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The Continuity With Respect To A General Displacemnt In The Orlicz And The Orlicz–Sobolev Spaces

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Abstract: In this paper necessary and sufficient conditions for the continuity with respect to a general displacement of functions from the Orlicz and the Orlicz-Sobolev spaces are obtained.

1. Introduction

A continuous convex function $\Phi(t)$ for $t \in [0, \infty)$ is called a Young's function if it satisfies the following conditions:

$$\lim_{t \to 0} \frac{\Phi(t)}{t} = 0,$$

$$\lim_{t \to \infty} \frac{\Phi(t)}{t} = \infty.$$

Its complementary function $\Psi(t)$ is also a Young's function (see [1,2]).

Let Ω be a measurable set in $\mathbb{R}^n$. $\widetilde{L}_\Phi(\Omega)$ denotes the set of all measurable functions f, defined almost everywhere in Ω such that

$$\rho_\Phi(f\,;\Omega) = \int_\Omega \Phi(|f(x)|)\,dx < \infty.$$

The set $\widetilde{L}_\Phi(\Omega)$ is called the Orlicz class. The Orlicz space $L_\Phi(\Omega)$ with the Orlicz norm

$$\|f\|_{L_\Phi(\Omega)} = \sup_{g,\,\rho_\Phi(g\,;\Omega) \le 1} \int_\Omega |f(x)\,g(x)|\,dx < \infty$$

is defined to be the linear hull of the Orlicz class $\widetilde{L}_\Phi(\Omega)$. For a vector–valued function $f = (f^{(1)}, \ldots, f^{(m)})$ we define

$$\rho_\Phi(f\,;\,\Omega) = \sum_{i=1}^{m} \rho_\Phi(f^{(i)}\,;\,\Omega)$$

and

$$\|f\|_{L_\Phi(\Omega)} = \sum_{i=1}^{m} \|f^{(i)}\|_{L_\Phi(\Omega)}.$$

The number

$$\| f \|_{L_\Phi(\Omega)} = \inf \{ K > 0 : \int_\Omega \Phi(\frac{1}{k} |f(x)|)\, dx \le 1 \}$$

is called the Luxemburg norm [3]. The Orlicz and the Luxemburg norms are equivalent and have the relation

$$\| f \|_{L_\Phi(\Omega)} \le \| f \|_{L_\Phi(\Omega)} \le 2 \| f \|_{L_\Phi(\Omega)} .$$

Now let Ω be a bounded measurable set in R^n. $E_\Phi(\Omega)$ is defined to be the closure in $L_\Phi(\Omega)$ of the set of all bounded measurable functions in Ω. Note that

$$E_\Phi(\Omega) \subset \tilde{L}_\Phi(\Omega) \subset L_\Phi(\Omega).$$

The following theorem on continuity holds.

Theorem 1.1 (See [2]). *For any function f in $E_\Phi(\Omega)$*

$$\lim_{h \to 0} \| f(x+h) - f(x) \|_{L_\Phi(\Omega \cap (\Omega - h))} = 0 .$$

A Young's function Φ is said to satisfy a Δ_2-condition if there exist a positive constant k and a non-negative constant T such that for all $t \ge T$

$$\Phi(2t) \le k\, \Phi(t)$$

If Φ satisfies a Δ_2-condition, then

$$E_\Phi(\Omega) = \tilde{L}_\Phi(\Omega) = L_\Phi(\Omega)$$

and consequently theorem 1.1 is also true for functions from $L_\Phi(\Omega)$. The theorem 1.1 is not true in general for $L_\Phi(\Omega)$. However in [4] the following theorem has been proved.

Theorem 1.2 *If Ω be bounded open set in R^n, then for every function f from $L_\Phi(\Omega)$ there exists a positive number λ such that*

$$\lim_{h \to 0} \rho_\Phi(\lambda (f(x+h) - f(x)); \Omega \cap (\Omega - h)) = 0 .$$

Now let Ω be bounded open set in R^n and $\ell \in \mathbb{N}$. The Orlicz–Sobolev space $W^\ell L_\Phi(\Omega)$ ($W^\ell E_\Phi(\Omega)$) is the set of all functions f all of whose generalised derivatives $\mathcal{D}^\alpha f$ exist and $\mathcal{D}^\alpha f \in L_\Phi(\Omega)$ ($\mathcal{D}^\alpha f \in L_\Phi(\Omega)$) where, $\alpha = (\alpha^{(1)}, \ldots, \alpha^{(n)})$ and $|\alpha| \le \ell$, and also

$$\| f \|_{W^\ell L_\Phi(\Omega)} = \sum_{|\alpha| \le \ell} \| \mathcal{D}^\alpha f \|_{L_\Phi(\Omega)} < \infty .$$

In this paper instead of linear displacement $x + h$ we consider a general displacement $B_h(x)$ to obtain the results analogous to theorems 1.1 and 1.2 for the Orlicz and the Sobolev spaces.

A transformation $B_h : \Omega \to R^n$ is said to be almost everywhere one -to- one if for every two subsets G_1, G_2 of Ω

$$G_1 \cap G_2 = \phi \Rightarrow mes\ (B_h(G_1) \cap B_h(G_2)) = 0.$$

The transformation B_h is said to have N-property if for any subset G of Ω

$$mes\ G = 0 \Rightarrow mes\ B_h(G) = 0$$

We use also the following notations :

$$\{\Omega\}_h = \{ x \in \Omega : B_h(x) \in \Omega \} = B_h^{-1}\ ((\Omega \cap B_h(\Omega)) .$$

In particular, if $B_h(x) = x + h$, then

$$\{\Omega\}_h = \Omega \cap (\Omega - h) .$$

2. The Orlicz Spaces

Theorem 2.1 *Let Ω be a bounded measurable set in R^n* and $\delta > 0$. *Assume that for all $h \in R^n$ satisfying $|h| < \delta$ there is a transformation $B_h : \Omega \to R^n$ which is almost everywhere one–to–one and has the N–property and its density function μ_h satisfies the following condition* :

(2.1) $$\exists\ C_1 > 0 : \forall\, h \in R^n \quad |h| < \delta \ \Rightarrow\ \mu_h(x) \geq C_1$$
for almost all $x \in \Omega$.
Then

(A) for every function f from $E_\Phi(\Omega)$

(2.2) $$\lim_{h \to 0} \ \| f \circ B_h - f \|_{L_\Phi(\{\Omega\}_h)} = 0$$

if, and only if,

(2.3) $$\lim_{h \to 0} \ \| B_h(x) - x \|_{L_\Phi(\{\Omega\}_h)} = 0 ;$$

(B) for every f from $L_\Phi(\Omega)$ there exists $\lambda > 0$ such that

$$\lim_{h \to 0} \ \rho_\Phi\ (\lambda (f \circ B_h(x) - f) : \{\Omega\}_h) = 0.$$

If and only if, there exists $\nu > 0$ such that

$$\lim_{h \to 0} \ \rho_\Phi\ (\nu\, (B_h(x) - x)\ ; \{\Omega\}_h) = 0 .$$

Proof : (A) Since for every $i \in \{1, \ldots n\}$, $x^{(i)} \in E_\Phi(\Omega)$, it is obvious that (2.2) implies (2.3)

To prove the sufficient condition, let f be a function in $E_\Phi(\Omega)$ and let $\mathcal{E}, \mathcal{E}_1$ and $\mathcal{E}_2$ be arbitrary positive numbers. From (2.1) we have

$$\exists\ \gamma > 0 : \forall\, h \in R^n \quad |h| < \gamma \Rightarrow \| B_h(x) - x \|_{L_\Phi(\Omega_h)} < \varepsilon_1 \tag{2.4}$$

Since $C_0^\infty(R^n)$ is dense in $E_\Phi(\Omega)$ (See [2]), there exists a function g in $C_0^\infty(R^n)$ such that

$$\| f - g \|_{L_\Phi(\Omega)} < \varepsilon_2 \tag{2.5}$$

Now,

$$\begin{aligned} \| f \circ B_h - f \|_{L_\Phi(\Omega_h)} &\le \| f \circ B_h - g \circ B_h \|_{L_\Phi(\Omega_h)} \\ &\quad + \| g \circ B_h - g \|_{L_\Phi(\Omega_h)} + \| f - g \|_{L_\Phi(\Omega_h)} . \end{aligned} \tag{2.6}$$

By (2.1) and Radon–Nikodym theorem [5] we have

$$\begin{aligned} &\int_{(\Omega)_h} \Phi(k^{-1} |f(B_h(x)) - g(B_h(x))|)\, dx \\ &\le C_1^{-1} \int_{\Omega \cap B_h(\Omega)} \Phi(k^{-1} |f(y) - g(y)|)\, dy. \end{aligned} \tag{2.7}$$

Now if $C_1 \le 1$, then by convexity of the function Φ we have

$$C_1^{-1}\ \Phi(|u|) \le \Phi(C_1^{-1} |u|)$$

and if $C_1 \ge 1$, then

$$C_1^{-1}\ \Phi(|u|) \le \Phi(C_2^{-1} |u|).$$

Thus, in both cases we assert that there exists $C_2 = \min(C_1, 1)$, such that

$$C_1^{-1}\ \Phi(|u|) \le (C_2^{-1} |u|).$$

Hence from (2.7) we get

$$\begin{aligned} &\| f \circ B_h - g \circ B_h \|_{L_\Phi((\Omega)_h)} \\ &\quad \le C_2^{-1}\ \| f - g \|_{L_\Phi(\Omega \cap B_h(\Omega))} \\ &\quad \le C_2^{-1}\ \| f - g \|_{L_\Phi(\Omega)} . \end{aligned}$$

Thus,

$$\| f \circ B_h - g \circ B_h \| \le 2\, C_2^{-1}\ \| f - g \|_{L_\Phi(\Omega)} . \tag{2.8}$$

Now since $g \in C_0^\infty(R^n)$, g satisfies the Lipshitz condition with a constant C_3. So,

$$\| g \circ B_h - g \|_{L_\Phi(\{\Omega\}_h)} \leq C_3 \| B_h(x) - x \|_{L_\Phi(\{\Omega\}_h)} \tag{2.9}$$

Choosing $\varepsilon_1 = \dfrac{\varepsilon}{2} \; C_3^{-1}$ and $\varepsilon_2 = (2C_2^{-1} + 1)\dfrac{\varepsilon}{2}$

from (2.1) –(2.6), (2.8), (2.9) we get

$$\| f \circ B_h - f \|_{L_\Phi(\{\Omega\}_h)} < \varepsilon$$

which implies (2.2).

The scheme of the proof of the part (B) is analogous to that of the previous proof.

Remark : In the proof of the sufficiency we have used the assertion that $C_0^\infty(R^n)$ is dense in $E_\Phi(\Omega)$, but to prove it with the help of modifiers we use theorem 1.1. Hence in the above proof we used indirectly theorem 1.1 But under stronger assumptions the sufficiency of theorem 2.1 can be proved by another method which does not use theorem 1.1.

Let an open set $G \supset \Omega$. Assume that

1. A transformation $B_h : G \to R^n$, $h \in R^n$ is one–to–one, $B_h \in C^1(G)$ and its Jacobian satisfies the following conditions

$$0 < C_4 \leq \left| \frac{\partial B_h(x)}{\partial x} \right| \leq C_5 < \infty$$

for all $x \in G$, where C_4 and C_5 are constants not depending on x and h;

2. $B_h(x)$ uniformly converges to x on Ω as h tends to zero. Then for every $f \in E_\Phi(\Omega)$ the property (2.2) holds.

The proof of the assertion is analogous to that of the assertion for Lebesgue spaces L_p given in [6].

3. The Orlicz–Sobolev Spaces

Lemma 3.1. *Let $\{g_k\}$ be a sequence of measurable functions defined on a bounded measurable set Ω such that.*

(a) $\lim\limits_{k\to\infty} \| g_k \|_{L_\Phi(\Omega)} = 0$ and (b) $\{ \| g_k \|_{L_\infty(\Omega)} \}$ is bounded.

Then for any function $\psi \in L_\Phi(\Omega)$, $\lim\limits_{k\to\infty} \| \psi g_k \|_{L_\Phi(\Omega)} = 0$

Proof : Since the function ψ is measurable, by Lusin's theorem for any $\delta > 0$ there exists a compact subset Ω_δ of Ω such that

$$mes\,(\Omega \setminus \Omega_\delta) < \delta$$

and ψ is continuous on Ω_δ. Then for any $\delta > 0$ we have

$$\|\psi g_k\|_{L_\Phi(\Omega)} \le \|\psi g_k\|_{L_\Phi(\Omega\setminus\Omega_\delta)} + \|\psi g_k\|_{L_\Phi(\Omega_\delta)}$$

$$\le \sup_{m\in N} \|g_m\|_{L_\Phi(\Omega)} \|\psi\|_{L_\Phi(\Omega\setminus\Omega_\delta)}$$

$$+ \|\psi\|_{C(\Omega_\delta)} \|g_k\|_{L_\Phi(\Omega)}$$

As k tends to infinity, for any $\delta > 0$ we get the upper limit

$$\overline{\lim_{k\to\infty}} \|\psi g_k\|_{L_\Phi(\Omega)} \le \sup_{m\in N} \|g_m\|_{L_\Phi(\Omega)} \|\psi\|_{L_\Phi(\Omega\setminus\Omega_\delta)}.$$

Now as $\delta \to 0$ we have

$$\overline{\lim_{k\to\infty}} \|\psi g_k\|_{L_\Phi(\Omega)} \le 0$$

which implies the assertion.

Theorem 3.2 *Let Ω be a bounded open set in R^n, $\ell \in N$, $\delta > 0$. Assume that for all $h \in R^n$ satisfying $|h| < \delta$ there is a transformation $B_h : \Omega \to R^n$ which is one-to-one and has the following properties :*

(a) $B_h \in C^\ell(\Omega)$

(b) $\exists\, C_6 > 0 : \forall i \in \{1, \dots, n\}, \forall \alpha, 1 \le |\alpha| \le \ell$

$$\|\partial^\alpha B_h^{(i)}\|_{C(\Omega)} \le C_6$$

(c) $\exists\, C_7 > 0 : \forall x \in \Omega$

$$\left|\frac{\partial B_h(x)}{\partial x}\right| \ge C_7.$$

Then

(A) for every function f from $W^\ell E_\Phi(\Omega)$

$$\lim_{h\to 0} \|f \circ B_h - f\|_{W^\ell L_\Phi(\Omega_h)} = 0 \tag{3.1}$$

If and only if,

$$\lim_{h\to 0} \|B_h(x) - x\|_{W^\ell L_\Phi(\Omega_h)} = 0, \tag{3.2}$$

(B) for every function f from $W^\ell E_\Phi(\Omega)$ there exists $\lambda > 0$ such that

$$\lim_{h\to 0} \sum_{|\alpha|\le\ell} \rho_\Phi\left(\lambda \partial^\alpha (f \circ B_h - f)\,;\, \{\Omega\}_h\right) = 0$$

if and only if, there exists $\nu > 0$ such that

$$\lim_{h \to 0} \sum_{|\alpha| \le \ell} \rho_{\Phi} (\nu \mathcal{D}^{\alpha} (B_h(x) - x) ; \{\Omega\}_h) = 0 .$$

Proof (A) The necessity is obvious. To prove sufficiency, let $f \in W^{\ell} E_{\Phi}(\Omega)$. For generalized derivatives of the composite function $f \circ B_h$ under the assumptions of the theorem the rule of differentiation is the same as that for usual derivatives [7]. So $f \circ B_h \in W^{\ell} E_{\Phi}(\{\Omega\}_h)$ and for almost all x of $\{\Omega\}_h$ the generalised derivative $\mathcal{D}^{\alpha} f(B_h(x))$ as $|\alpha| \le \ell$ takes the following form.

$$\mathcal{D}^{\alpha} f(B_h(x)) = (\mathcal{D}^{\alpha} f)(B_h(x)) \tag{3.3}$$
$$+ \sum_{1 \le |\beta| \le |\alpha|} \varphi_{\alpha,\beta,h}(x) (\mathcal{D}^{\beta} f)(B_h(x))$$

where $\beta = (\beta^{(1)}, \ldots, \beta^{(n)})$ and functions $\varphi_{\alpha,\beta,h}$ are expressed in terms of $B_h^{(1)}, \ldots, B_h^{(n)}$ according to the rule of differentiation. Now we show that

$$\lim_{h \to 0} \| \varphi_{\alpha,\beta,h} \|_{L_{\Phi}(\Omega)} = 0 \tag{3.4}$$

In fact, since for any real numbers z_j

$$| z_1^{\alpha^{(1)}} \ldots z_n^{\alpha^{(n)}} - 1 | \le \sum_{j=1}^{n} \alpha^{(j)} \, |z_j - 1| \, (|z_j| + 1)^{|\alpha| - 1}$$

$$\| \varphi_{\alpha,\beta,h} \|_{L_{\Phi}(\Omega)} = \left\| \bigcap_{j=1}^{n} \left(\frac{\partial B_h^{(j)}}{\partial x^{(j)}}(x) \right)^{\alpha^{(j)}} - 1 \right\|_{L_{\Phi}(\Omega)}$$

$$\le \sum_{j=1}^{n} \alpha^{(j)} \left\| \frac{\partial B_h^{(j)}}{\partial x^{(j)}} - 1 \right\|_{L_{\Phi}(\Omega)} \left(\left\| \frac{\partial B_h^{(j)}}{\partial x^{(j)}} \right\|_{L_{\Phi}(\Omega)} + 1 \right)^{|\alpha| - 1}$$

Hence because of (b) and (3.2) we get (3.4) if $\beta = \alpha$. Now if $\beta \ne \alpha$ bearing in mind the structure of $\varphi_{\alpha,\beta,h}$ and discussing in the similar manner again we get (3.4). Note that because of (b) there exists $M > 0$ such that for all $h \in R^n$ satisfying $|h| < \delta$ we get

$$\| \varphi_{\alpha,\beta,h} \|_{L_{\infty}(\Omega)} \le M . \tag{3.5}$$

Further, (3.3) implies that

$$\| \partial^\alpha (f \circ B_h) - \partial^\alpha f \|_{L_\Phi(\Omega_h)}$$
$$\leq \| (\partial^\alpha f) \circ B_h - \partial^\alpha f \|_{L_\Phi(\Omega_h)}$$
$$+ \sum_{1 \leq |\beta| \leq |\alpha|} \| \varphi_{\alpha,\beta,h} [(\partial^\beta f) \circ B_h] \|_{L_\Phi(\Omega_h)}$$

In the last inequality the first part of the right side tends to zero by theorem 2.1. And the second part also tends to zero. In fact, changing the variable in the norm as we did in theorem 2.1 to get (2.8) we have

$$\| \varphi_{\alpha,\beta,h} [(\partial^\beta f) \circ B_h] \|_{L_\Phi(\Omega_h)}$$
$$\leq C_8 \| \partial^\beta f \cdot [\mathring{\varphi}_{\alpha,\beta,h} \circ B_h^{-1}] \|_{L_\Phi(\Omega)},$$

where C_8 depends only on the constant C_7 and $\mathring{\varphi}_{\alpha,\beta,h} (B_h^{-1})$ is an extension of $\varphi_{\alpha,\beta,h} (B_h^{-1})$ by zero outside of $\Omega \cap B_h(\Omega)$. Now we prove that the right side of the last inequality tends to zero as h tends zero, using lemma 3.1 with $\psi = \partial^\beta f \in L_\Phi(\Omega)$ and

$$g_k = \mathring{\varphi}_{\alpha,\beta,h} \circ B_h^{-1}$$

(here $h \neq 0$, $h \to 0$ as $k \to \infty$).
For it bearing in mind (3.5), it is sufficient to prove that

$$\lim_{h \to 0} \| \varphi_{\alpha,\beta,h} \circ B_h^{-1} \|_{L_\Phi(\Omega \cap B_h(\Omega))} = 0. \tag{3.6}$$

Since again changing the variable in the norm we get

$$\| \varphi_{\alpha,\beta,h} \circ B_h^{-1} \|_{L_\Phi(\Omega \cap B_h(\Omega))} \leq C_9 \| \varphi_{\alpha,\beta,h} \|_{L_\Phi((\Omega))},$$

where C_9 depends on C_6 and h, (3.4) implies (3.6). This completes the proof of the part (A). The scheme of the proof of the part (B) is analogous to that of the above proof.

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Braid Types Of Periodic Orbits

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Abstract : The last decades have seen an explosion of interest in the study of non-linear dynamical systems. Many scientists realise the power of qualitative techniques developed during this period. The main idea is that the gross behaviour of all (at least main) solutions of the system is more important than the local behaviour of particular, analytically-precise solutions.

Among these results is the wellknown and surprising Sarkovskii's theorem discussed below. In this paper, which is a summary of a talk given at Tribhuvan University in December 1993, we try to present an overview of the framework introduced these last 20 years to study the structure of the set of periodic orbits of discrete dynamical systems in dimensions 1 and 2.

1. Introduction

A lot of natural phenomenon can be described in terms of discrete dynamical systems. That is, given a map $f: X \to X$, we iterate this map, beginning with a point $x \in X$ and we are interested in the properties of the set:

$$x, f(x), f^2(x) \ldots$$

which describes the successive states of an evolution process $x_{n+1} = f(x_n)$, $x_0 = x$. The set of all these points is the trajectory or the orbit of the point x under the action of f. In the case where f is a bijection, we will include, of course, the iterates of x under f^{-1}

$$\ldots, f^{-2}(x), f^{-1}(x), x$$

Among all the orbits, some of them are more remarkable. For example, they may form a cycle:

$$x, f(x), \ldots, f^n(x) = x$$

These are called the periodic points of f, We write *Per* (f), for the set of all these points.

In practice, it is often difficult, if it is not impossible to describe the behaviour of all the points of X under the action of f. Usually, we do not know explicitly f and even in that case, the algebraic expression of the iterates of f may be too ugly to be able to undertake a serious investigation. Most of the time, we know only a sample of the set of all the trajectories of f, for example a periodic orbit. The natural question which arises then is whether we can deduce from this little source of information more predictions about the dynamic of f : which means about orbits of other points of X under f.

This problem has been extensively studied these last years. From all the results which were obtained one main idea arised: *some types of orbits force coexistence of other types of orbits*. A classical examples is the very famous Sarkovskii theorem [18].

Theorem 1 (*Sarkovskii*) *Define the following linear ordering on the integers*:

$3 \succ 5 \succ \ldots \succ 3.2 \succ 5.2 \succ \ldots \succ 3.2^2 \succ 5.2^2 \succ \ldots \succ 2^3 \succ 2^2 \succ 2 \succ 1$

Then if $f : \mathbb{R} \to \mathbb{R}$ *is a continuous map which has a periodic point of period* n, *it has also a periodic point of period* m *for all* $n \succ m$

This beautiful theorem shows that some of the orbits, namely periodic orbits of period $q \cdot 2^d$ (q odd), lead to a very complex dynamic (infinity of periodic points of distinct periods) and that some others (those of period 2^d) just force the coexistence of a finite number of periods.

There are similar results for orientation reversing homeomorphisms of compact orientable surfaces [10, 21]. However a moment of reflexion is enough to realize that the period alone is not enough to characterize a periodic orbit in dimension two. A euclidean rotation by angle $2\pi/n$ around the origin has periodic orbits of period n, one fixed point and nothing else.

Actually, this phenomenon already exists in dimension 1. Some periodic orbits of period 4 of a continuous map of the interval force the existence of all periods but some of them force the coexistence of just one fixed point and a periodic point of period two as required by Sarkovskii's theorem. To take into account these considerations, it is necessary to characterize a periodic orbit with a little more information.

2. Dynamical Order

2.1 Cycle Associated To A Periodic Orbit In Dimension 1

In the case of continuous map of the unit interval I, it is the permutation induced by f on the points of the orbit, naturally ordered on the real axis which permit us to refine the ordering defined previously [1,4,5,25]. Let C_n be the subset of all the n–cycles of S_n, the symmetric group and define

$$C = \bigcup_{n \in \mathbb{N}^*} C_n .$$

Given a continuous map f, we define the set of all the cycles of f, noted $C(f)$ in the following. An *n–cycle*, θ, belongs to $C(f)$ if there exist reals

$$x_1 < x_2 < \ldots < x_n$$

such that $f(x_i) = x_{\theta(i)}$ The following relation :

$$\theta \succ \eta \quad \text{if and only if} \quad \forall f \in C^0(I,I), \theta \in C(f) \Rightarrow \eta \in C(f)$$

defines a partial order over $C_n[1,2,4,6]$. It is obviously reflexive and transitive and it can be shown that it is antisymmetric [2]. The main tool in the study of this order

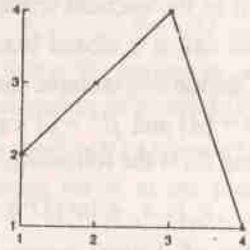

Figure 1: $\theta = (1234)$

is the construction for each $\theta \in C$ of a map f_θ which is *simplest* inside the class of maps such that $\theta \in C(f)$; in fact, a piecewise linear map (see Figure 1).

This map is such that $C(f_\theta) = \{\eta \ ; \theta \succ \eta\}$, which justify the qualification simplest given to it. There is a partition of I into a finite number of intervals $I_1, ..., I_{n-1}$ such that if $f(I_k)$ meets the interior of I_1, it covers I_1 once and only once. We say that f_θ possesses a Markov partition. Associated to this Markov partition there is a $(n-1, n-1)$ transition matrix $(a_{i,j})$ defined in the following way: $a_{i,j} = 1$ if $f(I_i)$ covers I_j and $a_{i,j} = 0$ otherwise. It is then easy to compute the successors of θ using its transition matrix and its Markov graph.

$$M(\theta) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

In the case of a homeomorphism $f : D^2 \to D^2$, the first problem which arises is to characterise periodic orbits. It is not longer possible to associate a permutation to an orbit $P = \{P_1, ..., P_n\}$ of f (there is no natural way to order the points of P as in the one–dimensional case). It is therefore necessary to introduce a new object : the braid type of P which will play the role of the cycle.

2.2 Artin Braid Group

For $i \in \mathbb{N}_n = \{1,2, ..., n\}$, let $Q_i = \{2i/(n+1)-1, 0)$, $Q = \{Q_1, Q_2, ..., Q_n\}$ and $A_i = (\gamma_i(t), t) \in D^2 \times I$ be an arc such that $\gamma_i(0)$ and $\gamma_i(1)$ belong to Q and $\gamma_i(t) \neq \gamma_j(t)$ if $i \neq j$. The set $A = A_1 \cup A_2 \cup ... \cup A_n$ is called a geometric braid.

We define on the set of all the geometric braids the following equivalence relation : $A \sim A'$ if there exists a continuous family $A(s) = A_1(s) \cup A_2(s) \cup ... \cup A_n$

(s) $(s \in I)$ of geometric braids such that $A(0) = A$ and $A(1) = A'$. The usual operation of stacking together two geometric braids is compatible with this relation and induce on the quotient set B_n a group structure. B_n is the *Artin braid group*. An element of B_n is just called a braid.

We can close a geometric braid A by gluing together its endpoints (which means looking at the projection of A in the quotient space $D^2 \times I/(x, 0) \sim (x, 1)$ which is solid torus). What we obtain we call it a closed braid and write $\widetilde{A}$. Two closed braids $\widetilde{A}$ *et* $\widetilde{A}'$ are equivalent (i.e. define two isotopic simple closed curve in the solid torus $(D^2 \times S^1)$ if and only if $\beta = [A]$ and $\beta' = [A']$ are conjugate in B_n [9].

Another way of constructing B_n is the following. Let

$$W_n = \{\{x_1, x_2, \dots, x_n\};\ x_i \in \operatorname{int}(D^2),\ x_i \neq x_j\},$$

be the set of non ordered n–tuples of distinct points in $D^2 \setminus \partial D^2$. Then $B_n = \pi_1(W_n, Q)$.

B_n is a group of finite type with generators $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ and relations $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| \geq 2$ and $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

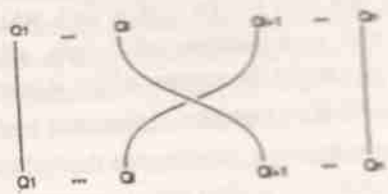

Figure 2: generator σ_i.

The centre of B_n, $Z(B_n)$ is the infinite cyclic group generated by $\theta_n = (\sigma_1, \sigma_2, \dots, \sigma_{n-1})^n$ (which corresponds to a full twist)

2. 3 Braid Type of Periodic Orbits

It is Rufus Bowen [12] (see also [13, 14] who first characterised a periodic orbit P of a homeomorphism f of a surface M by the isotopy class of f relative to the orbit P. Let $f \in Homeo^+(D^2)$, be an orientation preserving homeomorphism and

$$P = \{P_1, P_2, \dots, P_n\} \subset D^2 \setminus \partial D^2$$

a periodic orbit of f. In order to compare several orbits, it is necessary to fix a standard model for (D^2, P). The choice of a homeomorphism $h : (D^2, P) \to (D^2, Q)$ determines an element $f_P^h = h \circ f \circ h^{-1}$ of $Homeo^+(D^2, Q)$, the subgroup of $f \in Homeo+(D^2)$ such that $f(Q) = Q$. Changing h is the same as replacing f_P^h by one of its conjugate

in $Homeo^+(D^2, Q)$. Therefore, the conjugacy class $<\alpha>$ of the isotopy class $\alpha = [f_P^h]$ of f_P^h in the mapping class group $Aut(D^2,Q)$, of the homeomorphisms of D^2 which preserve Q, is independent of h.

The link with the braid group B_n is the following. Let

$$\in : \begin{cases} Homeo^+(D^2) \to & W_n \\ f & \mapsto \{f(Q_1), \ldots, f(Q_n)\} \end{cases}$$

It is not difficult to see that $\in$ is surjective and that $\in(f) = \in(g)$ if and only $f^{-1} g \in Homeo^+(D^2, Q)$. This permit us to identify W_n with the quotient space :

$$Homeo^+(D^2) / Homeo^+(D^2, Q).$$

The existence of a local section for $\in$ at the point $Q = \in(Id)$, implies that $\in$ is a fibration:

$$Homeo^+(D^2, Q) \xrightarrow{i} Homeo^+(D^2) \xrightarrow{\in} W_n$$

Using the long exact sequences on homotopy groups for this fibration and remembering that $B_n = \pi_1(W_n, Q)$ and $Aut(D^2, Q) = \pi_0(Homeo^+(D^2, Q))$, we get an epimorphism m: $B_n \to Aut(D^2, Q)$ whose kernel $\ker(m)$ is exactly $Z(B_n)$. We summarise this fact in the following theorem [9].

Theorem 2 $Aut(D^2, Q) \cong B_n / Z(B_n)$

This isomorphism induces a canonic bijection between conjugacy classes of the two groups. Therefore, we can identify $<\alpha>$, the conjugacy class of $[f_P^h]$ in *$Aut(D^2, Q)$ with a conjugacy class* $\widetilde{\beta}$ in $B_n / Z(B_n)$.

Definition 1 $bt(P, f) = \widetilde{\beta}$ is the braid type of the periodic orbit P.

Let CB_n, be the subset of braids whose closure possesses only one component. We define BT_n as the set of all conjugacy class of CB_n and we write:

$$BT = \bigcup_{n \geq 1} BT_n.$$

BT is the set of all braid types of periodic orbits of homeomorphisms of D^2. If $f \in Homeo^+(D^2)$, we set $BT(f) = \{bt(P, f) \, ; \, P \in Per(f)\}$ and we define on BT the following forcing relation [13, 14, 15]

$$\widetilde{\beta}_1 \succ \widetilde{\beta}_2 \text{ if and only if } \forall f \in Homeo^+(D^2), \, \widetilde{\beta}_1 \in BT(f) \Rightarrow \widetilde{\beta}_2 \in BT(f)$$

Theorem 3. *The relation $\succ$ is a partial order over BT.*

The problem is now to be able to work with this order. In particular:

- Given $\widetilde{\beta}_1, \widetilde{\beta}_2 \in BT$, does $\widetilde{\beta}_1 \succ \widetilde{\beta}_2$ or $\widetilde{\beta}_2 \succ \widetilde{\beta}_1$?
- Construct $\{\widetilde{\alpha} : \widetilde{\beta} \succ \widetilde{\alpha}\}$

These problems are still open ones. There are however some interesting partial results which have been obtained if we restrict ourselves on the induced order on a subset of BT. T. Matsuoka [23] has solved that question for three braid types $(BT_3, \succ)$ and T. Hall [20] did it for $(BT(H), \succ)$ the set of braid types which appear in Smale horseshoe (one main model of two dimensional dynamics).

3. Application of Nielsen–Thurston Theory to the Study of Braid Types

3.1 Nielsen–Thurston Theory of Mapping Class

Thurston [11, 19, 22, 24, 28] (completing a program initiated by Nielsen [27, 26] gave a beautiful description of surfaces homeomorphisms of a compact surface M up to isotopy. Two types of homeomorphisms are used as bricks to build a canonical representative in each isotopy class.

- The first type is constituted by the isometries for a given hyperbolic metric of constant curvature -1 on M. Given a fixed hyperbolic metric, the group of isometry is finte [29]. Therefore, for each isometry $f : M \to M$, there exists an integer $n > 0$ such that $f^n = Id$.
- The other type has been called pseudo-Anosov by analogy with Anosov diffeomorphisms of the torus. A pseudo-Anosov homeomorphism f of a closed surface is a homeomorphism which preserves two transverse foliations F^s and F^u which singularities are p–prongs singularities with $p \geq 3$. These foliations have transverse measures μ^s and μ^u such that $f(F^s, \mu^s) = (F^s, \lambda^{-1}\mu^s)$ and $f(F^u, \mu^u) = (F^u, \lambda \mu^u)$ where λ is a positive real number > 1. In the case where $\partial M \neq \phi$ each component of ∂M is a cycle of leaves of both foliations F^s and F^u separated by singularities of type spine and each component of ∂M has at least one singularity of each foliation.
- A homeomorphism $f : M \to M$ is reducible by a system of curves Γ of distinct simple closed curves $\Gamma_1, \Gamma_2, \dots, \Gamma_n$ if:
  1. Γ_i is not homotopic to a point nor to a component of ∂M.
  2. Γ_i is not homotopic to Γ_j if $i \neq j$.
  3. Γ is invariant under f

Theorem 4 Let $f : M \to M$ *a homeomorphism then* f *is isotopic to* $\phi : M \to M$, *where* ϕ *satisfies one of the following properties.*

1. *ϕ is an isometry for hyperbolic metric of constant curvature -1 on M, or*
2. *ϕ is pseudo-Anosov, otherwise*
3. *ϕ is reducible by a system of curves Γ .*

In that latest case, the following properties hold : Γ has a tubular neighbourhood $\eta(\Gamma)$ invariant under ϕ and if $S_1,\ldots, S_p$ are the component of $M\backslash\eta(\Gamma)$ and n_i is the smallest positive integer such that $\phi^{n_i}(S_i) = S_i$, then ϕ^{n_i}/S_i satisfies (1) or (2). If $\eta_1, \ldots, \eta_q$ are the component of $\eta(\Gamma)$ and m_j is the smallest positive integer such that $\phi^{m_j}/(\eta_j)=\eta_j$, then $\phi^{m_j}/(\eta_j)$ is a twist. (this twist is not necessary a full twist for $f^{m_i}/\partial\eta_i$ is not necessarily the identity)

A class is said to be irreducible if it has a representative of type (1) or (2), both of them, being mutually exclusive. In particular, if a class is of finite order, then it possesses a representative of type (1) [30]. We say that this class is periodic. If a class is of infinite order, then it has a pseudo–Anosov representative and we say that it is of type pseudo-Anosov, or it has a representative of type (3) and we say that it is reducible, both cases being mutually exclusive.

3.2 Classification of Braid Types

Let $\overline{D}_n$ be the compact surface obtained by replacing each puncture Q_i of $D_n = D^2 \setminus Q$ by a circle S_i (the circle of tangent vectors at this point) and let $p : \overline{D}_n \to D^2$ be a surjective continuous map which sends S_i onto Q_i and which is a diffeomorphism outside the circles.

Let f be a C^1–diffeomorphism. We can then extend the restriction f_Q of f to D_n into a homeomorphism $\bar{f}_Q$ of $\overline{D}_n$ in the following way ;

$$\bar{f}_Q(u) = D_{Q_i} f . u / \| D_{Q_i} f . u \| \quad u \in S_i$$

We can therefore characterise (Q, f) by the isotopy class of $\bar{f}_Q$ on $\overline{D}_n$

Proposition 1. *The projection map $p : \overline{D}_n \to D^2$ induces an isomorphism between $Aut(\overline{D}_n)$ and $Aut(D^2, Q)$.*

Let $\widetilde{\beta} \in BT_n$, a conjugacy class in $Aut(D^2,Q) \cong Aut(\overline{D}_n)$. Since the Thurston type of an element of $Aut(\overline{D}_n)$ is invariant under conjugacy, we can speak of a pseudo-Anosov (resp. periodic, reducible) conjugacy class $\widetilde{\beta}$.

Let $f \in Homeo^+(D^2, Q)$ and $\widetilde{\beta}(f, Q)$ its braid type. If $n \geq 2$ then the Euler characteristic of $\overline{D}_n$, $\chi(\overline{D}_n) = 1 - n$ is negative. A canonic representative $\overline{\phi}_\beta \in Homeo^+(\overline{D}_n)$ induces a homeomorphism $\phi_\beta \in Homeo^+(D^2, Q)$ which is isotopic to f (rel, to Q) and such that $p \circ \overline{\phi}_\beta = \phi_\beta \circ p$. In particular, if $\overline{\phi}_\beta$ is pseudo-Anosov, we will say that ϕ_β is pseudo-Anosov (rel. to Q), even if there is no really pseudo-Anosov homeomorphism on D^2. We write IBT_n for the set of irreducible elements of BT_n and $IBT = \bigcup_n IBT_n$

If $\widetilde{\beta}$ is periodic, there exists a smallest integer $m \in \mathbb{N}^*$ such that $\phi_\beta^m = Id$ and ϕ_β is conjugate to a euclidean rotation by an angle $2k\pi/m$ $(0 < k < m)$ around the origin [16, 17]. Therefore $n = m$ (since $\phi_\beta(Q) = Q$) and β is conjugate $\alpha_{k/n} = (\sigma_1 \sigma_2 \cdots \sigma_{n-1})^k$. There are only $\varphi(n)$ periodic elements in BT_n($\varphi(n)$ being the number of integers which are coprime with n and lower than n)

Suppose now that $\widetilde{\beta}$ is reducible. Let $\{S_0, S_1, \ldots, S_m\}$, the decompositon of D^2 into invariant (by ϕ_β) subsurfaces, S_0 being the component which contains ∂D^2 and $\{S_1, \ldots, S_p\}$ the components which boundary meets S_0. Up to conjugacy, we can suppose that $S_1, S_2, \ldots, S_p$ are like on figure 3.

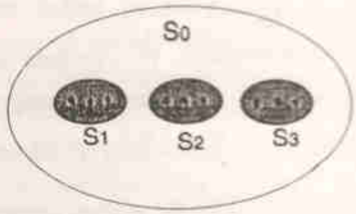

Figure 3 : invariant subsurfaces by a reducible element

Define $q = \frac{n}{p}$. We get $\phi_\beta(S_0) = S_0$ and $\phi_\beta^q(\Delta_i) = \Delta_i$ (notice that ϕ_β^q / Δ_i and ϕ_β^q / Δ_j are conjugate). Let $\widetilde{\gamma}_1 \in BT_q$ be the conjugacy class of $[\phi_\beta / S_0]$ in $Aut\,(D^2 \setminus \bigcup_{i=1}^{q} \Delta_i)$ and $\widetilde{\beta}'$ the conjugacy class of $[\phi^q_{\beta/\Delta_i}]$ in $Aut\,(D^2, Q \cap \Delta_i)$

We write:

$$\widetilde{\beta} = [\widetilde{\gamma}_1, \widetilde{\beta}']$$

(notice that $\widetilde{\gamma}_1$ and $\widetilde{\beta}'$ uniquely determined). By definition, $\widetilde{\gamma}_1$ is irreducible.

We can then undertake the same decomposition for $\widetilde{\beta}'$ (see figure 4). this process is finite

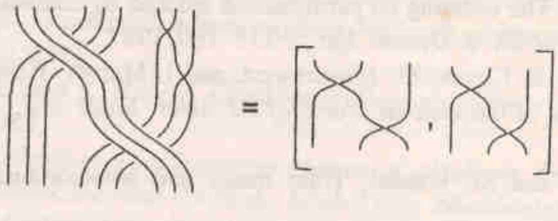

Figure 4: decomposition into irreducible factors

Theorem 5. *Each $\widetilde{\beta} \in BT$ has therefore a unique decomposition*

$$\widetilde{\beta} = [\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_r]$$

where $\widetilde{\gamma}_i$ is irreducible.

Notice that if $\widetilde{\beta} = [\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_r]$ and $\widetilde{\gamma}_i \in BT_{q_i}$ then $n = q_1\, q_2 \dots q_r$. In particular, if n is prime, we have $BT_n = IBT_n$

Theorem 6. *Let $\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_r, \widetilde{\alpha}, \widetilde{\beta} \in IBT$, then*

1. $[\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_r] \succ [\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_{r-1}]$
2. $[\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_r, \widetilde{\alpha}] \succ [\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_r, \widetilde{\beta}] \Leftrightarrow \widetilde{\alpha} \succ \widetilde{\beta}$.

Concerning explicit computations in $(BT, \succ)$, we can suggest the following steps:

1. Given a braid type $\widetilde{\beta} \in BT$, determine whether it is periodic, pseudo-Anosov or reducible and in that later case, compute the decomposition into irreducible factors. An algorithm to answer this question exists [3]
2. Given $\gamma \in IBT$ compute $\{\alpha \,;\, \gamma \succ \alpha\}$. This question is still the object of contemporary research [7, 8]

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Orbit Space Over Commutative Rings And Projectivities As Semi-isomorphism

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Abstract: For a free module of finite rank over a commutative (local) ring, the group of projectivities of the associated projective space is shown to be isomorphic to the group of geometric semi-isomorphisms of the associated geometric space of orbits.

1. Introduction

Orbit spaces of groups and vector spaces (in particular projective spaces) are studied as Pasch geometries ([1],[2],[3]). Different groups of semi-isomorphisms and isomorphisms of orbit spaces of vector space are related to classical groups of semi-linear and linear transformations [4]. Group of projectivities also arise from the projective space of a free module over commutative (local) ring ([5],[6]). Here it is the purpose to show that the projective spaces of free modules can also realized as orbit spaces and the group of projectivities is isomorphic to the group of geometric semi-isomorphisms of the associated orbit space.

2. Preliminaries

The concept of pasch geometry and related developments can be found in ([1],[2],[3]). For convenience we give a brief discussions here.

2.1 Pasch Geometry : Suppose A is a non-empty set, $e \in A$, $\Delta = \Delta_A \subseteq A \times A \times A$. Then the system (A, e, Δ) is said to form a Pasch geometry if the following hold : (i) For each $a \in A$, $\exists$ unique $b \in A$ with $(a,b,e) \in \Delta$. b is denoted by $a^{\#}$, (ii) $e^{\#} = e$ and $(a^{\#})^{\#} = a \ \forall \ a \in A$, (iii) $(a,b,c) \in \Delta \Rightarrow (b,c,a) \in \Delta$,(iv) $(a_1, a_2, a_3), (a_1, a_4, a_5) \in \Delta \Rightarrow \exists a_6 \in A$ with $(a_6, a_4^{\#}, a_2)$, $(a_6, a_5, a_3^{\#}) \in \Delta$. As a consequence one gets (v) $(a,b,c) \in \Delta \Rightarrow (c^{\#}, b^{\#}, a^{\#}) \in \Delta$, (vi) for $a,b \in A$, there exists $c \in A$ with $(a,b,c) \in \Delta$. The geometry is called sharp if (a,b,c) , $(a,b,d) \in \Delta \Rightarrow c = d$. In such, letting $ab = c^{\#}$, a group results with e as identity and $c^{\#}$ as the inverse of c. The geometry is called abelian iff $(a,b,c) \in \Delta \Rightarrow (b,a,c) \in \Delta$. Together with (iii) it gives every permutation of (a,b,c) in Δ. If $S \subseteq A$, then S is called a subgeometry of A if $e \in S$ and $(a,b,x) \in \Delta$, $a, b \in S \Rightarrow x \in S$. With $\Delta_s = S \cap (S \times S \times S)$, one gets a geometry (S, e, Δ_s). For an abelian geometry, a useful consequence is.

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2.2 Lemma : If A is an abelian geometry (with a given Δ), then $(x_1, s_1, t_1), (x_2, s_2, t_2), (x_1, x_2, x_3) \in \Delta \Rightarrow \exists s_3, t_3 \in A$ with $(s_1, s_2, s_3), (t_1, t_2, t_3), (s_3, t_3, x_3) \in \Delta$
In abelian geometry e will be usually denoted by 0.

2.3. Extension of Δ: Let A with a Δ be a geometry. Let $\Delta_1 = \{(e)\} \subset A^1$, $\Delta_2 = \{(x, x^{\#}) : x \in A\} \subset A \times A$, and $\Delta_3 = \Delta$. For any positive integer $n > 3$, we use induction to define $\Delta_n \subset A \times A \times ... \times A$ (n factors) such that $(x_1, x_2, ..., x_n) \in \Delta_n$ iff $\exists x \in A$ with $(x_1, x_2, ..., x_{n-2}, x) \in \Delta_{n-1}$ and $(x^{\#}, x_{n-1}, x_n) \in \Delta$. Thus for each n, one gets Δ_n (extended collinearity).

2.4 Geometry Morphisms : Let (A, e_A, Δ_A) and (B, e_B, Δ_B) be geometries. A map f: $A \mapsto B$ is called a geometry morphism if $f(e_A) = e_B$ and $(a,b,c) \in \Delta_A \Rightarrow (f(a), f(b), f(c)) \in \Delta_B$. Easily checks that $f(a^{\#}) = f(a)^{\#}$. A geometry morphism f is called strict if $(f(a), f(b), x) \in \Delta_B \Rightarrow \exists c \in A$ with $f(c) = x$ and $(a,b,c) \in \Delta$. A strict geometry morphism will be called a geometry homomorphism. A bijective geometry morphism whose inverse is also a morphism is a geometry isomorphism. If the morphism is a homomorphism, then bijective implies isomorphism. For a morphism f, $(x_1, x_2, ..., x_n) \in \Delta_n$ (for A) implies $(f(x_1), f(x_2) ..., f(x_n)) \in \Delta_n$ (for B). We simple write Δ for A or B letting the context distinguish.

2.5 Geometric Rings and Modules: Let $(R,0,\Delta)$ be an abelian geometry. Suppose a multiplication . is defined in R which is associative and $0. a = a. 0 = 0 \ \forall \ a \in R$. We call $(R, 0, \Delta, .)$ a geometric ring if $(a_1, a_2, a_3) \in \Delta$, $a \in R \Rightarrow (aa_1, aa_2, aa_3)$, $(a_1 a, a_2 a, a_3 a) \in \Delta$. The geometric ring is called strict if $(aa_1, aa_2, x) \in \Delta \Rightarrow x = aa_3$ with $(a_1, a_2, a_3) \in \Delta$ and similarly for $(a_1 a, a_2 a, x) \in \Delta$. An element 1 in R is the unit element if $a.1 = 1.a = a \ \forall \ a \in R$. If $ab = ba \ \forall \ a,b$, then it is a commutative geometric ring. It is called sharp if the geometry of R is sharp in which case one naturally gets a usual ring.

Let $(M, 0_M, \Delta_M)$ be an abelian geometry and R be a geometric ring with 1. Suppose R acts on M : $(r, x) \rightarrow rx \in M$. Then M is a (left) geometric module over R, if the following hold : (i) $a(bx) = (ab)x$; $0_R . x = 0_M = a . 0_M$ and $1.x = x \forall a,b \in R, x \in M$ (ii) $(a,b,c) \in \Delta_R$, $x \in M \Rightarrow (ax, bx, cx) \in \Delta_M$ and $a \in R$, $(x, y, z) \in \Delta_M \Rightarrow (ax, ay, az) \in \Delta_M$. We call the module strict if R is strict and $(ax, bx, w) \in \Delta_M$, $a,b \in R$, $x, w \in M \Rightarrow w = cx$ for some $c \in R$ with $(a,b,c) \in \Delta$. We simply write Δ for Δ_R or Δ_M. When R and M are sharp, one gets usual module over a ring. Conversely, every module over a ring is a strict geometric module. All geometric rings and modules will be strict.

Let M be a geometric module over R and $N \subset M$. Then N is a submodule of M if N is a subgeometry of M and $a \in R$, $x \in N \Rightarrow ax \in N$. Note that for $x \in M$, $Rx = (ax : a \in R)$ is a submodule for all x if M is strict. Also, for the strict module M and $x, y \in M$ the submodule $sp\,(x,y)$ generated by x and y is given by the set

$\{z : (z, ax, by) \in \Delta, a,b \in R\}$. More generally, if $x_1, x_2, \ldots, x_n \in M$, one uses induction to show that $sp(x_1, x_2, \ldots, x_n) = \{w \in M : (w, a_1x_1, a_2x_2, \ldots, a_nx_n) \in \Delta_{n+1}, a_i \in R\}$.

2.6 Homomorphisms and semi-homomorphisms : Let M and N be geometric R–modules and $\phi : M \mapsto N$. We call ϕ R-homomorphism if ϕ is a geometry homomorphism and $\phi(ax) = a\phi(x)\ \forall\ a \in R, x \in M$. If ϕ^{-1} is also a homomorphism, then ϕ is an R–isomorphism. Suppose $\sigma : R \mapsto R$ is a homomorphism of the geometric ring R. Then $\phi : M \mapsto N$ is a σ–semi-homomorphism if ϕ is a homomorphism of geometries and $\phi(ax) = \sigma(a)\phi(x)\ \forall\ a \in R, x \in M$. If ϕ is a geometry isomorphism and σ is a geometric automorphism of R, then ϕ is called σ–semi-isomorphism. Letting $M = N$, we get the set $SI_R(M) = SI(M)$, the set of all geometric semi-isomorphisms of M. It clearly forms a group with $I(M)$ the set of all R-isomorphisms as a subgroup. In particular, when R is a sharp geometric skewfield (so a usual skewfield), then $M = V$ is a vector space and $SI(V)$ is the group of all semi linear transformations.

Remarks: An element $x \in M$ is R–free iff $R \mapsto Rx$, $a \mapsto ax$ is an isomorphism of geometries. Thus for R–free x, $(ax, bx, cx) \in \Delta, a,b,c \in R \Rightarrow (a,b,c) \in \Delta$. In [2] it was assumed for every element $x \neq 0$, since R considered is a geometric skewfield.

3. Orbit Spaces of Free Modules

3.1 Modules over local rings and projectivities : Here we briefly recall facts about (free) modules over (local) rings and the associated group of projectivities ([5],[6]).

Let (R, m, k) be a commutative local ring with maximal ideal m and $k = R/m$ the quotient field. Let V be a free module over R of finite rank $n \geq 3$. Then mV is a submodule of V and V/mV is a k–vector space of dimension n. Let $\pi : V \mapsto V/mV$ be the natural map. Then $x_1, x_2, \ldots, x_n \in V$ form a R-basis for V, iff $\pi(x_1), \pi(x_2), \ldots, \pi(x_n)$ is a vector space basis for V/mV. Basis elements are linearly independent over R. If $x \in V$, x is unimodular if $\pi(x) \neq 0$. In this case x is R–free and Rx is a direct summand of V. Let $P(V) = \{Rx : x \text{ is unimodular}\}$, the set of all 1-dimensional direct summands of V. The elements (lines) of $P(V)$ form the points of the associated projective space. A projectivity is a map $\alpha : P(V) \to P(V)$ such that $Rx \subseteq Ry + Rz$ iff $\alpha(Rx) \subseteq \alpha(Ry) + \alpha(Rz)$. Given an automorphisms $\sigma : R \to R$ and σ-semi linear isomorphism $\phi : V \mapsto V$, it induces a projectivity $P(\phi) : P(V) \to P(V)$ given by $P(\phi)Rx = R\phi(x)$. The generalisation of fundamental theorem of projective geometry states that given a projectivity $\alpha : P(V) \to P(V)$, there exists a σ- semi linear $\phi : V \mapsto V$ with $P(\phi) = \alpha$. Let R^* denote the group of units of R ; $SI(V)$ the group of all semi linear isomorphisms of V and $SP(V)$, the group of all projectivities. Then one gets a surjective map $SI(V) \to SP(V), \phi \to P(\phi)$ whose kernel consists

of multiplication by unit scalars and so isomorphic to R^*. Thus there is an exact sequence.

$$1 \to R^* \to SI(V) \to SP(V) \to 1$$

giving $SI(V)/R^* \cong SP(V)$.

3.2 Orbit Space and the Semi-isomorphisms: Now, consider the free module V over R as above with R^* as the group of units. The R^* acts on R giving rise to the set of orbits $R/R^* = \overline{R}$. If $a \in R$, then $\overline{a} = \{\alpha a : \alpha \in R^*\}$ is the element of $\overline{R}$. $(\overline{a}, \overline{b}, \overline{c}) \in \Delta$ iff $a + \alpha b + \beta c = 0, \alpha, \beta \in R^*$ makes $\overline{R}$ into an abelian geometry with $\overline{a}^{\#} = \overline{a}$ $\forall \overline{a} \in \overline{R}$. Also, $\overline{a} . \overline{b} = \overline{ab}$ is well defined and makes $\overline{R}$ into a strict commutative geometric ring with $\overline{1}$. Similarly R^* acts on V and gives a geometry of orbits $\overline{V} = V/R^*$ on which the well defined action $\overline{a} . \overline{v} = \overline{av}$ makes it into a strict geometric module over $\overline{R}$. Here also $\overline{v}^{\#} = \overline{v}\ \forall \overline{v} \in \overline{V}$. If $x \in V$ is unimodular, we call the point $\overline{x}$ projective. Two projective points $\overline{x}$ and $\overline{y}$ are independent if x and y are independent in V. It easily checks that if $\overline{x}$ is projective, then $\overline{x} = \overline{y}$ iff $Rx = Ry$. So if $P(\overline{V}) = \{\overline{x} : \overline{x}$ is projective$\}$, then we have a bijection $\overline{x} \to Rx$ between $P(\overline{V})$ and $P(V)$, the set of one dimensional direct summands of V. Also, $(\overline{x}, \overline{y}, \overline{z}) \in \Delta \Rightarrow$ $Rx \subset Ry + Rz$. Conversely, $Rx \subset Ry + Rz \Rightarrow (\overline{x}, \overline{\alpha}\,\overline{y}, \overline{\beta}\,\overline{z}) \in \Delta$ where $\overline{\alpha} = \overline{1}$ or $\overline{\beta} = \overline{1}$. If $\overline{x}_1, \overline{x}_2, \ldots \overline{x}_r \in \overline{V}$ and $\overline{W}$ is the geometric submodule generated by the elements, then $\overline{W} = \{\overline{x} : (\overline{x}, \overline{a}_1\,\overline{x}_1, \ldots \overline{a}_i\,\overline{x}_r) \in \Delta_{r+1}, \overline{a}_i \in \overline{R}\}$ So if $e_1, e_2, \ldots e_n$ is a basis for V over R, then $\overline{e}_1, \overline{e}_2, \ldots \overline{e}_n$ is a basis of $\overline{V}$ over $\overline{R}$ in an obviously defined sense.

Now let $\phi : V \mapsto V$ be a σ-semi linear isomorphism of V so that $\sigma : R \mapsto R$ is an automorphism of the ring. Since every automorphism satisfies $\sigma(R^*) = R^*$, it induces a map $\overline{\sigma} : \overline{R} \to \overline{R}$ by $\overline{\sigma}(\overline{a}) = \overline{\sigma(a)}$ which is well defined. It checks that $\overline{\sigma}$ is a geometric automorphism of $\overline{R}$. Similarly ϕ induces a well defined map $\overline{\phi} : \overline{V} \mapsto \overline{V}$ by $\overline{\phi}(\overline{x}) = \overline{\phi(x)}$. $\overline{\phi}$ is a strict geometric homomorphism satisfying $\overline{\phi}(\overline{a}\,\overline{x}) = \overline{\sigma}(\overline{a})\overline{\phi}(\overline{x})$. Thus $\overline{\phi}$ is a geometric $\overline{\sigma}$-semi-isomorphism of $\overline{V}$. If $\phi_1 : V \mapsto V$ is another σ_1-semi linear isomorphism with $\overline{\phi} = \overline{\phi}_1$, then $\overline{\phi(x)} = \overline{\phi_1(x)}\ \forall x \in V$ easily checks to give $\phi = s\phi_1$ for some constant $s \in R^*$ and $\sigma_1 = \sigma$. Let $SI(V)$ be the group of all geometric semi-isomorphisms of $\overline{V}$ and $SI(V)$ as above. Then we have a map $\phi \mapsto \overline{\phi}$ from $SI(V)$ to $SI(\overline{V})$, a group homomorphism with kernel isomorphic to R^*. Thus we get an exact sequence.

$$1 \to R^* \to SI(V) \to SI(\overline{V})$$

For the exactness on the right, one gets the generalisation of the fundamental theorem as for $SP(V)$ [5]

3.3 Theorem : *Suppose V is a free module over a commutative local ring R of finite rank $n \geq 3$. Let R^* be the group of units of R and $\overline{V} = V/R^*$ be the geometric module of the orbits over the geometric ring $\overline{R}$. Suppose $\tau : \overline{V} \mapsto \overline{V}$ is a geometric η-semi-isomorphism of $\overline{V}$. Then there exists a σ-semi linear isomorphism : $V \mapsto V$ such that $\overline{\phi} = \tau$.*

Proof : Let $e_1, e_2, \dots e_n$ be a basis of V over R so that $\overline{e}_1, \overline{e}_2, \dots \overline{e}_n$ is a basis of $\overline{V}$ over $\overline{R}$. Then, $\tau(\overline{e}_1), \dots \tau(\overline{e}_n)$ is a basis for $\overline{V}$ over $\overline{R}$. For, if $\overline{y} \in \overline{V}$, $\exists\ \overline{x} \in \overline{V}$ with $\tau(\overline{x}) = \overline{y}$. Since $(\overline{x}, \overline{a}_1\overline{e}_1, \dots \overline{a}_n\overline{e}_n) \in \Delta$, $\overline{a}_i \in \overline{R}$, so $(\tau(\overline{x}), \eta(\overline{a}_1)\tau(\overline{e}_1), \dots \eta(\overline{a}_n)\tau(\overline{e}_n)) \in \Delta$, showing that $\tau(\overline{x}) = \overline{y}$ is in the span of $\tau(\overline{e}_1), \dots \tau(\overline{e}_n)$ and these have the right numbers. Let $\tau(\overline{e}_i) = \overline{f}_i$, $i = 1, \dots n$. We obviously have $(\overline{e}_1, \overline{e}_2, \overline{e_1 + e_2}) \in \Delta$, so $(\tau(\overline{e}_1), \tau(\overline{e}_2), \tau(\overline{e_1 + e_2})) \in \Delta$ i.e., $(\overline{f}_1, \overline{f}_2, \overline{f}) \in \Delta$ where $\tau(\overline{e_1 + e_2}) = \overline{f}$. Then $f = \alpha f_1 + \beta f_2$, $\alpha, \beta \in R^*$. By replacing αf_1 by f_1 and βf_2 by f_2, we still get $\tau(\overline{e}_i) = \overline{f}_i$, $i = 1,2$ and $\tau(\overline{e_1 + e_2}) = \overline{f} = \overline{f_1 + f_2}$. Doing this for every $i \geq 2$, we get a basis $f_1, f_2, \dots f_n$ of V such that $\tau(\overline{e_1 + e_i}) = \overline{f_1 + f_i}$, $\forall i \geq 2$. Now let $a \in R$. Clearly $(\overline{e}_1\ \overline{ae_2}, \overline{e_1 + ae_2}) \in \Delta$, so $(\overline{f}_1, \eta(\overline{a})\overline{f_2}, \overline{y}) \in \Delta$, $\overline{y} = \tau(\overline{e_1 + ae_2})$. If $\eta(\overline{a}) = \overline{b}$, it gives $(\overline{f}_1, \overline{bf}_2, \overline{y}) \in \Delta$ so $\exists\ s_1, s_2 \in R^*$ with $s_1 y = f_1 + s_2 b f_2$. Clearly for given a, $s_2 b$ is unique and so defines a map $\sigma : R \mapsto R$ by $\sigma(a) = s_2 b$ with the properties that $\tau(\overline{e_1 + ae_2}) = \overline{f_1 + \sigma(a) f_2}$ and $\eta(\overline{a}) = \overline{b} = \overline{\sigma(a)}$. Note that $\sigma(1) = 1$ and $\sigma(0) = 0$. Similarly choose any $i > 2$ and define $\sigma_1 : R \mapsto R$ by $\tau(\overline{e_1 + ae_i}) = \overline{f_1 + \sigma_1(a) f_i}$ and $\overline{\sigma_1(a)} = \eta(\overline{a})$. It will be shown that $\sigma = \sigma_1$.

First, the following two relations are easily checked:

(i) $\tau(\overline{e_1 + ae_2 + a'e_i}) = \overline{f_1 + \sigma(a) f_2 + \sigma_1(a') f_i}$ (ii) $\tau(\overline{ae_2 + e_i}) = \overline{\sigma(a) f_2 + f_i}$

For (i), take $(\overline{e_1 + ae_2 + a'e_i}, \overline{e_1 + ae_2}, \overline{a'e_i}), (\overline{e_1 + ae_2 + a'e_i}, \overline{e_1 + a'e_i}, \overline{ae_2}) \in \Delta$

If $\tau(\overline{e_1 + ae_2 + a'e_i}) = \overline{z}$, then applying τ gives $(\overline{z}, \overline{f_1 + \sigma(a) f_2}, \eta(\overline{a}')\overline{f}_i)$,

$(\bar{z}, \overline{f_1 + \sigma_1(a')f_i}, \eta(\bar{a})\bar{f}_2 \in \Delta$. Since $\eta(\bar{a}) = \overline{\sigma(a)}$, it gives $(\bar{z}, \overline{f_1 + \sigma(a)f_2}, \overline{\sigma(a')f_i}), (\bar{z}, \overline{f_1 + \sigma_1(a')f_i}, \overline{\sigma(a)f_2}) \in \Delta$. These give $z = s_1(f_1 + \sigma(a)f_2) + s_2(\sigma(a')f_1 = s_3(f_1 + \sigma_1(a')f_i) + s_4\sigma(a)f_2, s_i \in R^*$. By solving one gets $z = s_1(f_1 + \sigma(a)f_2 + \sigma_1(a')f_i)$ and hence (i). For (ii), we take $(\overline{ae_2 + e_i}, \overline{e_1 + ae_2 + e_i}, \bar{e}_1) \in \Delta$, proceed similarly using (i).

Now, we show σ is a homomorphism of rings. Let $a_1, a_2 \in R$. Using (i) and (ii) for $(\overline{e_1 + (a_1 + a_2)e_2 + e_i}, \overline{e_1 + a_1e_2}, \overline{a_2e_2 + e_i}) \in \Delta$, we get $(\overline{f_1 + \sigma(a_1 + a_2)f_2 + f_i}, \overline{f_1 + \sigma(a_1)f_2}, \overline{\sigma(a_2)f_2 + f_i}) \in \Delta$ giving $f_1 + \sigma(a_1 + a_2)f_2 + f_i + t_1(f_1 + \sigma(a_1)f_2) + t_2(\sigma(a_2)f_2 + f_i) = 0, t_1, t_2 \in R^*$. By linear independence, $t_1 = t_2 = -1$ and $\sigma(a_1 + a_2) = \sigma(a_1) + \sigma(a_2)$. Also we take $(\overline{e_1 + a_1a_2e_2 + a_1e_i}, \bar{e}_1, \bar{a}_1(\overline{a_2e_2 + e_i})) \in \Delta$ to get $(\overline{f_1 + \sigma(a_1a_2)f_2 + \sigma_1(a_1)f_i}, \bar{f}_1, \eta(\bar{a}_1)(\overline{\sigma(a_2)f_2 + f_i})) \in \Delta$. Since $\eta(\bar{a}_1) = \overline{\sigma(a_1)}$, it gives $f_1 + \sigma(a_1a_2)f_2 + \sigma_1(a_1)f_i + sf_1 + s'\sigma(a_1)(\sigma(a_2)f_2 + f_i) = 0, s, s' \in R^*$. Again by linear independence, $s = -1, \sigma_1(a_1) = -s'\sigma(a_1)$ and $\sigma(a_1a_2) = -s'\sigma(a_1)\sigma(a_2) = \sigma(a_1)\sigma(a_2)$. Since these are true for all a_1, a_2, taking $a_2 = 1$ gives $\sigma = \sigma_1$ and then $\sigma(a_1a_2) = \sigma(a_1)\sigma(a_2)$ follows. Thus, σ is a homomorphism of the ring. Using τ^{-1}, η^{-1} we can define σ^{-1} by $\tau^{-1}(\overline{f_1 + bf_2}) = \overline{e_1 + \sigma^{-1}(b)e_2}$, where σ^{-1} indeed gives the inverse of σ. Hence σ is an automorphism of R such that $\overline{\sigma(a)} = \eta(\bar{a})$ i.e., σ induces the geometric automorphism η.

Now, $\phi : V \to V$ is defined by $(a_1e_1 + \dots + a_ne_n) = \sigma(a_1)f_1 + \dots + \sigma(a_n)f_n$. ϕ is σ- semi-linear. To show that ϕ induces τ, it is necessary to show that $\tau(\overline{a_1e_1 + \dots + a_ne_n}) = \overline{\sigma(a_1)f_1 + \dots + \sigma(a_n)f_n}$. This is done as in [5] by first showing $\tau(\overline{e_1 + a_2e_2 + \dots + a_ne_n}) = \overline{f_1 + \sigma(a_2)f_2 + \dots + \sigma(a_n)f_n}$ and $\tau(\overline{a_2e_2 + \dots + a_ne_n}) = \overline{\sigma(a_2)f_2 + \dots + \sigma(a_n)f_n}$ For example, to show first equality, we use induction where we already have $\tau(\overline{e_1 + a_2e_2}) = \overline{f_1 + \sigma(a_2)f_2}$. Suppose $\tau(\overline{e_1 + a_2e_2 + \dots + a_ie_i}) = \overline{f_1 + \sigma(a_2)f_2 + \dots + \sigma(a_i)f_i}$ and. let $\tau(\overline{e_1 + a_2e_2 + \dots + a_{i+1}e_{i+1}}) = \bar{u}$. Then, we take $(\overline{e_1 + a_2e_2 + \dots + a_{i+1}e_{i+1}}, \overline{e_1 + a_2e_2 + \dots + a_ie_i}, \overline{a_{i+1}e_{i+1}}) \in \Delta$ to get $(\bar{u}, \overline{f_1 + \sigma(a_2)f_2 + \dots + \sigma(a_i)f_i}, \overline{\sigma(a_{i+1})f_{i+1}} \in \Delta$ and $(\overline{e_1 + a_2e_2 + \dots + a_{i+1}e_{i+1}}, \overline{e_1 + a_{i+1}e_{i+1}},$

$\bar{a}_2\,\bar{e}_2, \ldots \bar{a}_i\,\bar{e}_i) \in \Delta_{i+1}$ to get $(\bar{u}, \overline{f_1 + \sigma(a_{i+1})\, f_{i+1}}, \overline{\sigma(a_2) f_2}, \ldots \overline{\sigma(a_i) f_i}\ ) \in \Delta_{i+1}$.

These give two relations

$$u = s_1(f_1 + \sigma(a_2) f_2 + \ldots + \sigma(a_i) f_i) + s_2\ \sigma(a_{i+1})\ f_{i+1} \quad \text{and}$$

$$u = s_1'(f_1 + \sigma(a_{i+1})\ f_{i+1}) + s_2'\ \sigma(a_2)\, f_2 + \ldots + s_i'\ \sigma(a_i)\, f_i\,.$$

Solving these, we get the required. Similarly, the second easily checks. For the final result, one can even follow [5]. Thus the theorem is completely proved.

Using the surjectivity given by this theorem, we complete the exact sequenc

$$1 \to R^* \to SI(V) \to \mathrm{SI}(\bar{V}) \to 1.$$

It gives $SI(V)/R^* \cong SI(\bar{V})$. Together with $SI(V)/R^* \cong SP(V)$, we get

3.4 Theorem : *Let V be a free module of finite rank $n \geq 3$ over a commutative local ring R with R^* as the group of units and $SP(V)$ the associated group of projectivities. Let $SI(\bar{V})$ be the group of all geometric semi-isomorphisms of the geometric space of orbits $\bar{V} = V/R^*$ over the geometric ring $\bar{R} = R/R^*$.Then $SI(\bar{V}) \cong SP(V)$.*

Remark (1). Here the ring considered is local as in [5]. However the generalized fundamental theorem is available for arbitrary commutative rings [6] and even over some special non-commutative rings [7].The above may go through for commutative case but one must check how far it works for non-commutative rings. Also, one could take different spaces V and W over rings R and R' with an isomorphism $\sigma : R \mapsto R'$ as in [6].

(2).We believe that a reformulation of the generalized fundamental theorem in the terminology of an orbit space geometry (i.e. Pasch geometry) have advantages. Since, then, one could try to obtain the representation of an arbitrary geometry as an orbit space as done in classical situation, for example, an abstract projective space of high enough dimension is given by the orbit space V/F^* of a vector space over the skewfield F under the action of F^*. If an abstract space is a geometric space over a geometric skewfield, we have the generalisation that it is given as an orbit space V/Γ of vector space over F by the action of a normal subgroup Γ of F^*. On this line, one could ask: given an abstract geometric module W over a (commutative) geometric ring A, when does there exist a (free) module V over a (commutative) ring R with the group of units R^* such that V/R^* over R/R^* be (semi) isomorphic to W over A.

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Post-Widder Inversion Operator Of Generalized Functions

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Abstract: A.H. Zemanian and many others have extended Laplace transformation to distributions considering the Post-Widder inversion operator as a differential operator. In this paper, the Post-Widder inversion operator itself has been extended to generalized functions.

1. Introduction

Since L.Schwartz [11] published '*Transformation de Laplace des distributions*' in 1952, various integral transformations have been extended to distributions and generalized functions. But it seems very little works are being done in the extension of the real inversion operator to distributions and generalized functions. In this paper, we present some works in this direction.

If the Post-Widder inversion operator $L_{k,t}[f]$ ([15], p. 280–283) is given by

$$(1.1)\qquad L_{k,t}[f] = \frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} \int_0^\infty e^{-ku/t}\, u^k F(u)\, du \qquad t > 0,\ k = 0, 1, \ldots$$

where

$$(1.2)\qquad f(x) = \int_0^\infty e^{-xt}\, F(t)\, dt \qquad x > 0,$$

we define as in ([6],[7],[8]) a real inversion operator in the sense of Rooney [10] by

$$(1.3)\qquad P_{n,t}^{(\beta,\nu)}\left[L_{k,t}^{(\cdot)}[f]\right] = A(k,n) \int_0^\infty x^{\upsilon}\, Q_{\upsilon\,[1+2x^{-1}]}^{(0,\beta+2n)}\, L_{k,t}^{(x)}[f]\, dx \qquad n = 0,1,2,3 \quad \upsilon \neq 0,$$

where

$$(1.4)\qquad L_{k,t}^{(x)}[f] = \frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} \int_0^\infty e^{-kux/t}\, u^k\, F(u)\, du,$$

$A(k,n)$ is given by

$$(1.5)\qquad \frac{\Gamma(\upsilon+1)\, A(k,n)}{2(k-1)!} = \frac{\Gamma(\beta+\upsilon+2n+1)}{[\Gamma(k-\upsilon-1)]^2\, \Gamma(\beta+2\upsilon+2n-k+2)} = B(k,n)$$

and $Q_{\upsilon}^{(\alpha,\beta)}[x]$ is the Jacobi function of the second kind ([3],p.170) and defined by

$$(1.6)\quad Q_{\upsilon}^{(\alpha,\beta)}[x]=\frac{2^{\alpha+\beta+\upsilon}\Gamma(\alpha+\upsilon+1)\Gamma(\beta+\upsilon+1)}{\Gamma(\alpha+\beta+2\upsilon+2)(x-1)^{\alpha+\upsilon+1}}(1+x)^{-\beta}\ {}_2F_1\left[\begin{matrix}\upsilon+1,\alpha+\upsilon+1, \\ \alpha+\beta+2\upsilon+2;\end{matrix}\ 2(1-x)^{-1}\right]$$

It has been shown in [6], under certain conditions, that

$$(1.7)\qquad \lim_{n\to\infty} P_{n,t}^{(\beta,\upsilon)}\ [L_{k,t}^{(\cdot)}[f]]=F(t)$$

for all $t>0$.

The purpose of the present paper is to extend the inversion formula (1.7) to a class of generalized functions interpreting the convergence in the weak distributional sense.

2. Testing Functions Space

In this paper, we adopt Zemanian's method [18] to construct testing functions space $\mathcal{J}_{a,b,n}$. The dual space $\mathcal{J}'_{a,b,n}$ consists of the Post-Widder inversion operator transformable generalized functions to which (1.7) will be extended in the sense of weak distributions.

Let I denote an open interval on the real line. A function is said to be infinitely smooth or simply smooth on I if it has continuous derivatives of all orders at all points of I. Let $\mathcal{D}(I)$ be the space of all infinitely smooth complex valued functions having compact support in I. we assign to $\mathcal{D}(I)$ the customary topology that makes the dual space $\mathcal{D}'(I)$ the space of Schwartz's distributions [12].

For any two real numbers a and b with $0<a<b$, n and k, non-negative integers such that $n\geq k$, and u a real variable, define

$$(2.1)\qquad k_{a,b,n}(u)=\begin{cases}(2e)^{-2n^2}\ e^{-(\frac{2n^2b}{k}-a)u}\ u & 0<u<\infty \\ (2e)^{-2n^2}\ e^{(\frac{2n^2b}{k}+a)bu}\ u & -\infty<u<0\end{cases}$$

and

$$(2.2)\qquad k_{a,b,n}(u)=\begin{cases}e^{au} & 0<u<\infty \\ e^{bu} & -\infty<u<0\end{cases}$$

Clearly, $k_{a,b,n}(u)$ is an infinitely smooth functions for all u in $-\infty<u<\infty$ and has the property that, if $a\leq c<d\leq b$ and $m\leq n$, $0<k_{a,b,n}(u)/k_{c,d,m}(u)\leq 1$.

Define $\mathcal{J}_{a,b,n}$ to the space of all infinitely smooth complex valued functions $\phi(u)$ such that

(2.3) $$\gamma_{a,b,n}(\phi)=\gamma_n(\phi)=\sup_{-\infty<u<\infty}\left|K_{a,b,n}(u)\,u^n\,D_u^n\,\phi(u)\right|,\ D_u=\frac{d}{du}$$

is bounded for all u and n, and tends to zero as $n \to \infty$.

Clearly, $\mathcal{J}_{a,b,n}$ is a linear space under the customary definition of additions and multiplication by a complex number, the zero element being the identically zero function. The topology of $\mathcal{J}_{a,b,n}$ is generated by a system of semi-norms $\{\gamma_n\}$. Since γ_0 is a norm, the system is separating ([18], p.8), thereby making it countably multinormed space.

If $\phi(v, t)$ is defined over the euclidean space of the variables v and t, and if it is smooth with respect to (v, t), we denote, in particular, by $\mathcal{J}_{a,b,n,v}$ and $\mathcal{J}_{a,b,n,t}$ the classes of all complex valued functions $\phi(v, t)$ which are differentiable with respect to v and t and satisfy the seminorm conditions (2.3) with D_u^n replaced by D_v^n and D_t^n respectively. If $\mathcal{J}_{a,b,n}$ is in $\mathcal{J}_{a,b,n,v}$ and $\mathcal{J}_{a,b,n,t}$ both, we denote it by $\mathcal{J}_{a,b,n,t,v}$ Similarly, $\mathcal{J}'_{a,b,n,v}$, $\mathcal{J}'_{a,b,n,t}$ and $\mathcal{J}'_{a,b,n,v,t}$ will denote the corresponding spaces of generalized functions.

We say that a sequence $\{\phi_\upsilon\}$ of testing functions converges to ϕ in $\mathcal{J}_{a,b,n}$ if for each $\phi_\upsilon \in \mathcal{J}_{a,b,n}$ and for every n, $\gamma_n(\phi_\upsilon-\phi) \to 0$ as $\upsilon \to \infty$. By (2.3), the sequence $\{k_{a,b,n}(u)\,u^n D_u^n \phi_\upsilon(u)\}$ comprises a uniform Cauchy sequence on $-\infty < u < \infty$. Then by ([18], lemma 3.8.1), $\mathcal{J}_{a,b,n}$ is complete and therefore Frechet space.

We observe that $\mathcal{D}(I)$ is a subspace of $\mathcal{J}_{a,b,n}$ .Therefore, the topology of $\mathcal{D}(I)$ is stronger than the topology induced on $\mathcal{D}(I)$ by $\mathcal{J}_{a,b,n}$ and as such the restriction of any member of $\mathcal{J}_{a,b,n}$ to $\mathcal{D}(I)$ is in $\mathcal{D}'(I)$.

If $a \le c < d \le b$, the inequality

$$\left|K_{a,b,n}(u)\,u^n\,D_u^n\,\phi(u)\right| \le \left|K_{c,d,n}(u)\,u^n\,D_u^n\,\phi(u)\right|$$

implies $\gamma_{a,b,n}(\phi) \le \gamma_{c,d,n}(\phi)$ from which and by ([18], lemma 1.6.3) it follows that $\mathcal{J}_{c,d,n} \subseteq \mathcal{J}_{a,b,n}$.

Throughout the paper we assume that the parameters β and υ are real and satisfy $-3 < \upsilon < -2$ and $-1 < \beta + \upsilon < 0$ for some positive β

3. Generalized Post-Widder Inversion Operator

Let $F \in \mathcal{J}'_{a,b,n}$ for any two real numbers a and b with $0 < a < b$. We define the Post–Widder inversion operator of generalized functions as an application of F to the kernel function.

$$\frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} e^{-ku/t}\, u^k$$

by the equation

$$L_{k,t}[f] = \left\langle F(u),\ \frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} e^{-ku/t}\, u^k \right\rangle. \tag{3.1}$$

Then we say that F is the Post-Widder inversion operator transformable generalized function or the $L_{k,t}[f]$–transformable generalized function.

If $F(u)$ is locally integrable function such that $\int_0^\infty \left|F(u)/k_{a,b,n}(u)\right| du < \infty$, $F(u)$ generates a regular member of $\mathcal{J}'_{a,b,n}$ and

$$L_{k,t}[f] = \int_0^\infty \frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} e^{-ku/t}\, u^k\, F(u)\, du. \tag{3.2}$$

In this section we establish certain results concerning the testing functions space $\mathcal{J}_{a,b,n}$ and its dual space $\mathcal{J}'_{a,b,n}$

Lemma 3.1 and lemma 3.2 are simple and straightforward and can be easily verified and hence their proofs are omitted.

Lemma 3.1. For each pair of real variables u and t, and k, a non-negative integer, define

$$h_k(u,t) = \frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} e^{-ku/t}\, u^k. \tag{3.3}$$

Then for any non-negative integer n

(a) $(tD_t)^n\ h_k(u,t) = (-uD_u - 1)^n\ h_k(u,t)$

(b) $(-tD_t - 1)^n\ h_k(u,t) = (uD_u)^n\ h_k(u,t)$

(c) $t^n D_t^n\, h_k(u,t) = (-1-uD_u)(-2-uD_u)\ldots(-n-uD_u)\ h_k(u,t)$.

Lemma 3.2. For any arbitrary infinitely differentiable function ϕ

(a) $(tD_t)^n\, t^n\ \phi(t) = t^n\,(n + tD_t)^n\ \phi(t) = t^n\, Q_n(tD_t)\,\phi(t)$

(b) $Q_n(tD_t)\ P_n(-tD_t - 1)\ \phi(t) = P_n(-tD_t - 1)\ Q_n(tD_t)\,\phi(t)$,

where P_n and Q_n are polynomials of degree n.

Lemma 3.3. Let $h_k(u,t)$ be defined by (3.3). For any two real numbers a and b with $0 < a < b$, define Ω_t as the set $\{t : a < k/t < b\}$. Then for $n \ge k$, $h_k(u,t)$ belongs to $\mathcal{J}_{a,b,n}$ for every t in Ω_t and all u in $-\infty < u < \infty$.

Proof: Leibniz's rule of differentiation yields

$$u^n D_u^n h_k(u,t) = u^n D_u^n \left[\frac{1}{k!} \left(\frac{k}{t}\right)^{k+1} e^{-ku/t} u^k \right]$$

$$= \sum_{p=0}^{k} C_{n,k}(p) (ku/t)^{n+k-p+1} e^{-ku/t} u^{-1},$$

where

$$C_{n,k}(p) = \binom{n}{p} \frac{k(k-1)\dots(k-p+1)}{k!} (-1)^{n-p} \qquad p \le k$$

Then, if $u > 0$, $b > k/t$ and $n \ge k$, it is easy to see that

$$|k_{a,b,n}(u) u^n D_u^n h_k(u,t)|$$

$$\le 2^{-n} e^{-2n(nbu/k+1)} \sum_{p=0}^{k} C_{n,k}(p) (ku/t)^{n+k-p+1} e^{-(k/t-a)u}$$

$$\le e^{-2n(nbu/k+1)} (nu/t+1)^{2n} e^{-(k/t-a)u} (ku/t+1)$$

$$(3.4) \qquad < e^{-2n[(nu/t+1)-\log(nu/t+1)]} (ku/t+1) e^{(k/t-a)u},$$

where we have used the following inequality

$$2^{-n} \sum_{p=0}^{k} |C_{n,k}(p)| \le 1 \qquad k \le n$$

Clearly, the right hand side of (3.4) is bounded for all $u, t > 0$ and all n, and tends to zero as $n \to \infty$ for each t in Ω_t.

Proceeding similarly it can be shown that the lemma holds for $u < 0$. This proves the lemma.

Corollary 3.3. If $h_k(u,t)$ and Ω_t are defined as in lemma 3.3, $h_k(h,t)$ belongs to $\mathcal{J}_{a,b,n,t}$ for every t in Ω_t and for all u in $-\infty < u < \infty$

Using the identity (c) of lemma 3.1, we have

$$|k_{a,b,n}(u) t^n D_k^n h_k(u,t)|$$

$$= \left| k_{a,b,n}(u) \prod_{r=1}^{n} (-r - uD_u) h_k(u,t) \right|$$

$$\le \sum_{r=0}^{n} A_r(n) k_{a,b,n}(u) |(uD_u)^{n-r} h_k(u,t)|$$

$$= \sum_{r=0}^{n} \{ A_{n-r}(n) \frac{k_{a,b,n}(u)}{k_{a,b,r}(u)} \} \left| k_{a,b,r}(u)(uD_u)^r h_k(u,t) \right|$$

where $A_r(n)$ is the sum of the combinations of the first n positive integers by taking $r\,(r \leq n)$ at a time, $r = 0,1,2 \ldots n$ and $A_0(n) = 1$.

Since $n^{n-r} < 2^n (n-r)^n$ for $r < n$, we see that

$$A_{n-r} \frac{k_{a,b,n}(u)}{k_{a,b,r}(u)} < (2n-2r)^n (2e)^{-2(n^2-r^2)} e^{-2b(n^2-r^2)u/k} \qquad r<n$$

$$= 1 \qquad r=n$$

Then for every t in Ω_t

$$A_{n-r} \frac{k_{a,b,n}(u)}{k_{a,b,r}(u)} < (2n-2r)^n (2e)^{-2(n-r)} e^{2(n^2-r^2)u/t} \qquad r<n$$

$$= 1 \qquad r=n$$

Therefore,

$$\left| k_{a,b,n}(u)\, t^n D_t^n h_k(u,t) \right| < \left| k_{a,b,n}(u)\,(uDu)^n h_k(u,t) \right| + 2^{-2(n-r)} \sum_{r=0}^{n-1} \left| k_{a,b,r}(u)(uD_u)^r h_k(u,t) \right|$$

Now in view of ([9], lemma 1) and lemma 3.3, it is clear that each term on the right hand side is bounded for all u in $-\infty < u < \infty$, all n and tends to zero as $n \to \infty$. This establishes the corollary.

Lemma 3.4. If $\upsilon + 1$ is not a negative integer and if

$$M_{n,t}[\theta(u)] = u^{-k-n} D_u^{-n} u^{\upsilon+n+1} D_u^n u^{k-\upsilon-n-1} \theta(u)$$

and

$$\theta(t,x) = x^{-1} \theta(t/x),$$

then

(3.5) $$M_{n,t}[\theta(t,x)] = C(k,n)\, x^{-1} (t/x)^{-k+\upsilon} Q_{\upsilon+n}^{(0,\beta)} [1+2(t/x)^{-1}]$$

where

(3.6) $$\theta(t,x) = C(k,n)\, x^{-1} (t/x)^{-k+n+\upsilon} Q_{\upsilon}^{(n,\beta)} [1+2(t/x)^{-1}],$$

and

(3.7) $$C(k,n) = \frac{A(k,n)\,\Gamma(\upsilon+1)\,\Gamma(\beta+\upsilon+n+1)}{\Gamma(\upsilon+n+1)\,\Gamma(\beta+\upsilon+2n+1)}$$

with $A(k,n)$ is given by (1.5). Further, let $-3 < \upsilon < -2$ and $\beta + \upsilon + 1 > 0$ for some $\beta > 0$. Then for all $n \geq k$ and all $n \geq 3$ the function

(3.8) $$E_n(u, t) = \int_0^\infty M_{n,t}[\theta(t, x)]\, h_k(u, x)\, dx$$

belongs to $\mathcal{J}_{a,b,n}$ for every t in Ω_t and all u in $-\infty < u < \infty$.

Proof: Let

(3.9) $$h_k^{(v)}(u,t) = \frac{1}{k!}\,(\frac{k}{t})^{k+1}\, e^{-kuv/t}\, u^k.$$

Then using (3.5), after a simple change of variable, (3.8) becomes

$$E_n(u,t) = \int_0^\infty C(k,n)\, \mathrm{v}^\upsilon\, Q_{\upsilon+n}^{(\sigma,\beta)}\, [1 + 2\mathrm{v}^{-1}]\, h_k^{(v)}(u,t)\, d\mathrm{v}$$

$$= (\int_0^1 + \int_1^\infty) \ldots$$

$$= I'(u,t) + I''(u,t).$$

To prove the lemma, it suffices to show that each $I'(u,t)$ and $I''(u,t)$ belongs to $\mathcal{J}_{a,b,n}$. Clearly.

$$\left| I''(u,t) \right| \le h_k(u,t)\, C(k,n) \int_1^\infty \left| \mathrm{v}^\upsilon\, Q_{\upsilon+n}^{(\sigma,\beta)}\, [1 + 2\mathrm{v}^{-1}] \right| d\mathrm{v}.$$

If $-3 < \upsilon < -2$, $\beta + \upsilon + 1 > 0$ for some $\beta > 0$ and $n \ge 3$, Euler's representation of hypergeometric function ([2], p.114) gives

$${}_2F_1\left[\begin{matrix} \upsilon+n+1, & \upsilon+n+1, \\ \beta+2\upsilon+2n+2; & -\mathrm{v}\end{matrix}\right] > 0$$

(3.10)

$$< \frac{\Gamma(\beta + 2\upsilon + 2n + 2)}{\Gamma(\upsilon + n + 1)\, \Gamma(\beta + \upsilon + n + 1)}\, \mathrm{v}^{-n-\upsilon}$$

for all $\mathrm{v} \ge 0$.

Then by (1.6) with (3.10), it follows that

$$k_{a,b,n}(u)\, u^n\, D_u^n\, I''(u,t)$$

$$< k_{a,b,n}(u)\, u^n\, D_u^n\, h_k(u,t) \int_1^\infty c(k,n)\, \mathrm{v}^{\beta+\upsilon+1}\, (1+\mathrm{v})^{-\beta}\, d\mathrm{v},$$

where the integral on the right hand side is bounded. Then by lemma 3.3 it is clear that $I''(u,t)$ belongs to $\mathcal{J}_{a,b,n}$ for every t in Ω_t and all u in $-\infty < u < \infty$.

Next, $\quad k_{a,b,n}(u)\ u^n\ D_u^n\ I'(u,t)$

$$= C(k,n)\ K_{a,b,n}(u)\ u^n \int_0^1 v^{\upsilon}\ Q_{\upsilon+n}^{(o,\beta)}\ [1+2v^{-1}]$$

$$\times \sum_{p=0}^{k} C_{n,k}(p)\ v^{n-p}\ (ku/t)^{n+k-p+1}\ e^{-kuv/t}\ dv$$

$$= C(k,n)\ K_{a,b,n}(u) u^{-1} \sum_{p=0}^{k} C_{n,k}(p)\ (ku/t)^{n+k-p+1}$$

$$\times \int_0^1 e^{-kuv/t}\ v^{n-p+\upsilon}\ Q_{\upsilon+n}^{(0,\beta)}\ [1+2v^{-1}]\ dv$$

Let $u > 0$. The use of the inequality of the proof of lemma 3.3 and the relations (3.10) yield

$$k_{a,b,n}(u)\ u^n\ D_u^n\ I'(u,t)$$

$$< u^n\ k_{a,b,n}(u) u^{-1}\ (ku/t+1)^{2n+1}\ C(k,n) \int_0^1 e^{kuv/t}\ v^{\beta+\upsilon+1}\ dv$$

$$< e^{-2n[(nu/t+1)-\log(nu/t+1)]}\ (ku/t+1)\ e^{-(k/t-a)u}$$

$$\times \frac{C(k,n)}{(\beta+\upsilon+2)}\ {}_1F_1\,[1,\beta+\upsilon+3;\,ku/t]$$

Since ${}_1F_1\ [1,\beta.+\upsilon+3;\ ku/t]$ converges by ([13], p.2) and $C(k,n)$ is bounded. Lemma 3.3 shows that $I'(u,t)$ belongs to $\mathcal{J}_{a,b,n}$ .The case for $u<0$ follows similarly. The lemma is thus completely proved.

Lemma 3.5. Let $\phi(u) \in \mathcal{J}_{a,b,n}$ and let $k_{a,b,n}(u)$ be defined by (2.1) and (2.2). Then for $a \le c < d \le b$ and $m \le n$, $k_{c,d,m}(u)$ belongs to $\mathcal{J}_{a,b,n}$. If $\{\phi_\upsilon\}$ converges to zero in $\mathcal{J}_{a,b,n}$ the sequence $\{k_{c,d,m}(u)\ \phi_\upsilon(u)\}$ converge to zero functions in $\mathcal{J}_{a,b,n}$.

The proof follows easily from the following identity

$$k_{a,b,n}(u) u^n\ D_u^n\ k_{c,d,m}(u)\ \phi_\upsilon(u)$$

$$= \sum_{p=0}^{n} [\,K_{a,b,p}(u) u^p\ D_u^p \phi_\upsilon(u)]\ [\frac{k_{a,b,n}(u) u^{n-p}\ D_u^{n-p}\ k_{c,d,m}(u)}{k_{a,b,p}(u)}].$$

Since for every p, the quantity inside the first brace converges to zero and the quantity inside the second brace is always bounded.

Lemma 3.6. If $\phi \in \mathcal{J}_{a,b,n}$ and $F \in \mathcal{J}'_{a,b,n}$ then $F(u)/k_{a,b,n}(u)$ belongs to $\mathcal{J}'_{a,b,n}$.

It is a direct consequence of the lemma 3.5.

Lemma 3.7. Let $F \in \mathcal{J}'_{a,b,n}$ and let Ω_t be defined as in lemma 3.3. Then the Post-Widder inversion operator

$$L_{k,t}[f] = \langle\, F(u),\, h_k(u,t)\,\rangle \tag{3.11}$$

is infinitely smooth and belongs to $\mathcal{J}_{a,b,n,t}$ for every t in Ω_t, and for any non-negative integer m

$$D_t^m L_{k,t}[f] = \langle\, F(u),\, \frac{\partial^m}{\partial t^m} h_k(u,t)\,\rangle \tag{3.12}$$

and

$$\begin{aligned} L_{k,t}[f] &= o(t^{-1} e^{-\gamma/t}) & t \to 0 \\ &= 0(t^{-1}) & t \to \infty \end{aligned} \tag{3.13}$$

where γ is a positive number.

Proof: Since $F \in \mathcal{J}'_{a,b,n}$ and $h_k(u,t) \in \mathcal{J}_{a,b,n}$ for every t in Ω_t, (3.11) has sense For some fixed t is Ω_t, consider

$$\frac{L_{k,t+\Delta t}[f] - L_{k,t}[f]}{\Delta t} - \langle\, F(u),\, \frac{\partial}{\partial t} h_k(u,t)\,\rangle = \langle\, F(u),\, \varphi_t(u)\,\rangle \tag{3.14}$$

where

$$\begin{aligned} \varphi_t(u) &= \frac{h_k(u,t+\Delta t) - h_k(u,t)}{\Delta t} - \frac{\partial}{\partial t} h_k(u,t) & t \neq 0 \\ &= 0 & t = 0 \end{aligned} \tag{3.15}$$

We shall first show that as $\Delta t \to 0$, $\varphi_t(u)$ converges in $\mathcal{J}_{a,b,n,u}$ to zero. By arguments similar to those given in ([17], p.112), it readily follows that $\varphi_t(u)$ converges uniformly to zero over every finite u interval as $\Delta t \to 0$. A similar argument shows that $\varphi_t^{(n)}(u)$ converges uniformly to zero over every finite u interval as $\Delta t \to 0$.

To prove the above assertion we must show that $k_{a,b,n}(u)\ u^n D_u^n\ \varphi_t(u)$ is bounded for all u in $-\infty < u < \infty$ and for Δt in any finite interval in Ω_t tends to zero as $n \to \infty$. Indeed, rewriting (3.15) as

$$\varphi_t(u) = \frac{1}{\Delta t} \int_t^{t+\Delta t} D_y\ h_k(u,t)\,dy - D_t\,h_k(u,t)$$

$$= \frac{1}{\Delta t} \int_{t}^{t+\Delta t} dy \int_{t}^{y} D_z^2 \, h_k(u, z) dz$$

we see that

$$\left| k_{a,b,n}(u) u^n D_u^n \varphi_t(u) \right| \leq |\Delta t| \sup_{z \in [t, t+\Delta t]} \left| k_{a,b,n}(u) u^n D_u^n D_z^2 h_k(u, z) \right|$$

which, in view of the fact that for any non-negative integer N

$$k_{a,b,n}(u) u^{N+n} \; D_u^n \; h_k(u, t)$$

is bounded for each t in Ω_t and for all u in $-\infty < u < \infty$,establishes the assertion as $\Delta t \to 0$.

Then since $F \in \mathfrak{I}'_{a,b,n}$ the right hand side of (3.14) is zero and we get

$$D_t \; L_{k,t}[f] = \langle \; F(u), \frac{\partial}{\partial t} \; h_k(u, t) \; \rangle$$

Upon repeated application of this argument, it is clear that $L_{k,t}[f]$ is infinitely smooth and that (3.12) holds.

Now we shall show that $L_{k,t}[f] \in \mathfrak{I}_{a,b,n,t}$. For this,set

$$\lambda_t(u) = k_{a,b,n}(u) t^n \; D_t^n \; h_k(u, t).$$

As t varies through all real values in Ω_t, an infinite set $\{ \lambda_t(u) \}$ of testing functions in $\mathfrak{I}_{a,b,n,u}$ is generated. Since $h_k(u,t) \in \mathfrak{I}_{a,b,m,u,t}$ for some fixed t in Ω_t and by lemma 3.6, $F(u)/\, k_{a,b,n}(u)$ is in $\mathfrak{I}'_{a,b,n,u}$,

$$k_{a,b,n}(t) \;\; t^n \;\; D_t^n \;\; L_{k,t}[f]$$
$$= k_{a,b,n}(t) \; \langle \; F(u) / k_{a,b,n}(u) \, , k_{a,b,n}(u) \; t^n \;\; D_t^n \;\; h_k(u, t) \rangle$$

has sense. Then it follows from the boundedness property of generalized functions that there exist a constant C and fixed positive number r such that

$$\begin{aligned} \left| k_{a,b,n}(t) t^n D_t^n L_{k,t}[f] \right| &= k_{a,b,n}(t) \; \left| \langle F(u)/k_{a,b,n}(u), \lambda_t(u) \rangle \right| \\ &\leq C \; k_{a,b,n}(t) \sup_{-\infty < u < \infty} \left| (1+u^2)^r \, \lambda_t^{(r)}(u) \right| \\ &< C \; N_{nr} \; , \end{aligned}$$

where N_{nr} is a constant independent of t and u, from which it readily follows that $L_{k,t}[f]$ belongs to $\mathfrak{I}_{a,b,n,t}$.

Finally, an application of the boundeness property of generalized functions on $L_{k,t}[f]$ yields (3.13). This completes the lemma.

4. Main Result

Theorem 4.1. Let F be a $L_{k,t}[f]$– tranformable generalized function and let

$$(4.1) \qquad W_{n,t}[\theta(t,x)] = x^{-1}\ u^{n-k+\upsilon+1}\ D_u^n\ u^n\ D_u^n\ u^{-1-\upsilon}\ D_u^{-n}\ u^{n+\upsilon+1}\ D_u^n\ u^{k-n-1-\upsilon}\ \theta(u)\Big|_{u=t/x},$$

where $\theta(t,x) = x^{-1}\ \theta(t/x)$ and $\theta(u)$ is given by

$$(4.2) \qquad \theta(u) = C(k,n)\ u^{n-k+\upsilon}\ Q_\upsilon^{(n,\beta)}[1+2u^{-1}]$$

be an integro-differential operator. Further, let $-3 < \upsilon < -2$, $\beta + \upsilon + 1 > 0$ for some $\beta > 0$, $n \geq 4$ and $n \geq k$. Then in the sense of convergence in $\mathcal{J}'_{a,b,n}$

$$F(t) = \lim_{n\to\infty} p_{n,t}^{(\beta,\upsilon)}\ [L_{k,t}^{(\cdot)}[f]]$$

for every t in Ω_t. That is, for any $\phi \in \mathcal{B}(I)$

$$\langle\ F(t),\ \theta(t)\ \rangle = \lim_{n\to\infty}\ \langle\ p_{n,t}^{(\beta,\upsilon)}\ [L_{k,t}^{(\cdot)}[f]],\ \theta(t)\ \rangle.$$

First we need the following result.

Lemma 4.1. Let

$$(4.3) \qquad S_n(u,t) = \int_0^\infty W_{n,t}\ [\theta(t,x)]\ h_k(u,x)\,dx,$$

where

$$(4.4) \qquad W_{n,t}\ [\theta(t,x)] = A(k,n)\ x^{-1}\ (t/x)^{\upsilon-k}\ Q_\upsilon^{(0,\beta+2n)}\ [1+2(t/x)^{-1}]$$

with $A(k,n)$ as given by (1.5) and $\theta(t,x)$ as defined in theorem 4.1. If $-3 < \upsilon < -2$, $-1 < \beta + \upsilon < 0$ for some $\beta > 0$ and $n \geq 4$. then

$$(4.5) \qquad S_n(u,t) = \frac{B(k,n)}{ku} \int_0^\infty \frac{(ku/t)^{\upsilon-k}\ e^{-kuy/t}\ y^{\beta+\upsilon+2n}}{(1+y)^{\beta+\upsilon+2n+1}}\ dy$$

and

$$(4.6) \qquad \int_0^\infty S_n(u,t)\ dt = 1.$$

Proof: After a simple change of variable (4.3) becomes,

$$S_n(u,t) = A(k,n) \int_0^\infty \mathrm{v}^\upsilon\ Q_\upsilon^{(0,\beta+2n)}\ [1+2\,\mathrm{v}^{-1}]\ h_k^{(\mathrm{v})}(u,t)\,d\mathrm{v}$$

which on evaluating by the known result ([14, p.205) yields

$$S_n(u,t) = \frac{B(k,n)}{ku\Gamma(B+\upsilon+2n+1)} (ku/t)^{-(\beta+2\upsilon+2n-k+1)} E\left[{}^{\beta+\upsilon+n+1}_{\beta+\upsilon+n+1} :: ku/t\right]$$

where E is MacRobert function ([5], p.203).The use of integral representation of E function gives (4.5) and then (4.6) follows easily from (4.5).

Proof of theorem 4.1. If $\theta(u)$ is given by (4.2), it can be readily verified by formulae ([2], p.102–103) that the integro–differential operator $W_{n,t}[\theta(t,x)]$ defined by (4.1) is the relation given by (4.4). Moreover, if $-3 < \upsilon < -2$, $-1 < \beta + \upsilon < 0$ for some $\beta > 0$ and some fixed $n > k$ and $n \geq 4$,

$$(4.7) \qquad \begin{aligned} (t/x)^{-1} W_{n,t}[\theta(t,x)] &= o(x^{-1}) \quad x \to \infty, \ o(1) \ \ t \to 0 \\ (t/x)^2 W_{n,t}[\theta(t,x)] &= o(1) \qquad x \to 0, \ \ o(t^{-1}) \ t \to \infty. \end{aligned}$$

Now if we denote by Δ_x, the set $\{x\,;\, ta/k < x < tb/k\}$ and by ${}_0\Delta_x$ and ${}_x\Delta_\infty$, the sets of x satisfying $x \leq ta/k$ and $x \geq tb/k$ respectively with t is Ω_t, then

$$\begin{aligned} P_{n,t}^{(\beta,\upsilon)}[L_{k,t}^{(\cdot)}[f]] &= A(k,n) \int_0^\infty x^\upsilon Q_\upsilon^{(0,\beta+2n)} [1+2x^{-1}]\ L_{k,t}^{(x)}[f]\,dx \\ &= \int_0^\infty W_{n,x^2}[\theta(x^2,x)]\ x^{k+1} L_{k,t}^{(x)}[f]\,dx \\ &= \left(\int_{{}_0\Delta_x} + \int_{\Delta_x} + \int_{{}_x\Delta_\infty}\right) \ldots \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Now,

$$\begin{aligned} |I_1| &\leq t \int_{k/a}^\infty v^{-2}\ W_{n,t^2/v^2}\ [\theta(t^2/v^2, t/v)]\ L_{k,v}[f]\ dv \\ &= \int_{k/a}^\infty W_{n,t}[\theta(t,x)]\ L_{k,v}[f]\ dv. \end{aligned}$$

By the use of (3.13) and (4.7) it is readily seen that I_1 can be made less than an arbitrary positive number by choosing a small. Similarly, choosing b large I_3 becomes also less than an arbitrary positive number. Now I_2 is actually an integral with finite limits whose integrand is continuous with respect to (t,v) and has partial derivatives with respect to t that is also continuous with respect to (t,v). Hence, by ([14], p.5), we may differentiate I_2 under the integral sign with respect to t. Since $L_{k,t}[f]$ is a testing function in $\mathcal{J}_{a,b,n,t}$ for every t is Ω_t, $P_{n,t}^{(\beta,\upsilon)}\ [L_{k,t}^{(\cdot)}[f]]$ is infinitely smooth for every t

in Ω_t by lemma 3.7. Hence for every $\phi \in \mathcal{D}(I)$

$$\langle P_{n,t}^{(\beta,\upsilon)} [L_{k,t}^{(.)} [f]], \phi(t) \rangle \quad \text{is an integral} .$$

We observe that the integro-differential operator can be broken up into an integro-differential operator

$$M_{n,t}[\theta(t,x)] = x^{-1} u^{-k-n} D_u^{-n} u^{\upsilon+n+1} D_u^n u^{k-\upsilon-n-1} \theta(u) \Big|_{u=t/x}$$

and a differential operator

$$N_{n,t}[\zeta(t,x)] = x^{-1} u^{n-k+\upsilon+1} D_u^n u^n D_u^n u^{k-\upsilon-1} \zeta(u) \Big|_{u=t/x},$$

where θ (u) is given by (4.2) and ζ (u) by

$$\zeta(u) = C(k,n) u^{n+\upsilon-k} Q_{\upsilon+n}^{(0,\beta)} [1+2u^{-1}]$$

such that the later forms a polynomial $P_n(tD_t)$ of degree n and that

$$N_{n,t} [t^n M_{n,t} [\theta(t,x)]] = W_{n,t} [\theta(t,x)] \tag{4.8}$$

Now, formally we have

$$\langle P_{n,t}^{(\beta,\upsilon)} [L_{k,t}^{(.)} [f]], \phi(t) \rangle \tag{4.9}$$

$$= \langle \int_0^\infty N_{n,t} [t^n M_{n,t} [\theta(t,x)] L_{k,x}[f] dx, \phi(t) \rangle \tag{4.10}$$

$$= \langle \phi(t), P_n(-tD_t-1) t^n \int_0^\infty M_{n,t} [\theta(t,x)] L_{k,x}[f] dx \rangle \tag{4.11}$$

$$= \langle t^n P_n(-tD_t-1) \phi(t), \int_0^\infty M_{n,t} [\theta(t,x)] \langle F(u), h_k(u,x) \rangle dx \rangle \tag{4.12}$$

$$= \langle t^n P_n(-tD_t-1) \phi(t), \langle F(u), \int_0^\infty M_{n,t} [\theta(t,x)] h_k(u,x) dx \rangle \rangle \tag{4.13}$$

$$= \langle F(u), \langle t^n P_n(-tD_t-1) \phi(t), \int_0^\infty M_{n,t} [\theta(t,x)] h_k(u,x) dx \rangle \rangle \tag{4.14}$$

$$= \langle F(u), \langle \phi(t), \int_0^\infty W_{n,t} [\theta(t,x)] h_k(u,x) dx \rangle \rangle \tag{4.15}$$

$$= \langle F(u), \phi(u) \rangle \quad \text{as } n \to \infty \tag{4.16}$$

The theorem will be proved if we justify the above manipulations.
(4.9) is equal to (4.10) by virtue of (4.8). (4.11) is obtained from (4.10) by term by term integration within the integral sign. This is valid since the integral appearing in (4.10) with the integrand continuous with respect to t and x converges uniformly. Indeed, by arguments similar to those used to show that I_1 is less than an arbitrary positive number ε there exists a positive number R_0 large enough such that

$$\left| \int_R^\infty W_{n,t}\,[\theta(t,x)]\; L_{k,x}\,[f]\,dx \right| < \varepsilon\,, \quad R \geq R_0\,.$$

By the fact that ϕ has a compact support, integration by parts yields (4.12) from (4.11) and (4.13) is equal to (4.12) since the integral appearing in (4.13) belongs to $\mathcal{J}_{a,b,n}$ by lemma 3.4. Now to show (4.13) equal to (4.14) we use the technique of Riemann sum. For this, $\xi(t) = t^n P_n(-tD_t - 1)\,\phi(t)$ and

$$E_n(u,t) = \int_0^\infty M_{n,t}\,[\theta(t,x)]\; h_k\,\{u,x\}\,dx\,.$$

Then we must show that $\langle\, \xi(t), E_n(u,t)\,\rangle$ belongs to $\mathcal{J}_{a,b,n}$.

Assume that the support of ϕ is contained in the closed finite interval $[A,B]$, then $\xi(t)$ has also support contained in $[A,B]$. For each $m = 1,2,3,4$ ...., we partition $[A,B]$ into m subintervals whose lengths are $\Delta x_{\upsilon,m}$ ($\upsilon = 1,2,3,\ \ldots,\ m$). Let $\eta_{\upsilon,m}$ be a point in the υth sub-interval and assume that the maximum of $\Delta x_{\upsilon,m}$ tends to zero as m tends to ∞. Then we need only to show that

$$\int_A^B \xi(t)\; E_n(u,t)\,dt - \sum_{\upsilon=1}^{m} \xi(\eta_{\upsilon,m})\; E_n(u,\eta_{\upsilon,m})\;\Delta x_{\upsilon,m} \to 0$$

in $\mathcal{J}_{a,b,n}$ as $m \to \infty$. That is, we must show that

$$\text{(4.17)}\quad k_{a,b,n}(u)\,u^n\,D_u^n \Big[\int_A^B \xi(t)\,E_n(u,t)\,dt - \sum_{\upsilon=1}^{m} \xi(\eta_{\upsilon,m})\,E_n(u,\eta_{\upsilon,m})\,\Delta x_{\upsilon,m}\Big]$$

$$= \int_A^B \xi(t)\,[\,K_{a,b,n}(u)\,u^n\,D_u^n\,E_n(u,t)\,]\,dt$$

$$- \sum_{\upsilon=1}^{m} \xi(n_{\upsilon,m})\; k_{a,b,n}(u)\,u^n\,D_u^n\,E_n(u,\eta_{\upsilon,m})\,\Delta x_{\upsilon,m}$$

tends to zero uniformly as $m \to \infty$ for all u in $-\infty < u < \infty$.

By virtue of uniform continuity of the functions involved we can find a positive integer m_0 sufficiently large and a positive number R such the absolute value of the expression in (4.17) is less than an arbitrary $\varepsilon' > 0$ for all $m > m_0$ and $|u| \leq R$.

Also, since $k_{a,b,n}(u)\ u^n D_u^n\ E^n\ (u,t)$ tends to zero as u tends to infinity by lemma 3.4, we choose R sufficiently large such that

$$\left| k_{a,b,n}(u)\, u^n\ D_u^n\ E_n(u,\, t) \right| < \varepsilon'$$

for all $|\, u\,| > R$ and for all $t \in [A\,, B\,]$.Then the absolute value of the expression (4.17) is less than

$$\varepsilon'\ [\ \int_A^B |\xi(t)|\ dt + \sum_{\upsilon=1}^{m} \xi\,(\eta_{\upsilon,m})\ ]$$

which prove the assertion because ε' is arbitrary.

The equality between (4.14) and (4.15) is justified by the uniform convergence of the integral appearing in (4.15) and by the compact support of ϕ.

Now to complete the proof. it remains to show that how (4.15) tends to (4.16) as $n \to \infty$. For this, we first note that

$$(uD_u)^n\ E_n(u,t) = (-tD_t{-}1)^n\, E_n\,(u,t)$$

which is, indeed, true since

$$(uD_u)^n\ h_k^{(v)}\ (u,t) = (-tD_t{-}1)^n\ h_k^{(v)}\ (u,t).$$

Then, integration by parts yields

$$
\begin{aligned}
(uD_u)^n \langle\ E_n(u,t),\ t^n\ p_n(-tD_t{-}1)\,\phi\,(t)\ \ \rangle & \\
&= \langle\ (-tD_t{-}1)^n\ E_n\,(u,t)\,,\, t^n\ p_n(-tD_t{-}1)\,\phi\,(t)\ \ \rangle \qquad \text{(lemma 3.1(b))}\\
&= \langle\ E_n(u,t)\,,\,(tD_t)^n\, t^n\ p_n(-tD_t{-}1)\,\phi\,(t)\ \ \rangle\\
&= \langle\ E_n(u,t)\ ,\ t^n\ Q_n(tD_t)\,p_n(-tD_t{-}1)\,\phi\,(t)\ \ \rangle \qquad \text{(lemma 3.2(a))}\\
&= \langle\ E_n(u,t)\ ,\ t^n\ p_n(-tD_t{-}1)\ Q_n(tD_t)\,\phi\,(t)\ \rangle \qquad \text{(lemma 3.2(b))}\\
&= \langle\ p_n\,(tD_t)\, t^n\, E_{n,}\,(u,t),\, Q_n\,(tD_t)\ \phi\,(t)\ \rangle\\
&= \langle\, P_n\,(tD_t)\, t^n \int_0^\infty M_{n,t}\ [\theta\,(t,x)]\ h_k(u,x)\ dx,\, Q_n(tD_t)\,\phi\,(t)\ \rangle\\
&= \langle\ \int_0^\infty W_{n,t}\ [\theta\ (t,x)]\ h_k\,(u,x)\ \ dx,\ Q_n\,(tD_t)\ \phi\,(t)\ \rangle\\
&= \langle\ Q_n(tD_t)\ \phi\,(t),\ S_n(u,t)\ \ \rangle \qquad \text{(by (4.3))}
\end{aligned}
$$

Therefore, using (4.6) we have

$$
\begin{aligned}
(uD_u)^n\ \langle\ E_n(u,t),\ t^n\ p_n\,(-tD_t{-}1)\ \phi\,(t)\ \rangle - Q_n\,(uD_u)\ \phi\,(u) & \\
&= \langle\ [Q_n^*\,(t) - Q_n^*\,(u)]\ ,\ S_n\,(u,t)\ \ \rangle.
\end{aligned}
$$

where $Q_n^*(t) = Q_n(tD_t)\,\phi(t)$.

Now in view of ([9], lemma 1), we need only to show that

$$k_{a,b,n}(u)\ \langle\ Q_n^*(t) - Q_n^*(u)\,, S_n(u,t)\ \rangle \to 0 \text{ as } n \to \infty.$$

For this, assume that $\delta > 0$ be arbitrary. Split up the integration $(0, \infty)$ into the integration over $0 < t < u - \delta$, $u - \delta < t < u + \delta$ and $u + \delta < t < \infty$ and denote the corresponding integrals by $J_1(n,u)$, $J_2(n,u)$ and $J_3(n,u)$ respectively.

We first dispose of $J_2(n,u)$. For this, we suppose that the support of ϕ is contained in the finite interval $[A,B]$, $0 < A < B < \infty$.

If $B < u - \delta$ or $A > u + \delta$, obviously $J_2(n,u) = 0$. Therefore, restricting t on $u - \delta \le t \le u + \delta$, we have

$$|J_2(n,u)| \le k_{a,b,n}(u) \sup_{u-\delta \le y \le u+\delta} \left| \frac{d}{dy} Q_n(u) \right| \int_0^\infty S_n(u,t)\,dt$$

$$\le \delta \sup_{[A,B]} k_{a,b,n}(u)\,u^{-1} \left| \sum_{r=1}^{n} a_r(n)\,(uD_u)^{r+1}\,\phi(u) \right|$$

$$\le \delta \sup_{[A,B]} \left| \sum_{r=1}^{n} a_r(n) \sum_{p=0}^{r+1} b_p(r)\ k_{a,b,n}(u)\ u^{p-1}\ D_u^p\,\phi(u) \right|,$$

where $a_r(n)$ are the coefficients of the expansion $(n+uD_u)^n$ and $b_p(r)$ the coefficients obtained in transforming $(uD_u)^{r+1}$ into the polynomial in $u^p D_u^p$.

Since ϕ is infinitely smooth function having compact support in $[A,B]$, $0 < A < B < \infty$, by ([1], p.9) it always possible to choose A such that

$$\left| D_u^p\,\phi(u) \right| \le \left| k(p) \right|\ |A|^{-p}$$

satisfying

$$\sum_{p=0}^{r+1} \left| \frac{k(p)\,b_p(r)}{A^p} \right| < C.$$

Therefore,

$$|J_2(n,u)| \begin{cases} < C\delta \displaystyle\sum_{r=1}^{n-1} \binom{n}{r} (n-r)^{-(N-n)}\ 2^{-N} & r \ne n \\ = C\,\delta. & r = n \end{cases}$$

for some positive N. Then, choosing $N \ge n$, we have

$$J_2(n,u) < C\delta$$

from which it follows that $J_2(n,u) \to 0$ uniformly as δ is arbitrary.

Next,

$$J_3(n,u) = k_{a,b,n}(u) \int_{u+\delta}^{\infty} [Q_n^*(t) - Q_n^*(u)]\, S_n(u,t)\, dt$$

$$= k_{a,b,n}(u) \int_{u+\delta}^{\infty} Q_n^*(t)\, S_n(u,t)\, dt$$

$$- k_{a,b,n}\, Q_n^*(u) \int_{u+\delta}^{\infty} S_n(u,t)\, dt$$

$$= J_3'(n,u) - J_3''(n,u)$$

We only show that $J_3''(n,u) \to 0$ uniformly as $n \to \infty$, since $J_3'(n,u) \to 0$ uniformly as $n \to \infty$ easily. Indeed,

$$|J_3''(n,u)| \le \sup_{-\infty<u<\infty} \left| k_{a,b,n}(u)\, Q_n^*(u) \right|$$

$$\le \sup_{-\infty<u<\infty} \sum_{r=0}^{n} \left| a_r^*(n) \frac{k_{a,b,n}(u)}{k_{a,b,r}(u)} \right| \left| k_{a,b,r}(u)\, (uD_u)^r\, \phi(u) \right|$$

which tends to zero as $n \to \infty$ as in corollary 3.1.

Let $T(y) = e^{\frac{kuy}{2nt}(\beta+\upsilon)}(1+y)^{-1}$. Then $T(y)$ is an integral function of y for $u, t \ge 0$ if $\beta + \upsilon \le 0$.

Let $G(y) = \log \dfrac{y}{1+y} - \dfrac{kuy}{2nt}$. If $H(u,t) = \dfrac{1}{2u}\,[-u + (u^2 + 8ntu/k)^{\frac{1}{2}}]$,

$$G'(H(u,t)) = 0 \text{ and } G''(H(u,t)) \leqslant 0.$$

Then we may apply ([15], p.278) with $\phi = T$ to (4.5) to get

$$(4.18)\quad |S_n(u,t)| < \frac{n^{\frac{1}{2}}\, B(k,n)}{k\sqrt{\beta+\upsilon+2n}}\, u^{-1} \left(\frac{ku}{t}\right)^{k-\upsilon} \left[\frac{H(u,t)}{1+H(u,t)}\, e^{-\frac{knH(u,t)}{2nt}}\right]^{\beta+\upsilon+2n}$$

$$\times \frac{|T(H(u,t))|}{(-G''(H(u,t)))^{\frac{1}{2}}}.$$

Since

$$\frac{|T(H(u,t))|}{(-G''(H(u,t)))^{\frac{1}{2}}} = 0(t^{\frac{1}{2}}) \quad t \to \infty,$$

there exist a positive constant M and a t_0 such that for all $t \geq t_0 \geq u+\delta$

$$\frac{|T(H(u,t)|}{(-G''(H(u,t)))^{\frac{1}{2}}} < Mt^{\frac{1}{2}}.$$

Then, writing u_0 for $u+\delta$, we have

$$\left|J_3'(u,t)\right| < \frac{M\pi^{\frac{1}{2}} B(k,n)}{\sqrt{\beta+\upsilon+2n}}\, k_{a,b,n}(u)\,(ku)^{k-\upsilon-1}\, [\frac{H(u,u_0)}{1+H(u,u_0)}$$

$$\times\, e^{-\frac{kuH(u,u_0)}{2nu_0}}\,]^{\beta+\upsilon+2n} \int_0^\infty Q_n^*(t)\, t^{-k+\upsilon+\frac{1}{2}}\, dt$$

$$< \{MM^*\ B^*(k,n)\,(\frac{u}{u+\delta})^{k-\upsilon-3/2}\}\{k_{a,b,n}(u)u^{\frac{1}{2}}$$

$$[\frac{H(u,u_0)}{1+H(u,u_0)}\, e^{-\frac{kuH(u,u_0)}{2nu_0}}\,]^{\beta+\upsilon+2n}\},$$

where $M^* = \sup_{t\in[A,B]} \left|Q_n^*(t)\right|$ and $B^*(k,n) = \dfrac{M\,\pi^{\frac{1}{2}} B(k,n)\, k^{k-\upsilon-1}}{\sqrt{\beta+\upsilon+2n}\ (k-\upsilon-3/2)}$.

The right hand side tends to zero as $n\to\infty$ since the terms in the first brace are bounded and those in the second brace tend to zero as $n\to\infty$. This proves that $J_3'(n,u)\to 0$ uniformly as $n\to\infty$.

Now to complete the theorem, it remains to show that $J_1(n,u)\to 0$. uniformly as $n\to\infty$. For this, we break up $J_1(n,u)$ into $J_1'(n,u)$ and $J_1''(n,u)$. As in the previous case, we only show that $J_1'(n,u)\to 0$ uniformly as $n\to\infty$ since $J_1''(n,u)\to$ o uniformly $n\to\infty$. exactly in the same way as $J_3''(n,u)\to 0$ uniformly as $n\to\infty$.

We observe that for any real number a in $1<a<2$ the function

$$\frac{t^{-\frac{k}{an}}\,H(u,t)}{1+H(u,t)}\ e^{-\frac{kuH(u,t)}{2nt}}$$

is strictly increasing for $t<u$ provided $n\geq k$. Indeed

$$\frac{d}{dt}\,[\frac{t^{-\frac{k}{an}}H(u,t)}{1+H(u,t)}\ e^{-\frac{kuH(u,t)}{2nt}}] = \frac{t^{-\frac{k}{an}}\,H(u,t)}{1+H(u,t)}\ e^{-\frac{kuH(u,t)}{2nt}}\ [kt^{-2}\{uH(u,t)-\frac{2t}{a}\}$$

$$+H'(u,t)\,\{\frac{1}{H(u,t)\,(1+H(u,t))}-\frac{ku}{2nt}\}],$$

where the first term is positive for all $t < u$ and the second term is always zero.

Then, using (4.18) and writing v_o for $u - \delta$, we have

$$\left|J_1'(n,u)\right| \le \frac{M^* \pi^{\frac{1}{2}} B(k,n)}{\sqrt{\beta+\upsilon+2n}} (ku)^{k-\upsilon-1} k_{a,b,n}(u) \left[\frac{v_0^{-\frac{k}{an}} H(u,v_0)}{1+H(u,v_0)} e^{-\frac{kuH(u,v_0)}{2nv_0}}\right]^{\beta+\upsilon+2n} \times \int_0^{u-\delta} t^{\upsilon+k(\frac{2}{a}-1)+\frac{k}{an}(\beta+\upsilon)} \frac{\left|T(H(u,t))\right|}{(-G''(H(u,t)))^{\frac{1}{2}}} dt$$

Noting

$$\frac{|T(H(u,t))|}{(-G''(H(u,t)))^{\frac{1}{2}}} < \frac{1}{2}\left(\frac{9n}{k}\right)^{\frac{1}{4}} \text{ for } t < u$$

and choose n and k such that $\upsilon + k\left(\frac{2}{a}-1\right) + \frac{k}{an}(\beta+\upsilon) + 1 > 0$ for $1 < a < 2$, we have

$$\left|J_1'(n,u)\right| < \left\{\overline{B}(k,n)\left(\frac{u}{u-\delta}\right)^{k-\upsilon-1}\right\} \left\{k_{a,b,n}(u)\left[\frac{H(u,v_0)}{1+H(u,v_0)} e^{-\frac{kuH(u,v_0)}{2nv_0}}\right]^{\beta+\upsilon+2n}\right.,$$

where

$$\overline{B}(k,n) = \frac{M^* \pi^{\frac{1}{2}} B(k,n) k^{k-\upsilon-1}}{2\left[\upsilon+1+k\left(\frac{2}{a}-1\right)+\frac{k}{an}(\beta+\upsilon)\right]} \frac{\left(\frac{9n}{k}\right)^{\frac{1}{4}}}{\sqrt{\beta+\upsilon+2n}}.$$

The terms in the first brace are bonded and those in the second brace tend to zero as $n \to \infty$. This completes the proof of the theorem.

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On The Existence Of Desarguesian Planes Satisfying A Condition Of Suetake

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Abstract: The affine plane of the order 4 does not admit a transitive collineation group G that partitions 1_∞ into sets Δ, $1_\infty \backslash \Delta$ with $|\Delta| = 2$ and $|G(1_\infty, 1_\infty)| = 4$, $|G(P, 1_\infty)| = 2$ for $P = 1_\infty \backslash \Delta$ and $|G(P, 1_\infty)| = 1$ $(\forall p \in \Delta)$

1. Introduction

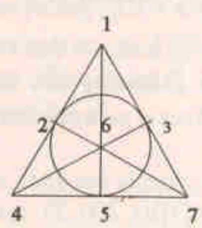
Let π be a finite affine plane of order $n = 2^m$ where $m \geq 1$. Let $G(1_\infty, 1_\infty)$ denote the subgroup of translations which are in a group G of collineations of π. Let G act transitively on the points of π and partition the line 1_∞ at infinity into two subsets Δ, $1_\infty \backslash \Delta$ with $|\Delta| = 2$. Then, according to Suetake [4], one of the following holds: (i) π is a translation plane and all the translations of π are in G, (ii) $|G(1_\infty, 1_\infty)| = 2^m$, $|G(P, 1_\infty)| = 1$ $(\forall P \in \Delta)$, $|G(P, 1_\infty)| = 2$ $(\forall P \in 1_\infty \backslash \Delta)$ It is not known yet that the two conclusions are mutually exclusive or that if affine planes exist that admit conclusion (ii). But it is known that the affine planes of orders 2 and 4 are translation planes. The purpose of this paper is to show the following

Theorem *The affine plane of order* 2 *admits conclusion* (ii) *of Suetake while the affine plane of order* 4 *does not*

In ξ 2, a figure of the affine plane of order 2 and a transitive group of collineations admitting conclusion (ii) of Suetake are given outright. In ξ 3, the non-existence of a desired group of collineations of the affine plane of order 4 is shown. Henceforth, G will refer to a group of collineations of π satisfying condition:

(*) G is transitive on the affine points has two orbits Δ, $1_\infty \backslash \Delta$ on 1_∞ with $|\Delta| = 2$ and admits conclusion (ii) of Suetake

2. Affine plane of order 2



The group $G = \langle (2, 5, 4, 6)(3, 7) \rangle$ where $l_\infty = \{1, 3, 7\}$, $\Delta = \{3, 7\}$, $l \setminus \Delta = \{1\}$.

3. Affine Plane of Order 4

Lemma 1, *Let π be an affine plane of order 2^m for $m \geq 1$. If G is a group of collineations of π and have property* (*), *then G has none of the following collineations*

(a) Involuntary homology,
(b) nontrivial homology with an affine point as its centre,
(c) nontrivial homology $\alpha - (p, l)$ where $p \in l_\infty \setminus \Delta$ and l an affine line,
(d) nontrivial elation $\alpha - (p, l)$ where P and l are affine point and affine line

Proof : (a) Use Lemma 4.9 in Kallaher [3, p.103]. (c) If α is in G then α is an involutory homology by showing that α^2 fixes a line through any point ot Δ. (d) Consider the effect of α on a line through P and any point on l_∞.

Now, since each collineation in G acts on the partitions of l_∞, we use the following notation to show the kind of action: $\varphi(1(x), 2(y), \#)$ which means that collineation φ acts (x) on Δ and (y) on $l_\infty \setminus \Delta$ and has order # choices for (x) are (a) "fixes Δ pointwise" and (b) "permutes the two points of Δ cyclically". Choices for (y) are (i) "fixes l_∞ pointwise", (ii) "fixes a point and permutes the other two cyclically, and (iii) "permutes $l_\infty \setminus \Delta$ cyclically".

Lemma 2. *Let π be an affine plane of order* 4. *If G is a group of collineations of π and has property* (*), *then the orders of G are bounded above by* 6 *and G has no collineation of order* 5.

Proof : Argue using the fact that an appropriate power of φ is a translation and must be of order 2 or 1. E.g. for $\# \geq 5$ and $\# \neq 6$, $\varphi(1(a), 2(iii), \#) \notin G$ by Lemma $1(b)$ because φ^3 fixes l_∞ pointwise and is of order greater than 2. In the case of $\varphi(1(b), 2(\text{iii}), 12)$, φ^2 fixes Δ pointwise and so φ^2 permutes the four lines through each of the two points of Δ. Since the highest order of a permutation on 4 letters is 4; one of φ^2, $(\varphi^2)^2$, $(\varphi^2)^3$, $(\varphi^2)^4$ fixes each of the points of Δ linewise. Then by duality one of them is the identity, a contradiction to the order of φ.

Lemma 3 *Let π be an affine plane of order* 4. *Let G be a group of collineation of π and have property* (*). *Then the following are the possible collineations in G together with an indication that it fixes a point or not:*

A. Does not fix an affine point:

$t(1(\text{a}), 2(\text{i}), 2)$, $m(1(\text{a}), 2(\text{ii}), 4)$, $p(1(\text{b}), 2(\text{i}), 2)$, $c(1(\text{b}), 2(\text{ii}), 4)$, $f(1(\text{a}), 2(\text{iii}), 6)$

B. Fixes an affine point :

d (1 (a) , 2 (ii), 2) , b (1 (b) , 2 (i) , 2), r(1 (b) , 2 (ii), 2) , a (1(a), 2(iii), 3), h(1 (a), 2(iii), 3), e(1(b), 2(iii),6).

Proof

A. If t, m, p, c, or f fixes an affine point, then t, m^2, p^2, c^2, or f^3 is an involutory homology, a contradiction.

B. Since b is of order 2, there are lines through each of the two points in Δ that are interchanged by b. The intersection of these two lines is a fixed point for b. Same arguments for d and r.

There are 16 affine points in π , Collineations a and h are sure to fix an affine point because $16 = 1 \pmod 3$). If e has a transposition in its cyclic decomposition, then fixes an affine point using the same arguments for collineation b. If it does not have a transposition, then e partitions the 16 affine points in either of the two ways: 6, 6, 3, 1 or 6, 3, 3, 3, 1. In each case, e fixes an affine point.

Lemma 4. *Let π be an affine plane of order* 4, *let G be a group of collineations of π and have property* (*). *Let m be a collineation as stated in Lemma* 3. *Then there is a translation $t \in G$ such that tm fixes an affine point.*

Proof: There is a common orbit between m and $G(1_\infty, 1_\infty)$

Proof of the theorem : Suppose that there exists such a group G. By Suetake's condition, $|G(1_\infty, 1_\infty)| = 4$. Since a translation does not fix any affine point, no two different translations send a point to the same point. $G(1_\infty, 1_\infty)$ partitions the 16 affine points into four orbits, each containing four points. According to Lemma 5 [3.p.3], $G(1_\infty, 1_\infty) \Delta G$. So $G/G(1_\infty, 1_\infty)$ acts on four blocks (which are orbits of $G(1_\infty, 1_\infty)$) . $G/G(1_\infty, 1_\infty)$ is transitive on the four blocks and $G/G(1_\infty, 1_\infty) \cong S_4$ by proposition 7.1 [6.p.13].

$G/G(1_\infty, 1_\infty) \cong \mathbb{Z}_4$, or V (Klein 4–group) or A_4 and S_4. Hence $G/G(1_\infty, 1_\infty) \geq 4$. Since $V \leq A_4, S_4$, we just consider the minimal transitive subgroup of S_4 .Thus, we take $|G/G(1_\infty, 1_\infty)| = 4$. Consequently, $|G| = 4.4 = 16$. Now, $16 = |P^G| \; |G_P|$. (P an affine point). Since G is transitive on π, $|P^G| = 16$ and so $|G_p| = 1$, that is, no nontrivial collineation in G fixes an affine point G. Taking note of the order of G and Lemma 3 a collineation of G can only be of

order 1 identity

order 2 only translations, they are already in $G(1_\infty, 1_\infty)$, or

order 4 m, p, c.

G cannot contain m because for some translation $t \in G(1_\infty, 1_\infty)$ tm is of type (1(a), 2(ii), #) which fixes an affine point by Lemma 4. If p(1(b), 2(i), 4) and c(1(b),

2 (ii), 4) are in G, then pc (1(a), 2 (ii), 4) is of type d or m, hence c and $P \notin G$. So the only possibility is either $c \in G$ or $p \in G$ but not both p , $c \in G$. In either case, there are 12 collineation c_i, (or p_i) $i = 1, \ldots 12$, of the same type. The collineations $c_i\ c_j$, $i \neq j$ are either of type (1(a), 2(i), 2) or of type (1(a), 2(iii), 3) which cannot be because it means G has 12 translations, G has a homology or G has a collineation of order 3.

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On The Mean Values Of An Entire Function In Several Complex Variables Represented By Multiple Dirichlet Series

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1. Introduction

The properties of mean values of an entire function represented by Dirichlet series in one complex variable have been studied by various authors to a considerable extent. But the properties of mean values in the case of several complex variable have not yet been studied. The purpose of this paper is to extend the concepts of the mean values of an entire function represented by Dirichlet series in one complex variable to an entire function of several complex variables represented by multiple Dirichlet series and to study some of their properties. For the sake of simplicity we consider here the case of two variables instead of several variables.

Consider the double Dirichlet series

$$f(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{m,n} \exp\,(s_1 \lambda_m + s_2 \mu_n) \tag{1.1}$$

$(s_j = \sigma_j + it_j,\ j = 1, 2)$ where $a_{m,n} \in C$, the field of complex numbers, $\lambda_m' s$, $\mu_n' s$ are real:

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_m \to \infty,\ 0 < \mu_1 < \mu_2 < \ldots < \mu_n \to \infty \ .$$

A.I. Janusauskas in his paper (Janusauskas 1977) has shown that if

$$\lim_{m \to \infty} \frac{\log m}{\lambda_m} = 0, \quad \lim_{n \to \infty} \frac{\log n}{\mu_n} = 0, \tag{1.2}$$

then the domain of convergence of the series (1.1) coincides with its domain of absolute convergence.

Also, Sarkar [1, pp. 99] has shown that the necessary and sufficient condition that the series (1.1) satisfying (1.2) to be entire is that

$$\lim_{(m,n) \to \infty} \frac{\log |a_{m,n}|}{\lambda_m + \mu_n} = -\infty \ . \tag{1.3}$$

Throughout F stands for the family of all double Dirichlet series of the form (1.1) satisfying (1.2) and (1.3). Then $f \in F$ denotes an entire function over C^2, the Cartesian product of two copies of the complex plane.

Corresponding to a $f \in F$, the maximum modulus $M = M_f$ and the maximum term $\mu = \mu_f$ on R^2 are defined [1, pp 100] as

$$M(\sigma) = M_f(\sigma_1,\sigma_2) = \max \{|f(s_1,s_2)| : s_1, s_2 \in C, \text{ Re } s_1 = \sigma_1 \text{ Re } s_2 = \sigma_2\}$$

$$\mu(\sigma) = \mu_f(\sigma_1,\sigma_2) = \max_{(m,n) \in N^2} \{|a_{m,n}| \exp(\sigma_1 \lambda_m + \sigma_2 \mu_n)\},$$

where N is the set of natural numbers.

We define the mean values of $|f(s_1,s_2)|$ as

(1.4) $$I_2(\sigma_1,\sigma_2,f) = I_2(\sigma_1,\sigma_2)$$

$$= \lim_{T_1,T_2 \to \infty} \frac{1}{4T_1T_2} \int_{-T_1}^{T_1} \int_{-T_2}^{T_2} |f(\sigma_1 + it_1, \sigma_2 + it_2)|^2 \, dt_1 \, dt_2$$

(1.5) $$m_{2,k}(\sigma_1,\sigma_2,f) = m_{2,k}(\sigma_1,\sigma_2)$$

$$= \lim_{T_1,T_2 \to \infty} \frac{1}{4T_1T_2 \, e^{k\sigma_1} e^{k\sigma_2}} \int_0^{\sigma_1} \int_0^{\sigma_2} \int_{-T_1}^{T_1} \int_{-T_2}^{T_2} |f(x_1 + it_1, x_2 + it_2)|^2$$

$$\times e^{kx_1} e^{kx_2} \, dx_1 \, dx_2 \, dt_1 \, dt_2$$

From (1.4) and (1.5) we can write

(1.6) $$m_{2,k}(\sigma_1,\sigma_2) = \frac{4}{e^{k\sigma_1} e^{k\sigma_2}} \int_0^{\sigma_1} \int_0^{\sigma_2} I_2(x_1,x_2) \, e^{kx_1} e^{kx_2} \, dx_1 \, dx_2 \;,$$

where k is a positive number.

2. Theorem 1 : *For the Dirichlet series $f(s_1, s_2), f \in F, I_2(\sigma_1,\sigma_2)$ is an increasing function of σ_1 and σ_2*

Proof : We have

$$|f(s_1,s_2)|^2 = f(s_1,s_2) \, \overline{f(s_1,s_2)}$$

$$= \sum_{m,n=1}^{\infty} a_{m,n} \exp\{(\sigma_1 \lambda_m + \sigma_2 \mu_n) + i(t_1 \lambda_m + t_2 \mu_n)\} \times$$

$$\sum_{M,N=1}^{\infty} \bar{a}_{M,N} \exp\{(\sigma_1 \lambda_M + \sigma_2 \mu_N) - i(t_1 \lambda_M + t_2 \mu_N)\}$$

$$= \sum_{m,n=1}^{\infty} |a_{m,n}|^2 \exp\{2(\sigma_1 \lambda_m + \sigma_2 \mu_n)\}$$

$$+\sum_{m\neq M}\sum_{n\neq M} a_{m,n}\ \bar{a}_{M,N} \exp\{\sigma_1(\lambda_m+\lambda_M)+\sigma_2(\mu_n+\mu_N)\}$$
$$+it_1(\lambda_m-\lambda_M)+it_2(\mu_n-\mu_N)\}.$$

Since both the series on the right are absolutely convergent, the resulting series is uniformly convergent for any finite t_1 and t_2 ranges, therefore we may integrate term by term for finite t_1 and t_2. Hence on integration, all the terms for which $m \neq M$, $n \neq N$, vanish as $T_1, T_2 \to \infty$ and we obtain.

$$(2.1) \qquad I_2(\sigma_1,\sigma_2) = \sum_{m,n=1}^{\infty} |a_{m,n}|^2 \exp\{2(\sigma_1\lambda_m+\sigma_2\mu_n)\}.$$

It is clear from the value of $I_2(\sigma_1,\sigma_2)$ that it is an increasing function of σ_2 for a fixed value of σ_1 and vice-versa. Hence $I_2(\sigma_1,\sigma_2)$ is an increasing function of both σ_1 and σ_2.

Corollary 1 : *For the Dirichlet series* $f(s_1,s_2)$, $f\in F$

$$\{\mu(\sigma_1,\sigma_2)\}^2 \le I_2(\sigma_1,\sigma_2) \le \{M(\sigma_1,\sigma_2)\}^2$$

This follows from the definitions of $M(\sigma_1,\sigma_2)$, $\mu(\sigma_1,\sigma_2)$ and (2.1)

3. Theorem 2 : *For the Dirichlet series* $f(s_1, s_2)$, $f\in F$, $m_{2,k}(\sigma_1,\sigma_2)$ *is an increasing function of* σ_1 *and* σ_2

Proof : We have from (1.6)

$$m_{2,k}(\sigma_1,\sigma_2) = \frac{4}{e^{k\sigma_1}\,e^{k\sigma_2}} \int_0^{\sigma_1}\!\!\int_0^{\sigma_2} I_2(x_1,x_2)\, e^{kx_1+kx_2}\, dx_1\, dx_2$$

Using (2.1) we obtain

$$m_{2,k}(\sigma_1,\sigma_2) = \frac{4}{e^{k\sigma_1}\,e^{k\sigma_2}} \int_0^{\sigma_1}\!\!\int_0^{\sigma_2} \sum_{m,n=1}^{\infty} |a_{m,n}|^2 \exp\{2(\sigma_1\lambda_m+\sigma_2\mu_n)\}$$
$$\times\ e^{kx_1+kx_2}\, dx_1\, dx_2$$

$$= \frac{4}{e^{k\sigma_1}\,e^{k\sigma_2}} \sum_{m,n=1}^{\infty} |a_{m,n}|^2 \int_0^{\sigma_1}\!\!\int_0^{\sigma_2} e^{(2\lambda_m+k)x_1}\ e^{(2\mu_n+k)x_2}\, dx_1\, dx_2$$

$$= \frac{4}{e^{k\sigma_1}\,e^{k\sigma_2}} \sum_{m,n=1}^{\infty} \frac{|a_{m,n}|^2 (e^{(2\lambda_m+k)\sigma_1}-1)(e^{(2\mu_n+k)\sigma_2}-1)}{(2\lambda_m+k)(2\mu_n+k)}$$

(3.1) $m_{2,k}(\sigma_1,\sigma_2)$

$$= 4 \sum_{m,n=1}^{\infty} \left[\frac{|a_{m,n}|^2 \, (e^{2\lambda_m \sigma_1} - e^{-k\sigma_1})\,(e^{2\mu_n \sigma_2} - e^{-k\sigma_2})}{(2\lambda_m + k)\,(2\lambda_n + k)} \right]$$

Thus $m_{2,k}(\sigma_1,\sigma_2)$ is an increasing function of σ_1 for a fixed value of σ_2 and vice-versa. Hence $m_{2,k}(\sigma_1,\sigma_2)$ is an increasing function of σ_1 and σ_2

4. Theorem 3 : *For the Dirichlet series $f(s_1, s_2), f \in F$. We have*

$$\text{(4.1)} \quad \limsup_{\sigma_1,\sigma_2 \to \infty} \frac{m_{2,k}(\sigma_1,\sigma_2)}{\{M(\sigma_1,\sigma_2)\}^2} \leq \limsup_{\sigma_1,\sigma_2 \to \infty} \frac{m_{2,k}(\sigma_1,\sigma_2)}{I_2(\sigma_1,\sigma_2)} \leq \frac{4}{k^2}$$

Proof : Since $I_2(x_1, x_2)$ is an increasing function of x_1 and x_2 and therefore from (1.6) we have

$$m_{2,k}(\sigma_1,\sigma_2) \leq \frac{4I_2(\sigma_1,\sigma_2)}{e^{k\sigma_1 + k\sigma_2}} \int_0^{\sigma_1}\int_0^{\sigma_2} e^{kx_1}\, e^{kx_2}\, dx_2\, dx_2$$

$$= \frac{4I_2(\sigma_1,\sigma_2)}{e^{k\sigma_1 + k\sigma_2}} \left[\left(\frac{e^{k\sigma_1}-1}{k}\right)\left(\frac{e^{k\sigma_2}-1}{k}\right) \right]$$

$$= \frac{4}{k^2} I_2(\sigma_1,\sigma_2)\,(1 - e^{-k\sigma_1})\,(1 - e^{-k\sigma_2})$$

Taking limits on both the sides we get

$$\text{(4.2)} \quad \limsup_{\sigma_1,\sigma_2 \to \infty} \frac{m_{2,k}(\sigma_1,\sigma_2)}{I_2(\sigma_1,\sigma_2)} \leq \frac{4}{k^2}$$

Also from corollary 1, we have

$$\text{(4.3)} \quad I_2(\sigma_1,\sigma_2) \leq \{M(\sigma_1,\sigma_2)\}^2$$

Therefore, from (4.2) and (4.3), it follows that

$$\limsup_{\sigma_1,\sigma_2 \to \infty} \frac{m_{2,k}(\sigma_1,\sigma_2)}{\{M(\sigma_1,\sigma_2)\}^2} \leq \limsup_{\sigma_1,\sigma_2 \to \infty} \frac{m_{2,k}(\sigma_1,\sigma_2)}{I_2(\sigma_1,\sigma_2)} \leq \frac{4}{k^2}$$

5. Theorem 4: *For $f(s_1, s_2), f \in F$, we have*

$$\lim_{(\sigma_1,\sigma_2)\to\infty} \left\{ \frac{1}{e^{k\sigma_1(1-\alpha_1)}\, e^{k\sigma_2(1-\alpha_2)}\; m_{2,k}(\sigma_1,\sigma_2) - m_{2,k}(\alpha_1\sigma_1,\alpha_2\sigma_2)} \right\} = 0,$$

where α_1, α_2 $(0 < \alpha_1, \alpha_2 < 1)$ are constants.

We first prove the following lemma.

Lemma 1. *Let $f(s_1, s_2)$ be an entire function, then for*

$$(0 < \sigma_1' < \bar{\sigma}_1 < \sigma_1) \text{ and } (0 < \sigma_2' < \bar{\sigma}_2 < \sigma_2)$$

$$I_2(\bar{\sigma}_1, \sigma_2')(e^{k\sigma_1} - e^{k\bar{\sigma}_1})(e^{k\bar{\sigma}_2} - e^{k\sigma_2'}) + I_2(\sigma_1', \bar{\sigma}_2)(e^{k\bar{\sigma}_1} - e^{k\sigma_1'})(e^{k\sigma_2} - e^{k\bar{\sigma}_1})$$
$$+ I_2(\bar{\sigma}_1, \bar{\sigma}_2)(e^{k\sigma_1} - e^{k\bar{\sigma}_1})(e^{k\sigma_2} - e^{k\bar{\sigma}_2})$$
$$\leq (\frac{k}{2})^2 \{e^{k\sigma_1 + k\sigma_2} m_{2,k}(\sigma_1, \sigma_2) - e^{k\bar{\sigma}_1 + k\bar{\sigma}_2} m_{2,k}(\bar{\sigma}_1, \bar{\sigma}_2)\}$$
$$\leq I_2(\sigma_1, \bar{\sigma}_2)(e^{k\sigma_1} - e^{k\bar{\sigma}_1})(e^{k\bar{\sigma}_2} - 1) + I_2(\bar{\sigma}_1, \sigma_2)(e^{k\bar{\sigma}_1} - 1)(e^{k\sigma_2} - e^{k\bar{\sigma}_2})$$
$$+ I_2(\sigma_1, \sigma_2)(e^{k\sigma_1} - e^{k\bar{\sigma}_1})(e^{k\sigma_2} - e^{k\bar{\sigma}_2})$$

where k is any positive number.

Proof of Lemma 1 : Since $I_2(x_1, x_2)$ is an increasing function of x_1 and x_2 and therefore from (1.6) we have

$$e^{k\sigma_1 + k\sigma_2} m_{2,k}(\sigma_1, \sigma_2) - e^{k\bar{\sigma}_1 + k\bar{\sigma}_2} m_{2,k}(\bar{\sigma}_1, \bar{\sigma}_2)$$
$$= 4 \int_{\bar{\sigma}_1}^{\sigma_1} \int_{0}^{\bar{\sigma}_2} I_2(x_1, x_2)\, e^{kx_1 + kx_2}\, dx_1\, dx_2 + 4 \int_{0}^{\bar{\sigma}_1} \int_{\bar{\sigma}_2}^{\sigma_2} I_2(x_1, x_2)\, e^{kx_1 + kx_2}\, dx_1\, dx_2$$
$$+ 4 \int_{\bar{\sigma}_1}^{\sigma_1} \int_{\bar{\sigma}_2}^{\sigma_2} I_2(x_1, x_2)\, e^{kx_1 + kx_2}\, dx_1\, dx_2$$

Hence

$$(5.2) \qquad e^{k\sigma_1 + k\sigma_2} m_{2,k}(\sigma_1, \sigma_2) - e^{k\bar{\sigma}_1 + k\bar{\sigma}_2} m_{2,k}(\bar{\sigma}_1, \bar{\sigma}_2)$$
$$\leq \frac{4}{k^2} I_2(\sigma_1, \bar{\sigma}_2)(e^{k\sigma_1} - e^{k\bar{\sigma}_1})(e^{k\bar{\sigma}_2} - 1)$$
$$+ \frac{4}{k^2} I_2(\bar{\sigma}_1, \sigma_2)(e^{k\bar{\sigma}_1} - 1)(e^{k\sigma_2} - e^{k\bar{\sigma}_2})$$
$$+ \frac{4}{k^2} I_2(\sigma_1, \sigma_2)(e^{k\sigma_1} - e^{k\bar{\sigma}_1})(e^{k\sigma_2} - e^{k\bar{\sigma}_2})$$

Also

$$(5.3) \qquad e^{k\sigma_1 + k\sigma_2} m_{2,k}(\sigma_1, \sigma_2) - e^{k\bar{\sigma}_1 + k\bar{\sigma}_2} m_{2,k}(\bar{\sigma}_1, \bar{\sigma}_2)$$

$$\geq 4 \int_{\bar{\sigma}_1}^{\sigma_1} \int_{\sigma_2'}^{\bar{\sigma}_2} I_2(x_1, x_2)\, e^{kx_1 + kx_2}\, dx_1\, dx_2$$

$$+ 4 \int_{\sigma_1'}^{\bar{\sigma}_1} \int_{\bar{\sigma}_2}^{\sigma_2} I_2(x_1, x_2)\, e^{kx_1 + kx_2}\, dx_1\, dx_2$$

$$+ 4 \int_{\bar{\sigma}_1}^{\sigma_1} \int_{\bar{\sigma}_2}^{\sigma_2} I_2(x_1, x_2)\, e^{kx_1 + kx_2}\, dx_1\, dx_2$$

$$\geq \frac{4}{k^2} I_2(\bar{\sigma}_1, \sigma_2')(e^{k\sigma_1} - e^{k\bar{\sigma}_1})(e^{k\bar{\sigma}_2} - e^{k\sigma_2'})$$

$$+ \frac{4}{k^2} I_2(\sigma_1', \bar{\sigma}_2)(e^{k\bar{\sigma}_1} - e^{k\sigma_1'})(e^{k\sigma_2} - e^{k\bar{\sigma}_2})$$

$$+ \frac{4}{k^2} I_2(\bar{\sigma}_1, \bar{\sigma}_2)(e^{k\sigma_1} - e^{k\bar{\sigma}_1})(e^{k\sigma_2} - e^{k\bar{\sigma}_2})$$

Combining (5.2) and (5.3) we obtain the lemma .

Proof of Theorem 4: If we put

$$\bar{\sigma}_1 = \alpha_1 \sigma_1, \sigma_1' = \beta_1 \sigma_1$$
$$\bar{\sigma}_2 = \alpha_2 \sigma_2, \sigma_2' = \beta_2 \sigma_2$$

where $\beta_1 < \alpha_1$, $\beta_2 < \alpha_2$ in Lemma 1, we get

$$4I_2(\alpha_1\sigma_1, \beta_2\sigma_2)(e^{k\sigma_1} - e^{k\alpha_1\sigma_1})(e^{k\alpha_2\sigma_2} - e^{k\beta_2\sigma_2})$$

$$+ 4I_2(\beta_1\sigma_1, \alpha_2\sigma_2)(e^{k\alpha_1\sigma_1} - e^{k\beta_1\sigma_1})(e^{k\sigma_2} - e^{k\alpha_2\sigma_2})$$

$$+ 4I_2(\alpha_1\sigma_1, \alpha_2\sigma_2)(e^{k\sigma_1} - e^{k\alpha_1\sigma_1})(e^{k\sigma_2} - e^{k\alpha_2\sigma_2})$$

$$\leq k^2 \left[e^{k\sigma_1 + k\sigma_2} m_{2,k}(\sigma_1, \sigma_2) - e^{k\alpha_1\sigma_1 + k\alpha_2\sigma_2} m_{2,k}(\alpha_1\sigma_1, \alpha_2\sigma_2)\right]$$

$$\leq 4I_2(\sigma_1, \alpha_2\sigma_2)(e^{k\sigma_1} - e^{k\alpha_1\sigma_1})(e^{k\alpha_2\sigma_2} - 1)$$

$$+ 4I_2(\alpha_1\sigma_1, \sigma_2)(e^{k\alpha_1\sigma_1} - 1)(e^{k\sigma_2} - e^{k\alpha_2\sigma_2})$$

$$+ 4I_2(\sigma_1, \sigma_2)(e^{k\sigma_1} - e^{k\alpha_1\sigma_1})(e^{k\sigma_2} - e^{k\alpha_2\sigma_2})$$

Dividing by $e^{k\alpha_1\sigma_1} e^{k\alpha_2\sigma_2}$, we get

$$4I_2(\alpha_1\sigma_1, \beta_2\sigma_2)(e^{k\sigma_1(1-\alpha_1)} - 1)(1 - e^{k\sigma_2(\beta_2 - \alpha_2)})$$

$$+ 4I_2(\beta_1\sigma_1, \alpha_2\sigma_2)(e^{k\sigma_2(1-\alpha_2)} - 1)(1 - e^{k\sigma_1(\beta_1 - \alpha_1)})$$

$$+ 4I_2(\alpha_1\sigma_1, \alpha_2\sigma_2)(e^{k\sigma_1(1-\alpha_1)} - 1)(e^{k\sigma_2(1-\alpha_2)} - 1)$$

$$\leq k^2 \,[e^{k\sigma_1(1-\alpha_1)}\, e^{k\sigma_2(1-\alpha_2)}\, m_{2,k}(\sigma_1,\sigma_2) - m_{2,k}(\alpha_1\sigma_1,\alpha_2\sigma_2)]$$

$$\leq 4I_2(\sigma_1,\alpha_2\sigma_2)\,(e^{k\sigma_1(1-\alpha_1)}-1)\,(1-e^{-k\alpha_2\sigma_2})$$
$$+\,4I_2(\alpha_1\sigma_1,\sigma_2)\,(e^{k\sigma_2(1-\alpha_2)}-1)\,(1-e^{-k\alpha_1\sigma_1})$$
$$+\,4I_2(\sigma_1,\sigma_2)\,(e^{k\sigma_1(1-\alpha_1)}-1)\,(e^{k\sigma_2(1-\alpha_2)}-1)\,.$$

Taking limits on both the sides, Theorem 4 follows from the above inequalities.

Corollary 2 : *For* $f(s_1, s_2)$, $f \in F$, *we have*

$$[\{\mu(\overline{\sigma}_1,\sigma_2')\}^2\,(e^{k\sigma_1}-e^{k\overline{\sigma}_1})\,(e^{k\overline{\sigma}_2}-e^{k\sigma_2'})$$
$$+\,\{\mu(\sigma_1',\overline{\sigma}_2)\}^2\,(e^{k\overline{\sigma}_1}-e^{k\sigma_1'})\,(e^{k\sigma_2}-e^{k\overline{\sigma}_2})$$
$$+\,\{\mu(\overline{\sigma}_1,\overline{\sigma}_2)\}^2\,(e^{k\sigma_1}-e^{k\overline{\sigma}_1})\,(e^{k\sigma_2}-e^{k\overline{\sigma}_2})]$$

$$\leq [(\frac{k}{2})^2\,\{e^{k\sigma_1+k\sigma_2}\, m_{2,k}(\sigma_1,\sigma_2) - e^{k\overline{\sigma}_1+k\overline{\sigma}_2}\, m_{2,k}(\overline{\sigma}_1,\overline{\sigma}_2)\}]$$

$$\leq [\{M(\sigma_1,\overline{\sigma}_2)\}^2\,(e^{k\sigma_1}-e^{k\overline{\sigma}_1})\,(e^{k\overline{\sigma}_2}-1)$$
$$+\,\{M(\overline{\sigma}_1,\sigma_2)\}^2\,(e^{k\overline{\sigma}_1}-1)\,(e^{k\sigma_2}-e^{k\overline{\sigma}_2})$$
$$+\,\{M(\sigma_1,\sigma_2)\}^2\,(e^{k\sigma_1}-e^{k\overline{\sigma}_1})\,(e^{k\sigma_2}-e^{k\overline{\sigma}_2})]\,.$$

The result follows by using Corollary 1 and Lemma 1.

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Group –Theoretic Study Of Certain Generating Function Of Brenke Type Polynomials

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Abstract: Making suitable interpretation to the parameter and the index of the Brenke type polynomial $G_n^{(\alpha,p)}(x)$ in order to derive the elements of Lie algebra, we have considered a four parameters. Lie group for this polynomial. Applying the group-theoretic technique some new generating functions for $G_n^{(\alpha,p)}(x)$ have been derived.

1. Introduction

The Brenke type polynomial $G_n^{(\alpha,p)}(x)$ is defined by

$$(1.1)\qquad G_n^{(\alpha,p)}(x) = \sum_{k=0}^{n} \frac{(1+\alpha)_n\ (p)_k\ x^k}{(1+\alpha)_k\ k!\ (n-k)!}$$

is the solution of the differential equation

$$(1.2)\qquad x(1+x)\ \frac{d^2 w}{dx^2} + \{\alpha+1+(-n+p+1)x\}\ \frac{dw}{dx} - npw = 0$$

Generating functions of the polynomial $G_n^{(\alpha,p)}(x)$ in different forms were studied by the authoress [1] through analytic and group-theoretic approach.

The object of this paper is to derive some generating functions for $G_n^{(\alpha,p)}(x)$ by applying Weisner-group theoretic method [2] which consist in constructing a partial differential equation form the ordinary differential equation by giving suitable interpretation to the parameter α and the index n of the polynomial $G_n^{(\alpha,p)}(x)$ and finding a Lie group admitted by the partial differential equation.

The following generating functions will be derived for the polynomial $G_n^{(\alpha,p)}(x)$

$$(1.3)\qquad (1-t_1)^n\ G_n^{(\alpha,p)}\left(\frac{x+t_1}{1-t_1}\right) = \sum_{r=0}^{\infty} \frac{1}{r!}\ (\alpha+p+1)_r\ G_{n-r}^{(\alpha+r,p)}(x)\ t_1^r$$

$$(1.4) \quad (1+t_2)(1-x t_2)^{-p}\, G_n^{(\alpha,p)}\left(\frac{x(1+t_2)}{1-t_2^{-1}}\right) = \sum_{s=0}^{\infty} \frac{1}{s!}(n+1)_s\, G_{n+s}^{(\alpha-s,p)}(x)\, t_2^s$$

$$(1.5) \quad (t_3-x)^{-p}\, t_3^p\, (t_3+1)\left(1+\frac{1}{w}(t_3+1)\right)^n G_n^{(\alpha,p)}\left(\frac{x(t_3+1)(1+\frac{1}{w})-\frac{1}{w}t_3-\frac{1}{w}}{(t_3-x)\{1+\frac{1}{w}(t_3+1)\}}\right)$$

$$= \sum_{s=0}^{\infty}\sum_{r=0}^{\infty} \frac{1}{s!}\,\frac{(-1/w)^r}{r!}\,(n+1)_s\,(\alpha+p+1)_r\; G_{n-r+s}^{(\alpha+r-s,\,p)}(x)\; t_3^{\alpha+r-s}$$

2. Group of operators

We replace $\frac{d}{dx}$ by $\frac{\partial}{\partial x}$, α by $z\frac{\partial}{\partial z}$, n by $t\frac{\partial}{\partial t}$ and w by $u(x,z,t)$ in (1.2) to get the partial differential equation

$$(2.1) \quad x(1+x)\frac{\partial^2 u}{\partial x^2} - xz\frac{\partial^2 u}{\partial x \partial z} + t\frac{\partial^2 u}{\partial t \partial x} + (p+1)\,x\frac{\partial u}{\partial x} - zp\frac{\partial u}{\partial z} = 0$$

Thus $u = (x,t,z) = G_n^{(\alpha,p)}(x)\, z^\alpha\, t^n$ is a solution of the equation (2.1), since $G_n^{(\alpha,p)}(x)$ is a solution of (1.1).

We now use the first order partial differential operators

$$A_3 = (1+x)\,\frac{z}{t}\,\frac{\partial}{\partial x} - z\frac{\partial}{\partial t}$$

and

$$A_4 = x(1+x)\,\frac{t}{z}\,\frac{\partial}{\partial x} + t\frac{\partial}{\partial z} + \frac{pxt}{z}$$

such that

$$A_3\left[G_n^{(\alpha,p)}(x)\, z^\alpha\, t^n\right] = (\alpha+p+1)\, G_{n-1}^{(\alpha+1,p)}(x)\; z^{\alpha+1}\; t^{n-1}$$

and

$$A_4\left[G_n^{(\alpha,p)}(x)\, z^\alpha\, t^n\right] = (n+1)\; G_{n+1}^{(\alpha-1,p)}(x)\; z^{\alpha-1}\; t^{n+1}$$

To find the group of operators, let us write $A_1 = t\frac{\partial}{\partial t}$ and $A_2 = z\frac{\partial}{\partial z}$ such that

$$A_1\left[G_n^{(\alpha,p)}(x)\, z^\alpha\, t^n\right] = n\; G_n^{(\alpha,p)}(x)\; z^\alpha\; t^n$$

and

$$A_2 \,[\, G_n^{(\alpha\,.\,p)}(x)\, z^{\alpha}\, t^n \,] = G_n^{(\alpha\,.\,p)}(x)\; z^{\alpha}\; t^n$$

We shall find the commutator relations by using the commutator notation with $[A, B\,]\, u = [AB - BA\,]\, u$ as

$$\text{(2.2)} \qquad \begin{aligned} &[A_1\,, A_2] = 0, \quad [A_2\,, A_3] = A_3\,, \quad [A_2\,, A_4] = -A_4 \\ &[A_1\,, A_3] - A_3\,, \ [A_1\,, A_4] = A_4\,, \quad [A_3\,, A_4] = A_1 - A_2 + P. \end{aligned}$$

3. Lie algebra

The commutator relations show that the set of operators 1, A_i $(i = 1, 2, 3, 4)$ generate a Lie algebra. The partial differential operator L given by

$$\text{(3.1)} \qquad L = x(1+x)\,\frac{\partial^2}{\partial x^2} + z\,\frac{\partial^2}{\partial x \partial z} - tz\,\frac{\partial^2}{\partial t \partial x} + (p+1)\,x\frac{\partial}{\partial x} - zp\frac{\partial}{\partial z} = 0$$

can be expressed as

$$\text{(3.2)} \qquad (1+x)\,L = A_4\; A_3\; + (1-p)\,A_1\; + A_1\; A_2\,.$$

We can easily show that the operator $(1 + x)L$ commutes with each of the operators A_i $(i = 1, 2, 3, 4)$ i.e.,

$$\text{(3.3)} \qquad [(1+x)\,L, A_i\,] = 0.$$

The extended form of the group generated by A_i $(i = 1, 2, 3, 4)$ are given by

(i) $\quad e^{a_1 A_1}\, u(x\,.z\,.t) = u(x\,.z\,,e^{a_1}\, t)$

(ii) $\quad e^{a_2 A_2}\, u(x\,.z\,\,t) = u(x\,.e^{a_2}\, z\,,t)$

(iii) $\quad e^{a_3 A_3}\, u(x\,.z\,.t) = u\left(\dfrac{x + a_3 \dfrac{z}{t}}{1 - a_3 \dfrac{z}{t}}\,, z\,, t - a_3 z \right)$

(iv) $\quad e^{a_4 A_4}\, u(x\,.z\,.t) = (1 - a_4\,\dfrac{xt}{z})^{\;p}\, u\left(\dfrac{x(1 + a_4\,\dfrac{t}{z})}{1 - a_4\,\dfrac{xt}{z}}\,, z + a_4\, t, t \right)$

where a_i $(i = 1, 2, 3, 4)$ are constants.
Making use of the above relations (i).. (iv), we obtain

$$\text{(3.4)} \qquad e^{a_4 A_4}\; e^{a_3 A_3}\; e^{a_2 A_2}\; e^{a_1 A_1}\; u(x\,, z\,, t) = (1 - a_4\,\frac{xt}{z})^{-p}\; u(\xi\,, \eta\,, \rho)\,,$$

where

$$\xi = \frac{x\,(1 + a_4 \frac{t}{z})\,(1 - a_3 a_4) + a_3 \frac{z}{t} + a_3 a_4}{(1 - a_4 \frac{xt}{z})\,(1 - a_3 \frac{z}{t} - a_3 a_4)}$$

$$\eta = e^{a_2}\,(z + a_4 t)$$

$$\rho = e^{a_1}\,[\,t - a_3\,(z + a_4\,t)\,]$$

From the commutation relation (2.2) we observe that

$$(3.5)\quad \exp\,(a_4 A_4 + a_3 A_3 + a_2 A_2 + a_1 A_1)$$
$$\neq \exp\,(a_4 A_4)\,\exp\,(a_3 A_3)\,\exp\,(a_2 A_2)\,\exp\,(a_1 A_1)$$

We have used the operator mentioned in the right side of (3.5) to get the relation (3.3).The order of $A_i\,(i = 1, 2, 3, 4)$ can be changed without changing its effect in the left side of (3.5), while that cannot be changed in the right side of (3.5). So if we change the order of $A_i\,(i = 1, 2, 3, 4)$ in the left side of (3.4) we get a different relation.

4. Generating Functions:

From the relation (2.1) we can easily show that $u(x, z, t) = G_n^{(\alpha\,,\,p)}\,(x)\,z^{\alpha}\,t^{n}$ is a solution of the systems

$$(4.1)\qquad \begin{cases} Lu = 0 \\ (A_1 - \alpha)\,u = 0\,; \end{cases} \begin{cases} Lu = 0 \\ (A_2 - n)\,u = 0\,; \end{cases} \begin{cases} Lu = 0 \\ (A_1 + A_2 - n)\;u = 0\,; \end{cases}$$

Since $[(1 + x)\,L, A_i] = 0$, $(i = 1, 2, 3, 4)$, we have the result

$$(4.2)\qquad SL\,(G_n^{(\alpha\,,\,p)}\,(x)\,z^{\alpha}\,t^{n}) = LS\,(G_n^{(\alpha\,,\,p)}\,(x)\,z^{\alpha}\,t^{n}) = 0,$$

where

$$S = e^{a_4 A_4}\;e^{a_3 A_3}\;e^{a_2 A_2}\;e^{a_1 A_1}$$

The transformation $S\,[\,G_n^{(\alpha , p)}\,(x)\,z^{\alpha}\,t^{n}\,]$ is annulled by the operator L, and thus it gives generating functions

We put $a_1 = a_2 = 0$ in (3.3) to arrive at

$$(4.3)\qquad e^{a_4 A_4}\;e^{a_3 A_3}\,[G_n^{(\alpha , p)}\,(x)\,z^{\alpha}\,t^{n}\,]$$

$$= (1 - a_4 \frac{xt}{z})^{-p}\,G\,(\,\frac{x\,(1 + a_4 \frac{t}{z})\,(1 - a_3 a_4) + a_3\,(\frac{z}{t} + a_4)}{(1 - a_4 \frac{t}{z})\,(1 - a_4 \frac{t}{z} - a_3 a_4)}\,)$$

$$\times\;(z + a_4\,t)\,[\,t - a_3\,(z + a_4\,t)\,]\,n.$$

$$= \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} \frac{a_3^r}{r!} \frac{a_4^s}{s!} (n+1)_s (\alpha+p+1)_r G_{n-r+s}^{(\alpha+r-s,\,p)}(x) \times z^{\alpha+r-s} t^{n-r+s}$$

To arrive at the desired results, we consider the following three cases

Case 1: We set $a_3 = 1$, $a_4 = 0$ and $\frac{z}{t} = t_1$ in (4.3) to get :

$$(4.4) \quad (1-t_1)^n \; G_n^{(\alpha,\,p)} \left(\frac{x+t_1}{1-t_1}\right) = \sum_{r=0}^{\infty} \frac{1}{r!} (\alpha+p+1)_r \; G_{n-r}^{(\alpha+r,\,p)}(x) t_1^r$$

which is the relation (1.3)

Case 2: We substitute $a_3 = 0$, $a_4 = 1$ and $\frac{t}{z} = t_2$ in (4.3) to obtain

$$(4.5) \quad (1+t_2)(1-xt_2)^{-p} \; G_n^{(\alpha,\,p)} \left(\frac{x(1+t_2)}{1-t_2}\right) = \sum_{s=0}^{\infty} \frac{1}{s!} (n+1)_s \; G_{n+s}^{(\alpha-s,\,p)}(x) t_2^s$$

which is (1.4)

Case 3: Finally we let $a_3 = -\frac{1}{w}$, $a_4 = 1$ and $\frac{t}{z} = t_3$, to arrive at

$$(4.6) \quad (t_3-x)^{-p} \; t_3^p \; (t_3+1)^{\alpha} \; \{1+\frac{1}{w}(t_3+1)\}^n \times G_n^{(\alpha,\,p)} \left(\frac{x(t_3+1)(1+\frac{1}{w}) - \frac{1}{w} t_3^2 - \frac{1}{w} t_3}{(t_3-x)\{1+\frac{1}{w}(t_3+1)\}} \right)$$

$$= \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-\frac{1}{w})^r}{s!\,r!} (n+1)_s (\alpha+p+1)_r G_{n-r+s}^{(\alpha+r-s,\,p)}(x) t_3^{\alpha+r-s}$$

which is the relation (1.5).

Thus we have derived all desired results.

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A Generalised Laplace Transform Of Generalised Functions

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1. The Integral Transform

$$F(s) = 2^{-\nu/2} \int_0^\infty (st)^\lambda \; e^{-\frac{1}{2}st} \; \mathcal{D}_\nu(\sqrt{2st}) \; f(t)\,dt$$

where $\mathcal{D}_\nu$ denotes Weber's parabolic cylinder function, studied byTiwari, B.M.L[4], has recently been extended by the authors to a class of generalised functions. This transform reduces to Laplace transform for $\lambda = \nu = 0$ and will be called Weber transform. In this paper we have proved an inversion formula for the generalised Weber transform and a uniqueness theorem for it. A structure formula for a class of Weber transformable generalized functions has also been obtained.

Let $u = x^\lambda \; e^{-\frac{1}{2}x} \; \mathcal{D}_\nu(\sqrt{2x})$,

where $\mathcal{D}_n(z) = e^{-\frac{1}{4}z^2} \; z^n \; \{1 - \frac{n(n-1)}{2z^2} + \frac{n(n-1)(n-2)(n-3)}{2.4z^4} - \ldots\}$

then, using a differential relation for Weber's parabolic cylinder function, we have

$$x^{\frac{1}{2}+\lambda} \; \frac{d}{dx} \, [x^{\frac{1}{2}} \; \frac{d}{dx} \; (x^{-\lambda} \; u)] = \frac{1}{2} \; x^\lambda \; e^{-\frac{1}{2}x} \; \mathcal{D}_{\nu+2} \; (\sqrt{2x})$$

since $\quad \frac{d^n}{dz^n} \; [e^{-\frac{1}{4}z^2} \; \mathcal{D}_\nu \; (z)] = (-1)^n \; e^{-\frac{1}{4}z^2} \; \mathcal{D}_{\nu+n} \; (z) \; ; \; n = 1, 2, 3 \ldots$

Further, if we define an operator A_λ by

$$A_{\lambda,x} \; \phi(x) = x^{\frac{1}{2}+\lambda} \; [D_x \, x^{\frac{1}{2}} \; D_x \; \{x^{-\lambda} \; \phi(x)\}] \, ,$$

where $D_x = \frac{d}{dx}$ then it has been shown in [2] that

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$$A_{\lambda,t}\ \{(st)^{\lambda}\ e^{-\frac{1}{2}st}\ \mathcal{B}_{\nu}(\sqrt{2st})\} = \frac{s}{2}\ (st)^{\lambda}\ e^{-\frac{1}{2}st}\ \mathcal{B}_{\nu+2}\ (\sqrt{2st}),$$

$$A^{n}_{\lambda,t}\ \{(st)^{\lambda}\ e^{-\frac{1}{2}st}\ \mathcal{B}_{\nu}(\sqrt{2st})\} = (\frac{s}{2})^{n}\ (st)^{\lambda}\ e^{-\frac{1}{2}st}\ \mathcal{B}_{\nu+2n}\ (\sqrt{2st})$$

$$n = 0, 1, 2\ \ldots\ldots$$

and that

$$A^{n}_{\lambda,t}\ \{(st)^{\lambda}\ e^{-\frac{1}{2}st}\ \mathcal{B}_{\nu}(\sqrt{2st})\} < \infty$$

for large and small t provided that Re $s > 0$.

Let α and β be real numbers and λ, a complex number with $Re\ \lambda > 0$.

Let $K_{\alpha,\beta}(I)$ be the set of all those complex valued smooth functions $\phi(t)$ defined on $I(0,\infty)$ for which the functionals.

$$\partial^{\lambda}_{\alpha,\beta,n}(\phi) = \sup_{0<t<\infty}\ |e^{\alpha t}\ t^{\beta+n}\ A^{n}_{\lambda,t}\phi(t)|$$

are finite for $n = 0, 1, 2, \ldots\ldots$

With the usual pointwise operations of addition of functions and multiplication by a complex number, $K_{\alpha,\beta}(I)$ is a linear space. The collection $M = \{\partial^{\lambda}_{\alpha,\beta,n}\}^{\infty}_{n=0}$ also forms a countable multinorm on $K_{\alpha,\beta}(I)$ and equipped with the topology generated by this multinorm, $K_{\alpha,\beta}(I)$ becomes a countably multinormed space which is also complete. It also satisfies all the conditions for being a testing function space and its dual $K'_{\alpha,\beta}(I)$ is also complete. We call f a Weber–transformable generalised function if it is a member of $K'_{\alpha,\beta}(I)$.

The Weber transform $F(s)$ of $f \ \varepsilon\ K'_{\alpha,\beta}(I)$ is defined by

$$F(s) = (\mathcal{B}_{\lambda,\nu}f)\ (s) = \langle f(t), \omega(st)\rangle,$$

where $\omega(st) = 2^{-\frac{\nu}{2}}\ (st)^{\lambda}\ e^{-\frac{1}{2}st}\ \mathcal{B}_{\nu}(\sqrt{2st})$; Re $> \alpha$, Re $\lambda + \beta > 0$ and $s\ \varepsilon\ \Omega_f$

The region Ω_f is defined by

$$\Omega_f = \{\, s \mid \text{Re}\ s > \sigma_f\ , s \neq 0\ , -\frac{3}{4}\pi < \arg\ s < \frac{3}{4}\pi\,\},$$

where σ_f is a real number (possibly $\sigma_f = -\infty$) such that $f \in K'_{\alpha,\beta}(I)$ for every $\alpha > \sigma_f$ and $f \notin K'_{\alpha,\beta}(I)$ for $\alpha > \sigma_f$.

$D(I)$ will denote the standard countable union space [Zemanian, 6, pp.32, 33] of the countably multinormed spaces $D_k(I)$, of all complex-valued smooth functions defined on $I(0, \infty)$ which vanish on those points of I which are not in a compact subset K of I, with seminorms defined by

$$\gamma_k(\phi) = \sup_{t \in I} \left| D^k \phi(t) \right|, \phi \in D_k(I)$$

and the topology generated by the countable multinorm $\{\gamma_k\}_{k=0}^{\infty}$ assigned to the corresponding linear space with usual pointwise operations of addition and multiplication of functions.

2. We now prove an inversion formula which determines the restriction to $D(I)$ of any $\vartheta_{\lambda,\nu}$ –transformable generalised function from its Weber transform and then give a weak version of a uniqueness theorem.

Let us first prove a few lemmas.

Lemma 2.1 : *If* $f \in K'_{\alpha,\beta}(I)$, *then*

$$\int_0^\infty x^{-s} \langle f(u), \omega(xu) \rangle \, dx = \langle f(u), \int_0^\infty x^{-s} \omega(xu) \, dx \rangle ,$$

where $\omega(xu) = 2^{-\frac{\lambda}{2}} (xu)^{\lambda} e^{-\frac{1}{2}xu} \vartheta_\nu(\sqrt{2xu})$, Re $s > \alpha$, Re $\lambda + \beta > 0$.

Proof : It is clear that

$$x^{-s} \langle f(u), \omega(xu) \rangle = \langle f(u), x^{-s} \omega(xu) \rangle$$

Let $1_{0,\infty}(x)$ denote both the functions

$$1_{0,\infty}(x) = 0, \ x \le 0$$
$$= 1, \ 0 < x < \infty$$

and the corresponding generalised function belonging to $K'_{\alpha,\beta}(I)$. Then using the definition of product of generalised functions [Zemanian [6], p.121], we have

(2.1) $$\langle 1_{0,\infty}(x) \, f(u), x^{-s} \, \omega(xu) \rangle = \langle 1_{0,\infty}(x), \langle f(u), x^{-s} \omega(xu) \rangle$$

(2.2) $$= \int_0^\infty \langle f(u), x^{-s} \, \omega(xu) \rangle \, dx$$

Step (2.1) is justified since $k^{-s} \omega(xu) \in K_{\alpha,\beta}(I)$.

Step (2.2) is obvious since $1_{0,\infty}(x)$ is a regular generalised function

Since the product of generalised functions is commutative, the left hand side of (2.1) can be written in the form

$$\langle f(u)\ 1_{0,\infty}(x)\ ,\ x^{-s}\ \omega(xu) \rangle \tag{2.3}$$

$$= \langle f(u), \langle 1_{0,\infty}(x), x^{-s}\ \omega(xu) \rangle \rangle \tag{2.4}$$

$$= \langle f(u), \int_0^{\infty} x^{-s}\ \omega(xu)\ dx \rangle \tag{2.5}$$

Thus, (2.2) = (2.5) which proves the lemma.

Lemma 2.2 *Let $\phi \in D(I)$ and r be a fixed real number. Let*

$$\psi(s) = \int_0^{\infty} t^{-s}\ \phi(t)\ dt,$$

where $s = c + iT$, c fixed, and $\max(\sigma_f, 1) < c < \infty, -\infty < T < \infty$

If $f \in K'_{\alpha,\beta}(I)$ and $\alpha < 0$, then

$$\frac{1}{2\pi i} \int_{-r}^{r} \langle f(u), u^{s-1} \rangle\ \psi(s)\ dT = \langle f(u), \frac{1}{2\pi i} \int_{-r}^{r} u^{s-1}\ \psi(s)\ dT \rangle \tag{2.6}$$

Proof : For $\phi(t) = 0$, the proof is trivial. Let $\phi(t) \neq 0$ and let

$$\langle f(u), u^{s-1} \rangle = \lambda(s) \tag{2.7}$$

(2.7) is justified since $u^{s-1} \in k_{\alpha,\beta}(I)$ for Re $s > 1-\beta$. It can be seen that $\lambda(s)$ is analytic for all s for which $\max(\sigma_f, 1) <$ Re $s < \infty$ and $\Psi(s)$ is also analytic for all finite values of s.

Thus the left hand side of (2.6) is an integral with an integrand analytic over a finite region and hence converges uniformly. Now

$$|\ e^{\alpha u}\ u^{\beta+n}\ A^n_{\lambda,u}\ \{ \frac{1}{2\pi i} \int_{-r}^{r} u^{s-1}\ \psi(s)\ dT \}\ |$$

$$= |\ e^{\alpha u}\ u^{\beta+n} \frac{1}{2\pi i} \int_{-r}^{r} \{ A^n_{\lambda,u}\ u^{s-1} \}\ \psi(s)\ dT\ |$$

$$= |\ e^{\alpha u}\ u^{\beta+n} \frac{1}{2\pi i} \int_{-r}^{r} (-\lambda+s-1)(-\lambda+s-\frac{3}{2}) \ldots (-\lambda+s-n)(-\lambda+s-\frac{2n+1}{2})$$

$$u^{s-n-1}\ \psi(s)\ dT\ |$$

$$\leq \frac{1}{2\pi} \int_{-r}^{r} \left| e^{\alpha u} u^{\beta + s - 1} \right| \; |(-\lambda + s - 1)(-\lambda + s - \frac{3}{2}) \ldots$$

$$(-\lambda + s - n)(-\lambda + s - \frac{2n+1}{2}) \; \psi(s) \; dT \,|$$

As $\int_{-r}^{r} (-\lambda + s - 1)(-\lambda + s - \frac{3}{2}) \ldots (-\lambda + s - n)(-\lambda + s - \frac{2n+1}{2}) \, \psi(s) \, dT$

is finite and $\sup\limits_{0 < u < \infty} | e^{\alpha u} \; u^{\beta + s - 1} | < \infty$ for Re $s > 1 - \beta$, $\alpha < 0$, we see that

$$\sup_{0 < u < \infty} \; | \; e^{\alpha u} \, u^{\beta + n} \, A^{n}_{\lambda, u} \; \{ \int_{-r}^{r} u^{s-1} \, \psi(s) \; ds \} \, | < \infty$$

proving that $\int_{-r}^{r} u^{s-1} \, \Psi(s) \, dT$ as a function of u belongs to $K_{\alpha \, . \, \beta} \, (I)$. Hence the right hand side of (2.6) is also meaningful.

Now to prove the equality, let us partition the path of integration on the straight line from $C - ir$ to $C + ir$ into m sub–intervals each of length $\frac{2r}{m}\cdot$ Let $s_P = C + iT_P$ be a point in the pth interval. We can write

$$\frac{1}{2\pi} \int_{-r}^{r} \langle f(u), u^{s-1} \rangle \; \psi(s) \; dT$$

$$= \lim_{m \to \infty} \sum_{p=1}^{m} \frac{1}{2\pi} \langle f(u), \, u^{s_p - 1} \rangle \; \psi(s_p) \; \frac{2r}{m}$$

$$= \lim_{m \to \infty} \; \langle \; f(u), \sum_{p=1}^{m} \frac{1}{2\pi} \; u^{s_p - 1} \; \psi(s_p) \; \frac{2r}{m} \; \rangle \tag{2.8}$$

Let us set

$$V_m(u) = \sum_{p=1}^{m} u^{s_p - 1} \; \psi(s_p) \; \frac{2r}{m}$$

If we can show that the sum within the last expression in (2.8) converges in $K_{\alpha \, . \, \beta} \, (I)$ to $\int_{-r}^{r} u^{s-1} \, \psi(s) \; dT$, the equality (2.6) will be proved.

Let us consider $B(u, m)$ where

$$B(u,m) = e^{\alpha u} u^{\beta+n} A^{n}_{\lambda,u} \left[V_m(u) - \int_{-r}^{r} u^{s-1} \psi(s)\, dT \right]$$

$$= e^{\alpha u} u^{\beta+n} \left[A^{n}_{\lambda,u} V_m(u) - A^{n}_{\lambda,u} \int_{-r}^{r} u^{s-1} \psi(s)\, dT \right]$$

$$= e^{\alpha u} u^{\beta+n} \sum_{p=1}^{m} \Big[(-\lambda+s_p-1)(-\lambda+s_p-\tfrac{3}{2})\ldots(-\lambda+s_p-n)$$

$$\times (-\lambda+s_p-\frac{2n+1}{2})\ u^{s_p-1-n}\ \psi(s_p)\ \frac{2r}{m} \Big]$$

$$- e^{\alpha u} u^{\beta+n} \int_{-r}^{r} (-\lambda+s-1)(-\lambda+s-\tfrac{3}{2})$$

$$\text{(2.9)} \qquad \ldots (-\lambda+s-n)(-\lambda+s-\frac{2n+1}{2})\ u^{s-1-n}\ \psi(s)\ dT$$

We have to show that $B(u, m)$ converges uniformly to zero on $0 < u < \infty$ as $m \to \infty$ For $\alpha < 0$, we see that

$$\left| e^{\alpha u}\ u^{\beta+s-1}\ (-\lambda+s-1)(-\lambda+s-\tfrac{3}{2})\ldots(-\lambda+s-n)(-\lambda+s-\frac{2n+1}{2}) \right|$$

tends uniformly to zero on $-r \le T \le r$ as $u \to \infty$. Consequently, given $\in > 0$, there exists a $u' > 0$ such that for $u > u' > 0$ and $-r \le T \le r$,

$$\left| e^{\alpha u}\ u^{\beta+s-1}\ (-\lambda+s-1)(-\lambda+s-\tfrac{3}{2})\ldots(-\lambda+s-n)(-\lambda+s-\frac{2n+1}{2}) \right|$$

$< \dfrac{\in}{3} \left[\int_{-r}^{r} \left| \psi(s)\ dT \right| \right]^{-1}$ as $\int_{-r}^{r} \left| \psi(s)\ dT \right|$ is finite and $\neq 0$ since $\phi(t) \neq 0$.

It follows that

$$\text{(2.10)} \qquad \sup_{u>u'} \left| e^{\alpha' u} u^{\beta+n} A^{n}_{\lambda,u} \frac{1}{2\pi} \int_{-r}^{r} u^{s-1}\ \psi(s)\, dT \right| < \frac{\in}{3}$$

Also for all m,

$$\text{(2.11)} \quad \sup_{u>u'} \left| e^{\alpha u} u^{\beta+n} A^{n}_{\lambda,u} \{V_m(u)\} \right| < \frac{\in}{3} \left[\int_{-r}^{r} \left| \psi(s)\, dT \right| \right]^{-1} \frac{2r}{m} \sum_{p=1}^{m} \left| \psi(S_p) \right|$$

Thus there exists an m_0 such that $m > m_0$, the right hand side is bounded by $\dfrac{2\in}{3}$

From (2.10) and (2.11) we see that for $m > m_0$, $u > u'$, $|B(u, m)| < \in$.

Let us now consider the range $0 < u \leq u'$ with $\in$ fixed in $\max(\sigma_f, 1) < \text{Re } s < \infty$.

We see that

$$u^{\beta+s-1}\ (-\lambda+s-1)(-\lambda+s-\frac{3}{2})\ldots(-\lambda+s-n)\ (-\lambda+s-\frac{2n+1}{2})\ \psi(s)$$

is an uniformly continuous function of (u,T) for $0 < u < u'$ and $-r \leq T < r$. This together with (2.9) shows that there exists m_1 such that for all $m > m_1$, $|B(u,m)| < \in$ on $0 < u < u'$ as well.

Thus when $m > \max(m_0, m_1)$, we have $|B(u,m)| < \in$ uniformly on $0 < u < \infty$. Hence the lemma.

Lemma 2.3 *Let* (i) $\phi \in D(I)$, (ii) α, β, c *and* r *be real numbers such that* $\max(\sigma_f, 1) < c < \infty$ *and* $\alpha < 0$. *Then*

$$\frac{1}{\pi}\int_0^\infty \phi(t)\ (\frac{u}{t})^c\ \frac{\sin\ r\ \log\frac{u}{t}}{u\ \log\frac{u}{t}}\ dt \to \phi(u)$$

in $K_{\alpha,\beta}'(I)$, as $r \to \infty$.

Proof: Let

$$I = \frac{1}{\pi}\int_0^\infty \phi(t)\ (\frac{u}{t})^c\ \frac{\sin r\log\frac{u}{t}}{u\log\frac{u}{t}}\ dt$$

Putting $u = te^x$ in I we have

$$I = \frac{1}{\pi}\int_{-\infty}^\infty \phi(ue^{-x})\ e^{(c-1)x}\ \frac{\sin rx}{x}\ dx$$

Hence

$$I - \phi(u) = \frac{1}{\pi}\int_{-\infty}^\infty [\,e^{(c-1)x}\ \phi(ue^{-x}) - \phi(u)\,]\ \frac{\sin rx}{x}\ dx$$

since $\displaystyle\int_{-\infty}^\infty \frac{\sin rx}{x}\ dx = \pi$.

Let

$$\theta_r(u) = e^{\alpha u}\ u^{\beta+n}\ A^n_{\lambda,u}\ \frac{1}{\pi}\int_{-\infty}^\infty [\,e^{(c-1)x}\ \phi(ue^{-x}) - \phi(u)\,]\ \frac{\sin rx}{x}\ dx.$$

Our lemma will be proved if we are able to prove that $Q_r(u) \to 0$ uniformly on $0 < u < \infty$ as $r \to \infty$; $n = 0, 1, 2, \ldots$ taking the differential operator inside the integral sign, we have

$$\theta_r(u) = \frac{1}{\pi}\, e^{\alpha u}\, u^{\beta+n} \int_{-\infty}^{\infty} [\, e^{(c-1)x}\, A^{n}_{\lambda,u}\, \phi(ue^{-x}) - A^{n}_{\lambda,u}\, \phi(u)\,]\, \frac{\sin rx}{x}\, dx$$

$$= \frac{1}{\pi}\, e^{\alpha u}\, u^{\beta+n}\, [\int_{-\infty}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\infty}]$$

$$= I_1 + I_2 + I_3, \text{ say}$$

Let us consider I_2 first and set

$$R(x,u) = e^{\alpha u}\, u^{\beta+n}\, [\frac{e^{(c-1)x}\, A^{n}_{\lambda,u}\, \phi(ue^{-x}) - A^{n}_{\lambda,u}\, \phi(x)}{x}].$$

By virtue of our supposition it is evident that $R(x, u)$ is a continuous function of (x, u) for all u in $0 < u < \infty$ and $x \neq 0$.
Also

$$\lim_{x \to 0} R(x,u) = \lim_{x \to 0} e^{\alpha u}\, u^{\beta+n}\, D_x\, [\, e^{(c-1)x}\, A^{n}_{\lambda,u}\, \phi(ue^{-x})\,]$$

by L' Hospital rule.
Hence assigning the value

$$e^{\alpha u}\, u^{\beta+n}\, D_x\, [\, e^{(c-1)x}\, A^{n}_{\lambda,u}\, \phi(ue^{-x})\,]\Big|_{x=0}$$

to $R(o,u)$, we see that $R(x,u)$ is a continuous function of (x,u) in $-\delta < x < \delta$; $0 < u < \infty$ and since $\phi(u)$ is smooth, $R(x,u)$ is bounded, say by k. Hence, for any $\in > 0$, there exists a $\delta > 0$ so small that

$$|I_2(u)| = |\, \frac{1}{\pi} \int_{-\delta}^{\delta} R(x,u) \sin rx\, dx\, |$$

$$\leq \frac{1}{\pi} \int_{-\delta}^{\delta} |\, R(x,u) \sin rx\, |\, dx$$

$$\leq \frac{1}{\pi} \int_{-\delta}^{\delta} |\, R(x,u)\, |\, dx$$

or, $I_2(u) \leq k\dfrac{2\delta}{\pi} < \in$, if δ is so fixed that it is less than $\dfrac{\pi \in}{2k}$. Now let us consider $I_1(u)$.

$$I_1(u) = \frac{1}{\pi}\, e^{\alpha u}\, u^{\beta+n} \int_{-\infty}^{-\delta} e^{(c-1)x}\, A_{\lambda,u}^{n}\, \phi(ue^{-x})\, \frac{\sin rx}{x}\, dx$$

$$- \frac{1}{\pi}\, e^{\alpha u}\, u^{\beta+n} \int_{-\infty}^{-\delta} A_{\lambda,u}^{n}\, \phi(u)\, \frac{\sin rx}{x}\, dx$$

$$= J_1(u) - J_2(u), \text{ say}$$

We have

$$J_2(u) = \frac{1}{\pi}\, e^{\alpha u}\, u^{\beta+n}\, [A_{\lambda,u}^{n}\, \phi(u)] \int_{-\infty}^{-r\delta} \frac{\sin z}{z}\, dz.$$

Now as $e^{\alpha u}\, u^{\beta+n}\, A_{\lambda,u}^{n}\, \phi(u)$ is bounded on $0 < u < \infty$ and $\int_{-\infty}^{0} \frac{\sin z}{z}\, dz$ is convergent, $J_2(u)$ tends uniformly to zero in $0 < u < \infty$ as $r \to \infty$, since

$$\lim_{r\to\infty} \int_{-\infty}^{-r\delta} \frac{\sin z}{z}\, dz = 0$$

On integration by parts,

$$J_1(u) = \frac{1}{\pi}\, e^{\alpha u}\, u^{\beta+n} \left[-\frac{e^{(c-1)x}}{x}\, A_{\lambda,u}^{n}\, \phi(ue^{-x})\, \frac{\cos rx}{r} \right]_{-\infty}^{-\delta}$$

$$+ \frac{1}{\pi r}\, e^{\alpha u}\, u^{\beta+n} \int_{-\infty}^{-\delta} \cos rx\, D_x \left\{ \frac{e^{(c-1)x}}{x}\, A_{\lambda,u}^{n}\, \phi(ue^{-x}) \right\} dx$$

Since $\phi(u) \in D(I)$ i.e., is of compact support, and $c > 1$

$$J_1(u) = \frac{1}{\pi r}\, e^{\alpha u}\, u^{\beta+n} \left[\frac{e^{-(c-1)\delta}\, A_{\lambda,u}^{n}\, \phi(ue^{\delta})}{\delta} \cos r\delta \right] \tag{2.12}$$

$$+ 0 + \frac{1}{\pi r}\, e^{\alpha u}\, u^{\beta+n} \int_{-\infty}^{-\delta} \cos rx\, D_x \left[\frac{e^{(c-1)x}}{x}\, A_{\lambda,n}^{n}\, \phi(ue^{-x}) \right] dx.$$

First term of (2.12) tends uniformly to zero in $0 < u < \infty$ as $r \to \infty$, since δ and c are fixed and $e^{\alpha u}\, A_{\lambda,u}^{n}\, \phi(ue^{-x})$ is a bounded function of u in $0 < u < \infty$.

Also

$$e^{\alpha u}\, u^{\beta+n}\, D_x \left[\frac{e^{(c-1)x}}{x}\, A^{n}_{\lambda,u}\, \phi(ue^{-x}) \right]$$

$$= e^{\alpha u}\, u^{\beta+n}\, \frac{(cx-x-1)\, e^{(c-1)x}}{x^2}\, A^{n}_{\lambda,u}\, \phi(ue^{-x})$$

$$+\, e^{\alpha u}\, u^{\beta+n}\, \frac{e^{(c-1)x}}{x}\, \frac{\partial}{\partial x} \left[A^{n}_{\lambda,u}\, \phi(ue^{-x}) \right].$$

Since each term is a bounded function of u and x in $0 < u < \infty$, $-\infty < x < -\delta$, the second term in the right hand side of (2.12) also goes uniformly to zero as $r \to \infty$.

Thus we see that $J_1(u) \to 0$ as $r \to \infty$ and combined with the fact that $j_2(u) \to 0$ uniformly in $0 < u < \infty$ as $r \to \infty$ as $r \to \infty$, we see that $I_1(u) \to 0$ uniformly in $0 < u < \infty$ as $r \to \infty$.

Similarly we can prove that $I_3(u) \to 0$ uniformly in $0 < u < \infty$ as $r \to \infty$.

Considering all these results we see that $\theta_r(u) < \in$, $r \to \infty$, $0 < u < \infty$, $\in > 0$ being arbitrary small. Hence the lemma.

3. Complex Inversion Formula

Theorem : Let (i) $f \in K'_{\alpha,\beta}(I)$, (ii) $F(x)$ be defined by $F(x) = \langle F(u), \omega(xu) \rangle$ and (iii) α, β, c be real numbers with max $(\sigma_f, 1) < c < \infty, \alpha < 0$ and $\operatorname{Re}(\lambda - s + 1) > 0$, $\operatorname{Re}\left(\lambda - s - \frac{\nu}{2} + \frac{3}{2}\right) > 0$, $s = c + iT$, $\operatorname{Re} s > \alpha$; $\operatorname{Re} \lambda + \beta > 0$

Then, for $\phi(y) \in D(I)$,

$$\left\langle \frac{1}{2\pi i} \int_{c-ir}^{c+ir} \frac{\Gamma\left(\lambda + \frac{3}{2} - \frac{\nu}{2} - s\right) t^{-s}}{\Gamma(\lambda + 1 - s)\, \Gamma\left(\lambda + \frac{3}{2} - s\right)}\, \Psi(s)\, ds,\ \phi(t) \right\rangle$$

$$\to \langle f, \phi \rangle \text{ as } r \to \infty,$$

where $$\Psi(s) = \int_0^{\infty} x^{-s}\, F(x)\, dx.$$

Proof : The theorem will be proved by justifying the following steps:

$$\langle \frac{1}{2\pi i} \int_{c-ir}^{c+ir} \frac{\Gamma(\lambda+\frac{3}{2}-\frac{\nu}{2}-s)\ t^{-s}}{\Gamma(\lambda+1-s)\ \Gamma(\lambda+\frac{3}{2}-s)} \Psi(s)\ ds,\ \phi(t)\ \rangle$$

(3.1) $$= \langle \frac{1}{2\pi i} \int_{c-ir}^{c+ir} M(s)\ t^{-s}\ \Psi(s)\ ds,\ \phi(t)\ \rangle$$

where

$$M(s) = \frac{\Gamma(\lambda+\frac{3}{2}-\frac{\nu}{2}-s)}{\Gamma(\lambda+1-s)\ \Gamma(\lambda+\frac{3}{2}-s)}$$

(3.2) $$= \int_0^\infty \frac{1}{2\pi i} \int_{c-ir}^{c+ir} M(s)\ \Psi(s)\ t^{-s}\ ds\ \phi(t)\, dt$$

(3.3) $$= \frac{1}{2\pi} \int_{-r}^{r} M(s)\ \Psi(s) \int_0^\infty t^{-s}\ \phi(t)\, dt\, dT \quad (s = c + iT)$$

(3.4) $$= \frac{1}{2\pi} \int_{-r}^{r} M(s)\ \{ \int_0^\infty x^{-s}\ F(x)\ dx\ \} \int_0^\infty t^{-s}\ \phi(t)\ dt\ dT$$

(3.5) $$= \frac{1}{2\pi} \int_{-r}^{r} M(s)\ \{ \int_0^\infty x^{-s} < f(u), \omega(xu) > ax\ \} \int_0^\infty t^{-s}\ \phi(t)\ dT$$

(3.6) $$= \frac{1}{2\pi} \int_{-r}^{r} M(s)\ \langle f(u), \int_0^\infty \omega(xu)\ dx \rangle \int_0^\infty t^{-s}\ \phi(t)\ dt\ dT$$

(3.7) $$= \frac{1}{2\pi} \int_{-r}^{r} M(s)\ \langle f(u), [M(s)]^{-1}\ u^{s-1} \rangle \int_0^\infty t^{-s}\ \phi(t)\ dt\ dT$$

(3.8) $$= \langle f(u), \frac{1}{2\pi} \int_{-r}^{r} u^{s-1} \int_0^\infty t^{-s}\ \phi(t)\ dt\ dT \rangle$$

(3.9) $$= \langle f(u), \frac{1}{2\pi} \int_0^\infty \phi(t) \int_{-r}^{r} u^{s-1}\ t^{-s}\ dT\ dt \rangle$$

$$(3.10) \qquad = \left\langle f(u), \frac{1}{2\pi}\int_0^\infty \phi(t)\left(\frac{u}{t}\right)^c \frac{\sin r \log \frac{u}{t}}{u \log \frac{u}{t}}\, dt \right\rangle$$

$$(3.11) \qquad = \langle f(u), \phi(u) \rangle.$$

Since the integral in (3.1) is a continuous function of t and $\phi(t)$ is a smooth function of compact support in $(0, \infty)$, (3.1) implies (3.2). As the integrand in (3.2) is continuous on a closed and bounded domain of integration, we can change the order of integration in (3.2) to obtain (3.3). (3.4) and (3.5) are obvious and (3.6) is justified by lemma 2.1

Now we have

$$\int_0^\infty 2^{-\nu/2}\, x^{-s}\, (xu)^{\lambda}\, e^{-\frac{1}{2}xu}\, \mathcal{B}_\nu(\sqrt{2xu})\, dx$$

$$= u^{s-1}\int_0^\infty (xu)^{(\lambda+\frac{5}{4}-s)-\frac{3}{2}}\, e^{-\frac{1}{2}xu}\, W_{\frac{1}{2}\nu+\frac{1}{4},-\frac{1}{4}}(xu)\, d(xu) \quad \text{by [5, p. 347]}$$

$$= u^{s-1}\, \frac{\Gamma(\lambda-s+1)\,\Gamma(\lambda-s+\frac{3}{2})}{\Gamma(\lambda-s-\frac{\nu}{2}+\frac{3}{2})}, \text{ provided}$$

$$\operatorname{Re}(\lambda-s+1)>0,\ \operatorname{Re}\left(\lambda-s-\frac{\nu}{2}+\frac{3}{2}\right)>0 \qquad \text{[Erdelyi , 1, p. 337]}$$

Hence (3.7) is a simplification of (3.6) . (3.8) is justified by lemma 2.2. As the integral in (3.8) converges uniformly we change the order of integration to obtain (3.9). After simplification (3.9) reduces to (3.10). Lemma 2.3 shows that the integral within (3.10) converges in $K_{\alpha,\beta}(I)$ uniformly in $0 < u < \infty$ to $\phi(u)$ as $r \to \infty$ and (3.10) implies (3.11). So the theorem is proved.

4. Uniqueness theorem

Let $f, g \in K'_{\alpha,\beta}(I)$ and

(i) $F(s) = (\mathcal{B}_{\lambda,\nu} f)(s),\ s \in \Omega_f$

(ii) $G(s) = (\mathcal{B}_{\lambda,\nu} g)(s),\ s \in \Omega_g$

(iii) $F(s) = G(s)$ for $s \in \Omega_f \cap \Omega_g$

Then, in the sense of equality in $D'(I)$, $f = g$.

The above weak version of uniqueness is an immediate consequence of the inversion theorem.

5. Structure formula

Now we will give a structure formula for the restriction of an element $f \in K'_{\alpha,\beta}(I)$.

Theorem 5.1 *Let f be an arbitrary element of $K'_{\alpha,\beta}(I)$. There exist bounded measurable functions $g_r(x)$ $(x > 0)$, $r = 0, 1, 2, \ldots, 2q + 1$, q being a non-negative integer depending on f, such that, for an arbitrary $\phi \in D(I)$, we have*

$$(5.1) \quad \langle f, \phi \rangle = \langle \sum_{r=0}^{2q+1} (-1)^r D^{(r+1)} \int_a^\tau g_r(t)\, e^{\alpha t}\, t^\beta\, Q_r(t)\, dt,\ \phi(\tau) \rangle,$$

where D indicates distributional derivative, $Q_r(t)$ is a polynomial in t and a is a positive number.

Proof : In view of the boundedness property of generalised functions there exists a positive constant C and a non-negative integer q such that for all $\phi \in D(I)$

$$|\langle f, \phi \rangle| \le C \max_{0 \le k \le q} \partial^{\lambda}_{\alpha,\beta,k}(\phi)$$

$$\le C \max_{0 \le k \le q} \sup_{0 \le t < \infty} \left| e^{\alpha t}\, t^{\beta + n}\, A^k_{\lambda,t}\, \phi(t) \right|$$

or

$$|\langle f, \phi \rangle| \le C \max_{0 \le k \le q} \sup_{0 \le t < \infty} \left| e^{\alpha t}\, t^\beta \sum_{r=0}^{2k} P_r(t)\, \phi^{(r)}(t) \right|$$

$$\le C \max_{0 \le k \le q} \sup_{0 \le t < \infty} \int_t^\infty D \sum_{r=0}^{2k} \left| e^{\alpha t}\, t^\beta\, P_r(t)\, \phi^{(r)}(t) \right| dt$$

$$< C \max_{0 \le k \le q} \sup_{0 \le t < \infty} \int_t^\infty \sum_{r=0}^{2k+1} \left| e^{\alpha t}\, t^\beta\, Q_r(t)\, \phi^{(r)}(t) \right| dt,$$

where $P_r(t)$ and $Q_r(t)$ are some polynomials in t of degree r. Hence,

$$(5.2) \qquad \langle f, \phi \rangle \le C \sum_{r=0}^{2q+1} \int_0^\infty \left| e^{\alpha t}\, t^\beta\, Q_r(t)\, \phi^{(r)}(t) \right| dt$$

Consequently, in view of the Riesz representation theorem and Hahn-Banach theorem, there exist bounded measurable functions $g_r(x)$, $r = 0, 1, 2, \ldots, 2q+1$, defined over $I(0, \infty)$, satisfying

$$(5.3) \qquad \langle f, \phi \rangle = \sum_{r=0}^{2q+1} \langle g_r(t), e^{\alpha t}\, t^{\beta}\, Q_r(t)\, \phi^{(r)}(t) \rangle$$

or

$$(5.4) \qquad \langle f, \phi \rangle = \langle \sum_{r=0}^{2q+1} (-1)^r D^{(r+1)} \int_a^{\tau} g_r(t)\, e^{\alpha t}\, t^{\beta}\, Q_r(t)\, dt, \phi(\tau) \rangle$$

Here the differentiation sign indicates differentiation in the distributional sense and a is some positive number. The expression (5.4) is obtained by integrating (5.3) by parts, since it can easily be shown that the function

$$\int_a^{\tau} g_r(t)\, e^{\alpha t}\, t^{\beta}\, Q_r(t)\, dt$$

corresponds to a regular distribution in $D'(I)$.

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Study Of Curvatures On Para–Kenmotsu Manifold

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Abstract: In this paper we have defined para-kenmotsu manifold and studied different curvature tensors on para-kenmotsu manifold and also the property of parallel π–null planes has been studied.

1. Introduction

Let $M = M^{m+1}$ be $(m+1)$–dim. almost para contact manifold with structure tensor $(\phi\ , \xi\ , \eta\ , g)$, where ϕ is a tensor field of type (1.1), ξ is a vector field, η is a 1–form and g is the associated metric on M. Then by def. [4] we have

(1.1) $$\phi^2 - I = -\eta \times \xi\ ,\ \eta(\xi) = 1\ ,\ \phi\,\xi = 0\ ,\ \eta\, o\, \xi = 0$$

(1.2) $$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\,\eta(Y), \eta(X) = g(\xi, X)$$

for all vector fields X, Y tangent to M and I is the identify tensor field.

Further, if we have vector field V, Y, Z tangent to M

(1.3) $$(\nabla_X\ \phi)Y = -\eta(Y)\ \phi\, X - g(X, \phi\, Y)\xi\ ,$$

where ∇ is the Riemannian connection in M, and

(1.4) $$\nabla_X\ \xi = X - \eta\,(X)\xi\ .$$

Then M is called as Para–Kenmotsu manifold,

2. Para–Kenmotsu Manifold of Constant ϕ–Holomorphic Sectional Curvature

Let R be the curvature tensor of the connection ∇. Then a para–kenmotsu manifold M is of constant ϕ –holomorphic sectional curvature C if

$$\begin{aligned} R(X,Y)Z = {} & (\frac{C-3}{4})\ [g(Y,Z)X - g(X,Z)Y] \\ & + (\frac{C+1}{4})\ [\eta\,(X)\,\eta\,(Z)Y - \eta\,(Y)\,\eta\,(Z)X \\ & + \eta\,(Y)\ g\,(X, Z)\xi\ - \eta\,(X)\,g(Y, Z)\xi \\ & + g(X, \phi\, z)\phi\, Y - g(Y, \phi z)\phi X\ +\ 2g(X, \phi\, Y)\phi\, Z\,] \end{aligned} \tag{2.1}$$

Now,

$$'R(X,Y,Z,W) = (\frac{C-3}{4})\,[g(Y,Z)\;g(X,W) - g(X,Z)\,g(Y,W)]$$

$$+\,(\frac{C+1}{4})\,[\eta(X)\,\eta(Z)\,g(Y,W) - \eta(Y)\,\eta(Z)\,g(X,W)$$

$$+\,\eta(Y)\;g(X,Z)\eta(W) - \eta(X)\,g(Y,Z)\eta(W)$$

$$+\,g(X,\phi Z)\;g(\phi Y,\,W) - g(Y,\phi Z)\,g(\phi X,W)$$

$$+\,2g(X,\phi Y)\;g(\phi Z,W)\,].$$

Then the sectional curvature R_M is

$$R_M(X,Y) = -'R(X,Y,X,Y), \text{ where } X, Y \text{ are orthonormal vectors}$$

$$= (\frac{C-3}{4}) - (\frac{C+1}{4})\,[\eta^2(X) + \eta^2(Y) - 3g(Y,\phi X)]$$

and the holomorphic sectional curvature H of para–Kenmotsu manifold.

$$H(X) = R_M(X,\phi X) = (\frac{C-3}{4}) - (\frac{C+1}{4})\,[7\eta^2(X) - 3 - 3\eta^4(X)].$$

From (1.3), we have

$$(2.2)\qquad (\nabla_X\,'\phi)(Y,Z) = g((\nabla_X\,\phi)Y,Z) = \,'R(X,Y,Z,\xi)$$

and therefore,

$$'R(X,Y,Z,\xi) = \eta(R(X,Y,Z)) = g(X,Z)\,\eta(Y) - g(Y,Z)\,\eta(X)$$

which gives

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X.$$

On contracting (2.1) we get Ricci tensor as

$$Ric(Y,Z) = (\frac{(C+1)(n+1)}{4})\;g(\phi Y,\phi Z) - (n-1)\,g(Y,Z)$$

and

$$(2.3)\qquad r = (\frac{n-1}{4})\;[(C+1)(n+1) - 4n].$$

From (2.1), we have

$$'R(X,Y,Z,\xi) = \eta\,[R(X,Y,Z)] = g(X,Z)\,\eta(Y) - g(Y,Z)\,\eta(X)$$

If we put ξ for Y in the above equation then

$$(2.4)\qquad 'R(X,\xi,Z,\xi) = g(X,Z) - \eta(X)\,\eta(Z).$$

Now if para-Kenmotsu manifold is flat then

$$[\, g(X, Z) - \eta(X)\, \eta(Z)\,] = 0\,,$$

which is not possible. Thus, we get a theorem

Theorem 2.1 *A para–kenmotsu manifold M of constant ϕ–holomorphic sectional curvature cannot be flat.*

If we suppose that $(\nabla_X \phi)Y = 0$, then from (1.3) we get $g(X, \phi Y) = -\eta(Y)\phi(X)$
Put ξ for Y in the above equation we get $\phi X = 0$, which is not possible and hence we get a theorem.

Theorem 2.2 *In a Para–Kenmotsu manifold of constant ϕ–holomorphic sectional curvature*

$$(\nabla_X \phi)(Y) \neq 0.$$

Differentiating (2.3) along V, we get

$$\begin{aligned}(\nabla_{ij} Ric)(Y, Z) + Ric(\nabla_{ij} Y, Z) + Ric(Y, \nabla_{ij} Z) \\ = \left(\frac{(C+1)(n+1)}{4}\right) [g(\nabla_{ij} Y, Z) + g(\phi Y, \nabla_{ij} Z)] \\ - (n-1)\,[g(\nabla_{ij} Y, Z) + g(Y, \nabla_{ij} Z)]\end{aligned} \tag{2.5}$$

In view of (2.3) and (2.5), we obtain

$$\begin{aligned}(\nabla_{ij} Ric)(Y, Z) = \left(\frac{(C+1)(n+1)}{4}\right) [(g(\nabla_{ij} \phi Y, \phi Z) - g(\nabla_{ij} Y, \phi Z)) \\ + g(\phi Y, \nabla_{ij} \phi Z) - g(\phi Y, \nabla_{ij} Z)]\end{aligned} \tag{2.6}$$

Let us assume that it admits parallel Ricci tensor then (2.6) gives,

$$'R(Y, \phi Z, U, \xi) + {}'R(Z, \phi Y, U, \xi) = 0$$

Putting ξ for Y in the above equation and using (2.4), we get $g(\phi Z, U) = 0$ which is not possible, hence we get a theorem.

Theorem 2.3 *In a para–Kenmotsu manifold of constant ϕ–holomorphic sectional curvature, the Ricci tensor is not parallel.*

3. Projective Tensor

The projective curvature tensor P is defined by

$$P(X, Y, Z) \overset{def}{=} R(X, Y, Z) - \frac{1}{n-1}\,[\, X\, Ric(Y, Z) - Y\, Ric(X, Z)\,] \tag{3.1}$$

which gives

$$R(X,Y,Z,\xi) = {}'R(X,Y,Z,\xi) - \frac{1}{n-1}[g(X,\xi)\ Ric(Y,Z) - g(Y,\xi)\ Ric(X,Z)]. \tag{3.2}$$

Theorem 3.1 *A projectively flat Para–Kenmotsu manifold is an Einstein manifold and also a manifold of constant Riemannean.*

Proof : For such a space $P(X, Y, Z) = 0$, consequently from (3.2) we have

$$(n-1)\ 'R(X, Y, Z, \xi) = g(X, \xi)\ Ric(Y, Z) - g(Y, \xi)\ Ric(X, Z)$$

$$\Rightarrow (n-1)[g(X, Z)\ \eta(Y) - g(Y, Z)\ \eta(X)] = g(X, \xi)\ Ric(Y, Z) - g(Y, \xi)\ Ric(X, Z). \tag{3.3}$$

Putting ξ for X in (3.3), we get

$$Ric(Y, Z) = -(n-1)\ g(Y, Z). \tag{3.4}$$

Hence the Para–Kenmotsu manifold is an Einstein manifold.

Now by virtue of (3.4) and (3.3), we have

$$'R(X, Y, Z, \xi) = g(Y, \xi)\ g(X, Z) - g(X, \xi)\ g(Y, Z).$$

Hence a manifold is of constant Riemannian curvature.

4. Parallel Field of Null Planes

It is easy to see that the tensor field ϕ satisfies

$$\phi^3 = \phi$$

and we can see that the rank of the matrix

$$((\gamma)) \overset{def}{=} ((\phi - I)) \tag{4.1}$$

is $(m + 1)$ over M where I is the identify map. Let λ be a vector field satisfying

$$\gamma(\lambda) = 0 \tag{4.2}$$

and take a field of m–planes π^m over M spanned by the vector field λ. We call this plane as π–plane field.

Now if π–plane field is parallel and λ's are basic vectors of π–plane then λ's satisfy

$$(\nabla_X \lambda) = \omega(X)\lambda \tag{4.3}$$

where ω is a covariant constant.

Theorem 4.1 *If a Para–Kenmotsu manifold admits π–plane field then its vectors are orthogonal to* ξ.

Proof : From (4.1) and (4.2) we have

$$\phi(\lambda) = \lambda \tag{4.4}$$

Operating both sides by (4.4) by ϕ and using (1.1), we get

$$\lambda - \eta(\lambda)\,\xi = \phi(\lambda) = \lambda$$

$$\Rightarrow \quad \eta(\lambda) = 0, \text{ that is}$$

$$g(\lambda, \xi) = 0 \tag{4.5}$$

which shows that the vectors of π–plane field are orthogonal to ξ.

Theorem 4.2 *In a para–Kenmotsu manifold, the π–plane field is parallel if the tensor* ϕ *is covariant constant over* M.

Proof: Let ϕ be covariant constant and hence

$$(\nabla_X \phi)(\lambda) = 0 \tag{4.6}$$

Taking covariant derivative of (4.4) w.r.t.∇ and using (4.2), we obtain

$$(\nabla_X \phi)(\lambda) + \gamma(\nabla_X \lambda) = 0,$$

which gives from (4.6)

$$\gamma(\nabla_X \lambda) = 0 \tag{4.7}$$

On comparing (4.7), (4.3) and (4.2), we get

$$(\nabla_X \lambda) = \omega(X)\,\lambda,$$

for some covariant vector ω and π–plane field is parallel, which proves our theorem..

Theorem 4.3 *In a Para–kenmotsu manifold of constant* ϕ*–holomorphic sectional curvature, the π–null planes are not parallel.*

Proof: Since, we know that a Para–Kenmotsu manifold of constant ϕ–holomorphic sectional curvature does not admit

$$(\nabla_X \phi)(Y) = 0; \quad \text{as theorem 2.2}$$

and hence π–null planes are not parallel.

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On Integral Operators Associated With Generalized Poisson Transform And The Operator H_α

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Abstract: Various interesting properties relating the integral operators associated with generalized Poisson transform and the operator H_α have been established. These properties have then been used in inversion by the limiting process.

1. Introduction

The Poisson transform studied by H.Pollard [2] is

$$F(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+(z-t)^2} f(t)\,dt. \tag{1.1}$$

The generalization of (1.1) studied by Charles Standish [3] is

$$F(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{A(z-t)+B}{c^2+(z-t)^2} f(t)\,dt. \tag{1.2}$$

The Poisson Operator and its conjugate for the transform (1.1) defined by G. O. Okikiolu [1] is as follows

$$\overline{P}_a(f(x)) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a}{a^2+(t-x)^2} f(t)\,dt. \tag{1.3}$$

and

$$\overline{Q}_a(f(x)) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{t-x}{a^2+(t-x)^2} f(t)\,dt. \tag{1.4}$$

In this paper, we study certain operators allied to the Poisson Operator and the transform $H_\alpha(f)$ defined by G.O. Okikiolu [1]

The Poisson Operator and its conjugate for (1.2) when $c = 1$ is

$$P_a(f(x)) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{A(x-t)+aB}{a^2+(x-t)^2} f(t)\,dt \tag{1.5}$$

and

$$Q_a(f(x)) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{aA - B(x-t)}{a^2 + (x-t)^2}\, f(t)\, dt . \tag{1.6}$$

Also, we define Hilbert transform $H(f)$ by

$$H(f(x)) = \frac{1}{\pi}\, P.\mathrm{v.}\int_{-\infty}^{\infty} \frac{f(t)}{t-x}\, dt . \tag{1.7}$$

The space $L^p(-\infty, \infty)$ will be denoted by L^p, and the pair of numbers p and p' will be connected by the equation $\frac{1}{p} + \frac{1}{p'} = 1$ and the norm

$$\left(\int_{-\infty}^{\infty} |f(t)|^{p'}\, dt\right)^{\frac{1}{p}} \text{ will be denoted by } \|f\|_p .$$

2. Preliminary Results

In this section, we obtain certain properties of operators P_a, Q_a, H_α and K_α which will be applied later.

Theorem 2.1 *Let $f \in L^p, p > 1$ Let $P_a(f)$ and $Q_a(f)$ be defined by* (1.5) *and* (1.6) *respectively, then $P_a(f) \in L^q$ and $Q_a(f) \in L^q$ for $q \geq p$. Further, for $g \in L^{q'}$, we have*

(i) $$\int_{-\infty}^{\infty} g(t)\, P_{-a}(f(t))\, dt = -\int_{-\infty}^{\infty} f(t)\, P_a(g(t))\, dt$$

(ii) $$\int_{-\infty}^{\infty} g(t)\, Q_{-a}(f(t))\, dt = -\int_{-\infty}^{\infty} f(t)\, Q_a(g(t))\, dt$$

Proof : The first part of the theorem follows from Okikiolu ([1], theorem 1). The second part can easily be obtained by changing the order of integration which is valid due to the absolute convergence of the integrals involved.

Theorem 2.2 *Let $f \in L^p, p > 1$ and a and b be positive numbers, then*

(i) $P_a\{H(f)\} = H\{P_a(f)\} = Q_a(f)$

(ii) $Q_a\{H(f)\} = H\{Q_a(f)\} = -P_a(f)$

(iii) $P_a\{P_b(f)\} = -Q_a\{Q_b(f)\} = P^*_{a+b}(f)$

(iv) $P_a\{Q_b(f)\} = Q_a\{P_b(f)\} = Q^*_{a+b}(f)$,

where $P_a(f)$, $Q_a(f)$ and $H(f)$ are defined by (1.5), (1.6) and (1.7) respectively. And $P^*_a(f)$ and $Q^*_a(f)$ are defined by (1.5) and (1.6) with constants A and B replaced by $2AB$ and $B^2 - A^2$ respectively.

Proof : (i) In view of (1.3) and (1.4) and using Okikiolu ([1], theorem 2), we have

$$\begin{aligned} P_a\{H(f)\} &= -A\ \overline{Q}_a(H(f)) + B\ \overline{P}_a(H(f)) \\ &= A\ \overline{P}_a(f) + B\ \overline{Q}_a(f) \\ &= Q_a(f). \end{aligned}$$

Similarly,

$$H\{P_a(f)\} = Q_a(f).$$

The proof of relation (ii) is exactly similar to (i). For (iii), using (1.3) and (1.4)

$$P_a\{P_b(f)\} = A^2\ \overline{Q}_a\{\overline{Q}_b(f)\} - AB\ \overline{Q}_a(\overline{P}_b(f)) - AB\ \overline{P}_a(\overline{Q}_b(f)) + B^2\ \overline{P}_a(\overline{P}_b(f))$$

Now, using Okikiolu([1], theorem 2) , we have

$$P_a\{P_b(f)\} = (B^2 - A^2)\ \overline{P}_{a+b}(f) - 2AB\ \overline{Q}_{a+b}(f) = P^*_{a+b}(f).$$

Similarly,

$$-Q_a\{Q_b(f)\} = P^*_{a+b}(f).$$

$$\therefore\ P_a\{P_b(f)\} = -Q_a\{Q_b(f)\} = P^*_{a+b}(f).$$

For (iv), in view of (i), we have

$$P_a\{Q_b(f)\} = P_a\{P_b(H(f))\} = P_a\{H(P_b(f))\} = Q_a\{P_b(f)\}.$$

Again in view of (i) and (iii),

$$P_a\{Q_b(f)\} = P_a\{P_b(H(f))\} = P^*_{a+b}(H(f)) = Q^*_{a+b}(f).$$

This completes the proof of the theorem.

Corollary 2.2 *Let $f \in L^p, p > 1$ and a and b be positive numbers. If $A = 0$ and $B = 1$, the theorem 2.2 reduces to the following theorem due to Okikiolu* [1]

(i) $\overline{P}_a\{H(f)\} = H\{\overline{P}_a(f)\} = \overline{Q}_a(f)$

(ii) $\overline{Q}_a\{H(f)\} = H\{\overline{Q}_a(f)\} = -\overline{P}_a(f)$

(iii) $\overline{P}_a\{\overline{P}_b(f)\} = -\overline{Q}_a\{\overline{Q}_b(f)\} = \overline{P}_{a+b}(f)$

(iv) $\overline{P}_a\{\overline{Q}_b(f)\} = \overline{Q}_a\{\overline{P}_b(f)\} = \overline{Q}_{a+b}(f)$

3. Representation Theorems for $\theta_\alpha^{(a)}$ and $\varphi_\alpha^{(a)}$

In this section, we express the generalized Poisson Operators $P_a(f)$ and its conjugate $Q_a(f)$ in terms of the integrals $\varphi_\alpha^{(a)}(f(x))$, $\theta_\alpha^{(a)}(f(x))$, $\overline{\varphi}_\alpha^{(a)}(f(x))$ and $\overline{\theta}_\alpha^{(a)}(f(x))$ which we shall define below. Lastly, we use these integrals to obtain f by the limiting process.

3.1 Definition

We define the integrals $\varphi_\alpha^{(a)}(f(x))$ and $\theta_\alpha^{(a)}(f(x))$ as follows

$$(3.1)\ \varphi_\alpha^{(a)}(f(x)) = \frac{1}{\varphi(1-\alpha)} \int_{-\infty}^{\infty} \left[\frac{A\cos(\alpha \arctan \frac{z}{a})}{(a^2+z^2)^{\frac{\alpha}{2}}} + \frac{B\sin(\alpha \arctan \frac{z}{a})}{(a^2+z^2)^{\frac{\alpha}{2}}} \right] f(z+x)dz$$

$$(3.2)\ \theta_\alpha^{(a)}(f(x)) = \frac{1}{\varphi(1-\alpha)} \int_{-\infty}^{\infty} \left[\frac{B\cos(\alpha \arctan \frac{z}{a})}{(a^2+z^2)^{\frac{\alpha}{2}}} - \frac{A\sin(\alpha \arctan \frac{z}{a})}{(a^2+z^2)^{\frac{\alpha}{2}}} \right] f(z+x)dz\,,$$

where $\varphi(\alpha) = 2\,\Gamma(\alpha)\sin\frac{\pi\alpha}{2}$ and the principal value of $\arctan x$ lying between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ is taken throughout. Further we define the integrals

$$(3.3)\quad \overline{\varphi}_\alpha^{(a)}(f(x)) = \frac{1}{\varphi(1-\alpha)} \int_{-\infty}^{\infty} \Big[A\Big\{ \frac{\cos(\alpha \arctan \frac{z-x}{a})}{(a^2+(z-x)^2)^{\frac{\alpha}{2}}} - \frac{\cos(\alpha \arctan \frac{z}{a})}{(a^2+z^2)^{\frac{\alpha}{2}}} \Big\}$$
$$+ B\Big\{ \frac{\sin(\alpha \arctan \frac{z-x}{a})}{(a^2+(z-x)^2)^{\frac{\alpha}{2}}} - \frac{\sin(\alpha \arctan \frac{z}{a})}{(a^2+z^2)^{\frac{\alpha}{2}}} \Big\}\Big]\, f(z)dz$$

$$(3.4)\quad \overline{\theta}_\alpha^{(a)}(f(x)) = \frac{1}{\varphi(1-\alpha)} \int_{-\infty}^{\infty} \Big[B\Big\{ \frac{\cos(\alpha \arctan \frac{z-x}{a})}{(a^2+(z-x)^2)^{\frac{\alpha}{2}}} - \frac{\cos(\alpha \arctan \frac{z}{a})}{(a^2+z^2)^{\frac{\alpha}{2}}} \Big\}$$
$$- A\Big\{ \frac{\sin(\alpha \arctan \frac{z-x}{a})}{(a^2+(z-x)^2)^{\frac{\alpha}{2}}} - \frac{\sin(\alpha \arctan \frac{z}{a})}{(a^2+z^2)^{\frac{\alpha}{2}}} \Big\}\Big]\, f(z)dz.$$

Next, we define the transform $H_\alpha(f(x))$ and the operators related to it. These are given by

(3.5) $$H_\alpha(f(x)) = \frac{1}{\varphi(\alpha)} \int_{-\infty}^{\infty} \frac{|t-x|^\alpha}{t-x} f(t)\,dt$$

(3.6) $$K_\alpha(f(x)) = \frac{1}{\varphi(\alpha)} \int_{-\infty}^{\infty} |t-x|^{\alpha-1} f(t)\,dt$$

(3.7) $$L_\alpha(f(x)) = \frac{1}{\varphi(\alpha)} \int_{-\infty}^{\infty} \left\{\frac{|t-x|^\alpha}{t-x} - \frac{|t|^\alpha}{t}\right\} f(t)\,dt$$

and

(3.8) $$M_\alpha(f(x)) = \frac{1}{\varphi(\alpha)} \int_{-\infty}^{\infty} \{|t-x|^{\alpha-1} - |t|^{\alpha-1}\} f(t)\,dt.$$

We shall now express $\varphi_\alpha^{(a)}(f(x))$, $\theta_\alpha^{(a)}(f(x))$, $\bar{\varphi}_\alpha^{(a)}(f(x))$ and $\bar{\theta}_\alpha^{(a)}(f(x))$, in terms of generalized Poisson Operator and H_α –transform. The following lemma will be employed frequently in later developments.

Lemma 3.1 *Let* $h_1(t) = \dfrac{-At+aB}{a^2+t^2}$ *and* $h_2(t) = \dfrac{aA+Bt}{a^2+t^2}$ *and let* $\alpha > 0$, *Then we have*

(i) $$H_{1-\alpha}(h_1(x)) = -\left(\cot\frac{1}{2}\pi\alpha\right) K_{1-\alpha}(h_2(x))$$

$$= -\Gamma(\alpha)\left[\frac{A\cos(\alpha\arctan\frac{x}{a})}{(a^2+x^2)^{\frac{\alpha}{2}}} + \frac{B\sin(\alpha\arctan\frac{x}{a})}{(a^2+x^2)^{\frac{\alpha}{2}}}\right]$$

(ii) $$H_{1-\alpha}(h_2(x)) = \left(\cot\frac{1}{2}\pi\alpha\right) K_{1-\alpha}(h_1(x))$$

$$= \Gamma(\alpha)\left[\frac{B\cos(\alpha\arctan\frac{x}{a})}{(a^2+x^2)^{\frac{\alpha}{2}}} - \frac{A\sin(\alpha\arctan\frac{x}{a})}{(a^2+x^2)^{\frac{\alpha}{2}}}\right]$$

Proof : Let f denote the Fourier transform of a function f. Then by Okikiolu ([1], lemma 2)

(3.9) $$H_\alpha(f(x)) = -\frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{|t|^{1-\alpha}}{t} f(t)\, e^{-ixt}\,dt$$

and

(3.10)
$$K_\alpha(f(x)) = \frac{\cot \frac{1}{2}\pi\alpha}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |t|^{-\alpha} f(t)\, e^{-ixt}\, dt.$$

Now, it is known that

$$\hat{h}_1(t) = -A\, i \sqrt{\frac{\pi}{2}}\, e^{-a|t|} + B \sqrt{\frac{\pi}{2}}\, e^{-a|t|}$$

and

$$\hat{h}_2(t) = A \sqrt{\frac{\pi}{2}}\, e^{-a|t|} + B\, i \sqrt{\frac{\pi}{2}}\, e^{-a|t|}\, \frac{|t|}{t},$$

where $h_1(t)$ and $h_2(t)$ are the Fourier transforms of $h_1(t)$ and $h_2(t)$ respectively.

Using (3.9) and (3.10), we have

$$H_{1-\alpha}(h_1(x)) = -\cot \tfrac{1}{2}\pi\alpha\ K_{1-\alpha}(h_2(x))$$

$$= \int_{-\infty}^{\infty} \Big[-\frac{A}{2} |t|^{\alpha-1} e^{-a|t|} e^{-ixt} - \frac{B i}{2} \frac{|t|^{\alpha}}{t} e^{-a|t|} e^{-ixt} \Big]\, dt$$

$$= -\Big[A \int_0^{\infty} e^{-at}\, t^{\alpha-1} \cos xt\, dt + B \int_0^{\infty} e^{-at}\, t^{\alpha-1} \sin xt\, dt \Big]$$

$$= -\Gamma(\alpha) \Big[\frac{A \cos(\alpha \arctan \frac{x}{a})}{(a^2+x^2)^{\frac{\alpha}{2}}} + \frac{B \sin(\alpha \arctan \frac{x}{a})}{(a^2+x^2)^{\frac{\alpha}{2}}} \Big].$$

Similarly, we can prove (ii).

Corollary 3.1 *When* $A = 0,\ B = 1$, *we have the following lemma due to Okikiolu* ([1], *lemma* 2)

(i) $H_{1-\alpha}(h_1(x)) = -\left(\cot \frac{1}{2}\pi\alpha\right) K_{1-\alpha}(h_2(x)) = -\Gamma(\alpha)\, \dfrac{\sin(\alpha \arctan \frac{x}{a})}{(a^2+x^2)^{\frac{\alpha}{2}}}$

(ii) $H_{1-\alpha}(h_2(x)) = \left(\cot \frac{1}{2}\pi\alpha\right) K_{1-\alpha}(h_1(x)) = \Gamma(\alpha)\, \dfrac{\cos(\alpha \arctan \frac{x}{a})}{(a^2+x^2)^{\frac{\alpha}{2}}}$

Theorem 3.1 Let $f \in L^p,\ p > 1$. Let $0 < 1-\alpha < \dfrac{1}{p}$ and $a > 0$. Then

(i) $\left(\sin \frac{1}{2}\pi\alpha\right) P_a\{H_{1-\alpha}(f)\} = \left(\cos \frac{1}{2}\pi\alpha\right) Q_a\{K_{1-\alpha}(f)\} = \varphi_\alpha^{(a)}(f(x))$

(ii) $-\left(\sin\frac{1}{2}\pi\alpha\right) Q_a\{H_{1-\alpha}(f)\} = \left(\cos\frac{1}{2}\pi\alpha\right) P_a\{K_{1-\alpha}(f)\} = \theta_\alpha^{(a)}(f(x))$,

where $\varphi_\alpha^{(a)}(f(x))$ and $\theta_\alpha^{(a)}(f(x))$ are defined by (3.1) and (3.2) respectively.

Proof: The results of lemma 3.1 can be written as

$$\frac{1}{\varphi(1-\alpha)}\int_{-\infty}^{\infty}\frac{|t-x|^{1-\alpha}}{t-x}\,\frac{-At+aB}{a^2+t^2}\,dt$$

$$(3.11)\qquad = -\frac{\cot\frac{1}{2}\pi\alpha}{\varphi(1-\alpha)}\int_{-\infty}^{\infty}|t-x|^{-\alpha}\,\frac{aA+Bt}{a^2+t^2}\,dt$$

$$= -\Gamma(\alpha)\left[\frac{A\cos(\alpha\arctan\frac{x}{a})}{(a^2+x^2)^{\frac{\alpha}{2}}}+\frac{B\sin(\alpha\arctan\frac{x}{a})}{(a^2+x^2)^{\frac{\alpha}{2}}}\right]$$

and

$$\frac{1}{\varphi(1-\alpha)}\int_{-\infty}^{\infty}\frac{|t-x|^{1-\alpha}}{t-x}\cdot\frac{aA+Bt}{a^2+t^2}\,dt$$

$$(3.12)\qquad = \frac{\cot\frac{1}{2}\pi\alpha}{\varphi(1-\alpha)}\int_{-\infty}^{\infty}|t-x|^{-\alpha}\,\frac{-At+aB}{a^2+t^2}\,dt$$

$$= \Gamma(\alpha)\left[\frac{B\cos(\alpha\arctan\frac{x}{a})}{(a^2+x^2)^{\frac{\alpha}{2}}}-\frac{A\sin(\alpha\arctan\frac{x}{a})}{(a^2+x^2)^{\frac{\alpha}{2}}}\right].$$

Now using (1.5), (3.5) and (3.11) and interchanging the order of integration which is valid by the absolute convergence of the integrals, we have

$$P_a\{H_{1-\alpha}(f)\} = \frac{\cot\frac{1}{2}\pi\alpha}{\pi\varphi(1-\alpha)}\int_{-\infty}^{\infty}\frac{aA+BT}{a^2+T^2}\,dT\int_{-\infty}^{\infty}|z-(T+x)|^{-\alpha}\,f(z)\,dz$$

$$= \cot\tfrac{1}{2}\pi\alpha\;Q_a\{K_{1-\alpha}(f(x))\}$$

which implies

$$\left(\sin\tfrac{1}{2}\pi\alpha\right)P_a\{H_{1-\alpha}(f)\} = \left(\cos\tfrac{1}{2}\pi\alpha\right)Q_a\{K_{1-\alpha}(f)\}.$$

Also, using (3.11)

$$P_a\{H_{1-\alpha}(f)\} = \frac{\Gamma(\alpha)}{\pi}\int_{-\infty}^{\infty}\left[\frac{A\cos(\alpha\arctan\frac{z-x}{a})}{(a^2+(z-x)^2)^{\frac{\alpha}{2}}}+\frac{B\sin(\alpha\arctan\frac{z-x}{a})}{(a^2+(z-x)^2)^{\frac{\alpha}{2}}}\right]f(z)\,dz$$

which gives,

$$(\sin\tfrac{1}{2}\pi\alpha)\ P_a\{H_{1-\alpha}(f)\} = \varphi_\alpha^{(a)}(f(x))$$

The proof of (ii) follows similarly.

Corollary 3.2 *When $A = 0$, $B = 1$, we get the following relations due to Okikiolu* ([1], *theorem* 5).

Let $f \in L^p$, $p > 1$. Let $0 < 1 - \alpha < \dfrac{1}{p}$ and $a > 0$. Then

(i) $(\sin\tfrac{1}{2}\pi\alpha)\ \overline{P}_a\{H_{1-\alpha}(f)\} = (\cos\tfrac{1}{2}\pi\alpha)\ \overline{Q}_a\{K_{1-\alpha}(f)\} = \varphi_\alpha^{(a)}(f(x))$

(ii) $-(\sin\tfrac{1}{2}\pi\alpha)\ \overline{Q}_a\{H_{1-\alpha}(f)\} = (\cos\tfrac{1}{2}\pi\alpha)\ \overline{P}_a\{K_{1-\alpha}(f)\} = \theta_\alpha^{(a)}(f(x))$.

Theorem 3.2 *Let $f \in L^p$, $p > 1$. Let $0 < 1 - \alpha < \dfrac{1}{p}$ and $a > 0$. then*

(i) $(\sin\tfrac{1}{2}\pi\alpha)\ L_{1-\alpha}\{P_a(f)\} = (\cos\tfrac{1}{2}\pi\alpha)\ M_{1-\alpha}\{Q_a(f)\} = \overline{\varphi}_\alpha^{(a)}(f(x))$

(ii) $-(\sin\tfrac{1}{2}\pi\alpha)\ L_{1-\alpha}\{Q_a(f)\} = (\cos\tfrac{1}{2}\pi\alpha)\ M_{1-\alpha}\{P_a(f)\} = \overline{\theta}_\alpha^{(a)}(f(x))$,

where $L_\alpha(f)$, $M_\alpha(f)$, $\overline{\varphi}_\alpha^{(a)}(f)$ and $\overline{\theta}_\alpha^{(a)}(f)$ are defined by (3.7), (3.8), (3.3) and (3.4) respectively.

Proof : Use of (1.5), (1.6), (3.7) and (3.8) and the arguements used similar to theorem 3.1, establishes theorem 3.2

Corollary 3.3 *When $A = 0$ and $B = 1$, we have the following results due to Okikiolu* ([1] ; *theorem* 6.).

Let $f \in L^p$, $p > 1$. Let $0 < 1 - \alpha < \dfrac{1}{p}$ and $a > 0$, then

(i) $\overline{\varphi}_\alpha^{(a)}(f(x)) = (\sin\tfrac{1}{2}\pi\alpha)\ L_{1-\alpha}\{\overline{P}_a(f)\} = (\cos\tfrac{1}{2}\pi\alpha)\ M_{1-\alpha}\{\overline{Q}_a(f)\}$

(ii) $\overline{\theta}_\alpha^{(a)}(f(x)) = -(\sin\tfrac{1}{2}\pi\alpha)\ L_{1-\alpha}\{\overline{Q}_a(f)\} = (\cos\tfrac{1}{2}\pi\alpha)\ M_{1-\alpha}\{\overline{P}_a(f)\}$.

Theorem 3.3 Let $f \in L^p$, $p > 1$. Let $0 < 1 - \alpha < \dfrac{1}{p}$ and $a > 0$. Then

(i) $H_{1-\alpha}\{P_a(f(x))\} = P_a\{H_{1-\alpha}(f(x))\}$

(ii) $H_{1-\alpha}\{Q_a(f(x))\} = Q_a\{H_{1-\alpha}(f(x))\}$

(iii) $K_{1-\alpha}\{P_a(f(x))\} = P_a\{K_{1-\alpha}(f(x))\}$

(iv) $K_{1-\alpha}\{Q_a(f(x))\} = Q_a\{K_{1-\alpha}(f(x))\}$.

Proof : The proof of the theorem 3.3 easily follows on using (1.5), (1.6), (3.5) and (3.6) and changing the order of integration twice which being valid due to the absolute convergence of the integrals involved.

4. Inversion Process

We shall now obtain results expressing P and Q operators in terms of $\varphi_\alpha^{(a)}$ and $\theta_\alpha^{(a)}$ operators and we try to obtain f by the limiting process.

Theorem 4.1 Let $f \in L^p$, $p > 1$. Let $1 - \dfrac{1}{p} < \alpha < 1$ and let a and b be positive numbers, then we have

(i) $$\bar{\varphi}_\alpha^{(a)}\ \{\varphi_{1-\alpha}^{(b)}\ (f(x))\} = -\bar{\theta}_\alpha^{(a)}\ \{\theta_{1-\alpha}^{(b)}\ (f(x))\}$$
$$= -\tfrac{1}{2}(\sin \pi\alpha) \int_0^x Q_{a+b}^{*}\ (f(t))\ dt$$

(ii) $$\bar{\varphi}_\alpha^{(a)}\ \{\theta_{1-\alpha}^{(b)}\ (f(x))\} = \bar{\theta}_\alpha^{(a)}\ \{\varphi_{1-\alpha}^{(b)}\ (f(x))\}$$
$$= -\tfrac{1}{2}(\sin \pi\alpha) \int_0^x P_{a+b}^{*}\ (f(t))\ dt.$$

Proof : Now $\bar{\varphi}_a\ \{\varphi_{1-\alpha}^{(b)}\ (f(x))\}$, $\bar{\varphi}_\alpha\ \{\theta_{1-\alpha}^{(b)}\ (f(x))\}$, $\bar{\theta}_\alpha^{(a)}\ \{\theta_{1-\alpha}^{(b)}\ (f(x))\}$ and $\bar{\theta}_\alpha^{(a)}\ \{\varphi_{1-\alpha}^{(b)}\ (f(x)\}$ can be obtained from theorem 3.2. Hence by substitution for $\varphi_{1-\alpha}^{(b)}\ (f(x))$ and $\theta_{1-\alpha}^{(b)}\ (f(x))$ from expressions similar to those given in theorem 3.1 and using theorems 2.2, 3.2, 3.3 we have the following relations

$$\begin{aligned}
\bar{\varphi}_\alpha^{(a)}\ \{\varphi_{1-\alpha}^{(b)}\ (f(x))\} &= (\cos \tfrac{1}{2}\pi(1-\alpha))\ \bar{\varphi}_\alpha^{(a)}\ \{\theta_b\ (K_\alpha\ f(x))\} \\
&= (\sin \tfrac{1}{2}\pi\alpha)^2\ L_{1-\alpha}\ [P_a\ \{Q_b\ (K_\alpha\ f(x))\}] \\
&= (\sin \tfrac{1}{2}\pi\alpha)^2\ L_{1-\alpha}\ \{Q_{a+b}^{*}\ (K_\alpha\ f(x))\} \\
&= (\sin \tfrac{1}{2}\pi\alpha)^2\ L_{1-\alpha}\ \{K_\alpha\ (Q_{a+b}^{*}\ (f(x)))\}.
\end{aligned}$$

Again,

$$\begin{aligned}
\bar{\varphi}_\alpha^{(a)}\ \{\varphi_{1-\alpha}^{(b)}\ (f(x)\} &= (\sin \tfrac{1}{2}\pi(1-\alpha))\ \bar{\varphi}_\alpha^{(a)}\ \{P_b\ (H_\alpha\ f(x))\} \\
&= (\cos \tfrac{1}{2}\pi\alpha)^2\ M_{1-\alpha}\ \{Q_a\ (P_b\ (H_\alpha\ (f(x)))\} \\
&= (\cos \tfrac{1}{2}\pi\alpha)^2\ M_{1-\alpha}\ \{Q_{a+b}^{*}\ (H_\alpha\ f(x))\} \\
&= (\cos \tfrac{1}{2}\pi\alpha)^2\ M_{1-\alpha}\ \{H_\alpha\ (Q_{a+b}^{*}\ (f(x)))\}.
\end{aligned}$$

Similarly, we can show that

$$- \bar{\theta}_{\alpha}^{(a)} \{\theta_{1-\alpha}^{(b)} (f(x))\} = (\sin \tfrac{1}{2} \pi \alpha)^2 \;\; L_{1-\alpha} \{K_\alpha (Q^*_{a+b} f(x))\}$$
$$= (\cos \tfrac{1}{2} \pi \alpha)^2 \;\; M_{1-\alpha} \{H_\alpha (Q^*_{a+b} (f(x))\}.$$

Hence,

$$\bar{\varphi}_{\alpha}^{(a)} \{\varphi_{1-\alpha}^{(b)} (f(x))\} = - \bar{\theta}_{\alpha}^{(a)} \{\theta_{1-\alpha}^{(b)} (f(x))\}$$
$$= (\sin \tfrac{1}{2} \pi \alpha)^2 \;\; L_{1-\alpha} \{K_\alpha (Q^*_{a+b} (f(x)))\}$$
$$= (\cos \tfrac{1}{2} \pi \alpha)^2 \;\; M_{1-\alpha} \{H_\alpha (Q^*_{a+b} (f(x))\}.$$

Now, in view of Okikiolu ([1]; theorem 4).

$$\bar{\varphi}_{\alpha}^{(a)} \{\varphi_{1-\alpha}^{(b)} (f(x))\} = - \bar{\theta}_{\alpha}^{(a)} \{\theta_{1-\alpha}^{(b)} (f(x))\}$$
$$= - \tfrac{1}{2} (\sin \pi \alpha) \int_0^x Q^*_{a+b} (f(t)) \, dt.$$

On the same way, we can show that

$$\bar{\varphi}_{\alpha}^{(a)} \{\varphi_{1-\alpha}^{(b)} (f(x))\} = - \bar{\theta}_{\alpha}^{(a)} \{\varphi_{1-\alpha}^{(b)} (f(x))\}$$
$$= (\sin \tfrac{1}{2} \pi \alpha)^2 \;\; L_{1-\alpha} \{K_\alpha (P^*_{a+b} (f(x)))\}$$
$$= (\cos \tfrac{1}{2} \pi \alpha)^2 \;\; M_{1-\alpha} \{H_\alpha (P^*_{a+b} f(x)))\}$$
$$= - \tfrac{1}{2} (\sin \pi \alpha) \int_0^x P^*_{a+b} (f(t)) \, dt$$

Remark 4.1 By letting a and b tend to zero in theorem 4.1, we have

(i) $$\lim_{a,b \to 0+} \bar{\varphi}_{\alpha}^{(a)} \{\varphi_{1-\alpha}^{(b)} (f(x))\}$$
$$= - \lim_{a,b \to 0+} \bar{\theta}_{\alpha}^{(a)} \{\theta_{1-\alpha}^{(b)} (f(x))\}$$
$$= - \lim_{a,b \to 0+} \tfrac{1}{2} (\sin \pi \alpha) \int_0^x H(f(t)) \, dt$$

(ii) $$\lim_{a,b \to 0+} \bar{\theta}_{\alpha}^{(a)} \{\varphi_{1-\alpha}^{(b)} (f(x))\}$$
$$= \lim_{a,b \to 0+} \bar{\varphi}_{\alpha}^{(a)} \{\theta_{1-\alpha}^{(b)} (f(x))\}$$
$$= - \lim_{a,b \to 0+} \tfrac{1}{2} (\sin \pi \alpha) \int_0^x P^*_{a+b} (f(t)) \, dt$$
$$= AB \sin \pi \alpha \int_0^x H (f(t)) \, dt.$$ This establishes the theorem

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On Characterisation Of A Generalised Directed Divergence Of Order α

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Abstract: A generalised directed divergence of order α, involving three discrete probability distributions $p = (p_1, \dots, p_n)$. $Q = (q_1, \dots, q_n)$. $R = (r_1, \dots, r_n)$, was defined by Nath [7]. A characterisation theorem for this directed divergence is proved here with the help of a functional equation.

1. Introduction

Shannon [9] entropy measure

$$H(P_1, \dots, P_n) = -\sum_{i=1}^{n} P_i \log P_i \tag{1.1}$$

of a discrete probability distribution $(P_1, \dots, P_n; P_i \geq 0, \sum_{i=1}^{n} P_i = 1)$ has been characterised in several ways (see Aczel and Daroczy [2]).Chaundi and McLeod [5] characterised it through the functional equation.

$$\sum_{i=1}^{m} \sum_{j=1}^{n} F(x_i\, y_j) = \sum_{i=1}^{m} F(x_i) + \sum_{j=1}^{n} F(y_j), \tag{1.2}$$

where

$$\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j = 1, x_i \geq 0, y_j \geq 0.$$

Renyi [8] generalised Shannon's entropy by introducing the additive entropy of order α

$$H_\alpha(P_1, \dots, P_n) = \frac{1}{1-\alpha} \log \left(\sum_{i=1}^{n} P_i^\alpha \right) \quad \alpha > 0\ \alpha \neq 1 \tag{1.3}$$

which reduces to Shannon's entropy when $\alpha \to 1$. This additive entropy of order α was characterised by Ahmad [3] through the functional equation

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$$(1.4)\qquad \sum_{i=1}^{m}\sum_{j=1}^{n} F(x_i\, y_j) = \Big(\sum_{i=1}^{m} F(x_i)\Big)\Big(\sum_{j=1}^{n} F(x_j)\Big),$$

where
$$\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j = 1,\ x_i \geq 0,\ y_j \geq 0.$$

Renyi [8] defined the directed divergence of order α

$$H_\alpha(p_1,\dots,p_n\,;\,q_1,\dots,q_n) = \frac{1}{\alpha-1}\log\Big(\sum_{i=1}^{n} p_i^{\alpha}\, q_i^{1-\alpha}\Big)$$

$$(1.5)\qquad \alpha > 0,\ \alpha \neq 1$$

of two given discrete probability distributions $P = (P_1, \dots, P_n)$ $Q = (q_1, \dots, q_n)$. Note that for $\alpha \to 1$, (1.5) reduces to the directed divergence

$$H(p_1,\dots,p_n\,;\,q_1,\dots,q_n) = \sum_{i=1}^{n} p_i \log(p_i / q_i)$$

of Kullback [6]. This directed divergence of order α was characterised by Ahmad [4] through the functional equation

$$(1.6)\qquad \sum_{i=1}^{m}\sum_{j=1}^{n} F(x_i\, y_j, u_i\, v_j) = \Big(\sum_{i=1}^{m} F(x_i, u_i)\Big)\Big(\sum_{j=1}^{n} F(y_j, v_j)\Big)$$

We generalise the functional equation (1.6) to

$$(1.7)\qquad \sum_{i=1}^{m}\sum_{j=1}^{n} F(x_i\, y_j, u_i\, v_j, s_i\, t_j) = \Big(\sum_{i=1}^{m} F(x_i, u_i, s_i)\Big)\Big(\sum_{j=1}^{n} F(y_j, v_j, t_j)\Big)$$

and use (1.7) to characterise the generalised directed divergence of order α

$$(1.8)\qquad H_\alpha(p_1,\dots,p_n\,;q_1,\dots,q_n\,;r_1,\dots,r_n) = \frac{1}{\alpha-1}\log\Big(\sum_{i=1}^{m} p_i\, q_i^{\alpha-1}\, r_i^{1-\alpha}\Big)$$

$$\alpha > 0,\ \alpha \neq 1,$$

defined by Nath [7], for three given discrete probability distributions $P = (P_1, \dots, P_n)$, $Q = (q_1, \dots, q_n)$, $R = (r_1, \dots, r_n)$. Note that for $\alpha \to 1$, (1.8) reduces to the generalised directed divergence

$$H(p_1,\dots,p_n\,;q_1,\dots,q_n\,;r_1,\dots,r_n) = \sum_{i=1}^{m} p_i \log(q_i / r_i)$$

of Theil [10].

2. Characterization

We first determine continuous solutions of the functional equation (1.7). The result is given in the following Lemma.

Lemma : *The continuous solution $F : I \times I \times I \to R$ of the functional equation* (1.7), *other than the trivial solution $F \equiv 0$, for all positive integers m and n, is given by*

(2.1)
$$F(x,y,z) = x^{\gamma}\, y^{\delta}\, z^{\nu},$$

where γ, δ and ν are arbitrary constants, and I denotes the closed unit interval [0,1].

Proof Let a, b, c, d, e, f and a', b', c', d', e', f' be positive integers such that $1 \le a' \le a, 1 \le b' \le b, 1 \le c' \le c, 1 \le d' \le d, 1 \le e' \le e, 1 \le f' \le f$
Taking

$$x_i = \frac{1}{a}, i = 1,\ldots,m-1,\ x_m = \frac{a'}{a}$$

$$y_j = \frac{1}{b}, j = 1,\ldots,n-1, y_n = \frac{b'}{b}$$

$$u_i = \frac{1}{c}, i = 1,\ldots,m-1,\ u_m = \frac{c'}{c}$$

$$v_j = \frac{1}{d}, j = 1,\ldots,n-1,\ v_n = \frac{d'}{d}$$

$$s_i = \frac{1}{e}, i = 1,\ldots,m-1,\ s_m = \frac{e'}{e}$$

$$t_j = \frac{1}{f}, j = 1,\ldots,n-1,\ t_n = \frac{f'}{f}$$

in (1.7) we get

$$(m-1)(n-1)\,F\left(\frac{1}{ab}, \frac{1}{cd}, \frac{1}{ef}\right) + (m-1)\ F\left(\frac{b'}{ab}, \frac{d'}{cd}, \frac{f'}{ef}\right)$$

(2.2)
$$+ (n-1)\ F\left(\frac{a'}{ab}, \frac{c'}{cd}, \frac{e'}{ef}\right) + F\left(\frac{a'b'}{ab}, \frac{c'd'}{ed}, \frac{e'f'}{ef}\right)$$

$$= \left[(m-1)\ F\left(\frac{1}{a}, \frac{1}{c}, \frac{1}{e}\right) + F\left(\frac{a'}{a}, \frac{c'}{c}, \frac{e'}{e}\right)\right]$$

$$\times \left[(n-1)\,F\left(\frac{1}{b}, \frac{1}{d}, \frac{1}{f}\right) + F\left(\frac{b'}{b}, \frac{d'}{d}, \frac{f'}{f}\right)\right]$$

and then putting $a' = b' = c' = d' = e' = f' = 1$ in (2.2), we obtain

$$F\left(\frac{1}{ab}, \frac{1}{cd}, \frac{1}{ef}\right) = F\left(\frac{1}{a}, \frac{1}{c}, \frac{1}{e}\right) F\left(\frac{1}{b}, \frac{1}{d}, \frac{1}{f}\right) \tag{2.3}$$

Again taking only $a' = c' = e' = 1$ in (2.2), we get

$$\begin{aligned} & m(n-1)\ F\left(\frac{1}{ab}, \frac{1}{cd}, \frac{1}{ef}\right) + mF\left(\frac{b'}{ab}, \frac{d'}{cd}, \frac{f'}{ef}\right) \\ & = \left[mF\left(\frac{1}{a}, \frac{1}{c}, \frac{1}{e}\right)\right] \left[(n-1)\ F\left(\frac{1}{b}, \frac{1}{d}, \frac{1}{f}\right) + F\left(\frac{b'}{b}, \frac{d'}{d}, \frac{f'}{f}\right)\right] \end{aligned} \tag{2.4}$$

Using (2.3), (2.4) gives

$$F\left(\frac{b'}{ab}, \frac{d'}{cd}, \frac{f'}{ef}\right) = F\left(\frac{1}{a}, \frac{1}{c}, \frac{1}{e}\right) F\left(\frac{b'}{b}, \frac{d'}{d}, \frac{f'}{f}\right) \tag{2.5}$$

Similarly taking $b' = d' = f' = 1$ in (2.2) and once again using (2.3), we have

$$F\left(\frac{a'}{ab}, \frac{c'}{cd}, \frac{e'}{ef}\right) = F\left(\frac{a'}{a}, \frac{c'}{c}, \frac{e'}{e}\right) F\left(\frac{1}{b}, \frac{1}{d}, \frac{1}{f}\right). \tag{2.6}$$

Now (2.2)with (2.3), (2.5) and (2.6) yields

$$F\left(\frac{a'b'}{ab}, \frac{c'd'}{cd}, \frac{e'f'}{ef}\right) = F\left(\frac{a'}{a}, \frac{c'}{c}, \frac{e'}{e}\right) F\left(\frac{b'}{b}, \frac{d'}{d}, \frac{f'}{f}\right)$$

or

$$F(xy, uv, st) = F(x, u, s)\ F(v, y, t) \tag{2.7}$$

for all rationals $x, y, u, v, s, t \in [0, 1]$, where $\frac{a'}{a} = x$, $\frac{b'}{b} = y$, $\frac{c'}{c} = u$, $\frac{d'}{d} = v$, $\frac{e'}{e} = s$ and $\frac{f'}{f} = t$ which is the functional equation (2.7) having the continuous solutions (refer Aczel [1])

$$F(x, y, z) = x^{\gamma}\ y^{\delta}\ z^{\nu}, \tag{2.8}$$

γ, δ, and ν being constants.

We are now in a position to give a characterisation of Nath's generalised directed divergence of order α.

Theorem : *Nath's generalised directed divergence of order α is given by*

$$\begin{aligned} & H_{\alpha}(p_1, \dots, p_n\ ;\ q_1, \dots, q_n\ ;\ r_1, \dots, r_n) \\ & = \frac{1}{\alpha - 1} \log \left[\sum_{i=1}^{n} F(p_i\, q_i\, r_i) \right], \end{aligned} \tag{2.9}$$

where $\sum_{i=1}^{m} \sum_{j=1}^{n} F(x_i y_j, u_i v_j, s_i t_j) = (\sum_{i=1}^{m} F(x_i, u_i, s_i)) \; (\sum_{i=1}^{m} F(y_j, v_j, t_j))$

with $F(1/2\,,1/2\,,1/2) = 1/2$, $F(1\,,1/2\,,1/2) = 1$ and $F(1,1,1/2) = \frac{1}{2^{1-\alpha}}$.

Proof : By the lemma proved above we have

$$(2.10) \qquad H_\alpha = (p_1, \ldots, p_n ; q_1, \ldots, q_n ; r_1, \ldots, r_n) = \frac{1}{\alpha - 1} \log (\sum_{i=1}^{n} p_i^\gamma \, q_i^\delta \, r_i^\nu)$$

for some constants γ , δ and ν. Using the condition

$F(1/2, 1/2, 1/2) = 1/2$ gives $(1/2)^\gamma \; (1/2)^\delta \; (1/2)^\nu = 1/2$ i.e. $\gamma + \delta + \nu = 1$.

Condition $F(1\,,1/2\,,1/2) = 1$ gives $\delta + \nu = 0$. Hence we have

$\gamma = 1$, $\delta + \nu = 0$. Finally, $F(1,1,1/2) = \frac{1}{2^{1-\alpha}}$ gives $(1/2)^\nu = \frac{1}{2^{1-\alpha}}$

$\nu = 1 - \alpha$. With these relations (2.10) reduces to generalised directed divergence of order α of Nath [7].

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On Viral Disease–A Simple Mathematical Model

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Abstract: This report highlights some important results related to an investigation of a simple mathematical model of infectious diseases.

1. Introduction

This section describes the main concept of immunology. By immunology we literally mean that the study of resistance to infectious disease. Immune response is related to the universal character of the organisms defence against bacterial and viral attacks, as well as against poisoning by products of viral–bacterial activity or intoxication by foreign agents of biological nature.

The immunocompetent cells (lymphocyte and leukocyte) and blood cells produced in bone marrow. One part of the cells traversing "Bursa" transform into "B lymphocyte". The second part of the cells passes through the "Thymus" and causes the production of "Th–lymphocyte helpers" (which interact with specific antigen and promote the transformation of B cells into plasmacytes), "Tk–lymphocyte killers" (which are responsible for the genetic purity of cells of the organism) and "Ts–lymphocyte suppressors" (which maintains the level of sensitivity and also play an important role in controlling the immune response). The remaining part remains in the bone marrow turns into matured "A leukocyte". In our model we are considering B and A are homogeneous population.

In a healthy organism, plasma cells are formed continuously producing immunoglobulins which are actually antibodies capable of binding and neutralising the antigens.

Bacterial and viral diseases first of all presupposes a latent period of the course of the disease when the antigen that have penetrated the organism multiply but have not encountered a sufficiently pronounced reaction on the part of the immune system. During this latent period the immune system adjust to the neutralisation reaction of the specific antigen. In the presence of Th–lymphocyte and of the inductor of the immunodiseases the B lymphocyte starts to divide and differentiate towards plasma cells. Such a cascade process of cloning plasma cells takes several hours to several days.

In this paper we are mainly examining the immune response to an antigen for "subclinical form", "acute form with recovery", "acute form with lethal out-come" and

"chronic form". In subclinical form the multiplying population of the viruses or the bacteria is suppressed by the available resources and the antigen is destroyed before it reaches the level of concentration to cause an observable immune or physiologic reactions of the organism. If the antigen is unfamiliar to the organism its concentration increases which accompanied by a process of recognition and of formation of B cells producing antibodies. This process takes several days during which the antigen concentration achieves the levels of surpassing the level of the appreciable physiological and pathological changes. This is the case of normal acute form of the disease. The cause of lethal outcome is when the immune response has been delayed for various reasons so that the organ attacked by antigens can no longer maintain the normal efficiency of organ's responsibility for the formation of antibodies. The chronic diseases are the most serious and debilitating forms of a disease and often last many years.

The layout of the paper is as follows. In section 2, a simple mathematical model is presented. The analytical results are carried out in section 3. The ensuing section comprises the numerical results. The concluding remarks are given in the final section.

2. Mathematical Model

The main factors of the infectious disease are the following :

$V(t)$ – Concentration of viruses. By viruses we mean the multiplying pathogenic antigens.

$F(t)$ – Concentration of antibodies. By antibodies we mean substrates of the immune system. neutralising viruses.

$C(t)$ – Concentration of plasma cells. This is the population of carriers and producers of antibodies.

$m(t)$ – Relative characteristics of the damaged organ.
(Let M be a characteristic of a healthy organ (mass or area) and let M' be the same characteristic of healthy part of the damaged organ. Then $m = 1 - M'/M$, and clearly $0 \leq m \leq 1$; $m = 0$ imply the healthy organ ; $m = 1$ implies entirely damaged organ)

The model is given by the following system of *ODE* with time delay:

(1) $$\frac{dV}{dt} = \alpha V - pVF,$$

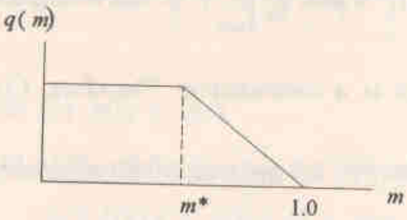
Where α–the rate of antigen multiplication, p–the rate of contact between antigen and antibodies.

$$\frac{dF}{dt} = \beta C - \gamma pVF - aF \tag{2}$$

where β –the rate of antibody production by one plasma cell, γ –the amount of antibodies necessary for the neutralisation of the antigen, a–mean life time of antibodies.

$$\frac{dC}{dt} = -\mu(C - C^*) + q(m).x.V_{t-\tau}.F_{t-\tau} \tag{3}$$

where $q(m)$ describes the dysfunction of immune system due to substantial organ damage: $0 \le q(m) \le 1$: $q(0) = 1$: $q(1) = 0$.



$$V_{t-\tau}.F_{t-\tau} = \begin{cases} 0 \text{ for } t < \tau \\ V_0F_0 \text{ for } t = \tau \\ V_{t-\tau}.F_{t-\tau} \text{ otherwise} \end{cases}$$

μ –mean life time of plasma cells. x–the coefficient of immune system stimulation. C^* –normal level of the plasma cells.

$$\frac{dm}{dt} = \sigma V - \eta m \tag{4}$$

where σ–the rate of organ injury by antigen, η –the rate of regeneration of the mass of damaged organ.

With the initial condition that at $t = t_0 = 0$

(5) $V(0) = V_0 \ge 0$: $F(0) = F_0 \ge 0$: $C(0) = C_0 \ge 0$ and $m(0) = m_0 \ge 0$.

3. Analytical Results

We shall see the validity of the following theorems.

Theorem 1: *If the initial values are non–negative, then the solution of the system* (*eqns.* (1)–(4)) *are also non–negative for all* $t \ge 0$

Proof : From the equation (1), we have

$$V = V_0 \exp \left[\int_0^t (\alpha - pF)\, dt \right] \geq 0 \text{ for all } t \geq 0. \tag{6}$$

For the solutions $F(t)$, $C(t)$ and $m(t)$ we divide the time intervals $[n\tau, (n+1)\tau)$ where $n = 0, 1, 2, \ldots$. Let $t = [0, \tau)$, then eqn. (3) becomes

$$\frac{dC}{dt} = -\mu(C - C^*). \tag{7}$$

Assume that $C(t) \leq 0$ for all $t > 0$. Since the solution is continuous there exists $t_1 \in [0, \tau)$ such that $C(t_1) = 0$ and $\left.\frac{dC}{dt}\right|_{t=t_1} < 0$. But from (7) it is clear that $\left.\frac{dC}{dt}\right|_{t=t_1} = \mu C^* > 0$ which is a contradiction. Therefore, $C(t) \geq 0$ for $t \in [0, \tau)$.

Similarly, we can show that $F(t) \geq 0$ and $m(t) \geq 0$ for all $t \in [0, \tau)$.

Now let $t \in [\tau, 2\tau)$. Noting that $F_{t-\tau} \geq 0$, $V_{t-\tau} \geq 0$ for $t \in [\tau, 2\tau)$. Then in the similar way as above we can prove the solutions are non–negative in the interval $[\tau, 2\tau)$ .By induction, continuing in the same way it can be proved for the subsequent intervals.

Theorem 2 *For all* $t \geq 0$, *the system* (*eqns.*(1)–(4)) *has unique solution satisfying the initial condition* (5).

Proof : From the above theorem it is seen that if the solution of the system exists it is non–negative. Here also we consider the sub–intervals $[n\tau, (n+1)\tau)$, $n = 0, 1, 2, \ldots$. On $[0, \tau)$ we have

$$\begin{cases} \dot{V} = \alpha V - pVF \leq \alpha V \\ \dot{F} = \beta C - \gamma pVF - aF \leq \beta C - aF \\ \dot{C} = -\mu(C - C^*) \\ \dot{m} = \sigma V - \eta m \end{cases} \tag{8}$$

On $[0, \tau)$.eqns.(1)–(4) satisfying the initial condition (5) is dominated by the linear system (8) subjected to the same initial condition whose solutions $y(t) = (y_1(t), y_2(t), y_3(t), y_4(t))^T$ is continuous and exists for all $t \geq 0$. Let $x(t) = (V, F, C, m)^T$ and let $f(x) = (f_1, f_2, f_3, f_4)^T$ be the *RHS* vector of (8).

On $[0,\tau)$ $\|x\| = \sum_i^4 |x_i|$, $\|f\| = \sum_i^4 |f_i|$. Since $f \in C^{0,1}$ $(0, x_0)$, Picard–Lindelof theorem guarentees the local existence and uniqueness of the solution.

(Picard–Lindelof Theorem : Let $D \subseteq \mathbb{R}^2$ open. $f: D \to R$ constant. Given $(x_0, y_0) \in D$ then there exists at least one integral curve of $y' = f(x, y)$ through (x_0, y_0) and the integral curve exists up to the boundary. In addition, if f is continuously differentiable *w.r.t.y*, the solution is unique).

Now we have the situation $x = f(x) \leq F(x)$ and $v = F(x)$ with $x(0) = y(0) = x_0$. So the solution can be written as $x = x_0 + \int_0^t f \, d\tau$, $y = y_0 + \int_0^t F \, d\tau$

$$\Rightarrow x - y = \int_0^t (f - F) \, d\tau \Rightarrow x(t) \leq y(t).$$

RHS vector dominated by the linear terms $\Rightarrow f(x) \leq A(t)\, x + B(t)$

$$\Rightarrow \| f(x) \| \leq A(t) \| x \| + B(t) \tag{9}$$

Since $f(x) = f(x^*) + (x - x^*) f'(\xi) \Rightarrow$

$$\| f(x) - f(x^*) \| = | f'(\xi) | \, \| x - x^* \| \leq \max | f'(\xi) | \, \| x - x^* \| = L(t) \, \| x - x^* \| \tag{10}$$

This estimation implies the existence and uniqueness of the solution for $t \geq 0$. Here, $A(t)$, $B(t)$ and $L(t)$ are the positive continuous functions for $t \geq 0$. Then the unique solution $x(t) = \phi(t) \leq y(t)$ exists on $[0, \tau)$. For $t \geq \tau$, the model is given by $x = f(x(t), x(t - \tau))$ and we have $x(\tau) = \phi(\tau)$. We can reduce this system to (8). i.e., $x = f(x(t) \,.\, \phi(t' - \tau) = \tilde{f}(x)$ for $x(\tau) = \phi(\tau)$. Taking into account that $0 \leq q\ (m) \leq 1$. $x(t)$ is non–negative and also the form (eqns.(1)–(4)). We can construct the linear system $x = kx + l(t) \geq \tilde{f}(x)$. So, establishing the conditions (9) and (10) the existence and uniqueness of the solution of $x = \tilde{f}(x)$ is implied if $x(\tau) = \phi(\tau)$ on $[\tau, 2\tau]$. In the same way we can extend for the interval $[n\tau, (n+1)\tau]$, $n = 1, 2, 3, \ldots$, i.e., for $t \geq \tau$. Hence, the unique solution on $[0, \tau]$ extends uniquely for all $t \geq \tau$:

In the stationary case we have the following equations for $q = 1$.

$$\begin{cases} \alpha V - pVF = 0 \\ \beta C - \gamma pVF - aF = 0 \\ -\mu(C - C^*) + xVF = 0 \\ \sigma V - \eta m = 0 \end{cases} \tag{11}$$

After solving (11), we get the following stationary solutions

(12) $$V_1\,\text{st} = 0 : F_1\,\text{st} = \beta\, C^*/\,a : \; C_1\,\text{st} = C^* : m_1\,\text{st} = 0$$

(13) $$V_2\,\text{st} = \mu\,(a\alpha - p\,\beta\, C^*)\,/\,(\alpha(\beta\, x - \mu p\gamma\,)) : F_2\,\text{st} = \alpha/p :$$

$$C_2 st = (\alpha xa - p^2\gamma\mu C^*)\,/\,(p(\beta x - \mu p\gamma)\,; m_2 st = \sigma\mu(a\alpha - p\beta C^*)\,/\,(\eta\alpha(\beta x - \mu p\gamma)$$

In (12) we have V_1 st = 0 and m_1 st = 0 which implies the state of healthy organism. But (13) implies the chronic form of the disease when V_2st > 0.The sufficient condition for the second stationary solution to be positive is

(14) $$a\alpha > p\beta C^* : \beta x > \mu p\gamma \;;\; \alpha xa > p^2\gamma\,\mu C^*$$

Now we shall discuss the stability of the stationary solutions. Let Xst = (Vst, Fst, Cst, mst)^T be the stationary solution of (1)–(4). We linearize this near the point $X = X$st. Let $x = X - X$st $=(V - V$st$, F - F$ st$, C - C$st$, m - m$st$)^T = (x_1, x_2, x_3, x_4)^T$. This yields the system

(15) $$\begin{cases} x_1 = (\alpha - pFst)x_1 - pVst\,.x_2 \\ x_2 = -\gamma\, pFst\,.x_1 - (\gamma\, pVst + a)x_2 + \beta\, x_3 \\ x_3 = xFst\,.x_1(t-\tau) + xVst\,.x_2(t-\tau) - \mu x_3 \\ x_4 = \sigma x_1 - \eta\, x_4 \end{cases}$$

In (15) we have the term $x_1(t-\tau)$ and $x_2(t-\tau)$.To express these in terms of $x_1(t)$ and $x_2(t)$ respectively. We use the Fourier transform. Note that $F\{f\} = \int_{-\infty}^{\infty} f(t)\,e^{i\alpha t}\,dt$; $F(\dot f) = -\,i\alpha\;F\{f\}$; $F\{f\,(t-\tau)\} = e^{i\alpha\tau}\;F(\tau)$. So, taking Fourier transform of (15), we get

(16) $$\begin{cases} -i\alpha\, F\{x_1\} = (\alpha - pFst)F\{x_1\} - pVstF\{x_2\} \\ -i\alpha\, F\{x_2\} = -\gamma\, pFst\,.F\{x_1\} - (\eta\, pVst + a)\,F\{x_2\} + \beta\, F\{x_3\} \\ -i\alpha\, F\{x_3\} = x\,Fst\,.e^{i\alpha\tau}\;F\{x_1\} + xVst\,.e^{i\alpha\tau}\;F\{x_2\} - \mu F\{x_3\} \\ -i\alpha\, F\{x_4\} = \sigma F\{x_1\}\,.\,\eta\, F\{x_4\} \end{cases}$$

$\Rightarrow$

(17) $$\begin{pmatrix} \alpha - pFst & -\;pVst & 0 & 0 \\ -\gamma\, pFst & -(\gamma\, pVst + a) & \beta & 0 \\ xe^{i\alpha\tau}\,Fst & xe^{-\alpha\tau}\;Vst & -\mu & 0 \\ \sigma & 0 & 0 & -\eta \end{pmatrix} \begin{pmatrix} F\{x_1\} \\ F\{x_2\} \\ F\{x_3\} \\ F\{x_4\} \end{pmatrix} = -i\alpha \begin{pmatrix} F\{x_1\} \\ F\{x_2\} \\ F\{x_3\} \\ F\{x_4\} \end{pmatrix}$$

$\Rightarrow -i\alpha$ is the eigenvalue of the matrix with eigenvector $(F\{x_1\}, F\{x_2\}, F\{x_3\}, F\{x_4\})^T$. Thus we have the characteristic quasi polynomial

$$(18) \quad P(\lambda) = \begin{vmatrix} \alpha - pFst - \lambda & -pVst & 0 & 0 \\ -\gamma\, pFst & -(\gamma\, pVst + a) - \lambda & \beta & 0 \\ xe^{-\lambda\tau}\, Fst & xe^{-\lambda\tau}\, Vst & -\mu - \lambda & 0 \\ \sigma & 0 & 0 & -\eta - \lambda \end{vmatrix} = 0\cdot$$

Consider the equilibrium solution (12), then

$$(19) \qquad P(\lambda) = -(\alpha - p\beta\, C^*/a - \lambda)\,(a + \lambda)\,(\mu + \lambda)\,(\eta + \lambda) = 0$$

$\Rightarrow \lambda_1 = -a$: $\lambda_2 = -\mu$: $\lambda_3 = -\eta$: $\lambda_4 = \alpha - p\beta\, C^*/a$: $\Rightarrow$ this solution is asymptotically stable if $\lambda_4 < 0 \Leftrightarrow \alpha < p\beta\, C^*/a$ which leads to the following theorem.

Theorem 3 *A sufficient condition for the asymptotic stability of the stationary solution* (12) *is that the inequality* $\alpha < p\beta\, C^*/a$ *is satisfied.*

Consider the second equilibrium solution (13). then

$$(20) \qquad P(\lambda) = -(\eta - \lambda)\,(-\lambda^3 + b\lambda^2 - d\lambda + e + f\,\lambda e^{-\lambda\tau} - qe^{-\lambda\tau}) = 0$$

where $b = \mu + \gamma\, p\, V_{2}\mathrm{st} + a > 0$; $d = \mu\,(\gamma\, p\, V_{2}\mathrm{st} + a) - \gamma\, p\alpha V_{2}\mathrm{st} > 0$; $e = p\,\mu\,\alpha\gamma\, V_{2}\mathrm{st} > 0$; $f = x\beta\, V_{2}\mathrm{st} > 0$ and $q = \alpha\, x\beta\, V_{2}\mathrm{st} > 0$. For the asymptotic stability of second equilibrium point, the validity of the following theorem is given in [1].

Theorem 4: *A sufficient condition for the stationary solution* (13) *to be asymptotically stable is that the inequality* $0 < (q - e)\,/\,(b - f\tau) < d - f - q\tau$ is satisfied for $\mu\,\tau \le 1$.

4. Numerical Results

We simulate here the infection of a healthy organism by a small doses of viruses V_0. The initial condition is taken as $V(0) = V_0$: $F(0) = F^*$: $C(0) = C^*$ and $m(0) = 0$ with $F^* = 1$ and $C^* = 1$. Also, we have taken in general that $\beta = a$. Thus we have the system

$$\begin{aligned} \dot{V} &= \alpha\, V - pVF \\ \dot{F} &= \beta\,(C - F) - \gamma\, pVF \\ \dot{C} &= -\mu(C - 1) + q(m)\,.\,x\,.\,V(t-\tau)\,.\,F(t-\tau) \\ \dot{m} &= \sigma V - \eta\, m \end{aligned}$$

We solve the above system using Runge–Kutta fourth order method [2]. In all the following cases we assume that $m^* = 0, 1$.

Case 1: (Subclinical Form)

In fig.1., the simulation shows that the nature of elimination of the viruses from the organism for sufficiently small infectious doses V_0 with stimulation. The condition of occurrence is when $\alpha < p$.From fig.2., it is observed that when $\alpha = p$ the stability of elimination of the viruses is not violated in the presence of stimulated immune system.

Case 2: (Acute Form with Normal Outcome)

In this case the viruses pass through the immunologic barrier and their concentration in the organism goes upward during several days. The viruses effectively stimulate the immune system and the quantity of antibodies becomes sufficient to eliminate the antigens completely. The damage of organ is not massive enough to qualitatively change the process. When the antigen concentration decline to zero, the antibody concentration and the characteristics of the damaged organ tend towards their normal levels (See figs. 3a, 3b, 3c). Figure 4 shows several acute form of a disease with distinct doses of infection. i.e., for $V_0 = 10^{-5}$, 10^{-7} and 10^{-9}.

Case 3: (Chronic Form)

Figures 5a, 5b, 5c show a typical chronic form of a disease for which the condition of occurrence is when $\alpha > p$ and $x\beta > \mu\gamma p$. Since the viruses have flaccid dynamics, the non-effective stimulation of the immune system can be a function of small infectious dose V_0 of viruses. In fig .6 we show the chronic form turned to acute form when the initial antigen dose is more, i.e. $V_0 = 10^{-2}$ (!).

Case 4: (Acute Form With lethal Outcome)

It occurs when immune response is delayed so much (i.e., for large τ) and that the complete damage of the organ becomes inevitable (Sec figs. 7a, 7b, 7c) under the condition $\alpha > p$ and $\beta f > \gamma pq$.

5. Concluding Remarks

With the above simple mathematical model, we are making the following conclusions.

The subclinical form of a disease is characterised by a stable elimination of viruses from the organism and resembles the vaccination by non–pathogenic

proliferate antigen which can be interpreted as utilisation of live vaccine. In this case the organ is practically intact and the antigen concentration tends to zero in time.

The acute form of a disease is characterised by expressed dynamics of the viruses: rapid growth and sharp decline to zero. An effective immune response occurs in the organism leading to recovery. Only considerable damage of the organ can lead to chronic form or the lethal outcome. Hence the treatment of acute form needs to be directed towards suppressions of pathogenic properties of viruses.

The presence of non-zero population of viruses possessing flaccid dynamics of the organism with the slight damage of the organ characterises the chronic form. This is caused by insufficiently effective stimulation of the immune system. Apparently, in some cases the chronic form of a disease should be treated by making the disease aggravated.

The lethal outcome of the disease is connected with severe damage of the organ which is no longer capable of securing a normal vital activity of the organism. Severe damage of the organ is caused either by high pathogeneity of viruses (i.e., the coefficient σ large), by weak stimulation (i.e., for small x) and untimely immune response (i.e., for τ large). Both acute and chronic forms can lead to lethal outcome in case of severe damage of the organ.

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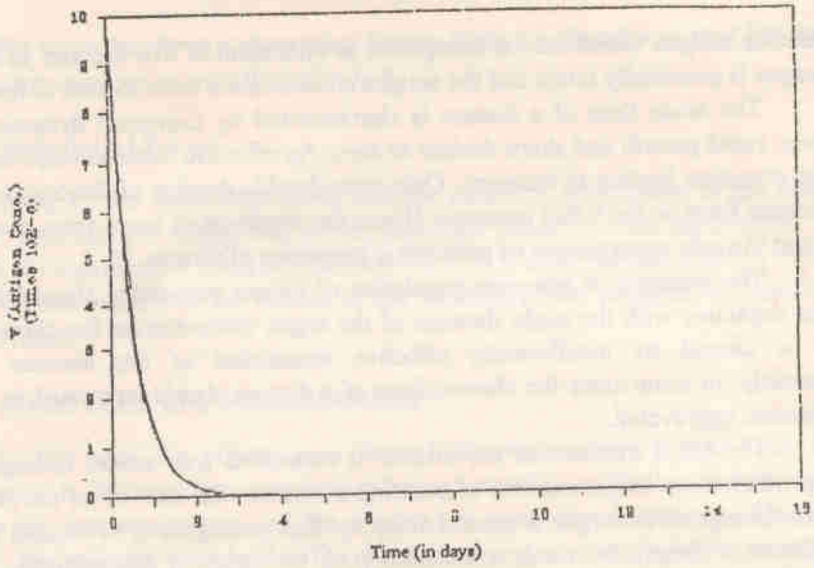

Figure 1 : Subclinical form

The parameters are :

$\alpha = 8$ $\quad p = 10$ $\quad \beta = a = 0.17$ $\quad \gamma p = 8$

$\mu = 0.5$ $\quad x = 10^4$ $\quad \sigma = 10$ $\quad 1\eta = 0.12$

$\tau = 0.5.$

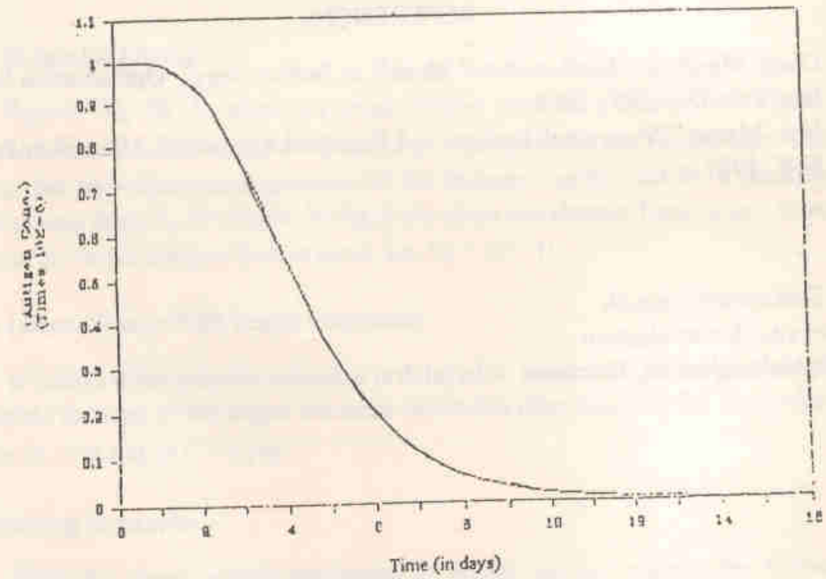

Figure 2 : Subclinical form

The parameters are :

$\alpha = 8$ $\quad p = 8$ $\quad \beta = a = 0.17$ $\quad \gamma p = 8$

$\mu = 0.5$ $\quad x = 10^4$ $\quad \sigma = 10$ $\quad \eta = 0.12$

$\tau = 0.5.$

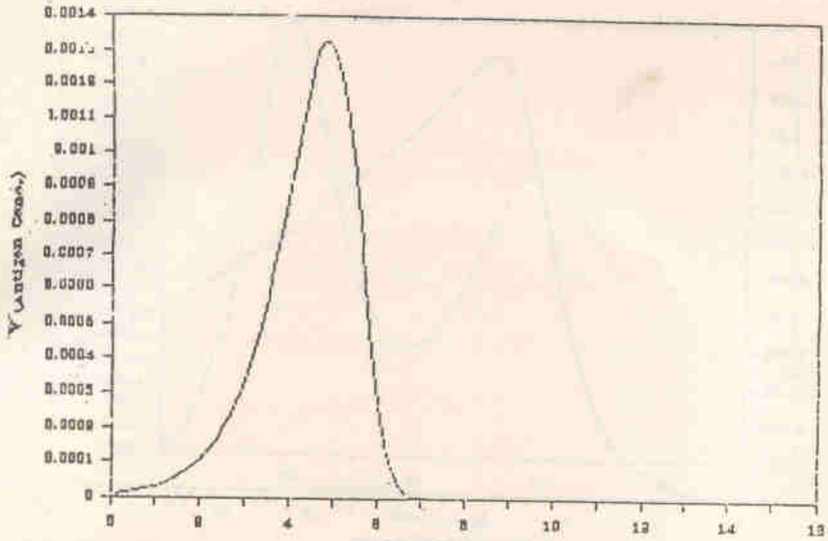

Figure 3a : Acute form (V)

The parameters are :

$\alpha = 2$ $\quad p = 0.8$ $\quad \beta = a = 0.17$ $\quad \gamma p = 8$

$\mu = 0.5$ $\quad x = 10^4$ $\quad \sigma = 10$ $\quad \eta = 0.12$

$\tau = 0.5.$

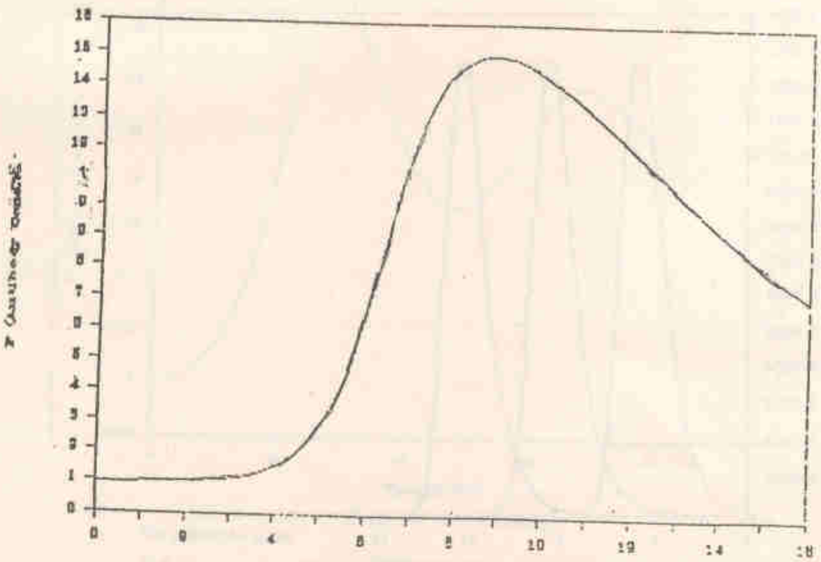

Figure 3b : Acute form (F)

The parameters are :

$\alpha = 2$ $\quad p = 0.8$ $\quad \beta = a = 0.17$ $\quad \gamma p = 8$

$\mu = 0.5$ $\quad x = 10^4$ $\quad \sigma = 10$ $\quad \eta = 0.12$

$\tau = 0.5.$

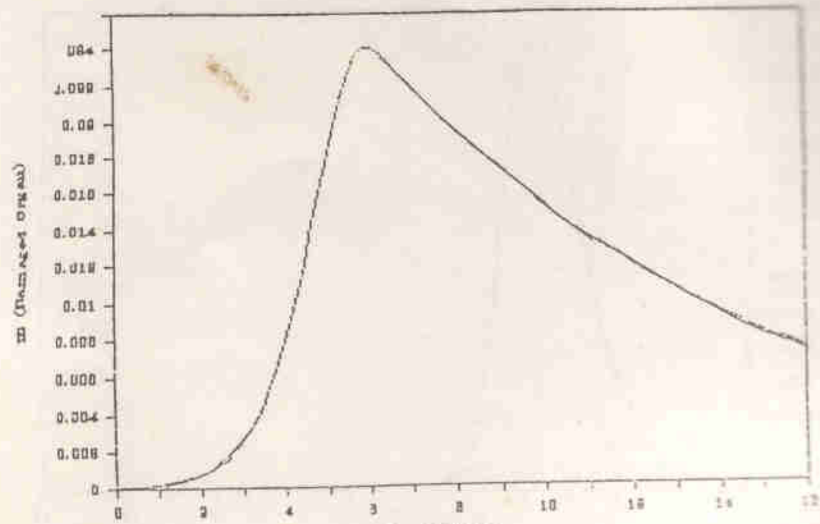

Figure 3c : Acute form (m)

The parameters are :

$\alpha = 2$	$p = 0.8$	$\beta = a = 0.17$	$\gamma p = 8$
$\mu = 0.5$	$x = 10^4$	$\sigma = 10$	$\eta = 0.12$
$\tau = 0.5.$			

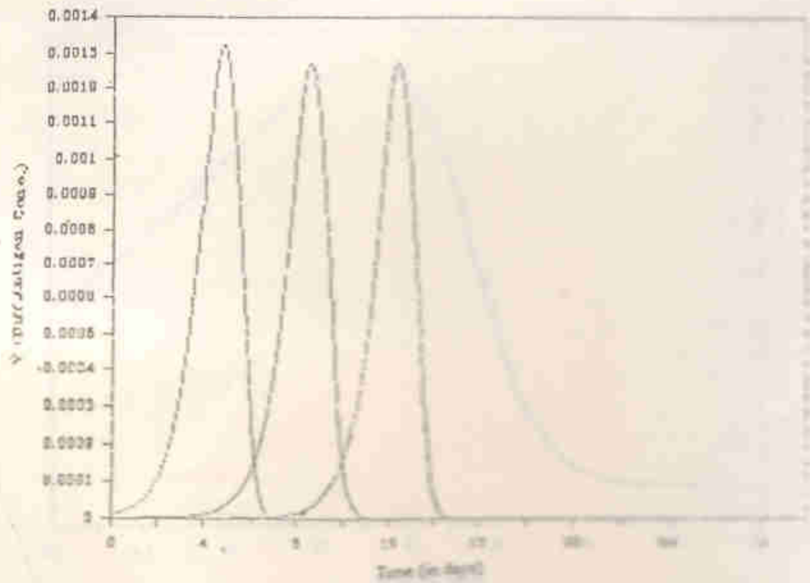

Figure 4 : Subclinical form (for different doses of infection (P))

The parameters are :

$\alpha = 2$	$p = 0.8$	$\beta = a = 0.17$	$\gamma p = 8$
$\mu = 0.5$	$x = 10^4$	$\sigma = 10$	$\eta = 0.12$
$\tau = 0.5$			

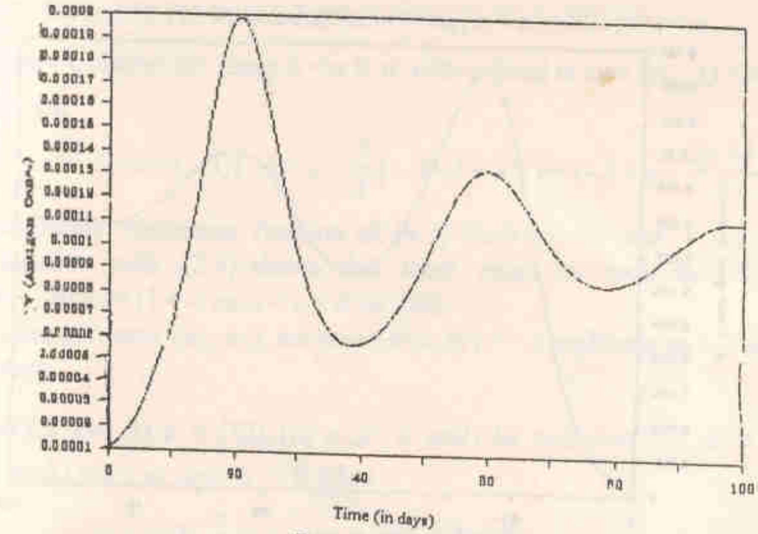

Figure 5a : Chronic form (V)

The parameters are :

$\alpha = 1$ $\quad p = 0.8$ $\quad \beta = a = 0.17$ $\quad \gamma p = 8$

$\mu = 0.5$ $\quad x = 10^3$ $\quad \sigma = 10$ $\quad \eta = 0.12$

$\tau = 0.5.$

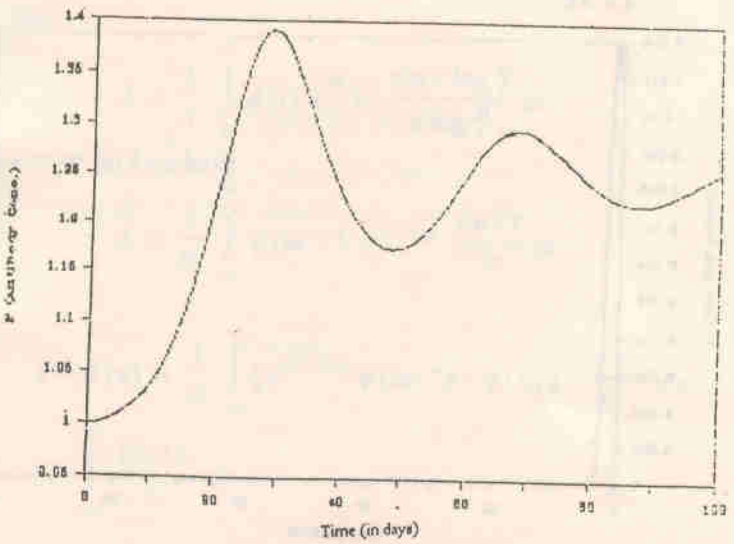

Figure 5b : Chronic form (F)

The parameters are :

$\alpha = 1$ $\quad p = 0.8$ $\quad \beta = a = 0.17$ $\quad \gamma p = 8$

$\mu = 0.5$ $\quad x = 10^3$ $\quad \sigma = 10$ $\quad \eta = 0.12$

$\tau = 0.5.$

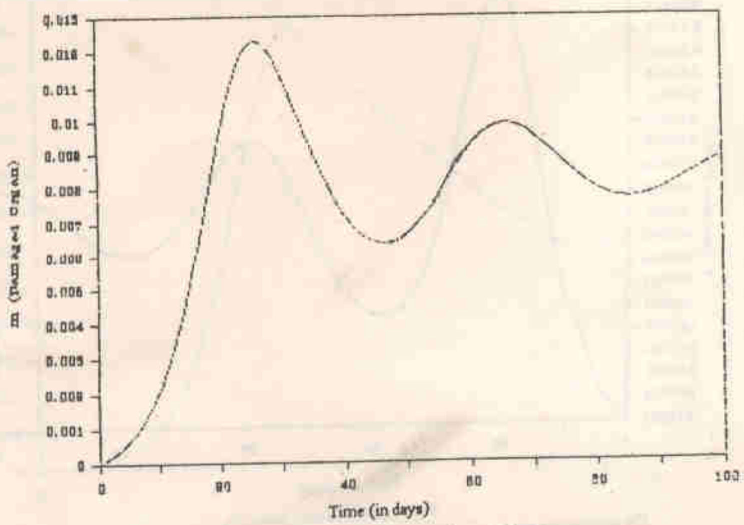

Figure 5c : Chronic form (m)

The parameters are :

$\alpha = 1$ $\quad p = 0.8$ $\quad \beta = a = 0.17$ $\quad \gamma p = 8$

$\mu = 0.5$ $\quad x = 10^3$ $\quad \sigma = 10$ $\quad \eta = 0.12$

$\tau = 0.5.$

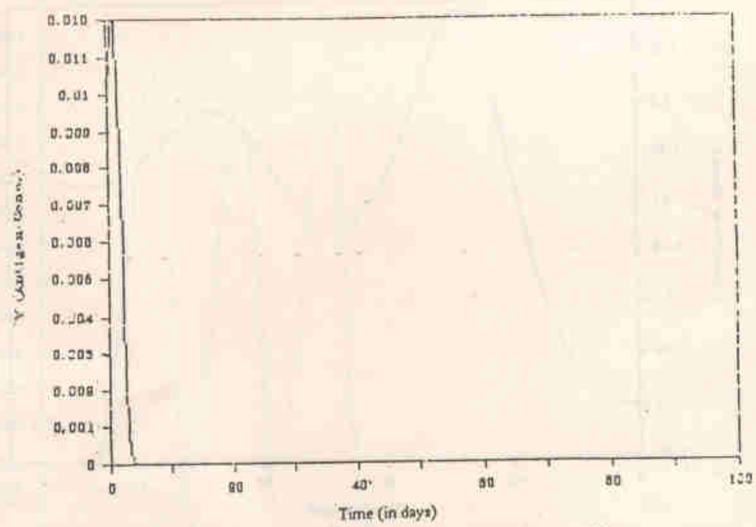

Figure 6 : Chronic form with Recovery ($V_0 = 10^{-2}$)

The parameters are :

$\alpha = 1$ $\quad p = 0.8$ $\quad \beta = a = 0.17$ $\quad \gamma p = 8$

$\mu = 0.5$ $\quad x = 10^3$ $\quad \sigma = 10$ $\quad \eta = 0.12$

$\tau = 0.5.$

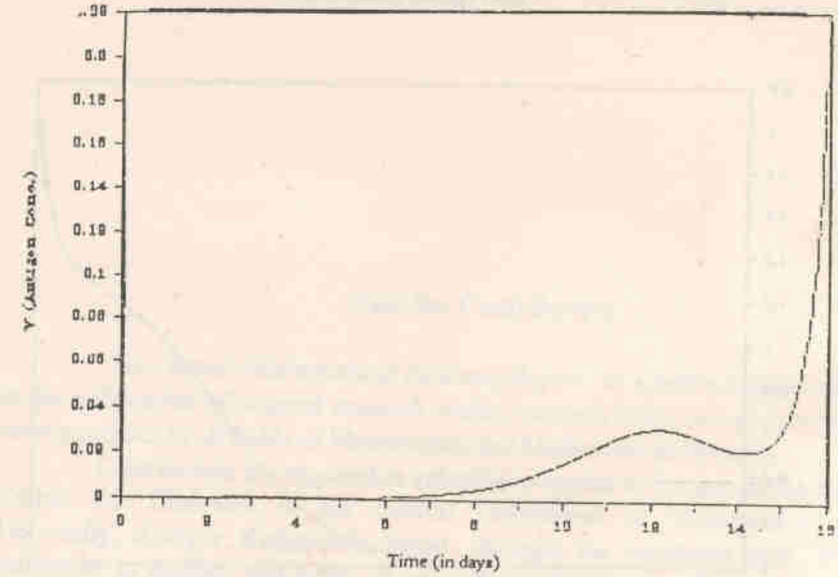

Figure 7a : Lethal outcome (V)

The parameters are :

$\alpha = 1.54$ $p = 0.77$ $\beta = a = 0.15$ $\gamma p = 8$
$\mu = 0.5$ $x = 880$ $\sigma = 12$ $\eta = 0.12$
$\tau = 2.12$

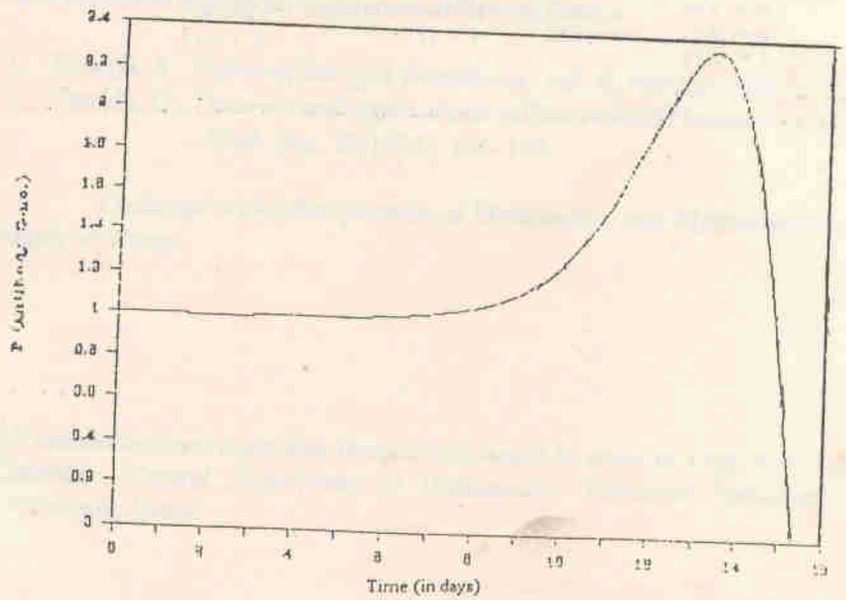

Figure 7b : Lethal outcome (F)

The parameters are :

$\alpha = 1.54$ $p = 0.77$ $\beta = a = 0.15$ $\gamma p = 8$
$\mu = 0.5$ $x = 880$ $\sigma = 12$ $\eta = 0.12$
$\tau = 2.12.$

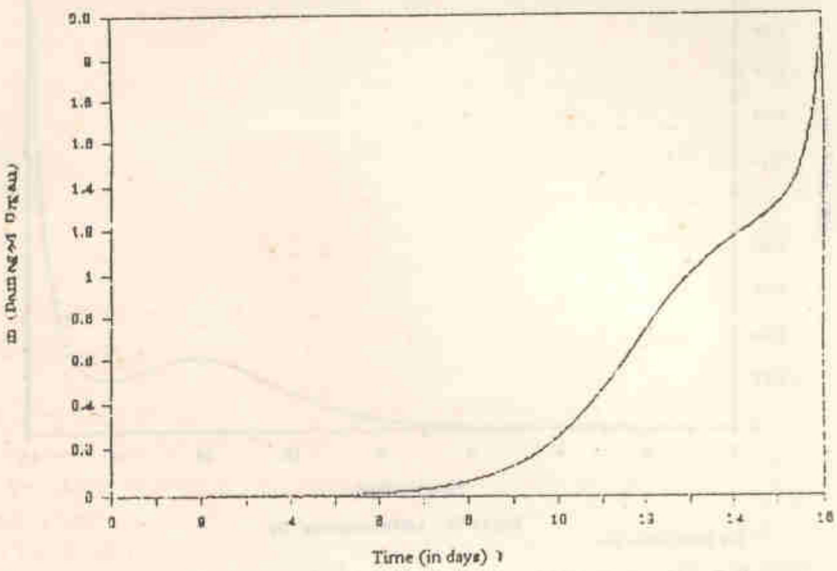

Figure 7c : Lethal outcome (m)

The parameters are :

$\alpha = 1.54$ $p = 0.77$ $\beta = a = 0.15$ $\gamma p = 8$
$\mu = 0.5$ $x = 880$ $\sigma = 12$ $\eta = 0.12$
$\tau = 2.12$..

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