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Contents

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On the Radius of Starlikeness of Schlicht Functions

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Abstract. In this paper, we find the region in which the function $f(z) = z + \sum_{p=1}^{\infty} a_{n+p} z^{n+p}$, such that $F(z) = \frac{c+\alpha}{z^{c-\alpha+1}} \int_0^z t^{c-\alpha} \{f(t)\}^{\alpha} dt$, is a starlike function of order β where $F(z)$ is a member of $S^*_{\alpha\beta}$. This result is sharp and contains the theorems of Livingston [6], Bernardi [4], Bajpai and Srivastava [3], Singh [2] and Rizvi [1] as special cases.

1. Let S denote the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic and univalent in the unit disc $D = \{|z| < 1\}$, while S^* denote the class of functions in S which map D onto a starlike region with respect to the origin. Let S^*_{β} denote the class of functions $f(z)$ in S^* having the additional property

$$\mathrm{Re}\,\{\frac{z\,f'(z)}{f(z)}\} \geq \beta;\ 0 \leq \beta \leq 1.$$

Here β is referred as the order of starlike function $f(z)$ and we identify $S^*_o = S^*$. Livingston [6] showed that if F is in S^*_o then $f(z) = \frac{1}{2}[z\,F(z)]'$ is univalent and starlike for $|z| < \frac{1}{2}$. This was generalised by Bernardi [4] who proved that if F is in S^*_o, then $f(z) = \frac{z^{1-c}}{(c+1)}[z^c F(z)]'$ is starlike for $|z| < \frac{-2+(3+c^2)^{1/2}}{(c-1)}$ where $c=2,3,4,\ldots$, and $|z| < \frac{1}{2}$ for C=1. This result was further extended by Bajpai and Srivastava [3] to cover the case when F is in S^*_{β}. The result of Bajpai and Srivastava has been further generalised by Rizvi [1].

Let $f(z) = z + \sum_{p=1}^{\infty} a_{n+p} z^{n+p}$ be regular and univalent in D. Singh [2] further extended the earlier results. In this paper, we find the region in which the function $f(z)$, such that $F(z) = \frac{c+\alpha}{z^{c-\alpha+1}} \int_0^z t^{c-\alpha} \{f(t)\}^{\alpha} dt$, is a starlike function of order β where $F(z)$ is a member of $S^*_{\alpha\beta}$. Our theorems further extend the earlier works of Livingston [6], Bernardi [4], Bajpai and Srivastava [3], Singh [2] and Rizvi [1].

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2. Theorem 1. If $f(z) = z + \sum_{p=1}^{\infty} a_{n+p}\, z^{n+p} \in S^*_\beta$

(2.1) and $F(z) = \dfrac{c+\alpha}{z^{c-\alpha+1}} \int_0^z t^{c-\alpha} \{f(t)\}^\alpha dt$, then $F(z) \in S^*_{\alpha\beta}$,

where $c = 1,2,3,\ldots$, $0 \le \alpha \le 1$ and $0 \le \beta \le 1$.

Proof Let $J(z) = \int_0^z t^{c-\alpha} \{f(t)\}^\alpha \, dt$.

Then

$$J'(z) = z^{c-\alpha} \{f(z)\}^\alpha$$

and

(2.3) $$F(z) = \frac{c+\alpha}{z^{c-\alpha+1}} J(z)$$

Now

$$F'(z) = (c+\alpha) \left[\frac{J'(z)}{z^{c-\alpha+1}} - \frac{c-\alpha+1}{z^{c-\alpha+2}} J(z)\right]$$

$$= (c+\alpha) \left[\frac{z^{c-\alpha}\{f(z)\}^\alpha}{z^{c-\alpha+1}} - \frac{(c-\alpha+1)}{z^{c-\alpha+2}} J(z)\right]$$

Hence,

$$z^{c-\alpha+2} F'(z) = (c+\alpha) \left[z^{c-\alpha+1} \{f(z)\}^\alpha - (c-\alpha+1) J(z)\right]$$

Now,

$$\left[z^{c-\alpha+2} F'(z)\right]' = (c+\alpha) \left[(c-\alpha+1)\, z^{c-\alpha} \{f(z)\}^\alpha + \right.$$

$$\left. + z^{c-\alpha+1} \alpha \{f(z)\}^{\alpha-1} f'(z) - (c-\alpha+1) J'(z)\right]$$

$$= (c+\alpha)\alpha\, z^{c-\alpha+1} \{f(z)\}^{\alpha-1} f'(z)$$

Hence,

$$\frac{\left[z^{c-\alpha+2} F'(z)\right]'}{J'(z)} = (c+\alpha)\alpha \; \frac{zf'(z)}{f(z)}$$

Therefore using (1.1), we get

(2.3) $$\text{Re}\, \frac{(z^{c-\alpha+2} F'(z))'}{J'(z)} = (c+\alpha)\alpha \;\; \text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \ge (c+\alpha)\alpha\beta$$

Therefore from ([5], p. 431) and (2.3) we get

$$\text{Re}\ \{\frac{z^{c-\alpha+2}\,F'(z)}{J(z)}\} \geq (c+\alpha)\alpha\beta$$

Hence,

$$\text{Re}\ \{\frac{zF'(z)}{F(z)}\} \geq \alpha\beta$$

which gives $F(z) \in S^*_{\alpha\beta}$.

3. <u>Theorem 2</u>. If $f(z) = z + \sum_{p=1}^{\infty} a_{n+p}\ z^{n+p}$ and

$F(z) = \frac{c+\alpha}{z^{c-\alpha+1}} \int_0^z t^{c-\alpha}\ \{f(t)\}^{\alpha}\ dt$ such that $F(z) \in S^*_{\alpha\beta}$, then $f(z) \in S^*_{\beta}$ for $|z| < r_o$, where $r_o = \min\ (r_1, r_2)$, r_1 and r_2 are the smallest Positive roots of the equations

$$(3.1) \qquad (c+2\alpha\beta-\alpha)r^{2n} + 2(n-1-\alpha\beta)r^n - (c-\alpha+2) = 0$$

and

$$(3.2) \qquad 2r^2-(1-r^2)[r^nP(r)+n-1]=0,\ P(r)=[\frac{(c-\alpha+2)+(c+2\alpha\beta-\alpha)r^n}{1+r^n}]$$

respectively. This result is sharp if $r_1 \leq r_2$.

<u>Proof</u>. By the hypothesis of Theorem 1, we have

$$(3.3) \qquad \frac{zF'(z)}{F(z)} = \frac{zJ'(z) - (c-\alpha+1)J(z)}{J(z)}$$

Further, since $F(z)$ is a starlike function of order $\alpha\beta$, so there exists a function $\omega(z)$ which is regular in the unit and satisfies conditions of [(2), Lemma 1] such that

$$(3.4) \qquad \frac{zF'(z)}{F(z)} = \frac{1-(1-2\alpha\beta)z^{n-1}\omega(z)}{1 + z^{n-1}\ \omega(z)}$$

From (3.3) and (3.4) it follows that

$$\{f(z)\}^{\alpha} = \frac{(c-\alpha+2)+\{(c-\alpha+2\alpha\beta)z^{n-1}\ \omega(z)\}\ J(z)}{[1+z^{n-1}\ \omega(z)]z^{c-\alpha+1}}$$

Differentiating logarithmically, we get

$$\frac{\alpha z f'(z)}{f(z)} = \frac{zJ'(z)}{J(z)} - (c-\alpha+1) + \frac{2z(\alpha\beta-1)[z^{n-1}\omega(z)]'}{[1+z^{n-1}\omega(z)][(c-\alpha+2)+(c-\alpha+2\alpha\beta)z^{n-1}\omega(z)]}$$

Making use of (3.4), we get

$$(3.5)\quad \frac{\alpha z f'(z)}{f(z)} - \alpha\beta = (1-\alpha\beta)\Big[\frac{1-z^{n-1}\,\omega(z)}{1+z^{n-1}\,\omega(z)} - \frac{2z[z^{n-1}\,\omega(z)]'}{\{1+z^{n-1}\omega(z)\}\,\{(c-\alpha+2)+(c-\alpha+2\alpha\beta)z^{n-1}\omega(z)\}}\Big]$$

But

$$(3.6)\quad \mathrm{Re}\,\Big\{\frac{1-z^{n-1}\,(z)}{1+z^{n-1}\,\omega(z)}\Big\} = \frac{1-|z|^{2n-2}|\omega(z)|^2}{|1+z^{n-1}\,\omega(z)|^2}$$

and

$$(3.7)\quad \mathrm{Re}\,\Big[\frac{2z\{z^{n-1}\,\omega(z)\}'}{\{1+z^{n-1}\,\omega(z)\}\,\{(c-\alpha+2)+(c-\alpha+2\alpha\beta)z^{n-1}\,\omega(z)\}}\Big]$$

$$= \mathrm{Re}\,\Big[\frac{2z^{n}\omega'(z)}{\{1+z^{n-1}\,\omega(z)\}\{(c-\alpha+2)+(c-\alpha+2\alpha\beta)z^{n-1}\,\omega(z)\}}\Big]$$

$$+ \mathrm{Re}\,\Big[\frac{2(n-1)z^{n-1}\,\omega(z)}{\{1+z^{n-1}\,\omega(z)\}\,\{(c-\alpha+2)+(c-\alpha+2\alpha\beta)z^{n-1}\,\omega(z)\}}\Big]$$

Now

$$\mathrm{Re}\,\Big[\frac{2z^{n}\omega'(z)}{\{1+z^{n-1}\omega(z)\}\,\{(c-\alpha+2)+(c-\alpha+2\alpha\beta)z^{n-1}\omega(z)\}}\Big]$$

$$\leq \frac{2|z|^{n}|\omega(z)|}{|1+z^{n-1}\omega(z)|\,\,|(c-\alpha+2)+(c-\alpha+2\alpha\beta)z^{n-1}\omega(z)|}$$

From a well known lemma ([7] p. 168), we have

$$(3.8)\quad |\omega'(z)| \leq \frac{1-|\omega(z)|^2}{1-|z|^2}$$

Hence,

$$(3.9)\quad \mathrm{Re}\,\Big[\frac{2z^{n}\omega'(z)}{\{1+z^{n-1}\omega(z)\}\,\{(c-\alpha+2)-(c-\alpha+2\alpha\beta)z^{n-1}\omega(z)\}}\Big]$$

$$\leq \frac{2|z|^n \ (1-|\omega(z)|^2)}{(1-|z|^2) \ |1+z^{n-1}\omega(z)| \ |(c-\alpha+2)+(c-\alpha+2\alpha\beta)z^{n-1}\omega(z)|}$$

and

$$(3.10) \quad \text{Re} \left[\frac{2(n-1) \ z^{n-1}\omega(z)}{\{1+z^{n-1}\omega(z)\} \ \{(c-\alpha+2)+(c-\alpha+2\alpha\beta)z^{n-1}\omega(z)\}}\right]$$

$$\leq \frac{2(n-1) \ |z|^{n-1}|\omega(z)|}{|1+z^{n-1}\omega(z)| \ |(c-\alpha+2)+(c-\alpha+2\alpha\beta) \ z^{n-1}\omega(z)|}$$

From (3.6), (3.9), (3.10) and (3.5), $f(z)$ is starlike of order β, if

$$(3.11) \quad \frac{2|z|^n \ (1-|\omega(z)|^2)}{1 - |z|^2} + 2(n-1) \ |z|^{n-1}|\omega(z)|$$

$$< \frac{(1-|z|^{2n-2}|\omega(z)|^2) \ |(c-\alpha+2)+(c-\alpha+2\alpha\beta)z^{n-1}\omega(z)|}{|1+z^{n-1}\omega(z)|}$$

Now, since $|\omega(z)| < |z|$ and $\frac{(c-\alpha+2\alpha\beta)}{(c-\alpha+2)} \leq 1$,

we have

$$(3.12) \quad \frac{|(c-\alpha+2)+(c-\alpha+2\alpha\beta)z^{n-1}\omega(z)|}{|1+z^{n-1}\omega(z)|} \geq \frac{(c-\alpha+2)+(c-\alpha+2\alpha\beta) \ |z|^n}{1 + |z|^n}$$

From (3.11) and (3.12) we obtain that $f(z) \in S^*_\beta$ if

$$\frac{2|z|^n \ (1-|\omega(z)|^2)}{1 - |z|^2} + 2(n-1) \ |z|^{n-1}|\omega(z)|$$

$$< (1-|z|^{2n-2}|\omega(z)|^2) \ \left\{\frac{(c-\alpha+2)+(c-\alpha+2\alpha\beta) \ |z|^n}{1 + |z|^n}\right\}.$$

Putting $|z| = r$, $|\omega(z)| = t$ and $P(r) = \left\{\frac{(c-\alpha+2)+(c-\alpha+2\alpha\beta)r^n}{1 + r^n}\right\}$

we get

$$(3.13) \quad \frac{2r^n(1-t^2)}{1-r^2} + 2(n-1)r^{n-1}t - (1-r^{2n-2}t^2) \ P(r) < 0$$

Let $q(t) = \frac{2r^n(1-t^2)}{1-r^2} + 2(n-1)r^{n-1}t - (1-r^{2n-2}t^2)\,P(r)$,

$$q'(t) = -\left[\frac{4r^n}{1-r^2} - 2r^{2n-2}\,P(r)\right]t + 2(n-1)r^{n-1}$$

Since $0 \le t \le r$, $q'(t) > 0$ if

$$\frac{2r^2}{1-r^2} - r^n\,P(r) - (n-1) < 0$$

It is satisfied if $r < r_2$ where r_2 is the smallest positive root of

$$2r^2 - [(n-1) + r^n\,P(r)]\,(1-r^2) = 0 .$$

Hence, if $r < r_2$ the maximum of $q(t)$ is attained for $t = r$. Now (3.13) holds if

$$2nr^n - (1-r^n)\,\{(c-\alpha+2) + (c-\alpha+2\alpha\beta)r^n\} < 0$$

or,

$$(c-\alpha+2\alpha\beta)r^{2n} + 2(1-\alpha\beta+n)r^n - (c-\alpha+2) < 0$$

It holds if $r < r_1$ where r_1 is the smallest positive root of

$$(c-\alpha+2\alpha\beta)r^{2n} - 2(1+n-\alpha\beta)r^n - (c-\alpha+2) = 0.$$

If $r_1 \le r_2$, sharpness of the result is shown by the function $F(z) = z(1+z^n)^{-2(1-\alpha\beta)/n} \in S^*_{\alpha\beta}$, for the corresponding function $f(z)$ we get,

$$\frac{\alpha z f'(z)}{f(z)} - \alpha\beta = (1-\alpha\beta)\left[\frac{(c-\alpha+2)-2(1-\alpha\beta+n)z^n - z^{2n}\,(c-\alpha+2\alpha\beta)}{(1+z^n)\,[(c-\alpha+2) + z^n\,(c-\alpha+2\alpha\beta)]}\right] .$$

Clearly $\frac{zf'(z)}{f(z)} - \beta = 0$ for $z = r_o$, hence, $f(z)$ cannot be starlike of order β in any circle of radius greater than r_o.

As a corollary of Theorem 2, we get the following Theorem of Rizvi [1] which includes a result of Bajpai and Srivastava [3] for $\alpha = 1$ and Bernardi [4] for $\alpha = 1$, $\beta = 0$.

Corollary 1 [Rizvi]. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and

$F(z) = \frac{c+\alpha}{z^{c-\alpha+1}} \int_0^z t^{c-\alpha} \{f(t)\}^{\alpha} dt$, where $c = 1,2,3\ldots$, $0 < \alpha < 1$ and if $F(z) \in S^*_{\alpha\beta}$ then $f(z)$ is starlike of order β in the region

$$|z| < r_o = \frac{-(2-\alpha\beta)+ \{3+\alpha^2\beta^2+ (c-\alpha+1)^2 + 2(c-\alpha+1)\ \alpha\beta - 2\alpha\beta\ \}^{1/2}}{(c-\alpha+2\alpha\beta)} \quad \text{if } c = 2,3,\ldots$$

$$= \frac{1}{2}, \text{ if } c = 1, \text{ and } \beta = 0$$

$$= \frac{-(2-\alpha\beta) + (4 + \alpha^2\beta^2)^{1/2}}{2\alpha\beta}, \text{ if } c = 1 \text{ and } 0 < \beta < 1, \alpha \neq 0.$$

This result is obtained by taking $n = 1$ in equation (3.1) and (3.2). These equations become identical, as can be easily verified, thus showing $r_o = r_1 = r_2$.

By choosing $\alpha = 1$ in Theorem 2 gives the following result of Singh [2].

Corollary 2 [Singh]. If $F(z) = z + \sum_{p=1}^{\infty} a_{n+p} z^{n+p} \in S^*_{\beta}$ then $f(z) = \frac{1}{(c+1)z^{c-1}} [z^c F(z)]'$ is univalent and starlike of order β for $|z| < r_0$ where $r_o = \min(r_1, r_2)$, r_1 and r_2 are the smallest positive root of the equations $(c+2\beta-1)\ r^{2n} - 2(n+1-\beta)r^n - (1+c) = 0$

$$2r^n - (1-r^2)\ [r^n P(r) + (n-1)] = 0,\ P(r) = \left[\frac{(1+c) + (c+2\beta-1)r^n}{1+r^n}\right]$$

respectively. This result is sharp if $r_1 \leq r_2$.

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Maximum Likelihood Estimation in Branching Evolutionary Markov Processes in Discrete-Time

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Abstract

The individuals in a certain population reproduce according to a Galton-Watson (GW) branching process and disperse spatially among states of a finite discrete Markov process. When the dispersals occur according to a Markov chain or an autoregressive chain, the process is observed for n generations and the dispersals of the individuals are observed at the time of branching. The maximum likelihood estimates (MLE) of the parameters of the dispersal component are derived and the asymptotic properties of these estimates as the number of generations increases to infinity are discussed.

1. Introduction

Dharmadhikari (1982) has studied the inferential aspects of a few branching differential processes where the particles grow according to linear birth or birth and death process and the dispersal processes have stationary independent increments. Shrestha (1984) studied the problem of estimation of the parameters of the dispersal components of some branching evolutionary Markov (BEM) processes where the particles grow according to some branching laws but have Markov dependent dispersals. In this paper, we shall study two discrete BEM processes where for each process the numerical increase of the population of individuals (particles) is described by a GW branching process while the dispersals of the particles follow a discrete Markov chain (MC) for the first process and an autoregressive chain for the other process. Inference in BEM processes where the dispersals occur according to semi-Markov chain and compound geometric distribution may be found in Shrestha (1984).

Consider the GM branching process whose n consecutive generation sizes are $\{\xi_0, \xi_1, \dots \xi_n\}$. Let the offspring distribution be given by

$\{a_k, k = 0,1,2,\ldots\}$, then $a_k = P\{\xi_j = k\}$ for all $j = 1,2,\ldots$ We assume that at the zeroth generation there is only one ancestral particle. Let p denote the probability of branching of a particle. The probability of k offsprings at the first generation is

$$(1.1) \qquad \begin{aligned} b_k &= a_k \cdot p & k \neq 1 \\ &= a_k \cdot p + (1-p), & k = 1. \end{aligned}$$

Since the probability of branching of a particle does not depend on the generation time or the generation number, we may assume that the offspring distribution is given by $\{b_k, k = 1,2,\ldots\}$ where b_k is given by (1.1) and b_k's are known.

The problem of estimation of parameters of the dispersal component of a discrete BEM process may be studied by exploiting the underlying structure of independent and identically distributed offspring sizes of the successive generations. Also the numerical increase in the population represented by the GW branching process and the spatial distribution of the particles are governed by independent probability laws. Because of this assumption of independence of the two components, parameters underlying the former can be studied independently of the parameters underlying the latter. We also assume that the life span of all the particles are same e.g., the life span of some organism, bacterium, neutron etc. At the end of the life span, each one of the particles produces progeny in accordance with the probability laws of the particles of the previous generation. The offspring particle occupies the same state occupied by the parent at the time of its birth. The particles simultaneously move about some appropriate state space to be defined later on for each dispersal process.

In the next Section 2, we define the GW branching MC and derive the likelihood function for the process. We obtain the MLE of the parameters and show that the MLE's are strongly consistent and asymptotically normal. In Section 3, similar analysis is carried out for the GW branching autoregressive chain.

The MLE's of the parameters of the GW branching process have been studied by Dion (1972). Since her results are applicable to the branching component of the present process, we shall not discuss them. We assume that they are known. Because of the assumption of the independence of the branching and dispersal components, this additional assumption will not affect the inference problem concerning the parameter of the dispersal component of the observed processes.

2. GW Branching Markov Chain

A GW branching MC describes a situation wherein the numerical change in the population is described by a GW branching process and each one of the particles in the population disperses simultaneously and independently over the countable Section I of integers according to a discrete MC. Such a process is defined by an offspring distribution $\{b_k, k = 1,2,\ldots\}$ of the GW branching process and the transition probability matrix $P = ((p(a,b)))$, $a,b \in I$ of the MC. It is assumed that the MC has no instantaneous state and that it is ergodic with $\{\pi_i\}$ as the stationary distribution, $\pi_i > 0$ for all i. We also assume that $b_0 = 0$ so that the population does not become extinct. The branching process is observed for n generations and the dispersals of the particles are observed only at the times of branching.

Suppose the process is observed till the nth generation. Let ξ_i denote the number of particles in the ith generation. Let $m_n(a,b) = \sum_{i=1}^{n} \sum_{j=1}^{\xi_{i-1}} \delta_{ij}(a,b)$ denote the total number of transitions from state a to state b during the n generations, and where $\delta_{ij} = 1$ or 0 according as whether or not the stated transition occurs. Since the particles move independently of each other, the likelihood function describing the dispersals of the particles may be described as

$$(2.1) \qquad L_n(P) \propto \prod_{a,b\in I} \{p(a,b)^{m_n(a,b)}\}.$$

Thus was see that $\{m_n(a,b), a,b\in I\}$ is a sufficient statistics for P. The parameter space is given by $\{0 < p(a,b) < 1, \Sigma\, p(a,b) = 1, a,b\in I\}$. The MLE of p(a,b) is given by

(2.2) $\hat{p}_n(a,b) = \{m_n(a)\}^{-1} \quad m_n(a,b)$,

where $m_n(a) = \sum_{b\in I} m_n(a,b)$. Let $m_n = \sum_{a\in I} m_n(a) = N_{n-1}$. Since there is no extinction, $n\to\infty$ implies $m_n\to\infty$. This is so because there are no instantaneous states in the MC and P $\{m_n(a,b) > 0,\ a,b \in I\}\to 1$ as $n\to\infty$. This also implies that for each $a \in I$, $m_n(a)\to\infty$ a.s., because as $n\to\infty$,

(2.3) $(m_n)^{-1}\ m_n(a) \to \pi(a) > 0.$

For given $m_n(a)$, $\{m_n(a,b),\ b \in I\}$ is multinomial with index $m_n(a)$ and parameters $\{p(a,b),\ b \in I\}$. Hence $\{m_n(a,b) - p(a,b)\ m_n(a)\}$ is a zero mean martingale sequence. Therefore as $n \to \infty$,

(2.4) $\hat{p}_n(a,b) \to p\ (a,b)$ a.s.

thereby proving the strong consistency of the MLE's. Also using the central limit theorem for random sums, we have

(2.5) $m_n(a)^{\frac{1}{2}}\ (\hat{p}_n(a,b)-p(a,b)) \to N(0,p(a,b)(1-p(a,b))) = Z_1$ (say)

where $\to$ denotes the convergence in distribution. This establishes the asymptotic normality of the MLE's. Further the joint distribution of $\{m_n(a)^{\frac{1}{2}}\ (\hat{p}_n(a,b) - p(a,b)),\ a,b \in I\}$ is asymptotically multivariate normal with the $\{(a,b),\ (a',b')\}$th element of the covariance matrix given by

(2.6) $\delta(a,a')\ p(a,b)\ (\delta\ (b,b') - p(a,b'))$ for $a,b,a',b' \in I$.

To study the asymptotic distribution of the MLE's when non-random norms are used, we need the following results. We have

(2.7) $E\ m_n(a) = E \sum_{b\in I} \sum_{i=1}^{n} \sum_{j=1}^{\xi_{i-1}} \delta_{ij}(a,b) = \pi(a)\ E\ N_{n-1}$.

Also from Athreya and Ney (1972) it follows that $E\ N_{n-1} \to \infty$ a.s. as $n \to \infty$. Therefore as $n \to \infty$, $E\ m_n(a) \to \infty$ a.s.

Also let $\delta_{ij}(a)$ be 1 or 0 according as the jth particle of the ith generation is in state a or not then $m_n(a) = \sum_{i=1}^{n} \sum_{j=1}^{\xi_{i-1}} \delta_{ij}(a)$.

Since $n \infty$ implies $N_{n-1} \to \infty$ a.s., we have

$$\lim_{n \to \infty} N_{n-1}^{-1} m_n(a) = \pi(a) \text{ a.s.}$$

If we recall from Athreya and Ney (1972) that as $n \to \infty$, $N_{n-1}/E\, N_{n-1} \to W$ a.s., where $W > 0$ is a non-degenerate random variable with $E(W) = 1$, then it follows that, as $n \to \infty$,

(2.8) $\quad m_n(a)/E\, N_{n-1} \to \pi(a)\, W$

and hence that

(2.9) $\quad m_n(a)/E\, m_n(a) \to W$ a.s.

Therefore,

$$\{E\, m_n(a)\}^{\frac{1}{2}} (\hat{p}_n(a,b) - p(a,b)) \to Z_1/\sqrt{W}$$

where $\to$ holds stably. The asymptotic distribution of the MLE's is thus weighted normal (or mixed normal) when normed by $E\, m_n(a)^{-\frac{1}{2}}$. The joint asymptotic distribution of $\{Em_n(a)^{-\frac{1}{2}} (\hat{p}_n(a,b)-p(a,b)),\ a,b \varepsilon I\}$ is weighted normal in the sense that its limiting covariance matrix is W^{-1} times (2.6).

In the particular case when the offspring distribution is geometric, W is exponentially distributed with $E\, W = 1$ and we find that the asymptotic distribution of the estimate of the parameter $p(a,b)$ converges to Student's t-distribution with 2 degrees of freedom.

3. GW Branching Auto-regressive Chain

In this section, we consider a situation wherein particles are born into the population according to a GW branching process and the dispersals of these particles on the real line $\mathbb{R}$ occur according to the auto-regressive scheme of first order. The data for such a process consist of the family tree of the particles in the population and the position vector of the particles existing at the end of each generation. Consider the ith generation. Let Y_{ij} denote the magnitude of dispersal of the jth particle of this generation. Let $X_{i-1,j} = \sum_{k=1}^{i-1} Y_{kj}$ denote the position of the jth

particle at the $(i-1)$th generation with X_{01} denoting the position of the ancestor particle at the zeroth generation. Let the dispersals of the particles be expressed as follows.

(3.1) $\alpha Y_{ij} = X_{i-1,j} + Z_{ij}, \; 1 \le i \le n, \; 1 \le j \le \xi_{i-1}, \; \alpha > 1$

where $\{Z_{ij}\}$ are i.i.d. random variables with zero mean and variance unity. We may write

$$X_{ij} = Z_{ij} + (1+\alpha) Z_{i-1,j} + \ldots\ldots + (1+\alpha)^{i-1} Z_{1j}$$

so that $\{X_{ij}\}$ is a first order Markov autoregressive scheme. Given $(Y_{1j},\ldots,Y_{i-1,j})$, Y_{ij} is normal with mean $\alpha X_{i-1,j}$ and variance unity for all $1 \le j \le \xi_{i-1}$ and $1 \le i \le n$. The likelihood function of the dispersal component of the observed process up to the nth generation may thus be written as

$$(3.2) \qquad L_n(\alpha) \propto \exp\Big\{ \sum_{i=1}^{n} \sum_{j=1}^{\xi_{i-1}} Y_{ij} X_{i-1,j} - \frac{\alpha^2}{2} \sum_{k=1}^{n} \sum_{j=1}^{\xi_{i-1}} X^2_{i-1,j} \Big\}$$

The sequence $\{Y_{ij};\ 1 \le j \le \xi_{i-1},\ 1 \le i \le n\}$ is a sufficient statistics to estimate α. The MLE of α is given by

$$(3.3) \qquad \hat{\alpha}_n = \Big\{ \sum_{i=1}^{n} \sum_{j=1}^{\xi_{i-1}} X^2_{i-1,j} \Big\}^{-1} \Big\{ \sum_{i=1}^{n} \sum_{j=1}^{\xi_{i-1}} Y_{ij} X_{i-1,j} \Big\} .$$

To establish the strong consistency of $\hat{\alpha}_n$ as $n \to \infty$, we use (3.1) to rewrite (3.3) as

$$\hat{\alpha}_n - \alpha = \Big\{ \sum_{i=1}^{n} \sum_{j=1}^{\xi_{i-1}} X^2_{i-1,j} \Big\}^{-1} \Big\{ \sum_{i=1}^{n} \sum_{j=1}^{\xi_{i-1}} X_{i-1,j} Z_{ij} \Big\} .$$

We note that $X_{i-1,j}$ denotes the position of the jth particle of the ith generation prior to a displacement by Y_{ij} during the ith generation. Therefore,

$$P\,\{X_{ij} \mid X_{1j},\ldots\ldots, X_{i-1,j}\} = X_{i-1,j}$$

and $\{X_{ij}\}$ is a martingale sequence. Also $\{Z_{ij}\}$ is a sequence of standard normal variates which are distributed independently of the X_{ij}'s. Thus $Z_{ij} X_{i-1,j}$ denotes a zero mean martingale with the conditional variance $X^2_{i-1,j}$. Also $\sum_{j=1}^{\xi_{i-1}} Z_{ij} X_{i-1,j}$ is a random sum of martingales with zero mean and conditional variance $\sum_{j=1}^{i-1} X^2_{i-1,j}$. By Lemma A.2 of Shrestha (1984), the sum of the random sums of martingales $\sum_{i=1}^{n} \sum_{j=1}^{\xi_{i-1}} Z_{ij} X_{i-1,j}$ is thus a martingale with zero mean and conditional variance $\sum_{i=1}^{n} \sum_{j=1}^{\xi_{i-1}} X^2_{i-1,j} = I_n$ (say).

Also since $X_{i-1,j}$ is a $N(0,A_{i-1})$ variable such that $A_{i-1}^{-\frac{1}{2}} X_{i-1,j} \to Z$ a.s., where $A_{i-1} = ((1+\alpha)^2-1)^{-1}((1+\alpha)^{2(i-1)}-1)$ is a sequence of non-negative random variables for $\alpha > 1$ and Z is a N(0,1) variable. Thus we have

$$A_{i-1}^{-1} X^2_{i-1,j} \to Z^2 \quad \text{a.s.}$$

By Toeplitz lemma,

$$\left(\sum_{i=1}^{n} \sum_{j=1}^{\xi_{i-1}} A_{i-1}\right)^{-1} I_n = \left(\sum_{i=1}^{n} \sum_{j=1}^{\xi_{i-1}} A_{i-1}\right)^{-1} \left(\sum_{i=1}^{n} \sum_{j=1}^{\xi_{i-1}} (A_{i-1}^{-1} X^2_{i-1,j}).A_{i-1}\right)$$

$$\to Z^2 \quad \text{a.s.}$$

Since $n \to \infty$ implies $\xi_{n-1} \to \infty$ a.s., for $\alpha > 1$ as $n \to \infty$,

$$\sum_{i=1}^{n} \sum_{j=1}^{\xi_{i-1}} A_{i-1} = \sum_{i=1}^{n} \xi_{i-1} A_{i-1} \to \infty \text{ a.s.}$$

Hence $I_n \to \infty$ a.s. as $n \to \infty$ and I_n is the increasing function associated with the martingale $\sum_{i=1}^{n} \sum_{j=1}^{\xi_{i-1}} Z_{ij} X_{i-1,j}$. An application of the martingale strong law of large numbers yields $\hat{\alpha}_n \to \alpha$ a.s. Also

$E I_n = \sum_{i=1}^{n} E \xi_{i-1} A_{i-1} \to \infty$ a.s. as $n \to \infty$. Next

$$I_n / E\, I_n = \Big(\sum_{i=1}^{n} E\, \xi_{i-1}\, A_{i-1}\Big)^{-1} \Big(\sum_{i=1}^{n} \sum_{j=1}^{\xi_{i-1}} X_{i-1,j}^{2}\Big)$$

$$\to \lim_{n \to \infty} \Big(\sum_{i=1}^{n} E\, \xi_{i-1}\, A_{i-1}\Big)^{-1} \Big(\sum_{i=1}^{n} \xi_{i-1}\, A_{i-1}\Big)\Big\} \; Z^2 \quad \text{a.s.}$$

$$\to cZ^2 \quad \text{a.s.}$$

where $\left(\sum_{i=1}^{n} E\, \xi_{i-1}\, A_{i-1}\right)^{-1} \left(\sum_{i=1}^{n} \xi_{i-1}\, A_{i-1}\right) \to c$ a.s. as $n \to \infty$ by Kolmogorov 0-1 law. Therefore we have the following

Theorem. As $n \to \infty$, $\hat{\alpha}_n \to \alpha$ a.s. Further $\sqrt{I_n}(\alpha_n - \alpha)$ has asymptotically a standard normal distribution and $\sqrt{EI_n}(\alpha_n - \alpha)$ has asymptotically a weighted normal distribution.

Remarks. When $|\alpha| < 1$, the auto-regressive scheme would be a stationary one. In such a model, the MLE of the serial correlation of the auto-regressive scheme and its asymptotic properties would be of interest. The inference aspects for such a process would follow on the lines of argument presented herein.

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Analyticity Theorem for a Distributional Generalized Laplace Transform

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1.1 Introduction

A generalization of the Laplace transform,

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt \qquad \dots (1.1\text{-}1)$$

was given by Saksena [1] in the form

$$F(s) = \int_0^\infty e^{-(p-q/2)st} (qst)^{c-1/2} W_{k,m} (qst) f(t)\, dt \qquad \dots (1.1\text{-}2)$$

where $W_{k,m}$ is Whittaker function [2].

By analogy with this we define the generalized Laplace transform of a distribution $f(t)$, whose support is bounded on the left, by

$$F(s) = L f(t)$$
$$= < f(t),\ e^{-(p-q/2)st} (qst)^{c-1/2}\ W_{k,m} (qst) > \qquad \dots (1.1\text{-}3)$$

In this paper, we have established the Analyticity Theorem for the generalized Laplace transform (1.1-3). Firstly, we shall define some testing function spaces and shall quote some theorems and properties relating to them.

1.2 The Testing Function Spaces $W_{\alpha,\beta}(I)$, $\bar{W}_{\alpha,\beta}(I)$

Definition 1.2.1

We define $W_{\alpha,\beta}$ (I) as the set of all those complex-valued smooth functions $\phi(t)$ defined on $(0,\infty)$ for which the expression,

$$\sup_{0<t<\infty} \left| e^{\beta t}\ t^{\alpha+r} \frac{d^r \phi(t)}{dt^r} \right|$$

where α , β are suitably fixed real numbers, is finite for each non-negative integer r,

i.e. $W_{\alpha,\beta}$ (I) = $\{\emptyset : \emptyset \in C^{\infty}$ (0,∞) and

$$\sup_{0<t<\infty} \left| e^{\beta t} \; t^{\alpha+r} \; \frac{d^r \emptyset(t)}{dt^r} \right| < \infty \} \qquad \text{... (1.2-1)}$$

$W_{\alpha,\beta}$ (I) is a complex-linear space. It can be seen with the usual pointwise operations of addition of functions and multiplication by a complex-number.

For $\emptyset \in W_{\alpha,\beta}$ (I) and r = 0, 1, 2, ----,

let us define functionals $Y^{c,k,m}_{\alpha,\beta,r}$ ($\emptyset$),

by $$Y^{c,k,m}_{\alpha,\beta,r}(\emptyset) = \sup_{0<t<\infty} \left| e^{\beta t} \; t^{\alpha+r} \; \frac{d^r \emptyset(t)}{dt^r} \right| \qquad \text{... (1.2-2)}$$

(1.2-2) clears that for each r = 0,1,2, ---- ,

$Y^{c,k,m}_{\alpha,\beta,r}$ is a semi-norm on $W_{\alpha,\beta}$ (I), while

$Y^{c,k,m}_{\alpha,\beta,o}$ is a norm. Hence, the collection

$$W_{\alpha,\beta}(I) = \quad Y^{c,k,m}_{\alpha,\beta,r} : r = 0,1,2,\text{-----} \qquad \text{... (1.2-3)}$$

is a countable multinormed space. Sequential convergence and cauchy-sequence in $W_{\alpha,\beta}$ (I) [defined as in Zamanian 3, p: 5-6] clears that if $\{\Phi_n\}_{n=1}^{\infty}$ converges to Φ in $W_{\alpha,\beta}$ (I), then $\{\Phi_n\}_{n=1}^{\infty}$ is a Cauchy-sequence in $W_{\alpha,\beta}$ (I). But it is not immediately obvious whether the converse is true.

Theorem: 1.2.1

$W_{\alpha,\beta}$ (I) is complete and therefore a Ferchet-space.

Proof: The proof is trivial.

Corollary: 1.2.1

If $\{\Phi_n\}_{n=1}^{\infty}$ be a sequence of functions in $W_{\alpha,\beta}$ (I) converging to zero [the zero function of $W_{\alpha,\beta}$ (I)] when $n \to \infty$, then for each non-negative integer r, $\{D^r \Phi_n\}_{n=1}^{\infty}$ converges to zero uniformly on every compact subset of $(0, \infty)$ as $n \to \infty$.

Thus, $W_{\alpha,\beta}$ (I) satisfies all the three conditions for being a testing functions space, [Zemanian 3, p. 39]. We call elements of $W_{\alpha,\beta}$ (I) as testing function.

Definition: 1.2.2

$\bar{W}_{\alpha,\beta}$ (I) is the set of all those complex-valued functions ϕ (t) defined on I($0<t<\infty$) such that, for each non-negative integer r, the expression

$$\sup_{0<t<\infty} \left| e^{\beta t} \; t^{\alpha} \left(t\frac{d}{dt}\right)^r \phi(t) \right|$$

where α, β are suitably fixed real numbers, is finite.

i.e. $\bar{W}_{\alpha,\beta}$ (I) = $\{\phi : \phi \in C^{\infty}$ and

$$\sup_{0<t<\infty} \left| e^{\beta t} \; t^{\alpha} \left(t\frac{d}{dt}\right)^r \phi(t) \right| < \infty \quad \}$$

$\bar{W}_{\alpha,\beta}$ (I) is a complex-linear space with the usual pointwise operations and multiplication of a function by a complex number.

For $\phi \in \bar{W}_{\alpha,\beta}$ (I) and r = 0,1,2,-----; we define semi-norms

$P_{\alpha,\beta,r}(\phi)$, by $P_{\alpha,\beta,r}(\phi) = \sup\limits_{0<t<\infty} \left| e^{\beta t} \; t^{\alpha} \left(t\frac{d}{dt}\right)^r \phi(t) \right|$ ... (1.2-6)

Since $P_{\alpha,\beta,0}$ is a norm, the collection $\bar{W}_{\alpha,\beta} = (P_{\alpha,\beta,r} : r = 0,1,2,-----)$ is a countable multinorm and $W_{\alpha,\beta}$ (I) with the topology generated by the the countable multinorm $\bar{W}_{\alpha,\beta}$, is a countable multinormed space. The concept of convergence and the completeness of $\bar{W}_{\alpha,\beta}$ (I) are defined in a way similar to those of $W_{\alpha,\beta}$ (I). Hence, the space $W_{\alpha,\beta}$ (I) is complete and is seen to be a testing function space.

Lemma 1.2.1

Suppose $W_{\alpha,\beta}$ (I) and $\bar{W}_{\alpha,\beta}$ (I) are testing function spaces as given in definitions 1.2.1 and 1.2.2, α, β being same fixed real numbers for both.

(a) $W_{\alpha,\beta}$ (I) = $\bar{W}_{\alpha,\beta}$ (I) in store of elements,

(b) the topology T_1 generated by

$W_{\alpha,\beta} = \{Y^{c,k,m}_{\alpha,\beta,r}\}^{\infty}_{r=o}$ in $W_{\alpha,\beta}$ (I) is the same as the topology T_2 generated by

$\bar{W}_{\alpha,\beta} = \{P_{\alpha,\beta,r}\}^{\infty}_{r=o}$ in $\bar{W}_{\alpha,\beta}$ (I).

Proof:

Suppose $\delta = t\,\frac{d}{dt}$

By the method of induction it can be observed that for k = 1,2,-----

$$t^k D^k_t \phi(t) = \delta(\delta-1) -- (\delta-k+1)\phi(t) \qquad \ldots (1.2\text{-}7)$$

so that $t^k D^k \phi$ is a linear combination of

$\delta\phi, \delta^2\phi, \delta^3\phi,$ ----- $\delta^k\phi$. Also, we see that

$$\delta^k \phi = b_1 \left(t\frac{d\phi}{dt}\right) + b_2 \left(t^2 \frac{d^2\phi}{dt^2}\right) + -- b_k \left(t^k \frac{d^k\phi}{dt^k}\right) \qquad \ldots (1.2\text{-}8)$$

where b_i 's are integral constants. Now, (a) is clear from (1.2-7) and (1.2-8). Also, from these relations we can see that

$$Y^{c,k,m}_{\alpha,\beta,r}(\phi) \le A \max \{P_{\alpha,\beta,o}(\phi), ---, P_{\alpha,\beta,r}(\phi)\} \qquad \ldots (1.2\text{-}9)$$

and

$$P_{\alpha,\beta,\ell}(\phi) \le B \max \{Y^{c,k,m}_{\alpha,\beta,o}(\phi), -----, Y^{c,k,m}_{\alpha,\beta,\ell}(\phi)\} \qquad \ldots (1.2\text{-}10)$$

for some suitable constants A and B, r and ℓ depending upon the choice of $Y^{c,k,m}_{\alpha,\beta,r}$ and $P_{\alpha,\beta,\ell}$ respectively [Zemanian 3, Lemma 1.6-3]. This proves (b).

Theorem: 1.2.2

Suppose α , β be such that

(i) Re $(\alpha + c \pm m) > 0$, Re $m \geq 0$;

(ii) Re $s > \beta$, s not lying on the negative real axis, then

$w(qst) \varepsilon W_{\alpha,\beta}$ (I), where,

$$w(qst) = (qst)^{c-1/2}\, e^{-(p-q/2)st}\, W_{k,m}\,(qst)$$

$W_{k,m}$ being the Whittaker function as mentioned in sec. 1.1.

Proof:

We have

$$\frac{d^r w(x)}{dx^r} = \frac{d^r}{dx^r}\left[x^{c-1/2}\, e^{-1/2x}\, W_{k,m}\,(x)\right]$$

$$= \frac{d^r}{dx^r}\left[x^{c+m}\, e^{-x}\, \Psi\,(-k+m+1/2\,;\ 2m+1;x)\right]$$

[from Erdelyi, 5 p. 264]

or

$$\frac{d^r w(x)}{dx^r} = \sum_{n=0}^{r}\ {}^rC_n\ (-1)^n\ e^{-x}\ \Psi\ (-k+m+1/2;\ 2m+1;x)$$

$$.(c+m)\text{-----}(c+m+r+n+1)\ x^{c+m-r+m}$$

[from Erdelyi, 5, p.258]

$$= \sum_{n=0}^{r}\ {}^rC_n\ (-1)^n\ (c+m)_{r-n}\ x^{c-1/2-r+n/2}\ e^{-x/2}\,.W_{k+\frac{n}{2}\ ,\ m+\frac{n}{2}}\ (x)$$

where, $(c+m)_{r-n} = (c+m)$ ----- $(c+m-r+n+1)$.

Thus,

$$\frac{\partial^r w(qst)}{\partial t^2} = (qs)^r\ \frac{\partial^r w(qst)}{\partial (qst)^r}$$

$$= (qs)^r \sum_{r=o}^{n} {}^nC_r \, (-1)^{n-r+u} \, (p/q-1)^{n-r} \, e^{-(p-q/2)\, st}$$

$$\sum_{u=o}^{r} {}^rC_u \, \frac{(c-m)!}{(c-m-u+u)!} \, (qst)^{c-n+\frac{u}{2} - \frac{1}{2}} \, (qs)^u . W_{k+\frac{u}{2}\, ,\, m - \frac{u}{2}} (qst) \, ,$$

where $c+u > m+n$.

$$Y^{c,k,m}_{\alpha,\beta,r} [w(qst)] = \sup_{0<t<\infty} \left| e^{\beta t} \, t^{\alpha+r} \, \frac{\partial^r}{\partial t^r} w(qst) \right|$$

$$= \sup_{0<t<\infty} \Big| e^{\beta t} \, t^{\alpha+r} \, (qs)^{r+u} \sum_{r=o}^{n} {}^nC_r (-1)^{n-r+u} \, (p/q-1)^{n-r}$$

$$e^{-(p-q/2)st} \sum_{u=o}^{r} {}^rC_u \, \frac{(c-m)!}{(c-m-n+u)!} \, (qst)^{c-n+\frac{u}{2}-\frac{1}{2}} . W_{k+\frac{u}{2},\, m-\frac{u}{2}} (qst) \Big|$$

$$\leq \sup_{0<t<\infty} \Big| e^{\beta t} \, t^{\alpha} \sum_{r=o}^{n} (qs)^u \, (qst)^{c+u/2-1/2} \, e^{-(p-q/2)st}$$

$$W_{k+u/2,\, m-u/2} (qst) \Big| \left| \sum_{u=o}^{r} {}^nC_r \, (p/q-1)^{n-r} \, {}^rC_u \, \frac{(c-m)!}{(c-m-n+u)!} \right| \quad \ldots(1.2\text{-}11)$$

Now for large t, i.e. for $|qst| \to \infty$, s fixed,
$(c-m-n+u)! > 0$, $p > q$, we have

$$\left| e^{\beta t} \, t^{\alpha} \, (qs)^u \, (qst)^{c-1/2+u/2} \, e^{-(p-q/2)st} \, W_{k+u/w,m-u/2}(qst) \right|$$

$$= \left| e^{\{\beta - (p-q/2)s\}\, t} \, t^{\alpha+c+k+u-1/2} \, (qs)^{c+k+2u-1/2} \right|$$

$\to 0$, as $t \to \infty$, s fixed with Re $s > \frac{2\,\beta}{2\,p-q}$

for small t,

$$\left| e^{\beta t}\, t^{\alpha} (qs)^{u}\, (qst)^{c-1/2+u/2}\, e^{-(p-q/2)st}\, W_{k+u/2,m-u/2}(qst) \right|$$

$$= \left| t^{\alpha+c\pm m+\text{ (u or o)}}\, (qs)^{c-m}\, e^{\beta t}\, e^{-(p-q/2)st} \right| < \infty$$

as $t \to \infty$, s fixed, for Re $(\alpha + c\pm m) > 0$.

Hence, from (1.2-11) we have that under the assumed conditions of α and β,

$$\gamma^{c,k,m}_{\alpha,\beta,r}\ [w(qst)] < \infty\ ,\ r = 0,1,2,\text{-----}$$

Now, it becomes clear that

$w(qst) \in W_{\alpha,\beta}\ (I)$.

Corollary: 1.2.2:

Under the same conditions of the theorem 1.2.2;

$$\frac{\partial^{l}\, w(qst)}{\partial (qs)^{l}} \in W_{\alpha,\beta}\ (I).$$

Corollary: 1.2.3:

Let r and n be negative integers and Ω a compact set of complex plane which does not contain any point of negative real axis. Then,

$$\gamma^{c,k,m}_{\alpha,\beta,r} \left\{ \frac{\partial^{n} w(qst)}{\partial (qs)^{n}} \right\} \leq B_{\Omega} < \infty$$

uniformly for all s lying in Ω, B_{Ω} is a real constant depending upon Ω.

1.3 The Generalized Function Spaces $W'_{\alpha,\beta}(I)$:

We denote the dual of $W_{\alpha,\beta}$ (I) by $W'_{\alpha,\beta}$ (I). Members $W'_{\alpha,\beta}$ (I) are generalized functions [Zemanian 3, p. 39]. Concepts of convergence and completeness in $W'_{\alpha,\beta}$ (I) are defined as before. Since $W'_{\alpha,\beta}$(I) is complete we have from [Zemanian 3, Theorem 1.8.3],

Theorem: 1.3.1:

$W'_{\alpha,\beta}$ (I) is complete.

1.4 Properties of $W_{\alpha,\beta}(I)$ and its dual $W'_{\alpha,\beta}(I)$.

Property: 1.4.1:

Since $\beta_1 < \beta_2$

$$|e^{\beta_1 t}\, t^{\alpha+r}\, D^r\, \phi(t)| < |e^{\beta_2 t}\, t^{\alpha+r}\, D^r\, \phi(t)|$$

we have

$$Y^{c,k,m}_{\alpha,\beta_1,r}(\phi) \leq Y^{c,k,m}_{\alpha,\beta_2,r}(\phi) \qquad \ldots (1.4\text{-}1)$$

(1.4-1) clears that whenever $\beta_1 < \beta_2$

(i) $W_{\alpha,\beta_2}(I) \subseteq W_{\alpha,\beta_1}(I)$,

(ii) the topology of $W_{\alpha,\beta_2}(I)$ is stronger than the topology induced on it by $W_{\alpha,\beta_1}(I)$ [Zemanian 3, Lemma 1.6-3] and thus, the restriction of any $f \in W'_{\alpha,\beta_1}(I)$ to $W_{\alpha,\beta_2}(I)$ is in $W'_{\alpha,\beta_2}(I)$, [i.e. $f \in W'_{\alpha,\beta_2}(I)$].

PropertY; 1.4.2:

By the definitions of the spaces D(I) and E(I) [Zemanian 3, p.33-35] it is clear that $D(I) \subset W_{\alpha,\beta}(I) \subset E(I)$. Since D(I) is dense in E(I), $W_{\alpha,\beta}(I)$ is dense in E(I).

We can easily prove that if $\{\phi_n\}$ converges to ϕ in D, then $\{\phi_n\}$ converges to ϕ in $W_{\alpha,\beta}(I)$. For the support of ϕ_n and ϕ are all contained in some closed interval $[a,b]$, $0 < a < b < \infty$, so that

$$Y^{c,k,m}_{\alpha,\beta,r}(\phi_n - \phi) = \sup_{0<t<\infty} \left| e^{\beta t}\, t^{\alpha+r}\, D^r\, (\phi_n - \phi) \right|$$

$$= \sup_{a<t<b} \left| e^{\beta t}\, t^{\alpha+r}\, D^r\, (\phi_n - \phi) \right|$$

$$\leq C \sup_{a<t<b} \left| D^r (\phi_n - \phi) \right|$$

where, $C = \sup_{a<t<b} \left| e^{\beta t} \, t^{\alpha + r} \right|$.

It follows that the restriction of any $f \in W'_{\alpha,\beta}(I)$ to D is in D', so that $W'_{\alpha,\beta}(I) \subseteq D'$.

Property: 1.4.3:

For each $f \in W'_{\alpha,\beta}(I)$ there exists a positive constant C and a non-negative integer n such that for all $\phi \in W_{\alpha,\beta}(I)$,

$$|\langle f, \phi \rangle| \leq C \max_{o<r<n} Y^{c,k,m}_{\alpha,\beta,r} (\phi).$$

It follows from the Theorem 1.8.1 of Zemanian [3].

1.5: The $W_{c,k,m}$ Transform of Generalized Functions:

Definition: 1.5.1:

We call 'f' a $W_{c,k,m}$- transformable generalized function if it is a member of $W'_{\alpha,\beta}(I)$ for some fixed real numbers α, β.

From the property 1.4.1, we have that $f \in W_{\alpha,\beta}(I)$ then f is a member of $W'_{\alpha,\beta}(I)$ for every $\beta' > \beta$, which implies that there exists a real number σ_f (possibly $\sigma_f = -\infty$) such that $f \in W'_{\alpha,\beta}(I)$ for every $\beta > \sigma_f$ and $f \notin W_{\alpha,\beta}(I)$ for $\beta < \sigma_f$.

Definition: 1.5.2:

Let $f \in W'_{\alpha,\beta}(I)$. We define the $W_{c,k,m}$ - transform of f denoted by $W_{c,k,m}$ (f) by the relation

$$F(S) = (W_{c,k,m} f)(s)$$
$$= \langle f(t), w(qst) \rangle \qquad \ldots(1.5\text{-}1)$$

where, $w(qst) = (qst)^{c-1/2} \cdot e^{-(p-1/2)st}\, W_{k,m}(qst)$ and $s \in \Omega_f$.

The region Ω_f is defined as below,

$$f = \{s \,/\, \text{Re}\, s > f,\ s \neq 0,\ -\pi < \arg s < \pi\} \qquad \ldots(1.5\text{-}2)$$

If $\sigma_f > 0$, Ω_f is called a cut-half plane and is obtained by deleting all real non-positive values of s.

The definition 1.5.1 has a sense as an application of $f \in W'_{\alpha,\beta}(I)$ to $w(qst) \in W_{\alpha,\beta}(I)$ for fixed real numbers, α and β with Re $s < \beta$, Re $(\alpha + c \pm m) > 0$.

We call the number σ_f as abscissa of definition and Ω_f the region of definition for the generalized $W_{c,k,m}$ - transform.

Theorem:1.5.1 (Analyticity Theorem):

Let $F(s) = W_{c,k,m}(f)$ for $s \in \Omega_f$. Then F(s) is an analytic function of s and

$$D_s^n\, f(s) = \left\langle f(t), \frac{\partial^n w(qst)}{\partial (qst)^n} \right\rangle, \quad s \in \Omega_f \qquad .(1.5\text{-}3)$$

where, $w(qst) = (qst)^{c-1/2}\, e^{-(p-q/2)st}\, W_{k,m}(qst)$

and $D_s^n = \dfrac{d^n}{d(qs)^n}$, $n = 1,2,$ ------

Proof:

From Cor. 1.2.2, we observe that the right hand side of (1.5-3) is well defined, we shall show (1.5-3) for n = 1 and the theorem will follow for every n = 2,3, ----- by the method of induction.

Let s be an arbitrary but fixed point in Ω_f. We choose real positive numbers β', r and r_1 such that

$$\sigma_f < \beta < \beta' = \text{Re}\, s - r_1 < \text{Re}\, s - r < \text{Re}\, s.$$

Also, we denote the circle centred at s by C whose radius is r_1. We restrict r_1 (and thereby β' and r) still further by requiring that C lies entirely within Ω_f (i.e. C does not intersect the non positive real axis). Finally, suppose Δs be a non-zero increment such that $|\Delta s| < r$ and consider the expression,

$$\frac{F(s+\Delta s) - F(s)}{\Delta s} - \langle f(t),\, D_s\, w(qst) \rangle$$

$$= \langle f(t),\ \Psi_{\Delta s}(t) \rangle \qquad \ldots (1.5\text{-}4)$$

where

$$\Psi_{\Delta s}(t) = \frac{w\,(\overline{qs+\Delta s}\,t) - w(qst)}{\Delta s} - D_s . w(qst) .$$

From Cor. 1.2.2 we have that

$$\frac{\partial^n w(qst)}{\partial (qs)^n},\ n = 0,1,2, \text{-----}$$

are members of $W_{\alpha,\beta}$ (I), hence (1.5-3), (1.5-4) are well defined. Now we shall prove that $\Psi_{\Delta s}(t) \to 0$ in $W_{\alpha,\beta}$ (I) as $\Delta s \to 0$, consequently it will prove that

$$\langle f(t),\ \Psi_{\Delta s}(t) \rangle \to 0 \text{ as } \Delta s \to 0,$$

[from the continuity property of $f \in W'_{\alpha,\beta}$ (I)] and hence proving the equality (1.5-3) for n = 1.

The order of differentiation can be changed to that with respect to qs in w(qst). Applying Cauchy's integral formula which states that "If f(z) is analytic on and within a closed contour C and 'a' is any interior point of C then

$$f^n(a) = \frac{n\,!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}}\ dz."$$

where C is traversed in the positive direction to $\Psi_{\Delta s}(t)$, we have

$$\frac{d^n}{dt^n}\{\Psi_{\Delta s}(t)\}$$

$$= \frac{\partial^n}{\partial n^t}\left\{\frac{w(\overline{qs+\Delta st}) - w(qst)}{\Delta s} - \frac{\partial w(qst)}{\partial(qs)}\right\}$$

$$= \frac{1}{\Delta s}\left[w_{n_t}(\overline{qs+\Delta st}) - w_{n_t}(qst)\right] - \frac{\partial w_{n_t}(qst)}{\partial(qs)}$$

where, the suffix n_t indicates n times partial derivatives with respect to t.

$$= \frac{1}{\Delta s}\left[\frac{1}{2\pi i}\int_C \left\{\frac{1}{z-\overline{qs+\Delta s}} - \frac{1}{z-qs}\right\} w_{n_t}(qst)\right.$$

$$- \frac{1}{2\pi i}\int_C \frac{w_{n_t}(zt)}{(z-qs)^2}\, dz$$

(using Cauchy's integral formula).

$$= \frac{\Delta s}{2\pi i}\int_C \frac{w_{n_t}(zt)}{(z-\overline{qs+\Delta s})(z-qs)^2}\, dz$$

Now, $\left| e^{\beta t}\, t^{\alpha+n} \frac{d^n}{dt^n} \Psi_{\Delta s}(t)\right|$

$$= \left|\frac{\Delta s}{2\pi i}\int_C \frac{w_{n_t}(zt)\, e^{\beta t}\, t^{\alpha+n}}{(z-\overline{qs+\Delta s})(z-qs)^2}\, dz\right|$$

$$\leq \frac{|\Delta s|}{2\pi}\ \frac{2\pi r_1}{(r-r_1)\, r_1^2} = \frac{|\Delta s|\, M}{(r-r_1)\, r_1} \qquad \ldots (1.5\text{-}5)$$

In the light of Cor. 1.2.3, $|e^{\beta t}\, t^{\alpha+n}\, w_{n_t}(qst)|$ is bounded on any compact subset of Ω_f, let us suppose that this bond be $\leq M$, hence (1.5-5) is justified. From (1.5-5), we have

$$\sup_{0<t<\infty} \left| e^{\beta t} \, t^{\alpha+n} \, \frac{d^n}{dt^n} \, \Psi_{\Delta s}(t) \right| \to 0$$

as $\Delta s \to 0$.

This proves the theorem.

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Hydromagnetic Flow of Visco-Elastic Fluid Near an Oscillating Porous Flat Plate

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1. Introduction

The exact solution for the flow of a viscous incompressible fluid near an infinite oscillating flat plate has been obtained by Stokes [7] and later by Rayleigh [4]. The solution for the flow of an incompressible visco-elastic (Revlin-Ericksen model) fluid near an infinite porous flat plate has been obtained by the author [6] when there is a uniform suction imposed over the plate. The solution for the flow of an incompressible viscous conducting fluid near an oscillating non-conducting flat plate in its own plane has been obtained by Ong and Nicholls [3] in the presence of uniform transverse magnetic field. This analysis has been extended by Chowdhary [1] when there is a uniform suction imposed over the plate. This analysis has further been extended by Kishan and Sharma [2] taking into account the effect of the action of body force.

In this paper we obtain the solution for the flow of an incompressible visco-elastic (Revlin-Ericksen model) conducting fluid near an oscillating non-conducting porous flat plate in its own plane when there is a uniform suction imposed over the plate in the presence of uniform transverse magnetic field.

2. Formulation of the Problem

Consider the flow of an incompressible visco-elastic conducting fluid about an infinite porous flat plate which executes linear harmonic oscillation with velocity U cosnt in its own plane along the x-axis. A magnetic field of uniform strength H_o is applied in the direction of y-axis taken perpendicular to the plate. Since the plate is infinite in length and uniform suction is imposed over it, all the physical variables depend on y and t only. The pressure p in the fluid is assumed constant. If v_s represents the velocity of suction or injection at the plate, the equation of continuity

(1) $$\frac{\partial v}{\partial y} = 0$$

with the condition $v = v_s$ at $y = 0$ yields $v = v_s$ everywhere.

The governing equation describing the flow of an incompressible visco-elastic conducting fluid in the presence of uniform transverse magnetic field [5] is

(2) $$\frac{\partial u}{\partial t} + v_s \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} + \beta \frac{\partial^2}{\partial y^2} \left(\frac{\partial u}{\partial t} + v_s \frac{\partial u}{\partial y}\right) \frac{\sigma}{\rho} B_o^2 u$$

with the boundary conditions

(3) $$\left.\begin{aligned} &u = U \text{ cosnt} \quad \text{at } y = 0, \\ &u = 0 \text{ as } y \to \infty, \end{aligned}\right\} t > 0$$

where ν is the kinematic viscosity, β the kinematic visco-elasticity, ρ the density of the fluid, σ the electrical conductivity and $\bar{B} = \mu_e H_o$ (= constant), the component of electromagnetic induction.

Introducing

$$\bar{t} = nt,$$

$$\eta = y\sqrt{\frac{n}{\nu}},$$

$$\lambda = \frac{v_s}{\sqrt{n\nu}}; \qquad \begin{aligned} &\lambda > 0 \text{ (injection)} \\ &\lambda < 0 \text{ (suction)} \end{aligned}$$

$$S = \frac{\beta n}{\nu},$$

$$m = \frac{\sigma B_o^2}{R},$$

$$m_1 = \frac{m}{n} = R_m^2$$

in (2) and (3) we get

(4) $$\frac{\partial u}{\partial t} + \lambda \frac{\partial u}{\partial \eta} = \frac{\partial^2 u}{\partial \eta^2} + S \frac{\partial^2}{\partial \eta^2} \left(\frac{\partial u}{\partial t} + \lambda \frac{\partial u}{\partial \eta}\right) - m_1 u$$

with the conditions

(5) $$u = U \cos \bar{t} \text{ at } \eta = 0,$$
$$u = 0 \text{ as } \eta \to \infty.$$

3. Solution

Assume

(6) $$u = W(\eta) \exp(i\bar{t})$$

and substituting in (4) we get

(7) $$S\lambda W''' + (1+iS) W'' - \lambda w' - (i + m_1) W = 0,$$

where dashes denote differentiation with respect to η.

The corresponding boundary conditions are

(8) $$W = U \text{ at } \eta = 0,$$
$$W = 0 \text{ as } \eta \to \infty.$$

Equation (7) is solved by perturbation technique and for this we assume

(9) $$W = W + SW_1 + 0(S^2),$$

where the elastic parameter $S << 1$ and W_j's are independent of S.

Substituting (9) in (7) and equating the coefficients of like powers of S, we get

(10) $$W_o'' - \lambda W_o' - (i + m) W_o = 0$$

(11) $$W_1'' - \lambda W_1' - (i + m)W_1 = - \lambda W_o''' - iW_o''.$$

The corresponding boundary conditions are

(12) $$W_o = U, W_1 = 0 \text{ at } \eta = 0,$$
$$W_o = W_1 = 0 \text{ as } \eta \to \infty.$$

Solving (10) and (11) with the boundary conditions (12) we get

(13) $$W_o = U \exp(b\eta)$$

$$W_1 = \frac{(\lambda b + i + m_1)b^2}{\sqrt{\lambda^2 + 4\,(i + m_1)}}\,\eta\,\exp(b\eta) \tag{14}$$

where

$$b = \frac{1}{2}\,(\lambda - \sqrt{\lambda^2 + 4(i + m_1)}).$$

Hence from (6), (9), (13) and (14) we get

$$\frac{u}{U} = [\exp(b\eta) + S\,\{\frac{(\lambda b + i + m_1)b^2}{\sqrt{\lambda^2 + 4(i + m_1)}}\,\eta\,\exp(b\eta)\}]\,\exp(i\bar{t}) \tag{15}$$

$$= (F_r + iF_i)\,\exp(i\bar{t}),$$

where

$$F_r = \{\cos\frac{q\eta}{2} + S\eta(M\cos\frac{q\eta}{2} + N\sin\frac{q\eta}{2})\}\,\exp(\frac{\lambda - p}{2}\eta),$$

$$F_i = \{-\sin\frac{q\eta}{2} + S\eta(N\cos\frac{q\eta}{2} - M\sin\frac{q\eta}{2})\}\,\exp(\frac{\lambda - p}{2}\eta);$$

$$M = \frac{Cp + Dq}{p^2 + q^2}, \quad N = \frac{Dp - Cq}{p^2 + q^2},$$

$$C = \frac{(\lambda - p)\,\{(\lambda - p)^2 - q^2\}}{8} - \frac{(2 - \lambda q)\,(\lambda - p)q}{4},$$

$$D = \frac{(2 - \lambda q)\,\{(\lambda - p)^2 - q^2\}}{8} + \frac{\lambda(\lambda - p)q}{4},$$

$$p = \frac{1}{\sqrt{2}}\,[\sqrt{(\lambda^2 + 4m_1)^2 + 16} + (\lambda^2 + 4m_1)]^{\frac{1}{2}},$$

$$q = \frac{1}{\sqrt{2}}\,[\sqrt{(\lambda^2 + 4m_1)^2 + 16} - (\lambda^2 + 4m_1)]^{\frac{1}{2}}.$$

Therefore from (15), the real part of $u(\eta,t)$ is

$$u = U\,[\cos(\bar{t} - \frac{q\eta}{2}) + S\eta\{M\cos(\bar{t} - \frac{q\eta}{2}) + N\sin(\bar{t} - \frac{q\eta}{2})\}]\,\exp(\frac{\lambda - p}{2})\eta. \tag{16}$$

For $S \neq 0$, $\lambda \neq 0$, $m_1 = 0$, we get

$$u = U \,[\cos(\bar{t} - \frac{\hat{q}\eta}{2}) + S\eta\{\hat{M}\cos(\bar{t} - \frac{\hat{q}\eta}{2}) + \hat{N}\sin(t - \frac{\hat{q}\eta}{2})\}]\exp(\frac{\lambda - \hat{p}}{2})\eta$$

where

$$\hat{p} = \frac{1}{\sqrt{2}}(\sqrt{\lambda^4 + 16} + \lambda^2)^{\frac{1}{2}},$$

$$\hat{q} = \frac{1}{\sqrt{2}}(\sqrt{\lambda^4 + 16} - \lambda^2)^{\frac{1}{2}},$$

$$\hat{M} = \frac{\hat{C}\hat{p} + \hat{D}\hat{q}}{\hat{p}^2 + \hat{q}^2}, \quad \hat{N} = \frac{\hat{D}\hat{p} - \hat{C}\hat{q}}{\hat{p}^2 + \hat{q}^2},$$

$$\hat{C} = \frac{\lambda(\lambda - \hat{p})\{(\lambda - \hat{p})^2 + \hat{q}^2\}}{8} - \frac{(2 - \lambda\hat{q})(\lambda - \hat{p})\hat{q}}{4},$$

$$\hat{D} = \frac{(2 - \lambda\hat{q})\{(\lambda - \hat{p})^2 - \hat{q}^2)}{8} + \frac{\lambda(\lambda - \hat{p})\hat{q}}{4}$$

which is the solution for the flow of non-conducting visco-elastic fluid near an oscillating porous flat plate [6].

For $S = 0$, $\lambda \neq 0$, $m_1 \neq 0$, we get

$$u = U \exp\frac{\eta}{2}\,[\lambda - \frac{1}{\sqrt{2}}\{\sqrt{(\lambda^2 + 4m_1)^2 + 16} + (\lambda^2 + 4m)\}^{\frac{1}{2}}]$$

$$\cdot \cos[\,\bar{t} - \frac{\eta}{2\sqrt{2}}\{\sqrt{(\lambda^2 + 4m_1) + 16} - (\lambda^2 + 4m_1)\}^{\frac{1}{2}}]$$

which is the solution for hydromagnetic flow near an oscillating porous flat plate [1].

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Semi-Geodesic Correspondence in a Riemannian Space II

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Abstract

A curve whose differential equation is $d^2x^h/ds^2 + \Gamma^h_{ji}\, dx^j/ds\, dx^i/ds = 0$, where Γ^h_{ji} is a semi-symmetric connection (Yano [6]), has been called a semi-geodesic curve in a Riemannian space by the author [4]. The correspondence of Riemannian spaces such that semi-geodesics of one are also semi-geodesics of other has been defined and studied by the author [4,5] and called semi-geodesic correspondence. In this paper we have studied semi-geodesic correspondence based on certain known connections and obtained certain entities which are invariant with respect to semi-geodesic correspondence. Some new curvature tensors have been introduced and their properties studied.

1. Preliminaries

Let M^n be a Riemannian space with metric tensor g_{ji}, Christoffel symbols $\{^h_{ji}\}$, curvature tensor $K_{jih}{}^k$ and the Ricci tensor K_{ji}. Let $p_i = g_{hi}p^h$ be a covector then the semi-symmetric metric connection is expressed as [3]:

$$(1.1) \qquad \Gamma^h_{ji} = \{^h_{ji}\} + \delta^h_j\, p_i - g_{ji}\, p^h.$$

A curve whose differential equation is given by $d^2x^h/ds^2 + \Gamma^h_{ji}\, dx^j/ds\, dx^i/ds = 0$, is called a semi-geodesic curve [4].

Similar to (1.1) author [5] has also studied semi-geodesic curves based on Γ^h_{ji} defined by

$$(1.2) \qquad \Gamma^h_{ji} = \{^h_{ji}\} - p_j\ \ \delta^h_i\ .$$

If M^n and $\bar{M}^n$ are two Riemannian spaces, then a mapping $f: M^n \to \bar{M}^n$ is called semi-geodesic correspondence if semi-geodesics of M^n are transferred to semi-geodesics of $\bar{M}^n$.

In [4] author has proved that if M^n and $\bar{M}^n$ are in semi-geodesic correspondence then Γ^h_{ji} given by (1.1), L^h_{ji} and T^h_{ji} given by

$$(1.3) \qquad L^h_{ji} = \left\{ {h \atop ji} \right\} + \tfrac{1}{2}\, \delta^h_j\, p_i + \tfrac{1}{2}\, \delta^h_i\, p_j - g_{ji}\, p^h$$

and

$$(1.4) \qquad T^h_{ji} = \tfrac{1}{2}\, (\delta^h_{ji}\, p_i - i|j),$$

transform as follows:

$$(1.5) \qquad \bar{\Gamma}^h_{ji} = \Gamma^h_{ji} + \{\delta^h_j(2n\, \phi_i - \partial_i \log \bar{g}/g) - \delta^h_i(2\phi_j - \partial_j \log \bar{g}/g)\}/2(n-1),$$

$$(1.6) \qquad \bar{L}^h_{ji} = L^h_{ji} + \tfrac{1}{2}\, (\delta^h_j\, \phi_i + i|j)$$

and

$$(1.7) \qquad \bar{T}^h_{ji} = T^h_{ji} + \{\delta^h_j\, ((n+1)\, \phi_i - \partial_i \log \bar{g}/g) - i|j\}\, /2(n-1),$$

where $- i|j$ means interchange of indices i, j and substraction and ϕ_i is an arbitrary covector field [4].

In [5] author has proved that if M^n and $\bar{M}^n$ are in semi-geodesic correspondence then Γ^h_{ji} given by (1.2), M^h_{ji} and T^h_{ji} given by

$$(1.8) \qquad M^h_{ji} = \left\{ {h \atop ji} \right\} - \tfrac{1}{2}\, (\delta^h_j\, p_i + i|j)$$

and (1.4), transform as follows:

$$(1.9) \qquad \bar{\Gamma}^h_{ji} = \Gamma^h_{ji} + (\delta^h_j\, \partial_i \log \bar{g}/g - j|i)/2(n+1) + \phi_j\, \delta^h_i\,,$$

$$(1.10) \qquad \bar{M}^h_{ji} = M^h_{ji} + \tfrac{1}{2}\, (\delta^h_j\, \phi_i + i|j)$$

and

$$(1.11) \qquad \bar{T}^h_{ji} = T^h_{ji} + \tfrac{1}{2}\, [\delta^h_j((n+1)^{-1}\, \partial_i \log \bar{g}/g - \phi_i) - i|j].$$

2. Semi-geodesic Correspondence

From equation (1.2) by virtue of (1.9) we get

$$(2.1)\qquad \overline{\{{}^{h}_{ji}\}} = \Gamma^{h}_{ji} + (\delta^{h}_{j}\ \partial_{i} \log \bar{g}/g - i|j)/2(n+1) + \emptyset_{j}\ \delta^{h}_{i} + \bar{p}_{j}\ \delta^{h}_{i}\ .$$

Now using [5]:

$$(2.2)\qquad \bar{p}_{j} = p_{j} + (\partial_{j} \log \bar{g}/g - (n+1)\ \emptyset_{j})/(n+1)$$

in (2.1) we get on simplification

$$(2.3)\qquad \overline{\{{}^{h}_{ji}\}} = \{{}^{h}_{ji}\} + (\delta^{h}_{j}\ \partial_{i} \log \bar{g}/g + i|j)/2(n+1).$$

Theorem (2.1). Under a semi-geodesic correspondence [5] $\{{}^{h}_{ji}\}$ given by (1.2) transforms as in (2.3).

Now we consider the effect of semi-geodesic correspondence on the connection defined by

$$(2.4)\qquad N^{h}_{ji} \overset{\text{def.}}{=\!=\!=} \{{}^{h}_{ji}\} - g_{ji}\ p^{h}.$$

From (1.3) we can see that

$$(2.5)\qquad N^{h}_{ji} = L^{h}_{ji} - \tfrac{1}{2}(\delta^{h}_{j}\ p_{i} + i|j).$$

Applying (1.6) in (2.5) we can get

$$(2.6)\qquad \bar{N}^{h}_{ji} = N^{h}_{ji} + \tfrac{1}{2}(\delta^{h}_{j}\ (\emptyset_{i} - \bar{p}_{i} + p_{i}) + i|j).$$

Since it is known that [4]

$$(2.7)\qquad \bar{p}_{j} = p_{j} + ((n+1)\ \emptyset_{j} - \delta_{j} \log \bar{g}/g)/(n-1),$$

therefore from (2.6) and (2.7) we get on simplification

$$(2.8)\qquad \bar{N}^{h}_{ji} = N^{h}_{ji} - (\delta^{h}_{j}(2\emptyset_{i} - \partial_{i} \log \bar{g}/g) + i|j)/2(n-1).$$

Hence:

Theorem (2.2). <u>Under a semi-geodesic correspondence</u> [4] N^h_{ji} <u>defined by (2.4) transforms as in (2.8)</u>.

If $*g_{ji} = e^{2p} g_{ji}$, i.e., two Riemannian spaces M^n and $*M^n$ are in conformal correspondence, then it is known that their Christoffel symbols are related as [2]:

$$(2.9) \qquad *\left\{{h \atop ji}\right\} = \left\{{h \atop ij}\right\} + \delta^h_j\, p_i + \delta^h_i\, p_j - g_{ji}\, p^h.$$

With the help of (2.4), equation (2.9) can be expressed as

$$(2.10) \qquad *\left\{{h \atop ji}\right\} = N^h_{ji} + (\delta^h_i\, p_j + i|j),$$

which when transformed under a semi-geodesic correspondence [4] takes the following form by virtue of (2.7), (2.8) and (2.9)

$$(2.11) \qquad \overline{*\left\{{h \atop ji}\right\}} = *\left\{{h \atop ji}\right\} + (\delta^h_j\, (2n\phi_i - \partial_i \log \bar{g}/g) + i|j)/2(n-1).$$

Hence:

Theorem (2.3). <u>Under a semi-geodesic correspondence</u> [4] <u>the conformal Christoffel symbols given by (2.9) transform as in (2.11)</u>.

If in equation (2.4) $p^h = 0$, then from (2.6) we get

$$(2.12) \qquad \overline{\left\{{h \atop ji}\right\}} = \left\{{h \atop ji}\right\} + \tfrac{1}{2}(\delta^h_j\, \phi_i + i|j).$$

Conversely if $\left\{{h \atop ji}\right\}$ transforms as in (2.12), then from (2.4) by virtue of (2.6) we get

$$(2.13) \qquad \overline{\left\{{h \atop ji}\right\}} = \left\{{h \atop ji}\right\} + (\bar{g}_{ji}\, \bar{p}^h - g_{ji}\, p^h)$$
$$- (\delta^h_j\, (2\phi_i - \partial_i \log \bar{g}/g) + i|j)/2(n-1).$$

Comparing (2.12) and (2.13) we get

$$(2.14)\qquad \bar{g}_{ji}\,\bar{p}^h - g_{ji}\,p^h = (\delta_j^h\,((n+1)\,\phi_i - \partial_i \log \bar{g}/g + i|j)/2(n-1),$$

which when contracted for h and i, by virtue of (2.7) leads to

$$(2.15)\qquad (n-1)\,(\bar{p}_j - p_j)/2 = 0,$$

i.e., p_j is invariant. Hence we have

Theorem (2.4). <u>If</u> $p^h = 0$, <u>then under a semi-geodesic correspondence</u> *[4]*, <u>Christoffel symbols transform as in (2.12). Conversely if they transform in (2.12), then the vector</u> p_j <u>is invariant under a semi-geodesic correspondence</u>.

<u>Remark</u>. Equation (2.12) can also be obtained from (2.3) with the help of (2.2) for $p_j = 0$.

From equations (2.3), (2.8) and (2.11) we can easily obtain the following entities

$$(2.16)\qquad \left\{{h \atop ji}\right\}^* \overset{\text{def.}}{=\!=\!=} \left\{{h \atop ij}\right\} - (\delta_j^h\,\partial_i \log g + i|j)/2(n+1),$$

$$(2.17)\qquad N^{*h}_{ji} \overset{\text{def.}}{=\!=\!=} N^h_{ji} - (\delta_j^h\,N^r_{ri} + i|j)/(n+1)$$

and

$$(2.18)\qquad B^h_{ij} \overset{\text{def.}}{=\!=\!=} {}^*\left\{{h \atop ji}\right\} - (\delta_j^h\,{}^*\left\{{r \atop ri}\right\} + i|j)/(n+1),$$

which are invariant under the semi-geodesic correspondence. We call them semi-projective invariants.

3. <u>Some Curvature Tensors</u>

Let $N_{kji}{}^h$ be the curvature tensor based on N^h_{ji} then we can easily obtain

$$(3.1)\qquad N_{kji}{}^h = K_{kji}{}^h - (g_{ji}\,{}'\nabla_k\,p^h - j|k),$$

where ${}'\nabla_j$ denotes covariant differentiation with respect to N^h_{ji}.

If $g_{ah} N_{kji}{}^{a} = N_{kjih}$, then this tensor obviously satisfies

(3.2)a $$N_{kji}{}^{h} = - N_{kji}{}^{h}, \quad N_{kjih} = - N_{jkih},$$

(3.2)b $$N_{kjih} + i|h = [g_{ki} (\nabla_j p_h - p_h p_j) + i|h] - j|k$$

and

(3.2)c $$N_{kji}{}^{h} + \text{cycl. } (k,j,i) = 0,$$

where ∇_k denotes covariant derivative with respect to Christoffel symbols.

Putting $N_{ji} = N_{hji}{}^{h}$ and $n_{kj} = N_{kji}{}^{i}$, we get from (3.1)

(3.3)a $$N_{ji} = K_{ji} + (\nabla_j p_i - p_i p_j - g_{ji}\, {}'\nabla_h p^h)$$

and

(3.3)b $$n_{ji} = \nabla_i p_j - i|j.$$

From (3.3)a and (3.3)b we easily obtain

(3.4) $$N_{ji} - N_{ij} = n_{ij}.$$

Hence

Theorem (3.1). <u>The tensor</u> N_{ji} <u>is symmetric in</u> i <u>and</u> j <u>iff</u> p_i <u>is a gradient vector</u>, $\nabla_j p_i \neq 0$.

The Bianchi-identity for $N_{kji}{}^{h}$ can be obtained in the following form

(3.5) $${}'\nabla_l N_{kji}{}^{h} + \text{cycl. } (l,k,j) = 0.$$

Contracting (3.5) for h and k we get

(3.6) $${}'\nabla_l N_{ji} - {}'\nabla_j N_{li} = {}'\nabla_h N_{lji}{}^{h}.$$

Hence:

Theorem (3.2). <u>The tensor</u> ${}'\nabla_l N_{ji}$ <u>is symmetric in</u> l <u>and</u> j <u>iff</u> ${}'\nabla_h N_{lji}{}^{h}$ <u>is identically zero</u>.

If we assume $\nabla_l N_{kji}{}^h = 0$, then from equation (3.1) we can obtain

$$(3.7) \qquad \nabla_l K_{kji}{}^h = (g_{ji} \nabla_l {}'\nabla_k p^h - j|k),$$

which gives on contraction for h and k

$$(3.8) \qquad \nabla_l K_{ji} = g_{ji} \nabla_l {}'\nabla_h p^h - g_{hi} \nabla_l {}'\nabla_j p^h.$$

Multiplying (3.8) by g^{jl} we get

$$(3.9) \qquad \nabla_l K^l_i = \nabla_i {}'\nabla_h p^h - g_{hi} g^{jl} \nabla_l {}'\nabla_j p^h.$$

Also from (3.8) we can obtain

$$(3.10) \qquad \nabla_l \tilde{K} = 0,$$

where $\tilde{K} \overset{\text{def.}}{\equiv} K - (n-1) {}'\nabla_h p^h$.

Hence:

Theorem (3.3). <u>If $N_{kji}{}^h$ has a vanishing covariant derivative then there exists a scalar $\tilde{K}$ whose covariant derivative is zero.</u>

Further if we contract (3.7) for h and i and note that $K_{kji}{}^i = 0$, we can obtain

$$(3.11) \qquad \nabla_l (\nabla_k p_j - j|k) = 0.$$

Hence:

Theorem (3.4). <u>If $N_{kji}{}^h$ has a vanishing covariant derivative then equation (3.11) is identically satisfied.</u>

Further comparing (3.9) and (3.10) and using /2/:

$2\nabla_a K^a_i = \nabla_i K$, we obtain

$$(3.12) \qquad g^{ji} \nabla_l ({}'\nabla_j p^t) + \tfrac{1}{2}(n-3) g^{lt} \nabla_l ({}'\nabla_j p^j) = 0.$$

A Riemannian space is said to be symmetric if it satisfies Cartan [1]

$$(3.13) \qquad \nabla_1 K_{kji}{}^{h} = 0,$$

therefore from (3.1) for a symmetric Riemannian space we get

$$(3.14) \qquad \nabla_1 N_{kji}{}^{h} + g_{ji} \nabla_1 {}'\nabla_k p^h - g_{ki} \nabla_1 {}'\nabla_j p^h = 0,$$

which by virtue of $N_{kji}{}^{h} g^{ji} = N_k^h$ gives

$$(3.15) \qquad \nabla_1 (N_k^h + (n-1) {}'\nabla_k p^h) = 0.$$

Hence:

Theorem (3.5). In a symmetric Riemannian space M^n the tensor $N_k^h + (n-1) {}'\nabla_k p^h$, has a vanishing covariant derivative.

Similarly contracting (3.14) for h and k we get

$$(3.16) \qquad \nabla_1 N_{ji} + g_{ji} \nabla_1 {}'\nabla_h p^h - g_{hi} \nabla_1 {}'\nabla_j p^h = 0,$$

which on multiplication by g^{ji} and simplification leads to

$$(3.17) \qquad \nabla_1 {}'\nabla_j p^j = -(n-1)^{-1} \nabla_1 N,$$

where $N = N_{ji} g^{ji}$.

Substituting from (3.17) in (3.16) we get

$$(3.18) \qquad \nabla_1 {}'\nabla_j p^h = \nabla_1 N_j^h - (n-1)^{-1} \delta_j^h \nabla_1 N,$$

which together with (3.14) gives

$$(3.19) \qquad \nabla_1 N_{kji}{}^{1} = g_{ki} (\nabla_1 N_j^1 - (n-1)^{-1} \nabla_j N) - j|k.$$

Hence

Theorem (3.6). In a symmetric Riemannian space M^n $\nabla_1 N_{kji}{}^{1} = 0$ iff $\nabla_1 N_j^1 = (n-1)^{-1} \nabla_j N$.

It is known that the curvature tensor based on $*\{{}^{h}_{ji}\}$ is given by [2]:

$$(3.20) \quad *K_{kji}{}^{h} = K_{kji}{}^{h} - [(\delta^h_k\, p_{ji} + p^h_k\, g_{ji}) - j|k],$$

where $p_{ji} = \nabla_j\, p_i - p_j\, p_i + \tfrac{1}{2} p_t\, p^t\, g_{ji}$, $p^h_j = g^{ih}\, p_{ji}$.

Also we know that $*K_{jki}{}^{i} = 0$ and $*K_{ji} = *K_{kji}{}^{k}$ is given by

$$(3.21) \quad *K_{ji} = K_{ji} - (n-2)\, p_{ji} - p^t_t\, g_{ji}.$$

If $\nabla_l\, *K_{kji}{}^{h} = 0$, then from (3.20) we can obtain

$$(3.22) \quad \nabla_l\, K_{kji}{}^{h} = (\delta^h_k\, \nabla_l\, p_{ji} + g_{ji}\, \nabla_l\, p^h_k) - j|k.$$

Contracting (3.22) for h and k we get

$$(3.23)\text{a} \quad \nabla_l\, p_{ji} = (\nabla_l\, K_{ji} - g_{ji}\, \nabla_l\, p^t_t)/(n-2),$$

which easily gives

$$(323)\text{b} \quad \nabla_l\, p^h_j = (\nabla_l\, K^h_j - \delta^h_j\, \nabla_l\, p^t_t)/(n-2)$$

and

$$(3.23)\text{c} \quad \nabla_l\, p^t_t = \nabla_l\, K/(2n-1).$$

Substituting from (3.23) in (3.22) and contracting the resulting equation for h and l and using

$$(3.24)\text{a} \quad \nabla_l\, K_{kji}{}^{l} = \nabla_k\, K_{ji} - j|k$$

and

$$(3.24)\text{b} \quad \nabla_l\, K^l_k = \tfrac{1}{2}\, \nabla_k\, K,$$

we get on simplification

$$(3.25) \quad \nabla_k\, K_{ji} - j|k = (2n-5)\, (g_{ji}\, \nabla_k\, K - j|k)/2(2n-1)(n-3).$$

Multiplying (3.25) by g^{ji} we get $\nabla_k\, K = 0$. Hence:

Theorem (3.7). <u>If in a Riemannian space</u> M^n, $\nabla_l *K_{kji}{}^h = 0$, <u>then the scalar</u> K <u>is covariantly constant and the tensor</u> $\nabla_k K_{ji}$ <u>is symmetric in</u> j <u>and</u> k.

In a symmetric Riemannian space from (3.20) we can obtain

$$(3.26)\qquad \nabla_l *K_{kji}{}^h = (\delta_j^h \nabla_l P_{ki} + g_{ki} \nabla_l P_j^h) - j|k,$$

which easily leads to

$$(3.27)a\qquad \nabla_l P_{ji} = (g_{ji} \nabla_l P_t^t + \nabla_l *K_{ji})/(2-n),$$

$$(3.27)b\qquad \nabla_l P_j^h = (\nabla_l *K_j^h + \delta_j^h \nabla_l P_t^t)/(2-n)$$

and

$$(3.27)c\qquad \nabla_l P_t^t = \nabla_l *K/2\,(1-n).$$

Substituting from (3.27) in (3.26) and contracting for h and l we obtain on simplification

$$(3.28)\qquad \nabla_l *K_{kji}{}^l = [\nabla_k *K_{ji} + g_{ji}\,(\nabla_l *K_k^l - \nabla_k *K/(n-1))]/(n-2) - j|k.$$

If $\nabla_j *K = (n-1)\,\nabla_l *K_j^l$, then equation (3.28) gives

$$(3.29)\qquad \nabla_l *K_{kji}{}^l = (\nabla_k *K_{ji} - j|k)/(n-2),$$

while if (3.29) is satisfied then $\nabla_j *K = (n-1)\,\nabla_l *K_j^l$.

Hence:

Theorem (3.8). <u>In a symmetric Riemannian space</u> M^n, <u>the necessary and sufficient condition for (3.29) to be satisfied is given by</u> $\nabla_j *K = (n-1)\,\nabla_l *K_j^l$.

4. <u>Correspondence of curvature tensors</u>

If $\bar{K}_{kji}{}^h$ denotes the curvature tensor based on $\{{}^{h}_{ji}\}$, then from (2.3) we can obtain

$$(4.1)\qquad \bar{K}_{kji}{}^h = K_{kji}{}^h + (\delta_j^h B_{ki} - j|k)/2(n+1),$$

where

$$B_{ji} \overset{\text{def.}}{=\!=\!=} \partial_j\, \partial_i \log \bar{g}/g - \{^{a}_{ji}\}\, \partial_a \log \bar{g}/g -$$
$$(\partial_j \log \bar{g}/g)\cdot(\partial_i \log \bar{g}/g)/2(n+1).$$

Contracting (4.1) we can easily obtain

$$(4.2)\text{a} \qquad \bar{K}_{ji} = K_{ji} - (n-1)\, B_{ji}/2(n+1)$$

and

$$(4.2)\text{b} \qquad \bar{K}_{kji}{}^{i} = 0.$$

If $\bar{N}_{kji}{}^{h}$ denotes the curvature tensor based on $\bar{N}^{h}_{ji}$, then from equation (2.8) we can obtain

$$(4.3) \qquad \bar{N}_{kji}{}^{h} = N_{kji}{}^{h} - [\delta^{h}_{i}\, \nabla_k\, \phi_j - \delta^{h}_{k}\, \{q_{ji} + (2\phi_i - \partial_i \log \bar{g}/g)\cdot$$
$$\cdot(2\phi_j - \partial_j \log \bar{g}/g)/2(n-1)\}]/2(n-1) - j|k,$$

where

$$q_{ji} \overset{\text{def.}}{=\!=\!=} 2\nabla_j\phi_i - \partial_j\, \partial_i \log \bar{g}/g + N^{a}_{ji}\, \partial_a \log \bar{g}/g.$$

From (4.3) on contraction for h and k and h and i respectively we obtain

$$(4.4)\text{a} \qquad \bar{N}_{ji} = N_{ji} - (\nabla_i\phi_j - \nabla_j\phi_i)/2(n-1)$$
$$+\tfrac{1}{2}\{q_{ji} + (2\phi_i - \partial_i\log \bar{g}/g)\ (2\phi_j - \partial_j\log \bar{g}/g)/2(n-1)\}$$

and

$$(4.4)\text{b} \qquad \bar{n}_{ji} = n_{ji} - (n+2)\ (\nabla_j\phi_i - i|j)/2(n-1).$$

If $*\bar{K}_{kji}{}^{h}$ denotes the curvature tensor based on $*\overline{\{^{h}_{ji}\}}$ then from (2.11) we get

$$(4.5) \qquad *\bar{K}_{kji}{}^{h} = *K_{kji}{}^{h} + (2n\delta^{h}_{i}\, \nabla_k\phi_j + \delta^{h}_{j}\, D_{ki})/2(n-1) - j|k,$$

where

$$D_{ki} \overset{\text{def.}}{=\!=\!=} (2n\ \partial_k\phi_i - \partial_k\, \partial_i \log \bar{g}/g) - \{^{a}_{ki}\}\ (2n\phi_a - \partial_a \log \bar{g}/g)$$
$$-\ (2n\phi_k - \partial_k \log \bar{g}/g)\ (2n\phi_i - \partial_i \log \bar{g}/g)/2(n-1).$$

From (4.5) we can easily obtain

$$(4.6)\text{a} \qquad {}^*\bar{K}_{ji} = {}^*K_{ji} + n(\nabla_i \emptyset_j - j|i)/(n-1) - \tfrac{1}{2} D_{ji}$$

and

$$(4.6)\text{b} \qquad \bar{\mu}_{ji} = n(n+1)\ (\nabla_j \emptyset_i - i|j)/(n-1),$$

where $\bar{\mu}_{ji} = {}^*K_{jik}{}^{k}$.

5. Semi-projective invariants

By eliminating B_{ji} from (4.1) and (4.2)a we easily obtain the following semi-projectively invariant curvature tensor

$$(5.1) \qquad P_{kji}{}^{h} = K_{kji}{}^{h} + (\delta_j^h K_{ki} - j|k)/(n-1),$$

which is well known projective curvature tensor [2]. Hence:

Theorem (5.1). The Weyl's projective curvature tensor $P_{kji}{}^{h}$ is also invariant under the semi-geodesic correspondence [5], in a Riemannian space M^n.

Eliminating $(\nabla_j \emptyset_i - i|j)$ from (4.4)a, b we get

$$(5.2) \qquad q_{ji} + (2\emptyset_i - \partial_i \log \bar{g}/g)(2\emptyset_j - \partial_j \log \bar{g}/g)/2(n-1)$$

$$= 2[\bar{N}_{ji} - N_{ji} + (\bar{\eta}_{ji} - \eta_{ji})/n+2)].$$

Substituting from (4.4)b and (5.2) in (4.3) we obtain on simplification the following semi-projectively invariant curvature tensor

$$(5.3) \qquad S_{kji}{}^{h} \overset{\text{def.}}{=\!=\!=} N_{kji}{}^{h} - \delta_i^h\ \eta_{kj}\ (n+2)^{-1}$$

$$- \{\delta_k^h\ (N_{ji} + (n+2)^{-1}\ \eta_{ji}) - j|k\}/(n-1)$$

Contracting (5.3) and using $S_{ji} = S_{hji}{}^{h}$ and $\psi_{ji} = S_{kji}{}^{i}$, we get

$$(5.4) \qquad S_{ji} = 0, \quad \psi_{ji} = (n-2)(n-1)^{-1}(n+2)^{-1}\ \eta_{ij}.$$

From (5.3) we can observe that if $S_{kji}{}^{h} = 0$, then

$$(5.5) \qquad N_{kji}{}^{h} = (n+2)^{-1}\,\delta_i^h\,\eta_{kj} + (n-1)^{-1}\{\delta_k^h\,(N_{ji} + (n+2)^{-1}\,\eta_{ji}) - j|k\}.$$

Contracting (5.5) for h and i we obtain for $n > 4$, $\eta_{ji} = 0$. Hence we have

Theorem (5.2). <u>If in a Riemannian space</u> M^n, $S_{kji}{}^{h} = 0$, $(n>4)$, <u>then the tensor</u> N_{ji} <u>is a symmetric tensor</u>.

Also from (5.3) we can easily establish

Theorem (5.3). <u>If in a Riemannian space</u> M^n <u>any two of the following conditions</u> i) $S_{kji}{}^{h} = 0$, ii) $N_{kji}{}^{h} = 0$, iii) $N_{ji} = 0$, <u>are satisfied then the third is also satisfied</u>.

Further if in (5.3) $\eta_{kj} = 0$, then we can get

$$(5.6) \qquad S_{kji}{}^{h} = N_{kji}{}^{h} - (n-1)^{-1}\{\delta_k^h\,N_{ji} - j|k\}.$$

Hence:

Theorem (5.4). <u>If in a Riemannian space</u> M^n, $\eta_{ji} = 0$, <u>then the necessary and sufficient condition for</u> $S_{kji}{}^{h}$ <u>to be zero is given by</u>

$$N_{kji}{}^{h} = (n-1)^{1}\,(\delta_k^h\,N_{ji} - j|k).$$

From (5.3) by virtue of (3.2)c we can easily get

$$(5.7) \qquad [S_{kji}{}^{h} - \delta_k^h\,\eta_{ji}\,(n-1)^{-1}\,(n+2)^{-1}] + \text{cycl.}\,(k,j,i) = 0,$$

while from (3.5) and (3.6) equation (5.3) yields

$$(5.8) \qquad [{}'\nabla_l\,S_{kji}{}^{h} - (n-1)^{-1}\{\delta_l^h\,({}'\nabla_p\,N_{kji}{}^{p} - (n+2)^{-1}\,{}'\nabla_i\,\eta_{ki})\}]$$
$$+ \text{cycl.}(l,k,j) = 0.$$

Replacing h by l in (5.8) we get

$$(5.9) \qquad {}'\nabla_l\,S_{kji}{}^{l} = (n-1)^{-1}\,[(2n-3)\,({}'\nabla_k\,N_{ji} - j|k)$$
$$-(n-2)\,(n+2)^{-1}\,{}'\nabla_i\,\eta_{kj}].$$

If ${}'\nabla_i\,\eta_{kj} = 0$, equation (5.9) reduces to

$$(5.10) \qquad {}'\nabla_l\,S_{kji}{}^{l} = (2n-3)\,(n-1)^{-1}\,({}'\nabla_k\,N_{ji} - j|k).$$

Hence:

Theorem (5.5). In a Riemannian space M^n satisfying $'\nabla_i\, n_{kj} = 0$, the tensor $'\nabla_k\, N_{ji}$ is symmetric in j and k iff $'\nabla_m\, S_{kji}{}^m = 0$.

From equation (4.6)a we can easily obtain

$$(5.11)\text{a}\quad 2(*K_{ji} - *\bar{K}_{ji}) + 2n(n-1)^{-1}\,(\nabla_i \emptyset_j - j|i) = D_{ji}$$

and

$$(5.11)\text{b}\quad (*K_{ji} + *K_{ij}) - (*\bar{K}_{ii} + *\bar{K}_{ij}) = D_{ji}.$$

Comparing (5.11)a and (5.11)b we can obtain

$$(5.12)\qquad (*K_{ji} - *\bar{K}_{ji}) - (*K_{ij} - *\bar{K}_{ij}) = 2n(n-1)^{-1}\,(\nabla_j \emptyset_i - i|j).$$

Substituting the value of $(\nabla_j \emptyset_i - i|j)$ and D_{ji} from (5.11)b and (5.12) in (4.5) we obtain on simplification

$$(5.13)\qquad V_{kji}{}^h \overset{\text{def.}}{=\!=\!=} *K_{kji}{}^h + \tfrac{1}{2}\left[\delta_i^h\, *K_{kj} + (n-1)^{-1}\,\delta_j^h\,(*K_{ki} + *K_{ik})\right] - j|k.$$

From (5.13) for $V_{kji}{}^k = V_{ji}$ and $V_{kji}{}^i = \nu_{kj}$, we obtain

$$(5.14)\text{a}\quad V_{ji} = 0$$

and

$$(5.14)\text{b}\quad \nu_{ji} = \tfrac{1}{2}\, n\,(*K_{ji} - i|j).$$

From (5.13) analogous to (5.3) we easily obtain the following theorem:

Theorem (5.6). If in a Riemannian space M^n any two of the following conditions, i) $V_{kji}{}^h = 0$, ii) $*K_{kji}{}^h = 0$, iii) $*K_{kj} = 0$, are satisfied then the third is also satisfied.

If $V_{kji}{}^h = 0$, then from (5.13) and (5.14)b we can easily obtain $*K_{ji} = *K_{ij}$. Hence:

Theorem (5.7). If in a Riemannian space the tensor $V_{kji}{}^h = 0$, then the tensor $*K_{ji}$ is symmetric in j and i.

If p_i is a gradient vector, equation (5.13) by virtue of (3.20) and (5.1) can be expressed as

$$(5.15) \qquad V_{kji}{}^{h} = P_{kji}{}^{h} + [p_j^h g_{ki} + \delta_j^h (p_{ki} - p_t^t g_{ki})/(n-1)] - j|k,$$

which for $V_{kji}{}^{h} = 0$, gives

$$(5.16) \qquad P_{kji}{}^{h} = [p_k^h g_{ji} + \delta_k^h (p_{ji} - p_t^t g_{ji})/(n-1)] - j|k.$$

Hence:

Theorem (5.8). <u>If p_i is a gradient vector then the necessary and sufficient condition for the vanishing of $V_{kji}{}^{h}$ is that the projective curvature tensor is expressed as</u> (5.16).

If we assume $\nabla_l K_{kji}{}^{h} = 0$, then (5.13) implies

$$(5.17) \qquad \nabla_l {}^*K_{kji}{}^{l} = \tfrac{1}{2}[\nabla_i ({}^*K_{jk} - j|k) + \{\nabla_k ({}^*K_{ji} + i|j) + j|k\}/(n-1)].$$

Now if we assume that $\nabla_i ({}^*K_{jk} - j|k) = 0$, then equation (5.17) implies

$$(5.18) \qquad \nabla_l {}^*K_{kji}{}^{l} = (\nabla_k {}^*K_{ji} + j|k)/(n-1).$$

Since $\nabla_l {}^*K_{kji}{}^{l}$ is skew-symmetric in j and k and the right hand side of (5.18) is symmetric in j and k we easily obtain

$$(5.19) \qquad \nabla_k {}^*K_{ji} + j|k = 0.$$

Hence:

Theorem (5.9). <u>If in a Riemannian space M^n, the tensor $V_{kji}{}^{h}$ has a vanishing covariant derivative and the tensor $\nabla_j p_i - \nabla_i p_j$ is covariantly constant, then the tensor $\nabla_k {}^*K_{ji}$ is skew-symmetric in</u> k <u>and</u> j.

If in (5.17) ${}^*K_{ji} = -{}^*K_{ji}$, then we easily obtain

$$(5.20) \qquad \nabla_l {}^*K_{kji}{}^{l} = \nabla_i {}^*K_{jk},$$

which gives

Theorem (5.10). *If in a Riemannian space M^n, $\nabla_l V_{kji}{}^h = 0$, and the tensor $*K_{ji}$ is skew-symmetric in j and i, then the vanishing of $\nabla_i *K_{jk}$ implies the vanishing of $\nabla_l *K_{kji}{}^l$ and vice-versa.*

Remark

One can also obtain the properties of symmetric Riemannian spaces based on $V_{kji}{}^h$.

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"*Effect of Perturbative Force on Motion and Stability of Interconnected Satellite System*"

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Abstract

The effect of atmospheric resistance and magnetic forces on the relative motion of two charged satellites connected by a light flexible inextensible and non-conducting string in the earth's central gravitational field of force is discussed. It has been assumed that the centre of mass of the system moves along a keplerian orbit under Lorentz magnetic and atmospheric resistance. Under the assumption that the centre of mass moves in a circular orbit the line-arised and normalised equations of motion are derived. It is supposed that the satellites are subjected to the impacts absolutely non-elastic in nature, when the string becomes tight. Finally the motion of the system has been considered in equitorial plane of the earth, taking into account the non-dissipative part of the atmospheric resistance on the relative motion of the system. Although this amounts to over simplification of the problem, reducing the same to scalaronomic system and hence admitting the Jacobi's integral. But the existence of this single integral comes to our rescue in unveiling the behaviour of its motion relative to the centre of mass of the system to a great extent. With the help of this single integral, we have been able to obtain the sufficient conditions for the non-evolutional motion of the system in which the system always moves like a dumbbell satellite and the string is always taut.

The paper contains also discussion about regions in which the motion of the system remains always similar to the dumbbell satellite and the string never slackens up.

Introduction

The present study is devoted to the study of the effect of atmospheric resistance and the magnetic force on the motion of a system of two cable connected satellites in the central gravitational field of force.

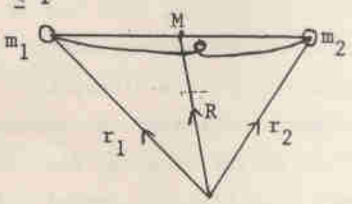
The satellites are considered as material particles moving in Lorentz force field and atmosphere. The motion of each of them relative to their centre of mass has been studied. It is supposed that the centre of moves along a given keplerian elliptical orbit. It is assumed throughout the work that the satellites are subjected to absolutely non-elastic impacts when the string tightens up. The cable connecting the two satellites is supposed to be light, flexible inextensible and non-conducting.

1. Motion of the Centre of Mass

Let us first consider the motion of the centre of mass of the system under the central gravitational force only. Let $\vec{r}_1$ and $\vec{r}_2$ denote the radius vector of the particle m_1 and m_2 respectively with respect to the attracting centre and 'l' denote the length of the string connecting them.

The constraint of the system is given by

$$|\vec{r}_1 - \vec{r}_2|^2 \leq l^2 \qquad \ldots (1.1)$$



(Fig. 1)

The Lagrange's equation of motion of the first kind for the particles m_1 and m_2 are respectively

$$m_1\ddot{\vec{r}}_1 + \frac{m_1\mu\vec{r}_1}{r_1^3} + 2\lambda\ (\vec{r}_1 - \vec{r}_2) = 0$$

$$m_2\ddot{\vec{r}}_2 + \frac{m_2\mu\vec{r}_2}{r_2^3} + 2\lambda\ (\vec{r}_2 - \vec{r}_1) = 0 \qquad \ldots (1.2)$$

Adding the equations in (1.2) we get

$$M\ddot{\vec{R}} + \mu\ \left(\frac{m_1\vec{r}_1}{r_1^3} + \frac{m_2\vec{r}_2}{r_2^3}\right) = 0 \qquad \ldots (1.3)$$

where $$R = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

and $$M = m_1 + m_2 \qquad \ldots (1.4)$$

Let ρ_1 and ρ_2 denote the radius vector of the particle m_1 and m_2 respectively relative to their common centre of mass then

$$\vec{r}_1 = \vec{R} + \rho_1$$

$$\vec{r}_2 = \vec{R} + \rho_2 \qquad \ldots (1.5)$$

It is but natural to consider that maximum distance 'ℓ' between the particles is infinitesimally small compared to the distance r_1 and r_2 of the particles m_1 and m_2 from the centre of attraction.

i.e. $\frac{1}{r_1} << 1$ and $\frac{1}{r_2} << 1$

Now $\rho_1 \leq \ell$ and $\rho_2 \leq \ell$ thus

$$\rho_1 << r_1 \text{ and } \rho_2 << r_2$$

But $r_1 \simeq r_2 \simeq R$. Hence

$$\frac{\rho_1}{R} << 1 \text{ and } \frac{\rho_2}{R} << 1 \qquad \ldots (1.6)$$

Substituting the values of $\vec{r}_1$ and $\vec{r}_2$ from (1.5) in the expression under bracket in (1.3) and then expanding in ascending power of the small quantities (1.6) we get

$$M\ddot{\vec{R}} + \frac{\mu M\vec{R}}{R^3} = F_1 + F_2 \qquad \ldots (1.7)$$

where $$F_1 = \frac{3\mu}{R^3}(m_1\vec{\rho}_1 + m_2\vec{\rho}_2) + \frac{3\mu\vec{R}}{R^5}\left[\vec{R}\,(m_1\rho_1 + m_2\rho_2)\right]$$

and $$F_2 = \frac{3}{2}\frac{\mu}{R^3}\left[m_1\left\{\left(\frac{\rho_1}{R}\right)^2 - 5\left(\frac{\vec{R}}{R}\cdot\frac{\vec{\rho}_1}{R}\right)^2\right\} + m_2\left\{\left(\frac{\rho_2}{R}\right)^2 - 5\left(\frac{\vec{R}}{R}\cdot\frac{\vec{\rho}_2}{R}\right)^2\right\}\right] \times \vec{R}$$

$$+ \frac{3\mu}{R^5} m_1 \, (\vec{R}\,\vec{\rho}_1)\, \vec{\rho}_1 + \frac{3\mu m_2}{R^5} \, (\vec{R}\,\vec{\rho}_2)\, \vec{\rho}_2 \qquad \dots (1.8)$$

Other terms in (1.7) are of higher order in the infinitesimals. From (1.4) and (1.5) we can easily obtain

$$\vec{\rho}_1 = \frac{m_2}{m_1+m_2} \, (\vec{r}_1 - \vec{r}_2)$$

$$\vec{\rho}_2 = \frac{m_1}{m_1+m_2} \, (\vec{r}_2 - \vec{r}_1) \qquad \dots (1.8A)$$

From (1.8A) we get

$$m_1\rho_1 + m_2\rho_2 = 0 \qquad \dots (1.9)$$

Obviously the first order perturabation term F_1 is identically zero. Thus the motion of the centre of mass upto the first order infinitesimal can be given by

$$M\ddot{\vec{R}} + \frac{\mu M}{R^3} \vec{R} = 0 \qquad \dots (1.10)$$

We have simply neglected the second and higher order terms in the infinitesimal

$$\frac{\rho_1}{R} \text{ and } \frac{\rho_2}{R}$$

This shows that the motion takes place along keplerian orbit to a good degree of approximation.

2. Equation of Motion of the System Relative to its Centre of Mass

Let us now consider the motion of the system relative to the centre of mass taking into account the atmospheric resistance and Lorentz magnetic force. The equations of relative motion can be deduced with the help of Lagrange's equations of motion of the first kind as there is a constraint imposed on the system.

The Lagrange's equation of motion of the first kind for the particles m_1 and m_2 are

$$m_1\ddot{\vec{r}}_1 + \frac{m_1\mu\vec{r}_1}{r_1^3} = \lambda(\vec{r}_1-\vec{r}_2) + \theta_1\ (\vec{r}_1 \wedge H) + \rho_a c_1 m_1\ |\dot{\vec{r}}_1|\ \dot{\vec{r}}_1$$

$$m_2\ddot{\vec{r}}_2 + \frac{m_2\mu\vec{r}_2}{r_2^3} = \lambda(\vec{r}_2-\vec{r}_1) + \theta_2\ (\vec{r}_2 \wedge H) + \rho_a c_2 m_2\ |\dot{\vec{r}}_2|\ \dot{\vec{r}}_2 \quad \ldots(2.1)$$

Where

λ = Legrange's undetermined multiplier

μ = Product of the gravitational constant with the mass of the attracting centre.

$\theta_i = \dfrac{\text{charge } q_i \text{ on the particle}}{\text{velocity of light } c}$ $(i = 1, 2)$

H The intensity of the earth's magnetic field for equatovial satellites $= -\nabla\ (\dfrac{\vec{M}\vec{r}}{r^3})$

M = Magnetic moment of the earth. Subtracting the two equations in (2.1) we get

$$(\ddot{\vec{r}}_1-\ddot{\vec{r}}_2) = \mu\ [\frac{\vec{r}_1}{r_1^3} - \frac{\vec{r}_2}{r_2^3}] + (\vec{r}_1-\vec{r}_2)\ (\frac{\lambda}{m_1} - \frac{\lambda}{m_2})$$

$$- [\frac{\theta_1}{m_1}\ (\vec{r}_1 \wedge \nabla\ \frac{M\vec{r}_1}{r_1^3}) - \frac{\theta_2}{m_2}\ (\vec{r}_2 \wedge \nabla\ \frac{M\vec{r}_2}{r_2^3})\ ]$$

$$+ \rho_a\ [\ c_1\ |\dot{\vec{r}}_1|\dot{\vec{r}}_1 - c_2\ |\dot{\vec{r}}_2|\ \dot{\vec{r}}_2\ ] \qquad \ldots (2.2)$$

Now since

$$\ddot{\vec{r}}_1 - \ddot{\vec{r}}_2 = -\ (\frac{m_1 + m_2}{m_1})\ \ddot{\vec{\rho}}_2 \qquad \ldots (2.3)$$

The equation of motion with non-dissipative part of the aerodynamic forces reduces to the form.

$$\ddot{\vec{\rho}}_2 + \frac{\mu}{R^3}\ \vec{\rho}_2 - \frac{3\mu\vec{R}}{R^5}\ (\vec{R}\ \vec{\rho}_2) - (\frac{m_1 + m_2}{m_1 m_2})\ \lambda\vec{\rho}_2$$

$$= .\ \frac{m_1}{m_1 + m_2}\ \dot{\vec{R}} \wedge \nabla\{(\frac{M\vec{R}}{R^3})\ \ (\frac{\theta_1}{m_1} - \frac{\theta_2}{m_2})\} + a_1\dot{\vec{R}} \qquad \ldots (2.4)$$

Again let us write

$\vec{K}_\varepsilon$ = Unit vector along the axis of the magnetic dipole of the earth.

μ_ε = The volume of the magnetic moment of the earth dipole.

$\vec{e}_r$ = Unit vector along the radius vector

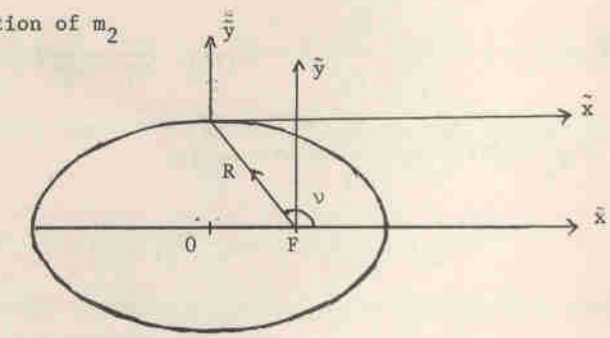
$$R = \frac{\vec{R}}{|R|}$$

So the equation (2.4) reduces to .....

$$\ddot{\vec{\rho}}_2 + \frac{\mu}{R^3}\vec{\rho}_2 - \frac{3\mu R}{R^5}(\vec{R}\,\vec{\rho}_2) - \frac{m_1+m_2}{m_1\,m_2}\lambda\,\vec{\rho}_2 = \frac{m_1}{m_1+m_2}\left(\frac{\theta_1}{m_1}-\frac{\theta_2}{m_2}\right)$$

$$\times\left[\frac{\mu_\varepsilon}{R^3}\dot{\vec{R}}\wedge K_\varepsilon - 3(\vec{K}_\varepsilon.e_r)\,\vec{R}\wedge\vec{e}_r\right] + a_1\dot{\vec{R}} \qquad \ldots (2.5)$$

This is the basic equation of motion of the system which describes the relative motion of m_2



(Fig. 2)

Now the cartesian system of coordinates $(\tilde{x}_1\tilde{y}_1 z)$ will be introduced with the origin at the centre of mass of the system and axis as shown in Fig. 2 (Zaxis being normal to the plane of the figure) though F not shown in the figure then

$$\rho_2 = \tilde{x}\,\vec{I} + \tilde{y}\,\vec{J} + \tilde{z}\,\vec{K}$$

$$\vec{R} = \vec{I}\,\vec{R}\cos\nu + \vec{J}\,\vec{K}\sin\nu$$

Therefore $\vec{e}_r = |\vec{R}| \;/\; |R| = i \cos\nu + j \sin\nu$

Where $\vec{i}$, $\vec{j}$, $\vec{k}$ are unit vector along x,y and z axis respectively and ν is the true anomally of the centre of mass.

i = inclination of the orbit with the equatorial plan of the earth.

W = Argument of perijee.

K_ε = (sinw sini) $\vec{i}$ + (cosw sini) $\vec{j}$ + co si $\vec{k}$

With this relation for $\vec{K\varepsilon}$ the cartesian equivalent of the vector equation (2.5) can now be easily obtained as --

$$\ddot{\tilde{x}} + \frac{\mu}{R^3}\tilde{x} - \frac{3\mu}{R^3}(\tilde{x}\cos\nu + \tilde{y}\sin\nu) - \left(\frac{m_1+m_2}{m_1\, m_2}\right)\lambda\tilde{x}$$

$$= \frac{-m_1}{m_1+m_2}\left(\frac{\theta_1}{m_1} - \frac{\theta_2}{m_2}\right)\left[\frac{\mu_\varepsilon}{R^3}\cos i\,(\sin\nu.\dot{\vec{R}} + \cos\nu\,\vec{R}.\dot{\nu})\right]$$

$$+ a_1(\dot{R}\cos\nu - R\dot{\nu}\sin\nu) \;\ldots\; (2.6A)$$

$$\ddot{\tilde{y}} + \frac{\mu}{R^3}\tilde{y} - \frac{3\mu}{R^3}\sin\nu\,(\tilde{x}\cos\nu + \tilde{y}\sin\nu) - \left(\frac{m_1+m_2}{m_1\, m_2}\right)\lambda\tilde{y}$$

$$= -\frac{m_1}{m_1+m_2}\left(\frac{\theta_1}{m_1} - \frac{\theta_2}{m_2}\right)\left[\frac{\mu_\varepsilon}{R^3}\{\cos i\,(\sin\nu\,\vec{R}\dot{\nu} - \cos\nu\dot{R})\}\right]$$

$$+ a_1(\dot{R}\sin\nu + R\dot{\nu}\cos\nu) \;\ldots \qquad (2.6B)$$

$$\ddot{\tilde{z}} + \frac{\mu}{R^3}\tilde{z} - \left(\frac{m_1+m_2}{m_1\, m_2}\right)\lambda\tilde{z} = -\frac{m_1}{m_1+m_2}\left(\frac{\theta_1}{m_1} - \frac{\theta_2}{m_2}\right).\ \sin i\ x$$

$$\left[\frac{\mu_\varepsilon}{R^3}\{\cos(\nu+\omega)\dot{R} - \sin(\nu+\omega)\,R\dot{\nu}\} + 3\sin(\nu+\omega)\,R\dot{\nu}\right] \ldots (2.6C)$$

and the equation of the constraint takes the form

$$\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2 \leq 1 \qquad \ldots (2.7)$$

In order to study the normal and transversal motion we shall introduce the rotating system of coordinates with the origin at the centre of mass of the system. Axis ξ along the radius vector axis η towards

the transversal to the orbit of the centre of mass in the direction of the motion and axis ξ being directed along the normal to the orbital plane of the centre of mass of the system.

In the rotating frame of reference the system of equation (2.6) will be transformed as -

$$\ddot{\xi} - 2\dot{\nu}\dot{\eta} - \ddot{\nu}\eta - \dot{\nu}^2\xi - \frac{2\mu}{R^3}\xi - \left(\frac{m_1+m_2}{m_1\, m_2}\right)\lambda\xi$$

$$= - \left(\frac{m_1}{m_1+m_2}\right) \left(\frac{\theta_1}{m_1} - \frac{\theta_2}{m_2}\right) \left[\frac{\mu\varepsilon}{R^3} (R\nu)\right] \cos i + a_1\dot{R} \qquad \ldots (2.8A)$$

$$\ddot{\eta} + 2\dot{\nu}\dot{\xi} - \dot{\nu}^2\eta + \ddot{\nu}\xi + \frac{\mu}{R^3}\eta - \left(\frac{m_1+m_2}{m_1\, m_2}\right)\lambda\eta$$

$$= - \left(\frac{m_1}{m_1+m_2}\right) \left(\frac{\theta_1}{m_1} - \frac{\theta_2}{m_2}\right) \left[\frac{\mu\varepsilon}{R^3} \cdot \dot{R}\right] \cos i + a_1 R\dot{\nu} \qquad \ldots (2.8B)$$

$$\ddot{\zeta} + \frac{\mu}{R^3}\zeta - \left(\frac{m_1+m_2}{m_1\, m_2}\right)\lambda\zeta = - \left(\frac{m_1}{m_1+m_2}\right) \left(\frac{\theta_1}{m_1} - \frac{\theta_2}{m_2}\right) \times$$

$$\operatorname{Sin} i \left[\frac{\mu\varepsilon}{R^3} R \cos (\nu+\omega) + \left(3 - \frac{\mu\varepsilon}{R^3}\right) \operatorname{Sin} (\nu+\omega) R\dot{\nu}\right] \ldots (2.8C)$$

and the inequality (2.7) giving the condition of constraint assumes the new form

$$\xi^2 + \eta^2 + {}^2 \le 1 \qquad \ldots (2.9)$$

The Nechvil's transformation given by the following equation

$$\xi = \rho x , \ \eta = \rho y , \ \zeta = \rho z \qquad \ldots (2.10)$$

where $\rho = R/p = \frac{1}{1+e \cos\nu}$ ... (2.11)

and p = focal parameter

e = eccentricity of the orbit.

transform the equation (2.8) into the following set of equations (2.13) where the true anomaly of the centre of mass is given by the relation

$$\overset{\circ}{\nu} = \frac{d\nu}{dt} = \frac{\sqrt{\mu p}}{p^2} \cdot \frac{1}{\rho^2} \qquad \ldots (2.12)$$

$$x'' - 2y' - 3x\rho = \lambda_{\alpha} x - \frac{A}{\rho} \cos i + f\rho\rho'$$

$$y'' + 2x' = \lambda_{\alpha} y - \frac{A\rho'}{\rho^2} \cos i + f\rho^2$$

$$z'' + z = \lambda_{\alpha} z - \frac{A}{\rho} [\frac{\rho'}{\rho} \cos (\nu+\omega) + \frac{A}{\mu_{\varepsilon}} .$$

$$(3p^3\rho^3 - \mu\varepsilon) \sin (\nu+\omega)] \sin i \qquad \ldots (2.13)$$

where $A = \frac{m_1}{m_1+m_2} (\frac{\theta_1}{m_1} - \frac{\theta_2}{m_2}) \frac{\mu_\varepsilon}{\sqrt{\mu p}}$ and $\lambda_{\alpha} = \frac{p^3\rho^4}{\mu} (\frac{m_1+m_2}{m_1 m_2})$

deshes representing differentiation with respect to ν .

The condition of the constraint (2.9) is now given by

$$x^2 + y^2 + z^2 \le \frac{1}{\rho^2} \qquad \ldots (2.14)$$

3. Motion of the System in the Case of Circular Orbit of the Centre of Mass

Let us assume that the centre of mass of the system moves along the circular orbit. Thus $e = 0$ and $\rho = 1$ i.e. $\rho^1 = o$. We shall assume that the motion of the system takes place in the plane of the orbit of the centre of mass and hence the Z-Co-ordinate will vanish. Thus the equations governing the motion are given by

$$x'' - 2y' \; 3x = \lambda_{\alpha} x - A \cos i$$

$$y'' + 2x' = \lambda_{\alpha} y + f \qquad \ldots (3.1)$$

With the condition of the constraint

$$x^2 + y^2 \le 1 \qquad \ldots (3.2)$$

As we are dealing with the unilateral constraint (3.2) there will arise three cases.

(i) The case of loose string (Free-motion)

(ii) The case of tight string (Constrained-motion)

(iii) The case of alternately loose and tight string (Evolutional motion)

Let us first consider the case of loose string. The motion of the system is free constraint and therefore $\lambda_{\ell} = 0$ and $x^2 + y^2 = 1$.

Thus m_2 moves inside the circle

$$x^2 + y^2 = 1 \qquad \ldots (3.3)$$

The equation of motion in this case takes the form

$$\begin{aligned} x'' - 2y' - 3x &= - A \cos i \\ y'' + 2x' &= f \end{aligned} \qquad \ldots (3.4)$$

Whose solutions can be easily obtained as

$$x = 2c_1 + A \cos i + 2f\nu + c_2 \cos\nu + c_3 \sin\nu$$

$$y = - (3c_1 + 2A \cos i)\nu - \frac{3}{2} f\nu^2 - 2 (c_2 \sin\nu - c_3 \cos\nu) + c_4 \qquad \ldots (3.5)$$

Because of the presence of the secular terms $2f\nu$ and $-(3c_1+2A\cos i)\nu - 3/_2 f\nu^2$ we see that the moving particle is bound to touch the circle $x^2 + y^2 = 1$ at certain epoch and after this moment the string will become tight and the particle will start moving on the circle and its motion will now be governed by the equation (3.1).

As the centre of mass is moving along a circular orbit and the time does not appear in the system of equations (3.1) explicitly. The energy equation must exist for the problem.

Now in order to obtain jacobi's integral for the motion of the particle m_2, we multiply first equation of (3.1) by x' and second by y' and then adding we get

$$x'x'' + y'y'' - 3xx' = \lambda_\alpha (xx' + yy') - Ax' \cos i + fy' \quad \ldots (3.6)$$

So for the unit circle

$$x^2 + y^2 = 1 \text{ and } xx' + yy' = 0$$

Hence

$$x'x'' + y'y'' - 3xx' = - Ax' \cos i + fy'$$

Integrating we get

$$\frac{1}{2} (x'^2 + y'^2) - \frac{1}{2} . 3x^2 = - Ax\cos i + fy + h/2$$

$$\text{i.e. } x'^2 + y'^2 - 3x^2 = - 2Ax\cos i + 2fy + h \quad \ldots (3.7)$$

Again multiplying the first and the second equation of (3.1) by x and y respectively and then adding we get

$$xx'' + yy'' = 2(xy'-x'y) + \lambda_\alpha (x^2+y^2) + 3x^2 - Ax\cos i + fy \ldots (3.8)$$

Differentiating (3.6)A with respect to ν we get

$$x'^2 + y'^2 = - (xx'' + yy'') \quad \ldots (3.9)$$

So (3.8) and (3.9) give

$$- \lambda_\alpha = (x'^2+y'^2) + 2(xy'-x'y) + 3x^2 - Ax\cos i + fy \quad \ldots (3.10)$$

Now to simplify the equation (3.7) and (3.10) further we switch on to the polar coordinates on the unit circle by substituting

$$x = \cos\Psi$$
$$x = \sin\Psi \quad \ldots (3.11)$$

Using these and their derivatives, the equation (3.7) and (3.10) reduce respectively to

$$\Psi'^2 = 3\cos^2\Psi - 2A\cos\Psi \cos i + h + 2f \sin\Psi \quad \ldots (3.12)$$

which is the Jacobian integral of the equation of motion in polar form, h being Jacobian constant.

$$-\lambda_\alpha = (\Psi'+1)^2 + 3\cos^2\Psi - A\cos\Psi \cos i + f \sin\Psi - 1 \quad \ldots (3.13)$$

From which the Lagrane's indetermined multiplier λ can be obtained.

The mechanical implication of Lagrange's indetermined multiplier is that the constraint is effective as long as λ_{α} (t) $\leq$ o and whenever this condition does not hold the constraint is not operative. Then the particle moves with loose string. Thus we obtain the condition of the constraint motion in the form

$$(\psi' + 1)^2 + 3\cos^2\psi - A\cos\psi \cos i - 1 + f \sin\psi \geq o \qquad \ldots (3.14)$$

Hence forth in our subsequent discussion we shall assume that the condition (3.14) is satisfied with the help of (3.12) and (3.14). The region of possible motion of the particle m_2 on the unit circle (3.3) can be determined. From (3.12) we obtain

$$h - 2A\cos\psi \cos i + 3\cos^2\psi + 2f \sin\psi \gg o \qquad \ldots (3.15)$$

obviously this holds if

$$h > 2A\cos\psi \cos i + 2f \sin\psi \qquad \ldots (3.16)$$

It is not very difficult to obtain from (3.15):

$$(f^2 + A^2\cos^2 i) \cos^2\psi - Ah\cos i \cos\psi + (h^2/4 - f^2) > o$$

or $$\cos\psi = \frac{Ah\cos i \pm \sqrt{A^2h^2\cos^2 i - 4(h^2/4 - f^2)(f^2 + A^2\cos^2 i)}}{2\,(f^2 + A^2\cos^2 i)}$$

$$= \frac{Ah\cos i \pm \sqrt{4f^4 + 4f^2A^2\cos^2 i - f^2h^2}}{2\,(f^2 + A^2\cos^2 i)}$$

i.e. lies between

$$\cos^{-1}\left[\frac{Ah\cos i + \sqrt{4f^4 + 4f^2A^2\cos^2 i - f^2h^2}}{2\,(f^2 + A^2\cos^2 i)}\right]$$

and

$$\cos^{-1}\left[\frac{Ah\cos i - \sqrt{4f^4 + 4f^2A^2\cos^2 i - f^2h^2}}{2\,(f^2 + A^2\cos^2 i)}\right] \qquad \ldots (3.17)$$

4. Condition Regarding Constrained Motion

Condition for motion of the particle m_2 under effective constraint is given by (3.14). This is obviously satisfied if

$$- 1 \geq A\cos\psi \cos i - f \sin\psi \qquad \dots (4.1)$$

In our subsequent discussion we shall use the term non-evolution motion if the string remains always tight during the motion and evolutional motion when the motion of the system is the combination of free motion and constrained motion.

From (4.1) it follows that we shall have evolutional motion if

$$- 1 < A\cos\psi \cos i - f \sin\psi \qquad \dots (4.2)$$

The inequality (3.14) can be written in the form

$$\psi'^2 + 2\psi^1 + 3\cos^2\psi + f \sin\psi > A\cos\psi \cos i \qquad \dots (4.3)$$

The evolutional motion will take place if

$$\psi'^2 + 2\psi^1 + 3\cos^2\psi + f \sin\psi < A\cos\psi \cos i \qquad \dots (4.4)$$

The inequality (4.4) represents a curve in the two dimensional phase space (ψ, ψ^1).

If the moving particle comes inside this curve in the phase space (ψ, ψ^1) the string will become loose and the motion of the particle in this case will be free from constraint.

The equation (3.13) represents also a curve in the same phase space on which the particle is moving for some fixed value of h.

Then the set of real points of intersection of these two curves in the two dimensional phase space are the regions of evolutional motion.

In order to obtain the points of intersection of these two curves, let us add (3.13) and (4.4)

$$2\psi'^2 + 2\psi^1 + f \sin\psi < h - A\cos\psi \cos i \qquad \dots (4.5)$$

This inequality will always hold if the weaker condition is
which is obtained by putting $\cos\psi = 1$ (max) in (4.5)

i.e. $\psi'^2 + \psi^1 - (h/_2 - A/_2 \cos i) = o$ ... (

The roots will be real if

$$h \geqslant - \frac{1}{2} + A\cos i \quad \ldots ($$

Thus if $h < - \frac{1}{2} + A\cos i$ ... (

The point of intersection will not be real and hence the moti
this case will be non-evolutional.

Examining the real roots of the equation (4.6) between which
evolutional motion may take place it can be seen that region of ev
tional motion for negative values of ψ' is greater than that for
values of ψ'. Hence all motions for which evolution does not take
in the case of $\psi' < o$ will remain non-evolutional for $\psi' > o$ as we

Subtracting (3.12) from (4.4) we have

$$2\psi' + \delta\cos^2\psi + h < 3A\cos\psi\cos i - 3f\sin\psi \quad \ldots ($$

In order that evolution may take place we must get real value
$\cos\psi$ and hence the inequality

$$- 2\psi' - h - 3f\sin\psi + \frac{3}{8} A^2 \cos^2 i \geq o \quad \ldots (4$$

Consequently the motion will be non-evolutional if

$$- 2\psi' - h - 3f\sin\psi + \frac{3}{8} A^2 \cos^2 i < o \quad \ldots (4$$

Substituting the roots of the equation (4.6) into (4.11) to el
nate ψ' we obtain

$$\mp \sqrt{1+2\ (h-A\cos i)} + 1-h + \frac{3}{8} A^2 \cos^2 i - 3f\sin\psi < o$$

which gives the condition for non-evolutional motion in the fo

(i) $(h-2-\frac{3}{8}A^2\cos^2 i)^2 > 4(1-\frac{A\cos i}{2}+\frac{3}{16}A^2\cos^2 i + f\sin\psi$

$+ hf\sin\psi + f^2\sin\psi + \frac{3}{8}A^2\cos^2 i\, f\sin\psi)$

for any ψ^1. ... (4.12)

and (ii) $(h-2-\frac{3}{8}A^2\cos^2 i)^2 < 4(1-\frac{A\cos i}{2}+\frac{3}{16}A^2\cos^2 i + f\sin\psi$

$+ hf\sin\psi + f^2 \sin + \frac{3}{8}A^2\cos^2 i\, f\sin\psi$

for $\psi^1 > o$ ... (4.13)

From the results obtained we conclude that the motion of the system will be non-evolutional and it will be similar to the motion of a dumb bell satellite if any one of the following condition is satisfied.

1. $-1 < A\cos\psi \cos i - f\sin\psi$
2. $h < -\frac{1}{2} + A\cos i$
3. $(h-2-\frac{3}{8}A^2\cos^2 i)^2 > 4(1-\frac{A\cos i}{2}+\frac{3}{16}A^2\cos^2 i + f\sin\psi$

   $+ hf\sin\psi + f^2\sin\psi + \frac{3}{8}A^2\cos^2 i\, f\sin\psi)$

   for any ψ^1.
4. $(h-2-\frac{3}{8}A^2\cos^2 i)^2 < 4(1-\frac{A\cos i}{2}+\frac{3}{16}A^2\cos^2 i + f\sin\psi$

   $+ hf\sin\psi + f^2\sin\psi + \frac{3}{8}A^2\cos^2 i\, f\sin\psi)$

   for $\psi^1 > o$ ... (4.14)

Now if we put A=0 and f=0 in the above in equalities we get

1. $h < -\frac{1}{2}$
2. $(h-2)^2 > 4$ for any ψ^1
3. $(h-2)^2 \leq 4$ for $\psi^1 > o$

... (4.15)

Which are the results for non-evolutional motion of two cable-connected satellites without any perturbation of atmospheric resistance or magnetic force in the central gravitational field of force.

Now the curve (3.12) and

$$(\psi^1 + 1)^2 = 1 + A\cos\psi \cos i - 3\cos^2\psi - f \sin\psi \qquad \ldots (4.16)$$

Which gives the boundary of evolutional and non-evolutional motion have been plotted for different values of h, A and i.

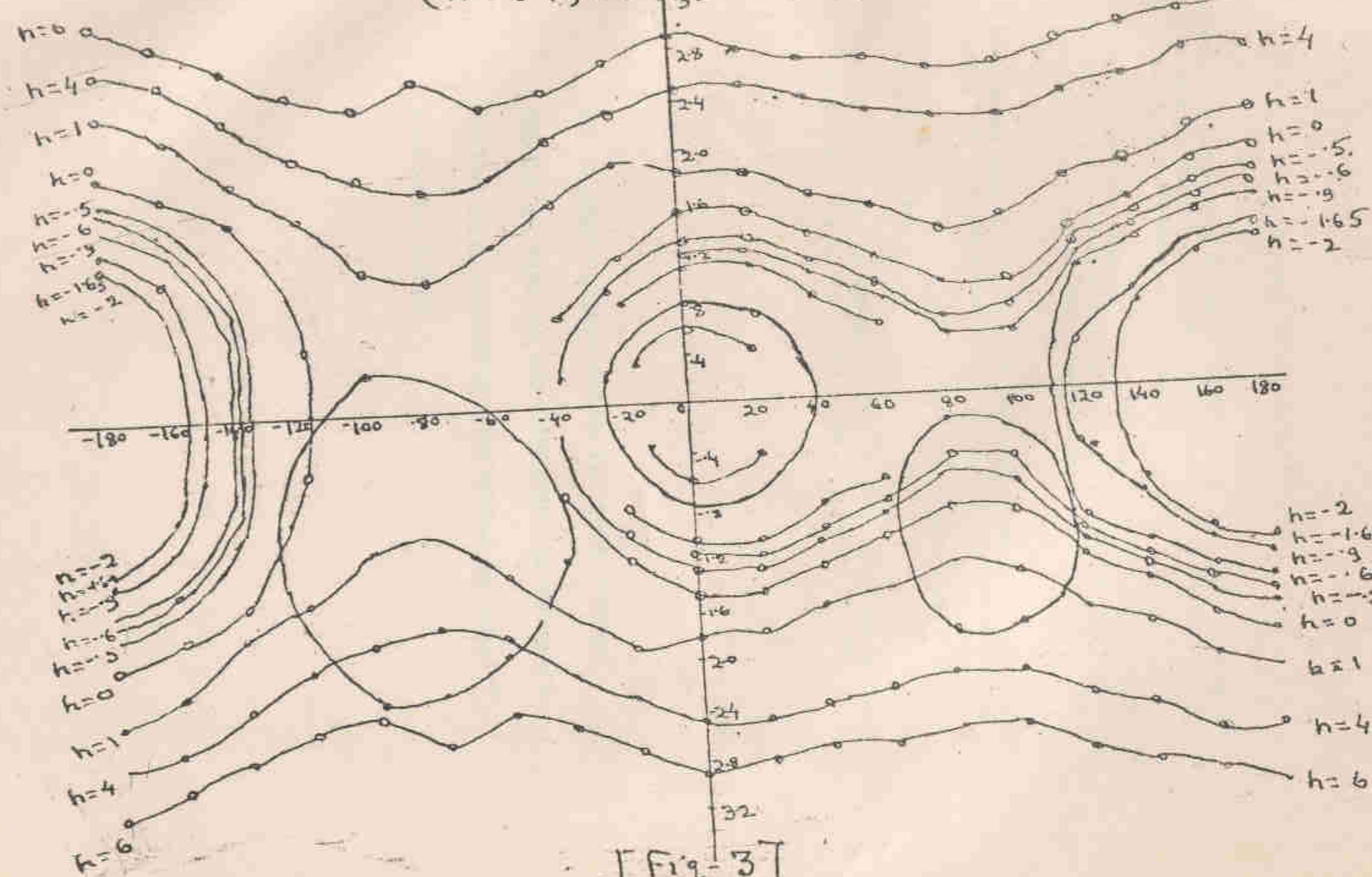
The curve depicting (3.12) are cerred integral curves. The integral curve which cross the boundary (4.16) represents the case of evolutional motion and evidently the curves which do not touch the boundary represent the non-evolutional motion of the system.

It has been seen from the Figure (3) that the integral curve $h < -1.65$ are bounded and donot enter the boundary (4.16) and hence they represent non-evolutional oscillatory motion of the system (like a dumbbell satellite). If $\psi^1 > o$ then all integral curves for $h > 1$ lie outside the boundary (4.16) and as they are not closed curves, the system is rotating like a dumbbell satellite; if $\psi^1 < o$ then it can be seen from the Figure (3) that all the integral curves for $h \geq 4$ donot enter the boundary (4.16) and hence the system is rotating like a dumbbell satellite.

It is worth mentioning that in case of motion of the system in the central gravitational force the integral curves for $h < -.5$ lie outside the boundary (4.16) and represent the non-evolutional motion but here in case of Lorentz force field and atmospheric resistance, the integral curves of $h < -.65$ remains outside the boundary (4.16) where as motions for $-1.65 < h < .5$ are evolutional which are non-evolutional in the case of central force only [1].

$\psi^2 = 3\cos^2\psi - 2A\cos\psi\cos i + h + 2f\sin\psi$; $(\psi' + 1)^2 = 1 + A\cos\psi\cos i - 3\cos^2\psi - f\sin\psi$

$(A = \cdot 3,\ f = \cdot 5$ and $i = 0)$



[Fig. 3]

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Contents

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