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General Radical for Γ-Rings

A.C. Paul

Abstract: General radical theory for Γ-rings are given. Some of the characterizations of radicals for Γ-rings are developed. We also describe briefly certain particular radicals of interest viz Jacobson radical, Nil radical, Levitzki nil radical and strongly nilpotent radical for Γ-rings.

1. Introduction

The idea of a Γ-ring as the generalization of a ring was introduced by Nobusawa /8/ and Barnes /3/. Barnes /3/, Kyuno /7/, Coppase and Luh /4/ have defined certain radicals for Γ-rings and they have studied them.

The general radical theory for rings have been introduced by Kurosh /6/ and Amitsur /1,2/. In this paper we obtain the general radical theory for Γ-rings. We study some of the characterizations of general radicals for Γ-rings. In this connection we have also discussed particular radicals such as Jacobson radical, Nil radical, Levitzki nil radical and strongly nilpotent radical. These radicals are the generalizations of the radicals of classical ring theory.

2. Definition

Let M and Γ be two obelian groups. Suppose that there is a mapping (composition) from $M \times \Gamma \times M \to M$ sending (x, α, y) into $(x\alpha y)$ such that

(i) $(x + y)\alpha z = x\alpha z + y\alpha z$,
$x(\alpha + \beta)z = x\alpha z + x\beta z$
$x\alpha(y + z) = x\alpha y + x\alpha z$

(ii) $(x\alpha y)\beta z = x\alpha(y\beta z)$,

where $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, Then M is called a Γ-ring.

Every ring M is a Γ-ring if we take $\Gamma = M$ and interpret the above operation in the natural way.

A subset A of the Γ-ring M is a right (left) ideal of M if A is an additive subgroup of M and $A \Gamma M$ ($M \Gamma A$) is contained in A. If A is both a right and a left ideal, then we say that A is an ideal or two sided ideal of M.

Let A be an ideal of M. Define $M/A = \{x + A : x \in M\}$ the set of cosets of A, is again a Γ-ring with respect to the operations.

$$(x + A) + (y + A) = (x + y) + A \text{ and}$$
$$(x + A) \alpha (y + A) = x \alpha y + A.$$

We call M/A the Γ-residue class ring of M with respect to A.

Let M and N be Γ-rings and let $\theta : M \to N$ be a mapping. Then θ is a Γ-homomorphism if and only if

$(x + y)\theta = x\theta + y\theta$ and $(x \alpha y)\theta = (x\theta) \alpha (y\theta)$ for all $x, y \in M$ and $\alpha \in \Gamma$.

If θ is one-to-one and onto, then θ is a Γ-isomorphism.

If θ is a Γ-homomorphism of M into N, then the kernel of θ

$$0\theta^{-1} = \{x \in M : x\theta = 0\} \text{ which is also an ideal of M.}$$

Now we state three theorems concerning Γ-homomorphisms due to Barnes [3] which are needed for our development of general radicals.

Theorem 2.1

Let A be an ideal of a Γ-ring M. The mapping $\theta : M \to M/_A$ is a Γ-homomorphism with kernel A.

Conversely, if θ is a Γ-homomorphism of M onto a Γ-ring N and A is the kernel of θ, then M/A is Γ-isomorphism to N.

Theorem 2.2

Let θ be a Γ-homomorphism of a Γ-ring M onto a Γ-ring N with kernel A. Then B' is an ideal of N iff $B'\theta^{-1} = B$ is an ideal of M containing A. In this case we have M/B, N/B' and (M/A)/(B/A) are all Γ-isomorphic.

Theorem 2.3

Let A and B be ideals of the Γ-ring M and $\theta : M \to M/B$, the canonical Γ-homomorphism then $A + B = (A\,\theta)\,\theta^{-1}$ and $\frac{A+B}{B}$ is Γ-isomorphic to $A/A \cap B$.

3. General radicals for Γ-rings

A general radical theory has been developed by Kurosh /6/ and Amitsur (/1/, /2/) for classical rings. In this section we have obtained the corresponding theory of radicals for Γ-rings.

Definition

Let $\mathcal{R}$ be the class of Γ-rings. A Γ-ring $M \in \mathcal{R}$ is called $\mathcal{R}$-Γ-ring. If I is an ideal of M such that $I \in \mathcal{R}$, then I is called an $\mathcal{R}$-ideal of a Γ-ring M. A Γ-ring which does not contain any nonzero $\mathcal{R}$-ideal will be called $\mathcal{R}$-semisimple.

A class $\mathcal{R}$ of Γ-rings is called a radical if

(A) $\mathcal{R}$ is Γ-homomorphically closed, i.e. if $M \in \mathcal{R}$, and I an ideal of M, then $M/_I \in \mathcal{R}$.

(B) Every Γ-ring M contains an $\mathcal{R}$-ideal R(M) which contains all other $\mathcal{R}$-ideal of M.

(C) M/R(M) contains no non-zero $\mathcal{R}$-ideal.

The $\mathcal{R}$-ideal R(M) is called the $\mathcal{R}$-radical of M and an $\mathcal{R}$-Γ-ring is called an $\mathcal{R}$-radical Γ-ring. If R(M) = 0, then M is called $\mathcal{R}$-semisimple Γ-rings.

It is clear from (B) that 0 is an $\mathcal{R}$-Γ-ring. Since 0 is an $\mathcal{R}$-Γ-ring, we may say that an $\mathcal{R}$-semisimple Γ-ring is one whose radical is zero. An $\mathcal{R}$-Γ-ring is its own radical and we call it a radical Γ-ring. Clearly 0 is the only Γ-ring which is both $\mathcal{R}$-radical Γ-ring and $\mathcal{R}$-semisimple.

Theorem 3.1

A nonempty class $\mathcal{R}$ of Γ-rings is a radical class if and only if

(A) $\mathcal{R}$ is Γ-homomorphically closed.

(D) If every nonzero Γ-homomorphic image of a Γ-ring M contains a nonzero $\mathcal{R}$-ideal, then M is in $\mathcal{R}$.

Proof

Suppose that $\mathcal{R}$ is a radical. Then (A) holds by definition. To prove (D), let M be a Γ-ring such that every nonzero Γ-homomorphic image of M contains a nonzero $\mathcal{R}$-ideal. If $M \neq R(M)$, then $\frac{M}{R(M)}$ is a nonzero Γ-homomorphic image of M. Hence $\frac{M}{R(M)}$ contains a nonzero $\mathcal{R}$-ideal, which is a contradiction. Therefore $M = R(M)$ and M is in $\mathcal{R}$.

Conversely, let R satisfies the conditions (A) and (D) of the theorem.

Let M be a Γ-ring. Let B be the union (sum) of all $\mathcal{R}$-ideals of M. Let $\bar{B}$ be a nonzero Γ-homomorphic image of B. Then $\bar{B} = B/I$, where I is an ideal of B and $B \neq I$. If $B = 0$, then $B \in \mathcal{R}$, so let $B \neq 0$, then there exists at least one $\mathcal{R}$-ideal I_o such that $I_o \not\subseteq I$. Then $\frac{I_o + I}{I}$ is a nonzero ideal of $\frac{B}{I}$. By Theorem 2.3, $\frac{I_o + I}{I} \cong \frac{I_o}{I_o \cap I}$. Therefore by (A) $\frac{I_o + I}{I} \in \mathcal{R}$. By (D) $B \in \mathcal{R}$. Thus the condition (B) of the definition is satisfied with $R(M) = B$.

If $\frac{M}{B}$ has nonzero $\mathcal{R}$-ideal. Let $I/_B$ with $I \neq B$, $I \supsetneq B$ where I is an ideal of M. Let K be an ideal of I with $K \neq I$. If $B \subseteq K$, then $I/_K \cong \frac{I/_B}{K/_B}$ by Theorem 2.2. Hence by (A) $I/_K \in \mathcal{R}$.

If $B \not\subseteq K$, then $\frac{B + K}{K}$ is a nonzero ideal of $I/_K$ and $\frac{B + K}{K} \cong \frac{K}{B \cap K}$ by Theorem 2.3. Since $B \in \mathcal{R}$, by (A) $\frac{B}{B \cap K} \in \mathcal{R}$, therefore $\frac{B + K}{K} \in \mathcal{R}$. Thus every nonzero Γ-homomorphic image of I contains a nonzero $\mathcal{R}$-ideal, then $I \in \mathcal{R}$. Hence $I \subseteq B$, which is a contradiction. Hence $\frac{M}{B}$ has no nonzero $\mathcal{R}$-ideal. Therefore (C) is satisfied. Thus the proof of the theorem is complete.

Theorem 3.2

A nonempty class $\mathcal{R}$ of Γ-rings is a radical class if and only if

(1) $\mathcal{R}$ is Γ-homomorphically closed.

(2) $\mathcal{R}$ is closed under existensions, i.e. for a Γ-ring M and an ideal I of M, $I \in \mathcal{R}$ and $M/_I \in \mathcal{R}$ together imply $M \in \mathcal{R}$.

(3) If $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots \subseteq$ is an ascending chain of $\mathcal{R}$-ideals of a Γ-ring M, then $\bigcup_\alpha I_\alpha \in \mathcal{R}$.

Proof

First suppose that $\mathcal{R}$ is a radical class. Then clearly (1) holds. Now let M be a Γ-ring and I be an ideal of M such that $I \in \mathcal{R}$ and $\frac{M}{I} \in \mathcal{R}$. Let R(M) be the $\mathcal{R}$-radical of M. Then $I \subseteq \mathcal{R}(M)$. Hence $\frac{M/I}{R(M)/I} = \frac{M}{R(M)}$ by Theorem 2.2. But $\frac{R/I}{R(M)/I}$ is a Γ-homomorphic image of M/I. Therefore $M/R(M) \in \mathcal{R}$. By the condition (C) of the definition $M/R(M) = 0$ i.e. $M = R(M)$. Hence $M \in \mathcal{R}$. Thus (2) holds.

Now, let M be a Γ-ring and let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$. be an ascending chain of $\mathcal{R}$-ideals of M. Let $B = \bigcup_\alpha I_\alpha$. Let R(B) be the R-radical of B. If $B \neq R(B)$, then $\exists\, I_{\alpha_0}$ such that $I_{\alpha_0} \not\subseteq R(B)$. Then $\frac{I_{\alpha_0} + R(B)}{R(B)}$ is a non-zero R-ideal of B/R(B). This contradicts the condition (C) of the definition. Hence $B = R(B)$. Hence $B \in \mathcal{R}$. Thus (3) holds.

We now prove the converse, so, let $\mathcal{R}$ satisfy conditions (1), (2) and (3), clearly the condition (A) of the definition of radical holds.

Let C be the class of $\mathcal{R}$-ideals of a Γ-ring M. Then C is nonempty, since $0 \in C$. C is partially ordered with respect to inclusion, and by condition (3) every totally ordered subset of C has a least upper bound in C. Hence by Zorn's Lemma, C has a maximal member say Z. Then $Z \in \mathcal{R}$. Let I be any $\mathcal{R}$-ideal of M. Then $\frac{I+Z}{Z} = \frac{I}{I \cap Z}$ by Theorem 2.3. Since $Z \in \mathcal{R}$ and $\frac{I}{I \cap Z} \in \mathcal{R}$, then by (2) $I + Z \in \mathcal{R}$. The maximality of Z implies $I + Z = Z$ i.e. $I \subseteq Z$. Thus the condition (B) of definition holds.

Let $\bar{I}$ be an $\mathcal{R}$-ideal of $\frac{M}{Z}$. Then $I = \frac{I}{Z}$ for some ideal I of M such that $Z \subseteq I$. Since $\bar{I} \in \mathcal{R}$ and $Z \in \mathcal{R}$, we have by condition (2) $I \in R$. Hence $I \subseteq Z$. Thus $\bar{I} = 0$. Hence the condition (C) of the definition holds. This proves the theorem.

Examples of radicals of Γ -rings are developed in the following.

Definition

A Γ -ring M is said to be locally nilpotent Γ -ring, if for any finite subset $S \subseteq M$ and any finite subset $\phi \subseteq \Gamma$, there exists a positive integer n such that $(S\phi)^n S = 0$.

An ideal I of a Γ -ring M is said to be locally nilpotent, if it is locally nilpotent as a Γ -ring.

Lemma 3.3

Every subring and every homomorphic image of a locally nilpotent Γ -ring is locally nilpotent Γ -ring.

Proof. Obvious.

Lemma 3.4

Let M be a Γ -ring and I be an ideal of M such that both I and M/I are locally nilpotent Γ -ring. Then M is locally nilpotent Γ -ring.

Let $S = \{s_1, \ldots, s_r\}$ be a finite subset of M. Consider M/I and finite number of coset $s_i + I$, $i = 1,2, \ldots r$. The subring generated by the cosets is S which is finite and also subset of M/I. Since M/I is locally nilpotent, then for any finite subset $\phi \subseteq \Gamma$, there exists a positive integer n such that

$(S\phi)^n S = 0$ in $\frac{M}{I}$. Therefore $(S\phi)^n S \subseteq I$. Now $(S\phi)^n S$ is generated by a finite set of elements namely the set of all products of n of $s_i\,\phi_i$ for each $\phi_i \in \Phi$ with s_i.

Since I is locally nilpotent Γ -ring, then $\exists$ a positive integer m such that

$$((S\phi)^n S\ \phi)^m (S\phi)^n S = 0$$

$$\text{i.e. } (S\phi)^{nm+m+n} S = 0.$$

Therefore M is a locally nilpotent Γ -ring.

Lemma 3.5

Let $I_1 \subseteq I_2 \subseteq \ldots$ is the ascending chain of locally nilpotent ideals of a Γ -ring M, then $\bigcup_{\alpha} I_\alpha$ is locally nilpotent Γ -ring.

Proof

Consider a finite subset $S \subseteq \bigcup_{\alpha} I_{\alpha}$. Then S is contained in some I_{α}. Since I_{α} is locally nilpotent, then for any finite subset $\phi \subseteq \Gamma$, there exists a positive integer n such that

$$(S\phi)^n S = 0.$$

Therefore $\bigcup_{\alpha} I_{\alpha}$ is a locally nilpotent Γ-ring.

Thus by Theorem 3.2 we have the following theorem which gives an example of a radical of Γ-rings,

Theorem 3.6

The class of locally nilpotent Γ-rings is a radical. This radical is known as Levitzkinil radical and is denoted by $\mathcal{L}$.

Definition

An element a of a Γ-ring is strongly nilpotent if there exists a positive integer n such that

$$(a\Gamma)^n a = (a\,\Gamma\, a\,\Gamma\, a\,\Gamma \,\ldots\ldots\, a\,\Gamma)\, a = 0.$$

A Γ-ring M is strongly nil if each of its elements is strongly nilpotent. A Γ-ring M is strongly nilpotent if there exists a positive integer n such that

$(M\Gamma)^n M = (M\,\Gamma\, M\,\Gamma \,\ldots\ldots\, M\,\Gamma)\, M = 0$. Clearly a strongly nilpotent Γ-ring is also strongly nil.

Theorem 3.7

The class of strongly nilpotent Γ-ring is a radical.

Proof

Let M be a strongly nilpotent Γ-ring and I be an ideal of M. Then clearly M/I is strongly nilpotent.

Let M be a Γ-ring and I be an ideal of M such that both I and M/I are strongly nilpotent, then we shall prove that M is strongly nilpotent. Since M/I is strongly nilpotent, there exists a positive

Definition

An element a of a Γ-ring M is said to be right quasi-regular (abbreviated rqr) if, for any $r \in \Gamma$, there exists $\delta_i \in \Gamma$, $x_i \in M$, $i = 1,2,3,\ldots n$ such that $xra + \sum_{i=1}^{n} x\,\delta_i\,x_i - \sum_{i=1}^{n} xra\,\delta_i\,x_i = 0$ for all $x \in M$.

A Γ-ring M is called right quasi-regular if every element of M are right quasi-regular. A subset S of M is rqr if every element in S is rqr.

Theorem 3.9

The class of right quasi-regular Γ-ring is a radical.

Proof

Let M be a rqr Γ-ring and I be an ideal of M. Then clearly M/I is rqr.

Let M be a rqr Γ-ring and I be an ideal of M such that both I and M/I are rqr. Then M is rqr. To prove this, let $a \in M$, since M/I is rqr, then for any $r \in \Gamma$, there exists $(x_i + I) \in M/I$ and $\delta_i \in \Gamma$, $i=1, 2, \ldots n$, we have

$xra + \sum_{i=1}^{n} x\,\delta_i\,x_i - \sum_{i=1}^{n} xra\,\delta_i\,x_i \subseteq I$ for all $x \in M$. Put $x = ara$.

Then $c = a\,(ra)^2 + \sum_{i=1}^{n} ara\,\delta_i\,x_i - \sum_{i=1}^{n} a\,(ra)^2\,\delta_i\,x_i$ is contained in I.

If $y \in M$, then $yra \in M$ and hence

$$(yra)rc + \sum_{j=1}^{n} yra\,\lambda_j\,Z_j - \sum_{j=1}^{n} (yra)\,rc\,\lambda_j\,z_j = 0,$$

substituting for c and rearranging terms, we obtain

$$yra + (-\,yra - y\,(ra)^2 - y\,(ra)^3 + \sum_{i} y\,(ra)^3\,\delta_i\,x_i$$
$$- \sum_{i,j} y\,(ra)^3\,\delta_i\,x_i\,\lambda_j z_j + \sum_{j} yra\,\lambda_j z_j + \sum_{j} y\,(ra)^2\,\lambda_j z_j$$
$$+ \sum_{j} y\,(ra)^3\,\lambda_j z_j) - (-\,y\,(ra)^2 - y\,(ra)^{\cdot} - y\,(ra)^4$$

$$+ \sum_i y\,(ra)^4\,\delta_i x_i - \sum_{i,j} y\,(ra)^4\,\delta_i x_i \lambda_j z_j + \sum_j y\,(ra)^2\,\lambda_j z_j$$

$$+ \sum_j y\,(ra)^3 \lambda_j z_j + \sum_j y\,(ra)^4\,\lambda_j z_j = 0.$$

Hence a is rqr. Therefore M is rqr.

Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots\ldots$ is an ascending chain of rqr ideals of a Γ-ring M such that each I_α is rqr, then $\bigcup_\alpha I_\alpha$ is rqr.

To prove this let $a \in \bigcup I_\alpha$. Then a is in I_α for some α. Since I_α is rqr, then for all $r \in \Gamma$, there exists $\delta_i \in \Gamma$ and $x_i \in M$ such that

$$xra + \sum_{i=1}^{n} x\,\delta_i\,x_i - \sum_{i=1}^{n} xra\,\delta_i\,x_i = 0\ , \text{ for all } x \in M\ .$$

Therefore a is rqr. Hence $\bigcup_\alpha I_\alpha$ is rqr.

The radical is known as Jacobson radical and is denoted by J.

Theorem 3.10

Every nilpotent element in a Γ-ring M is rqr.

Proof

Let $a \in M$ be nilpotent. Then for any $r \in \Gamma$ we have $(a\,r)^n a = 0$ for some n. Let $\delta_1 = \delta_2 = \ldots = \delta_n = r$ and let $x_1 = -a$, $x_i = -(ar)^{i-1}a$ for $i = 2,3, \ldots\ldots n$. Then $xra + \sum_{i=1}^{n} x\,\delta_i\,x_i - \sum_{i=1}^{n} xra\,\delta_i\,x_i$ $= xr\,(a\,r)^n a = 0$ for all $x \in M$. Hence a is rqr.

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Complex - Inversion Formula for a Distributional Generalized Laplace Transform

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1.1 Introduction

A generalization of the Laplace transform

$$F(s) = \int_0^\infty e^{-st} f(t)dt \qquad \dots (1.1\text{-}1)$$

was given by Saksena [1] in the form

$$F(s) = \int_0^\infty e^{-(p-q/_2)st} (qst)^{c-1/_2} W_{k,m}(qst) f(t)dt \qquad \dots (1.1\text{-}2)$$

where, $W_{k,m}$ is the Whittaker function [Whittaker & Watson (2)]. We define this transform as the generalized Laplace transform of a distribution f(t), whose support is bounded on the left, by

$$F(s) = Lf(t)$$
$$= \langle f(t), e^{-(p-q/_2)st} (qst)^{c-1/_2} W_{k,m}(qst) \rangle \qquad \dots (1.1\text{-}3)$$

In this paper we have extended the complex - inversion formula,

$$\frac{f(t+o) + f(t-o)}{2} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s+c-k+\frac{1}{2})}{\Gamma(s+c+m)\,\Gamma(s+c-m)} \cdot$$
$$\Phi(s) t^{s-1}\, ds \qquad \dots (1.1\text{-}4)$$

where, $\Phi(s) = \int_0^\infty (qs)^{s-1} F(s)ds$,

(under certain conditions of f(t) and the parameters involved) corresponding to the classical Whittaker ($W_{c,k,m}$) - transform (1.1-2) to generalized functions in the testing function spaces (defined as in

$$= \langle f(u), \int^{\infty} (ux)^{s-1} \, w(xux)dx \rangle$$

Akhaury, S.K. & Kumar, V. [5] $W'_{\alpha,\beta}(I)$. Several lemmas have been proved in Section 2 which are required for establishing the complex - inversion theorem.

1.2 Some Lemmas

Lemma 1.2.1 If $f \in W'_{\alpha,\beta}(I)$, then

$$\int_0^\infty (qs)^{s-1} < f(u), w(qsu) > ds$$

$$= < f(u), \int_0^\infty (qs)^{s-1} w(qsu) ds >$$

where, $w(qsu) = (qsu)^{c-\frac{1}{2}} e^{-(p-q/2)su} W_{k,m}(qsu)$

and Re $s > 1$.

Proof: It is obvious that

$$(qs)^{s-1} < f(u), w(qsu) > = < f(u), (qs)^{s-1} w(qsu) >$$

Let $1_{o,\infty}(qs)$ denotes both the functions,

$$\begin{aligned} 1_{o,\infty}(qs) &= 0 \qquad , s \le 0 \\ &= 1 \qquad , 0 < s < \infty, \end{aligned}$$

and the corresponding generalized function belonging to

$W'_{\alpha,\beta}(I)$, then we have from [Zemanian 3, p. 121].

$$< 1_{o,\infty}(qs) f(u), (qs)^{s-1} w(qsu) > \qquad \ldots (1.2\text{-}1)$$

$$= < 1_{o,\infty}(qs), f(qs)^{s-1} w(qsu) > \qquad \ldots (1.2\text{-}2)$$

Re $s > 1$

$$= \int_0^\infty < f(u), (qs)^{s-1} w(qsu) > \qquad \ldots (1.2\text{-}3)$$

we can easily observe that [Zemanian 3, p. 121]

$< f(u), (qs)^{s-1} w(qsu) > \in W_{\alpha,\beta}(I)$ and hence the step (1.2-2) is justified. (1.2-3) is clear as $1_{o,\infty}$ is a regular generalized function.

As the product of the generalized function is commutative, we can write (1.2-1) in the form

$$< f(u),\ 1_{o,\infty}\ (qs),\ (qs)^{s-1}\ w(qsu)\ > \qquad \ldots (1.2\text{-}4)$$

$$= < f(u),\ < 1_{o,\infty}\ (qs),\ (qs)^{s-1}\ w(qsu)\ >> \qquad \ldots (1.2\text{-}5)$$

$$= < f(u),\ \int_o^\infty (qs)^{s-1}\ w(qsu)\ > \qquad \ldots (1.2\text{-}6)$$

Hence, (1.2-3) = (1.2-6)

and this completes the proof of the lemma.

Lemma 1.2.2

Let $\phi \in D(I)$ and r be a fixed real number. If

$$\Psi(s) = \int_o^\infty z^{s-1}\ \Phi\ (z)dz$$

where, $s = \sigma + i\omega$, σ fixed and $\alpha > \sigma > \max(\sigma_f, 1)$,

$-\infty < \omega < \infty$ and if $f \in W'_{\alpha,\beta}(I)$, then

$$\frac{1}{2\pi} \int_{-r}^{r} < f(u),\ u^{-s} > \Psi\ (s)d\omega$$

$$= < f(u),\ \frac{1}{2\pi} \int_{-r}^{r} u^{-s}\ \Psi\ (s)d\omega > \qquad \ldots (1.2\text{-}7)$$

Proof: The proof is trivial for $\phi\ (z) \equiv 0$. Now we suppose $\phi\ (z) \not\equiv 0$ and let

$$< f(u),\ u^{-s}\ > = \ Y\ (s) \qquad \ldots (1.2\text{-}8)$$

(1.2-8) is justified as $u^{-s} \in W_{\alpha,\beta}$ (I) for Re $s < \alpha$. It can be seen that Y (s) is analytic for all s for which $\alpha > \sigma > \max(\sigma_f, 1)$ and Ψ (s) is analytic for all finite values of s [Carslaw 4, p. 196-198 and Titchmarsh 5, p. 99].

Thus, the left hand side of (1.2-7) is an integral with an integrand as an analytic function over a finite region and therefore converges uniformly. Now,

$$\left| e^{\beta u}\, u^{\alpha+n}\, \frac{d^n}{du^n}\left\{\frac{1}{2\pi}\int_{-r}^{r} u^{-s}\,\Psi(s)d\omega\right\}\right|$$

$$\leq \frac{1}{2\pi}\int_{-r}^{r} e^{\beta u}\, u^{\alpha-s}\, \left|(-1)^n (s)(s+1) --- (s+n-1)\,\Psi(s)d\omega\right|$$

clearly,

$$\int_{-r}^{r} s(s+1) ----- (s+n-1)\,\Psi(s)d\omega \text{ is finite and}$$

$$\sup_{0<u<\infty} \left|e^{\beta u}\, u^{\alpha-s}\right| < \infty \text{ for } \alpha > \operatorname{Re} s \;;\; \beta < 0 .$$

We observe that,

$$\sup_{0<u<\infty} \left|e^{\beta u}\, u^{\alpha+n}\, \frac{d^n}{du^n}\left\{\frac{1}{2\pi}\int_{-r}^{r} u^{-s}\,\Psi(s)d\omega\right\}\right| < \infty$$

which proves $\int_{-r}^{r} u^{-s}\,\Psi(s)d\omega$, as a function of u belonging to $W_{\alpha,\beta}(I)$. Thus, the right hand side of (1.2-7) is also meaningful.

Now, we make partition of the path of integration on the straight line from $\sigma - ir$ to $\sigma + ir$ into intervals each of length 2r/m. Let $s_\ell = \sigma + i\omega_\ell$ be a point in lth interval.

Let us construct

$$B_m(u) = \sum_{\ell=1}^{m} u^{-s_\ell}\,\Psi(s_\ell).\,\frac{2r}{m}$$

And, now we can write

$$\frac{1}{2\pi}\int_{-r}^{r} < f(u),\ u^{-s} > \Psi(s)d\omega$$

$$= \lim_{m\to\infty} < f(u),\ \sum_{\ell=1}^{m} \frac{1}{2\pi}\, u^{-s}\,\Psi(s_\ell)\,\frac{2r}{m} > \qquad \ldots (1.2\text{-}9)$$

We shall prove that the sum within the last expression converges in $W_{\alpha,\beta}$ (I) to $\frac{1}{2\pi} \int_{-r}^{r} u^{-s}\, \Psi(s)d\omega$, and this will establish the equality (1.2-7). We consider V(u,m), where

$$V(u,m) = e^{\beta u}\, u^{\alpha+n} \Big[\sum_{\ell=0}^{m} (-1)^n s_\ell (s_\ell+1) \text{-----} \text{---} (s_\ell + n-1)\, u^{-s_\ell-1}\, \Psi(s_\ell) . \frac{2r}{m}$$

$$- \int_{-r}^{r} (-1)^n s(s+1) \text{---} (s+n-1)\, u^{-s-n}\, \Psi(s)d\omega \Big] \quad \ldots (1.2\text{-}10)$$

We have to prove that V(u,m) converges uniformly to zero on $0<u<\infty$, as $m \to \infty$. For $\beta<0$, it can be seen that,

$$|e^{\beta u}\, u^{\alpha+n}\, s(s+1) \text{----} (s+n-1)| < \varepsilon/_3 \Big[\int_{-r}^{r} |\Psi(s)d\omega \Big]^{-1}$$

$\Big[\int_{-r}^{r} |\Psi(s)d\omega|$ is finite and $\neq 0$ because $\phi(z) \neq 0 \Big]$.,
which clears that

$$\sup_{u>u'} e^{\beta u}\, u^{\alpha+n} \frac{d^n}{du^n} \{ \int_{-r}^{r} u^{-s}\, \Psi(s)d\omega \} < \varepsilon/_3 \qquad \ldots (1.2\text{-}11)$$

also,

$$\sup_{u>u'} |e^{\beta u}\, u^{\alpha+n}\, D_u^n\, B_m(u)|$$

$$< \varepsilon/_3 \Big[|\Psi(s)d\omega| \Big]^{-1} . \frac{2r}{m} \sum_{\ell=1}^{m} |\Psi(s_\ell)| \text{, for all m.} \qquad \ldots (1.2\text{-}12)$$

Hence, there exists m_0 such that for $m > m_0$ the right-hand side is bounded by $2\,\varepsilon/_3$.

From (1.2-11) and (1.2-12) it can be seen that for $m > m_o$, $u > u'$

$$|V(u,m)| < \varepsilon .$$

Let us consider the range $0 < u \leq u'$, with σ fixed as $\alpha > \sigma > \max(\sigma_f, 1)$. It follows that $u^{\alpha - s}\, s(s+1) ---- (s+n-1)\, \Psi(s)$ is a uniformly continuous function of (u, ω) with $o < u \leq u'$ and $-r \leq \omega \leq r$. This together with (1.2-10) shows that for all $m > m_o$,

$$|V(u,m)| < \varepsilon$$

on $0 < u \leq u'$ as well.

Hence, when $m > \max(m_o, m_1)$, we get

$$|V(u,m)| < \varepsilon ,$$

uniformly on $0 < u < \infty$, which proves the lemma.

<u>Lemma 1.2-3</u>

Let (i) $\phi \in D(I)$

(ii) α, β, σ be real numbers such that

$$\text{Re}\,(\alpha + c \pm m) > 0, \alpha > \sigma > \max(\sigma_f, 1),\ \beta < 0,$$

then

$$\frac{1}{\pi} \int_0^\infty \phi(z)\, (z/_u)^{\sigma - 1}\, \frac{\sin r \log z/_u}{u \log z/_u}\, dz \to \phi(u) \qquad \dots (1.2\text{-}13)$$

in $W_{\alpha,\beta}(I)$ as $r \to \infty$.

<u>Proof:</u> We consider,

$$I = \frac{1}{\pi} \int_0^\infty \phi(z)\, (z/u)^{\sigma - 1}\, \frac{\sin r \log z/_u}{u \log z/_u}\, dz$$

and put $\log z/_u = t$ i.e. $z = u\, e^t$ in I and get

$$I = \frac{1}{\pi} \int_{-\infty}^{\infty} \phi\,(ue^{t})\, e^{\sigma t}\, \frac{\sin rt}{t}\, dt.$$

And,

$$[\, I - \phi(u)\,] = \frac{1}{\pi} \int_{-\infty}^{\infty} \{e^{\sigma t}\, \phi(u\, e^{t}) - \phi(u)\}\, \frac{\sin rt}{t}\, dt.$$

$$[\text{since} \int_{-\infty}^{\infty} \frac{\sin rt}{t}\, dt = \pi].$$

we suppose

$$Q_r\,(u) = e^{\beta u}\, u^{\alpha+n}\, \frac{d^n}{du^n} \frac{1}{\pi} \int_{-\infty}^{\infty} [e^{\sigma t}\, \phi(ue^{t}) - \phi(u)]\, \frac{\sin rt}{t}\, dt$$

Now, if we shall prove that $Q_r(u) \to 0$ uniformly on $0 < u < \infty$ as $r \to \infty$; $n = 0,1,2.$-----, our lemma will be established.

Taking the differential operator inside the integral sign, we get,

$$Q_r(u) = \frac{1}{\pi} e^{\beta u}\, u^{\alpha+n} \int_{-\infty}^{\infty} [e^{\sigma t}\, D_u^n\, \phi(ue^{t}) - D_u^D\, \phi(u)\,]\, \frac{\sin rt}{t}\, dt$$

$$= \frac{1}{\pi} e^{\beta u}\, u^{\alpha+n} \int_{-\infty}^{-\eta} [e^{\sigma t}\, D_u^n\, \phi(ue^{t}) - D_u^n\, \phi(u)\,]\, \frac{\sin rt}{t}\, dt$$

$$+ \frac{1}{\pi} e^{\beta u}\, u^{\alpha+n} \int_{-\eta}^{\eta} [e^{\sigma t}\, D_u^n\, \phi(ue^{t}) - D_u^n\, \phi(u)\,]\, \frac{\sin rt}{t}\, dt$$

$$+ \frac{1}{\pi} e^{\beta u}\, u^{\alpha+n} \int_{\eta}^{\infty} [e^{\sigma t}\, D_u^n\, \phi(ue^{t}) - D_u^n\, \phi(u)\,]\, \frac{\sin rt}{t}\, dt \qquad (3)$$

$$= I_1 + I_2 + I_3 \text{ (say)}.$$

Now, we set considering I_2 first

$$N(t,u) = e^{\beta u}\, u^{\alpha+n} \left[\frac{e^{\sigma t}\, D_u^n\, \phi\,(ue^{t}) - D_u^n\, \phi(u)}{t} \right]$$

By virtue of our supposition $N(t,u)$ is a continuous functions of (t,u) for, all u $(0 < u < \infty)$ and $t \neq 0$. Also, by L'Hospital's rule,

$$\lim_{t\to 0} N(t,u) = \lim_{t\to o} e^{\beta u} u^{\alpha+n} D_t [e^{\sigma t} D_u^n \phi(ue^t)].$$

Thus assigning the value

$$e^{\beta u} u^{\alpha+n} D_t [e^{\sigma t} D_u^n \phi (ue^{\sigma t})]_{t=o}$$

to $N(o,u)$, we have that $N(t,u)$ is a continuous function of (t,u) in $-\eta < t < \eta$, $0 < u < \infty$ and since $\phi(u)$ is smooth, $N(t,u)$ is bounded, say by K. Hence for any $\varepsilon > 0$, there is a η so small that

$$|I_2(u)| = \left|\frac{1}{\pi} \int_{-\eta}^{\eta} N(t,u) \sin rt \, dt \right|$$

$$\leq \frac{1}{\pi} \int_{-\eta}^{\eta} |N(t,u)| \, dt$$

$$\leq K \, 2\eta \frac{1}{\pi} < \varepsilon \text{ ; if } \eta \text{ is so fixed}$$

that $\eta \leq \frac{\varepsilon}{2K}$

Now, we consider $I_1(u)$,

$$I_1(u) = \frac{1}{\pi} e^{\beta u} u^{\alpha+n} \int_{-\infty}^{-\eta} \{e^{\sigma t} D_t^n \phi(ue^t) - D_u^n \phi(u)\} \frac{\sin rt}{t} dt$$

$$= \frac{1}{\pi} e^{\beta u} u^{\alpha+n} \int_{-\infty}^{-\eta} e^{\sigma t} D_u^n \phi(ue^t) \frac{\sin rt}{t} dt$$

$$- \frac{1}{\pi} e^{\beta u} u^{\alpha+n} \int_{-\infty}^{-\eta} e^{\sigma t} D_u^n \phi(u) \frac{\sin rt}{t} dt$$

$= P_1(u) - P_2(u)$, (say)

we have

$$P_2(u) = \frac{1}{\pi} e^{\beta u} u^{\alpha+n} \{D_u^n \phi(u)\} \int_{-\infty}^{-r\eta} \frac{\sin z}{z} dz .$$

Now, as $\{e^{\beta u} u^{\alpha+n} D_u^n \phi(u)\}$ is bounded in $0 < u < \infty$ and $\int_{-\infty}^{o} \frac{\sin z}{z} dz$ is convergent, $P_2(u)$ uniformly tends to zero in

$0 < u < \infty$, as $r \to \infty$ [since $\lim_{r\to\infty} \int_{-\infty}^{-r\eta} \frac{\sin z}{z}\, dz = 0$].

Integrating by parts, we have

$$P_1(u) = \frac{1}{\pi} e^{\beta u} u^{\alpha+n} \left[\frac{e^{\sigma t} D_u^n \phi(ue^t)}{t} \cdot \frac{\cos rt}{t} \right]_{-\infty}^{-\eta}$$

$$+ \frac{1}{\pi r} e^{\beta u} u^{\alpha+n} \int_{-\infty}^{-\eta} \cos rt\, D_t \{ \frac{e^{\sigma t}}{t} D_u^n \phi(ue^t) \}\, dt .$$

Since $\phi(u) \in D(I)$, i.e. is of compact support and $\sigma > 1$, $P_1(u)$ becomes

$$P_1(u) = \frac{1}{\pi} e^{\beta u} u^{\alpha+n} \left[- \frac{e^{\sigma\eta} D_u^n \phi\, (ue^{-\eta})}{-\eta} \cdot \frac{\cos r\eta}{r} \right]$$

$$+ \lim_{t\to\infty} \frac{1}{\pi} e^{\beta u} u^{\alpha+n} \left[- \frac{e^{\sigma t} D_u^n \phi\, (ue^t)}{t} \frac{\cos rt}{r} \right]$$

$$+ \frac{1}{\pi r} e^{\beta u} u^{\alpha+n} \int_{-\infty}^{-\eta} \cos rt\, D_t \{ \frac{e^{\sigma t}}{t} D_u^n \phi\, (ue^t) \}\, dt$$

$$= \frac{1}{\pi r} e^{\beta u} u^{\alpha+n} \left[- \frac{e^{-\sigma\eta} D_u^n \phi\, (ue^{-\eta})}{-\eta} \cos r\eta \right]$$

$$+ 0 + \frac{1}{\pi r} e^{\beta u} u^{\alpha+n} \int_{-\infty}^{-\eta} \cos rt\, D_t \{ \frac{e^{\sigma t}}{t} D_u^n \phi\, (ue^t) \}\, dt \qquad \ldots (1.2\text{-}14)$$

First term of (1.2-14) uniformly tends to zero in $0 < u < \infty$ as $r \to \infty$, since η and σ are fixed and $e^{\beta u} u^{\alpha+n} D_u^n (ue^{-\eta})$ is a bounded function of u in $0 < u < \infty$. Also,

$$e^{\beta u} u^{\alpha+n} D_t \left[\frac{e^{\sigma t}}{t} D_u^n \phi\, (ue^t) \right]$$

$$= e^{\beta u} u^{\alpha+n} \left(\frac{t\, \sigma\, e^{\sigma t} - e^{\sigma t}\, 1}{t^2} \right) D_u^n \phi\, (ue^t)$$

$$+ e^{\beta u} u^{\alpha+n} \frac{e^{\sigma t}}{t} \frac{d}{dt} (D_u^n \phi\, (ue^t)).$$

Since each term is a bounded function of u and t, $0<u<\infty$, $-\infty < t < -\eta$, hence the second term in the right hand side of (1.2-14) also tends uniformly to zero as $r \to \infty$.

Thus, we see that $P_1(u) \to 0$ as $r \to \infty$ and also $P_2(u) \to 0$ as $r \to \infty$. Hence, $I_1(u) < 0$ uniformly in $0 < u < \infty$, as $r \to \infty$. Similarly, we can show that $I_3(u)$ converges uniformly in $0 < u < \infty$, as $r \to \infty$. In view of all these facts, we see that $Q_r(u) < \varepsilon$, $r \to \infty$, $0 < u < \infty$, $\varepsilon > 0$, being arbitrary small.

This proves the lemma.

1.3 Complex Inversion Formula

Theorem 1.3.1: Let

(a) $f \in W'_{\alpha,\beta}(I)$

(b) F(s) be defined by

$$F(s) = \langle f(u),\ w(qsu) \rangle$$

$$w(qsu) = (qsu)^{c-\frac{1}{2}}\, e^{-(p-\frac{q}{2})su}\, W_{k,m}(qsu)$$

(c) α , β , σ be real numbers with

(i) $\mathrm{Re}\,(\alpha + c \pm m) > 0$, $\alpha > \sigma > \max(\sigma_f, 1)$

(ii) $\beta < 0$, and

(iii) $\mathrm{Re}\,(s + c - k + \frac{1}{2}) > 0$, $\mathrm{Re}\,(s + c \pm m) > -1$,

$$s = \sigma + i\omega$$

then for any $\phi(z) \in D(I)$

$$\left\langle \frac{1}{2\pi i} \int_{\sigma-ir}^{\sigma+ir} \frac{\Gamma(s+c-k+1/2)}{\Gamma(s+c+m)\,\Gamma(+c-m)}\, \phi(s)\, z^{s-1}\, ds,\ \phi(z) \right\rangle$$

$$\to \langle f, \phi \rangle \ , \text{ as } r \to \infty ,$$

where $\phi(s) = \int_0^\infty (qs)^{s-1}\, F(s)\, ds$.

Proof

We shall prove the theorem by justifying steps in the following manipulations;

$$\left\langle \frac{1}{2\pi i} \int_{\sigma-ir}^{\sigma+ir} \frac{1}{M}(s)\ \Phi(s)\ z^{s-1}\ ds,\ \phi(z) \right\rangle \qquad \ldots (1.3\text{-}1)$$

$$M(s) = \frac{\Gamma(s+c+m)\ \Gamma(s+c-m)}{\Gamma(s+c-k+\frac{1}{2})}$$

$$= \int_0^\infty \frac{1}{2\pi i} \int_{-r}^{r} M^{-1}(s)\ \Phi(s)\ z^{s-1}\ ds\ \phi(z)\ dz \qquad \ldots (1.3\text{-}2)$$

$$= \frac{1}{2\pi} \int_{-r}^{r} M^{-1}(s)\ \Phi(s) \int_0^\infty z^{s-1}\ \phi(z)\ dz\ d\omega\ \ (s=\sigma+i\omega) \qquad \ldots (1.3\text{-}3)$$

$$= \frac{1}{2\pi} \int_{-r}^{r} M^{-1}(s)\ \left\{\int_0^\infty (qs)^{s-1}\ F(s) ds\right\} \int_0^\infty z^{s-1}\ \phi(z) dz\ d\omega \qquad \ldots (1.3\text{-}4)$$

$$= \frac{1}{2\pi} \int_{-r}^{r} M^{-1}(s)\ \left\{\int_0^\infty (qs)^{s-1} \left\langle f(u),\ w(qsu) \right\rangle ds\right\} \int_0^\infty z^{s-1}\ \phi(z)\ dz d\omega \qquad \ldots (1.3\text{-}5)$$

$$= \frac{1}{2\pi} \int_{-r}^{r} M^{-1}(s) \left\langle f(u),\ \int_0^\infty (qs)^{s-1}\ w(qsu) ds \right\rangle \int_0^\infty z^{s-1}\ \phi(z) dz d\omega \qquad \ldots (1.3\text{-}6)$$

$$= \frac{1}{2\pi} \int_{-r}^{r} M^{-1}(s) \left\langle f(u),\ M(s)\ u^{-s+1} \right\rangle \int_0^\infty z^{s-1}\ \phi(z) dz d\omega \qquad \ldots (1.3\text{-}7)$$

$$= \left\langle f(u),\ \frac{1}{2\pi} \int_{-r}^{r} u^{-s+1} \int_0^\infty z^{s-1}\ \phi(z) dz d\omega \right\rangle \qquad \ldots (1.3\text{-}8)$$

$$= \Big\langle f(u),\ \frac{1}{2\pi} \int_0^\infty \phi(z) \int_{-r}^{r} u^{-s+1}\ z^{s-1}\ d\omega dz \qquad \ldots (1.3\text{-}9)$$

$$= \Big\langle f(u),\ \frac{1}{2\pi} \int_0^\infty \phi(z)\ (z/_u)^{\sigma-1}\ \frac{\sin \log z/_u}{u \log z/_u}\ dz \qquad \ldots (1.3\text{-}10)$$

$$= \left\langle f(u),\ \phi(u) \right\rangle,\ \text{as } r \to \infty. \qquad \ldots (1.3\text{-}11)$$

Now, since the integral in (1.3-1) is a continuous function of z and ϕ (z) is a smooth function of compact support in $(0,\infty)$, (1.3-1) is nothing but (1.3-2). As the integral in (1.3-2) is continuous on a closed bounded domain of integration, the order of integration can be changed in (1.3-2) to get (1.3-3). (1.3-4) and (1.3-5) are clear. (1.3-6) can be justified by Lemma 1.2.1.

From [Erdelyi 4, p. 336], we have

$$\int_0^\infty (qs)^{s-1}\,(qsu)^{c-1/2}\,e^{-(p-q/2)su}\,W_{k,m}\,(qsu)\,ds$$

$$= u^{-s+1}\,\frac{\Gamma(s+c+m)\,\Gamma(s+c-m)}{\Gamma(s+c-k+1/2)}$$

provided that Re $(s+c\pm m) > -1$, Re $(s+c-k+1/2) > 0$.

Thus, (1.3-7) is a simplification of (1.3-6) and (1.3-8) can be justified by Lemma 1.2.2. Since the integral in (1.3-8) converges uniformly, the order of integration can be changed to deduce (1.3-9) and (1.3-9) reduces to (1.3-10). Lemma 1.2.3 proves that the integral within (1.3-10) converges in $W_{\alpha,\beta}$ (I) uniformly to $\phi(u)$ in $0 < u < \infty$, as $r \to \infty$ and hence (1.3-10) converges in the sense of weak distributional convergence to (1.3-11).

This complete the proof.

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On the Maximum Real Part of an Integral Function Represented by Dirichlet Series

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1. Consider the Dirichlet series

$$(1.1)\qquad f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n} ,$$

where $s = \sigma + it$, $\lambda_1 \geq 0$, $\lambda_n < \lambda_{n+1} \to \infty$,

and

$$(1.2)\qquad \limsup_{n\to\infty} \frac{\log n}{\lambda_n} = D < \infty .$$

Let σ_c and σ_a be the abscissa of convergence and abscissa of absolute convervence, respectively, of $f(s)$. We know ([1], p. 4) that $0 \leq \sigma_c - \sigma_a < D$. If $\sigma_c = \infty$, then from this inequality $\sigma_a = \infty$, and $f(s)$ represents an integral function.

Let the maximum real part be as

$$A(\sigma) = \underset{-\infty < t < \infty}{\text{l.u.b.}} \left|\text{Re}\ f(\sigma + it)\right| .$$

Since $\log A(\sigma)$ is an increasing convex function of σ [2], therefore $\log A(\sigma)$ is differentiable almost everywhere with an increasing derivative. This enables us to write $\log A(\sigma)$ in the following form:

$$(1.3)\qquad \log A(\sigma) = \log A(\sigma_o) + \int_{\sigma_o}^{\sigma} \frac{A'(x)}{A(x)}\, dx.$$

This integral representation of $\log A(\sigma)$ helps us in deriving interesting properties of $\log A(\sigma)$. The results are given in the form of theorems.

2. In this article we estimate the ratio of $A(\sigma)$ for any two positive values of σ in terms of the ratio of $A'(\sigma)$ and $A(\sigma)$ for those values of σ.

Theorem 1. If $f(s)$ is an integral function and $o < \sigma_o < \sigma_1 < \sigma_2$, then

$$\frac{A'(\sigma_1)}{A(\sigma_1)} \leq \frac{1}{(\sigma_2 - \sigma_1)} \log \frac{A(\sigma_2)}{A(\sigma_1)} \leq \frac{A'(\sigma_2)}{A(\sigma_2)} \tag{2.1}$$

Proof. From (1.3), we have

$$\log \frac{A(\sigma_2)}{A(\sigma_1)} = \int_{\sigma_1}^{\sigma_2} \frac{A'(x)}{A(x)}\, dx$$

$$\geq \frac{A'(\sigma_1)}{A(\sigma_1)} (\sigma_2 - \sigma_1), \quad \sigma_1 > \sigma_2$$

and

$$\log \left\{\frac{A(\sigma_2)}{A(\sigma_1)}\right\} \leq \frac{A'(\sigma_2)}{A(\sigma_2)} (\sigma_2 - \sigma_1), \ \sigma_1 > \sigma_o ,$$

since $\log A(\sigma)$ is an increasing function of σ.

Corollary. If $f(s)$ is an integral function, other than a constant, and $0 < \alpha < 1$, then

$$\lim_{\sigma \to \infty} \frac{A(\alpha\sigma)}{A(\sigma)} = 0$$

If we put $\sigma_1 = \alpha\,\sigma$ and $\sigma_2 = \sigma$ in (2.1), then

$$e^{-\sigma(1-\alpha)} \frac{A'(\sigma)}{A(\sigma)} \leq \frac{A(\alpha\sigma)}{A(\sigma)} \leq e^{-\sigma(1-\alpha)} \frac{A'(\alpha\sigma)}{A(\alpha\sigma)}$$

The result follows on taking limits on both the sides.

3. Theorem 2. If $f(s)$ is an integral function other than an exponential polynomial and $\sigma > \sigma_o$, $\varepsilon > 0$, then

$$A'(\sigma) > \frac{A(\sigma) \log A(\sigma)}{(1 + \varepsilon)\sigma} ,$$

where $\varepsilon = \varepsilon(\sigma) \to \infty$ as $\sigma \to \infty$.

Proof. From (1.3), we have

$$\log A(\sigma) < \log A(\sigma_o) + \frac{A'(\sigma)}{A(\sigma)} (\sigma - \sigma_o) ,$$

and therefore

$$\lim_{\sigma\to\infty} \sup \frac{A(\sigma) \log A(\sigma)}{\sigma A'(\sigma)} \leq 1$$

giving

$$\frac{A(\sigma) \log A(\sigma)}{\sigma A'(\sigma)} < (1 + \varepsilon)$$

where $\sigma > \sigma_o$ and $\varepsilon = \varepsilon(\sigma) \to 0$ as $\sigma \to \infty$.

4. Theorem 3. If $f(s)$ is an integral function of Ritt order ρ $(0 < \rho < \infty)$ and lower order λ, then

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \{\frac{A'(\sigma)}{A(\sigma)}\}}{\sigma} = \begin{matrix}\rho\\ \lambda\end{matrix}$$

Proof. From (1.3), we have

$$\log A(\sigma) \leq \log A(\sigma_o) + (\sigma - \sigma_o) \frac{A'(\sigma)}{A(\sigma)} ,$$

and therefore using the result ([2], p. 122)

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \{\frac{A'(\sigma)}{A(\sigma)}\}}{\sigma} \tag{4.1}$$

$$\geq \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \log A(\sigma)}{\sigma} = \begin{matrix}\rho\\ \lambda\end{matrix}$$

Moreover, for an arbitrary fixed $h > 0$

$$\log A(\sigma + h) = \log A(\sigma) + \int_{\sigma}^{\sigma+h} \frac{A'(x)}{A(x)} dx \geq h \frac{A'(\sigma)}{A(\sigma)} ,$$

and therefore

$$(4.2) \quad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \{\frac{A'(\sigma)}{A(\sigma)}\}}{\sigma}$$

$$\leq \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log A(\sigma + h)}{\sigma + h} = \begin{matrix}\rho\\ \lambda\end{matrix}$$

Combining (4.1) and (4.2), we get the result.

5. <u>Theorem 4.</u> If $f^m(s)$ is the m-th derivative of $f(s)$ and $A'(\sigma, f^{(m)})$ is the derivative of $A(\sigma, f^{(m)})$, then for $\lambda \geq \delta > 0$

$$(5.1) \quad A(\sigma) < A'(\sigma) < A'(\sigma, f^{(1)}) < \ldots < A'(\sigma, f^{(p)})$$

almost everywhere for $\sigma > \sigma_o \geq 0$ and

$$A(\sigma, f^{(m)}) = \underset{-\infty<t<\infty}{\text{l.u.b.}} |\text{Re } f^{(m)} (\sigma + it)|$$

<u>Proof.</u> We have from Theorem 3

$$(5.2) \quad A(\sigma) e^{\sigma(\lambda-\varepsilon)} < A'(\sigma) < A(\sigma) e^{\sigma(\lambda+\varepsilon)}$$

almost everywhere for $\sigma > \sigma_1$

If $\lambda \geq \delta > 0$, then from (5.2)

$$A(\sigma) < A'(\sigma)$$

almost everywhere $\sigma > \sigma_1 \geq 0$.

Similarly, using (5.2) for the function $f^{(1)}(s)$, we have

$$A(\sigma, f^{(1)}) < A'(\sigma, f^{(1)})$$

almost everywhere for $\sigma > \sigma_1 \geq 0$.

But ([2], p. 126)

$$A'(\sigma) \leq A(\sigma, f^{(1)}).$$

Hence,

$$A'(\sigma) < A'(\sigma,\ f^{(1)})$$

almost everywhere for $\sigma > \max\ (\sigma_i,\ \sigma_1)$.

Similar results can be obtained for the higher derivatives and hence the theorem follows for almost all values of $\sigma > \sigma_o \geq 0$,

where $\sigma > \sigma_o = \max\ (\sigma_i,\ \sigma_1,\ \sigma_2,\ \ldots\ ,\ \sigma_p)$.

6. Theorem 5. If $\lambda \geq \delta > 0$ and $f(s)$ is not an exponential polynomial, then

$$A'(\sigma,\ f^{(m)}) > A'(\sigma) \left[\frac{\log A'(\sigma)}{(1+\epsilon)\sigma} \right]^m , \tag{6.1}$$

almost everywhere for $\sigma > \sigma_o \geq 0$, $\epsilon > 0$ and where $A'(\sigma, f^{(m)})$ is the derivative of $A(\sigma,\ f^{(m)})$.

Proof. Using Theorem 1 for $f^{(1)}(s)$, we have for $\sigma > \sigma_i$

$$A'(\sigma,\ f^{(1)}) > \frac{A(\sigma,f^{(1)})\log A(\sigma,f^{(1)})}{(1\ +\ \epsilon)\sigma}$$

$$\geq \frac{A'(\sigma)\ \log A'(\sigma)}{(1\ +\ \epsilon)\sigma} ,$$

since $A(\sigma,f^{(1)}) \geq A'(\sigma)$.

Writing the above inequality for the p-th derivative, we get

$$\frac{A'(\sigma,f^{(p)})}{A'(\sigma,f^{(p-1)})} > \frac{\log A'(\sigma,f^{(p-1)})}{(1\ +\ \epsilon)\sigma}$$

for $\sigma > \sigma'_p$

Taking $p = 1,2,\ \ldots,\ m$ and multiplying together, we have

$$A'(\sigma,f^{(m)}) > \frac{A'(\sigma)\ \prod_{p=1}^{m} \log A'(\sigma,f^{p-1})}{[(1+\epsilon)\sigma]^m}$$

for $\sigma > \max\,(\sigma_1', \sigma_2', \ldots, \sigma_m')$.

Hence, using (5.1), we have

$$A'(\sigma, f^{(m)}) > A'(\sigma) \left[\frac{\log A'(\sigma)}{(1+\varepsilon)\,\sigma} \right]^m ,$$

almost everywhere for $\sigma > \sigma_o \geq 0$,

where $\sigma_o = \max\,(\sigma_1, \sigma_2, \ldots, \sigma_{m-1}, \sigma_1', \sigma_2', \sigma_m')$.

7. Theorem 6. If $f(s)$ is an integral function of Ritt order $\rho\,(0<\rho<\infty)$, type τ and lower type ν, then

$$(7.1) \qquad \lim_{\sigma\to\infty} \substack{\sup \\ \inf} \frac{\left\{\frac{A'(\sigma)}{A(\sigma)}\right\}}{e^{\rho\sigma}} \leq \substack{e\rho\tau \\ e\rho\nu}$$

Proof. From (1.3) we have for $h > 0$

$$\log A(\sigma + h) = O(1) + \int_{\sigma_o}^{\sigma+h} \frac{A'(x)}{A(x)}\,dx$$

$$> \int_{\sigma}^{\sigma+h} \frac{A'(x)}{A(x)}\,dx \;, \quad \sigma > \sigma_o$$

$$\geq \frac{A'(\sigma)}{A(\sigma)}\,h \;.$$

Hence,

$$\lim_{\sigma\to\infty} \substack{\sup \\ \inf} \frac{\left\{\frac{A'(\sigma)}{A(\sigma)}\right\}}{e^{\rho h}} \leq \frac{e^{\rho h}}{h} \lim_{\sigma\to\infty} \substack{\sup \\ \inf} \frac{\log A(\sigma + h)}{e^{\rho(\sigma + h)}}$$

$$\leq \frac{e^{\rho h}}{h} \lim_{\sigma\to\infty} \substack{\sup \\ \inf} \frac{\log M(\sigma + h)}{e^{\rho(\sigma + h)}} = \substack{\tau e^{\rho h}/h \\ \nu e^{\rho h}/h} \;,$$

where $M(\sigma) = \underset{-\infty<t<\infty}{\text{l.u.b.}} \; |f(\sigma + it)|$.

Taking $h = \frac{1}{\rho}$ we get (7.1).

8. Let $L(e^{\sigma})$ be a slowly changing function, i.e.

(i) $L(e^{\sigma}) > 0$ and is continuous for $\sigma > \sigma_o$,

(ii) $L(\ell e^{\sigma}) \sim L(e^{\sigma})$ as $\sigma \to \infty$ for every constant $\ell > 0$.

Let, for $0 < \rho < \infty$

$$(8.1) \quad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log A(\sigma)}{e^{\rho\sigma} L(e^{\sigma})} = \begin{matrix}T\\ t\end{matrix}, \quad (0 < t \leq T < \infty);$$

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\left\{\frac{A'(\sigma)}{A(\sigma)}\right\}}{e^{\rho\sigma} L(e^{\sigma})} = \begin{matrix}p\\ q\end{matrix} \quad (0 < q \leq p < \infty)$$

Theorem 7. If $f(s)$ is an integral function of Ritt- order ρ $(0 < \rho < \infty)$, then

(i) $q/\rho \leq t \leq T \leq p/\rho$

(ii) $t \leq q/\rho \log (ep/q)$, and

(iii) $T \geq p/e\rho \ e^{q/p}$.

Proof. Writing (1.3) as

$$\log A(\sigma + h) = O(1) + \int_{\sigma_o}^{\sigma} \frac{A'(x)}{A(x)} dx + \int_{\sigma}^{\sigma+h} \frac{A'(x)}{A(x)} dx, \quad \sigma > \sigma_o$$

$$< O(1) + (p + \varepsilon) \int_{\sigma_o}^{\sigma} e^{\rho x} L(e^{x}) \, dx + \frac{A'(\sigma + h)}{A(\sigma + h)} h$$

$$= O(1) + (p + \varepsilon) \int_{e^{\sigma_o}}^{e^{\sigma}} x^{\rho-1} L(x) \, dx + \frac{A'(\sigma + h)}{A(\sigma + h)} h$$

$$\sim (p + \varepsilon) \frac{e^{\rho\sigma}}{\rho} L(e^{\sigma}) + \frac{A'(\sigma + h)}{A(\sigma + h)} h$$

by ([3], Lemma 5).

Dividing by $e^{\rho\sigma} L(e^{\sigma})$, taking limits and using (8.1) we get

(8.2) $\quad e^{\rho h}\, T \le \frac{p}{\rho} + h\, e^{\rho h}\, p$, and

(8.3) $\quad e^{\rho h}\, t \le \frac{p}{\rho} + h\, e^{\rho h}\, q$.

Similarly we obtain

(8.4) $\quad e^{\rho h}\, T \ge q/\rho + hp$, and

(8.5) $\quad e^{\rho h}\, t \ge q/\rho + hq$.

It can be seen that the minima of the right hand expressions of (8.2) and (8.3) occur at $h = 0$ and $e^{\rho h} = p/q$. Substituting $h = 0$ in (8.2) and $e^{\rho h} = p/q$ in (8.3), we get the second part of (i) and (ii) respectively. Taking $h = (p-q)/\rho p$ in (8.4) and $h = 0$ in (8.5), we get (iii) and the first part of (i) respectively.

9. Theorem 8. If $\log A(\sigma) \sim T\, e^{\rho\sigma}\, L(e^{\sigma})$, then

$$\left\{\frac{A'(\sigma)}{A(\sigma)}\right\} \sim T\, e^{\rho\sigma}\, L(e^{\sigma}) .$$

Proof. Suppose now $T = t$. If $0 < \eta < 1$, we have from (1.3) for $\sigma > \sigma_o$

$$\frac{A'(\sigma)}{A(\sigma)}\,\eta < \int_{\sigma}^{\sigma+\eta} \frac{A'(x)}{A(x)}\,dx = \log A(\sigma+\eta) - \log A(\sigma)$$

$$= T\, e^{\rho(\sigma+\eta)}\, L(e^{\sigma+\eta}) - Te^{\rho\sigma}\, L(e^{\sigma}) + o(e^{\rho\sigma}\, L(e^{\sigma}))$$

$$= T\, e^{\rho\sigma}\, \{1 + \rho\eta + o\,(\eta^2)\}\, \{1 + o(1)\}\, L(e^{\sigma})$$

$$- T\, e^{\rho\sigma}\, L(e^{\sigma}) + o(e^{\rho\sigma}\, L(e^{\sigma})).$$

Hence,

$$\limsup_{\sigma\to\infty} \frac{\left\{\frac{A'(\sigma)}{A(\sigma)}\right\}}{e^{\rho\sigma}\, L(e^{\sigma})} \le T(\rho + H\eta) ,$$

where H is a constant. Since η is arbitrary, we get

$$\lim_{\sigma\to\infty} \sup \frac{\{\frac{A'(\sigma)}{A(\sigma)}\}}{e^{\rho\sigma} L(e^{\sigma})} \le T\,\rho\ .$$

Considering $\log A(\sigma) - \log A(\sigma - \eta)$ and proceeding as above, we get

$$\lim_{\sigma\to\infty} \inf \frac{\{\frac{A'(\sigma)}{A(\sigma)}\}}{e^{\rho\sigma} L(e^{\sigma})} \ge T\,\rho\ ,$$

and hence,

$$\frac{A'(\sigma)}{A(\sigma)} \sim T\,\rho\, e^{\rho\sigma} L(e^{\sigma}).$$

Corollary: If $\frac{A'(\sigma)}{A(\sigma)} \sim p\, e^{\rho\sigma} L(e^{\sigma})$, then

$$\log A(\sigma) \sim \frac{p}{\rho}\ e^{\rho\sigma} L(e^{\sigma}).$$

From (i) of Theorem 7 if $p = q$, $T = t = p/\rho$.

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A Note on Estimating the Finite Population Mean Using Auxiliary Information

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Abstract

A class of estimators for estimating finite population mean using auxiliary information on a supplimentary variable is defined and studied its asymptotic properties.

1. Introduction

In many survey situations of practical importance the use of auxiliary information points out the use of ratio, dual to ratio, dual to product [See Srivenkataramana (1980)], regression and generalized product estimators, among others, in order to estimate population mean (or total) of the study character y. Consider a simple random sample without replacement (SRSWOR) $\{(y_i, x_i),\ i = 1,2, \ldots, n\}$ of size n from a bivariate population of size N.

Denote by

$$N\bar{X} = \sum_{i=1}^{N} x_i,\quad N\bar{Y} = \sum_{i=1}^{N} y_i,\quad (N-1)\, S_x^2 = \sum_{i=1}^{N} (x_i - \bar{x})^2$$

$$n\bar{x} = \sum_{i=1}^{n} x_i,\quad n\bar{y} = \sum_{i=1}^{n} y_i,\quad (N-1)\, S_y^2 = \sum_{i=1}^{N} (y_i - \bar{Y})^2$$

$$(N-1)\, S_{yx} = \sum_{i=1}^{N} (y_i - \bar{Y})(x_i - \bar{x}),\quad (n-1)s_{yx} = \sum_{i=1}^{n} (y_i - \bar{y})(x_i - \bar{x})$$

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$\beta = S_{yx}/S_x^2$: The population regression coefficient of y on x,

$\hat{\beta} = s_{yx}/s_x^2$: The sample regression coefficient of y on x,

$\bar{y}_{\ell r} = \bar{y} + \hat{\beta}(\bar{X} - \bar{x})$: The usual biased regression estimator of $\bar{Y}$,

$V(\bar{y}_{\ell r}) = [V(\bar{y}) - \frac{\{Cov(\bar{y}, \bar{x})\}^2}{V(\bar{x})}]$: The variance of $\bar{y}_{\ell r}$.

The problem here is to estimate population mean $\bar{Y}$ of the study character y using auxiliary information on a supplimentary variable x. Srivastava (1980) proposed a class of estimators for $\bar{Y}$ as

$$\bar{Y}_1 = h(\bar{y}, \bar{x}) \qquad \ldots (1)$$

where $h(\bar{y}, \bar{x})$ is a function of $\bar{y}$, $\bar{x}$ such that

$$h(\bar{Y}, \bar{X}) = \bar{Y}, \; h(\bar{Y}, \bar{X}) = \frac{\partial}{\partial \bar{y}} h(\bar{y}, \bar{x}) \Big|_{(\bar{Y}, \bar{X})} = 1$$

and also satisfy certain regularity conditions. The minimum variance of $\hat{\bar{Y}}_1$, to terms of order $0(n^{-1})$, is given by

$$\text{Min. } V(\hat{\bar{Y}}_1) = V(\bar{y}_{\ell r}) = [V(\bar{y}) - \frac{\{Cov(\bar{y}, \bar{x})\}^2}{V(\bar{x})}], \qquad \ldots (2)$$

where $V(\bar{y}) = \frac{(N-n)}{(N-1)n} S_y^2$, $V(\bar{x}) = \frac{N-n}{(n-1)n} S_x^2$,

$Cov(\bar{y}, \bar{x}) = \frac{(N-n)}{(N-1)n} S_{yx}$.

It is to be pointed out that the class of estimators $\hat{\bar{Y}}_1 = h(\bar{y}, \bar{x})$ is very vast but it fails to include the usual biased regression estimator $\bar{y}_{\ell r}$ while its minimum variance to terms of order $o(n^{-1})$ is same as that of $\bar{y}_{\ell r}$.

The main objective of this note is to provide a class of estimators for $\bar{Y}$ using information on supplimentary variable x so as to include regression estimator $\bar{y}_{\ell r}$ and Srivastava (1980). Asymptotic expressions for bias and variance of the proposed class of estimators are obtained. Optimum estimator in the class is also identified which has smaller variance than that of $y_{\ell r}$ and $\hat{\bar{Y}}_1$.

2. The Class of Estimators

Keeping the form of regression estimator $\bar{y}_{\ell r} = \bar{y} + \hat{\beta}\,(\bar{X}-\bar{x})$, in view, it is quite logical to suggest the following class of estimators

$$\hat{\bar{Y}}_h = h(\bar{y}, \hat{\beta}\ \bar{x}) \qquad \ldots (3)$$

for $\bar{Y}$, where $h(\bar{y}, \hat{\beta}, \bar{x})$ is a function of $(\bar{y}, \hat{\beta}, \bar{x})$ such that

$$h(\bar{Y}, \beta, \bar{X}) = \bar{Y}$$

$$h_1(\bar{Y}, \beta, \bar{X}) = \left.\frac{\partial h\ (\bar{y}, \hat{\beta}, \bar{x})}{\partial \bar{y}}\right|_{(Y,\ \hat{\beta},\ \bar{X})} = 1 \qquad \ldots (4)$$

and satisfies the following conditions.

(i) Whatever be the sample chosen, let $(\bar{y}, \hat{\beta}, \bar{x})$ assume values in a bounded closed convex subset P of the three dimensional real space containing the point $(\bar{Y}, \hat{\beta}, \bar{x})$;

(ii) The function h $(\bar{y}, \hat{\beta}, \bar{x})$ is continuous and bounded in P; and

(iii) The first and second partial derivatives of h $(\bar{y}, \hat{\beta}, \bar{x})$ exist and are continuous and bounded in P.

To find the bias and variance of $\hat{\bar{Y}}_h$ we will need the following expressions which are easily derived using the formulae developed by Sukhatme (1944), we have

$$V(\hat{\beta}) = \frac{(N-n)\beta^2}{(N-2)n}\ K\ \left[\frac{\mu_{22}}{\mu_{11}^2} + \frac{\mu_{o4}}{\mu_{o2}^2} - \frac{2\ \mu_{13}}{\mu_{o2}\ \mu_{11}}\right],$$

$$\mathrm{Cov}(\bar{x}, \hat{\beta}) = \frac{(N-n)\beta}{(N-2)n}\left[\frac{\mu_{12}}{\mu_{11}} - \frac{\mu_{o3}}{\mu_{o2}}\right],$$

$$\mathrm{Cov}(\bar{y}, \hat{\beta}) = \frac{(N-n)\beta}{(N-2)n}\left[\frac{\mu_{21}}{\mu_{11}} - \frac{\mu_{12}}{\mu_{o2}}\right],$$

$$V(\bar{y}) = \frac{(N-n)}{(N-1)} \mu_{2o},$$

$$V(\bar{x}) = \frac{(N-n)}{(N-1)n} \mu_{o2},$$

where $\mu_{a,b} = N^{-1} \sum_{i=1}^{N} (y_i - \bar{Y})^a (x_i - \bar{X})^b$, a,b = 0 to4 ,

and $K = \frac{(N-1)\ (Nn-N-n-1)}{(n-1)\ N\ (N-3)}$

To find the expectation and variance of $\hat{\bar{Y}}_h$, we expand $h(\bar{y},\beta,\bar{x})$ about the point $(\bar{Y},\beta,\bar{X})$ in a Second Order Taylor's series we have,

$$E(\hat{\bar{Y}}_n) = \bar{Y} + o\ (n^{-1}) \qquad \ldots (5)$$

which shows that the bias of $\hat{\bar{Y}}_n$ is of the order of n^{-1} and so upto order n^{-1} the mean squarred error and variance of $\hat{\bar{Y}}_n$ are same

The variance of $\bar{Y}_n$ is given by

$$\begin{aligned} V(\hat{\bar{Y}}_n) &= E(\hat{\bar{Y}}_n - \bar{Y})^2 \\ &= \big[V(\bar{y}) + V(\hat{\beta})\ h_2^2\ (\bar{Y},\beta,x,) + V(\bar{x})h_3^2\ (\bar{Y},\beta,\bar{X}) \\ &\quad + 2\ \mathrm{cov}\ (\bar{x},\hat{\beta})\ h_2\ (\bar{X},\beta,\bar{X})\ h_3\ (\bar{Y},\beta,\bar{X}) \\ &\quad + 2\ \mathrm{cov}\ (\bar{y},\hat{\beta})\ h_2\ (\bar{Y},\beta,\bar{X}) \\ &\quad + 2\ \mathrm{cov}\ (\bar{y},x)\ h_3\ (\bar{Y},\beta,\bar{X}) \big] \end{aligned} \qquad \ldots (6)$$

where

$$h_2\ (\bar{Y},\beta,\bar{X}) = \left.\frac{\partial h\ (\bar{y},\hat{\beta},\bar{x})}{\partial\hat{\beta}}\right|_{(\bar{Y},\beta,\bar{X})} \quad \text{and}$$

$$h_3\ (\bar{Y},\beta,\bar{X}) = \left.\frac{\partial h\ (\bar{y},\hat{\beta},\bar{x})}{\partial\bar{x}}\right|_{(\bar{Y},\beta,\bar{X})}$$

Any parametric function $h(\bar{y},\hat{\beta},\bar{x})$ satisfying (4) and conditions (i) to (iii) can generate estimator of the class (3).

The optimum values of the parameter, in $h(\bar{y},\hat{\beta},\bar{x})$ which minimizes the variance of $\hat{\bar{Y}}_n$ are given by

$$h_2\ (\bar{Y},\beta,\bar{X}) = \frac{[\mathrm{Cov}\ (\bar{x},\hat{\beta})\ \mathrm{Cov}\ (\bar{y},\bar{x}) - V(\bar{x})\ \mathrm{Cov}\ (\bar{y},\hat{\beta})]}{[V(\bar{x})\ V(\hat{\beta}) - \{\mathrm{Cov}\ (\bar{x},\hat{\beta})\}^2]} \quad \ldots (7)$$

$$h_3\ (\bar{Y},\hat{\beta},\bar{X}) = \frac{[\mathrm{Cov}\ (\bar{x},\hat{\beta})\ \mathrm{Cov}\ (\bar{y},\hat{\beta}) - V\ (\hat{\beta})\ \mathrm{Cov}\ (\bar{y},\bar{x})]}{[V(\bar{x})\ V(\hat{\beta}) - \{\mathrm{Cov}\ (\bar{x},\hat{\beta})\}^2]} \quad \ldots (8)$$

Hence the minimum variance of $\hat{\bar{Y}}_h$ is given by

$$\mathrm{Min}\ V\ (\hat{\bar{Y}}_n) = V\ (\bar{y}) - \frac{\{\mathrm{Cov}\ (\bar{y},\bar{x})\}^2}{V\ (\bar{x})} -$$

$$- \frac{[\mathrm{Cov}\ (\bar{y},\bar{x})\ \mathrm{Cov}\ (\bar{x},\hat{\beta}) - V\ (\bar{x})\ \mathrm{Cov}\ (\bar{y},\hat{\beta})]^2}{V\ (\bar{x})\ [V\ (\bar{x})\ V\ (\hat{\beta}) - \{\mathrm{Cov}\ (\bar{x},\hat{\beta})\}^2]} - \quad \ldots (9)$$

From (2) to (9) we have

$$\mathrm{Min}\ V(\hat{\bar{Y}}_1) = V\ (\bar{Y}_{\ell r}) - \mathrm{Min}\ V\ (\hat{\bar{Y}}_n)$$

$$= \frac{[\mathrm{Cov}\ (\bar{y},\bar{x})\ \mathrm{Cov}\ (\bar{x},\hat{\beta}) - V\ (\bar{x})\ \mathrm{Cov}\ (\bar{y},\hat{\beta})]^2}{V\ (\bar{x})\ [V\ (\bar{x})\ V\ (\hat{\beta}) - \{\mathrm{Cov}\ (\bar{x},\hat{\beta})\}^2]} \geq 0 \quad \ldots (10)$$

Also

$$V\ (\bar{y}) - \mathrm{Min}\ V\ (\hat{\bar{Y}}_1) = V\ (\bar{y}_{\ell r}) = \frac{[\mathrm{Cov}\ (\bar{y},\bar{x})]^2}{V\ (\bar{x})} \geq 0 \quad \ldots (11)$$

Thus from (10) and (11) we have the following inequality

$$\mathrm{Min}\ V\ (\hat{\bar{Y}}_n) \leq \mathrm{Min}\ V(\hat{\bar{Y}}_1) = V(\hat{\bar{Y}}_{\ell r}) \leq V(\bar{y}) \quad \ldots (12)$$

Hence it follows from (12) that the proposed estimator is wider as well as more efficient than that of Srivastava (1980) estimator $\hat{\bar{Y}}_1$, usual biased regression estimator $\hat{\bar{y}}_{\ell r}$ and sample mean $\bar{y}$.

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A Note on Boundary Value Problems

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1. Introduction

Boundary value problems play an important role in a variety of real world problems. In finding solutions to two point and three-point boundary value problems, the construction of Green's function is vital. It is sufficiently known the construction of Green's function for problems involving non-singular square matrices. However the theory for rectangular matrices involves significant difficulties as the inverse of matrix in the usual sense, does not exist. In this paper a new approach is adopted. By a suitable transformation, the rectangular matrices are transformed into a square non-singular matrix and solutions are finally expressed in terms of the rectangular matrices. This approach is substantially new and covers various classes of boundary value problems, which are not covered earlier.

In this paper we shall be concerned with the existence and uniqueness of solutions to two-point boundary value problems associated with the system of first order differential equations

$$Ly \equiv P(t)y' + Q(t)y = f(t) \qquad \ldots (1.1)$$

where $P(t) \varepsilon C^2[a,b]$, $Q(t) \varepsilon C^1[a,b]$ are rectangular matrices of order $m \times n$ and $y(t)$ is a column matrix with components $(y_1, y_2, \ldots, y_n)$ and we assume through out the paper that the rows of $P(t)$ are linearly independent on $[a,b]$. Section 3, deals with the existence and uniqueness of solutions to three-point boundary value problems. The results obtained in this paper are examplified at the end of this paper.

2. Two-point Boundary Value Problems

Definition 2.1 : If $P^T Y(t)\ C$ is a fundamental matrix for the equation (1.1) then the matrix D defined by

$$D = M P^T(a)\ Y(a) + N P^T(b)\ Y(b)$$

is called a characteristic matrix for the boundary value problem.

In this section, we consider the following two point boundary value problem:

$$Ly \equiv P(t)y' + Q(t)\ y(t) = f(t),\ a \le t \le b\ (' = \frac{d}{dt}) \qquad \ldots (2.1)$$

$$My(a) + Ny(b) = 0 \qquad \ldots (2.2)$$

Theorem 2.1 : Any solution of

$$Ly \equiv P(t)y' + Q(t)y = 0 \qquad \ldots (2.3)$$

is of the form $P^T Y(t)\ C$ where $Y(t)$ is a fundamental matrix of

$$Z' = -[\ A^{-1}(t)\ B(t)\ ]Z \qquad \ldots (2.4)$$

where $A(t) = P(t)\ P^T(t)$ and $B(t) = P(t)\ P^{T'}(t) + Q(t)\ P^T(t)$.

Proof : The transformation $y = P^T z$ transforms (2.3) into

$$L_1 z \equiv Az' + Bz\ .$$

Since A is non-singular, $L_1 z = 0$ implies $z'(t) = -[A^{-1}(t)\ B(t)]\ z(t)$ it follows that $z(t) = Y(t)\ C$ for some constant matrix C. Hence $y(t) = P^T\ Y(t)\ C$.

Theorem 2.2 : A particular solution $\bar{y}(t)$ of (2.1) is of the form

$$\bar{y}(t) = P^T(t)\ Y(t) \int_a^t Y^{-1}(s)\ [P(s)\ P^T(s)]^{-1}\ f(s)\ ds.$$

Proof : We seek a particular solution of

$$z'(t) + A^{-1}(t)\ B(t)\ z(t) = A^{-1}(t)\ f(t)$$

in the form $Y(t)\ K(t)$. Then it can be easily verified that

$$K(t) = \int_a^t Y^{-1}(s)\ [P(s)\ P^T(s)]^{-1}\ f(s)\ ds.$$

Hence the particular solution of (2.1) is of the form

$$\bar{y}(t) = P^T\ Y(t) \int_a^t Y^{-1}(s)\ [P(s)\ P^T(s)]^{-1}\ f(s)\ ds.$$

Theorem 2.3 : Any solution of (2.1) is of the form

$y(t) = P^T\ Y(t)\ C + \bar{y}(t)$, where $\bar{y}(t)$ is a particular solution of (2.1).

Proof : It can easily be verified that $P^T\ Y(t)\ C + \bar{y}(t)$ is a solution of (2.1) for any constant matrix C. Now to prove that every solution is of the form, let $y(t)$ be any solution of (2.1) and $\bar{y}(t)$ be a particular solution of (2.1). Then it can easily be verified that $y(t) - \bar{y}(t)$ is a solution of (2.3). Hence by

Theorem (2.1), we have $y(t) - \bar{y}(t) = P^T\ Y(t)\ C$ or $y(t) = P^T Y(t) C + \bar{y}(t)$.

Theorem 2.4 : Suppose the homogeneous boundary value problem is incompatible. Then there exists a unique solution to the boundary value problem (2.1) satisfying (2.2) and is given by

$$y(t) = \int_a^b G(t,s)\ f(s)\ ds$$

where $G(t,s)$ is the Green's function for the corresponding homogeneous boundary value problem.

Proof : From Theorem 2.2 and Theorem 2.3 any solution of (2.1) is

$$y(t) = P^T\ Y(t)\ C + P^T\ Y(t) \int_a^t Y^{-1}(s)\ [P(s)\ P^T(s)]^{-1}\ f(s)\ ds.$$

where $Y(t)$ is a fundamental matrix for the equation (2.3) and C is a constant matrix and will be determined uniquely from the fact that the solution $y(t)$ must satisfy the boundary conditions (2.2).

Substituting the general form of y(t) in the boundary condition matrix (2.2) we get

$$\left[M\, P^T Y(a) + N\, P^T Y(b)\right] C + N\, P^T Y(b) \int_a^b Y^{-1}(s) \left[P(s)\, P^T(s)\right]^{-1} f(s)\,ds$$

and thus

$$C = -\,D^{-1}\, N\, P^T Y(b) \int_a^b Y^{-1}(s) \left[P(s) P^T(s)\right]^{-1} f(s)\,ds$$

where D is a characteristic matrix for the boundary value problem. Thus

$$y(t) = \int_a^b G(t,s)\, f(s)\, ds$$

where

$$G(t,s) = \begin{cases} P^T(t)\, Y(t)\, D^{-1}\, M\, P^T(t)\, Y(a)\, Y^{-1}(s) \left[P(s)\, P^T(s)\right]^{-1} & a \le s < t \le b \\ -P^T(t)\, Y(t)\, D^{-1}\, N\, P^T(t)\, Y(b)\, Y^{-1}(s) \left[P(s)\, P^T(s)\right]^{-1} & a \le t \le s \le b \end{cases}$$

Theorem 2.5 : The Green's function G(t,s) has the following properties:

1) G(t,s) as a function of t with fixed s have continuous derivatives every where except at t = s. At the point t = s, G(t,s) has a jump discontinuity and its jump is

$$G(s^+,s) - G(s^-,s) = P^T(s) \left[P(s)\, P^T(s)\right]^{-1}$$

2) G(t,s) is a formal solution of the homogeneous boundary value problem Ly = 0 satisfying (2.2). G fails to be a true solution because of the discontinuity at t = s.

3) G(t,s) satisfying properties (1) and (2) is unique.

3. Three Point Boundary Value Problems

In this section, we consider the following three-point boundary value problem

$$Ly \equiv P(t)\ y'(t) + Q(t)\ y(t) = f(t) \qquad \ldots (3.1)$$

$$My(a) + Ny(b) + Ry(c) = 0 \qquad \ldots (3.2)$$

where $P(t) \in C^2[a,c]$, $Q(t) \in C^1[a,c]$ are rectangular matrices of order mxn and $y(t)$ is a column matrix with components $(y_1, y_2, \ldots, y_n)$ and we assume throughout the paper that the rows of $P(t)$ are linearly independent and M, N, R are mxn rectangular matrices.

Definition 3.1 : If $P^T\ Y(t)C$ is a fundamental matrix for the equation (3.1), then the matrix D defined by

$$D = M\ P^T(a)\ Y(a) + N\ P^T(b)\ Y(b) + R\ P^T(c)\ Y(c)$$

is called a characteristic matrix for the boundary value problem.

The proof of Theorem 3.1 of this section is analogous to the proof of Theorem 2.5, and hence omitted.

Theorem 3.1 : Suppose the homogeneous boundary value problem is incompatible. Then there exists a unique solution to three-point boundary value problem (3.1) satisfying (3.2) and is given by

$$y(t) = \int_a^c G(t,s)\ f(s)\ ds$$

where $G(t,s)$ is the Green's function for the corresponding homogeneous boundary value problem and is given by

$$\underset{t \in [a,b]}{G(t,s)} = \begin{cases} [P^TY(t)\ Y^{-1}(s) - P^TY(t)\ D^{-1}\ N\ P^TY(b)\ Y^{-1}(s) \\ \qquad - P^T\ Y(t)\ D^{-1}\ R\ P^TY(c)\ Y^{-1}(s)]\ (P\ P^T)^{-1} \\ \qquad\qquad a \le s < t \le b < c \\ [-\ P^TY(t)\ D^{-1}\ N\ P^TY(b)\ Y^{-1}(s) \\ \qquad - P^T\ Y(t)\ D^{-1}\ R\ P^TY(c)\ Y^{-1}(s)]\ (P(s)\ P^T(s))^{-1} \\ \qquad\qquad a \le t < s \le b < c \\ -\ P^T\ Y(t)\ D^{-1}\ R\ P^TY(c)\ Y^{-1}(s)\ (P(s)\ P^T(s))^{-1} \\ \qquad\qquad a < t < b < s < c \end{cases}$$

$$\underset{t\in[b,c]}{G(t,s)} = \begin{cases} \left[P^T\, Y(b)Y^{-1}(s) - P^T Y(t) D^{-1} R\, P^T Y(c) Y^{-1}(s)\right] \left[P(s)\, P^T(s)\right]^{-1} & a < b < s < t \le c \\ -P^T Y(t)\, D^{-1} R\, P^T\, Y(c)\, Y^{-1}(s) \left[P(s)\, P^T(s)\right]^{-1} & a < b \le t < s < c \\ \left[P^T Y(t)\, Y^{-1}(s) - P^T\, Y(t)\, D^{-1}\, N\, P^T\, Y(b)\, Y^{-1}(s) \right. & \\ \quad \left. - P^T\, Y(t)\, D^{-1} R\, P^T\, Y(c)\, Y^{-1}(s)\right] \left[P(s)\, P^T(s)\right]^{-1} & a < s < b < t < c \end{cases}$$

Theorem 3.2 : The Green's function $G(t,s)$ has the following properties.

(i) $G(t,s)$ as a function of t with fixed s have continuous derivatives everywhere except at $t = s$. At the point $t = s$, $G(t,s)$ has a jump discontinuity and its jump is

$$G(s^+,s) - G(s^-,s) = P^T\, (P(s)\, P^T(s))^{-1}$$

(ii) $G(t,s)$ is a formal solution of the homogeneous boundary value problem $Ly = 0$ satisfying (3.1). G fails to be a true solution because of the discontinuity at $t = s$.

(iii) $G(t,s)$ satisfying the properties (i) and (ii) is unique.

As an example, consider the boundary value problem

$$Ly \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} Y' + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} Y = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \ldots (3.3)$$

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} Y(o) + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} Y(1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \ldots (3.4)$$

Now by using the transformation $Y = P^T Z$. The equation (3.3) becomes

$$L_1 Z \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} Z' + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Z = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Now the fundamental matrix Y(t) for homogeneous equation

$$Z' = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} Z ,$$

is given by

$$Y(t) = \begin{pmatrix} \exp(t) & \exp(-t) \\ -\exp(-t) & \exp(-t) \end{pmatrix}$$

The characteristic matrix D is given by

$$D = M P^T Y(0) + N P^T Y(1)$$

$$= \begin{pmatrix} -1 - \exp(1) & 1 + \exp(-1) \\ -1 & -1 \end{pmatrix}$$

Now the solution will be in the form

$$Y(t) = \int_o^1 G(t,s) \begin{pmatrix} 2 \\ 1 \end{pmatrix} ds$$

where

$$G(t,s) = \begin{cases} \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \\ 0 & 0 \end{pmatrix} & a < s < t \leq b \\[2ex] \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \\ 0 & 0 \end{pmatrix} & a \leq t < s < b \end{cases}$$

where

$$A_1 = \exp(-s)\ [\exp(t)\ (\exp(1) + 2) + \exp(2)\ \exp(-t)]$$

$$- \exp(s)\ [\exp(t)\ (2\exp(1) + 1) + \exp(2)\ \exp(-t)].$$

$$B_1 = -\exp(-s)\left[\exp(t)\ (\exp(1)+2) + \exp(2)\exp(-t)\right]$$
$$-\exp(s)\left[\exp(t)\ (2\exp(1)+1) - \exp(2)\exp(-t)\right].$$

$$C_1 = -\exp(-s)\left[\exp(t)\ (2\exp(1)+1) - \exp(2)\exp(-t)\right]$$
$$+\exp(s)\left[\exp(t)\ (2\exp(1)+1) - \exp(2)\exp(-t)\right].$$

$$D_1 = \exp(-s)\left[\exp(t)\ (2\exp(1)+1) - \exp(2)\exp(-t)\right]$$
$$+\exp(s)\left[\exp(t)\ (2\exp(1)+1) - \exp(2)\exp(-t)\right].$$

$$A_2 = -\exp(1)\left[\exp(-t) - \exp(t)\right]\left[\exp(1-s) + \exp(s-1)\right].$$

$$B_2 = \exp(1)\left[\exp(-t) - \exp(t)\right]\left[\exp(1-s) + \exp(s-1)\right].$$

$$C_2 = -\exp(1)\left[\exp(t) + \exp(-t)\right]\left[\exp(1-s) + \exp(s-1)\right].$$

$$D_2 = \exp(1)\left[\exp(t) + \exp(-t)\right]\left[\exp(1-s) + \exp(s-1)\right].$$

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On Univalence of Certain Analytic Functions Associated with Starli e Functions II

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1. Let S be the class of functions $f(Z) = Z + \sum_{n=2}^{\infty} a_n Z^n$, which are regular and univalent in the unit disc D $\{|Z|<1\}$, while S* denotes the class of functions in S which maps D onto a Starlike region with respect to the origin. An equivalent analytic characterization for functions of S* having an additional property

$$(1.1) \qquad \text{Re}\left\{\frac{Zf'(Z)}{f(Z)}\right\} \geq \beta \ , \ Z \in D \ ; \ 0 \leq \beta \leq 1$$

Here β is referred as the order of Starlike functions f(Z), and we identify $S^*_o = S^*$.

In this paper we are mainly concerned with the radius of Starlikeness of the function F(Z). Incidently the results of Padmanabhan [2], Bajpai and Srivastava [3], Bernardi [4] and Rizvi [5] follows from ours

2. Theorem:-

Let $f(Z) = Z + \sum_{n=2}^{\infty} a_n Z^n$, $g(Z) = Z + \sum_{n=2}^{\infty} b_n Z^n$

and $F(Z) = \frac{(C+p+q)}{Z^{C-p-q+1}} \int_o^Z t^{C-p-q} \{f(t)\}^p \{g(t)\}^q \, dt \in S^*_{p\beta_1 + q\beta_2}$

where $o < p, q, \beta_1, \beta_2 < 1$

then,

$f(Z) \in S^*_{\beta_1}$ and $g(Z) \in S^*_{\beta_2}$ in the region ,

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$|Z| < r_o =$

$$-\frac{\{2-(p\beta_1+q\beta_2)\}+\{4+(p\beta_1+q\beta_2)^2+(C-p-q)^2+2(C-p-q)(p\beta_1+q\beta_2)+2(C-p-q)\}^{1/2}}{(C-p-q)+2(p\beta_1+q\beta_2)};$$

if $C=2,3,\ldots$

$$=\frac{-2+\{4+(1-p-q)^2+2(1-p-q)\}^{1/2}}{(1-p-q)}, \quad C=1,\ \beta_1=\beta_2=0$$

$$=\frac{-2-(p\beta_1+q\beta_2)+\{4+(p\beta_1+q\beta_2)^2+(1-p-q)^2+2(1-p-q)(p\beta_1+q\beta_2)+2(1-p-q)\}^{1/2}}{(1-p-q)+2(p\beta_1+q\beta_2)};$$

$,C=1,$

$0 < \beta_1,\beta_2 < 1$

<u>Proof:</u>

(2.1) $$F(Z) = \frac{(C+p+q)}{Z^{C-p-q+1}} \int_0^Z t^{C-p-1} \{f(t)\}^p \{g(t)\}^q dt .$$

Let,

(2.2) $$J(Z) = \int_0^Z t^{C-p-q} \{f(t)\}^p \{g(t)\}^q dt,$$

then,

(2.3) $$F(Z) = \frac{(C+p+q)}{Z^{C-p-q+1}} J(Z) .$$

Now,

$$ZF'(Z) = (C+p+q)\left[J'(Z)Z^{-C+p+q} + (p+q-C-1)\, Z^{-C+p+q+1} J(Z)\right] .$$

(2.4) $$= (C+p+q)Z^{-C+p+q-1}\left[ZJ'(Z) + (p+q-C-1)\, J(Z)\right].$$

Therefore,

(2.5) $$\frac{ZF'(Z)}{F(Z)} = \frac{ZJ'(Z) + (p+q-C-1)\, J(Z)}{J(Z)},$$

or

(2.6) $$\frac{ZF'(Z)}{F(Z)} = \frac{ZJ'(Z)}{J(Z)} + (p+q-C-1).$$

Further, since F(Z) is a starlike function of order $(p\beta_1 + q\beta_2)$, so there exists a function $\omega(Z)$, which is regular in unit disc and satisfies the conditions of Schwartz Lemma, such that

$$(2.7) \qquad \frac{ZF'(Z)}{F(Z)} = \frac{1-\{1-2\ (p\beta_1 + q\beta_2)\}\ \omega(Z)}{1+\omega(Z)}$$

From (1.6) and (1.7), it follows that

$$\frac{ZJ'(Z)}{J(Z)} + (p+q-C-1) = \frac{1-\{1-2\ (p\beta_1+q\beta_2)\}\ \omega(Z)}{1+\omega(Z)}$$

or

$$\frac{Z^{C-p-q+1}\{f(Z)\}^p\{g(Z)\}^q}{J(Z)} = \frac{(C-p-q+2)+\{(C-p-q-1)+2(p\beta_1+q\beta_2)-1\}\omega(Z)}{1+\omega(Z)}$$

or,

$$(2.8)\ f(Z)\}^p\{g(Z)\}^q = \frac{J(Z)\left[(C-p-q+2)+\{(C-p-q+1)+2(p\beta_1+q\beta_2)-1\}\omega(Z)\right]}{\left[1+\omega(Z)\right]Z^{C-p-1+1}}.$$

Differentiating equation (2.8) logarithmically and simplifying, finally we get,

$$(2.9) \quad \{\frac{pZf'(Z)}{f(Z)} + \frac{gZg'(Z)}{g(Z)}\} - (p\beta_1 + q\beta_2)$$

$$=\{1-(p\beta_1+q\beta_2)\}\left[\frac{1-\omega(Z)}{1+\omega(Z)} - \frac{2Z\omega'(Z)}{\left[1+\omega(Z)\right]\left[(C-p-q+2)+\{(C-p-q)+2(p\beta_1+q\beta_2)\}\omega(Z)\right]}\right].$$

But,

$$(2.10) \qquad \text{Re}\ \{\frac{1-\omega(Z)}{1+\omega(Z)}\} =. \frac{1-|\omega(Z)|^2}{|1+\omega(Z)|^2},$$

and

$$(2.11) \qquad \text{Re}\ \{\frac{2Z\omega'(Z)}{\left[1+\omega(Z)\right]\left[(C-p-q+2)\ +\ (C-p-q)+2(p\beta_1+q\beta_2)\}\omega(Z)\right]}\}$$

$$\leqslant \frac{2\ Z\ (1-\omega(Z)^2)}{(1-|Z|^2|1+\omega(Z)||(C-p-q+2)+\{(C-p-q)+2(p\beta_1+q\beta_2)\}\omega(Z)|}.$$

The last inequality is obtained by using the following well known inequality([6], p. 168)

$$|\omega'(Z)| \le (1-|\omega(Z)|^2) \;/\; (1-|Z|^2).$$

From (2.9) and using (2.10) and (2.11), it follows that f(Z) is a starlike function of order β_1 and g(Z) of order β_2, if

$$\frac{2|Z|\;(1-|\omega(Z)|^2)}{|1+\omega(Z)|\;(1-|Z|^2)\,|(C-p-q+2)+\{(C-p-q)+2\;(p\beta_1+q\beta_2\}\;\omega(Z)|}$$

$$\le \frac{1-|\omega(Z)|^2}{|1+\omega(Z)|^2}$$

or

$$(2.12)\quad \frac{2|Z|}{1-|Z|^2} \le (C-p-q+2)\left|1+\frac{(C-p-q)+2(p\beta_1+q\beta_2)}{(C-p-q+2)}\,\omega(Z)\right| \Big/ |1+\omega(Z)|.$$

Since,

$$|\omega(Z)| \le |Z|$$

and

$$\{(C-p-q)+2\;(p\beta_1+q\beta_2)\} \Big/ (C-p-q+2) \le 1.$$

We have,

$$(2.13)\qquad 1+\frac{(C-p-q)+2\;(p\beta_1+q\beta_2)}{(C-p-q+2)}\,|Z| \Big/ (1+|Z|)$$

$$\le \left|1+\frac{(C-p-q)+2\;(p\beta_1+q\beta_2)}{(C-p-q+2)}\;\omega(Z)\right| \Big/ |1+\omega(Z)|$$

Therefore from (2.12) and (2.13), we obtain that $f(Z) \in S^*_{\beta_1}$ and $g(Z) \in S^*_{\beta_2}$, if

$$2\,|Z| < [(C-p-q+2)+\{(C-p-q)+2(p\beta_1+q\beta_2)\}\;|Z|]\;(1-|Z|)$$

i.e. if

$$(C-p-q+2)-2\{2-(p\beta_1+q\beta_2)\}r-\{(C-p-q)+2(p\beta_1+q\beta_2)\}\;r^2 > 0.$$

Let $P(|z|)=P(r)=(C-p-q+2)-2\{2-(p\beta_1+q\beta_2)\}r-\{(C-p-q)+2(p\beta_1+q\beta_2)\}\quad r^2$

Since $P(c) = (C-p-q+2$ and $P'(r) < 0$, the positive root r_o, for which $P(r) > 0$ must be less than the root of the polynomial $P(r) = 0$, that gives the required value of r_o and proof of the theorem is complete.

Corollary 1:- Theorem G of Bernardi [4], follows by taking $p=1$, $q=o$, $\beta_1=o$, $\beta_2=o$ and $C=1,2,3,\ldots$

Corollary 2:- Theorem of Padmanabhan [2], follows by taking $p=1$. $q=o$, $o > \beta_1 < 1/2$, $\beta_2=o$ and $C=1$.

Corollary 3:- Theorem 1 of Bajpai and Srivastava [3], follows by taking $p=1$, $q=o$, and $\beta_2=o$

Corollary 4:- Theorem 2 of Rizvi [5], follows by taking $q=o$, $\beta=o$.

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Integrability Conditions of a Structure F Safisfying. $F^K + F = 0$

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Summary. Ishihara and Yano [2] have obtained the integrability conditions of a structure f satisfying $f^3 + f = 0$. Gouli-Andreou [1] has studied the integrability conditions of a structure f satisfying $f^5+f=0$. The purpose of the present paper is to establish the integrability conditions of a structure F satisfying $F^K + F = 0$, where K is a positive integer ≥ 2. Firstly, the Nijenhuis tensor of F(K,1)-structure has been studied and then the partial integrability conditions and integrability conditions of this structure have been deduced in terms of its Nijenhuis tensor.

1. Preliminaries

Let us consider an n-dimensional differentiable manifold M^n of class C^∞ equipped with a non-null tensor field F of type (1,1) and of class C^∞ satisfying

$$F^K + F = 0, \tag{1.1}$$

where K is a positive integer ≥ 2.

Let us put

$$s \overset{\text{def}}{=\!=\!=} - F^{K-1}, \qquad t \overset{\text{def}}{=\!=\!=} I + F^{K-1}, \tag{1.2}$$

where I denotes the unit tensor field. Then we have

Theorem 1.1. For a tensor field $F \neq 0$ satisfying (1.1), the operators s, t defined by (1.2) and applied to the tangent space at each point of the manifold are complementary projection operators.

Proof. In consequence of (1.1) and (1.2), we have

$$s + t = I, \tag{1.3}$$

$$(1.4)\qquad s^2 = F^{2K-2} = F^K.F^{K-2}$$
$$= - F.F^{K-2} = -F^{K-1} = s\ ,$$

$$(1.5)\qquad t^2 = I + F^{2K-2} + 2F^{K-1}$$
$$= I - F^{K-1} + 2F^{K-1} = t\ ,$$

$$(1.6)\qquad s.t = t.s = - F^{K-1} - F^{2K-2} = 0.$$

This proves the theorem.

Thus, if there is given a (1,1) tensor field $F \neq 0$ satisfying (1.1); then there exist two complementary distributions S and T corresponding to the projection operators s and t respectively. Let the rank of F be constant and be equal to r everywhere, then the dimensions of S and T are r and (n-r) respectively. We call such a structure a 'F(K,1)-structure of rank r' and the manifold M^n with this structure a 'F(K,1)-manifold.'

Theorem 1.2. For a tensor field $F \neq 0$ satisfying (1.1) and the operators s, t defined by (1.2), we have

$$(1.7)\qquad Fs = sF = F\ ,\quad Ft = tF = 0\ ;$$

$$(1.8)\qquad F^2 s = F^2\ ,\quad F^2 t = 0\ ;$$

$$(1.9)\qquad F^{K-2} s = sF^{K-2} = F^{K-2},\quad F^{K-2}\, t = tF^{K-2} = 0\ ;$$

$$(1.10)\qquad F^{K-1}\, s = -s,\quad F^{K-1}\, t = 0\ .$$

Proof. The proof of the theorem follows by virtue of the equations (1.1) and (1.2).

If the rank of F is maximal, then r = n. Thus t = 0 and F satisfies

$$F^{K-1} + I = 0\ .$$

2. Nijenhuis tensor of F(K,1)-structure

Let F be a F(K,1)-structure of rank r ; then the Nijenhuis tensor N(X,Y) of F is

$$(2.1) \qquad N(X,Y) = [FX,FY] - F[FX,Y] - F[X,FY] + F^2[X,Y].$$

Theorem (2.1). We have the following identities :

$$(2.2) \qquad F^{K-3}.N(tx,tY) = F^{K-3}s.N(tX,tY) = -s[tX,tY];$$

$$(2.3) \qquad t.N(X,Y) = t[FX,FY] ;$$

$$(2.4) \qquad t.N(sX,sY) = t[FX,FY] ;$$

$$(2.5) \qquad t.N(F^{K-2}X,\ F^{K-2}Y) = t[sX,sY]$$

Proof. In consequence of (1.2), (1.4), (1.7) and (2.1), we have

$$F^{K-3}.N(tX,tY) = F^{K-3}.F^2[tX,tY] = -s[tX,tY]$$

and

$$F^{K-3}s.N(tX,tY) = -s^2[tX,tY] = -s[tX,tY] ;$$

whence we get (2.2).

The proofs of identities (2.3), (2.4) and (2.5) follow by virtue of the equations (1.2), (1.7) and (2.1).

Theorem (2.2). For any vector fields X and Y, the following three conditions are equivalent to each other:

$$(2.6) \qquad t.N(X,Y) = 0 ,$$

$$(2.7) \qquad t.N(sX,sY) = 0 ,$$

$$(2.8) \qquad t.N(F^{K-2}X,\ F^{K-2}Y) = 0 .$$

Proof. Since the right sides of (2.3) and (2.4) are equal, the conditions (2.6) and (2.7) are equivalent. If we have $t.N(sX,sY) = 0$ for any vector fields X and Y; then from (1.2) and (1.7), $t.N(F^{K-2}X, F^{K-2}Y) = 0$ and conversely. Hence the theorem follows.

The Lie derivative $\mathcal{L}_Y F$ of the tensor field F with respect to a vector field Y is a tensor field of the same type as F given by [3]

$$(2.9) \qquad (\pounds_Y F)X \overset{\text{def}}{=\!=\!=} F[X,Y] - [FX,Y] .$$

In view of (1.7), (2.1) and (2.9), we have

$$(2.10) \qquad N(sX,tY) = F(\pounds_{tY}F)sX = F(s(\pounds_{tY}F)sX)$$

and

$$(2.11) \qquad s.N(sX,tY) = F(s(\pounds_{tY}F)sX) .$$

3. Integrability Conditions

In this section, we shall obtain the partial integrability conditions of $F(K,1)$-structure in terms of its Nijenhuis tensor.

Theorem (3.1). The distribution T is integrable if and only if

$$(3.1) \qquad N(tX,tY) = 0 ,$$

or equivalently

$$(3.2) \qquad s.N(tX,tY) = 0 ,$$

for any vector fields X and Y.

Proof. We know that [2] the distribution T is integrable if and only if $s[tX,tY] = 0$ for any vector fields X and Y. Thus in view of (2.2), we get (3.2). For any vector fields X and Y, we have $s.N(tX,tY) = N(tX,tY)$. So (3.2) is equivalent to (3.1).

Theorem (3.2). The distribution S is integrable if and only if one of the conditions (2.6), (2.7) or (2.8) of Theorem (2.2) is satisfied.

Proof. We know that [2] the distribution S is integrable if and only if $t[sX,sy] = 0$ for any vector fields X and Y. Thus in view of (2.5), the theorem follows.

Theorem (3.3). The two distributions S and T are both integrable if and only if

$$(3.3) \qquad N(X,Y) = s.N(sX,sY) + N(sX,tY) + N(tX,sY),$$

for any vector fields X and Y.

Proof. In consequence of (1.3), (1.7) and (2.1), the Nijenhuis tensor $N(X,Y)$ of F can be expressed as

(3.4) $N(X,Y) = N(sX,sY) + N(sX,tY) + N(tX,Sy) + N(tX,tY)$.

Equation (3.4), in consequence of (1.3) takes the form

(3.5) $$N(X,Y) = s.N(sX,sY) + t.N(sX,sY) + N(sX,tY) + N(tX,sY) + + N(tX,tY).$$

Now by virtue of (3.5) and Theorems (3.1) and (3.2), the result follows.

Definition (3.1). If the distribution S is integrable and moreover, if the structure $F' \overset{\text{def}}{=\!=\!=} F/S$ induced from F on each integral manifold of S is also integrable; then we say that the F(K,1)-structure is 'partially integrable'.

Theorem (3.4). A necessary and sufficient condition for a F(K,1)-structure to be partially integrable is that one of the following equivalent conditions be satisfied:

(3.6) $$N(sX,sY) = 0$$
and
(3.7) $$N(F^{K-2}X,\ F^{K-2}Y) = 0,$$

for any vector fields X and Y.

Proof. Suppose that the distribution S is integrable. Then F induces on each integral manifold of S a structure F' defined by $F' = F/S$ which satisfies $(F')^{K-1} = -I$. The Nijenhuis tensor of this structure is exactly the same as $N(sX,sY)$. The induced structure is integrable if and only if its Nijenhuis tensor vanishes identically.

Since for any vector fields X and Y, the conditions $N(sX,sY) = 0$ and $N(F^{K-2}X,\ F^{K-2}Y) = 0$ are equivalent. Therefore, in view of definition (3.1) and Theorem (3.2), the theorem follows.

Theorem (3.5). A necessary and sufficient condition for the distribution T to be integrable and the F(K,1)-structure to be partially integrable is that

(3.8) $$N(X,Y) = N(sX,tY) + N(tX,sY),$$
for any vector fields X and Y.

Proof. The proof of the theorem follows from definition (3.1) and theorems (3.3) and (3.4).

4. Condition $N(sX,tY) = 0$

In this section, we shall obtain the integrability conditions of $F(K,1)$-structure by means of the condition $N(sX,tY) = 0$.

Theorem (4.1). The tensor field $s(\mathcal{L}_{tY}F)s$ vanishes identically for any vector field Y, if and only if

$$N(sX,tY) = 0 , \tag{4.1}$$

for any vector fields X and Y.

Proof. The proof of the theorem follows by virtue of the equation (2.10).

Suppose that the distributions S and T are both integrable. Then we can choose a local coordinate system such that all S are represented by putting $(n-r)$ local coordinates constant and all T by putting the other r coordinates constant. Let us call such a coordinate system an 'adapted coordinate system'.

It can be supposed that in an adapted coordinate system, the projection operators s and t have the components of the form

$$s = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} , \qquad t = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} \tag{4.2}$$

respectively, where I_r and I_{n-r} are unit matrices of order r and $(n-r)$ respectively.

Since F satisfies (1.2) and (1.7), the tensor F has the components of the form

$$F = \begin{bmatrix} F_r & 0 \\ 0 & 0 \end{bmatrix} \tag{4.3}$$

in an adapted coordinate system, where F_r is a $r \times r$ square matrix. Thus the Lie derivative $\mathcal{L}_{tY}F$ has components of the form

$$(4.4) \qquad \mathcal{L}_{tY}F = \begin{bmatrix} L^* & 0 \\ 0 & 0 \end{bmatrix} ,$$

for any vector field tY on M^n, where $L^* = \mathcal{L}_{tY}F_r$.

Theorem (4.2). Suppose that the distributions S and T are both integrable and that an adapted coordinate system has been chosen. Then the components of F are independent of the coordinates which are constant along the integral manifold of S if and only if

$$(4.5) \qquad N(sX, tY) = 0 ,$$

for any vector fields X and Y.

Proof. Let us suppose that $N(sX,tY) = 0$ for any vector fields X and Y. Then from theorem (4.1), the tensor field $s(\mathcal{L}_{tY}F)s$ vanishes identically for any vector field Y. Thus we have $\mathcal{L}_{tY}F = 0$. This implies that the components of F are independent of the coordinates which are constant along the integral manifold of the distribution S in an adapted coordinate system.

Conversely, if the components of F are independent of these coordinates, then it can be easily shown that the tensor field $s(\mathcal{L}_{tY}F)s$ vanishes identically for any vector field Y. Hence $N(sX,tY) = 0$ for any vector fields X and Y.

Theorem (4.3). Suppose that the distributions S and T are both integrable and that an adapted coordinate system has been chosen. Then the components of F are independent of the coordinates which are constant along the integral maniflds of S if and only if

$$(4.6) \qquad N(X,Y) = s.N(sX,sY) ,$$

for any vector fields X and Y.

Proof. The proof of the theorem follows from theorems (3.3) and (4.2).

Definition (4.1). We say that the F(K,1)-structure is 'integrable' if

(i) the F(K,1)-structure is partially integrable;

(ii) the distribution T is integrable;

(iii) the components of the $F(K,1)$-structure are independent of the coordinates which are constant along the integral manifolds of S in an adapted coordinate system.

Theorem (4.4). In order that the $F(K,1)$-structure be integrable, it is necessary and sufficient that

$$N(X,Y) = 0 , \tag{4.7}$$

for any vector fields X and Y.

Proof. The proof of the theorem follows from definition (4.1) and theorems (3.4) and (4.3).

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Polynomial and Rational Approximation to the Legendre Function by τ-Method

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Abstract

In this article the Polynomial and Rational Approximation to the Legendre Function has been obtained by τ-Method. It has been observed that both the numerator and denominator polynomials of the rational approximation of hypergeometric function are polynomials of the hypergeometric type.

1. Introduction

The τ-method for polynomial and rational approximations has been suggested by Lonczos. Such approximations are also known as "Economised approximations" and the method obtaining as the "economisation process". In this process the function is approximated by the ratio of two polynomials and hence the name "Rational approximation". These approximations are of the form $\Psi_n(Z,\gamma)/h_n(\gamma)$ where $h_n(\gamma)$ and $\Psi_n(Z,\gamma)$ are polynomials of one and two variables respectively and γ is a free parameter.

2. We define the Legendre function in the hypergeometric forms as:

$$P_\nu^\mu(z) = 2F_1\left(-\nu, \nu+1; 1-\mu; \frac{(1-z)}{2}\right)$$

$$(X) = 2xF_1\lambda$$

which converges for $|z| < 1$ as we know that the generalised hypergeometric $_{q+1}F_q$ converges for $|z| < 1$ when $p = q+1$.

3. Let $F(z) = {}_2F_1(-\nu, \nu+1; 1-\mu; z) = \sum_{k=0}^{\infty} a_k z^k$

$$\text{where } a_k = \frac{(-\nu)_k\,(\nu+1)_k}{(1-\mu)_k\,k!}, \quad p = q+1 \text{ and } |z| < 1 \qquad \ldots (1)$$

Since $F(Z)$ satisfies the differential equation,

$$A_{1,2}(\delta) \quad F(Z) = 0$$

$$\text{where } A_{1,2}(\delta) = [\delta(\delta-\mu) - z\ (\delta-\nu)\ (\delta+\nu+1)] \qquad \ldots (2)$$

$$\delta = z\frac{zd}{dz}$$

We consider the related differential equation

$$A_{1,2}(\delta)\ F_n(z) = \tau.Z\ R_n\ (z/\gamma) \qquad \ldots (3)$$

$$F_n(Z) = \sum_{k=o}^{n} b_{n,k}\ Z^k \qquad \ldots (4)$$

Where $R_n(Z)$ is a polynomial in Z of degree n

$$\text{Let } R_n(Z) = \left[\frac{(-\mu)}{\gamma}\right] \sum_{k=o}^{n} (k+1)\ g_{n,k+1}Z^k \qquad \ldots (5)$$

Putting (4) and (5) in (3), we get

$$A_{1,2}(\delta) \sum_{k=o}^{n} b_{(k+1)}\ Z^k = \tau.Z\ \left[\frac{(-\mu)}{\gamma}\right] \sum_{k=o}^{n}, g_{n,k+1}\ (\tfrac{z}{\gamma})^k$$

$$\text{or} \quad \left[z\frac{d}{dz}\ (z\frac{d}{dz} -\mu) - z(z\frac{d}{dz} - \nu)\ (z\frac{d}{dz} +\nu+1)\right] \sum_{k=o}^{n} b_{n,k}Z^k$$

$$= \tau.Z(\frac{-\mu}{\gamma}) \sum_{k=o}^{n} (k+1)\ g_{n,k+1}(Z/\gamma)^k$$

on differentiating

$$\sum_{k=1}^{n} (b_{n,k}K^2Z^k) - \mu \sum_{k=1}^{n} (b_{n,k}kZ^k) - \sum_{k=1}^{n} (b_{n,k}k^2Z^{k+1}) - \sum_{k=1}^{n} (b_{n,k}kZ^{k+1})$$

$$+ \nu(\nu+1) \sum_{k=o}^{n} (b_{n,k}Z^{k+1}) = \tau(-\mu) \sum_{k=o}^{n} (k+1)\ g_{n,k+1}\ (Z/\gamma)^{k+1} \qquad \ldots (6)$$

Equating coefficients of Z^{k+1} from both sides

$$b_{n,k+1} = \frac{\{k^2 + k - \nu(\nu + 1)\}\ b_{n,k}}{(k+1)\ (k - \mu + 1)} + \frac{(-\mu)\ \tau.g_{n,k+1}}{(k - \mu + 1)\gamma^{k+1}}$$

But from (1)

$$\frac{a_{k+1}}{a_k} = \frac{k^2+k-\nu\ (\nu+1)}{(k+1)(1-\mu+k)}$$

Hence

$$b_{n,k+1} = \frac{a_{k+1}}{a_k}\, b_{n,k} + \frac{(-\mu)\tau}{(1-\mu+k)}\, g_{n,k+1} \qquad \ldots (7)$$

$$k = 0,1,2,\ldots,n;\ b_{n,n+1} = o$$

Let us take $b_{n,o} = 1 + \tau\ g_{n,o}$ ... (8)

To obtain the values of $b_{n,k}$ and τ we solve the equation (7) by successive reduction and multiplying both sides of each of these equations by $1, \frac{a_k}{a_{k-1}}, \frac{a_k}{a_{k-2}}, \ldots, \frac{a_k}{a_1}, \frac{a_k}{a_o}$

respectively and adding, get

$$b_{n,k} = a_k \left[1 + \sum_{r=o}^{k} \frac{(-\mu)\tau\ g_{n,r}}{a_r \{(1-\mu)+(r-1)\}\gamma^r} \right]$$

putting $k = n + 1$ and using $b_{n,n+1} = 0$, we get

$$\tau = - \left(\sum_{r=o}^{n+1} \frac{(-\mu)\ g_{n,r}}{a_r \{(1-\mu)+(r-1)\}\gamma^r} \right)^{-1} \qquad \ldots (9)$$

Also

$$b_{n,k} = a_k \left[1 + \sum_{r=o}^{n+1} \frac{(-\mu)\tau\ g_{n,r}}{a_r \{(1-\mu)+(r-1)\}\gamma^r} - \sum_{r=k+1}^{n+1} \frac{\tau(-\mu)\ g_{n,r}}{a_r \{(1-\mu)+(r-1)\}\gamma^r} \right]$$

which gives on putting the value of τ from (9),

$$b_{n,k} = \frac{a_k \displaystyle\sum_{r=k+1}^{n+1} \frac{(-\mu)\ g_{n,r}}{a_r \{(1-\mu)+(r-1)\}\gamma^r}}{\displaystyle\sum_{r=o}^{n+1} \frac{(-\mu)\ g_{n,r}}{a_r \{(1-\mu)+(r-1)\}\gamma^r}} \qquad \ldots (10)$$

Now let $F_n(Z)$ be an approximation to $F(Z)$ and $E_n(Z)$ be the error polynomial, then

$$F(Z) = F_n(Z) + E_n(Z) \qquad \ldots (11)$$

and

$$F_n(Z) = \frac{\Psi_n(Z,\gamma)}{h_n(\gamma)} \qquad \ldots (12)$$

Taking

$$h_n(\gamma) = -\frac{1}{\tau} = \sum_{r=o}^{n+1} \frac{(-\mu)_r \, g_{n,r} \, (1)_r}{(-\nu)_r (\nu+1)_r \gamma^r} \qquad \ldots (13)$$

From (12), we get

$$\Psi(Z,\gamma) = F_n(Z)\, h_n(\gamma) = \left(\sum_{k=o}^{n} b_{n,k} Z^k\right) \left(\sum_{r=o}^{n+1} \frac{g_{n,r} (-\mu)_r (1)_r}{(-\nu)_r (\nu+1)_r \gamma^r}\right)$$

$$= \sum_{k=o}^{n} a_k Z^k \sum_{r=k+1}^{n+1} \frac{g_{n,r} (-\mu)_r (1)_r}{(-\nu)_r (\nu+1)_r \gamma^r}$$

$$= \sum_{k=o}^{n} \frac{(-\nu)_k (\nu+1)_k}{(1-\mu)_k \, k!} \gamma^k \left(\frac{Z}{\gamma}\right)^k \sum_{r=k+1}^{n+1} \frac{(-\mu)_r \, g_{n,r} (1)_r}{(-\nu)_r (\nu+1)_r \gamma^r} \qquad \ldots (14)$$

Similar to $F_n(Z)$, let us take

$$E_n(Z) = \frac{X_n(Z,\gamma)}{h_n(\gamma)}, \qquad \ldots (15)$$

Now we proceed to determine $X_n(Z,Y)$. From (11), we have

$$E_n(Z) = F(Z) - F_n(Z) = \sum_{k=o}^{\infty} a_k Z^k - \sum_{k=o}^{n} b_{n,k} Z^k$$

$$= \sum_{k=o}^{\infty} a_k Z^k - \sum_{k=o}^{n} a_k Z^k \left\{ \frac{\sum_{r=k+1}^{n+1} \frac{g_{n,r} (-\mu)_r (1)_r}{(-\nu)_r (\nu+1)_r \gamma^r}}{h_n(\gamma)} \right\}$$

or, $$h_n(\gamma) E_n(Z) = h_n(\gamma) \sum_{k=o}^{\infty} a_k Z^k - \sum_{k=o}^{n} a_k Z^k \sum_{r=k+1}^{n+1} P_r$$

$$\text{where, } P_r = \frac{g_{n,r} (-\mu)_r (1)_r}{(-\nu)_r (\nu+1)_r \gamma^r}$$

From (15), we have

$$X_n(Z,\gamma) = \sum_{r=o}^{n+1} P_r \sum_{k=o}^{\infty} a_k Z^k - \sum_{k=o}^{n} a_k Z^k - \sum_{r=k+1}^{n+1} P_r$$

$$= \sum_{k=o}^{\infty} a_{k+r} Z^k \sum_{r=o}^{n+1} P_r Z^k$$

$$= \sum_{k=o}^{\infty} \frac{(-\nu)_k (\nu+1)_k Z^k}{(1-\mu)_k (1)_k} \sum_{r=o}^{n+1} \frac{g_{n,r} (-\mu)_r (1)_r (-\nu+k)_r (\nu+1+k)_r \, Z^r}{(-\nu)_r (\nu+1)_r (1-\mu+k)_r (1+k)_r \, \gamma^r} \qquad \ldots (16)$$

Again, we have

$$X_n(Z,\gamma) = \sum_{k=o}^{\infty} \frac{(-\nu)_{r+k} (\nu+1)_{r+k} Z^k}{(1-\mu)_{r+k} \lfloor k+r} \sum_{r=o}^{n+1} \frac{g_{n,r} (-\mu)_r (1)_r Z^r}{(-\nu)_r (\nu+1)_r \gamma^r}$$

which on interchanging r and k, gives

$$X_n(Z,\gamma) = \sum_{k=o}^{n+1} \frac{g_{n,r} (-\mu)_k Z^k}{(1-\mu)_k \gamma^k} \sum_{r=o}^{\infty} \frac{(-\nu+k)_r (\nu+1+k)_r Z^r (1)_r}{(1-\mu+k)_r (1+K)_r \lfloor r}$$

$$= \sum_{k=o}^{n+1} \frac{g_{n,r} (-\mu)_k Z^k}{(1-\mu)_k \gamma^k} \, {}_3F_2 \left(\begin{matrix} -\nu+k, \; \nu+k+1, 1 \\ 1-\mu+k, \; 1+k \end{matrix} \Big/ Z \right) \qquad \ldots (17)$$

We now specify $R_n(Z)$ as an extended Jacobi polynomial. We consider two special cases according as $g_{n,o}$ is zero or unity. We write

$$g_{n,o} = a, \quad a = o \text{ or } 1 \qquad \ldots (18)$$

and putting

$$g_{n,k+1} = \left[\frac{\gamma(-\nu)(\nu+1)}{(-\mu)}\right]^{1-a} \left[\frac{-n(n+\lambda)cf}{(\beta+1)dg}\right]^{a} \frac{(a-n)_k(n+\lambda+a)_k(cf+a)_k}{(\beta+1+a)_k(dg+a)_k(2)_k}$$

$$k = 0, 1, 2, \ldots \qquad \ldots (19)$$

Now from (13), we get

$$h_n(\gamma) = \sum_{r=o}^{n+1} \frac{g_{n,r}(-\mu)_r(1)_r}{(-\nu)_r(\nu+k)_r\gamma^r}$$

$$= g_{n,o} + \sum_{k=o}^{n} \frac{(-\mu)_{k+1}(1)_{k+1}}{(-\nu)_{k+1}(\nu+1)_{k+1}\gamma^{k+1}} g_{n,k+1}.$$

Putting the value of $g_{n,k+1}$ from (19) and $g_{n,o} = a$ we get

$$h_n(\gamma) = a + \left[\frac{\gamma(-\nu)(\nu+1)}{(-\mu)}\right]^{1-a} \left[\frac{-n(n+\lambda)\ cf}{(\beta+1)dg}\right]^{a}$$

$$\times \sum_{k=o}^{n} \frac{(a-n)_k(n+\lambda+a)_k(cf+a)_k(-\mu)_{k+1}(1)_{k+1}}{(\beta+1+a)_k(dg+a)_k(2)_k(-\nu)_{k+1}(\nu+1)_k\gamma^{k+1}} \quad (E_1)$$

Let $a = 0$ and $a = 1$, then from (E_1)

$$h_n(\gamma) = \sum_{k=o}^{n} \frac{(-n)_k(n+\lambda)_k(cf)_k(1-\mu-a)_k}{(\beta+1)_k(dg)_k(1-\nu-a)_k(\nu+2-a)_k\gamma^k} \quad (E_2)$$

and $$h_n(\gamma) = \sum_{k=o}^{n} \frac{(-n)_k(n+\lambda)_k(cf)_k(1-\mu+a)_k}{(\beta+1)_k(dg)_k(1-\nu-a)_k(\nu+2-a)_k\gamma^k} \quad (E_3)$$

Thus from (E_1), (E_2) and (E_3), we see that either $a = 0$ or 1.

$$h_n(\gamma) = \sum_{k=o}^{n} \frac{(-n)_k(n+\lambda)_k(1-\mu-a)_k(cf)_k(1)_k}{(\beta+1)_k(dg)_k(1-\nu-a)_k(\nu+2-a)_k\ \underline{|k}\ \gamma^k}$$

$$= {}_{4+f}F_{3+g}\left(\begin{matrix}-n,\ n+\lambda,\ 1-\mu-a,\ cf,\ 1\\ \beta+1,\ 1-\nu-a,\ \nu+2-a, dg\end{matrix} \Big/ 1/\gamma\right) \qquad \ldots (20)$$

Now from (14)

$$\Psi_n(Z,\gamma) = \sum_{k=o}^{n} a_k \gamma^k \left(\frac{Z}{\gamma}\right)^k \sum_{r=k}^{n} \frac{g_{n,r+1}(-\mu)_{r+1}(1)_{r+1}}{(-\nu)_{r+1}(\nu+1)_{r+1}\gamma^{r+1}}$$

on putting the value of $g_{n},_{r+1}$ from (19), we get

$$\Psi_n(Z,\gamma) = \sum_{k=o}^{n} a_k \gamma^k (Z/\gamma)^k \sum_{r=o}^{n-k} \left(\frac{\gamma(-\nu)(\nu+1)}{(-\mu)}\right)^{1-a} \left(\frac{-n(n+\lambda)cf}{(\beta+1)dg}\right)^a$$

$$X \frac{(-n+a)_{r+k}(n+\lambda+a)_{r+k}(cf+a)_{k+r}(-\mu)_{r+k+1}}{(\beta+1+a)_{r+k}(dg+a)_{r+k}(-\nu)_{r+k+1}(\nu+1)_{r+k+1}\gamma^{r+k+1}}$$

When $a = 0$, then we get

$$\Psi_n(Z,\gamma) = \sum_{k=o}^{n-a} a_k \gamma^k \left(\frac{Z}{\gamma}\right)^k \sum_{r=o}^{n-k-a} \frac{(-n)_{r+k+a}(n+\lambda-)_{r+k+a}(cf)_{r+k+a}(1-\mu-a)_{r+k+a}}{(\beta+1)_{r+k+a}(dg)_{r+k+a}(1-\nu-a)_{r+k+a}(\nu+2-a)_{r+k+a}\gamma^{r+k+a}}$$

when $a = 1$, we get

$$\Psi_n(Z,\gamma) = \sum_{k=o}^{n-a} a_k \gamma^k (Z/\gamma)^k$$

$$X \sum_{r=o}^{n-k-a} \frac{(-n)_{r+k+a}(n+\lambda)_{r+k+a}(cf)_{r+k+a}(1-\mu-a)_{r+k+a}}{(\beta+1)_{r+k+a}(dg)_{r+k+a}(1-\nu-a)_{r+k+a}(\nu+2-a)_{r+k+a}\gamma^{r+k+a}}$$

Thus we get either $a = 0$ or $a = 1$.

$$\Psi_n(Z,\gamma) = \left[\frac{-n(n+\lambda)(1-\mu-a)cf}{\gamma(\beta+1)(1-\nu-a)(\nu+2-a)dg}\right] \sum_{k=o}^{n-a} \frac{(-n+a)_k(n+\lambda+a)_k(-\nu)_k(\nu+1)_k(cf+a)_k}{(\beta+1+a)_k(1-\mu)_k(\nu+2)_k(dg+a)_k \lfloor k} \left(\frac{Z}{\gamma}\right)^k$$

$$X_{4+f}F_{3+g}\left(\begin{matrix} -n+a+k, & n+\lambda+a+k, & 1-\mu+k, & cf+a+k, & 1 \\ \beta+1+a+k, & -\nu+1+k, & \nu+2+k, & dg+a+k & \end{matrix} \middle/ \frac{1}{\gamma}\right) \quad \ldots (21)$$

Next we proceed to determine $X_n(Z,\gamma)$. We have (21) from (16)

$$X_n(Z,\gamma) = \sum_{k=o}^{\infty} a_k Z^k \{a + \sum_{r=o}^{n} \frac{g_{n,r+1}(-\mu)_{r+1}(1)_{r+1}(-\nu+k)_{r+1}(\nu+1+k)_{r+1}}{(-\nu)_{r+1}(\nu+1)_{r+1}(1-\mu+k)_{r+1}(1+k)_{r+1}} (\frac{Z}{\gamma})^{r+1}\}$$

putting the value of $g_{n,r+1}$ from (19), we get

$$X_n(Z,\gamma) = a \sum_{k=o}^{\infty} a_k Z^k + \left[\frac{-n(n+\lambda)cf(-\mu)}{\gamma(\beta+1)dg(-\nu)(\nu+1)}\right]^a \sum_{k=o}^{\infty} \frac{(-\nu)_{k+1}(\nu+1)_{k+1} Z^{k+1}}{(1-\mu)_{k+1}\,(1)_{k+1}}$$

$$X \sum_{r=o}^{n} \frac{(a-n)_r(n+\lambda+a)_r(cf+a)_r(1-\mu)_r(1-\nu+k)_r(\nu+2+k)_r}{(\beta+1+a)_r(dg+a)_r(1-\nu)_r(\nu+2)_r(2-\mu+k)_r(2+k)_r} (Z/\gamma)^r$$

Now since

$$(-n+a)_r = \frac{(-n)_{a+r}}{(-n)_a}; \quad (n+\lambda+a)_r = \frac{(n+\lambda)_{a+r}}{(n+\lambda)_a};$$

$$(cf+a)_r = \frac{(cf)_{a+r}}{(cf)_a}.$$

Also $(-n)_a = (-n)^a$; $a = 0$ or 1, we have

$$X_n(Z,\gamma) = a \sum_{k=o}^{\infty} a_k Z^k + (\frac{(-\mu)}{\gamma(-\nu)(\nu+1)})^a \sum_{k=o}^{\infty} a_{k+1} Z^{k+1}$$

$$X \sum_{r=o}^{n} \frac{(-n)_{a+r}(n+\lambda)_{a+r}(cf)_{a+r}(1-\mu)_r(1-\nu+k)_r(\nu+2+k)_r}{(\beta+1)_{a+r}(dg)_{a+r}(1-\nu)_r(\nu+2)_r(2-\mu+k)_r(2+k)_r} (Z/\gamma)^r \quad \dots (22)$$

When $a = 0$, then from (22)

$$X_n(Z,\gamma) = \sum_{k=o}^{\infty} a_{k+1-a} Z^{k+1-a}$$

$$X_{6+f}F_{5+g}\left(\begin{matrix}-n, n+\lambda, 1-\nu-a+k, \nu+2-a+k, 1-\mu-a, cf, 1\\ \beta+1, 2-\mu-a+k, 2-a+k, 1-\nu-a, \nu+2-a, dg\end{matrix} \Big/ Z/\gamma\right) \quad \dots (23)$$

when $a = 1$, then from (22)

$$X_n(Z,\gamma) = \sum_{k=o}^{\infty} a_k Z^k + \sum_{k=o}^{\infty} a_k Z^k$$

$$x \sum_{r=o}^{n} \frac{(-n)_{r+1}(n+\lambda)_{r+1}(cf)_{r+1}(-\mu)_{r+1}(-\nu+k)_{r+1}(\nu+1+k)_{r+1}}{(\beta+1)_{r+1}(dg)_{r+1}(-\nu)_{r+1}(\nu+1)_{r+1}(1-\mu+k)_{r+1}(1+k)_{r+1}} \left(\frac{Z}{\gamma}\right)^{r+1}$$

But since

$$\sum_{r=o}^{n} P_{r+1} = \sum_{r=1}^{n+1} P_r ,$$

$$X_n(Z,\gamma) = \sum_{k=o}^{\infty} a_k Z^k \sum_{r=o}^{n} \frac{(-n)_r (n+\lambda)_r (cf)_r (-\mu)_r (-\nu+k)_r (\nu+1+k)_r}{(\beta+1)_r (dg)_r (-\nu)_r (\nu+1)_r (1-\mu+k)_r (1+k)_r} (Z/\gamma)^r$$

$$= \sum_{k=o}^{\infty} a_{k+1-a} Z^{k+1-a} \quad \{\text{since } (-n)_{n+1} = 0\}$$

$$x \sum_{r=o}^{n} \frac{(-n)_r (n+\lambda)_r (cf)_r (1-\mu-a)_r (-\nu+1+k-a)_r (\nu+2+k-a)_r (1)_r}{(\beta+1)_r (dg)_r (-\nu+1-a)_r (\nu+2-a)_r (2-\mu+k-a)_r (2+k+a)_r \, \underline{|r}} \left(\frac{Z}{\gamma}\right)^r$$

$$= \sum_{k=o}^{\infty} a_{k+1-a} Z^{k+1-a}$$

$$X \; {}_{6+f}F_{5+g}\left(\begin{matrix} -n, n+\lambda, 1-\mu-a, \; 1-\nu-a+k, \nu+2-a+k, \; cf, 1 \\ \beta+1, 1-\nu-a, 2-\mu-a+k, 2+k-a, dg, \nu+2-a \end{matrix} \Big/ \; Z/\gamma \right) \qquad \ldots(24)$$

By determining $h_n(\Upsilon)$, $\Psi_n(Z,\gamma)$ and $X_n(Z,\gamma)$, we arrive at our approximation

$$F_n(Z) = \frac{\Psi_n(Z,\gamma)}{h_n(\gamma)}$$

and the error polynomial

$$E_n(Z) = \frac{X_n(Z,\gamma)}{h_n(\gamma)}$$

Obviously, the approximation $\Psi_n(Z,\gamma)/h_n(\gamma)$ converges uniformly to ${}_2F_1(-\nu, \nu+1 ; 1-\mu ; Z)$ as $n \to \infty$ for we have from a theorem that these approximations converge when $p = q + 1$.

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On Relative Modified Defects of Memorphic Functions

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1. Let $f(z)$ be a meromorphic function of order ρ. Let $N(r,f)$, $m(r,f)$, $T(r,f)$, $S(r,f)$, $\delta(a,f)$, $\Theta(a,f)$ have the usual meaning of Nevanlinna theory. Let α be a real number such that $o \le \alpha < \rho$ if $o < \rho \le \infty$ and $\alpha = o$ if $\rho = o$.

Definition: For $r_o > o$,

$$T_\alpha(r,f) = \int_{r_o}^{r} \frac{T(t,f)}{t^{1+\alpha}}\,dt$$

$$N_\alpha(r, \frac{1}{f-a}) = N_\alpha(r,a) = \int_{r_o}^{r} \frac{N(t,a)}{t^{1+\alpha}}\,dt,$$

$m_\alpha(r,f)$ and $S_\alpha(r,f)$ are defined similarly.

Let $$\delta_\alpha(a,f) = 1 - \limsup_{r\to\infty} \frac{N_\alpha(r,a)}{T_\alpha(r,f)}$$

$$\Theta_\alpha(a,f) = 1 - \limsup_{r\to\infty} \frac{\bar{N}_\alpha(r,a)}{T_\alpha(r,f)}$$

$T_\alpha(r,f)$ is called the modified characteristic function and $\delta_\alpha(a,f)$ the modified α- defect with respect to $f(z)$, see [6].

The advantages in modified characteristic function and the modified defect lie in the fact that the exceptional set for $\rho=\infty$ does not occur here. It is known that if $f(z)$ is a meromorphic function of infinite order, then $S(r,f) = o\,(T(r,f))$ for all r outside an exceptional set E of finite linear measure and such sets exist. An example of the existence of such a set has been given by Hayman [1,122]. But in modified characteristic function (for all functions of finite or infinite order) we have $S_\alpha(r,f) = o\,(T_\alpha(r,f))$ as $r \to \infty$.

Definition: Set $\delta_r^{(1)}(a,f) = 1 - \limsup_{r \to \infty} \frac{N(r, \frac{1}{f'-a})}{T(r,f)}$

$\delta_r^{(1)}(a,f)$ is called the relative defect of a with respect to $f'(z)$.

We extend the idea of relative defect to relative modified defect. We define

$$\delta_{\alpha r}^{(k)}(a,f) = 1 - \limsup_{r \to \infty} \frac{N_\alpha(r, \frac{1}{f^{(k)}-a})}{T_\alpha(r,f)}$$

We shall call $\delta_{\alpha r}^{(k)}(a,f)$ the relative modified defect of a with respect to $f^{(k)}(z)$.

In general $\delta_{\alpha r}^{(k)}(a,f) \neq \delta_\alpha(a,f^{(k)})$. As a matter of fact using the inequality

$$T(r,f^{(k)}) \leq (k+1)\, T(r,f) + S(r,f)$$

and

$$\int_{r_o}^{r} \frac{S(t,f)}{t^{1+\alpha}}\, dt = 0 \;\left(\int_{r_o}^{r} \frac{\log T(k,f)}{t^{1+\alpha}}\, dt \right)$$

we can deduce that

$$\delta_{\alpha r}^{(k)}(a,f) \geq (k+1) \quad \delta_\alpha(a,f^{(k)}) - k.$$

Also from the definition of $\delta_\alpha(a,f)$ and $\delta(a,f)$ if follows that

$$\delta_\alpha(a,f) \geq \delta(a,f) \qquad \ldots (1)$$

In general equality in (1) does not hold see Toda [6].

2. We prove the following theorems.

Theorem 1 : Let $f(z)$ be a transcendental meromorphic function and let $\delta_\alpha(a,f) = 1$ $(a \neq \infty)$ and $Ⓗ_\alpha(\infty,f) = 1$. Then for all $b \neq a$, $\delta_\alpha(b,f^{(k)}) = \delta_{\alpha r}^{(k)}(b,f)$.

Corollary : If $Ⓗ_\alpha(\infty,f) = \delta_\alpha(o,f) = 1$, then for all b $(b \neq o, \infty)$, $\delta_\alpha(b,f^{(k)}) = o$ $(k = 1,2,\ldots)$. In particular $\delta(b,f^{(k)}) = o$ since $\delta(b,f^{(k)}) \leq \delta_\alpha(b,f^{(k)})$.

Theorem 2 : Let $f(z)$ be a transcendental meromorphic function such that $Ⓗ_{\alpha}(\infty,f) = 1$. Then either $f(z)$ has no finite modified α-defect or $\Sigma\, \delta_{\alpha}(b,f') < 1$ $(b \neq o,\infty)$.

Theorem 3 : Let $f(z)$ be a transcendental meromorphic function such that $Ⓗ_{\alpha}(\infty,f) = 1$ and $\sum_{a\neq\infty} \delta_{\alpha}(a,f) = 1$. Then $f'(z)$ has no finite modified α-deficient value except possibly zero.

3. <u>Proof of Theorem 1</u> : $T(r,f') = m(r,f') + N(r,f')$

$$\leq m(r, \frac{f'}{f}) + m(r,f) + N(r,f) + \bar{N}(r,f)$$

$$= m(r, \frac{f'}{f}) + T(r,f) + \bar{N}(r,f).$$

Hence

$$T_{\alpha}(r,f') \leq m_{\alpha}(r, \frac{f'}{f}) + T_{\alpha}(r,f) + \bar{N}_{\alpha}(r,f) \qquad \ldots (2)$$

Since

$$m_{\alpha}(r, \frac{f'}{f}) = o(T_{\alpha}(r,f)), \quad \text{see} \quad [5,69]$$

and since $Ⓗ_{\alpha}(\infty,f) = 1$, we have

$$\limsup_{r\to\infty} \frac{T_{\alpha}(r,f')}{T_{\alpha}(r,f)} \leq 1 .$$

On the otherhand [4, 298]

$$T(r,f) < T(r,f') + N(r, \frac{1}{f-a}) - N(r, \frac{1}{f'}) + S(r,f) .$$

Hence since $\delta_{\alpha}(a,f) = 1$, we have

$$T_{\alpha}(r,f) < T_{\alpha}(r,f') + S_{\alpha}(r,f).$$

Also from [2, 33] for a_1, a_2, ... aq finite,

We have

$$\sum_{\nu=1}^{q} m(r,a_{\nu}) \leq m(r, \sum_{\nu=1}^{q} \frac{1}{f-a_{\nu}}) + o(1)$$

$$\leq m(r, \frac{1}{f'}) + m(r, \sum_{\nu=1}^{q} \frac{f'}{f-a_{\nu}}) + o(1)$$

Hence $\sum_{\nu=1}^{q} m_{\alpha}(r,a_{\nu}) \le m_{\alpha}(r, \frac{1}{f'}) + o(T_{\alpha}(r,f))$ ... (3)

Hence from (2) and (3) we deduce

$$\sum_{\nu=1}^{q} \delta_{\alpha}(a_{\nu},f) \le \delta_{\alpha}(o,f')\ \{2- \Theta_{\alpha}(\infty,f)\} .$$

Hence since q is arbitrary making $q \to \infty$

We get

$$\sum_{a\ne\infty} \delta_{\alpha}(a,f) \le \delta_{\alpha}(o,f')\ \{2- \Theta_{\alpha}(\infty,f)\} .$$

Hence $\delta_{\alpha}(o,f') = 1$.

Also since $\bar{N}_{\alpha}(r,f') = \bar{N}_{\alpha}(r,f)$ and $T_{\alpha}(r,f') \sim T_{\alpha}(r,f)$, using the first part we get $T_{\alpha}(r,f') \sim T_{\alpha}(r,f'')$.

By induction it follows that

$$T_{\alpha}(r,f) \sim T_{\alpha}(r,f^{(k)}) \qquad \text{... (4)}$$

Thus the result follows.

<u>Proof of the Corollary</u> : From [3, 671] ,

$$(q+1)\ T(r,f) < 2\ \bar{N}(r,f) + (q+1)\ N(r, \frac{1}{f}) + N(r, \frac{1}{f-a})$$

$$+ \sum_{\nu=1}^{q} N(r, \frac{1}{f^{(k)}-b_{\nu}}) + S(r,f)$$

where $a, b_1, b_2, \dots b_q$ are distinct, finite and non-zero.

Now since $N(r, \frac{1}{f-a}) \le T(r,f) + o(1)$, we get

$$q\ T_{\alpha}(r,f) < 2\ \bar{N}_{\alpha}(r,f) + (q+1)\ N_{\alpha}(r, \frac{1}{f})$$

$$+ \sum_{\nu=1}^{q} N_{\alpha}(r, \frac{1}{f^{(k)}-b_{\nu}}) + S_{\alpha}(r,f) .$$

Since $\delta_{\alpha}(o,f) = 1$, $\Theta_{\alpha}(\infty,f) = 1$, it follows that

$$N_{\alpha}(r,\frac{1}{f}) = o(T_{\alpha}(r,f)),\ \bar{N}_{\alpha}(r,f) = o(T_{\alpha}(r,f)).$$

Hence

$$(q+o(1))\ T_{\alpha}(r,f) < \sum_{\nu=1}^{q} N_{\alpha}(r, \frac{1}{f^{(k)}-b\nu}) .$$

Hence from (4),

$$q \leq \sum_{j=1}^{q} \limsup_{r \to \infty} \frac{N_{\alpha}(r, \frac{1}{f^{(k)}-b_j})}{T_{\alpha}(r, f^{(k)})}$$

Thus $\sum_{j=1}^{q} \delta_{\alpha}(b_j, f^{(k)}) = o$ for all b_j such that $o < |b_j| < \infty$.

This proves the corollary.

<u>Proof of Theorem 2</u> : Let $\sum_{o<|b|<\infty} \delta_{\alpha}(b, f') = M$ and

let $M \geq 1$, $\sum_{i=1}^{\infty} \delta_{\alpha}(a_i, f) = S$.

Since $S \leq 2$ and $M \leq 2$, we can choose p and q such that

$$\sum_{i=p+1}^{\infty} \delta_{\alpha}(a_i, f) < \varepsilon_1, \sum_{j=q+1}^{\infty} \delta_{\alpha}(b_j, f') < \varepsilon_2 .$$

Now from [4,299] we have

$$q\ T(r,f') < \bar{N}(r,f) + N(r, \frac{1}{f'}) + \sum_{j=1}^{q} N(r, \frac{1}{f'-b_j}) + S(r,f)$$

Hence $\quad q\ T_{\alpha}(r,f') < \bar{N}_{\alpha}(r,f) + N_{\alpha}(r, \frac{1}{f'})$

$$+ \sum_{j=1}^{q} \{1 - \delta_{\alpha}(b_j, f') + \varepsilon_3\}\ T_{\alpha}(r,f') + S_{\alpha}(r,f)$$

Hence

$$(M - \varepsilon_2 - q\ \varepsilon_3)\ T_{\alpha}(r,f') - N_{\alpha}(r, \frac{1}{f'}) < \bar{N}_{\alpha}(r,f) + S_{\alpha}(r,f) .$$

Since $\quad M \geq 1$,

$$(M - \varepsilon_2 - q\ \varepsilon_3)\ T_{\alpha}(r,f') - M\ N_{\alpha}(r, \frac{1}{f'})$$

$$< \{1 - \Theta_{\alpha}(\infty, f) + \varepsilon_4\}\ T_{\alpha}(r,f) + S_{\alpha}(r,f)$$

Hence

$$M\{T_\alpha(r,f') - N_\alpha(r,\tfrac{1}{f'})\} - \{1 - \text{Ⓗ}_\alpha(\infty,f) + \varepsilon_4\}\, T_\alpha(r,f)$$

$$< (\varepsilon_2 + q\,\varepsilon_3)\, T_\alpha(r,f') + S_\alpha(r,f) \qquad \dots (5)$$

Now $\quad S \leq \sum_{i=1}^{p} \delta_\alpha(a_i,f) + \varepsilon_1$.

Hence

$$\sum_{i=1}^{p} N_\alpha(r,\tfrac{1}{f-a_i}) \leq (p + \varepsilon_1 - S - \varepsilon_5)\, T_\alpha(r,f) \qquad \dots (6)$$

Also from [4, 298] we have

$$p\ \ T_\alpha(r,f) < T_\alpha(r,f') + \sum_{i=1}^{p} N_\alpha(r,\tfrac{1}{f-a_i}) .$$

Hence

$$p\,T_\alpha(r,f) < T_\alpha(r,f') + (p + \varepsilon_1 - S - \varepsilon_3)\, T_\alpha(r,f)$$

$$- N_\alpha(r,\tfrac{1}{f'}) + S_\alpha(r,f)$$

Hence

$$(S - \varepsilon_1 + \varepsilon_5)\, T_\alpha(r,f) < T_\alpha(r,f') - N_\alpha(r,\tfrac{1}{f'}) + S_\alpha(r,f) \quad \dots (7)$$

From (5) and (7) we get

$$M(S - \varepsilon_1 + \varepsilon_5)\, T_\alpha(r,f) < M\,\{T_\alpha(r,f') - N_\alpha(r,\tfrac{1}{f'})\} + S_\alpha(r,f)$$

$$< \{1 - \text{Ⓗ}_\alpha(\infty,f) + \varepsilon_4\}\, T_\alpha(r,f)$$

$$+ (\varepsilon_2 + \varepsilon_3\,q)\, T_\alpha(r,f) + S_\alpha(r,f).$$

Hence $SM < 1 - \text{Ⓗ}_\alpha(\infty,f)$.

Now by hypothesis $\text{Ⓗ}_\alpha(\infty,f) = 1$, hence $S = 0$.

Thus $f(z)$ has no modified α-deficient value.

<u>Proof of Theorem 3</u> : Let $\sum_{a\neq\infty} \delta_{\alpha}(a,f) = S.$

and $\sum_{b} \delta(1)_{\alpha r}(b,f) = N \; (b \neq o,\; b \neq \infty)$

Now from [4, 299] we have

$$pq\, T(r,f) < \bar{N}(r,f) + q \sum_{i=1}^{p} N(r, \frac{1}{f-a_i})$$

$$+ \sum_{j=1}^{p} N(r, \frac{1}{f'-b_j}) + S(r,f).$$

Hence

$$pq\, T_{\alpha}(r,f) < \bar{N}_{\alpha}(r,f) + q \sum_{i=1}^{p} N_{\alpha}(r, \frac{1}{f-a_i})$$

$$+ \sum_{j=1}^{q} N_{\alpha}(r, \frac{1}{f'-b_j}) + S_{\alpha}(r,f) \qquad \ldots (8)$$

Now choose p so large that

$$\sum_{p+1}^{\infty} \delta_{\alpha}(a_i f) < \varepsilon_1 ,$$

then

$$\sum_{i=1}^{p} N_{\alpha}(r, \frac{1}{f-a_i}) < (p + p\,\varepsilon_2 - S - \varepsilon_1)\, T_{\alpha}(r,f)$$

Also choose q such that $\sum_{q+1}^{\infty} \delta^{(1)}_{\alpha}(b_j,f) < \varepsilon_3.$

Then $\sum_{j=1}^{q} N_{\alpha}(r, \frac{1}{f'-b_j}) < (q + q\,\varepsilon_4 - N - \varepsilon_3)\, T_{\alpha}(r,f)$

and $\bar{N}_{\alpha}(r,f) < (1 - H_{\alpha}(\infty,f) + \varepsilon_5)\, T_{\alpha}(r,f).$

Hence from (8) we get

$$pq\, T_{\alpha}(r,f) < T_{\alpha}(r,f)\,\{1 - H_{\alpha}(\infty,f) + pq - q\, S + q - N\} + o(T_{\alpha}(r,f)).$$

Hence

$$N + \text{Ⓗ}_{\alpha}(\infty,f) \leq q(1-s) + 1 .$$

Thus $N + q\,s + \text{Ⓗ}_{\alpha}(\infty,f) < q + 1$.

Hence using the hypothesis, we get $N \leq 0$.

Also since $\text{Ⓗ}_{\alpha}(\infty,f) = 1, \sum_{|a|<\infty} \delta_{\alpha}(a,f) = 1$.

So from theorem (1), $\delta_{\alpha}(b,f') = \delta^{(1)}_{\alpha r}(b,f)$

Thus $o = N \sum_{o<|b|<\infty} \delta^{(1)}_{\alpha r}(b,f) = \sum_{o<|b|<\infty} \delta_{\alpha}(b,f')$

Hence $\delta_{\alpha}(b,f') = o$ for all $b \neq o$, $b \neq \infty$.

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On the Order and Type of Integral Functions of Several Complex Variables

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Let

$$(1.1)\qquad f(Z) = \sum_{m_1,\dots,m_n=0}^{\infty} a_{m_1,\dots,m_n} Z_1^{m_1} \dots Z_n^{m_n},$$

be an integral function of n complex variables $Z_1, Z_2, \dots, Z_n$ whose coefficients $a_{m_1\dots m_n}$ are complex numbers.

Dzrbaryan, M.M. [1, p.1] has shown that a necessary and sufficient condition for the series (1.1) to represent an integral function of variables $Z_1, \dots, Z_n$ is

$$(1.2)\qquad \limsup_{a_{m_1+,\dots,+m_n} \to \infty} \left| a_{m_1,\dots,m_n} \right|^{1/(m_1+\dots+m_n)} = 0.$$

Order ρ and type T of an integral function f have been defined as [2, p. 146]

$$(1.3)\qquad \rho = \limsup_{m_1+\dots+m_n \to \infty} \frac{(m_1+\dots+m_n)\log(m_1+\dots+m_n)}{-\log \left|a_{m_1,\dots,m_n}\right|}$$

and

$$(1.4)\qquad (e\rho T)^{1/\rho} = \limsup_{m_1+\dots+m_n \to \infty} \left[(m_1+\dots+m_n)^{1/\rho} \left\{\varphi \left|a_{m_1,\dots,m_n}\right|\right\}^{1/(m_1+\dots+m_n)}\right]$$

where

$$(1.5)\qquad \varphi = \max |Z_1|^{m_1} \dots |Z_n|^{m_n}.$$

Dalal, S.S. [3, p. 216] has defined the lower order $\underline{\lambda}$ and the lower bar $\underline{t}$ for integral function as

(1.6) $$\underline{\lambda} = \liminf_{m_1+\dots+m_n \to \infty} \frac{(m_1+\dots+m_n)\log(m_1+\dots+m_n)}{-\log |a_{m_1,\dots,m_n}|}$$

and

(1.7) $$(e\rho\underline{t})^{1/\rho} = \liminf_{m_1+\dots+m_n \to \infty} \left[(m_1+\dots+m_n)^{1/\rho}\{\varphi|a_{m_1+\dots,m_n}|\}^{1/(m_1+\dots+m_n)}\right]$$

Srivastava, R.K. and Vinod Kumar [4, pp. 161-166], Singh, J.P. [5, pp. 111-121] and Dalal, S.S. [3, pp. 215-220] have obtained relations between orders and types of two or more integral functions. In this paper we have obtained some more relations between orders, types, lower bar orders and lower bar types of two or more integral functions when their coefficients are asymptotically connected.

2. Theorem 1

Let
$$f_k = \sum_{m_1,\dots,m_n=o}^{\infty} (a_{m_1,\dots,m_n})_k z_1^{m_1}\dots z_n^{m_n}, \quad (k = 1,2,..p)$$

be p integral functions of the same finite non zero order ρ and finite non-zero types $T_1, T_2, .., T_p$ respectively. Then the integral function

$$f = \sum_{m_1+\dots+m_n = o}^{\infty} a_{m_1,\dots,m_n} z_1^{m_1}\dots z_n^{m_n},$$

where

(2.1) $$\sum_{k=1}^{p} \alpha_k \left|(a_{m_1,\dots,m_n})_k\right|^{\rho/(m_1+\dots+m_n)} \sim \left|a_{m_1,\dots,m_n}\right|^{\rho'/(m_1+\dots+m_n)}$$
$$(0 < \alpha_k < 1, \ \sum_{k=1}^{p} \alpha_k = 1),$$

is such that

(2.2) $$\rho' T \le \sum_{k=1}^{p} \alpha_k \rho T_k,$$

where ρ' and T are the order and type of f respectively.

Proof: Since f_k is an integral function, therefore, using (1.2), we have

$$\limsup_{m_1+\ldots+m_n\to\infty} \left|(a_{m_1,\ldots m_n})_k\right|^{1/(m_1+\ldots+m_n)} = 0 \quad (k=1,2,\ldots p).$$

Also from (2.1), we get

$$\limsup_{m_1+\ldots+m_n\to\infty} \left|a_{m_1,\ldots,m_n}\right|^{\rho'/(m_1+\ldots+m_n)} \le$$

$$\le \sum_{k=1}^{p} \limsup_{m_1+\ldots+m_n\to\infty} \alpha_k \left|(a_{m_1,\ldots,m_n})_k\right|^{\rho/m_1+\ldots+m_n)}$$

$$= 0.$$

Therefore, f is an integral function.

Using the relation (1.4) for the function f_k, we get

$$\limsup_{m_1+\ldots+m_n\to\infty} (m_1+\ldots+m_n)\{\varphi\left|(a_{m_1,\ldots,m_n})_k\right|\}^{\rho/(m_1+\ldots+m_n)} = e\rho T_k.$$

Hence,

$$\sum_{k=1}^{p} \limsup_{m_1+\ldots+m_n\to\infty} \alpha_k(m_1+\ldots+m_n)\{\varphi\left|(a_{m_1,\ldots,m_n})_k\right|\}^{\rho/(m_1+\ldots+m_n)}$$

$$= \sum_{k=1}^{p} e\rho\alpha_k T_k.$$

or

$$(2.3) \quad \limsup_{m_1+\ldots+m_n\to\infty} (m_1+\ldots+m_n)\,\varphi^{\rho/(m_1+\ldots+m_n)}$$

$$\sum_{k=1}^{p} \alpha_k \left|(a_{m_1,\ldots,m_n})_k\right|^{\rho/(m_1+\ldots+m_n)} < e\rho\sum_{k=1}^{p} \alpha_k T_k.$$

Using (2.1) in (2.3), we get

$$(2.4) \quad \limsup_{m_1+\ldots+m_n\to\infty} (m_1+\ldots+m_n)\,\varphi\left|a_{m_1,\ldots,m_n}\right|^{\rho'(m_1+\ldots+m_n)} \le$$

$$\le e\rho\sum_{k=1}^{p} \alpha_k T_k.$$

Again using (1.4) in (2.4), we get

$$e\rho' T \leq e\rho \sum_{k=1}^{p} \alpha_k T_k$$

or

$$\rho' T \leq \rho \sum_{k=1}^{p} \alpha_k T_k .$$

<u>Theorem 2:</u> Under the hypothesis of Theorem 1, if

$$(2.5) \quad \sum_{k=1}^{p} \alpha_k \left|(a_{m_1,\ldots,m_n})_k\right|^{-\rho/(m_1+\ldots+m_n)} \sim \left|a_{m_1,\ldots,m_n}\right|^{-\rho'(m_1+\ldots+m_n)}$$

$$\left(0 < \alpha_k < 1, \ \sum_{k=1}^{p} \alpha_k = 1\right),$$

then

$$(2.6) \qquad \frac{\rho'^{-1}}{T} \geq \rho^{-1} \sum_{k=1}^{p} \alpha_k / T_k ,$$

where ρ' and T are the order and type of f respectively.

<u>Proof:-</u> Since f_k is an integral function, therefore using (1.2), we have for an arbitrary $\varepsilon > 0$ and large R

$$\left|(a_{m_1,\ldots,m_n})_k\right|^{-1} > (R-\varepsilon)^{m_1+\ldots+m_n}, \text{ for } m_1+\ldots+m_n > h_k$$

therefore, for $m_1+\ldots+m_n > h = \max(h_1,\ldots,h_p)$

$$(2.7) \quad \sum_{k=1}^{p} \alpha_k \left|(a_{m_1,\ldots,m_n})_k\right|^{-\rho/(m_1+\ldots+m_n)} \geq (R-\varepsilon)^{\rho}.$$

Therefore, using (2.5) in (2.7), we get

$$\limsup_{m_1+\ldots+m_n \to \infty} \left|a_{m_1,\ldots,m_n}\right|^{-1/m_1+\ldots+m_n} = 0.$$

Hence f is an integral function.

Now using the relation (1.4) for the function f_k, we get

$$\liminf_{m_1+\dots+m_n\to\infty} (m_1+\dots+m_n)^{-1}\{\varphi\left|(a_{m_1,\dots,m_n})_k\right|\}^{-\rho(m_1+\dots+m_n)} = (e\rho T_k)^{-1}.$$

Therefore,

$$\sum_{k=1}^{p} \liminf_{m_1+\dots+m_n\to\infty} \alpha_k (m_1+\dots+m_n)^{-1}\{\varphi\left|(a_{m_1+\dots+m_n})_k\right|\}^{-\rho/(m_1+\dots+m_n)}$$

$$= \sum_{k=1}^{p} \alpha_k (e\rho T_k)^{-1}$$

or,

$$\liminf_{m_1+\dots+m_n\to\infty} (m_1+\dots+m_n)^{-1} \sum_{k=1}^{p} \alpha_k\{\varphi\left|(a_{m_1,\dots,m_n})_k\right|\}^{-\rho/(m_1+\dots+m_n)}$$

$$\geq (e\rho)^{-1} \sum_{k=1}^{p} \alpha_k T_k^{-1}.$$

Now using the relation (2.5) in (2.8), we get

$$(2.9)\quad \liminf_{m_1+\dots+m_n\to\infty} (m_1+\dots+m_n)^{-1}\{\varphi\left|a_{m_1,\dots,m_n}\right|\}^{-\rho/(m_1+\dots+m_n)}$$

$$\geq (e\rho)^{-1} \sum \alpha_k T_k^{-1}.$$

Again using (1.4) in (2.9), we get

$$(\rho' T)^{-1} \geq \rho^{-1} \sum_{k=1}^{p} \alpha_k T_k^{-1}.$$

3. Theorem 3:- Under the hypothesis of Theorem 1, if $\underline{t}_1, \underline{t}_2, \dots, \underline{t}_p$ be the lower bar types of p integral functions $f_1, f_2, \dots f_p$ respectively, then

$$(3.1)\quad \rho' \underline{t} \geqslant \rho \sum_{k=1}^{p} \alpha_k \underline{t}_k$$

where ρ' and $\underline{t}$ are the order and lower bar type of the integral function f.

Proof is similar to that of Theorem 1.

Theorem 4: Under the hypothesis of Theorem 2, if $\underline{t}_1, \underline{t}_2, \ldots, \underline{t}_p$ the lower bar types of p integral functions $f_1, f_2, \ldots, f_p$ respectively then

$$(\rho'\underline{t})^{-1} \leq \rho^{-1} \sum_{k=1}^{p} \alpha_k \underline{t}_k^{-1},$$

where ρ' and $\underline{t}$ are the order and lower bar type of f respectively.

Proof is similar to that of Theorem 2.

4. Theorem 5: Let

$$f_k = \sum_{m_1,\ldots,m_n=0}^{\infty} (a_{m_1,\ldots,m_n})_k \, z_1^{m_1} \ldots z_n^{m_n} \quad (k=1,2,\ldots,p),$$

be p integral function of finite, positive lower bar types order $\underline{\lambda}, \ldots, \underline{\lambda}_p$ respectively, then the integral function

$$f = \sum_{m_1,\ldots,m_n=0}^{\infty} a_{m_1,\ldots,m_n} z_1^{m_1} \ldots z_n^{m_n},$$

where

$$\{\log(1/|a_{m_1,\ldots,m_n}|)\}^{-1} \sim \sum_{k=1}^{p} \alpha_k \{\log(1/|(a_{m_1,\ldots,m_n})_k|)\}^{-1}$$

$$\left(0 < \alpha_k < 1, \ \sum_{k=1}^{p} \alpha_k = 1\right),$$

is such that

$$\underline{\lambda} > \sum_{k=1}^{p} \alpha_k \underline{\lambda}_k,$$

where $\underline{\lambda}$ is the lower bar order of f.

Proof: Using the relation (1.6) for the integral function f_k, we get

$$\liminf_{m_1+\ldots+m_n \to \infty} \frac{(m_1+\ldots+m_n)\log(m_1+\ldots+m_n)}{\log(1/|(a_{m_1,m_2,\ldots,m_n})_k|)} = \underline{\lambda}_k .$$

Therefore for any $\varepsilon > 0$, we get

$$(4.2) \qquad \sum_{k=1}^{p} \alpha_k\{\log(1/|(a_{m_1,\dots,m_n})_k|)\}^{-1} >$$

$$> \{(m_1+\dots+m_n)\log(m_1+\dots+m_n)\}^{-1} \sum_{k=1}^{p} \alpha_k(\underline{\lambda}_k - \varepsilon)$$

for $m_1+\dots+m_n > h_k$.

Therefore, for $m_1+\dots+m_n > h = \max\,(h_1,\dots h_p)$ and using (4.1) in (4.2), we get

$$(4.3) \qquad \liminf_{m_1+\dots+m_n\to\infty} \frac{(m_1+\dots+m_n)\log(m_1+\dots+m_n)}{\log(1/|a_{m_1,\dots,m_n}|)} \geq \sum_{k=1}^{p} \alpha_k\underline{\lambda}_k .$$

Again using (1.6) in (4.3), we get

$$\underline{\lambda} \geqslant \sum_{k=1}^{p} \alpha_k\underline{\lambda}_k .$$

The result for ρ has been obtained by Singh, J.P. [5].

<u>Theorem 6:</u> Let

$$f_k = \sum_{m_1,\dots,m_n=0}^{\infty} (a_{m_1,\dots,m_n})_k\, z_1^{m_1}\dots z_n^{m_n} \quad (k=1,\dots,p)$$

be p integral functions having finite positive lower bar order $\underline{\lambda}_1,\dots,\underline{\lambda}_p$ respectively, then the integral function

$$f = \sum_{m_1,\dots,m_n=0}^{\infty} a_{m_1,\dots,m_n}\, z_1^{m_1}\dots z_n^{m_n} ,$$

where

$$(4.4) \qquad \prod_{k=1}^{p} \{\log(1/|(a_{m_1,\dots,m_n})_k|)\}^{k} \sim \log(1/|a_{m_1,\dots,m_n}|).$$

$$(0 < \alpha_k < 1, \ \sum_{k=1}^{p} \alpha_k=1),$$

is such that

$$\underline{\lambda} \geq \prod_{k=1}^{p} \underline{\lambda}_k^{\alpha_k} ,$$

where $\underline{\lambda}$ is the lower bar order of f.

Proof: Using the relation (1.6) for the integral function f_k and for any $\varepsilon > 0$, we have

$$\{\log(1/|(a_{m_1,\ldots m_n})_k|)\}^{-\alpha_k} > (\underline{\lambda}_k - \varepsilon)^{\alpha_k} \{(m_1+\ldots+m_n)\log(m_1+\ldots+m_n)\}^{-\alpha_k}$$

$$\text{for } m_1+\ldots+m_n > h_k,$$

hence,

$$(4.4) \quad \prod_{k=1}^{p} \{\log(1/|(a_{m_1,\ldots,m_n})_k|)\}^{-\alpha_k} > \prod_{k=1}^{p} (\underline{\lambda}_k - \varepsilon)^{\alpha_k} \{(m_1+\ldots+m_n)\log(m_1+\ldots+m_n)\}^{-\alpha_k} .$$

Therefore, for $m_1+\ldots+m_n \geq h_k = \max(h_1,\ldots,h_p)$ and using (4.1) in (4.4), we get

$$(4.5) \qquad \liminf_{m_1+\ldots+m_n \to \infty} \frac{(m_1+\ldots+m_n)\log(m_1+\ldots+m_n)}{\log\{1/|a_{m_1,\ldots,m_n}|\}} \geq \prod_{k=1}^{p} \underline{\lambda}_k^{\alpha_k} .$$

Again using (1.6) in (4.5), we get

$$\underline{\lambda} \geq \prod_{k=1}^{p} \underline{\lambda}_k^{\alpha_k} .$$

The result for ρ has been obtained by Srivastava, R.K. and Vinod Kumar [4].

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Inverse Topologies in a Quasigroup

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A quasigroup G can be defined to be a system of three compositions viz. product ('.'), right division('/') and left division ('\\') such that for every a, b, c ϵ G, $ab=c \Longleftrightarrow c/b = a \Longleftrightarrow a\backslash c = b$

Unless otherwise stated throughout the present paper G will denote an arbitrarily given quasigroup.

Every element 'a' in G has four inverses viz. $a_{rr}=a\backslash(a\backslash a)$, $a_{rL} = (a\backslash a)/a$, $a_{Lr}=a\backslash(a/a)$, $a_{LL}=(a/a)/a$ which are respectively called the right - right inverse, right - left inverse, left - right inverse, left-left inverse of a. If G is commutative, then $a_{rr} = a_{rL} = a_{Lr} = a_{LL}$.

Let N be the set of all positive integers including zero.

Let us now make a convention that $a = a_{(rr,o)}$ and $a_{(rr,n+1)}$ is the right - right inverse of $a_{(rr,n)}$ for every $n \ \epsilon$ N.

Let A be any subset of G. Let $\mu_{rr}(A) = \{a_{rr};\ a \ \epsilon\ A\}$. Also let $A = \mu^{o}_{rr}(A)$, $\mu^{n+1}_{rr}(A) = \mu_{rr}(\mu^{n}_{rr}(A))$ for every $n\ \epsilon$ N and $\bar{A}_{rr} = \bigcup_{n \epsilon N} \mu^{n}_{rr}(A)$. The sets $\bar{A}_{rr}$ are characterised by the property that $a\ \epsilon\ \bar{A}_{rr} \Longrightarrow a_{rr}\ \epsilon \bar{A}_{rr}$. Sets which are closed w.r. to right - right inverse elements will be called RRI - sets. $\bar{A}_{n}$ is an RRI - set. G has at least one RRI - subset viz. G itself.

$w_{rr} : G \to N$ is defined by the relation: $w_{rr}(a)=n$ iff a is the right - right inverse of n elements in G for every $a\ \epsilon$ G.

<u>Theorem 1</u> : If A, B be any two subsets of G, then

(i) $\bar{\phi}_{rr} = \phi$, $\bar{G}_{rr} = G$;

(ii) $A \subset \bar{A}_{n}$;

(iii) $A \subset B \Longrightarrow \bar{A}_{rr} \subset \bar{B}_{rr}$;

(iv) $\overline{(\bar{A}_{rr})}_{rr} = \bar{A}_{rr}$;

(v) $\overline{(A \cap B)}_{rr} \subset \bar{A}_{rr} \cap \bar{B}_{rr}$;

(vi) $\overline{(A \cup B)}_{rr} = \bar{A}_{rr} \cup \bar{B}_{rr}$.

Proof: (i), (ii), (iii), (iv) are obvious.

(v) Follows from (iii) .

(vi) By (iii), $\bar{A}_{rr} \cup \bar{B}_{rr} \subset \overline{(A \cup B)}_{rr}$ ... (1)

Now, $x \in \mu_{rr}\ (A \cup B) \Longrightarrow x = a_{rr}$ for some $a \in A \cup B \Longrightarrow x \in \mu_{rr}(A)$ or $x \in \mu_{rr}\ (B) \Longrightarrow x \in \mu_{rr}\ (A) \cup \mu_{rr}(B)$. So $\mu_{rr}\ (A \cup B) \subset \mu_{rr}(A) \cup \mu_{rr}(B)$.

Therefore $\overline{(A \cup B)}_{rr} \subset \bar{A}_{rr} \cup \bar{B}_{rr}$ ... (2)

Now (vi) follows from (1) and (2).

Remark 1 : $\overline{(A \cap B)}_{rr}$ may not be equal to $\bar{A}_{rr} \cap \bar{B}_{rr}$ as shown by

Example 1 : Consider the quasigroup whose multiplication table is

	a	b	c	d
a	d	a	c	b
b	c	d	b	a
c	a	b	d	c
d	b	c	a	d

Let $A = \{a,c\}$, $B = \{c,d\}$. Then

$\bar{A}_{rr} \cap \bar{B}_{rr} = \{a,c,d\} \cap \{c,d\} = \{c,d\} \neq \{c\} = \overline{(A \cap B)}_{rr}$.

Remark 2 : $\overline{(A \cap B)}_{rr} = \bar{A}_{rr} \cap \bar{B}_{rr}$ ... (3)

if $w_{rr}\ (a) = 1$ for every $a \in \mu_{rr}\ (A \cap B)$.

Remark 3 : The relation (3) may hold even if there exists an element a in $\mu_{rr}\ (A \cap B)$ such that $w_{rr}\ (a) > 1$ as shown by

Example 2 : For the quasigroup defined in Example 1, let $A = \{a, b, d\}$, $B = \{b, c, d\}$. Then $\mu_{rr}\ (A \cap B) = \{d\}$ and $w_{rr}(d) = 2$. But $\bar{A}_{rr} \cap B_{rr} = \overline{(A \cap B)}_{rr}$.

It follows from Theorem 1 that there exists a topology τ_{rr} in G having the family of all RRI - sets as the family of all closed sets. τ_{rr} is said to be the right - right inverse fore - topology in G.

Similarly one can define RLI - sets, LRI - sets, LLI - sets and obtain three other topologies τ_{rL}, τ_{Lr}, τ_{LL} having the family of all RLI - sets, LRI - sets, LLI - sets as the family of all closed sets respectively. τ_{rL}, τ_{Lr}, τ_{LL} are respectively said to be the right - left inverse fore - topology, left - right inverse fore - topology, left - left inverse fore - topology in G.

Another four topologies can be defined in G in the following way. Let a be any element of G and let $\nu_{rr}\ (a) = \mu^{-1}_{rr}\ (a)$. If $A \subset G$, let $\nu_{rr}\ (A) = \bigcup_{a \in A} \nu_{rr}\ (a)$. Let $A = \nu^{o}_{rr}\ (A)$, $\nu^{n+1}_{rr}\ (A) = \nu_{rr}\ (\nu^{n}_{rr}\ (A))$ for every $n \in N$ and $A'_{rr} = \bigcup_{n \in N} \nu^{n}_{rr}\ (A)$.

Theorem 1 is valied if '—' is replaced by '/' and μ by ν . Therefore there exists a topology $\bar{\tau}_{rr}$ in G which is defined to be the right-right inverse hind - topology in G. Closed sets of this topology are said to be RRH - sets.

Similarly one can define the right - left inverse hind - topology $\bar{\tau}_{rL}$, left - right inverse hind - topology $\bar{\tau}_{Lr}$, left - left inverse hind - topology $\bar{\tau}_{LL}$ in G. Closed sets of the topologies $\bar{\tau}_{rL}$, $\bar{\tau}_{Lr}$, $\bar{\tau}_{LL}$ are said to be RLH - sets, LRH - sets, LLH - sets respectively.

The right fore - topology, left fore - topology, right hind - topology, left hind - topology in G introduced by Choudhury [2] will be denoted by τ_r, τ_L, $\bar{\tau}_r$, $\bar{\tau}_L$ respectively.

The right identity of $a \in G$ will be denoted by a_r. Also let $a = a_{(r,o)}$ and let $a_{(r,n+1)}$ be the right identity of $a_{(r,n)}$ for every $a \in G$ and $n \in N$.

Proposition 1 : If every element in G be idempotent, then $\tau_r = \tau_L = \tau_{rr} = \tau_{rL} = \tau_{Lr} = \tau_{LL} = \bar{\tau}_r = \bar{\tau}_L = \bar{\tau}_{rr} = \bar{\tau}_{rL} = \bar{\tau}_{Lr} = \bar{\tau}_{LL} =$ discrete topology.

The straightforward proof is omitted.

Proposition 2 : If any one of τ_r, $\bar{\tau}_r$, τ_L, $\bar{\tau}_L$ be a T_1 - topology, then every element in G is idempotent.

Proof : If τ_r be a T_1 - topology, then $\{a\}$ is closed w.r. to τ_r for every $a \in G$. So $a\backslash a = a$ ie, $a.a=a$ ie. a is idempotent for every $a \in G$. The cases of τ_L, $\bar{\tau}_r$, $\bar{\tau}_L$ can be similarly disposed of.

Remark 4 : τ_{rr}, τ_{rL}, τ_{Lr}, τ_{LL}, $\bar{\tau}_{rr}$, $\bar{\tau}_{rL}$, $\bar{\tau}_{Lr}$, $\bar{\tau}_{LL}$ may be discrete topologies even if no element in G is idempotent as shown by

Example 3 : Consider the quasigroup given by the multiplication table

	a	b	c
a	b	a	c
b	a	c	b
c	c	b	a

No element of this quasigroup is idempotent. But $\tau_{rr} = \tau_{rL} = \tau_{Lr} = \tau_{LL} = \bar{\tau}_{rr} = \bar{\tau}_{rL} = \bar{\tau}_{Lr} = \bar{\tau}_{LL} =$ discrete topology.

Proposition 3 : If H be a subquasigroup of G, then

(i) H is closed w.r.to each of the topologies τ_r, τ_L, τ_{rr}, τ_{rL}, τ_{Lr}, τ_{LL} ;

(ii) the relativization of τ_{rr} to H is the right - right inverse - fore topology in H.

The proof is obvious.

Remark 5 : H may not be closed w.r.to the topologies $\bar{\tau}_r$, $\bar{\tau}_L$, $\bar{\tau}_{rr}$, $\bar{\tau}_{rL}$, $\bar{\tau}_{Lr}$, $\bar{\tau}_{LL}$ as shown by

Example 4 : The set of all rational numbers R when distorted by the law $a \circ b = -a - b$ becomes a commutative quasigroup. The set of all even integers E forms a subquasigroup of this quasigroup. But E is not closed w.r.to $\bar{\tau}_r$, $\bar{\tau}_L$, $\bar{\tau}_{rr}$, $\bar{\tau}_{rL}$, $\bar{\tau}_{Lr}$, $\bar{\tau}_{LL}$, since $1 \in R-E$ but $1_L = 1_r = -2 \in E$ and $\frac{2}{3} \in R-E$ but $(\frac{2}{3})_{rr} = (\frac{2}{3})_{rL} = (\frac{2}{3})_{Lr} = (\frac{2}{3})_{LL} = -2 \in E$.

Proposition 4 : If G be a finite quasigroup and if $w_{rr}(a) = 1$ for every $a \in G$, then $\tau_{rr} = \bar{\tau}_{rr}$.

Proof : If $w_{rr}(a) = 1$ for every $a \in G$, then μ_{rr} is bijective.

So F is closed w.r.to $\tau_{rr} \Longleftrightarrow \mu_{rr}(F) = F \Longleftrightarrow \nu_{rr}(F) = \mu_{rr}^{-1}(F) = F \Longleftrightarrow$ F is closed w.r.to $\bar{\tau}_{rr}$. Therefore $\tau_{rr} = \bar{\tau}_{rr}$.

Remark 6 : Proposition 4 may not be true if G be an infinite quasigroup as shown by

Example 5 : Consider the quasigroup (R, ϕ) where R is the set of all rational numbers and ϕ is defined by the relation: $\phi(a, b) = -a + b$ for every $a, b \in R$. For this quasigroup $w_{rr}(a) = 1$ for every $a \in R$. $\{\bar{1}\}_{rr} = \{1, 3, 3^2, \ldots\}$ is closed w.r.to τ_{rr} . But since $(\{1\})\,\mu_{rr}^{-1} = \frac{1}{3} \in \{\bar{1}\}_{rr}$ it is not closed w.r.to $\bar{\tau}_{rr}$. So $\tau_{rr} \neq \bar{\tau}_{rr}$.

Proposition 5 : If α is a homomorphism of G into a quasigroup G', then α is a continuous mapping of the topological space (G, τ_{rr}) into the topological space (G', τ'_{rr}) where τ'_{rr} is the right - right inverse fore - topology in G'.

Proof : Let F be any closed set w.r.to τ'_{rr} . Now, $x \in \alpha^{-1}(F) \Longrightarrow \alpha(x) \in F \Longrightarrow [\alpha(x)]_{rr} = \alpha(x_{rr})$ (since α is a homomorphism) $\in F$ (since F is closed) $\Longrightarrow x_{rr} \in \alpha^{-1}(F) \Longrightarrow \mu_{rr}(\alpha^{-1}(F)) \subset \alpha^{-1}(F) \Longrightarrow \alpha^{-1}(F)$ is closed w.r.to τ_{rr} . So α is continuous.

Remark 7 : The converse of the above result is not true as shown by

Example 6 : Let G be the quasigroup defined in Example 3 and let G' be the quasigroup given by the multiplication table

	a	b	c
a	c	a	b
b	b	c	a
c	a	b	c

Then τ_{rr} = discrete topology in {a, b, c} and the closed sets of τ'_{rr} are ϕ, {a, b, c} , {a, c} , {b, c} , {c} The identity mapping of the set {a, b, c} is a continuous mapping of (G, τ_{rr}) onto (G', τ'_{rr}). But it is not a homomorphism of G onto G'.

Definition 1 : A topology τ defined in a quasigroup G is said to be a quasigroup topology if product, right division and left division in G are continuous w.r.to τ .

Theorem 2 : A necessary and sufficient condition that τ_{rr} may be a quasigroup topology in G is that for every x, y ε G and p, q ε N, there exist k, m, n ε N such that

$$x_{(rr,p)} \cdot y_{(rr,q)} = (x.y)_{(rr,k)} \qquad \dots (4)$$

$$x_{(rr,p)} / y_{(rr,q)} = (x/y)_{(rr,m)} \qquad \dots (5)$$

$$x_{(rr,p)} \setminus y_{(rr,q)} = (x\backslash y)_{(rr,n)} \qquad \dots (6)$$

Proof : The conditions are equivalent to the following conditions:

$$\overline{\{x\}}_{rr} \cdot \overline{\{y\}}_{rr} \subset \overline{\{x.y\}}_{rr} \qquad \dots (7)$$

$$\overline{\{x\}}_{rr} / \overline{\{y\}}_{rr} \subset \overline{\{x/y\}}_{rr} \qquad \dots (8)$$

$$\overline{\{x\}}_{rr} \setminus \overline{\{y\}}_{rr} \subset \overline{\{x\backslash y\}}_{rr} \qquad \dots (9)$$

for every x, y ε G .

If τ_{rr} be a quasigroup topology in G, then since product, right division and left division are continuous w.r.to τ_{rr}, (7), (8), (9) hold.

Conversely let the given conditions be satisfied. Then (7), (8), (9) hold. Let A be any subset of G x G. If the operation of multiplication in G be denoted by 'φ', then

$$\begin{aligned}\varphi(\bar{A}_{rr}) &= \bigcup \{\{\bar{x}\}_{rr} \cdot \{y\}_{rr} \; ; \; (x, y) \in A\} \\ &\subset \bigcup \{\overline{\{x.y\}}_{rr} \; ; \; x.y \in \varphi(A)\} \\ &= \overline{(\varphi(A))}_{rr} .\end{aligned}$$

So φ is continuous w.r.to τ_{rr}. Similarly one can verify the continuity of right division and left division in G w.r.to τ_{rr}. Therefore τ_{rr} is a quasigroup topology in G .

<u>Theorem 3</u> : $\bar{\tau}_{rr}$ is a quasigroup topology in G if, and only if for every x, y $\in$ G and p, q $\in$ N such that $\nu^p_{rr}(x) \neq \phi$, $\nu^q_{rr}(y) \neq \phi$ there exist k, m, n $\in$ N such that

$$\nu^p_{rr}(x) \cdot \nu^q_{rr}(y) \subset \nu^k_{rr}(x.y) ,$$

$$\nu^p_{rr}(x) \; / \; \nu^q_{rr}(y) \subset \nu^m_{rr}(x/y) ,$$

$$\nu^p_{rr}(x) \setminus \nu^q_{rr}(y) \subset \nu^n_{rr}(x \backslash y) .$$

The proof is similar to that of Theorem 2 .

<u>Remark 8</u> : There are analogous results for the topologies τ_r, τ_L, τ_{rL}, τ_{Lr}, τ_{LL}, $\bar{\tau}_r$, $\bar{\tau}_L$, $\bar{\tau}_{rL}$, $\bar{\tau}_{Lr}$, $\bar{\tau}_{LL}$.

<u>Proposition 6</u> : τ_r, $\bar{\tau}_r$ (resp. τ_L, $\bar{\tau}_L$) are not quasigroup topologies if G be a right (resp. left) loop. Moreover, if there exists at least one element a in G such that $a^2 \neq e$ where e is the right (resp. left) identity in G, then τ_{rr}, τ_{rL}, $\bar{\tau}_{rr}$, $\bar{\tau}_{rL}$ (resp. τ_{Lr}, τ_{LL}, $\bar{\tau}_{Lr}$, $\bar{\tau}_{LL}$) are not quasigroup topologies in G .

<u>Proof</u> : Let G be a right loop with the right identity e .

Let x,y ($\neq$ e) $\in$ G. Then $x_{(r,o)} \cdot y_{(r,1)} = x.e = x \neq xy$ or $(x.y)_{(r,1)} = e$. So τ_r is not a quasigroup topology in G .

Next let $x\ (\neq e) \in G$. Then $(e)\ \nu_r^1 . (x)\ \nu_r^o = G.x = G \not\subset e.x$ or $(e.x)\ \nu_r^1 = \phi$. So $\bar{\tau}_r$ is not a quasigroup topology in G.

To show that τ_{rr}, $\bar{\tau}_{rr}$ are not quasigroup topòlogies, let $a \in G$ be such that $a^2 \neq e$. Then $a_{rr} \neq a$ and $a_{rL} \neq a$. Now let $x = a$ and $y = a_{rr}$. Then $x_{(rr,1)} . y_{(rr,o)} = a_{rr} . a_{rr} \neq e = (x.y)_{(rr,n)}$ for every $n \in N$. So τ_{rr} is not a quasigroup topology in G. Next let $x = a_{rL}$, $y = a$. Then $(x)\ \nu_{rr}^o . (y)\ \nu_{rr}^1 = a_{rr} . a_{rr} \neq e = (x.y)\ \nu_{rr}^n$ for every $n \in N$. So $\bar{\tau}_{rr}$ is not a quasigroup topology in G.

Similarly one can show that τ_{rL}, $\bar{\tau}_{rL}$ are not quasigroup topologies in G.

The case of a left loop can be similarly disposed of.

<u>Corollary</u> : If G be a loop, then τ_r, τ_L, $\bar{\tau}_r$, $\bar{\tau}_L$ are not quasigroup topologies in G. Moreover if there exists an element in G, the square of which is not the identity element, then $\bar{\tau}_{rr}$, $\bar{\tau}_{rL}$, $\bar{\tau}_{Lr}$, $\bar{\tau}_{LL}$, τ_{rr}, τ_{rL}, τ_{Lr}, τ_{LL} are not quasigroup topologies in G.

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On Fixed Point Theorems for Mapping on a 2-Metric Space Involving Four Points of the Space

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The results on fixed point for mappings on a complete bounded 2-metric space involving four points are studied here. The results obtained here generalize those of Achari [1,2,3], Patil and Achari [12], Pittnauer [13].

I. Introduction

Pittnauer [13] has studied fixed point theorems for a mapping involving four points of the space and recently Achari [3] has extended this idea further to establish some fixed point theorems. Further, Som and Mukherjee [16] extend their results using a mapping which is not necessarily continuous and satisfies a generalized condition involving five or six points.

The notion of 2-metric space was introduced by Gahler in his series of papers [6] to [9]. For further literature we refer to White [17], Ehret [5], Diminie et al. [4], Iseki et al. [10], Khan [11], Singh [14] and Singh and Ram [15].

The aim of this note is to establish some results on fixed point for generalized contraction type mapping on a 2-metric space involving four points. The results obtained unify the existence of fixed point in the space.

Before stating our results we give some definitions:

Definition 1. A 2-metric space is a space X in which for each triple points a,b,c, there exists a real function $d(a,b,c)$ such that

(1a) to each pair of points a,b $(\neq b)$ from X, there is $c \in X$ satisfying $d(a,b,c) \neq 0$,

(1b) $d(a,b,c) = 0$ only when at least two of three points are equal,

(2) $d(a,b,c) = d(a,c,b) = d(b,c,a)$

(3) $d(a,b,c) \leq d(a,b,e) + d(a,e,c) + d(e,b,c)$.

A 2-metric space is called bounded, if there exists a constant M such that $d(a,b,c) \leq M$ for $a,b,c \in X$.

Definition 2. A sequence $\{x_n\}$ in X is called Cauchy if lim $d(x_m,x_n,a) = 0$ for all $a \in X$.

Definition 3. A sequence $\{x_n\}$ in X is convergent to $x \in X$ if $\lim d(x_n,x,a) = 0$ for each $a \in X$.

Here the point x is called the limit of $\{x_n\}$.

Definition 4. A 2-metric space in which every Cauchy sequence converges is called a complete 2-metric space.

Then we give below our main theorems.

Theorem 1. Let X be a compact,bounded 2-metric space. Let $f:X \to X$ satisfy the following condition

$$\text{(A)} \quad d(fu_1, fu_2, a) \leq p_1\, d(u_1,u_2,a) + p_2\, d(u_1,fu_3,a) + p_3\, d(u_2,fu_4,a) + p_4\, d(u_1,fu_4,a) + p_5\, d(u_2,fu_3,a) + p_6\, d(fu_3,fu_4,a)$$

for all $u_1,u_2,u_3,u_4 \in X$ and for each $a \in X$ where $0 \leq p_i < 1$, $i = 1,2,3,4,5,6$ such that $\Sigma\, p_i < 1$.

Then f has a unique fixed point.

Pfoof. Let $x,y \in X$. Define $u_1 = fx$, $u_2 = fy$, $u_3 = x$, $u_4 = y$. Then the expression (A) becomes

$$\text{(B)} \quad d(f^2x,f^2y,a) \leq (p_1+p_4+p_5+p_6)\, d(fx,fy,a)$$

Let x_o be an arbitrary. Putting $fx_{n-2} = x_{n-1}$, $fx_{n-1} = x_n$, $fx_n = x_{n+1}$, $n = 1,2....$ and writing x_{n-1} for x and x_{n-2} for y in (B) we have

(C) $\quad d(x_{n+1},x_n,a) \leq p\, d(x_n,x_{n-1},a)$

where $p = p_1 + p_4 + p_5 + p_6$

From (C) we have for $i < j$

$$d(x_i,x_j,a) \leq d(x_i,x_{i+1},x_j) + d(x_{i+1},x_{i+2},x_j)$$

$$+ \ldots\ldots + d(x_{j-2},x_{j-1},x_j) + d(x_i,x_{i+1},a)$$

$$+ d(x_{i+1},x_{i+2},a) + \ldots\ldots + d(x_{j-1},x_j,a)$$

$$\leq (p^i+p^{i+1} + \ldots +p^{j-2})\, d(x_o,x_1,x_j)$$

$$+ (p^i+p^{i+1} + \ldots +p^{j-1})\, d(x_o,x_1,a)$$

i.e. (D) $\quad d(x_i,x_j,a) \leq 2(p^i+p^{i+1} + \ldots. + p^{j-1})\, M$ since X is bounded where M is a constant.

As $i,j \to \infty$, $d(x_i,x_j,a) \to 0$ which shows that $\{x_n\}$ is Cauchy. Since X is complete, there exists a point $u \in X$ such that

(E) $\quad \lim x_n = u$ as $n \to \infty$.

Now we claim that

(F) $\quad fu = u$.

Putting $u_1 = u_3 = u$, $u_2 = x_{n-1}$, $u_4 = x_{n-2}$ in (A) we have

$$d(fu,x_n,a) \leq (p_1 + p_4)\, d(u_1,\, x_{n-1},a) + p_2\, d(u,fu,a)$$

$$+ (p_5 + p_6)\, d(x_{n-1},\, fu,\, a)$$

As $n \to \infty$, we have

$$(1 - p_2 - p_5 - p_6)\, d(fu,\, u,\, a) \leq 0$$

Therefore $u = fu$.

If possible, let $v(\neq u)$ be another fixed point of f. Then from (A) we get (letting $u_1 = u_3 = u$, $u_2 = u_4 = v$)

$$d(fu, fv, a) \le p_1 d(u, v, a) + p_2 d(u, fu, a) + p_3 d(v, fv, a)$$
$$+ p_4 d(u, fv, a) + p_5 d(v, fu, a) + p_6 d(fu, fv, a)$$

$$\therefore \quad (1-p_1-p_4-p_5-p_6)\, d(u, v, a) \le 0$$

$$\therefore \quad u = v$$

Hence uniqueness. This completes the proof.

Corollary 1. If f and g be mappings from a complete, bounded 2-metric space X into itself, satisfying the following condition

$$\text{(G)} \quad d(fu_1,gu_2,a) \le p_1\, d(u_1,u_2,a) + p_2\, d(u_1,fu_3,a)$$
$$+ p_3\, d(u_2,gu_4,a) + p_4\, d(u_1,gu_4,a)$$
$$+ p_5\, d(u_2,fu_3,a) + d(fu_3,gu_4,a)$$

for all $u_1,u_2,u_3,u_4 \in X$ and for each $a \in X$ where $0 \le p_i < 1$, $i = 1,2,3,4,5,6$ such that $\Sigma\, p_i < 1$.

Then f and g have a unique common fixed point.

Corollary 2. Considering a complete 2-metric space and making $p_6 = 0$, the above corollary reduces to the theorem of Patil and Achari [12] as a second corollary.

Theorem 2. Let X be a complete, bounded 2-metric space. Let $f : X \to X$ satisfy the following condition

$$\text{(H)} \quad d(f^2u_1,f^2u_2,a) \le p_1\, d(fu_1,fu_2,a) + p_2\, d(fu_1,f^2u_3,a)$$
$$+ p_3\, d(fu_2,f^2u_4,a) + p_4\, d(fu_1,f^2u_4,a)$$
$$+ p_5\, d(fu_2,f^2u_3,a) + p_6\, d(f^2u_3,f^2u_4,a)$$

for all $u_1, u_2, u_3, u_4 \in X$ and for each $a \in X$ where $0 \leq p_i < 1$, $i = 1,2,3,4,5,6$ such that $\Sigma\, p_i < 1$.

Then f has a unique fixed point.

Its proof follows from that of Theorem I.

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