 \end{pmatrix}\right]\right)^3 = <(xx^{\theta}x^{\theta^2}, yy^{\theta}y^{\theta}, 0>, <(1, P, 0)>$$

So

$$\mathrm{cls}\left[\theta, \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ c & d & 1 \end{pmatrix}\right] \text{ fix the point } <(1, y^{-1}x, 0)> .$$

II(a) : Let $<(1, P, 0)> \in \Gamma$. Then

$$<(1, P, 0)> \mathrm{cls}\left[\begin{pmatrix} 0 & x & 0 \\ x & 0 & 0 \\ c & d & 1 \end{pmatrix}\right] = <(xP, x, 0)>$$

$$= <(1, p^{-1}, 0)>.$$

II(b) : Let $<(1, P, 0)> \in \Gamma$. Then

$$<(1, P, 0)> \mathrm{cls}\left[\begin{pmatrix} 0 & 1 & 0 \\ x & 0 & 0 \\ c & d & 1 \end{pmatrix}\right] = <(xP, 1, 0)>$$

$= <(1, x^{-1}P^{-1}, 0)>$, which means that the collineation fixed the point $<(1, F, 0)> \in \Gamma$ when $F^2 = x^{-1}$.

II(c) II(d) : The proofs of II (c) and II(d) are similar to that of II (b)

II(e) : Let $<(1, P, 0)> \in \Gamma$. Then

$$<(1, P, 0)> \mathrm{cls}\left[\theta, \begin{pmatrix} 0 & x & 0 \\ x & 0 & 0 \\ c & d & 1 \end{pmatrix}\right]$$

$$= <(1, P^{\theta}, 0)> \mathrm{cls}\left[\begin{pmatrix} 0 & x & 0 \\ x & 0 & 0 \\ c & d & 1 \end{pmatrix}\right] = (xP^{\theta}, x, 0)>$$

$$= (P^{\theta}, 1, 0)> = <(1, (P^{\theta})^{-1}, 0)>$$

$$= <(1, P, 0)> \left(\mathrm{cls}\left[\theta, \begin{pmatrix} 0 & x & 0 \\ x & 0 & 0 \\ c & d & 1 \end{pmatrix}\right]\right)^2 = <(1, P^{\theta^2}, 0)>$$

$$= <(1, P, 0)> \left(\mathrm{cls}\left[\theta, \begin{pmatrix} 0 & x & 0 \\ x & 0 & 0 \\ c & d & 1 \end{pmatrix}\right]\right)^3 = <(1, P^{-1}, 0)>.$$

Similarly, after another three steps, we get

$$= <(1, P^{-1}, 0)> \left(\text{cls}\left[\theta, \begin{pmatrix} 0 & x & 0 \\ x & 0 & 0 \\ c & d & 1 \end{pmatrix} \right] \right)^3 = <(1, (P^{-1})^{-1}, 0)>$$

$$<(1, P, 0)> \cdot$$

II(f): Let $<(1, P, 0) \in \Gamma$. Then

$$<(1, P, 0)> \text{cls}\left[\theta, \begin{pmatrix} 0 & x & 0 \\ y & 0 & 0 \\ c & d & 1 \end{pmatrix} \right]$$

$$= <(1, P^{\theta}, 0)> \text{cls}\left[\theta, \begin{pmatrix} 0 & x & 0 \\ y & 0 & 0 \\ c & d & 1 \end{pmatrix} \right] = <(yP^{\theta}, x, 0)>,$$

$$= <(1, P, 0)> \left(\text{cls}\left[\theta, \begin{pmatrix} 0 & x & 0 \\ y & 0 & 0 \\ c & d & 1 \end{pmatrix} \right] \right)^2$$

$$= <(yP^{\theta}, x, 0)> \text{cls}\left[\theta, \begin{pmatrix} 0 & x & 0 \\ y & 0 & 0 \\ c & d & 1 \end{pmatrix} \right]$$

$$= <(y^{\theta} p^{\theta^2}, x^{\theta}, 0)> \text{cls}\left[\begin{pmatrix} 0 & x & 0 \\ y & 0 & 0 \\ c & d & 1 \end{pmatrix} \right] = <(yx^{\theta}, xy^{\theta}p^{\theta^2}, 0)>,$$

$$<(1, P, 0)> \left(\text{cls}\left[\theta, \begin{pmatrix} 0 & x & 0 \\ y & 0 & 0 \\ c & d & 1 \end{pmatrix} \right] \right)^3$$

$$= <(yx^{\theta}, xy^{\theta}p^{\theta^2}, 0)> \text{cls}\left[\theta, \begin{pmatrix} 0 & x & 0 \\ y & 0 & 0 \\ c & d & 1 \end{pmatrix} \right]$$

$$= <(y^{\theta} x^{\theta^2}, x^{\theta} y^{\theta^2} p^{\theta^2}, 0)> \text{cls}\left[\begin{pmatrix} 0 & x & 0 \\ y & 0 & 0 \\ c & d & 1 \end{pmatrix} \right]$$

$$= <(yx^{\theta} y^{\theta^2} p^{\theta^2}, xy^{\theta} x^{\theta^2}, 0)>$$

$$= <(yx^{\theta} y^{\theta^2} p, xy^{\theta} x^{\theta^2}, 0)> = <(1, x^3 y^{-3} p^{-1}, 0)> \cdot$$

Similarly,

$$< (1, x^3 y^{-3} p^{-1}, 0) > \left(\text{cls}\left[\theta , \begin{pmatrix} 0 & x & 0 \\ y & 0 & 0 \\ c & d & 1 \end{pmatrix} \right] \right)^3$$

$$= < (1, (x^3 y^{-3})(x^3 y^{-3} p^{-1})^{-1}, 0) >$$

$$= < (1, x^3 y^{-3})(x^3 y^{-3})^{-1})(p^{-1})^{-1}, 0) >$$

$$= < (1, P, 0) >$$

Lemma 1 tells us how a collineation of a desired group partitions the points of Γ (hence of l_∞).

Remark 2 :

Let G be a transitive collineation group of the affine plane of order 8 satisfying Suetake's condition.

Then a collineation of G partitions l_∞ in either of the following ways:

Type	Partitions of l_∞	Type	Partitions of l_∞
I(a)	(1, 1, 7)	II(a)	(2, 1, 2, 2, 2)
I(b)	(1, 1, 1, 1, 1, 1, 1, 1, 1)	II(b)	(2, 1, 2, 2, 2)
I(c)	(1, 1, 1, 3, 3)	II(c)	(2, 1, 2, 2, 2)
I(d)	(1, 1, 1, 3, 3)	II(d)	(2, 1, 2, 2, 2)
		II(e)	(2, 1, 6)
		II(f)	(2, 1, 6)

Now we put a bound to the orders of the collineations of the desired group.

Lemma 2 : Let π be an affine plane of order 8 and $P. = \pi$ »1• be the extended projective plane of π. Let G be a collineation group of π which is transitive on π and satisfies Suetake's condition. Then there is no collineation of order greater than 14.

Proof : The possible number or partitions of l_∞ by a collineation of π are as follows :

(1) (1, 1, 1, 1, 1, 1, 1, 1, 1)
(2) (2, 1, 2, 2, 2)
(3) (1, 1, 1, 3, 3)
(4) (2, 1, 6)
(5) (1, 1, 7).

Let $\alpha \in G$ and $|\alpha| > 14$. All the above cases cannot happen since α, α^2, α^3 α^6 and α^7 fix l_∞ pointwise in 1), 2), 3), 4), 5) respectively. But in any case the orders of α, α^2, α^3, α^6, α^7 are greater than 2, so they cannot be a translation of G since they are translations of order 2.

Lemma 3 : Let π be an affine plane of orders 8 and let $\mathbb{P} = \pi \cup l_\infty$ be the projective extension.Let G be a transitive collineation group of π such that G has two orbits Δ and $l_\infty \backslash \Delta$ with $|\Delta| = 2$, $|G(l_\infty, l_\infty)| = 8$, $|G(P, l_\infty)| = 1$ for all $P \in \Delta$ and $|G(P, l_\infty)| = 2$ for all $P \in l_\infty \backslash \Delta$. If G contains a cls$\left[\begin{pmatrix} 0 & x & 0 \\ x^{-1} & 0 & 0 \\ 0 & z & 1 \end{pmatrix}\right]$, then there are no collineations of type cls$\left[\begin{pmatrix} 0 & x & 0 \\ x^{-1} & 0 & 0 \\ 0 & y & 1 \end{pmatrix}\right]$ and cls$\left[\begin{pmatrix} 0 & x & 0 \\ x^{-1} & 0 & 0 \\ w & 0 & 1 \end{pmatrix}\right]$ where $y \neq z$ and $w \neq x^{-1}z$ in G.

Proof : Let the collieation cls$\left[\begin{pmatrix} 0 & x & 0 \\ x^{-1} & 0 & 0 \\ 0 & z & 1 \end{pmatrix}\right]$ be in G.

Clearly, its inverse cls$\left[\begin{pmatrix} 0 & x & 0 \\ x^{-1} & 0 & 0 \\ x^{-1}z & 0 & 1 \end{pmatrix}\right]$ is in G.

Then the product cls$\left[\begin{pmatrix} 0 & x & 0 \\ x^{-1} & 0 & 0 \\ 0 & z & 1 \end{pmatrix}\right]$. cls$\left[\begin{pmatrix} 0 & x & 0 \\ x^{-1} & 0 & 0 \\ w & 0 & 1 \end{pmatrix}\right]$

$= \text{cls}\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x^{-1}z+w & 0 & 1 \end{pmatrix}\right]$ and cls$\left[\begin{pmatrix} 0 & x & 0 \\ x^{-1} & 0 & 0 \\ x^{-1}z & 0 & 1 \end{pmatrix}\right]$. cls$\left[\begin{pmatrix} 0 & x & 0 \\ x^{-1} & 0 & 0 \\ 0 & y & 1 \end{pmatrix}\right]$

$= \text{cls}\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & z+y & 1 \end{pmatrix}\right]$ are not elements of G by [Lemma 3[4]].

Remark 3:

From lemma 3, we see that if G contains cls$\left[\begin{pmatrix} 0 & k & 0 \\ k^{-1} & 0 & 0 \\ 0 & z & 1 \end{pmatrix}\right]$, then G cannot contain 14

collineations of the form cls $\left[\begin{pmatrix} 0 & x & 0 \\ x^{-1} & 0 & 0 \\ 0 & y & 1 \end{pmatrix}\right]$, cls $\left[\begin{pmatrix} 0 & x & 0 \\ x^{-1} & 0 & 0 \\ w & 0 & 1 \end{pmatrix}\right]$ $y \neq z$ and $w \neq x^{-1} z$,

so G can contain at most 50 collineations of the form cls $\left[\begin{pmatrix} 0 & x & 0 \\ x^{-1} & 0 & 0 \\ c & d & 1 \end{pmatrix}\right]$ for a particular value of $x \in GF(8)^*$.

Theorem: *Let π be an affine plane of order 8 and let $P = \pi \cup l_\infty$ be its projective extension. There exists no collineation group G of π which acts transitively on the affine points of π, has two orbits Δ and $l_\infty \backslash \Delta$ with $|\Delta| = 2$ and which satisfies Suetake's condition*

(i) $|G(l_\infty, l_\infty)| = 8$.

(ii) $|G(P, l_\infty)| = 1$ *for all* $P \in \Delta$

(iii) $|G(P, l_\infty)| = 2$. *for all* $P \in l_\infty \backslash \Delta$.

Proof: Suppose that there exist such a group G. By hypothesis $|G(l_\infty, l_\infty)| = 8$ and since no two different translations send a point to the same point. $|G(l_\infty, l_\infty)|$ divides the 64 affine points into eight orbits each consisting of eight affine points. By Result 3 (ii), $G(l_\infty, l_\infty)$ is a normal subgroup of G. Since $G(l_\infty, l_\infty)$ is intransitive in affine points of π G is imprimitive and $G/G(l_\infty, l_\infty) \cong \overline{G} \leq S_8$. Then $\overline{G}$ is transitive on the set of blocks of $G(l_\infty, l_\infty)$. [see [7] page 13].

Now, we need only to consider the minimal transitive subgroup of S_8. The minimal transitive subgroup of S_8 are regular and there are just 5 of them (furnished by Professor C.E. Praeger, The University of Western Australia). Since the Sylow 2–subgroup of S_8 are transitive, the minimal transitive subgroup are 2–groups. They are as follows :

Z_8, $Z_4 \times Z_2$, $Z_2 \times Z_2 \times Z_2$, D_4 (dihedral group of order 8) and Q_8 (quaternion group of order 8). They are regular.

Hence, we take $|G/G(l_\infty, l_\infty)| = 8$ and conclude that $|G| = |G/G(l_\infty, l_\infty)|$ $|G(l_\infty, l_\infty)| = 8.8 = 64$. Since, $|G| = 64$, the collineations of G cannot be of orders multiple of 3 or 7. So they must be of type I(b),II(a) II(b), II(c) or II(d). (see Remark 2)

They are all of form cls $\left[\begin{pmatrix} 0 & x & 0 \\ y & 0 & 0 \\ c & d & 1 \end{pmatrix}\right]$,

Where $0 \neq x, y \in GF(8)$ except for I(b) which are translations of the form

cls $\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & d & 1 \end{pmatrix}\right]$ where c and d are both zero or both non zero.

If $\text{cls}\left[\begin{pmatrix} 0 & x & 0 \\ y & 0 & 0 \\ c & d & 1 \end{pmatrix}\right]$ is in G, then its square $\text{cls}\left[\begin{pmatrix} 0 & x & 0 \\ y & 0 & 0 \\ c & d & 1 \end{pmatrix}\right]\text{cls}\left[\begin{pmatrix} 0 & x & 0 \\ y & 0 & 0 \\ c & d & 1 \end{pmatrix}\right]$

$$= \text{cls}\left[\begin{pmatrix} xy & 0 & 0 \\ 0 & xy & 0 \\ dy+c & cx+d & 1 \end{pmatrix}\right]$$

is in G and must then be of type 1(b) (See Remark 2), which is a translation. So $xy = 1$ which implies $y = x^{-1}$. Hence, the nontranslation of G are type II(d) for particular value $y = x^{-1}$.

Suppose there exist collineations $\text{cls}\left[\begin{pmatrix} 0 & x & 0 \\ x^{-1} & 0 & 0 \\ c & d & 1 \end{pmatrix}\right]$ for two different values of x say p and q in G. Then the product $\text{cls}\left[\begin{pmatrix} 0 & p & 0 \\ p^{-1} & 0 & 0 \\ c & d & 1 \end{pmatrix}\right]\text{cls}\left[\begin{pmatrix} 0 & q & 0 \\ q^{-1} & 0 & 0 \\ c & d & 1 \end{pmatrix}\right]$

$= \text{cls}\left[\begin{pmatrix} pq^{-1} & 0 & 0 \\ 0 & qp^{-1} & 0 \\ e & f & 1 \end{pmatrix}\right]$ with $pq^{-1} \neq 1$ so it is not a translation. From which, we conclude that there exists only one fixed value $x = k$. Since G has 8 translations, the remaining $64 - 8 = 56$ are of the form $\text{cls}\left[\begin{pmatrix} 1 & k & 0 \\ k^{-1} & 1 & 0 \\ c & d & 1 \end{pmatrix}\right]$. Not one of them shall be of the form $\text{cls}\left[\begin{pmatrix} 0 & k & 0 \\ k^{-1} & 0 & 0 \\ 0 & z & 1 \end{pmatrix}\right]$ by Remark 3 nor shall they be of the form $\text{cls}\left[\begin{pmatrix} 0 & k & 0 \\ k^{-1} & 0 & 0 \\ zk^{-1} & 0 & 1 \end{pmatrix}\right]$ which are of the inverse of $\text{cls}\left[\begin{pmatrix} 0 & k & 0 \\ k^{-1} & 0 & 0 \\ 0 & z & 1 \end{pmatrix}\right]$. Thus, from the 64 choices of collineations of the form $\text{cls}\left[\begin{pmatrix} 0 & k & 0 \\ k^{-1} & 0 & 0 \\ c & d & 1 \end{pmatrix}\right]$, 15 cannot be in G. Hence, the remaining 49 cannot make up the 56 nontranslations in G. Therefore G does not exist.

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Some New Identities of The Roger's –Ramanujan Type

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1. Introduction:

Bailey [3] showed that by starting with the simple identity

(1.1) $$\sum_{n=0}^{\infty}\alpha_n\gamma_n=\sum_{n=0}^{\infty}\beta_n\delta_n$$

where

(1.2) $$\beta_n=\sum_{n=0}^{\infty}\frac{\alpha\gamma}{(q)_{n-r}(dq)_{n+r}}$$

and taking

$$\delta_r=(\rho_1)_r(\rho_2)_r\left(\frac{dq}{\rho_1\rho_2}\right)^r$$

and using Gauss' q–summation formula

$$_2\phi_1\left[{a,b \atop c};q,\frac{c}{ab}\right]=\frac{(c/a,c/b;q)_\infty}{(c,c/ab;q)_\infty},\ |c/ab|<1,$$

we get

$$\sum_{n=0}^{\infty}(\rho_1,\rho_2;q)_n\left(\frac{dq}{\rho_1\rho_2}\right)^n\beta_n$$

(1.3) $$=\frac{(dq/\rho_1,dq/\rho_2;q)_\infty}{(dq,dq/\rho_1\rho_2;q)_\infty}\sum_{n=0}^{\infty}\frac{(\rho_1,\rho_2;q)_n(d_q/\rho_1\rho_2)^n\alpha_n}{(dq/\rho_1,dq/\rho_2;q)_n}$$

subject to the convergence of the infinite series and products and $|q|<1$ always.
We shall take

(1.4) $$\delta_r=\frac{(a)_r(b)_r(cq)_r}{(c)_r}\left(\frac{d}{ab}\right)^r$$

and shall use the summation formula due to Andrews [1]

$$
{}_3\phi_2\left({a,b,cq \atop c,d} ; q , \frac{d}{abq} \right)
$$

$$
(1.5) \qquad = \frac{(d/a, d/b ; q)_\infty}{(d, d/abq ; q)_\infty} \left[1 - \frac{d}{aq} - \frac{d}{bq} + \frac{d}{q} + \frac{cd(1-a)(1-b)}{abq(1-c)} \right]
$$

to prove the main Theorem.

The notation used is

$$
{}_r\phi_s\left({a_1, a_2, \ldots, a_r ; q^k, t \atop b_1, \ldots, b_s} \right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r ; q^k)_n \, t^n}{(q, b_1, \ldots, b_s ; q^k)_n},
$$

where

$$
(A_1, A_2, \ldots, A_r ; q^k)_n = \prod_{i=1}^{r} \prod_{j=0}^{n-1} (1 - A_i q^{kj})
$$

and

$$
(A_1, A_2, \ldots, A_r ; q)_\infty = \prod_{i=1}^{r} \prod_{j=0}^{n-1} (1 - A_i q^{kj})
$$

when not indicated the base will be q, bases other than q will be written.

2. Theorem:

$$
\sum_{n=0}^{\infty} \frac{(a)_n (b)_n (cq)_n}{(c)_n} \left(\frac{d}{ab} \right)^n \beta_n
$$

$$
= \frac{(dq/a, dq/b ; q)_\infty}{(dq, d/ab ; q)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (cq)_n}{(c)_n (dq/a)_n (dq/b)_n} \left(\frac{d}{ab} \right)^n \alpha_n
$$

$$
(2.1) \qquad \times \left[1 - \frac{dq^n}{a} - \frac{dq^n}{b} + dq^{2n} + \frac{cdq^n (1 - aq^n)(1 - bq^n)}{ab(1 - cq^n)} \right]
$$

Proof : Taking δ as given in (1.4), the r. h. s. of (1.1) is

$$
\sum_{n=0}^{\infty} \frac{(a)_n (b)_n (cq)_n}{(c)_n} \left(\frac{d}{ab} \right)^n \beta_n
$$

$$
= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (cq)_n}{(c)_n} \left(\frac{d}{ab} \right)^n \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r} (dq)_{n+r}}
$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (b)_r (cq)_r}{(c)_r (cq)_{2r}} \left(\frac{d}{ab}\right)^r \alpha_r \sum_{n=0}^{\infty} \frac{(aq^r)_n (bq^r)_n (cq^{r+1})_n}{(q)_n (cq^r)_n (dq^{2r+1})_n} \left(\frac{d}{ab}\right)^n$$

Using the summation formula (1.5) to sum the n–series on the r.h. s., we finally have

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n (cq)_n}{(c)_n} \left(\frac{d}{ab}\right)^n \beta_n$$

$$= \frac{(dq/a, dq/b\,;q)_\infty}{(d, c/abq\,;q)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (cq)_n}{(c)_n (dq/a)_n (dq/b)_n} \left(\frac{d}{ab}\right)^n \alpha_n$$

$$\times \left[1 - \frac{dq^n}{a} - \frac{dq^n}{b} + dq^{2n} + \frac{cdq^n (1-aq^n)(1-bq^n)}{ab(1-cq^n)}\right]$$

which prove the theorem.

3. We shall take the following six sets of Bailey pairs and insert them in (2.1) to obtain some intersecting identities, the first five are due to Bailey [2] and sixth due to Slater [4]

(i) $\alpha_n = (-1)^n (1-dq^{2n}) \dfrac{(dq)_{n-1}}{(q)_n} d^n q^{\frac{n(3n-1)}{2}}$ $\qquad n \geq 1$

$\qquad = 1$ $\qquad n = 0$

(3.1) $\beta_n = \dfrac{1}{(q)_n}$

(ii) $\alpha_{3n} = \dfrac{(-1)^n (dq^3; q^3)_{n-1} (1-dq^{6n})}{(q^3; q^3)_n} d^n q^{\frac{3n(3n-1)}{2}}$

$\alpha_{3n-1} = \alpha_{3n-2} = 0, \qquad n \geq 1$

$\alpha_0 = 1$

(3.2) $\beta_n = \dfrac{(dq^3\,; q^3)_{n-1}}{(q\,;q)_n (dq\,;q)_{2n-1}}, \; n \geq 1$

(3.3) (iii) $\alpha_n = (1-dq^{2n}) \dfrac{(dq)_{n-1} (e)_n}{(q)_n (dq/b)_n} \dfrac{d^n q^{n^2}}{e^n}$ $\qquad n \geq 1$

$\alpha_0 = 1$

$$(3.4) \qquad \beta_n = \frac{\left(\frac{-dq^{n+1}}{e};q\right)_n}{(q^2;q^2)_n(-dq;q)_{2n}(dq/e;q)}$$

$$\text{(iv)} \qquad \alpha_n = \frac{(-1)^n(1-dq^{2n})(dq)_{n-1}}{(q)_n} d^n q^{\frac{n(3n-1)}{2}} \qquad n \geq 1$$

$$\alpha_0 = 1$$

$$(3.5) \qquad \beta_n = \frac{(dq;q)_{3n}}{(q^3;q^3)_n(d^3q^3;q^3)_{2n}}.$$

$$\text{(v)} \qquad \alpha_{2n} = \frac{(dq^2;q^2)_{n-1}(f;q^2)_n(1-dq^{4n})}{(q^2;q^2)_n(dq^2/f;q^2)_n}\left(\frac{d}{f}\right)^2 q^{2n^2}$$

$$\alpha_{2n-1} = 0 \qquad n \geq 1$$

$$(3.6) \qquad \beta_n = \frac{(dq/f;q^2)_n}{(q;q)_n(dq;q^2)_n(dq/f;q)_n}$$

$$\text{(vi)} \qquad \alpha_{3n-1} = -q^{6n^2-5n+1}$$

$$\alpha_{3n} = -q^{6n^2-n} + q^{6n^2+n}$$

$$\alpha_{3n-2} = -q^{6n^2-7n+2} \qquad n \geq 1$$

$$(3.7) \qquad \beta_n = \frac{1}{(q)_{2n}}$$

4. Inserting the Bailey pair in (3.1) and in (2.1), we have

$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n(cq)_n}{(c)_n(c)_n}\left(\frac{d}{ab}\right)^n$$

$$= \frac{\left(\frac{dq}{a}, \frac{dq}{b};q\right)_\infty}{\left(dq;\frac{d}{ab};q\right)_\infty}\left[\left\{1-\frac{d}{a}-\frac{d}{b}+d+\frac{cd(1-a)(1-b)}{ab(1-c)}\right\}\right.$$

$$\left. + \sum_{n=1}^{\infty} \frac{(a)_n(b)_n(cq)_n}{(c)_n\left(\frac{dq}{a}\right)_n\left(\frac{dq}{b}\right)_n}\left(\frac{d}{ab}\right)^n \frac{(-1)^n(1-dq^{2n})(dq)_{n-1}d^nq^{\frac{n(3n-1)}{2}}}{(q)_n}\right.$$

$$\times\left\{1-\frac{dq^n}{a}-\frac{dq^n}{b}+dq^{2n}+\frac{cd\,q^n(1-aq^n)(1-bq^n)}{ab(1-cq^n)}\right\}\Bigg]\Bigg] \tag{4.1}$$

Letting $a, b \to \infty$, we have

$$(dq)_\infty \sum_{n=0}^{\infty}\frac{(1-cq^n)\,d^nq^{n^2-n}}{(q)_n}$$

$$=(1-c+d)+\sum_{n=1}^{\infty}\frac{(-1)^nq^{\frac{5n^2-3n}{2}}(1-cq^n)\,d^{2n}(1-dq^{2n})(dq)_{n-1}}{(q)_n} \tag{4.2}$$

$$\times\left\{1+dq^{2n}+\frac{cd\,q^{3n}}{1-cq^n}\right\}$$

Case 1. Putting $d=1$ and using Jacobi's Triple Product Identity we have

$$\sum_{n=0}^{\infty}\frac{q^{n^2-n}(1-cq^n)}{(q)_n}=\prod_{n\neq 1,4(\text{mod}\,5)}(1-q^n)^{-1}+(1-c)\prod_{n\neq 0,2,3(\text{mod}\,5)}(1-q^n)^{-1}$$

Dividing by c and then taking the limit as $c\to\infty$, we have the famous .Roger's–Ramanujan identity

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q)_n}=\prod_{n\neq 0,2,3(\text{mod}\,5)}(1-q^n)^{-1}$$

Case 2. Putting $c=1$, we have

$$\sum_{n=0}^{\infty}\frac{q^{n^2-1}(1-q^n)}{(q)_n}=\prod_{n\neq 0,1,4(\text{mod}\,5)}(1-q^n)^{-1}$$

Case 3. Putting $d=q$, we have

$$\sum_{n=0}^{\infty}\frac{q^{n^2}(1-cq^n)}{(q)_n}=\prod_{n\neq 0,2,3(\text{mod}\,5)}(1-q^n)^{-1}-c\prod_{n\neq 0,1,4(\text{mod}\,5)}(1-q^n)^{-1}$$

Dividing by c and taking the limit as $c\to\infty$, we have another famous Roger's s–Ramanujan identity.

$$\sum_{n=0}^{\infty}\frac{q^{n^2-n}}{(q)_n}=\prod_{n\neq 0,1,4(\text{mod}\,5)}(1-q^n)^{-1}$$

Case 4. Letting $b \to \infty$ and putting $a = -\sqrt{dq}$ and then putting $d = q$ and $c = 1$ in (4.1) we have

$$\sum_{n=0}^{\infty} \frac{q^{\frac{n^2-n}{2}}(-q)_n(1-q^n)}{(q)_n} = \frac{(-q)_\infty}{(q)_\infty}(q^2;q^4)_\infty^2(q^4;q^4)_\infty.$$

In this case if we put $c = 0$ we get

$$\sum_{n=0}^{\infty} \frac{q^{\frac{n^2-n}{2}}(-q)_n}{(q)_n} = \frac{(-q)_\infty}{(q)_\infty}\left[(q^2;q^4)_\infty^2(q^4;q^4)_\infty + (q;q^4)_\infty(q^3;q^4)_\infty(q^4;q^4)_\infty\right]$$

Combining the above two identities we get

$$\sum_{n=0}^{\infty} \frac{q^{\frac{n^2+n}{2}}(-q)_n}{(q)_n} = \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty}$$

an identity attribute to Lebesque.

Case 5. Letting $b \to \infty$ and putting $a = -1$, $d = 1$ and $c = 1$ in (4.1)

$$\sum_{n=0}^{\infty} \frac{q^{\frac{n^2-n}{2}}(-1)_n(1-q^n)}{(q)_n} = \frac{(-q)_\infty}{(q)_\infty}2(q^4;q^4)_\infty(q^3;q^4)_\infty(q;q^4)_\infty.$$

5. Inserting the Bailey Pair in (3.1) and (2.1) we have

$$(5.1)\qquad 1 + \sum_{n=1}^{\infty} \frac{(a)_n(b)_n(cq)_n}{(c)_n}\left(\frac{d}{ab}\right)^n \frac{(dq^3;q^3)_{n-1}}{(q)_n(dq)_{2n-1}}$$

$$= \frac{\left(\frac{dq}{a},\frac{dq}{b},q\right)_\infty}{\left(dq,\frac{d}{ab};q\right)_\infty}\left[\left\{1-\frac{d}{a}-\frac{d}{b}+d+\frac{cd(1-a)(1-b)}{ab(1-c)}\right\}\right.$$

$$+\sum_{n=1}^{\infty}\frac{(a)_{3n}(b)_{3n}(cq)_{3n}q^{\frac{3n(3n-1)}{2}}\left(\frac{d}{ab}\right)^{3n}(-1)^n(dq^3;q^3)_{n-1}(1-dq^{6n})d^n}{(c)_{3n}\left(\frac{dq}{a}\right)_{3n}\left(\frac{dq}{b}\right)_{3n}(q^3;q^3)_n}$$

$$\left.\times\left\{1-\frac{dq^{3n}}{a}-\frac{dq^{3n}}{b}+dq^{6n}+\frac{cdq^{3n}(1-aq)^{3n}(1-bq)^{3n}}{ab(1-cq^{3n})}\right\}\right]$$

Case 1. Letting $a, b \to \infty$ in (5.1) we have

(5.2) $$(dq)_\infty \left[1+\sum_{n=0}^{\infty} \frac{q^{n^2-n}(1-cq^n)d^n(dq^3;q^3)_{n-1}}{(1-c)(q)_n(dq)_{2n-1}}\right.$$

$$= 1+d+\frac{cd}{1-c}+\sum_{n=1}^{\infty}(-1)^n q^{\frac{27n^2-9n}{2}}(1-cq^{3n})d^{4n}(dq^3;q^3)_{n-1}(1-dq^{6n})$$

$$\left.\times\left\{1+dq^{6n}+\frac{cdq^{9n}}{1-cq^{3n}}\right\}\right]$$

Case 2 Putting $d=q^3$, in (5.2), we have

(5.3) $$(q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2+2n}(1-cq^n)(q^3;q^3)_n}{(q)_n(q)_{2n+2}}$$

$$= \prod_{n\neq 0,6,21(\text{mod }27)}(1-q^n)^{-1} - c \prod_{n\neq 0,3,24(\text{mod }27)}(1-q^n)^{-1}$$

For $c \neq 0$ and dividing by c and then taking limit as $c \to \infty$ we get Rogers–Ramanujan type identities due to Slater [4,eq.(91) and eq.(90)].

Case 3 Putting $d=1$ and $c=1$ in (5.2) we have

$$\sum_{n=1}^{\infty} \frac{q^{n^2-n}(q^3;q^3)_{n-1}}{(q)_{n-1}(q)_{2n-1}} = \prod_{n\neq 0,9,18(\text{mod }27)}(1-q^n)^{-1}$$

Case 4. Letting $b\to\infty$ and putting $a=-q^2$, $d=q^3$ in (5.1) we have

$$\sum_{n=0}^{\infty} \frac{q^{\frac{n^2+n}{2}}(1-cq^n)(-q)_{n+1}(q^3;q^3)_n}{(q)_n(q)_{2n+2}}$$

$$= (-q)_\infty\left[\prod_{n\neq 0,6,12(\text{mod }18)}(1-q^n)^{-1} + (q-c)\prod_{n\neq 0,3,15(\text{mod }18)}(1-q^n)^{-1}\right]$$

Case 5. Putting $c=q$ in case 4, we have

$$\sum_{n=0}^{\infty} \frac{q^{\frac{n^2+n}{2}}(1-q^n)^{n+1}(-q)_{n+1}(q^3;q^3)_n}{(q)_n(q)_{2n+2}} = (-q)_\infty \prod_{n\neq 0,6,12(\text{mod }18)}(1-q^n)^{-1}$$

Case 6. Letting $b\to\infty$ and putting $a=q^2$, $d=q^3$, $c=-q$, in (5.2) we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n^2+n}{2}}(q^3;q^3)_n}{(q)_{2n+2}} = 1+\sum_{n=1}^{\infty} q^{9n^2+3n} - \sum_{n=1}^{\infty} q^{9n^2-3n}$$

6. Similarly inserting the Bailey Pair in (3.3), (3.4), (3.5) and (3.6) in (2.1) and giving suitable values to a, b, c, d, we have the following identities

(i) $$\sum_{n=0}^{\infty}\frac{q^{2n^2-2n}}{(q^2;q^2)_n(-q)_{2n}}=\frac{(q^7,q^6,q;q^7)_\infty+(q^7,q^5,q^2;q^7)_\infty}{(q,-q;q)_\infty}$$

(ii) $$\sum_{n=0}^{\infty}\frac{q^{2n^2-2n}}{(q^2;q^2)_n(-q)_{2n-1}}=\frac{(q^7,q^6,q;q^7)_\infty+(q^7,q^5,q^2;q^7)_\infty+(q^7,q^4,q^3;q^7)_\infty}{(q,-q;q)_\infty}$$

(iii) $$\sum_{n=0}^{\infty}\frac{q^{3n^2-3n}}{(q;q^3)_n(q^3;q^3)_{2n}}=\frac{(q^9,q^8,q;q^9)_\infty+(q^9,q^7,q^2;q^9)_\infty}{(q^3;q^3)_\infty}$$

(iv) $$\sum_{n=0}^{\infty}\frac{q^{3n^2}(q)_{3n+1}}{(q;q^3)_n(q^3;q^3)_{2n+1}}=\frac{(q^9,q^5,q^4;q^9)_\infty-q(q^9,q^7,q^2;q^9)_\infty}{(q^3;q^3)_\infty}$$

(v) $$\sum_{n=0}^{\infty}\frac{q^{\frac{3n^2-3n}{2}}(1-q^{3n})(q^2)_{3n}(-q^3;q^3)_n}{(q;q^3)_n(q^3;q^3)_{2n+1}}=\frac{(-q;q^3)_\infty(q^6,q^4,q^2;q^6)_\infty}{(q^3;q^3)_\infty}$$

(vi) $$\sum_{n=0}^{\infty}\frac{(-1)^n q^{\frac{2n^2-3n}{2}}}{(-q^{1/2})_n(q)_n}=\frac{(1-q^{1/2})(q^{1/2};q)_\infty(q^6;q^4,q^2;q^6)_\infty}{(q,q)_\infty}$$

(vii) $$(q)_\infty\sum_{n=0}^{\infty}\frac{q^{n^2-n}}{(q)_{2n}}=(q^{30},-q^{19},-q^{11};q^{30})_\infty+(q^{30},-q^{17},-q^{13};q^{30})_\infty$$
$$-q^3(q^{30},-q^{29},-q;q^{30})_\infty-q(q^{30},-q^{23},-q^7;q^{30})_\infty.$$

Conclusion

Many more identities can be obtained by giving suitable values to a, b, c, d. Many Rogers–Ramanujan Type identities obtained earlier by Slater, Varma etc. follow as a special case of a general result.

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Mathematical Models and Algorithms For Certain Open shop Problems With Unit Processing Times

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Abstract: Objective is to revise and present blockmatrices model and give algorithms for open shop problems with unit processing times in scheduling theory. Considered problems are minimizing the completion and total completion times on machines. Furthermore, maximal idle-time and total idle-times on machines are minimized. polynomial algorithms are presented in the case of total operation set with unit processing times.

Key Words: Scheduling; Open shop problems; Sequence graph, Polynomial algorithms; Latin rectangles

1. Introduction

In classical open shop problems we have in machines $j \in J = \{1, 2, ..., m\}$ and n jobs $i \in I = \{1, 2, \ldots, n\}$. The machine order of job i is the order of all machines in which the job i is processed, and the job order on machine j is the order of all jobs on machine j. An example of such a graph is shown in figure (2.1). In this example we have the machine order $M_2 \to M_3 \to M_4 \to M_1$ for J_1, and on machine M_4 we have the job orders $J_4 \to J_5 \to J_1 \to J_2 \to J_3$. By sequence, we mean feasible combination of machine and job orders whereas by schedules we mean completion times of all operations. Let $\overline{C}_j$ denotes the maximum completion time and $\underline{C}_j$ denotes the minimum completion time of operation on machine j respectively, and p_{ij} denotes the processing time of job i on machine j.

With $SIJ = I \times J$, it is considered the following open shop problems

(P_1) $\quad O \mid p_{ij} = 1 \mid g_1$, $\qquad (P_3)$ $\quad O \mid p_{ij} = 1 \mid f_1$,

(P_2) $\quad O \mid p_{ij} = 1 \mid g_2$, $\qquad (P_4)$ $\quad O \mid p_{ij} = 1 \mid f_2$,

where the objective functions g_1, g_2, f_1, f_2 are defined by

$$g_1 := \max_{j \in J}\{\overline{C}_J\}, g_2 := \sum_{j \in J} \overline{C}_J, f_1 := \max_{j \in J}\{\overline{C}_J - \underline{C}_j\}, \text{ and } f_2 = \sum_{j \in J}(\overline{C}_J - \underline{C}_j).$$

The machine and job orders can be chosen arbitrarily (open shop problems) and the usual assumptions is made : each job can be processed on at most one machine at a time and each machine can process at most one job at a time.

The problems $[O \mid p_{ij}=1 \mid C_{\max}$ and $O \mid p_{ij}=1 \mid \Sigma_{i\in I} C_i]$ are solved in [2] with polynomial time algorithms and here we present modified polynomial algorithms and methods for the problems P_1, P_2, P_3 and P_4, where C_i denotes the completion time of job i. The used mathematical techniques and algorithms in order to solve considered problems are based on [5].

Various attempts have been made in scheduling theory by the help of blockmatrices model in which connection between latin rectangles [4] and sequence graph [3] is established.

2. On the Blockmatrices Model

We apply the blockmatrices model [1, 2] for modeling the considered open shop problems P_1, P_2, P_3 and P_4. For convenience we give its brief discussion here too. Firstly, the concept of latin matrix is introduced and then its connection with sequence graph is established.

A latin rectangle $LR[n, m, r] = [a_{ij}]$ is a matrix of size $n \times m$ with entries $a_{ij} \in S = \{1, 2, \ldots, r\}$ such that each integer of the insertion symbol set S occurs at most once in each column and at most once in each row of LR. If $n = m = r$ holds the matrix is called latin square of order n and is denoted by $LS[n]$.

Let $LR[n, m, r] = [a_{ij}]$ be any latin rectangle in r symbols $a_{ij} \in S = \{1, 2, \ldots, r\}$. We define $I = \{1, 2, \ldots, n\}$ as a set of jobs. $J = \{1, 2, \ldots, m\}$ as a set of machines, and $SIJ = I \times J$ as a set of operations for open shop problem. For a graph $G = (V, E)$ with vertex set $V = SIJ$, we define edge set E as follows:

The operation $(K\, l)$ is an immediate successor of operation (ij) if any one of the following conditions is satisfied

(a) $i = k$, $a_{ij} < a_{kl}$ and there is no $v \in J$ such that $a_{ij} < a_{iv} < a_{il}$, or

(b) $j = l$, $a_{ij} < a_{kl}$ and there is no $u \in I$ such that $a_{ij} < a_{uj} < a_{kj}$.

Then the vertex set V in the graph represents the set of operations and edge set E represents the set of job orders and machine orders. Since we can interpret each a_{ij} as level of vertex (ij), there only exist edges from lower level to higher level. Therefore, the constructed graph is acyclic and such acyclic graph corresponds to feasible sequence.

Let $MO = [b_{ij}]$ and $JO = [d_{ij}]$ be $n \times m$ matrices of given machine orders and job orders respectively : b_{ij} is the position of machine j in the machine order for job i and d_{ij} is the position of job i in the job order on machine j. Here, in each row of MO we have permutation of the integers $1, 2, \ldots, m$, in each column of JO we have permutation of the integers $1, 2, \ldots, n$.

For given machine orders and job orders and a set of operations $SIJ = I \times J$, we have a graph $G = (V, E)$, where $V = SIJ$ is the vertex set consisting of operations,

and E as edge set reflects machine orders and job orders. A combination of a given machine order with a given job order is not necessarily feasible, it is called feasible if and only if the graph G does not contain any cycle. In such a case the graph G is called sequence graph.

It is also possible to obtain an $n \times m$ matrix $[a_{ij}]$ satisfying the properties of latin rectangle for given sequence graph.

We define rank (source) = 1 and rank of other vertex (ij) by maximal number of vertices of a path from a source to it. Since the graph does not contain any cycle, we can determine the rank a_{ij} of each vertex (ij) in G. Also being only horizontal and vertical arcs in G, there do not exist any two vertices of the same rank in some row or column. Therefore the matrix $[a_{ij}] = LR\,[n, m, r]$ so formed by the rank of each vertices satisfies the following property (2.1) and indeed it is a latin rectangle.

Property 2.1 If $a_{ij} > 1$, then there exists integers $a_{ij} - 1$ in row i or column j or in both.

It is also very clear that a latin rectangle LR $[n, m, r]$ satisfying the property 2.1 easily produces a sequence graph for open shop problem.

Following theorem connects open shop problems and latin rectangles.

Theorem 2.1 *There exists a one-to-one correspondence between the set of latin rectangles $LR[n, m, r]$ with the property* 2.1 *and the set of sequence graph G for open shop problem.*

Example 2.1 Following example illustrates the fact in the case of V = SIJ = I × J

$$LR\,[5,4,6] = \begin{bmatrix} 6 & 1 & 3 & 4 \\ 1 & 2 & 4 & 5 \\ 3 & 4 & 2 & 6 \\ 4 & 5 & 6 & 1 \\ 5 & 6 & 1 & 2 \end{bmatrix}, n = 5, m = 4, r = 6$$

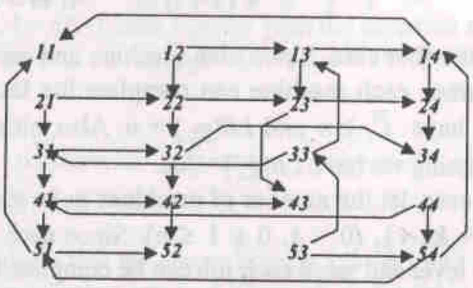

Figure 2.1 The Sequence Graph $G = (V, E)$

The horizontal edges represent the machine orders and the vertical edges represent the job orders respectively.

Note that we will consider here such latin rectangles satisfying the property 2.1 unless otherwise stated.

If we assign in a graph $G = (V, E)$, a vertex cost p_{ij} to each $(ij) \in SIJ$, we can consider different objective functions on the set of all sequence graphs and hence on the set of corresponding latin rectangles. For example, if $C = [c_{ij}]$ be the matrix of completion times ; $C_{\max} = \max \{c_{ij} \mid i \in I, j \in J\}$ is given by the weight of critical path in $G = (V, E)$ in the case of $p_{ij} = 1$. Then the problem here is to determine a latin rectangle with minimal cardinality of insertion set.. If we consider an objective function $\Sigma_{i \in I} C_i$, then the problem with $p_{ij} = 1$ is to determine a latin rectangle with minimal sum of the greatest elements of each row. In particular, if $p_{ij} = 1$for all i and for all j, then we have $C = LR[n, m, r]$ and the problem is relatively simple.

3. Minimizing the Completion and Total Completion Times.

In this section, the completion time and the total completion times on the machines are optimized in polynomial time algorithms if we have $p_{ij} = 1$ for all i and for all j. We consider the problems P_1 and P_2 with objective functions g_1 and g_2 of maximum completion time on machines and the sum of maximum completion times on each machines, respectively to be minimized. Here it is presented a polynomial time algorithm (see [5]) for the solutions of these open shop problems P_1 and P_2.

Let $LB(g_1)$ and $LB(g_2)$ denote the lower bounds for objective functions g_1 and g_2 respectively.

Lemma 3.1 Consider the open shop problems P_1 and P_2. Then, the lower bounds are

$$\text{(a) } LB(g_1) = \max \{m, n\}$$

$$LB(g_2) = \begin{cases} nm & \text{if} \quad m \le n \\ \frac{k(k+1)}{2} n^2 + (k+1)\, nl & \text{if } m = kn + l. \end{cases}$$

Proof: Let $m \le n$ be the first case. Since each machine and each job are available at zero time level and since each machine can complete the last job of its job order earliest at time n, we have $\overline{C}_j \ge n$ and $LB(g_1) = n$. Also, since there are exactly m machines all on processing we have $LB(g_2) = nm$.

As a second case, let the number of machines m be greater than the number of jobs n so that $m = kn + 1$, $(0 < k,\ 0 \le 1 \le n)$. Since each machine and job are available at zero time level and since each job can be completed earliest at time m we have, $LB(g_1) = m$, and at most n machines with $\overline{C}_j \ge m$. After removing n machines

there will be at most n machines with $\overline{C}_j \geq m-n$. After k steps the lower bound is $nm + n(m-n) + \ldots + n(m-(k-1)n)$. For the remaining 1 machines the trivial lower bound is $\overline{C}_j \geq n$. Since, there are exactly n jobs we have the lower bound is

$$LB(g_2) = \sum_{s=1}^{k}(nm-(s-1)n^2) + nl = \frac{k(k+1)}{2}n^2 + (k+1)nl.$$

Braesel /Kleinau presented in [2] a polynomial algorithm for solving the problems $[O \mid p_{ij} = 1 \mid C_{\max}$ and $O \mid p_{ij} = 1 \mid \Sigma_{i \in I} C_i]$. By changing the roles of machines and jobs we obtain the following algorithm for the open shop problem $[O \mid p_{ij} = 1 \| \overline{C}_{\max}, \mid \Sigma_{i \in J} C_j]$ with time complexity O(nm). Here, $\overline{C}_{\max}$ also denotes the objective function g_1

Algorithm 3.1 Solution of the problem $O \mid p_{ij} = 1 \mid \overline{C}_{\max}, \mid \Sigma_{j \in J} \overline{C}_j$.

Input: n, m, and $p_{ij} = 1$ for all i, j
Output: Matrix of completion times $C = [c_{ij}]$

SO: If $m \leq n$, then $C := LR[n, m, n]$
go to $S5$;
SI: Determine k and l with $m = kn + 1$, $(0 < k, 0 \leq 1 < n)$,
choose $K^* \in \{0, 1, 2, \ldots, k-1\}$ arbitrarily ;
S2: Insert in C, k^* latin squares with the insertion sets
$S_q = \{(q-1)n+1, \ldots, qn\}, q = 1, \ldots, k^*$,
S3: Insert in C one latin rectangle with n rows, $n+1$ columns, and the insertion set
$S = \{k^* n+1, \ldots\ldots, (k^*+1)n+1\}$
and the following two properties:
- in 1 columns the greatest integer is $(k^*+1)n$
- in n columns the greatest integer is $(k^*+1)n+1$;

S4: Insert in C, $k - k^*$ -1 latin squares with the insertion sets:
$Sq = \{qn+1+1, \ldots\ldots, (q+1)n+1\}, q = k^*+1, \ldots, k-1$,
S5 End

Theorem 3.1 *The polynomial algorithm presented above exactly solves the problem* $O \mid p_{ij} = 1 \mid g_1, g_2$.

Proof: In order to show that the lower bounds LB (g_1) and LB(g_2) calculate above are tight, we look the insertion sets in the presented algorithm 3.1.

Since the greatest integer in all insertion sets is given by max $\{n, m\}$, we have $\overline{C}_j = n$ for all j if $m \le n$ and $\overline{C}_j = m$ otherwise. Therefore we have LB(g_1) = max $\{n, m\} = \overline{C}_{\max}$.

Obviously, the lower bound satisfies LB(g_2) = $nm = \sum_{j\in J}\overline{C}_j$, if $m \le n$ since there are exactly m jobs on processing.

Finally, if $m = kn + l$, observing each insertion sets in the algorithm we get

$$\sum_{j\in J}\overline{C}_j = \sum_{j=1}^{k^*} njn + l(k^*+1)n + n[(k^*+1)n + l] + \sum_{j=k^*+1}^{k-1} n\,[(j+1)n + l]$$

$$= \sum_{j=1}^{k^*} jn^2 + l(k^*+1)n + n[(k^*+1)n + l] + \sum_{j=k^*+2}^{k} n\,[jn + l]$$

$$= n^2\sum_{j=1}^{k^*} j + (k+1)nl + n^2[(k^*+1) + n^2 \sum_{j=k^*+2}^{k} j$$

$$= \tfrac{k(k+1)}{2}n^2 + (k+1)nl = LB(g_2).$$

Thus obtaining the lower bounds as our objective function values the theorem is proved.

It is remarkable that all optimal schedules of the problem $O\,|\,p_{ij}=1\,|\,g_1, g_2$ can be constructed by the algorithm 3.1.

Example 3.1. Let there be $n = 3$ information sources and $m = 11$ receivers, where all receivers are required to receive all the informaions for unit time. It is assumed that no source will transmit the information to more than one receiver at the same time and no receiver can receive more than one information at a time. Then we have the objective value $g_1 = 11$ and $g_2 = 78$, and the following matrix C of completion times for the optimal schedules given by the above algorithm 3.1.

Here $k = 3$ and $l = 2$ so that we can choose $k^* \in \{0, 1, 2\}$.

$$C = \begin{pmatrix} 1 & 2 & 3 & : & 4 & 5 & 6 & : & 10 & 11 & 8 & : & 7 & 9 \\ 2 & 3 & 1 & : & 5 & 6 & 4 & : & 9 & 10 & 11 & : & 8 & 7 \\ 3 & 1 & 2 & : & 6 & 4 & 5 & : & 11 & 7 & 10 & : & 9 & 8 \end{pmatrix}, k^* = 2$$

$$C = \begin{pmatrix} 1 & 2 & 3 & : & 7 & 8 & 5 & : & 4 & 6 & : & 9 & 10 & 11 \\ 2 & 3 & 1 & : & 6 & 7 & 8 & : & 5 & 4 & : & 10 & 11 & 9 \\ 3 & 1 & 2 & : & 8 & 4 & 7 & : & 6 & 5 & : & 11 & 9 & 10 \end{pmatrix}, k^* = 1$$

$$C=\begin{pmatrix} 3 & 4 & 5 & : & 1 & 2 & : & 6 & 7 & 8 & : & 9 & 10 & 11 \\ 4 & 5 & 1 & : & 2 & 3 & : & 7 & 8 & 6 & : & 10 & 11 & 9 \\ 5 & 2 & 4 & : & 3 & 1 & : & 8 & 6 & 7 & : & 11 & 9 & 10 \end{pmatrix}, k^*=0$$

4. Minimizing the Idle-Time and Total Idle-Times

Let $\overline{S}_j$ be the idle-time of machine j. In this section we consider the problems P_3 and P_4 (see [5]) to minimize the idle -time $\overline{S}_{\max}$ of machines and total idle-times $\Sigma_{j\in J}\ \overline{S}_j$ of machines. respectively. The considered problems are solved in polynomials time algorithms in the case of processing times $p_{ij} = 1$ for all i and for all j.

Let LB (f_1) and LB (f_2) respectively denote the lower bounds for these problems.

Lemma 4.1 Consider the open shop problems P_3 and P_4 with SIJ = $I \times J$, then the lower bounds are LB (f_1) = n–1 and LB (f_2) = m (n–1).

Proof: Suppose that $m \le n$. Since each job can complete the last job of its job order earliest at time n units and since each machine and job can start its processing at zero time level, we have

$$\text{LB}\ (f_1) = n-1 \text{ and LB}\ (f_2) = m\ (n-1).$$

Suppose that the number of jobs n is less than the number of machines m so that $m = kn + 1$, where $0 < k$ and $0 \le 1 < n$. Since each machine and jobs are available at zero time level and the last operation completes of the job i earliest at time n units, the trivial lower bound LB (f_1) is n–1.

Because in each m machines exactly n time unit operations must be processed $\overline{C}_j - \underline{C}_j + 1 \ge n$, i.e., $\overline{C}_j - \underline{C}_j \ge n-1$ has to be satisfied. Therefore we have, LB $(f_2) \ge m\ (n-1)$.

Now we define the following objective functions

$$\overline{S}_{\max} := f_1 - (n-1) \text{ and } \Sigma_{j\in J}\ \overline{S}_j := f_2 - m\ (n-1).$$

The lower bounds for each of these objective functions is zero.

Now we present the following polynomial algorithm with slight modifications of algorithm 3.1. The time complexity of this algorithm is again O(nm).

Algorithm 4.1 Solution of the problem $O \mid p_{ij} = 1 \mid \overline{S}_{\max}, \mid \Sigma_{j\in J}\ \overline{S}_j$.

Input: n, m, and $p_{ij} = 1$ for all i, j
Output: Matrix of completion times $C = [c_{ij}]$

SO: If $m \le n$, then $C := LR[n, m, n]$,
go to $S5$;

SI: Determine k and l with $m = kn + l$, $(0, < k, 0 \le l < n)$
choose $k^* \in \{0, 1, 2, \ldots, k\}$ arbitrarily ;

S2: Insert in C, k^* latin squares with the insertion sets :
$M_q = \{(q-1)n+1, \ldots, qn\}, q = 1, \ldots, k^*$,

S3: Insert in C one latin rectangle $LR^*[n, l, n]$ with the insertion set
$M = \{k^*n+1, \ldots, (k^*+1)n\}$;

S4: Insert n C, k–k^* latin squares with insertion sets:
$Mq = \{qn+1, \ldots\ldots, (q+1)n\}, q = k^*+1, \ldots, k$;

S5 End

Theorem 4.1 *The algorithm presented above exactly solves the open shop problem* $O \mid p_{ij} = 1 \mid \overline{S}_{\max}, \mid \Sigma_{j \in J}\, \overline{S}_j$.

Proof: In order to show that the lower bounds calculated above are tight we look for the lower bounds in each insertion sets of given cardinality n.

For $m \le n$, since the latin rectangle LR[n, m, n] in algorithm contains exactly once n in each m columns (it is matrix of permutations) we have

$$\overline{S}_{\max} = 0 = \mathrm{LB}\,(f_1) - (n-1)\,,\ \Sigma_{j \in J}\ \overline{S}_j = 0 = \mathrm{LB}\,(f_2) - m\,(n-1).$$

Also, in order to show that the lower bounds are tight in the case of $m = kn + l$ with $0 < k$, $0 \le l < n$, we observe the insertion sets in the algorithm 4.1 again. Since we have k latin squares of size $n \times n$ and one latin rectangle of size $n \times l$ in n elements, obviously $\overline{S}_{\max} = 0 = \mathrm{LB}\,(f_1) - (n-1)$.

Finally we have,

$$\sum_{j \in J} \overline{S}_j = LB(f_2) - m(n-1)$$
$$= \sum_{j=1}^{k^*} (jn - jn + n - 1) + l\,[k^*n + n - k^*n - 1] + n \sum_{j=k^*+1}^{k} [jn + n - jn - 1] - m(n-1) = 0.$$

Example 4.1 Let be there 10 books of scheduling theory and 4 students willing to read these books in a library. Assume that each student can read at most one book at a time and on each book at a time only at most one student can read. All students are required to read all the books for unit time and no-idle time on books is accepted. Here, the students can be taken as jobs and the books can be taken as machines.

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Singularity Method For An Imperfect Spheroid In An Elastic Medium

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Abstract : In the present paper solution to the displacement problem of an imperfect rigid spheroidal particle in an infinite elastic medium is constructed by the help of singularities of the elastostatic equations. The analysis reveals that for small deformity while the force on the particle remains unchanged, a couple gets generated.

1. Introduction:

The elastostatic equations governing the displacement $\bar{u}$ are

$$(1.1) \qquad \mu\left[\frac{1}{1-2\nu}\text{grad div }\bar{u}+\nabla^2\bar{u}\right]+\bar{f}=0,$$

where μ is the shear modulus, ν the Poisson's ratio and $\bar{f}$ the body force per unit volume. The fundamental solution 'Kelvin Solution' of the above equation corresponds to the solution of (1.1) when $\bar{f}$ is the point force

$$(1.2) \qquad \bar{f}^k(\bar{x})=16\pi\mu(1-\nu)\delta(\bar{x})\bar{\alpha},$$

where $\bar{x}$ is the fluid position, $\delta(\bar{x})$ the three dimensional Dirac delta function and $\bar{\alpha}$ characterizes the strength of the force. Kanwal and Sharma [4] have extended the singularity method used by Chwang and Wu [1] for stokes equations to elastostatics and applied it for solving displacement problems involving a rigid spheroid. The problem of a spheroidal inclusion in an elastic medium under torsion was later considered by Datta and Kanwal [3]. In these problems the body is perfectly axial symmetric. But in practice it is seldom so, and in this paper we have investigated the effect of such imperfections.

For the sake of convenience we collect below the basic singularities that we shall exploit here and refer to the paper of Kanwal and Sharma [4] for details.

Kelvin Solution:

$$(1.3) \qquad \bar{U}^k(\bar{x};\bar{\gamma})=\frac{(3-4\nu)\bar{\alpha}}{R^3}+\frac{(\bar{\alpha}\times\bar{x})\bar{x}}{R^3}, \quad R=|\bar{x}|$$

Centre of Rotation:

$$\overline{U}^{r}(\overline{x};\overline{\gamma}) = \frac{\overline{\gamma} \times \overline{x}}{R^3} \tag{1.4}$$

Stresslet:

$$\overline{U}^{k.s}(\overline{x};\overline{\alpha},\overline{\beta}) = \frac{(1-2\nu)\,[(\overline{\beta}\,.\,\overline{x})\,\overline{\alpha} + (\overline{\alpha}\,.\,\overline{x})\overline{\beta}] - (\overline{\alpha}\,.\,\overline{\beta})\overline{x}}{R^3} + \frac{3\,(\overline{\alpha}\,.\,\overline{x})(\overline{\beta}\,.\,\overline{x})\overline{x}}{R^5} \tag{1.5}$$

Centre of Dilatation:

$$\overline{U}^{d}(\overline{x}) = \frac{\overline{x}}{R^3} \tag{1.6}$$

Doubtlet:

$$\overline{U}^{dd}(\overline{x};\overline{\alpha}) = -\frac{\overline{\alpha}}{R^3} + \frac{3(\overline{\alpha}\,.\,\overline{x})\,\overline{x}}{R^5}. \tag{1.7}$$

As explained in [4] the force on a control volume enclosing a Kelvin concentrated force of strength $\overline{\alpha}$ is given by

$$\overline{F} = -16\pi\mu\,(1-\nu)\,\overline{\alpha} \tag{1.8}$$

and the couple due to a centre of rotation of strength $\overline{\gamma}$ by

$$\overline{M} = -8\pi\,\mu\overline{\gamma}\,. \tag{1.9}$$

Let the imperfect spheroid be given by

$$S \equiv \frac{x^2}{a^2} + \frac{y - \in h^2(x)}{b^2} + \frac{z^2}{b^2} = 1\ ,\ (a \geq b) \tag{1.10}$$

for $\in = 0$, this represents a prolate spheroid with focii at $(\pm\, c, 0)$, where $c = ae$, $e = \sqrt{1 - b^2/a^2}$ being the eccentricity. In the sequel closed form solutions, when $h(x) = \frac{h_0}{a^2}(a^2 - \in^2 x^2)$, have been derived for small deformity parameter $\in$. This represents a non-axially symmetric body with centre line $y = \in h(x)$ in the xy plane $(z = 0)$ symmetric about the y-axis. But in a similar fashion, when the centre line is anti-symmetric about the y-axis, solution may be attempted and the scheme extended to more general situations.

The body S will be given the general uniform displacement

$$\overline{U} = U_x\overline{e}_x + U_y\overline{e}_y + U_z\overline{e}_z\ . \tag{1.11}$$

Observe that since the body is not axially-symmetric, unlike the case Kanwal and Sharma [4], there is need for introducing all the three components of displacement. On account of linearity we can find the solutions corresponding to the displacements U_x , U_y and U_z independently and than super impose them.

2. Displacement Parallel To x–Axis:

In the case we have to solve *eq.* (1.1) for the surface S given by (1.10) subject to the boundary conditions.

(2.1) $$\bar{u} = U_x \bar{e}_x \text{ on } S$$

and

(2.2) $$\bar{u} = 0 \text{ at infinity.}$$

Let us first consider the displacement $\bar{w}$ due to a uniform distribution of Kelvin solution and a parabolic distribution of doublets on x-axis between $(\pm c, 0, 0)$, i.e.; Set

(2.3) $$\bar{w} = A\bar{J}^k + B\bar{J}^d,$$

where

(2.4) $$\bar{J}^k = \int_{-c}^{c} \bar{U}^k (\bar{x} - \xi\,\bar{e}_x \,; \bar{e}_x)\, d\xi$$

$$= [(3-4\nu)\, A_{1,0} + x^2 A_{3,0} - 2xA_{3,1} + A_{3,2}]\, \bar{e}_x$$
$$+ (xA_{3,0} - A_{3,1})\, (y\bar{e}_y + z\bar{e}_z)$$

and

(2.5) $$\bar{J}^d = \int_{-c}^{c} (c^2 - \xi^2)\, \bar{U}^{dd} (\bar{x} - \xi\,\bar{e}_x \,; \bar{e}_x)\, d\xi$$

$$= [A_{3,2} - c^2 A_{3,0} - 3\{x^2 (A_{5,2} - c^2 A_{5,0}) - 2x(A_{5,3} - c^2 A_{5,1})$$
$$+ (A_{5,4} - c^2 A_{5,2})\}]\, \bar{e}_x$$
$$- 3\,[x(A_{5,2} - c^2 A_{5,0}) - (A_{5,3} - c^2 A_{5,1})]\, (y\bar{e}_y + z\bar{e}_z),$$

where

(2.6) $$A_{m,n} = \int_{-c}^{c} \frac{\xi^n d\xi}{\left[(x-\xi)^2 + y^2 + z^2\right]^{m/2}} \,.$$

Now, for points on *S*, we have the approximation

(2.7) $$\bar{J}^{k} = \bar{J}^{k_0} + \in \bar{J}^{k_1} + 0(\in^2),$$

(2.8) $$\bar{J}^{d} = \bar{J}^{d_0} + \in \bar{J}^{d_1} + 0(\in^2),$$

where, with $L = \ln\left(\dfrac{1+e}{1-e}\right)$,

(2.9) $$\bar{J}^{k_0} = \left[4(1-\nu)L - \frac{2e(a^2-x^2)}{a^2-e^2x^2} \right] \bar{e}_x + \frac{2ex}{a^2-e^2x^2}(y\bar{e}_y + z\bar{e}_z) ,$$

(2.10) $$\bar{J}^{d_0} = \left[-2L + \frac{4a^2e}{a^2-e^2x^2} \right] \bar{e}_x + \frac{4a^3x}{(1-e^2)(a^2-e^2x^2)}(y\bar{e}_y + z\bar{e}_z),$$

(2.11) $$\bar{J}^{k_1} = 2 \in y\, h(x) e \left[\left\{ \frac{a^4 + 3e^2a^2x^2 - 3a^2x^2 - e^2x^A}{(a^2-e^2x^2)^3} - \frac{2(3-4\nu)}{(1-e^2)(a^2-e^2x^2)} \right\} \bar{e}_x - \frac{(3a^2+e^2x^2)}{(a^2-e^2x^2)^3}(y\bar{e}_y + z\bar{e}_z) \right],$$

(2.12) $$\bar{J}^{d_1} = 4y\, h(x) e \left[\left\{ \frac{1}{(1-e^2)(a^2-e^2x^2)^3} - \frac{a^2+3e^2x^2}{(a^2+e^2x^2)^3} \right\} \bar{e}_x - \frac{e^2x}{(1-e^2)^2(a^2-e^2x^2)^3}\{(5-3e^2)a^2 - e^2(1+e^2)x^2\}(y\bar{e}_y + z\bar{e}_z) \right].$$

Next, it may be verified that the equation

(2.13) $$A\,\bar{J}^{k_0} + B\,\bar{J}^{d_0} = U_x\bar{e}_x$$

is satisfied provided we choose

(2.14) $$A = -\frac{2e^2\beta}{(1-e^2)} = \frac{e^2U_x}{\left[L(1+3e^2-4\nu e^2) - 2e\right]}$$

and this gives on surface S

$$A\bar{J}^{k_1} + B\bar{J}^{d_1} = \frac{8y\,h_0 eB}{a^2(1-e^2)} \left[\frac{1+2(1-2\nu)\,e^2}{(1-e^2)}\,\bar{e}_x - \frac{1}{a^2-e^2x^2}\cdot\frac{xe^2}{1-e^2}(y\bar{e}_y + z\bar{e}_z) \right],$$

where $$h(x) = \frac{h_0(a^2-e^2x^2)}{a^2}.$$

Further, consider the expression

$$\overline{q} = -N \int_{-c} (c^2 - \xi^2)\, \overline{U}^{ks}(\overline{x} - \xi\, \overline{e}_x\, , \overline{e}_y)\, d\xi$$

(1.15) $$+(1-2v)N + 2\left\{\frac{2e(3-2e^2)}{1-e^2} - 3L\right\} N^1 \int_{-c}^{c} (c^2 - \xi^2)\overline{U}^r(\overline{x} - \xi\,\overline{e}_x\, ; \overline{e}_z)d\xi$$

$$-(L-2e)N^1 \int_{-c}^{c} (c^2 - \xi^2)^2 \frac{\partial \overline{U}^{dd}}{\partial y}(x - \xi e_x\, ; e_x)d\xi ,$$

where

(2.16) $$N = \frac{6(1-e^2)^2 L^2 - 8e(3-e^2)(1-2e^2)L + 8e^2(3-4e^2)}{(1-e^2)[3(1-e^4)L^2 - 4e(3-3e^2-e^4)L + 4e^2(3-2e^2)]} \cdot \frac{h_0 B}{a^2}$$

and

(2.17) $$N^1 = \frac{3\{(2e-(1-e^2)L\} + 2(1-2v)\{2e+4e^3-(1-e^2)L\}}{3(1-e^4)L^2 - 4e(3-3e^2-e^4)L + 4e^2(3-2e^2)} \cdot \frac{h_0 B}{a^2}.$$

It may be shown that for points on S

(2.18) $$\overline{q} = A\,\overline{J}^{k_1} + B\,\overline{J}^{d_1} + 0(\in).$$

Now, we get

(2.19) $$\overline{u} = \overline{w} - \in \overline{q},$$

where $\overline{w}$ is given by (2.3) and $\overline{q}$ by (2.15). It may be verified that $\overline{u}$ as given by (2.19) is the required solution satisfying the boundary conditions (2.1) and (2.2) Adding up the contributions of Kelvin solution $\overline{U}^k$ occurring in (2.19), we find that the net force experienced by the body is along x-axis and is given by

(2.20) $$F_x = -16\pi\,\mu\,(1-v) \int_{-c}^{c} A d\xi$$

$$= 32\pi\,\mu\,(1-v)ae^3\,[((1+3e^2)-4ve^2)L-2e]^{-1}$$

and adding up the contributions of centre of rotations, the net couple along z-axis is

(2.21) $$M_z = -8\pi\in\mu \int_{-c}^{c}\left[(1-2v)N + 2\left\{\frac{2e(3-2e^2)}{1-e^2} - 3L\right\}\right] N^1(c^2 - \xi^2)d\xi$$

$$= 32 \in \pi \mu h_0 a e^3 \left[3\{2e-(1-e^2)L\}\{2e-(3-2e^2)-3(1-e^2)L\}\right.$$
$$+(1-2\nu)\left\{9(1-e^2)^2 L^2 - 4e(9-9e^2-2e^4)L + 4e^2(9-8e^4)\right\}\Big] \times$$
$$\Big[3\left\{3(1-e^4)L^2 - 4e(3-3e^2-e^4)L + 4e^2(3-2e^2)\right\}$$
$$\left.\left\{L(1+3e^2-4\nu e^2)-2e\right\}\right]^{-1}$$

It is seen that while net force is same as given by Kanwal and Sharma [4] for the underformed spheroid, a couple M_z given by (2.21) gets generated on account of the imperfection.

In the limiting process $\nu \to \frac{1}{2}$ gives the results, for the corresponding hydrodynamical solution involving stokes flow studied by Datta [2], with some corrections in the values of N and N^1. For $e = 0$, (2.20) provides the well known value of force on a sphere, while the couple M_z is seen to vanish, this confirms to the fact that for $\in = 0$ the equation (1.10) still represents a sphere.

The results for a slend^r body are obtained from the foregoing ones on applying the limiting process $c \to 1$. The force f_x reduces to that obtained by Kanwal and Sharma while the couple M_z has the asymptotic form

$$(2.22) \qquad M_z = 16 \in \pi \mu h_0 a \frac{3+(1-2\nu)\left[1+4\ln\left(\frac{2}{\delta}\right)\right]}{\left[1+2\ln\left(\frac{2}{\delta}\right)\right]\left[4(1-\nu)\ln\left(\frac{2}{\delta}\right)-1\right]},$$

where $\delta = \frac{b}{a}$ is the slenderness parameter.

3. Displacements Parallel To z and y Axes

Following the above procedure, we can construct the solutions for displacements parallel to z and y axes. Thus, we have for displacement parallel to z-axis

$$(3.1) \quad \bar{u} = A \int_{-c}^{c} \bar{U}^k(\bar{x}-\xi\bar{e}_x ; \bar{e}_z)\, d\xi + B \int_{-c}^{c} (c^2-\xi^2)\, \bar{U}^{dd}(\bar{x}-\xi\bar{e}_x ; \bar{e}_z)\, d\xi$$

$$- \frac{\in h_0 B e^3}{a^2}\left[\frac{4(1-2\nu)}{\{2e-(1-e^2)L+(1-e^2)\}} \int_{-c}^{c} (c^2-\xi^2)\, \bar{U}^r(\bar{x}-\xi\bar{e}_x ; \bar{e}_x)\, d\xi\right.$$

$$\left. - \frac{2\left[3(1-e^2)^2 L - 2e(3-5e^2) - 8(1-2\nu)e^5\right]}{(1-e^2)\left[3e^2(1-e^2)^2 L - 2e^4(3-5e^2) + 4(1-2\nu)e^5\{2e-L(1-e^2)\}\right]}\right]$$

$$\times \int_{-c}^{c} (c^2-\xi^2)\,\overline{U}^{ks}(\overline{x}-\xi\,\overline{e}_x\,;\overline{e}_y,\overline{e}_z)\,d\xi$$

$$+\frac{2(1-2\nu)\left[2e-(1-e^2)L+2e^3\right]}{3e^2(1-e^2)^2L-2e^4(3-5e^2)+4(1-2\nu)e^5\{2e-L(1-e^2)\}}$$

$$\times\int_{-c}^{c}(c^2-\xi^2)^2\frac{\partial\overline{U}^{dd}}{\partial y}(\overline{x}-\xi\overline{e}_x\,;\overline{e}_z)d\xi\ ,$$

where

$$(3.2)\qquad A=-\frac{2e^3B}{1-e^2}=2U_z e^2\left[2e-(1-7e^2+8\nu e)L\right]^{-1}.$$

For displacement parallel to y-axis:

$$(3.3)\qquad \overline{u}=A\int_{-c}^{c}\overline{U}^{k}(\overline{x}-\xi\,\overline{e}_x\,;\overline{e}_y)\,d\xi+B\int_{-c}^{c}(c^2-\xi^2)\,\overline{U}^{dd}(\overline{x}-\overline{e}_x\,;\overline{e}_y)\,d\xi$$

$$-\frac{8\in h_0Be^3}{(1-e^2)^2a^2}\left[C_1\int_{-c}^{c}(c^2-\xi^2)\,\overline{U}^{ks}(\overline{x}-\xi\,\overline{e}_x\,;\overline{e}_y,\overline{e}_y)\,d\xi\right.$$

$$+C_2\int_{-c}^{c}(c^2-\xi^2)^2\frac{\partial\overline{U}^{dd}}{\partial y}(\overline{x}-\xi\,\overline{e}_x\,;\overline{e}_y)d\xi$$

$$+C_3\int_{-c}^{c}(c^2-\xi^2)^2\frac{\partial\overline{U}^{dd}}{\partial z}(\overline{x}-\xi\overline{e}_x\,;\overline{e}_z)d\xi$$

$$+C_4\int_{-c}^{c}(c^2-\xi^2)\,\overline{U}^{ks}(\overline{x}-\xi\,\overline{e}_x;\overline{e}_x;\overline{e}_x)\,d\xi$$

$$+C_5\int_{-c}^{c}(c^2-\xi^2)^2\frac{\partial\overline{U}^{dd}}{\partial x}(\overline{x}-\xi\overline{e}_x\,;\overline{e}_x\,;\overline{e}_x)d\xi$$

$$\left.+C_6\int_{-c}^{c}(c^2-3\xi^2)\,\overline{U}^{d}(\overline{x}-\xi\,\overline{e}_x)\,d\xi\right],$$

where

$$A = -\frac{2e^3}{1-e^2}B = 2U_y e^2\left[2e-(1-7e^2+8\nu e^2)L\right]^{-1},$$

$$C_1 = D_5 / 2D_7 ,$$

$$C_2 = [DD_6 + (1-e^2)^2 (DD_5 + D_3 D_6)] / 2DD_7 ,$$

$$C_3 = [D_1 D_5 + D_3 D_6] / 2DD_7 ,$$

$$C_4 = [D_2 D_5 + D_4 D_6] / 2DD_7 ,$$

$$C_5 = (1-e^2)[D_2 D_5 + D_4 D_6] / 8e^2 DD_7 ,$$

$$C_6 = 8\,[D_1 D_5 + D_3 D_6] / 2DD_7 ,$$

$$D_1 = 2(1-2\nu)\,(L-2e)\left(\frac{2e}{1-e^2}-L\right)+\frac{4e}{1-e^2}[(3-e^2)L-6e],$$

$$D_2 = 36L^2(1+2e^2-e^4)-\frac{16eL}{1-e^2}(9+3e^2-26e^4+10e^5)$$

$$+\frac{16eL}{(1-e^2)^2}(9-12e^2-23e^4+32e^6-8e^8),$$

$$D_3 = -12(1-2\nu)\,(L-2e)\left[L-\frac{2e(3-5e^2)}{3(1-e^2)}\right]$$

$$+\frac{6}{e^2}[(3-e^2)L-6e]\left[L-\frac{2e(3-5e^2+4e^4)}{3(1-e^2)^2}\right],$$

$$D_4 = 252\,(1+2e^2-e^4)L^2-\frac{24eL}{(1-e^2)}(30-108e^2+121e^4-52e^6+3e^8)$$

$$+\frac{16e^2}{(1-e^2)^2}(90-288e^2+275e^4-43e^6-32e^8),$$

$$D_5 = (1-2\nu)\,\frac{8e^5}{(1-e^2)^2}-3\left[L-\frac{2e(3-5e^2)}{3(1-e^2)^2}\right],$$

$$D_6 = (1-2\nu)\,\frac{2e^3}{(1-e^2)}+(1-\nu)\left[\frac{2e}{1-e^2}-L\right],$$

$$D_7 = (1-\nu)\frac{8e^5}{(1-e^2)^2}\left[\frac{2e}{1-e^2} - L\right] + \frac{6e^3}{1-e^2}\left[L - \frac{2e(3-5e^2)}{3(1-e^2)^2}\right],$$

$$D = 16\,(1-2\nu)\,(L-2e)\left[\frac{2e}{1-e^2}(3+6e^2-5e^4) - 3(1+2e^2-e^4)L\right]$$

(3.4) $$+\frac{16e}{1-e^2}[(3-e^2)L-6e][5+e^2-2e^4].$$

The result (3.1) shows that the deformity in the spheroid does not produce any additional force for a displacement parallel to z direction but produces a couple M_x, along the x-axis given by

(3.5) $$M_x = \frac{64\pi \in \mu a(1-2\nu)\,e^6}{3\left[2e-(1-e^2)L\right]\left[2e-(1-7e^2+8\nu e^2)L\right]}$$

It may be verified that the result (3.1) agrees with the corresponding hydrodynamical case obtained by Datta [2] under the limiting process $\nu \to \frac{1}{2}$. It is also interesting to note that for stokes flow parallel to z-axis there is no couple on the deformed particle in contrast to the couple M_x in the elastostatic situation as given by (3.5).

The result (3.3) shows that for displacement along the y-axis there is no additional force or couple. It may also be pointed but that since the corresponding stokes flow result as given in [2] is in error, expression (3.3) does not correspond, with it in the limiting process $\nu \to \frac{1}{2}$. Actually the value of $\bar{u}$ as given in [2] is for the situation when the deformed spheroid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z-\in h^2(x)}{b^2}=1$$

is placed in a stream along y-direction. Further it is defined that drag coefficient of the net force experienced by the body along x-axis is given as

$$\frac{F_x}{6\pi\,\mu\,U_x a} = CF_x = \frac{16}{3}\,\frac{(1-\nu)e^3}{[(1+3e^2-4\nu e^2)L-2e]}.$$

The behaviour of this drag coefficient and aspect ratio of the body is given in fig (1), in which it is clear that CF_x increases gradually as aspect ratio $\delta = \frac{b}{a}$ and material (For which Poisson's ratio $0 < \nu < \frac{1}{2}$) changes

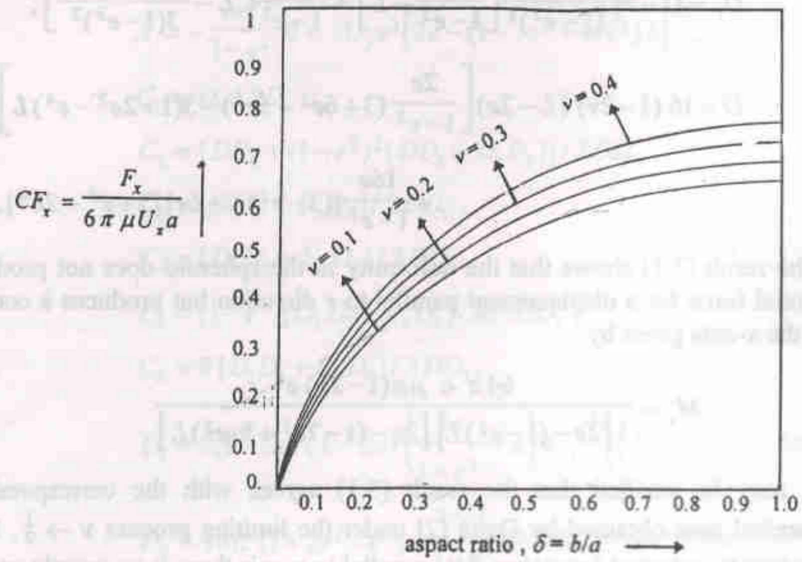

Fig (1)

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Min–constant For Polynomials*

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Abstract: Let $\mathcal{G}$ be some set of analytic functions f in the unit disk D normalized by the conditions $f(0) = 0$, $f'(0) = 1$. Let $\mathcal{G}_u$ denote the set of univalent functions in $\mathcal{G}$. We introduce the quantity

$$\lambda(\mathcal{G}) = \sup_{f \in \mathcal{G} \setminus \mathcal{G}_u} \{\inf_{|z|<1} |f'(z)|\},$$

which we call the min-constant of $\mathcal{G}$. We discuss $\lambda(\mathcal{T}_2)$, where $\mathcal{T}_2$ consists of the trinomials.

$$z + az^2 + bz^3$$

and obtain $\lambda(\mathcal{T}_2) = 0.032\ldots$.

1. Introduction:

Let $\mathcal{A}$ be the set of non-constant functions f analytic in the unit disk $|z| < 1$ and normalized by the conditions $f(0) = 0$, $f'(0) = 1$. Let $\mathcal{A}_u \subset \mathcal{A}$ be the subclass of functions which are univalent in the unit disk. Let $\mathcal{G}_n \subset \mathcal{A} \setminus \mathcal{A}_u$ consist of all polynomials of degree at most n, which are properly normalized and non-univalent in D. For $f \in \mathcal{A}$, let

$$m_f = \inf_{|z|<1} |f'(z)|, \tag{1}$$

and set

$$I_n = \sup_{f \in \mathcal{G}_n} m_f \tag{2}$$

We call I_n the min-constant for polynomials of degree n. The value of I_n could serve as a univalent criterion: if $M_f > I_n$ for some normalized polynomial f of

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degree n, then f is univalent in the unit disk (and hence the name minimum condition constant, or min-constant, for short). Clearly, I_n is an increasing function of n and $\lim_{n\to\infty} I_n = I$, where

$$I = \sup_{f \in \mathcal{A} \setminus \mathcal{A}_u} m_f .$$

It is easily verified that $I_2 = 0$, The main objective of the present paper is to establish the following identification of I_3.

Theorem 1. *We have*

$$I_3 = \frac{1}{18\sqrt{3}} = 0.032\ldots .$$

Furthermore, $I_3 = m_f$, where f is the (univalent) polynomial

$$f(z) = z + \frac{\sqrt{65}}{9} z^2 + \frac{5}{18} z^3 .$$

The extremal polynomial f is unique in the following sense : if $f_j \in \mathcal{G}_3$ satisfy $m_{f_j} \to I_3$, then there is a subsequence f_{jm} and an $x \in \mathbb{C}$ with $|x| = 1$ such that $f_{jm}(z) \to x^{-1} f(xz)$ in $\mathbb{D}$.

Our proof of Theorem 1 admits a generalization to the sets $\mathcal{T}_k$ of trinomials of the form

$$z + az^k + bz^{2k-1}, \qquad k = 2, 3, \ldots .$$

However, the formulas involved become so complicated with increasing k that no closed expressions for the extremal polynomials nor the values of

$$\text{(3)} \qquad \Lambda_k := \lambda(\mathcal{T}_k) = \sup_{f \in \mathcal{T}_k \setminus \mathcal{A}_u} m_f$$

are available, Numerical results can be achieved [2] such as

$$\text{(4)} \qquad \lambda(\mathcal{T}_3) = 0.055\ldots,\ \lambda(\mathcal{T}_4) = 0.069\ldots, \text{ and } \lim_{k\to\infty} \lambda(\mathcal{T}_k) = 0.119\ldots .$$

The following sections are organized as follows : in Sec. 2 we reduce the set of candidates in $\mathcal{T}_k$ which are extremal for Λ_k in the sense described in Theorem 1.. In Sect. 3 we prove Theorem 1 for polynomials with real coefficients, and Sect. 4 extends this to the general complex case. Lemmas 4, 6, 8 and Relation (13) have now been published as part of the author's [3] paper on the John Constants for Polynomials. We reproduce the short proofs, however, for completeness and to make for independent reading.

2. Reduction of the Problem

In this section we derive a general result which in fact replaces the search area for the supremum in (3).

Let $\mathcal{N}_k = \mathcal{T}_k \setminus \mathcal{A}_u$, i.e. the set of non-univalent functions in $\mathcal{T}_k$. We make a convention that whenever $\mathcal{V}_k$ is any subset of $\mathcal{T}_k$, then

$$\mathcal{V}_k^{\mathbf{R}} = \{ p = z + az^k + bz^{2k-1} \in \mathcal{V}_k \mid a, b \text{ are real} \}$$

and

$$\mathcal{V}_k^{\mathbf{C}} = \{ p = z + az^k + bz^{2k-1} \in \mathcal{V}_k \mid a \text{ is complex and } b \geq 0 \}.$$

Now set

(5) $$\Lambda_k^{\mathbf{R}} = \sup_{p \in \mathcal{N}_k^{\mathbf{R}}} m_p, \ \ \Lambda_k^{\mathbf{C}} = \sup_{p \in \mathcal{N}_k^{\mathbf{C}}} m_p$$

Then $\Lambda_k \geq \Lambda_k^{\mathbf{R}}$ as also $\Lambda_k \geq \Lambda_k^{\mathbf{C}}$. In fact

(6) $$\Lambda_k \geq \Lambda_k^{\mathbf{C}},$$

for given any polynomial p in $\mathcal{N}_k$ we can find, by rotation, a polynomial g in $\mathcal{N}_k^{\mathbf{C}}$ such that $m_p = m_g$.

The following is a particular case of a theorem of Quine [4].

Lemma 1 For $r > 0$, let

$$S_r = \{ (a,b) \in \mathbf{C}^2 \mid z + az^k + bz^{2k-1} \textit{ is univalent in } |z| < r \}$$

and $S = S_1$. If p is a polynomial whose coefficient pair (a, b) belongs to the boundary ∂S_r of S_r, then $p(x) = p(y)$ for some points $x, y, |x| = |y| = r$, or $p'(z) = 0$ for some point $z, |z| = r$.

We now prove

Lemma 2 Let $\mathcal{H}_k$ be the subset of polynomials $p(z) = z + az^k + bz^{2k-1}$ in $\mathcal{T}_k$ for which $(a, b) \in \partial S$ and $p'(z) \neq 0, |z| = 1$. Then

$$\sup_{p \in \mathcal{H}_k} m_p = \sup_{p \in \mathcal{N}_k} m_p.$$

Here the left side is taken to be zero if $\mathcal{H}_k$ is empty.

Remark 1 In view of Lemma 1, each polynomial in $\mathcal{H}_k$ assigns the same value to some two points which lie on the unit circle. The pairs (a, b), $p \in \mathcal{H}_k$, are said to form the double point curve in $\mathbf{C}^2$.

Proof of Lemma 2. First we prove the inequality.

$$(7) \qquad \sup_{p \in \mathcal{H}_k} m_p \geq \sup_{p \in \mathcal{N}_k} m_p \; (= \Lambda_k).$$

Case I. Suppose $m_p = 0$ for all p in $\mathcal{N}_k$. The right side is then zero, and we are done.

Case II. Suppose $m_f \neq 0$ for some f in $\mathcal{N}_k$. Then $f'(z) \neq 0$, $|z| < 1$. We set

$$r = \sup \{0 < t < 1 \mid f \text{ is univalent in } |z| < t\},$$

then $r > 0$, and $|z| < r$ is the largest disk centered in the origin where f is univalent. Hence r < 1, and the coefficient pair of f belongs to ∂S_r . We define

$$p(z) = \frac{1}{r} f(rz).$$

Then the coefficient pair of p belongs to ∂S. Moreover $p \in \mathcal{H}_k$, since $p'(z) \neq 0$, $|z| < \frac{1}{r}$. Now again, as r is less than 1,

$$\inf_{|z|<1} |p'(z)| = \inf_{|z|<1} |f'(rz)| \geq \inf_{|z|<1} |f'(z)|.$$

That is, $m_p \geq m_f$. Given f in $\mathcal{N}_k$, $m_f \neq 0$, we have thus found a p in $\mathcal{H}_k$ for which $m_p \geq m_f$. This completes the proof of (7).

To prove the opposite inequality consider first the possibility that $\mathcal{H}_k$ is empty. The left side of (7) is then zero, thus bringing the proof to an end. So suppose $p \in \mathcal{H}_k$. By Remark 1, $p(x) = p(y)$ for some points $x, y, x \neq y, |x| = |y| = 1$. For small $\epsilon > 0$, let

$$p_\varepsilon(z) = \frac{1}{1+\varepsilon} p[(1+\varepsilon)z].$$

The inequality

$$p_\varepsilon\left(\frac{x}{1+\varepsilon}\right) = p_\varepsilon\left(\frac{y}{1+\varepsilon}\right)$$

shows that p_ε is not univalent in $|z| < 1$, and so is an element of $\mathcal{N}_k$. Hence

$$\Lambda_k \geq m_{p\varepsilon} = \inf_{|z|<1} |p'[(1+\varepsilon)z)]|.$$

Letting $\varepsilon \to 0$, we find $\Lambda_k \geq m_p$. As $p \in \mathcal{H}_k$ is arbitrary, we have

$$\Lambda_k \geq \sup_{p \in \mathcal{H}_k} m_p.$$

The proof of the lemma is now complete.

Remark 2. A similar result holds when all the polynomials considered have either real coefficients, or complex first coefficient and nonnegative second coefficient. That is

$$\sup_{p \in \mathcal{H}_k^{\mathbb{R}}} m_p = \sup_{p \in \mathcal{N}_k^{\mathbb{R}}} m_p (= \Lambda_k^{\mathbb{R}}) \tag{8}$$

and

$$\sup_{p \in \mathcal{H}_k^{\mathbb{C}}} m_p = \sup_{p \in \mathcal{N}_k^{\mathbb{C}}} m_p (= \Lambda_k^{\mathbb{C}}) .$$

The last relation in conjunction with (6) yields

$$\Lambda_k = \sup_{p \in \mathcal{H}_k^{\mathbb{C}}} m_p \tag{9}$$

3. Mini–constant for Third degree Polynomials with Real Coefficients

Polynomials considered in this section all have only real coefficients. Our aim here is to determine $\Lambda_2^{\mathbb{R}}$. As (8) suggests, our first step should be to determine the set $\mathcal{H}_2^{\mathbb{R}}$. The following theorem of Brannan [1, Theorem 2(b)] serves this purpose.

Lemma 3. Suppose

$$p(z) = z + az^2 + bz^3 , \tag{10}$$

where a and b are real. Then p is univalent in $|z| < 1$ *if and only if*

$$|a| \le \begin{cases} (1+3b)/2, & -\frac{1}{3} \le b \le \frac{1}{5} \\ 2\sqrt{b(1-b)}, & \frac{1}{5} \le b \le \frac{1}{3}. \end{cases}$$

Lemma 4. The set $\mathcal{H}_2^{\mathbb{R}}$ consists of polynomials (10) such that

$$|a| = 2\sqrt{b(1-b)}, \quad \frac{1}{5} < b < \frac{1}{3}. \tag{11}$$

Proof : By Lemma 3, the coefficient pair (a, b) of p belongs to ∂S if and only if one of the following holds :

$$|a| = (1+3b)/2, \quad -\frac{1}{3} \le b \le \frac{1}{5};$$

$$|a| = 2\sqrt{b(1-b)}, \quad \frac{1}{5} < b < \frac{1}{3};$$

$$|a| = 2\sqrt{b(1-b)}, \quad b = \frac{1}{3}.$$

It follows from Lemma 6(b) (see below) that $p'(z) = 0$ for some point z, $|z| = 1$, if and only if a, b satisfy the first or third relation above. Therefore, numbers a, b which satisfy the second relation determine the polynomials in $\mathcal{H}_2^{\mathbb{R}}$. This prove the lemma.

Now suppose p, given by (10), is a polynomial in $\mathcal{H}_2^{*}$. Then (11) holds by the above lemma. For definiteness, we take $a \geq 0$; the case $a \leq 0$ is symmetric. To determine m_p we note that for any polynomial p, if $p'(z) = 0$ for some point z, $|z| \leq 1$, then $m_p = 0$. Otherwise, by the Maximum Principle,

$$m_p = \min_{|z|=1} |p'(z)| \tag{12}$$

If $z = e^{i\theta}$ is a point on the unit circle, let $p(\theta)$ denote $|p'(z)|^2$. Then

$$P(\theta) = 1 + 4a^2 + 9b^2 + 4a\cos\theta + 6b\cos 2\theta + 12ab\cos\theta$$

Its derivative

$$P'(\theta) = -4\sin\theta(a + 6b\cos\theta + 3ab)$$

is zero when $\sin\theta = 0$ or $\cos\theta = -a(3b+1)/(6b)$. Clearly, $P(\theta)$ attains its maximum value when $\theta = 0$. If $a(3b+1) \geq 6b$, $P(\theta)$ is minimized when $\theta = \pi$; otherwise when $\cos\theta = -a(3b+1)/(6b)$. Consider the polynomial

$$\begin{aligned} F(b) &= \frac{1}{4b}\left(a^2(3b+1)^2 - 36b^2\right) \\ &= -9b^3 + 3b^2 - 4b + 1. \end{aligned}$$

We see that $F(b_1) = 0$, where $b_1 = 0.261\ldots$ is a point in $\left(\frac{1}{5}, \frac{1}{3}\right)$. Also $F'(b) < 0$ for $b > 0$; so $F(b)$ is a decreasing function. As a result

$$\begin{aligned} &a(3b+1) \geq 6b, \quad \tfrac{1}{5} \leq b \leq b_1, \\ &a(3b+1) \geq 6b, \quad b_1 \leq b \leq \tfrac{1}{3}. \end{aligned}$$

Accordingly

$$m_p = \begin{cases} 1 + 3b - 2a, & \frac{1}{5} \leq b \leq b_1, \\ \dfrac{(1-3b)\sqrt{3b - a^2}}{\sqrt{3b}}, & b_1 \leq b \leq \frac{1}{3} \end{cases} \tag{13}$$

Using the relation $a = 2\sqrt{b(1-b)}$, we finally obtain

$$m_p = \begin{cases} 1 + 3b - 4\sqrt{b(1-b)}, & \frac{1}{5} \leq b \leq b_1, \\ (1-3b)\dfrac{\sqrt{4b-1}}{3}, & b_1 \leq b \leq \frac{1}{3} \end{cases}$$

m_p is increasing on the interval $\left(\frac{1}{5} \cdot b_1\right]$, its derivative there being positive. The maximum value of m_p on $\left[b_1, \frac{1}{3}\right)$ is hence the maximum on the whole interval $\left(\frac{1}{5}, \frac{1}{3}\right)$. In $\left[b_1, \frac{1}{3}\right)$ the derivative m_p' is zero when

$$54b^2 - 33b + 5 = 0$$

or

$$(3b-1)(18b-5) = 0.$$

m_p attains its maximum value at $b = \frac{5}{18}$. The coefficient $a = \frac{\sqrt{65}}{9}$ and $m_p = 0.032\ldots$ We come now to

Proposition 1. *The min-constant $\Lambda_2^{\mathbb{R}}\left(= I_3^{\mathbb{R}}\right)$ for third degree polynomials with real coefficients is equal to*

$$m_f = \frac{1}{18\sqrt{3}} = 0.032\ldots,$$

where f is the polynomial

$$f(z) = z + \frac{\sqrt{65}}{9}z^2 + \frac{5}{18}z^3.$$

Proof: By the very construction of f,

$$m_f = \sup_{p \in \mathcal{H}_2^{\mathbb{R}}} m_p.$$

But the right side is equal to $\Lambda_2^{\mathbb{R}}$ by (8).

4. Min-constant for Third Degree Polynomials

In view of (9), our first task this time should be to determine $\mathcal{H}_2^{\mathbb{C}}$. We quote another result of Brannan [1, Theorem 2(a)].

Lemma 5. Suppose $p(z) = z + (a+id)z^2 + bz^2$, where a, b, and d are real. Then:

(a) If $0 \le b \le \frac{1}{5}$, *p is univalent in* $|z| < 1$ *if and only if*

$$\frac{a^2}{(1+3b)^2} + \frac{a^2}{(1-3b)^2} \le \frac{1}{4}.$$

(b) If $\frac{1}{5} \le b \le \frac{1}{3}$, *p is univalent in* $|z| < 1$ *if and only if*

$$\frac{a^2}{(1+tb)^2} + \frac{d^2}{(1-tb)^2} \le \frac{1}{1+t}$$

is satisfied for all t such that $\frac{1-2b}{b} \le t \le 3$; *in particular, p is univalent in* $|z| < 1$ *if*

$$\frac{a^2}{4b(1-b)} + \frac{4d^2}{(1-3b)^2} \le 1.$$

Remark 3. Let us denote by S^b the section of the coefficient region S with a fixed value of b. If $\frac{1}{5} \le b \le \frac{1}{3}$, S^b is the intersection of the ellipses.

$$\frac{x^2}{(1+tb)^2}+\frac{y^2}{(1-tb)^2} \le \frac{1}{1+t}, \qquad \frac{1-2b}{b} \le t \le 3,$$

and ∂S^b is the closed curve enclosing it. If $0 \le b \le \frac{1}{5}$, then ∂S^b is the ellipse

$$\frac{x^2}{(1+3b)^2}+\frac{y^2}{(1-3b)^2}=\frac{1}{4}. \tag{14}$$

Lemma 6. *Suppose* $p(z)=z+(a+id)z^2+bz^3$, *where* a, d *are real and* $-\frac{1}{3} \le b \le \frac{1}{3}$.

(a) If $z=e^{i\theta}$, *then*

$$|p'(z)|=2\sqrt{\left(a+\frac{1+3b}{2}\cos\theta\right)^2+\left(d-\frac{1-3b}{2}\sin\theta\right)^2}$$

(b) $p'(z)=0$ for some point z, $|z|=1$, if and only if (a, d) lies on the ellipse (14).

(c) If $p'(z)\neq 0$, $|z|\le 1$, then m_p is equal to two times the minimum distance form (a, d) to the ellipse (14).

Proof : Clearly

$$\frac{p'(z)}{z}=\bar{z}+2(a+id)+3bz$$

if $|z|=1$. Put $z=e^{i\theta}$ and separate real and imaginary parts. Then

$$\frac{p'(z)}{z}=\cos\theta+2a+3b\cos\theta+i(-\sin\theta+2d+3\sin\theta).$$

Part (a) is now obtained if we take absolute values.
Part (b) follows from (a) by eliminating θ between the equations

$$a+\frac{1+3b}{2}\cos\theta=0,$$

$$d-\frac{1-3b}{2}\cos\theta=0.$$

To prove (c) suppose $p'(z)\neq 0$, $|z|\le 1$. We can then use (12) to evaluate m_p. The required result now readily follows from (a) if we note that

$$x=-\frac{1+3b}{2}\cos\theta, \quad y=\frac{1-3b}{2}\sin\theta, \quad 0\le\theta\le 2\pi,$$

are parametric equations for the ellipse (14).

A simple method for evaluating m_p is to use Part (c) in conjunction with

Lemma 7. *Let*

$$\frac{x^2}{A^2}+\frac{y^2}{D^2}=1, \quad 0<D\le A,$$

by the equation to an ellipse and $(q,0)$, $0\le q\le A$, *a point on the major axis. Then the minimum distance from the point* $(q,0)$ *to the ellipse is given by*

$$m=\begin{cases} D\sqrt{\dfrac{A^2-D^2-q^2}{A^2-D^2}}, & 0\le q\le \frac{A^2-D^2}{A},\\ A-q, & \frac{A^2-D^2}{A}\le q\le A. \end{cases}$$

The proof, which is elementary, is omitted. if we take $A=\frac{1}{2}(1+3b)$, $D=\frac{1}{2}(1-3b)$, and $q=a$, we obtain (13) as required.

Lemma 8. *Suppose* $p(z)=z+(a+id)z^2+bz^3$ *is a polynomial in* $\mathcal{H}_2^{\mathbb{C}}$. *Then* $a+id \in \partial S^b$, *where* $\frac{1}{5}<b<\frac{1}{3}$.

Proof: The point (a, d) lies on ∂S^b, by the very definition of $\mathcal{H}_2^{\mathbb{C}}$. By Remark 3, (a, d) lies also on the ellipse (14) if $0\le b\le\frac{1}{5}$ or if $b=\frac{1}{3}$. In view of Lemma 6(b), $p\notin \mathcal{H}_2^{\mathbb{C}}$ in such cases. Hence $\frac{1}{5}<b<\frac{1}{3}$, which completes the proof of the lemma.

Proposition 2. $\Lambda_2=\Lambda_2^{\mathbb{R}}$.

Proof : Obviously, $\Lambda_2\ge\Lambda_2^{\mathbb{R}}$. Assume that, given any p in $\mathcal{H}_2^{\mathbb{C}}$. we can find an f in $\mathcal{H}_2^{\mathbb{R}}$ such that

$$m_f \ge m_p \tag{15}$$

It would then follow from (8) and (9) that $\Lambda_2^{\mathbb{R}}\ge\Lambda_2$, and the proposition would proved. So let

$$p(z)=z+(a+id)z^2+bz^3$$

be a polynomial in $\mathcal{H}_2^{\mathbb{C}}$. Lemma 8 shows that $a+id\in\partial S^b$, where $\frac{1}{5}<b<\frac{1}{3}$. Consider the polynomial

$$f(z)=z+qz^2+bz^3,$$

where $q=2\sqrt{b(1-b)}$. Then $f\in\mathcal{H}_2^{\mathbb{R}}$ by Lemma 4. It remains to show that (15) holds.

Put $A=\frac{1+3b}{2}$ and $D=\frac{1-3b}{2}$, and consider the ellipses

$$\frac{x^2}{q^2}+\frac{y^2}{D^2}=1, \tag{16}$$

$$\frac{x^2}{A^2}+\frac{y^2}{D^2}=1. \tag{17}$$

Lemma 5 (b) shows that the ellipse (16) lies inside or on ∂S^b, while ∂S^b in turn lies inside or on the ellipse (17). For definiteness, we take $a,\ d \geq 0$, other cases being symmetric. Let a_1, a_2 be non-negative numbers such that the point $P' = (a_1, d)$ (see Figure 1) lies on the ellipse (16) and the point $Q' = (a_2, d)$ lies on the ellipse (17). Then $a_1 \leq a \leq a_2$ and

$$\frac{a_1^2}{q^2}+\frac{d^2}{D^2}=1, \quad \frac{a_2^2}{A^2}+\frac{d^2}{D^2}=1.$$

Hence

$$\frac{a_1}{q}=\frac{a_2}{A}=\frac{a_2-a_1}{A-q}. \tag{18}$$

We note that $a_2 - a_1 = 0$ if and only if $a = 0$, which implies that (a, d) lies on the ellipse (17). By Lemma 6(b), $m_p = 0$ in this case and (15) holds trivially. So assume $a > 0$.

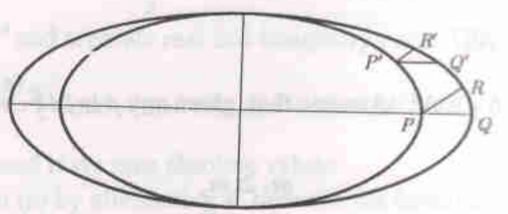

Figure 1

Let R be the point on the ellipse (17) which lies nearest to the point $P = (q, 0)$. The point R may be different from the vertex $Q = (A, 0)$ of the ellipse. In any case dist $(P, R) = h$ dist (P, Q), where $h > 0$.

The least distance from the point $P' = (a_1, d)$ to the ellipse (17) is not greater than dist (P', R'), where R' is the point on the ellipse such that $\overline{P'R'}$ is parallel to $\overline{PR}$. Suppose dist $(P', R') = h'$ dist (P', Q'), where $h' > 0$. It is easy to see that $h' \leq h$.

In view of Lemma 6(c), we now have

$$m_f = \text{dist}\,(P, R) = h(A-q)$$

whereas

$$m_p \leq \text{dist}\,(P', R') = h'\ \text{dist}\,(P', Q') = h'(a_2 - a_1)\ .$$

Hence

$$\frac{m_p}{m_f} \leq \frac{h'}{h} \cdot \frac{a_2 - a_1}{A - q}$$

$$= \frac{h'}{h} \cdot \frac{a_1}{q}, \quad \text{by (18)},$$

$$\leq\ 1,$$

since $a_1 \leq q$ and $h' \leq h$. This proves (15), and so completes the proof of the proposition.

Propositions 1 and 2 together finally prove Theorem 1.

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A Study of Special Function with Lie Theory

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Abstract: In this paper we give new proof of some properties of the parabolic cylindrical special functions using a few operators defined on a Lie algebra. We also give a new treatment of the Parabolic cylindrical special functions and some of its properties, which will be done by using some properties of operators defined on Lie-algebra.

Let End V be the Lie-algebra of endomorphisms of the vector space V endowed with the Lie bracket [,] defined by

$[A, B] = AB - BA$, for every $A, B \in$ End V. We denote by I the identity operator of V.

Theorem 1. *Let $A, B \in$ End V be such that $[A, B] = -I$. We define the sequence $\{y_n\} \subset V$ as follows : $Ay_0 = 0$ and $By_{n-1} = -y_n$ for every $n \geq 1$. Then y_n is an eigen vector of eigen value n for BA for every $n \geq 1$.*

Proof : We shall show firstly that

$$Ay_n = ny_{n-1} \text{ for every } n \geq 1.$$

For $n = 1$, this equality is evident because

$$[A, B]\, y_0 = ABy_0 - BAy_0 = -Ay_1 = -y_0.$$

We suppose that $Ay_n = ny_{n-1}$.

We may write similarity,

$$[A, B]\, y_n = -y_n \text{ or, } ABy_n - BAy_n = -y_n \text{ or, } Ay_{n+1} + nBy_{n-1} = y_n$$

$$\text{or, } Ay_{n+1} = (n+1)\, y_n.$$

Thus, $$BAy_n = Bny_{n-1} = -ny_n.$$

This completes the proof.

Let $V = C^{\infty}(R)$. We define the operators $A, B \in$ End.V by

$$(Af)\,x = f'(x) + \frac{xf(x)}{2}, \quad (Bf)\,x = f'(x) - \frac{xf(x)}{2} \text{ for every } x \in R.$$

We prove that these operators satisfy the commutation relation $[A, B] = -I$.

$$[A, B\,]\, f(x) = AB\, f(x) - BA\, f(x) = A\left[f'(x) - \frac{x f(x)}{2}\right] - B\left[f'(x) + \frac{x f(x)}{2}\right]$$
$$= -f(x).$$

Therefore, $[\,A, B\,] = -I$.

The Parabolic cylindrical function $D_n(x) = 2^{-n/2} \exp(-x^2/4) H(2^{-1/2} x)$, where $H_n(x) = (-1)^n \exp(x^2)\,(d^n/dx^n) \exp(-x^2)$ is the Hermite polynomial of degree n, is a solution of the differential equation

$$y'' + (n + 1/2 - x^2/4)\,y = 0.$$

Therefore, $$y'' + (1/2 - x^2/4)\,y = -ny.$$

Now, we have $Ay = y' + \dfrac{xy}{2}$. Also, $By = y' - \dfrac{xy}{2}$

or, $$\left(B + \frac{x}{2}\right)\left(Ay - \frac{xy}{2}\right) = y'' \quad \text{or} \quad BAy = y'' + (1/2 - x^2/4)\,y = -ny.$$

By theorem 1, it follows that y_n is a solution of the differential equation of the Parabolic cylindrical function. Again, from the definitions of the operators A and B it follows that $Ay_n = ny_{n-1}$ and $(Af)\,x = y' + \dfrac{xy}{2}$, we have

$$y_n' = ny_{n-1} - \frac{xy_n}{2} \quad \text{i.e.} \quad D_n'(x) = nD_{n-1}(x) - \frac{xD_n(x)}{2}.$$

which is a differential recursion relation for the Parabolic cylindrical functions.

Also, from $By_n = -y_{n+1}$ and $(Bf)\,x = y' - \dfrac{xy}{2}$, we obtain

$$y_n' = \frac{xy_n}{2} - y_{n+1}. \quad \text{i.e.} \quad D_n'(x) = \frac{xD_n(x)}{2} - D_{n+1}(x).$$

This gives another differential recursion relation for $D_n(x)$.

From the two recursion relations obtained above we readily obtain

$$D_{n+1}(x) - xD_n(x) + nD_{n-1}(x) = 0.$$

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On Normal Quasi–Sasakian Manifold

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Abstract: Recently Mishra [1] defined normal quasi-Sasakian (*nqs*) manifold and nearly nqs manifold. The purpose of this paper is to study some properties of these manifolds.

1. Introduction

Let M be an n–dimensional G^{∞}–manifold and let there exist on M a vector valued linear function ϕ of type (1,1), a vector field ξ and a 1–form η such that

(1.1) a) $\phi^2 = -\mathrm{I} + \eta \otimes \xi$
b) $\eta(\xi) = 1$,

for arbitrary vector field X. Then M is called an almost contact manifold and structure (ϕ, ξ, η) is called an almost contact structure [2].

It follows from (1.1) that the following hold in M: rank $(\phi) = n-1$, n is odd i.e., $n = 2m + 1$ and

(1.2) a) $\eta(\phi X) = 0$,
b) $\phi \xi = 0$.

In addition, if in M there exist a metric tensor g satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\,\eta(Y), \tag{1.3}$$

which is equivalent to $g(\phi X, \phi Y) = -g(\phi^2 X, Y)$ and $g(X, \xi) = \eta(X)$, then M is called an almost contact metric manifold and (ϕ, ξ, η, g) an almost contact metric structure [2].

The fundamental 2-form $'F$ of an almost contact metric is defined by

$$'F(X, Y) = g(\phi X, Y). \tag{1.4}$$

Thus we have

(1.5) a) $'F(\phi X, \phi Y) = 'F(X, Y)$,
b) $'F(X, Y) = -'F(Y, X)$

It can be easily proved that on an almost contact metric manifold [3].

(1.6) a) $(D_X\,'F)(Y,\xi) = -(D_X\,\eta)\,(\phi Y)$

b) $g((D_X\,\phi)\,Y, Z) = (D_X\,'F)\,(Y, Z)$,

(1.7) $(D_X\,'F)(\phi Y,\,\phi Z) = -(D_X\;\;'F)(Y, Z) + \eta(Y)(D_X\,\eta)(\phi Z) - \eta(Z)(D_X\,\eta)(\phi Y)$.

Further in an almost contact metric manifold the Jijenhuis tensor is given by

(1.8) a) $N(X, Y) = (D_{\phi X}\;\phi)(Y) - (D_{\phi Y}\;\phi)(X) - \phi(D_X\;\phi)(Y) + \phi(D_Y\;\phi)(X)$,

whence

b) $'N(X, Y, Z) = (D_{\phi X}\;\;'F)(Y, Z) - (D_{\phi Y}\,'F)(X, Z) + (D_X\;'F)(Y, \phi Z) - (D_Y\;'F)(X, \phi Z)$,

where

$$'N(X, Y, Z) = g(N(X, Y), Z),$$

and D is the Riemannian connection of g.

An almost co:tact metric manifold is said to be quasi-Sasakian manifold if

(1.9) $(D_X\,'F)(Y, Z) + (D_y\,'F)(Z, X) + (D_Z\;'F)\;(X, Y) = 0$.

An almost contact metric manifold satisfying

(1.10) $(D_X\;\eta)(\phi Y) = -(D_{\phi X}\;\eta)(Y) = (D_Y\;\eta)\;(\phi X) \Leftrightarrow$
$(D_X\;\eta)(Y) = -(D_{\phi X}\;\eta)(\phi Y) = -(D_Y\;\eta)\;(X)$.

(a) is called nqs manifold if

(1.11) $(D_X\;'F)(Y, Z) = \eta(Y)(D_Z\;\eta)\;(\phi X) + \eta(Z)(D_{\phi X}\;\eta)\;(Y)$, and

(b) is called nearly nqs manifold if

(1.12) $(D_X\;'F)(Y, Z) = \eta(Y)(D_{\phi X}\;\;\eta)(Z) + 2\eta(Z)(D_Y\;\eta)(\phi X)$
$= (D_Y\;\;'F)(Z, X) - = \eta(X)(D_{\phi Y}\;\;\eta)(Z)$.

In this class of manifolds the following results are true [1]

(1.13) a) $D_\xi\;\eta = 0$, b) $D_\xi\;\xi = 0$, c) $D_\xi\;'F = 0$.

2. Properties:

Theorem (2.1) *nqs manifold is quasi–Sasakian manifold.*

Proof: In consequence of (1.11), we find

$$(D_X \ 'F)(Y,Z) + (D_Y \ 'F)(Z,X) + (D_Z \ 'F)(X,Y)$$
$$= \eta(Y)[(D_Z \ \eta)(\phi X) + (D_{\phi Z} \ \eta)(X)]$$
$$+ \eta(Z)[(D_{\phi X} \ \eta)(Y) + (D_X \ \eta)(\phi X)]$$
$$+ \eta(X)[(D_Y \ \eta)(\phi Z) + (D_{\phi Y} \ \eta)(Z)].$$

Using (1.10) in above equation, we get Theorem (2.1).

Theorem (2.2). *nqs manifold is necessarily a nearly nqs manifold.*

Proof : Again by virtue of (1.11), we get

$$(D_X \ 'F)(Y,Z) + (D_X \ 'F)(X,Z) = \eta(Y)(D_Z \ \eta)(\phi X) + \eta(X)(D_Z \ \eta)(\phi Y) + \eta(Z)[(D_{\phi X} \ \eta)(Y) + (D_{\phi Y} \ \eta)(X)]. \tag{2.1}$$

Using (1.10) in (2.1), we get

$$(D_X \ 'F)(Y,Z) + (D_Y \ F)(X,Z) + 2\eta(Z)(D_Y \ \eta)(\phi X) + \eta(Y)(D_{\phi X} \ \eta)(Z) + \eta(X)[(D_{\phi Y} \ \eta)(Z) = 0,$$

and hence nqs manifold is a nearly nqs manifold.

Theorem (2.3). *nqs manifold is completely integrable.*

Proof: The condition for an almost contact metric manifold to be completely integrable [3] is

$$'N(\phi X, \phi Y, \phi Z) = 0. \tag{2.2}$$

By virtue of (1.8) and (1.11), we get

$$N(X,Y,Z) = 2\eta(Y)(D_Z \ \eta)(X) - 2\eta(X)(D_Z \ \eta)(Y) - 2\eta(Z)(D_X \ \eta)(Y). \tag{2.3}$$

Operation ϕ in (2.3), we get the result.

Theorem 2.4. *The necessary condition for nearly nqs manifold to be completely integrable is that*

$$2(D_{\phi Y} \ 'F)(\phi X, \phi Z) = (D_{\phi^2 Y} \ 'F)(\phi X, \phi^2 Z) - (D_{\phi^2 X} \ 'F)(\phi Y, \phi^2 Z). \tag{2.4}$$

Proof: From (1.8)b), we have

$$N(\phi X, \phi Y, \phi Z) = (D_{\phi^2 X} \ 'F)(\phi Y, \phi Z) - (D_{\phi^2 Y} \ 'F)(\phi X, \phi Z). + (D_{\phi X} \ 'F)(\phi Y, \phi^2 Z) - (D_{\phi Y} \ 'F)(\phi X, \phi^2 Z). \tag{2.5}$$

In consequence of (1.7) and (1.10), (2.5) yields

(2.6) $$N(\phi X, \phi Y, \phi Z) = (D_X\ 'F)(Y, Z) - (D_Y\ 'F)(X, Z) - (D_{\phi X}\ 'F)(Y, \phi Z) + (D_{\phi Y}\ 'F)(X, \phi Z).$$

Now from (2.6) and (1.12), we obtain

(2.7) $$N(\phi X, \phi Y, \phi Z) = -2(D_Y\ 'F)(X, Z) - (D_{\phi X}\ 'F)(Y, \phi Z) + (D_{\phi Y}\ 'F)(X, \phi Z) - \eta(X)(D_{\phi Y}\ \eta)(Z) - \eta(Y)(D_{\phi X}\ \eta)(Z) - 2\eta(Z)(D_Y\ \eta)(\phi X)$$

Thus, from (2.7), (2.2), we obtain (2.4).

Theorem 2.5. *On a nearly nqs manifold,*

(2.8) $$N(\phi X, \phi Y) = 4[(D_{\phi Y}\ \phi)(\phi X) + d\eta(\phi X, \phi Y)\ \xi]$$

Proof : From (1.12), we have

(2.9) $$(D_X\ \phi)(Y) = -(D_Y\ \phi)(X) - \eta(Y)(D_{\phi X}\ \xi) - \eta(X)(D_{\phi Y}\ \xi) - 2(D_{\phi X}\ \eta)(Y)\xi.$$

Now, $$(D_Y\ \phi)(\phi X) = D_Y\ \phi^2 X - \phi D_Y\ \phi X$$
$$= D_Y\ \phi^2 X - \phi D_Y\ \phi X + \phi(\phi D_Y\ X) - \phi(\phi D_Y\ \phi).$$

Thus,

(2.10) $$(D_Y\ \phi)(\phi X) = (D_Y\ \eta(X))\ \xi + \eta(X)(D_Y\ \xi) - \phi(D_Y\ \phi)(X) - \eta(D_Y\ X)\ \xi.$$

Using (2.10) in (2.9), we find

(2.11) $$(D_{\phi X}\ \phi)(Y) = -(D_Y\ \eta(X))\ \xi - \eta(X)(D_Y\ \xi) + \phi(D_Y\ \phi)(X) + \eta(D_Y\ X)\ \xi + \eta(Y)(D_X\ \xi) + 2(D_X\ \eta)(Y)\ \xi.$$

Hence inconsequence of (2.11) and (1.8a), we get

(2.12) $$N(X, Y) = 2\phi((D_Y\ \phi)X - (D_X\ \phi)(Y)) + 2\eta(Y)(D_X\ \xi) - 2\eta(X)(D_Y\ \xi) + 4d\eta(X, Y)\ \xi.$$
$$= 2\phi((D_Y\ \phi)(X)) + (D_Y\ \phi)(X) + \eta(Y)(D_{\phi X}\ \xi) + \eta(X)(D_{\phi Y}\ \xi) + 2(D_{\phi X}\ \eta)(Y)\xi) + 2\eta(Y)(D_X\ \xi) - 2\eta(X)(D_Y\ \xi) + 4d\eta\ (X, Y)\ \xi.$$

(2.13) $$N(X, Y) = 4\phi(D_Y\ \phi)(X) + \eta(Y)\ \phi\ (D_{\phi X}\ \xi) + \eta(X)\ \phi(D_{\phi Y}\ \xi) + 2\eta(Y)(D_X\ \xi) - 2\eta(X)\ (D_Y\ \xi) + 4d\eta\ (X, Y)\ \xi.$$

Now operating on (2.13), we get (2.8).

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Mathematical Models and Algorithms For Certain Open shop Problems With Unit Processing Times

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Abstract: Objective is to revise and present blockmatrices model and give algorithms for open shop problems with unit processing times in scheduling theory. Considered problems are minimizing the completion and total completion times on machines. Furthermore, maximal idle-time and total idle-times on machines are minimized. polynomial algorithms are presented in the case of total operation set with unit processing times.

Key Words: Scheduling; Open shop problems; Sequence graph, Polynomial algorithms; Latin rectangles

1. Introduction

In classical open shop problems we have in machines $j \in J = \{1, 2, ..., m\}$ and n jobs $i \in I = \{1, 2, . . . , n\}$. The machine order of job i is the order of all machines in which the job i is processed, and the job order on machine j is the order of all jobs on machine j. An example of such a graph is shown in figure (2.1). In this example we have the machine order $M_2 \to M_3 \to M_4 \to M_1$ for J_1, and on machine M_4 we have the job orders $J_4 \to J_5 \to J_1 \to J_2 \to J_3$. By sequence, we mean feasible combination of machine and job orders whereas by schedules we mean completion times of all operations. Let $\overline{C}_j$ denotes the maximum completion time and $\underline{C}_j$ denotes the minimum completion time of operation on machine j respectively, and p_{ij} denotes the processing time of job i on machine j.

With $SIJ = I \times J$, it is considered the following open shop problems

$$(P_1) \quad O\,|\,p_{ij} = 1\,|\, g_1 \,, \qquad (P_3) \quad O\,|\,p_{ij} = 1\,|\, f_1 \,,$$
$$(P_2) \quad O\,|\,p_{ij} = 1\,|\, g_2 \,, \qquad (P_4) \quad O\,|\,p_{ij} = 1\,|\, f_2 \,,$$

where the objective functions $g_1,\ g_2,\ f_1,\ f_2$ are defined by

$$g_1 := \max_{j\in J}\{\overline{C}_J\}, g_2 := \sum_{j\in J}\overline{C}_J, f_1 := \max_{j\in J}\{\overline{C}_J - \underline{C}_j\}, \text{ and } f_2 = \sum_{j\in J}(\overline{C}_J - \underline{C}_j).$$

The machine and job orders can be chosen arbitrarily (open shop problems) and the usual assumptions is made : each job can be processed on at most one machine at a time and each machine can process at most one job at a time.

The problems $[O \mid p_{ij} = 1 \mid C_{\max}$ and $O \mid p_{ij} = 1 \mid \Sigma_{i \in 1} C_i]$ are solved in [2] with polynomial time algorithms and here we present modified polynomial algorithms and methods for the problems P_1, P_2, P_3 and P_4, where C_i denotes the completion time of job i. The used mathematical techniques and algorithms in order to solve considered problems are based on [5].

Various attempts have been made in scheduling theory by the help of blockmatrices model in which connection between latin rectangles [4] and sequence graph [3] is established.

2. On the Blockmatrices Model

We apply the blockmatrices model [1, 2] for modeling the considered open shop problems P_1, P_2, P_3 and P_4. For convenience we give its brief discussion here too. Firstly, the concept of latin matrix is introduced and then its connection with sequence graph is established.

A latin rectangle $LR[n, m, r] = [a_{ij}]$ is a matrix of size $n \times m$ with entries $a_{ij} \in S = \{1, 2, \ldots, r\}$ such that each integer of the insertion symbol set S occurs at most once in each column and at most once in each row of LR. If $n = m = r$ holds the matrix is called latin square of order n and is denoted by $LS[n]$.

Let $LR[n, m, r] = [a_{ij}]$ be any latin rectangle in r symbols $a_{ij} \in S = \{1, 2, \ldots, r\}$. We define $I = \{1, 2, \ldots, n\}$ as a set of jobs. $J = \{1, 2, \ldots, m\}$ as a set of machines, and $SIJ = I \times J$ as a set of operations for open shop problem. For a graph $G = (V, E)$ with vertex set $V = SIJ$, we define edge set E as follows:

The operation $(K\,l)$ is an immediate successor of operation (ij) if any one of the following conditions is satisfied

(a) $i = k$, $a_{ij} < a_{kl}$ and there is no $v \in J$ such that $a_{ij} < a_{iv} < a_{il}$, or

(b) $j = l$, $a_{ij} < a_{kl}$ and there is no $u \in I$ such that $a_{ij} < a_{uj} < a_{kj}$.

Then the vertex set V in the graph represents the set of operations and edge set E represents the set of job orders and machine orders. Since we can interpret each a_{ij} as level of vertex (ij), there only exist edges from lower level to higher level. Therefore, the constructed graph is acyclic and such acyclic graph corresponds to feasible sequence.

Let $MO = [b_{ij}]$ and $JO = [d_{ij}]$ be $n \times m$ matrices of given machine orders and job orders respectively : b_{ij} is the position of machine j in the machine order for job i and d_{ij} is the position of job i in the job order on machine j. Here, in each row of MO we have permutation of the integers $1, 2, \ldots, m$, in each column of JO we have permutation of the integers $1, 2, \ldots, n$.

For given machine orders and job orders and a set of operations $SIJ = I \times J$, we have a graph $G = (V, E)$, where $V = SIJ$ is the vertex set consisting of operations,

and E as edge set reflects machine orders and job orders. A combination of a given machine order with a given job order is not necessarily feasible, it is called feasible if and only if the graph G does not contain any cycle. In such a case the graph G is called sequence graph.

It is also possible to obtain an $n \times m$ matrix $[a_{ij}]$ satisfying the properties of latin rectangle for given sequence graph.

We define rank (source) = 1 and rank of other vertex (ij) by maximal number of vertices of a path from a source to it. Since the graph does not contain any cycle, we can determine the rank a_{ij} of each vertex (ij) in G. Also being only horizontal and vertical arcs in G, there do not exist any two vertices of the same rank in some row or column. Therefore the matrix $[a_{ij}] = LR\,[n, m, r]$ so formed by the rank of each vertices satisfies the following property (2.1) and indeed it is a latin rectangle.

Property 2.1 If $a_{ij} > 1$, then there exists integers $a_{ij} - 1$ in row i or column j or in both.

It is also very clear that a latin rectangle LR $[n, m, r]$ satisfying the property 2.1 easily produces a sequence graph for open shop problem.

Following theorem connects open shop problems and latin rectangles.

Theorem 2.1 *There exists a one-to-one correspondence between the set of latin rectangles* $LR[n, m, r]$ *with the property* 2.1 *and the set of sequence graph G for open shop problem.*

Example 2.1 Following example illustrates the fact in the case of $V = SIJ = I \times J$

$$LR\,[5,4,6] = \begin{bmatrix} 6 & 1 & 3 & 4 \\ 1 & 2 & 4 & 5 \\ 3 & 4 & 2 & 6 \\ 4 & 5 & 6 & 1 \\ 5 & 6 & 1 & 2 \end{bmatrix}, n = 5, m = 4, r = 6$$

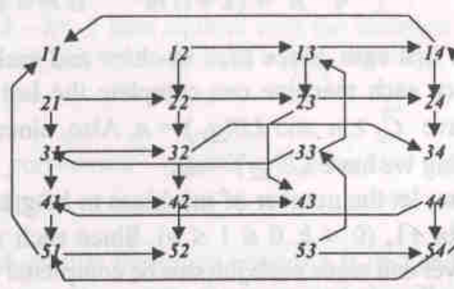

Figure 2.1 The Sequence Graph $G = (V, E)$

The horizontal edges represent the machine orders and the vertical edges represent the job orders respectively.

Note that we will consider here such latin rectangles satisfying the property 2.1 unless otherwise stated.

If we assign in a graph $G = (V, E)$, a vertex cost p_{ij} to each $(ij) \in$ SIJ, we can consider different objective functions on the set of all sequence graphs and hence on the set of corresponding latin rectangles. For example, if $C = [c_{ij}]$ be the matrix of completion times ; $C_{\max} = \max \{c_{ij} \mid i \in I, j \in J\}$ is given by the weight of critical path in $G = (V, E)$ in the case of $p_{ij} = 1$. Then the problem here is to determine a latin rectangle with minimal cardinality of insertion set.. If we consider an objective function $\Sigma_{i \in I} C_i$, then the problem with $p_{ij} = 1$ is to determine a latin rectangle with minimal sum of the greatest elements of each row. In particular, if $p_{ij} = 1$for all i and for all j, then we have $C = LR\,[n, m, r]$ and the problem is relatively simple.

3. Minimizing the Completion and Total Completion Times.

In this section, the completion time and the total completion times on the machines are optimized in polynomial time algorithms if we have $p_{ij} = 1$ for all i and for all j. We consider the problems P_1 and P_2 with objective functions g_1 and g_2 of maximum completion time on machines and the sum of maximum completion times on each machines, respectively to be minimized. Here it is presented a polynomial time algorithm (see [5]) for the solutions of these open shop problems P_1 and P_2.

Let $LB(g_1)$ and $LB(g_2)$ denote the lower bounds for objective functions g_1 and g_2 respectively.

Lemma 3.1 Consider the open shop problems P_1 and P_2. Then, the lower bounds are

(a) $LB\,(g_1) = \max\,\{m, n\}$

$$LB(g_2) = \begin{cases} nm & \text{if} \quad m \leq n \\ \frac{k(k+1)}{2}\,n^2 + (k+1)\,nl & \text{if } m = kn + l. \end{cases}$$

Proof: Let $m \leq n$ be the first case. Since each machine and each job are available at zero time level and since each machine can complete the last job of its job order earliest at time n, we have $\overline{C}_j \geq n$ and $LB(g_1) = n$. Also, since there are exactly m machines all on processing we have $LB(g_2) = nm$.

As a second case, let the number of machines m be greater than the number of jobs n so that $m = kn + 1$, $(0 < k,\ 0 \leq 1 \leq n)$. Since each machine and job are available at zero time level and since each job can be completed earliest at time m we have, $LB(g_1) = m$, and at most n machines with $\overline{C}_j \geq m$. After removing n machines

there will be at most n machines with $\overline{C}_j \geq m-n$. After k steps the lower bound is $nm + n(m-n) + \ldots + n(m-(k-1)n)$. For the remaining 1 machines the trivial lower bound is $\overline{C}_i \geq n$. Since, there are exactly n jobs we have the lower bound is

$$LB(g_2) = \sum_{s=1}^{k}(nm-(s-1)n^2) + nl = \tfrac{k(k+1)}{2}n^2 + (k+1)nl.$$

Braesel /Kleinau presented in [2] a polynomial algorithm for solving the problems $[O \mid p_{ij} = 1 \mid C_{\max}$ and $O \mid p_{ij} = 1 \mid \Sigma_{i\in I} C_i]$. By changing the roles of machines and jobs we obtain the following algorithm for the open shop problem $[O \mid p_{ij} = 1 \| \overline{C}_{\max}$, $|\Sigma_{i\in J} C_j]$ with time complexity O(nm). Here, $\overline{C}_{\max}$ also denotes the objective function g_1

Algorithm 3.1 Solution of the problem $O \mid p_{ij} = 1 \mid \overline{C}_{\max},\ |\Sigma_{j\in J}\overline{C}_j$.

Input: n, m, and $p_{ij} = 1$ for all i, j
Output: Matrix of completion times $C = [c_{ij}]$

SO: If $m \leq n$, then $C := LR[n, m, n]$
go to $S5$;
SI: Determine k and l with $m = kn + 1$, $(0 < k, 0 \leq 1 < n)$,
choose $K^* \in \{0, 1, 2, \ldots, k-1\}$ arbitrarily ;
S2: Insert in C, k^* latin squares with the insertion sets
$S_q = \{(q-1)n+1, \ldots, qn\}, q = 1, \ldots, k^*$,
S3: Insert in C one latin rectangle with n rows, $n+1$ columns, and the insertion set
$S = \{k^* n + 1, \ldots\ldots, (k^*+1)n + 1\}$
and the following two properties:
- in 1 columns the greatest integer is $(k^*+1)n$
- in n columns the greatest integer is $(k^*+1)n + 1$;

S4: Insert in C, $k - k^*$ -1 latin squares with the insertion sets:
$Sq = \{qn+1+1, \ldots\ldots, (q+1)n+1\}, q = k^*+1, \ldots, k-1$,
S5 End

Theorem 3.1 *The polynomial algorithm presented above exactly solves the problem* $O \mid p_{ij} = 1 \mid g_1, g_2$.

Proof: In order to show that the lower bounds LB (g_1) and LB(g_2) calculate above are tight, we look the insertion sets in the presented algorithm 3.1.

Since the greatest integer in all insertion sets is given by max $\{n, m\}$, we have $\overline{C}_j = n$ for all j if $m \le n$ and $\overline{C}_j = m$ otherwise. Therefore we have LB(g_1) = max $\{n, m\} = \overline{C}_{\max}$.

Obviously, the lower bound satisfies LB(g_2) = $nm = \sum_{j\in J} \overline{C}_j$, if $m \le n$ since there are exactly m jobs on processing.

Finally, if $m = kn + l$, observing each insertion sets in the algorithm we get

$$\sum_{j\in J} \overline{C}_j = \sum_{j=1}^{k^*} njn + l(k^*+1)n + n[(k^*+1)n + l] + \sum_{j=k^*+1}^{k-1} n\,[(j+1)n + l]$$

$$= \sum_{j=1}^{k^*} jn^2 + l(k^*+1)n + n[(k^*+1)n + l] + \sum_{j=k^*+2}^{k} n\,[jn + l]$$

$$= n^2 \sum_{j=1}^{k^*} j + (k+1)nl + n^2[(k^*+1) + n^2 \sum_{j=k^*+2}^{k} j$$

$$= \tfrac{k(k+1)}{2} n^2 + (k+1)nl = LB(g_2).$$

Thus obtaining the lower bounds as our objective function values the theorem is proved.

It is remarkable that all optimal schedules of the problem $O\,|\,p_{ij} = 1\,|\,g_1, g_2$ can be constructed by the algorithm 3.1.

Example 3.1. Let there be $n = 3$ information sources and $m = 11$ receivers, where all receivers are required to receive all the informaions for unit time. It is assumed that no source will transmit the information to more than one receiver at the same time and no receiver can receive more than one information at a time. Then we have the objective value $g_1 = 11$ and $g_2 = 78$, and the following matrix C of completion times for the optimal schedules given by the above algorithm 3.1.

Here $k = 3$ and $l = 2$ so that we can choose $k^* \in \{0, 1, 2\}$.

$$C = \begin{pmatrix} 1 & 2 & 3 & : & 4 & 5 & 6 & : & 10 & 11 & 8 & : & 7 & 9 \\ 2 & 3 & 1 & : & 5 & 6 & 4 & : & 9 & 10 & 11 & : & 8 & 7 \\ 3 & 1 & 2 & : & 6 & 4 & 5 & : & 11 & 7 & 10 & : & 9 & 8 \end{pmatrix}, k^* = 2$$

$$C = \begin{pmatrix} 1 & 2 & 3 & : & 7 & 8 & 5 & : & 4 & 6 & : & 9 & 10 & 11 \\ 2 & 3 & 1 & : & 6 & 7 & 8 & : & 5 & 4 & : & 10 & 11 & 9 \\ 3 & 1 & 2 & : & 8 & 4 & 7 & : & 6 & 5 & : & 11 & 9 & 10 \end{pmatrix}, k^* = 1$$

$$C = \begin{pmatrix} 3 & 4 & 5 & : & 1 & 2 & : & 6 & 7 & 8 & : & 9 & 10 & 11 \\ 4 & 5 & 1 & : & 2 & 3 & : & 7 & 8 & 6 & : & 10 & 11 & 9 \\ 5 & 2 & 4 & : & 3 & 1 & : & 8 & 6 & 7 & : & 11 & 9 & 10 \end{pmatrix}, k^* = 0$$

4. Minimizing the Idle-Time and Total Idle-Times

Let $\overline{S}_j$ be the idle-time of machine j. In this section we consider the problems P_3 and P_4 (see [5]) to minimize the idle -time $\overline{S}_{\max}$ of machines and total idle-times $\Sigma_{j\in J}\ \overline{S}_j$ of machines. respectively. The considered problems are solved in polynomials time algorithms in the case of processing times $p_{ij} = 1$ for all i and for all j.

Let LB (f_1) and LB (f_2) respectively denote the lower bounds for these problems.

Lemma 4.1 Consider the open shop problems P_3 and P_4 with SIJ = $I \times J$, then the lower bounds are LB (f_1) = $n-1$ and LB (f_2) = $m\,(n-1)$.

Proof: Suppose that $m \leq n$. Since each job can complete the last job of its job order earliest at time n units and since each machine and job can start its processing at zero time level, we have

LB (f_1) = $n-1$ and LB (f_2) = $m\,(n-1)$.

Suppose that the number of jobs n is less than the number of machines m so that $m = kn + 1$, where $0 < k$ and $0 \leq 1 < n$. Since each machine and jobs are available at zero time level and the last operation completes of the job i earliest at time n units, the trivial lower bound LB (f_1) is $n-1$.

Because in each m machines exactly n time unit operations must be processed $\overline{C}_j - \underline{C}_j + 1 \geq n$, i.e., $\overline{C}_j - \underline{C}_j \geq n-1$ has to be satisfied. Therefore we have, LB (f_2) $\geq m\,(n-1)$.

Now we define the following objective functions

$$\overline{S}_{\max} := f_1 - (n-1) \text{ and } \Sigma_{j\in J}\ \overline{S}_j := f_2 - m\,(n-1).$$

The lower bounds for each of these objective functions is zero.

Now we present the following polynomial algorithm with slight modifications of algorithm 3.1. The time complexity of this algorithm is again O(nm).

Algorithm 4.1 Solution of the problem $O \mid p_{ij} = 1 \mid \overline{S}_{\max},\ \mid \Sigma_{j\in J}\ \overline{S}_j$.

Input: n, m, and $p_{ij} = 1$ for all i, j
Output: Matrix of completion times $C = [c_{ij}]$

SO: If $m \leq n$, then $C := LR\,[n, m, n]$,
go to $S5$;
SI: Determine k and l with $m = kn + l$, $(0, < k, 0 \leq l < n)$
choose $k^* \in \{0, 1, 2, \ldots, k\}$ arbitrarily ;
S2: Insert in C, k^* latin squares with the insertion sets :
$M_q = \{(q-1)n + 1, \ldots, qn\}, q = 1, \ldots, k^*$,
S3: Insert in C one latin rectangle $LR^*\,[n, l, n]$ with the insertion set
$M = \{k^* n + 1, \ldots, (k^*+1)n\}$;
S4: Insert n C, $k-k^*$ latin squares with insertion sets:
$Mq = \{qn + 1, \ldots \ldots, (q+1)n\}, q = k^* + 1, \ldots, k$;
S5 End

Theorem 4.1 *The algorithm presented above exactly solves the open shop problem* $O\,|\,p_{ij} = 1\,|\,\overline{S}_{\max}, |\Sigma_{j\in J}\,\overline{S}_j$.

Proof: In order to show that the lower bounds calculated above are tight we look for the lower bounds in each insertion sets of given cardinality n.

For $m \leq n$, since the latin rectangle LR[n, m, n] in algorithm contains exactly once n in each m columns (it is matrix of permutations) we have

$$\overline{S}_{\max} = 0 = \text{LB}\,(f_1) - (n-1)\,,\ \Sigma_{j\in J}\ \overline{S}_j = 0 = \text{LB}\,(f_2) - m\,(n-1).$$

Also, in order to show that the lower bounds are tight in the case of $m = kn + l$ with $0 < k$, $0 \leq l < n$, we observe the insertion sets in the algorithm 4.1 again. Since we have k latin squares of size $n \times n$ and one latin rectangle of size $n \times l$ in n elements, obviously $\overline{S}_{\max} = 0 = \text{LB}\,(f_1) - (n-1)$.

Finally we have,

$$\sum_{j\in J} \overline{S}_j = LB\,(f_2) - m(n-1)$$
$$= \sum_{j=1}^{k^*} (jn - jn + n - 1) + l\,[k^* n + n - k^* n - 1] + n \sum_{j=k^*+1}^{k} [jn + n - jn - 1] - m(n-1) = 0.$$

Example 4.1 Let be there 10 books of scheduling theory and 4 students willing to read these books in a library. Assume that each student can read at most one book at a time and on each book at a time only at most one student can read. All students are required to read all the books for unit time and no-idle time on books is accepted. Here, the students can be taken as jobs and the books can be taken as machines.

Required is an optimal schedule with its optimal value for the considered problem P_4. We have optimal objective value $f_1 = 3$ and $f_2 = 30$, and examples of optimal schedules are given by

$$C = \begin{pmatrix} 1 & 2 & : & 5 & 6 & 7 & 8 & : & 9 & 10 & 11 & 12 \\ 2 & 3 & : & 6 & 7 & 8 & 5 & : & 10 & 11 & 12 & 9 \\ 3 & 4 & : & 7 & 8 & 5 & 6 & : & 11 & 12 & 9 & 10 \\ 4 & 1 & : & 8 & 5 & 6 & 7 & : & 12 & 9 & 10 & 11 \end{pmatrix}, k^* = 0$$

$$C = \begin{pmatrix} 1 & 2 & 3 & 4 & : & 5 & 6 & : & 9 & 10 & 11 & 12 \\ 2 & 3 & 4 & 1 & : & 6 & 7 & : & 10 & 11 & 12 & 9 \\ 3 & 4 & 1 & 2 & : & 7 & 8 & : & 11 & 12 & 9 & 10 \\ 4 & 1 & 2 & 3 & : & 8 & 5 & : & 12 & 9 & 10 & 11 \end{pmatrix}, k^* = 1$$

$$C = \begin{pmatrix} 1 & 2 & 3 & 4 & : & 5 & 6 & 7 & 8 & : & 9 & 10 \\ 2 & 3 & 4 & 1 & : & 6 & 7 & 8 & 5 & : & 10 & 11 \\ 3 & 4 & 1 & 2 & : & 7 & 8 & 5 & 6 & : & 11 & 12 \\ 4 & 1 & 2 & 3 & : & 8 & 5 & 6 & 7 & : & 12 & 9 \end{pmatrix}, k^* = 2.$$

In order to show that the algorithm presented above constructs all possible solutions, we consider some further restrictions. Let $f_3 = \max \{C_i\}$ be the maximum completion time of job i with respect to no-idle time. Then if LB(f_3) denotes the corresponding lower bound,

$$LB(f_3) = \begin{cases} n & \text{if } m \le n \\ m + n - l & \text{if } m = kn + l \end{cases}$$

which is tight. Therefore we have the following optimal solution

$$C_{\max} = \begin{cases} n & \text{if } m \le n \\ m + n - l & \text{if } m = kn + l \end{cases}$$

for the open shop problem $O \mid p_{ij} = 1$, no-idle $\mid C_{\max}$.

Proof of the following theorem can be found in [5].

Theorem 4.2 *All optimal schedules of the problem* $O \mid p_{ij} = 1 \mid f_1, f_2, f_3$ *can be constructed by the algorithm* 4.1.

Remark 4.1 Optimal solutions of the problems $O \mid p_{ij} = 1 \mid \overline{S}_{\max}, O \mid p_{ij} = 1 \mid \sum_{j \in J} \overline{S}_i$, and $O \mid p_{ij} = 1$ no-wait $\mid \overline{C}_{\max}$ can be obtained immediately using the algorithms of this section by changing the roles of machines and jobs.

5. Concluding Remarks

It is known from literature that many scheduling problems are very hard with respect to their computations. In this paper the polynomial algorithms are presented for the considered open shop problems with unit processing times $P_{ij} = 1$ for all i and j. But the corresponding problems with partial operation sets $SIJ \subset 1 \times J$ have been studied and a heuristic insertion algorithms based on branch and bound method are presented in [5]. The concept of partial latin matrix is very important in this case. In such insertion algorithm the insertion of one new operation is equivalent to fill a new non-occupied cell in a partial latin matrix. Such insertion algorithms are verified with small number of n an m in [5], but their computer implementation for large number of machines and jobs could be a topic for further research.

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Some Constructions of Group Divisible and Rectangular Designs

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Abstract: In this research paper, an attempt has been made to construct the group divisible and rectangular designs through balanced incomplete block design and otherwise. Tables for $r, k \leq 15$ have also been prepared.

1. Introduction

Constructions on group divisible designs, which is the simplest of 2–class association scheme, started with two research papers; one by John and Turner [4], who gave solution of 15 designs not listed by Clatworthy [2] and another by Freeman [3] who constructed them by cyclically generating an initial block. Later work on such constructions are due to Bhagwandas & Parihar [1], Kageyama and Tanaka [6] etc. The constructions on rectangular designs is mainly due to Vartak [7], Kageyama [5] etc. In this paper we have given the patterned construction of semi-regular group divisible (SRGD), regular group divisible (RGD) and rectangular designs (RD) through balanced incomplete block designs (BIBD) and otherwise in the section-3. Some of these designs are believed to be new non-isomorphic solutions of the existing designs.

2. Definitions and Notations

Definition 2.1 An incomplete block design with v^* treatments, arranged into b^* blocks, equi-replicates r^* and equ-blick sized k^* ($k^* < v^*$), no treatment occurs more than once in a block and each pair of treatments occurs together in λ^* blocks is called balanced incomplete block (BIB) design.

The symbols v^*, b^*, r^*, k^* and λ^* are called the parameters of the design. They satisfies the following relations:

i) $r^* = b^*\, k^*$

ii) $\lambda^* (v^*-1) = r^* (k^*-1)$

iii) $b^* \geq v^*$.

A BIB design with $v^*= b^*$ and hence $r^*= k^*$ is called symmetric BIB design, or SBIB design and is denoted by (v^*, k^*, λ^*).

Definition 2.2. A group divisible (GD) scheme is an arrangement of $v = mn$ treatments divided into m groups of n treatments each, such that any two treatments in the same group are first associates and any two treatments from different groups are second associates.

We denote a group divisible design of two associate classes by GD $(v, b, r, k ; \lambda_1, \lambda_2 ; m, n)$. the parameters of GD scheme are $v = mn$, $n_1 = n-1$, $p_{11}^1 = n-2$, $p_{11}^2 = 0$.

The group divisible designs are classified as :

i) A GD is singular or SGD if $r = \lambda_1$

ii) Semi-regular or SRGD if $rk = \lambda_2 v$

iii) Regular or RGD if $r > \lambda_1$ and $rk > \lambda_2 v$.

Definition 2.3. *The rectangular scheme of Vartak* [7] *is defined as follows* :

Let there be $v = mn$ treatments arranged in a rectangular form in m rows and n columns. For any treatment θ, $n-1$ other treatments in the same row as θ are its first associates, $m-1$ other treatments in the same column as θ are its second associates and $(n-1)(m-1)$ remaining treatments are its third associates i.e. $n_1 = n-1$, $n_2 = m-1$, $n_2 = (n-1)(m-1)$ with

$$P_1 = \begin{bmatrix} n-2 & 0 & 0 \\ 0 & 0 & m-1 \\ 0 & m-1 & (m-1)(n-2) \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 0 & 0 & n-1 \\ 0 & m-2 & 0 \\ n-1 & 0 & (m-2)(n-1) \end{bmatrix}$$

and

$$P_3 = \begin{bmatrix} 0 & 1 & n-2 \\ 1 & 0 & m-2 \\ n-2 & m-2 & (m-2)(n-2) \end{bmatrix}$$

3. Main Results

Construction of Group Divisible Designs:

Theorem 3.1 *If N is the incidence matrix of order $v^* \times b^*$ and $\bar{N}$ is its compliment of a BIBD with the parameters $V^* = 2k^*; r^*, k^*, \lambda^* = (k^*-1)$, then the incidence pattern*

$$S=\begin{bmatrix} \overline{N} & \overline{N} & N & \overline{N} & \overline{N} & N \\ \overline{N} & \overline{N} & \overline{N} & \overline{N} & N & \overline{N} \end{bmatrix}$$

is the incidence matrix of order $v \times b$ of a resolvable semi-regular group divisible design (SRGD) with parameters.

$$V = 4k^*, b = 12r^*, r = 6\,(2k^*-1), k = 2k^*\,;\ \lambda_1 = 6\,(k^*-1)$$

$$\lambda_2 = 3(2k^*-1);\ m = 2,\ n = v^*.$$

Proof: Let S be the incidence matrix of order $v \times b$ having row (column) sum $r(k)$ of a *GD* design with parameters given above and S' be the transpose of S, then

$$SS' = \begin{bmatrix} 6NN & 3rj \\ 3rj & 6NN' \end{bmatrix},$$

where $NN' = (r^* - \lambda^*)\, I_v + \lambda^*\, J_v$, we get $\lambda_1 = 6\lambda^*$ and $\lambda_2 = 3r^*$. The other parameters viz. v, b, r, k are obvious. Since $rk - \lambda_2\, v = 0$, the resulted design is a SRGD. Hence the theorem.

Remark 3.1 In the design constructed in Theorem 3.1, the relation $b \geq v - m + r$ holds, therefore resulting design is a resolvable SRGD design.

In particular, if we consider BIBD (2,1,0) then the resulting resolvable SRGD design has the parameters $v = 4$, $b = 12$, $r = 6$, $k = 2$; $\lambda_1 = 0$, $\lambda_2 = 3$; $m = 2$, $n = 2$. Our solution is a new non-isomorphic solution to the design SR 3 of Clatworthy [2].

Theorem 3.2 *If N is a* BIB $(v^*, b^*, r^*, k^*, \lambda^*)$ *design with $v^* = 2k^*$, then*

$$S=\begin{bmatrix} \overline{N} & N \\ \overline{N} & \overline{N} \end{bmatrix}$$

in the incidence matrix of a SRGD *design* $(2v^*, 2b^*, 2r^*, 2k^*; 2\lambda^*, r^*; 2, v^*)$.

As an illustration of this theorem, starting from

i) BIBD (2, 2, 1, 1, 0) yields SRGD (4, 4, 2, 2 ; 0, 1; 2, 2)

ii) BIBD (4, 6, 3, 2, 1) yields SRGD (8, 12, 6, 4 ; 2, 3 ; 2, 4)

iii) BIBD (6, 10, 5, 3, 2) yields SRGD (12, 20, 10, 6 ; 4, 5 ; 2, 6).

Remark 3.2 The designs are the new non-isomorphic solution of SR 1, SR 38 and SR 71 of Clatworthy [2].

Theorem 3.3 *If N is the incidence matrix of a* BIB *design* $(v^*, b^*, r^*, k^*, \lambda^*)$ *with $v^*=2k^*$, then*

$$S=\begin{bmatrix} N & \overline{N} & N & \overline{N} & N & \overline{N} & N & \overline{N} \\ N & N & \overline{N} & \overline{N} & N & N & \overline{N} & \overline{N} \\ N & \overline{N} & \overline{N} & N & N & \overline{N} & \overline{N} & N \\ N & N & N & N & \overline{N} & \overline{N} & \overline{N} & \overline{N} \\ N & \overline{N} & N & \overline{N} & \overline{N} & N & \overline{N} & N \end{bmatrix}$$

is the incidence matrix of a SRGD design ($10k^*$, $16r^*$, $8r^*$, $5k^*$, $8\lambda^*$, $4r^*$; 5, v^*).

As an illustration of this theorem starting from a BIBD (2,2,1,1,0), we get a SRGD (10, 16, 8, 5 ; 0, 4 ; 5, 2) a new isomorphic solution of design SR 54 of Clatworthy [2].

Remark 3.3 Compliment of the above design also yields a SRGD design.

If N be the incidence matrix of BIB design with parameters $v^*=2k^*$, $b^*=2r^*$, $r^*=t\,(2k^*-1)$, k^* and $\lambda^*=t\,(k^*-1)$ for a positive integer t, then incidence structure

$$S^*=\begin{bmatrix} \overline{N} & N & \overline{N} & N & \overline{N} & N & \overline{N} & N \\ \overline{N} & \overline{N} & N & N & \overline{N} & \overline{N} & N & N \\ \overline{N} & N & N & \overline{N} & \overline{N} & N & N & \overline{N} \\ \overline{N} & \overline{N} & \overline{N} & \overline{N} & N & N & N & N \\ \overline{N} & N & \overline{N} & N & N & \overline{N} & N & \overline{N} \end{bmatrix}$$

yields SRGD design with parameters :

$$v=10\,k^*,\ b=16\,t\,(2k^*-1),\ r=8\,t\,(2\,k^*-1),\ k=5k^*;$$
$$\lambda_1=8t\,(k^*-1)\ ;\ \lambda_2=4t\,(2\,k^*-1)\ ;\ m=5,\ n=2k^*.$$

Now, if N is the incidence matrix of resolvable BIBD, then S^* is also the incidence matrix of a resolvable SRGD design.

Theorem 3.4 *If N denote* (0, 1) *incidence matrix of order* $v^*\times b^*$ *of a* BIBD (v^*, b^*, r^*, k^*, λ^*) *and J is a* $v^*\times b^*$ *matrix of unities, then the matrix*

$$S=\begin{bmatrix} J & J & - & - & - & J & J & N \\ J & J & - & - & - & J & N & J \\ | & | & | & & & | & | & | \\ J & N & - & - & - & J & J & J \\ N & J & - & - & - & J & J & J \end{bmatrix}_{t\times t},$$

where $t \geq 2$, a positive integer, is an incidence matrix of a regular group divisible (RGD) design with the parameters:

$$v = tv^*, b = tb^*, r = r^* + (t-1)b^*, k = k^* + (t-1)v^* ;$$
$$\lambda_1 = \lambda^* + (t-1)\, b^*, \;\; \lambda_2 = 2r^* + (t-2)b^* ; m = t, n = v^*.$$

An illustration, for $t = 2$, if we consider a BIBD (4,6,3,2,1), we get a regular GD design (8, 12, 9, 6 ; 7 , 6) which is a new-isomorphic solution for R 164 given in Clatworthy [2].

Remark 3.4 Replace N by $\overline{N}$ in Theorem 3.4, we again get a incidence matrix of RGD design (tv^*, tb^*, $tb^*- r^*$, tv^*-k^* ; $tb^*-2r^* + \lambda^*$, $tb^* -2r^*$; $m = t$, $n = v^*$.

Remark 3.5 If N is an incidence matrix of a symmetric BIBD (v^*, k^*, λ^*), then S will be an incidence matrix of a regular GD design (tv^*, $k^*+ (t-1)v^*$; $\lambda^* + (t-1)\, v^*$, $2k^* + (t-2)v^*$; $m = t$, $n = v^*$.

We have the following corollaries :

Corollary 3.1 Existence of a symmetric BIB design (n, 1, 0), $n > 1$ implies the existence of a symmetric RGD design $\{nt, n\,(n-1) +1, n\,(t-1), n\,(t-2) +2\}$.

Corollary 3.2 Existence of a symmetric BIB design ($4p$+3, $2p$+1, p} implies the existence of a symmetric RGD design $\{t(4p+3), (4pt-2p+3t-2) \;;\; (4pt+3t-3p-3), (4pt - 4p + 3t - 4)\}$.

Corollary 3.3 Existence of a symmetric BIB design ($4p^2 -1$, $2p^2$, p^2) implies the existence of a symmetric RGD design $\{t\,(4p^2 -1), (4p^2\, t-2p^2 - t +1) \;;\; (4p^2\, t-3p^2 -t +1), (4p^2\, t - 4p^2 + t +2)\}$.

Corollary 3.4 Existence of a symmetric BIB design ($8m + 7$, $4m + 3$, $2m + 1$) implies the existence of a symmetric RGD design $\{t\,(8m + 7), (8mt - 4m + 7t - 4), (8mt - 6m + 7t -6), (8mt - 8m + 7t + 8)\}$,

As an illustration, if we take symmetric BIB design (4, 3, 2) we get a symmetric RGD design (8, 8, 5, 5 ; 4, 2), a new non-isomorphic solution to R 133 of Clatworthy [2].

Theorem 3.5 *If N is the incidence matrix of a* BIBD (v^*, b^*, r^*, k^*, λ^*) *and 0 is the* $v^* \times b^*$ *matrix of zeroes, then matrix*

$$S = \begin{bmatrix} N & 0 & - & - & - & - & 0 & 0 & 0 \\ 0 & N & - & - & - & - & 0 & 0 & 0 \\ | & | & | & & & & | & | & | \\ 0 & 0 & - & - & - & - & 0 & 0 & N \end{bmatrix}_{t \times t}$$

is an incidence matrix of a regular group divisible (RGD) *design with the parameters:*

$$v = tv^*,\ \ b = tb^*,\ \ r = r^*,\ \ k = k^*\ ;\ \ \lambda_1 = \lambda^*,\ \ \lambda_2 = 0\ ;\ \ m = t,\ \ n = v^*.$$

Since, for this design $rk > v\lambda_2$ and $r > \lambda_1$, hence the resulting design is regular GD design.

As an illustration, for $t = 2$ from a BIBD (3, 2, 1), we get S, the incidence matrix of a symmetric RGD design with parameters $v = b = 6$, $r = k = 2$; $\lambda_1 = 1$, $\lambda_2 = 0$; 2, 3. The blocks of the design are (1, 2), (1, 3), (2, 3), (4, 5), (4, 6) and (5, 6).

Corollary 3.5 Symmetric BIBD $(p, p-1, p-2)$ yields a symmetric RGD $v = b = pt$, $r = k = p-1\ ;\ \lambda_1 = p-2,\ \lambda_2 = 0\ ;\ \ m = t,\ \ n = p.$

Construction of Rectangular Designs :

Theorem 3.6 *Let us arrange n^2 treatments in a nxn square. Write all possible blocks of rows and also of columns, taking $\varnothing$, $2 \le \varnothing \le n-1$ of them at a time, then we get a PBIB design based on rectangular association scheme with parameters*:

$$v = n^2,\ b = 2\binom{n}{\varnothing},\ \ r = 2\binom{n-1}{\varnothing-1},\ k = \varnothing\, n,$$

$$\lambda_1 = \lambda_2 = \binom{n-2}{\varnothing-2}\left(\frac{n+\varnothing-2}{\varnothing-1}\right),\ \lambda_3 = 2\binom{n-2}{\varnothing-2}.$$

As an illustration, for $\varnothing = 4$ and $n = 6$, we get the parameters of the design $v = 36,\ b = 30\ \ r = 20,\ \ \ k = 24,\ \ \ \lambda_1 = \lambda_2 = 16,\ \ \ \lambda_3 = 12.$

The blocks of the design are $(R_1\ R_2\ R_3\ R_4)$, $(R_1\ R_2\ R_3\ R_5)$, $(R_1\ R_2\ R_3\ R_6)$, $(R_1\ R_2\ R_4\ R_5)$, $(R_1\ R_2\ R_4\ R_6)$, $(R_1\ R_2\ R_5\ R_6)$, $(R_1\ R_3\ R_4\ R_5)$, $(R_1\ R_3\ R_5\ R_6)$, $(R_1\ R_3\ R_4\ R_6)$, $(R_1\ R_4\ R_5\ R_6)$, $(R_2\ R_3\ R_4\ R_5)$ and $(R_3\ R_4\ R_5\ R_6)$. Similarly, for columns replacing R by C with same suffix.

Corollary 3.6 If $\varnothing = 2$, then parameters of rectangular design are $v = n^2$, $b = n\,(n-1)$, $r = 2\,(n-1)$, $k = 2n$; $\lambda_1 = \lambda_2 = n$, $\lambda_3 = 2$.

Corollary 3.7 If $\varnothing = 3$, then parameters of rectangular design are $v = n^2$, $b = 2\binom{n}{3}$,

$r = (n-1)\,(n-2),\ \ k = 3n\ ;\ \lambda_1 = \lambda_2 = \{(n+1)\,(n-2)/2\},\ \lambda_3 = 2\,(n-2).$

Remark 3.6. If $n = \varnothing$, then rectangular design reduces to complete block design.

Theorem 3.7 If N is the incidence matrix of order $v^* \times b^*$ of a BIBD (v^*, b^*, r^*, k^*, λ^*), then

$$S = \begin{bmatrix} \overline{N} & \overline{N} & \overline{N} & \overline{N} & N \\ \overline{N} & \overline{N} & \overline{N} & N & \overline{N} \\ \overline{N} & \overline{N} & N & \overline{N} & \overline{N} \\ \overline{N} & N & \overline{N} & \overline{N} & \overline{N} \\ N & \overline{N} & \overline{N} & \overline{N} & \overline{N} \end{bmatrix}$$

is the incidence matrix of a three associate class rectangular PBIB design with parameters :

$$v = 5v^*, b = 5b^*, r = 4b^* - 3r^*, k = 4v^* - 3k^* ; \lambda_1 = 4b^* - 8r^* + 5\lambda^*,$$
$$\lambda_2 = 3(b^* - r^*), \lambda_3 = 3b^* - 4r^* + \lambda^*.$$

As an illustration, we have

(i) BIBD (4, 6, 3, 2, 1) yields 3–PBIB design (20, 30, 15, 6 ; 5, 9, 7)

(ii) BIBD (4, 4, 3, 3, 2) yields 3–PBIB design (20, 20, 7, 7 ; 2, 3, 2)

Remark 3.7 Replace N by $O_{v^* \times b^*}$ (null matrix) then, we get a three associate PBIB design with parameters:

$$v = 5v^*, \quad b = 5b^*, \quad r = 4(b^* - r^*), \quad k = 4(v^* - k^*) ; \quad \lambda_1 = 4(b^* - 2r^* + \lambda^*),$$
$$\lambda_2 = 3(b^* - r^*), \quad \lambda_3 = 3(b^* - 2r^* + \lambda^*).$$

Remark 3.8 If $\overline{N}$ is replaced by N and N by $O_{v^* \times b^*}$ (null matrix), then again we get a three associate rectangular PBIB design with parameters :

$$v = 5v^*, \quad b = 5b^*, \quad r = 4r^*, \quad k = 4k^* ; \quad \lambda_1 = 4\lambda^*, \quad \lambda_2 = 3r^*, \quad \lambda_3 = 3\lambda^*. .$$

Remark 3.9 If in Remark 3.7 $r^* = \lambda^*$, we get a RGD design with parameters :

$$v = 5v^*, \quad b = 5\lambda^*, \quad r = 4\lambda^*, \quad k = 4k^* \; ; \quad \lambda_1 = 4\lambda^*, \quad \lambda_2 = 3\lambda^* ;$$
$$m = 5, \quad n = k^*.$$

Theorem 3.8 *If N is the incidence matrix of a* BIBD (v^*, b^*, r^*, k^*, λ^*), *and* O *is the $v^* \times b^*$ matrix of zeroes, then*

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & N & N & N & N \\ 0 & 0 & 0 & N & N & N & 0 & 0 & 0 & N \\ 0 & N & N & 0 & 0 & N & 0 & 0 & N & 0 \\ N & 0 & N & 0 & N & 0 & 0 & N & 0 & 0 \\ N & N & 0 & N & 0 & 0 & N & 0 & 0 & 0 \end{bmatrix}$$

is the incidence matrix of a three associate PBIB design with parameters :

$$v = 5v^*, \quad b = 10b^*, \quad r = 4r^*, \quad k = 2k^* \ ; \ \lambda_1 = 4\lambda^*, \quad \lambda_2 = r^*, \quad \lambda_3 = \lambda^*.$$

As an illustration, BIBD (4, 6, 3, 2, 1) yields 3-PBIB design (20, 60, 12, 4 ; 4, 3, 1).

Remark 3.10 If we put $r^* = \lambda^*$ ($r^* = k^*$, $b^* = r^*$, i.e. CBD) then above parameters will represent a singular GD design

$$v = 5k^*, \quad b = 10r^*, \quad r = 4r^*, \quad k = 2k^* \ ; \ \lambda_1 = 4r^*, \quad \lambda_2 = r^* \ ;$$

$$m = 5, \quad n = k^*.$$

Theorem 3.9 : *If N is the incidence matrix of a* BIBD ($v^*, b^*, r^*, k^*, \lambda^*$), *then*

$$S = \begin{bmatrix} N & N & N & 0 & 0 & 0 \\ N & 0 & 0 & 0 & N & N \\ 0 & N & 0 & N & 0 & N \\ 0 & 0 & N & N & N & 0 \end{bmatrix}$$

is the incidence matrix of a three associate PBIB *design with parameters* :

$$v = 4v^*, \quad b = 6b^*, \quad r = 3r^*, \quad k = 2k^* \ ; \ \lambda_1 = 3\lambda^*, \quad \lambda_2 = r^*, \quad \lambda_3 = \lambda^*.$$

As an illustration, we have

(i) BIBD (4,6,3,2,1) yields 3-PBIB design (16, 36, 9, 4 ; 3, 3, 1)

(ii) BIBD (7,7,3,3,1) yields 3-PBIB design (28, 42, 9, 6 ; 3, 3, 1).

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Table 1 : Group Divisible Designs for $r, k \leq 15$

Sl	*v*	*r*	*k*	*b*	*m*	*n*	λ_1	n_2	*E*	Remark	
										Based on	Listed as*
1.	4	6	2	12	2	2	0	3	0.60	Th. 3.1	SR–3
2.	4	2	2	4	2	2	0	1	0.60	Th. 3.2	SR–1
3.	8	6	4	12	2	4	2	3	0.85	Th. 3.2	SR–38
4.	12	10	6	20	2	6	4	5	0.91	Th. 3.2	SR–71
5.	16	14	8	28	2	8	6	7	0.93	Th. 3.2	
6.	10	8	5	16	5	2	0	4	0.88	Th. 3.3	SR–54
7.	8	12	4	24	2	4	4	6	0.85	Th. 3.3	
8.	4	3	3	4	2	2	2	2	0.89	Th. 3.4	
9.	8	9	6	12	2	4	7	6	0.95	Th. 3.4	R–164
10.	12	15	9	20	2	6	12	10	0.97	Th. 3.4	
11.	8	7	7	8	2	4	6	6	0.98	Th. 3.4	
12.	10	14	7	20	2	5	11	8	0.95	Th. 3.4	
13.	10	9	9	10	2	5	8	8	0.99	Th. 3.4	
14.	12	11	11	12	2	6	10	10	0.99	Th. 3.4	
15.	14	10	10	14	2	7	8	6	0.97	Th. 3.4	R–204
16.	14	11	11	14	2	7	9	8	0.98	Th. 3.4	
17.	14	13	13	14	2	7	12	12	0.99	Th. 3.4	
18.	6	5	5	6	3	2	4	4	0.96	Th. 3.4	
19.	12	15	10	18	3	4	13	12	0.98	Th. 3.4	
20.	12	11	11	12	3	4	10	10	0.99	Th. 3.4	
21.	15	14	14	15	3	5	13	13	0.99	Th. 3.4	
22.	8	5	5	8	2	4	4	2	0.90	Th. 3.4	R–133
23.	10	6	6	10	2	5	5	2	0.90	Th. 3.4	R–166
24.	10	14	7	20	2	5	11	8	0.95	Th. 3.4	
25.	12	7	7	12	2	6	6	2	0.90	Th. 3.4	R–173
26.	14	8	8	14	2	7	7	2	0.90	Th. 3.4	R–187
27.	16	9	9	16	2	8	8	2	0.90	Th. 3.4	R–195
28.	20	11	11	20	2	10	10	2	0.90	Th. 3.4	

* In Clatworthy (1973)

Table 2 : Rectangular Designs for $r, k \leq 15$

Sl	v	r	k	b	λ_1	λ_2	λ_3	n_1	n_2	n_3	Based	on
1.	10	5	5	10	0	3	2	1	4	4	Th.	3.7
2.	20	15	10	30	5	9	7	3	4	12	Th.	3.7
3.	20	7	7	20	2	3	2	3	4	12	Th.	3.7
4.	25	8	8	25	3	3	2	4	4	16	Th.	3.7
5	30	9	9	30	4	3	2	5	4	20	Th.	3.7
6.	35	10	10	35	5	3	2	6	4	24	Th.	3.7
7.	40	11	11	40	6	3	2	7	4	28	Th.	3.7
8.	50	13	13	50	8	3	2	9	4	36	Th.	3.7
9.	55	14	14	55	9	3	2	10	4	40	Th.	3.7
10.	10	4	4	10	0	3	0	1	4	4	Th.	3.7
11.	20	12	8	30	4	9	3	3	4	12	Th.	3.7
12.	20	4	4	20	0	3	0	3	4	12	Th.	3.7
13.	25	4	4	25	0	3	0	4	4	12	Th.	3.7
14.	30	4	4	30	0	3	0	5	4	20	Th.	3.7
15.	35	12	12	35	4	9	3	6	4	24	Th.	3.7
16.	35	4	4	35	0	3	0	6	4	24	Th.	3.7
17.	40	4	4	40	0	3	0	7	4	28	Th.	3.7
18.	50	4	4	50	0	3	0	9	4	36	Th.	3.7
19.	20	12	12	20	8	9	6	3	4	12	Th.	3.7
20.	10	4	2	20	0	1	0	1	4	4	Th.	3.8
21.	20	12	4	60	4	3	1	3	4	12	Th.	3.8
22.	20	12	6	40	8	3	2	3	4	12	Th.	3.8
23.	35	12	6	70	4	3	1	6	4	24	Th.	3.8
24.	8	3	2	12	0	1	0	1	3	3	Th.	3.9
25.	16	9	4	36	3	3	1	3	3	9	Th.	3.9
26.	24	15	6	60	6	5	2	5	3	15	Th.	3.9
27.	16	9	6	24	6	3	2	3	3	9	Th.	3.9
28.	20	12	4	60	3	4	1	4	3	12	Th.	3.9
29.	20	12	8	30	9	4	3	4	3	12	Th.	3.9
30.	24	15	4	90	3	5	1	5	3	15	Th.	3.9
31.	24	15	10	36	12	5	4	5	3	15	Th.	3.9
32.	28	9	6	42	3	3	1	6	3	18	Th.	3.9
33.	28	12	8.	42	6	4	2	6	3	18	Th.	3.9
34.	36	12	6	72	3	4	1	8	3	24	Th.	3.9
35.	44	15	10	66	6	5	2	10	3	30	Th.	3.9
36.	52	12	8	78	3	4	1	12	3	36	Th.	3.9
37.	84	15	10	126	3	5	1	20	3	60	Th.	3.9
38.	84	15	8	120	3	5	1	15	3	45	Th.	3.9

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On A Differentiable Structure Satisfying $f^{2v+4} + f^2 = 0$, $f \neq 0$ and of Type (1,1)

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1. Introduction:

In [3] A. Petrakis has studied the subordinated hermitian structures of an $f(2v + 3, 1)$–structure. In this paper we intend to study $f(2v + 4,1)$–structure in order to find the above conditions in the title such that ([2], [3]) K.Yano did for the structure $\phi^4 \pm \phi^2 = 0$ and [1] for the structure $f^5 \pm f^2 = 0$. Here $v \neq 0$ but even when $v = 0$ we get [3].

2. Let M_n be an n–dim. differentiable manifold of class C^∞_∞ and $f(\neq 0)$ a tensor field of type (1.1) and of rank r such that

$$f^{2v+4} + f^2 = 0 \tag{2.1}$$

Let (1,1) tensor ℓ and m

$$\ell \overset{def}{=} -f^{2v+2},\ m \overset{def}{=} f^{2v+2} + I \tag{2.2}$$

where I being the identity operator, have the properties

$$\ell^2 = \ell,\ m^2 = m,\ \ell m = m\ell = 0,\ \ell + m = I.$$

Thus, the operators ℓ, m are complementary orthogonal projections.
At each point $x \in M_n$

$$T_x = Im\ \ell x\ \otimes \text{Ker}\ \ell x,\ \ell n(x) = x\ \forall\ x \in Im\ \ell x$$

and

$$T_x = Im\ Mx\ \otimes \text{Ker}\ mx,\ Mn(x) = x\ \forall\ x \in Im\ mx.$$

Consequently, if there is a tensor field $f \neq 0$ satisfying (2.1), then there exist on M_n two complementary distributions L and M. Corresponding to ℓ and m respectively, with dim $L = r$ and dim $M = n - r$. Such structure, we call as $f(2v + 4, 1)$–structure.

of rank r. If $r = n$ then $\ell = 1$, $m = 0$ and f satisfies $f^{2\nu+2} = -1$. Thus, $f^{\nu+1}$ defines on M_n an almost complex structure and n must be even.

Theorem 2.1. *For f satisfying* (2.1) *and* ℓ , m *defined by* (2.2), *we have*

$$f\ell = \ell f = -f^{2\nu+3}, \quad fm = mf = f^{2\nu+3} + f, \quad f^{2\nu+2}\ell = -\ell, f^{2\nu+2}m = mf^{2\nu+2} = 0, \tag{2.3}$$

that is $f^{\nu+1}$ acts on L as an almost complex structure operator and on M as an almost tangent structure operator.

Proof (2.3) follows easily in view of (2.1) and (2.2).

Theorem 2.2 *For f satisfying (2.1) and m defined by (2.2), we have*

$$(m + f^{\nu+1})(m - f^{\nu+1}) = 1, \ f^{2m} = mf^2 = 0. \tag{2.4}$$

Proof: Proof follows easily in view of (2.1), (2.2) and (2.3).

Theorem 2.3. *If m is a projection operator and* f a tensor field on M_n *such that* $f^2 m = mf^2 = 0$ *and* $(m + f^{\nu+1})(m - f^{\nu+1}) = 1$, *then*

$$f^{2\nu+4} + f^2 = 0 \tag{2.5}$$

Proof: Now $(m + f^{\nu+1})(m - f^{\nu+1}) = 1$

$$\Rightarrow m^2 - f^{2\nu+1} = 1$$

$$\Rightarrow m - f^{2\nu+1} = 1 \text{ (because } m^2 = m)$$

$$\Rightarrow f^2 m - f^{2\nu+4} = f^2$$

$$\Rightarrow f^{2\nu+1} + f^2 = 0 \qquad \text{(because } f^2 m = 0)$$

which proves the theorem.

Theorem 2.4. *For tensor p and q defined by*

$$p = f^{\nu+2} + f/\sqrt{2}, \quad q = f^{\nu+2} - f/\sqrt{2} \tag{2.6}$$

we have,

$$pq = qp = -(p+q)\sqrt{2}, \ p^2 + q^2 = 0. \tag{2.7}$$

conversely, if there are given in the manifold two distinct tensor fields p and q both of type (1.1) satisfying (2.7), then we can find out a $f \neq 0$ such that $f^{2\nu+4} + f^2 = 0$ and p and q coincides with those of given by (2.6).

Proof If there are given in the manifold two distinct tensor fields p and q both of type (1.1) satisfying (2.7) then,

$$f = (p-q)/\sqrt{2}$$

satisfying $f^{2v+4} + f^2 = 0$. Furthermore, since $f^{v+2} = p + q/\sqrt{2}$, we have

$$p = f^{v+2} + f/\sqrt{2} \text{ and } q = f^{v+2} - f/\sqrt{2}$$

The proof of the rest part is trivial.

Theorem 2.5 *For tensors p and q defined by*

$$p = m + f^{v+1} \text{ and } q = m - f^{v+1}$$

we have the relations

(2.8) $$pq = qp = 1, p^2 - p + 1 = q, q^2 - q + 1 = p$$

so that $p^2 = q^2$

Proof $$p^2 - p + 1 = (m + f^{v+1})^2 - m - f^{v+1} + 1$$

$$= f^{2v+2} - f^{v+1} + 1 = q$$

$$q^2 - q + 1 = (m - f^{vH})^2 - m + f^{v+1} = p.$$

From this we have the fact [5], that on M_n, there exists a tensor field f of type (1,1) satisfying $f^4 + f^2 = 0$ [3].

Theorem 2.6. *If there are given on M_n, two distinct tensor fields s and t of type* (1,1) *defined as*

(2.9) $$s = m + f \text{ and } t = m - f,$$

where $$m = f^{2v+2} + 1 \text{ and } f = \tfrac{1}{2}(s-t).$$

Then we can find out a tensor field $f \neq 0$ such that $f^{2v+4} + f^2 = 0$.

Proof From (2.9) after computation, we get

$$s^{2v+2} - s + 1 = t, \quad t^{2v+2} - t + 1 = s$$

(2.10) $$st = ts, s^2 = t^2 \text{ and } st^{2v+1} = s^{2v+1}t = 1.$$

Now, $$f = \tfrac{1}{2}(s-t)$$

$$\Rightarrow f^2 = \tfrac{1}{2}(s^2 + t^2 - 2st) = \tfrac{1}{2}s(s-t) = sf$$

$$\Rightarrow f^3 = sf^2 = s^2 f$$

$$\Rightarrow f^4 = s^2 f^2, \ldots, f^{2v+4} = s^{2v+2} f^2$$

but $s^{2\nu+2} = t + s - 1$ (by 2.10)

$$\Rightarrow f^{2\nu+4} = (t - s - 1)\tfrac{1}{4} s (s - t)^2$$

$$\Rightarrow f^{2\nu+4} = \tfrac{1}{2}(t + s - 1) s (s - t)$$

$$= -\tfrac{1}{2} s (s - t)$$

$$= -f^2$$

$$\Rightarrow f^{2\nu+4} + f^2 = 0, \text{ hence proved.}$$

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Relation Between Padé Approximations and Continued Fractions of Generalised Laguerre Functions

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Abstract: In this paper we have established a relation between Padé tables and continued fractions of generalised Lagurerre functions $L_{\nu}^{(\mu)}$, we have also obtained Padé approximations from continued fraction of $L_{-\frac{1}{2}}^{(0)}(x)$

1. Introduction:

1.1. The so-called 'Generalised Laguerre functions' which arises in the theory of paraboloidal reflectors, is denoted and defined by

$$L_{\upsilon}^{(\mu)}(x)=\frac{\Gamma(1+\mu+\nu)}{\Gamma(1+\mu)\,\Gamma(1+\nu)}\,{}_1F_1\left[\begin{matrix}-\nu\\1+\mu\end{matrix}:x\right]$$

$$=\frac{\Gamma(1+\mu+\nu)}{\Gamma(1+\mu)\,\Gamma(1+\nu)}\left[1+\frac{-\nu}{1+\mu}\cdot\frac{x}{\lfloor 1}+\frac{-\nu(-\nu+1)}{(1+\mu)(2+\mu)}\cdot\frac{x^2}{\lfloor 1}+\ldots\right]$$

where ν is not necessarily an integer. $L_{\nu}^{(\mu)}(x)$ is a solution of the differential equation

$$y''+\left(\frac{1+\mu}{x}-1\right)y'+\frac{\nu}{x}\cdot y=0.$$

1.2. Padé Approximations

The L, M Padé Approximations to a power series $A(x)$ is denoted and defined by

$$\left[\frac{L}{M}\right]=\frac{P_L(x)}{Q_M(x)}, \tag{1}$$

where $P_L(x)$ is a polynomial of degree at most L and $Q_M(x)$ is a polynomial of degree at most M. The formal power series

(2) $$A(x)=\sum_{j=0}^{\infty} a_j\, x^j$$

determines the coefficient of $P_L(x)$ and $Q_M(x)$ by the equation

(3) $$A(x)-\frac{P_L(x)}{Q_M(x)}=0\,(x^{L+M+1}).$$

1.3. Padé Tables of $L_\nu^{(\mu)}(x)$

For computation of Padé tables of $L_\nu^{(\mu)}(x)$ we proceed in the following manner:

$$A(x)=L_\nu^{\mu}(x)=1+\frac{-\nu}{1+\mu}\cdot\frac{x}{\lfloor 1}+\frac{-\nu(-\nu+1)}{(1+\mu)\,(2+\mu)}\cdot\frac{x^2}{\lfloor 2}+$$

Now putting $L=M=1$ in 1.2(3), we get

$$A(x)\,Q_1(x)-P_1\,(x)=Q_1\,(x)\,0(x^3)$$

i.e. $$\left[1+\frac{-\nu}{1+\mu}\,.\,x+\frac{-\nu(-\nu+1)}{(1+\mu)\,(2+\mu)}\cdot\frac{x^2}{\lfloor 2}+\dots\right](1+q_1\,x)-(P_0+p_1\,x)$$

$$=(1+q_1\,x)\,0\,(x^3).$$

From this identity we get the following equations

$$P_0=1$$

$$q_1-\frac{\nu}{1+\mu}-p_1=0$$

$$q_1\frac{-\nu}{1+\mu}+\frac{-\nu(-\nu+1)}{2(1+\mu)(2+\mu)}=0.$$

Solving these equations we get

$$P_0=1$$

$$p_1=-\frac{\mu+3\nu+\mu\nu+1}{2(1+\mu)\,(2+\mu)}$$

$$q_1=\frac{\nu-1}{2\,(2+\mu)}.$$

Thus $$\left[\frac{1}{1}\right]=\frac{P_0+P_1x}{1+q_1x}$$

$$=\frac{2(1+\mu)\,(2+\mu)-(\mu+3\nu+\nu\mu+1)x}{2(1+\mu)\,(2+\mu)+(1+\mu)(\nu-1)x}.$$

Proceeding in this manner we compute the following Padé Table 1.

In particular we put $\nu = -\frac{1}{2}$ and $\mu = 0$ and compute the Padé table 2.

1.4. Continued Fractions from Padé Table

Following are the relations between Padé table and continued fraction [Baker (1)]

$$(1)\qquad \begin{aligned} a_{2n+1} &= \frac{p_n\left(\frac{n+1}{n}\right) q_0\left(\frac{n}{n-1}\right)}{q_0\left(\frac{n+1}{n}\right) p_n\left(\frac{n}{n-1}\right)} \\ a_{2n} &= \frac{q_n\left(\frac{n}{n}\right) q_0\left(\frac{n-1}{n-1}\right)}{q_0\left(\frac{n}{n}\right) q_{n-1}\left(\frac{n-1}{n-1}\right)} \end{aligned}$$

q_0 being constant term in the Padé denomination.

$p_n\left(\frac{n+1}{n}\right)$ = coefficient of x^n in the numerator of Padé approximation $\left[\frac{n+1}{n}\right]$

$q_n\left(\frac{n}{n}\right)$ = coefficient of x^n in the denominator of Padé approximation $\left[\frac{n}{n}\right]$.

We proceed to obtain three different types of continued fractions for the Laguerre function as follows:

Type 1 : Corresponding or Stieltjes Type Continued Fractions

This type of continued fraction is given by

$$(2)\qquad f(x) = b_0 + \cfrac{a_1 x}{1 + \cfrac{a_2 x}{1 + \cfrac{a_3 x}{1 + \ldots\ldots\ldots}}}$$

Truncations of this give $\left[\frac{M}{M-1}\right]$ and $\left[\frac{M}{M}\right]$ Padé approximations of the function.

Type II: Associated or Jacobi Type Continued Fractions

$$f(x) = b_0 + \cfrac{a_1 x}{1 + a_2 x - \cfrac{a_2 a_3 x^2}{1 + (a_3 + a_4)x - \cfrac{a_4 a_5 x^2}{1 + (a_5 + a_6)x \ldots\ldots\ldots\ldots}}} \tag{3}$$

Truncations of this give the $\left[\dfrac{M}{M}\right]$ Padé approximations i.e., diagonal elements of the Padé table.

Type III: Equivalent or Euler Type Continued Fractions

This type of continued fraction is given by

$$f(x) = \cfrac{g_0}{1 - \cfrac{(g_1 / g_0)\, x}{1 + (g_1 / g_0)\, x - \cfrac{(g_2 / g_1)\, x}{1 + (g_2 / g_1)\, x - \ldots\ldots - \cfrac{(g_n / g_{n-1})\, x}{1 + (g_n / g_{n-1})\, x}}}},$$

where $f(x) = g_0 + g_1\, x + g_2\, x^2 + \ldots + g_n\, x^n$

Truncations of this give the first column of Padé Table.

1.5. Padé Approximation from Continued Fraction

The continued fraction introduced by Viskovatoff [3] is given by

$$f(x) = \frac{\sum_{k=0}^{\infty} t_{1k}\, x^k}{\sum_{k=0}^{\infty} t_{0k}\, x^k}$$

$$= \cfrac{t_{10}}{t_{00} + \cfrac{t_{20} x}{t_{10} + \cfrac{t_{30}\, x}{t_{20} + \cfrac{t_{40}\, x}{t_{30} + \ldots\ldots\ldots\ldots}}}}, \tag{1}$$

where

(2) $$t_{mn} = -\det \begin{vmatrix} t_{m-2,0} & t_{m-2,n+1} \\ t_{m-1,0} & t_{m-n,n+1} \end{vmatrix}$$

Truncations of this give the $\left[\dfrac{M}{M}\right]$ and $\left[\dfrac{M}{M+1}\right]$ Padé approximations.

2. Continued Fractions of Generalised Laguerre Function $L_\nu^{(\mu)}(x)$:

By using Padé table 1 and relation 1.4(1) we get

$$b_0 = 1$$

$$a_1 = -\frac{\nu}{1+\mu}$$

$$a_2 = \frac{q_1\,(1/1)\,q_0\,(0/0)}{q_0\,(1/1)\,q_0\,(0/0)} = \frac{\nu-1}{2(2+\mu)}$$

$$a_3 = \frac{p_1\,(2/1)\,q_0\,(1/0)}{q_0\,(2/1)\,q_1\,(1/0)} = \frac{\mu\nu+5\nu+\mu-1}{6(2+\mu)\,(3+\mu)}$$

$$a_4 = \frac{q_2\,(2/2)\,q_0\,(1/1)}{q_0\,(2/2)\,q_2\,(1/1)}$$

$$= \frac{(2+\mu)(\nu-2)(7\nu^2+\mu\nu^2-12\nu-\mu+5)}{6(3+\mu)\,(4+\mu)(\nu-1)(\mu\nu+5\nu+\mu-1)} \quad \text{etc.}$$

We have considered only the portion ${}_1F_1\left[\begin{matrix}-\nu\\1+\mu\end{matrix} : x\right]$ of $L_\nu^{(\mu)}(x)$.

Type I : Corresponding or Stieltjee type continued fraction is obtained by 1.4(2) as follows :

$$\frac{\Gamma(1+\mu)\,\Gamma(1+\nu)}{\Gamma(1+\mu+\nu)}\;L_\nu^{(\mu)}(x)$$

(1) $$= 1 - \cfrac{\dfrac{\nu}{1+\mu}\,x}{1+\cfrac{\dfrac{(\nu-1)}{2(2+\mu)}\,x}{1-\cfrac{\dfrac{(\mu\nu-5\nu+\mu-1)}{6(2-\mu)\,(3+\mu)}\,x}{1+\cfrac{\dfrac{(2+\mu)\nu-2)(7\nu^2+\mu\nu^2-12\nu-\mu+5)}{6(3+\mu)\,(4+\mu)\,(\nu\mu+5\nu+\mu-1)(\nu-1)}\,x}{1+}}}}$$

Type II : Associated or Jacobi Type continued fraction is obtained by 1.4(3) as follows

$$\frac{\Gamma(1+\mu)\,\Gamma(1+\nu)}{\Gamma(1+\mu+\nu)}\; L_{\nu}^{(\mu)}(x)$$

$$= 1 - \cfrac{\frac{\nu}{1+\mu}\,x}{1 + \frac{(\nu-1)}{2(2+\mu)}\,x + \cfrac{\frac{(\nu-1)\,(\mu\nu+5\nu+\mu-1)x^2}{12(2+\mu)^2(3+\mu)}}{\cfrac{(2+\mu)^2\,(\nu-2)\,(7\nu^2+\mu\nu^2-12\nu-\mu+5)}{1+\frac{-(4-\mu)\,(\nu-1)(\mu\nu+5\nu+\mu-1)^2}{6(2+\mu)\,(3+\mu)\,(4+\mu)(\nu-1)}x}}}$$

(2) $$(\mu\nu+5\nu+\mu-1)\cdots$$

Type III : Equivalent or Euler type continued fraction is obtained by 1.4(4) as follows :

$$\frac{\Gamma(1+\mu)\,\Gamma(1+\nu)}{\Gamma(1+\mu+\nu)}\; L_{\nu}^{(\mu)}(x) = \cfrac{1}{\cfrac{\frac{\nu}{1+\mu}x}{1+\cfrac{}{\left[1+\frac{\nu}{1+\mu}x-\frac{(-\nu+1)}{2(2+\mu)}x\right]}}}$$

(3) $$1+\frac{(-\nu+1)}{2(2+\mu)}x-\cdots$$

in Particular, if $\nu=-\frac{1}{2}$, $\mu=0$, the corresponding results obtained from Padé table 2 and 1.4(1) are as follows:

Using Padé table and relation 1.4(1), we get

$$b_0=1,\ a_1=\tfrac{1}{2}$$

$$a_2=\frac{q_1\,(1/1)\,q_0\,(0/0)}{q_0\,(1/1)\,q_0\,(0/0)}=-\frac{3}{8}$$

$$a_3 = \frac{p_2\,(2/1)\,q_0\,(1/0)}{q_0\,(2/1)\,p_1\,(1/0)} = \frac{7}{72}$$

$$a_4 = \frac{q_2\,(2/2)\,q_0\,(2/1)}{q_0\,(2/2)\,q_1\,(1/1)} = -\frac{85}{504}$$

$$a_5 = \frac{p_3\,(3/2)\,q_0\,(2/1)}{q_0\,(3/2)\,p_2\,(2/1)} = \frac{51}{23800}$$

$$a_6 = \frac{p_3\,(3/3)\,q_0\,(2/2)}{q_0\,(3/3)\,q_2\,(2/2)} = -\frac{216531}{2267800} \quad \text{etc.}$$

Type I: Corresponding or stieltypes type continued fraction is given by

$$L^{(0)}_{-\frac{1}{2}}(x) = 1 + \cfrac{\frac{1}{2}x}{1 - \cfrac{\frac{3}{8}x}{1 + \cfrac{\frac{7}{72}x}{1 - \cfrac{\frac{85}{504}x}{1 + \cfrac{\frac{51}{23800}x}{1 - 216531x/2267800.}}}}}$$

Type II: Associated or Jacobi type continued fraction is given by

$$L^{(0)}_{-\frac{1}{2}}(x) = 1 + \cfrac{\frac{1}{2}x}{1 - \frac{3}{8}x + \cfrac{\frac{7}{192}x^2}{1 - \frac{1}{14}x + \cfrac{\frac{51}{141120}x^2}{1 - \frac{251889}{2698682}x}}} \tag{2}$$

Type III: Equivalent or Euler type continued fraction is given by

$$L_{-\frac{1}{2}}^{(0)}(x)=\cfrac{1}{1-\cfrac{\frac{1}{2}x}{1+\frac{1}{2}x-\cfrac{\frac{3}{8}x}{1+\frac{3}{8}x-\dots}}} \tag{3}$$

3. Padé Approximations From Continued Fraction of $L_{-\frac{1}{2}}^{(0)}(x)$.

Using 1.5(1) and 1.5(2) we get the continued fraction of $L_{-\frac{1}{2}}^{(0)}(x)$ as follows :

$$L_{-\frac{1}{2}}^{(0)}(x)=\cfrac{1}{1-\cfrac{\frac{1}{2}x}{1-\cfrac{\frac{1}{16}x}{-\frac{1}{2}-\cfrac{\frac{7}{768}x}{-\frac{1}{16}+}}}} \tag{1}$$

Truncations of this continued fraction are:

$$\frac{2}{2-x},\ \frac{8+x}{8-3x},\ \frac{48-8x}{48-32x+7x^2},\ \frac{1344+72x+37x^2}{1344-600x+85x^2}\ \text{etc.,}$$

which are Padé approximations $\left[\frac{0}{0}\right], \left[\frac{0}{1}\right], \left[\frac{1}{1}\right], \left[\frac{1}{2}\right], \left[\frac{2}{2}\right]$.etc., are respectively of the function $L_{-\frac{1}{2}}^{(0)}(x)$.

Table 1: Padé Table of the Organised Laguerre Functions $\left[L_{\nu}^{(\mu)}(x)\right]$

L \ M	0	1	2
0	1	$\dfrac{(1+\mu)}{(1+\mu)+\nu x}$	$\dfrac{2(1+\mu)^2(2+\mu)}{2(1+\mu)^2(2+\mu)+2(1+\mu)(2+\mu)\nu x+(\mu\nu^2+\mu\nu+3\nu^2+\nu)x^2}$
1	$\dfrac{(1+\mu)-\nu x}{(1+\mu)}$	$\dfrac{2(1+\mu)(2+\mu)-(\mu+3\nu+\mu\nu+1)x}{2(1+\mu)(2+\mu)+(1+\mu)(\nu-1)x}$	$\dfrac{6(1+\mu)(2+\mu)(3+\mu)(3\nu+\mu+\mu\nu+1)-(19\nu^2+2\mu^2+8\mu\nu^2+3\mu^2\nu+\mu^2\nu^2+18\mu\nu+15\nu+4\mu+2)}{6(1+\mu)(2+\mu)(3+\mu)(3\nu+\mu+\mu\nu+1)+4(\nu-1)(1+\mu)(2+\mu)(4\nu+\mu\nu+1)x+\nu(-\nu+1)(-5\nu-10\mu\nu-\mu^2+3\mu^2\nu+1)x^2}$
2	$\dfrac{2(1+\mu)(2+\mu)-2\nu(2+\mu)x+\nu(\nu-1)x^2}{2(1+\mu)(2+\mu)}$	$\dfrac{6(1+\mu)(2+\mu)(3+\mu)-4(2+\mu)(4\nu+\mu+\mu\nu+1)x+\nu(5\nu+\mu+\mu\nu-1)x^2}{6(1+\mu)(2+\mu)(3+\mu)+2(1+\mu)(2+\mu)(\nu-2)x}$	$\dfrac{12(1+\mu)(2+\mu)(3+\mu)(4+\mu)(5\nu+\mu+\mu\nu-1)-6(2+\mu)(3+\mu)(34\nu^2+9\mu\nu+11\mu\nu^2+6\nu+\mu^2\nu+\mu^2\nu^2+2\mu-4)x+(\mu^3\nu^3+16\mu^2\nu^3+46\mu^2\nu^2+5\mu^3\nu+26\mu^2\nu+89\mu\nu^3-238\mu\nu^2+7\mu\nu+158\mu^3+20\nu^2-14\nu+4\mu^3\nu^2+2\mu^3-4\mu^2-26\mu-20)x^2}{12(1+\mu)(2+\mu)(3+\mu)(4+\mu)(5\nu+\mu+\mu\nu-1)+6(1+\mu)(2+\mu)(\nu-2)(3+\mu)(6\nu+\mu+\mu\nu-2)x+(1+\mu)(2+\mu)(\nu-2)(7\nu^2+\mu\nu^2-12\nu-\mu+5)x}$

N. B. : Each approximation [L/M] must be multiplied by $\dfrac{\Gamma(1+\mu+\nu)}{\Gamma(1+\mu)\,\Gamma(1+\nu)}$

Table 2: Padé Table of the Laguerre Functions $L^{(0)}_{-\frac{1}{2}}(x)$ *

L \ M	0	1	2	3
0	1	$\frac{2}{2-x}$	$\frac{16}{16-8x+x^2}$	$\frac{96}{96-48x+6x^2+x^3}$
1	$\frac{2+x}{2}$	$\frac{8+x}{8-3x}$	$\frac{48-8x}{48-32x+7x^2}$	$\frac{1536-336x}{1536-432x-72x^2+37x^3}$
2	$\frac{16+8x+3x^2}{16}$	$\frac{144+32x+7x^2}{144-40x}$	$\frac{1344+72x+37x^2}{1344-600x+85x^2}$	$\frac{71040-768x-2718x^2}{71040-36288x+7542x^2+667x^3}$
3	$\frac{96+43x+18x^2+5x^3}{96}$	$\frac{1536+432x+120x^2+17x^3}{1536-336x}$	$\frac{163200+22464x+7542x^2+667x^2}{163200-59136x+6510x^2}$	$\frac{5122560+216000x+210528x^2+7471x^2}{5122560-2345280x+422688x^2-30933x^3}$

$$*L^{(0)}_{-\frac{1}{2}}(x)=1+\frac{1}{2}x+\frac{3}{16}x^2+\frac{5}{56}x^3+\frac{35}{3072}x^4+\frac{21}{10240}x^5+\frac{77}{245760}x^6+\frac{1001}{24084480}x^7+\frac{1001}{205520896}x^8+\ldots\ldots$$

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