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On Strongly Irresolute Mappings

Phullendu Das

Abstract

Replacing 'closure' by 'semi-closure' in the definition of a strongly continuous mapping, a strongly irresolute mapping is defined and some properties of strongly irresolute mappings analogous to those for strongly continuous mappings are obtained.

Let (X, τ) and (Y, τ') be any two topological spaces.

A set $A \subset X$ is said to be semi-open if there exists an open set 0 such that $0 \subset A \subset \bar{0}$ where $\bar{0}$ is the closure of 0 (Levine [3]). A is said to be semi-closed if its complement is semi-open (Crossley and Hildebrand [7]).

S.O. (τ) will denote the class of all semi-open sets of (X, τ).

In [7] Crossley and Hildebrand defined semi-closure $\underline{A}$ and in [4] Das defined semi-derived set A, of a set $A \subset X$ in a manner analogores to closure and derived set.

Unless otherwise mentioned in what follows α will denote a mapping of (X, τ) into (Y, τ').

α is said to be irresolute if $0 \in$ S.O.$(\tau') \Rightarrow \alpha^{-1}(0) \in$ S.O.(τ) (Crossley and Hildebrand [8]). In [6] Das called such mappings as demi-continuous mappings.

α is said to be strongly continuous if for every $A \subset X$, $\alpha(\bar{A}) \subset \alpha(A)$ (Levine [2]). In [9] Arya and Gupta obtained some properties of strongly continuous mappings.

The object of the present paper is to define strongly irresolute mappings by replacing 'closure' in the definition of a strongly continu-

mapping by 'semi-closure' and to examine some properties of strongly irresolute mappings analogous to those for strongly continuous mappings.

Definition 1: α is said to be strongly irresolute if for every $A \subset X$, $\alpha(\underline{A}) \subset \alpha(A)$.

Thus α is strongly irresolute iff $\alpha(A_{,}) \subset \alpha(A)$ for every $A \subset X$.

Theorem 1: α is strongly irresolute iff $\alpha^{-1}(B)$ is semi-closed for every $B \subset Y$.

Proof: Let α be strongly irresolute. Let $B \subset Y$. Let $p \in [\alpha^{-1}(B)]_{,}$. Then $\alpha(p) \in \alpha\{[\alpha^{-1}(B)]_{,}\} \subset \alpha(\alpha^{-1}(B))$ ($\because \alpha$ is strongly irresolute) $\subset B$. $\therefore p \in \alpha^{-1}(B)$. $\therefore \alpha^{-1}(B)$ is semi-closed.

Conversely let the given condition be satisfied. Let $A \subset X$. Then $A_{,} \subset (\alpha^{-1}(\alpha(A))_{,} \subset \alpha^{-1}(\alpha(A))$ by the given condition. $\therefore \alpha(A_{,}) \subset \alpha(A)$. $\therefore \alpha$ is strongly irresolute.

Corollary 1: α is strongly irresolute iff $\alpha^{-1}(B)$ is semi-open for every $B \subset Y$.

Corollary 2: α is strongly irresolute iff $\alpha^{-1}(B)$ is both semi-open and semi-closed for every $B \subset Y$.

Note 1: α is strongly continuous $\Rightarrow$ α is strongly irresolute $\Rightarrow$ α is irresolute.

Note 2: α is irresolute $\not\Rightarrow$ α is strongly irresolute as shown by

Example 1: Let $X = \{a,b,c\}$,

$$\tau = \{\phi, X, \{a\}, \{b\}, \{a,b\}\}$$

Then $$S.O.(\tau) = \tau \cup \{\{a,c\}, \{b,c\}\}.$$

The identity mapping I_x of X is irresolute but not strongly irresolute.

<u>Note 3</u>: α is strongly irresolute $\not\Rightarrow$ α is strongly continuous as shown by

<u>Example 2</u>: Consider the topological space (X, τ) defined in Example 1.

$\alpha : X \to X$ defined

$\alpha(a) = a = \alpha(c)$, $\alpha(b) = b$

is strongly irresolute but it is not strongly continuous.

<u>Theorem 2</u>: α is strongly irresolute iff $\alpha^{-1}(y)$ is semi-open for every $y \in Y$.

<u>Note 4</u>: If $\alpha^{-1}(Y)$ be semi-closed for every $y \in Y$, then α is not necessarily strongly irresolute as shown by

<u>Example 3</u>: The identily mapping I_X defined in Example 1 has the desired property.

<u>Corollary 3</u>: Corresponding to each decomposition of a space X into disjoint semi-open sets, there exists a strongly irresolute mapping on X.

<u>Note 5</u>: Restriction of a strongly irresolute mapping α to any subset of X is not always strongly irresolute as shown by

<u>Example 4</u>: Let $X = \{a,b,c,d\}$, $Y = \{x,y,z\}$,

$\tau = \{0, X, \{a\}, \{b\}, \{a, b\}\}$,

$\tau' = \{\phi, Y, \{x\}, \{y\}, \{x,y\}\}$.

Then $S.O.(\tau) = \tau \cup \{\{a,c\}, \{a,d\}, \{a,c,d\}, \{b,c\}, \{b,d\}, \{b,c,d\}, \{a,b,c\}, \{a,b,d\}\}$

and $S.O.(\tau') = \tau' \cup \{\{x,z\}, \{y,z\}\}$.

$\alpha : X \to Y$ defined by

$\alpha(a) = x$, $\alpha(b) = \alpha(c) = \alpha(d) = y$

is strongly irresolute.

Let $A = \{a,c,d\}$. Then $\tau_A = \{\phi, A, \{a\}\}$.

$\therefore S.O(\tau_A) = \tau_A \cup \{\{a,c\}, \{a,d\}\}$.

$\beta = \alpha|A : A \to Y$ is not strongly irresolute since $\beta^{-1}(y) = \{c,d\} \notin S.O.(\tau_A)$.

<u>Theorem 3</u>: If α be strongly irresolute and $\beta:(Y,\tau') \to (Z, \tau'')$ be any map, then $\beta\alpha:(X, \tau) \to (Z, \tau'')$ is strongly irresolute.

<u>Proof</u>. Let $A \subset Z$. Then $\beta^{-1}(A) \subset Y$. Since α is strongly irresolute $\alpha^{-1}(\beta^{-1}(A)) = (\beta\alpha)^{-1}(A)$ is semi-closed. $\therefore$ $\beta\alpha$ is strongly irresolute.

<u>Corollary 4</u>: The composite of two strongly irresolute mappings is strongly irresolute.

<u>Theorem 4</u>: If α be irresolute and β: $(Y, \tau') \to (Z, \tau'')$ be strongly irresolute, then $\beta\alpha$: $(X, \tau) \to (Z, \tau'')$ is strongly irresolute.

<u>Proof</u>: Let $A \subset Z$. Since β is strongly irresolute, $\beta^{-1}(A)$ is both semi-open and semi-closed. $\therefore$ $\underline{((\beta\alpha)^{-1}(A))} = \underline{\alpha^{-1}(\beta^{-1}(A))} \subset \alpha^{-1}(\underline{\beta^{-1}(A)})$ ($\because$ α is irresolute) $= \alpha^{-1}(\beta^{-1}(A))$ ($\because$ $\beta^{-1}(A)$ is semi-closed) $= (\beta\alpha)^{-1}(A)$. $\therefore$ $(\beta\alpha)^{-1}(A)$ is semi-closed. $\therefore$ $\beta\alpha$ is strongly irresolute.

<u>Theorem 5</u>: Let γ: $(X, \tau) \to \prod_{\alpha \in A}(X_\alpha, \tau_\alpha)$ be strongly irresolute. Let γ_α : $(X, \tau) \to (X_\alpha, \tau_\alpha)$ be defined by $\gamma_\alpha(x) = x_\alpha$ if $\gamma(x) = (x_\alpha)$ for every $\alpha \in A$. Then γ_α is strongly irresolute for every $\alpha \in A$.

The result follows from Theorem 3 since $\gamma_\alpha = P_\alpha\gamma$ where P_α is the projection of $\prod_{\alpha \in A} X_\alpha$ onto X_α.

<u>Theorem 6</u>: Let α_i: $(X_i, \tau_i) \to (Y_i, \tau_i')$ be strongly irresolute for $i = 1, 2$. Let $X_o = X_1 \times X_2$, $Y_o = Y_1 \times Y_2$, $\tau_o = \tau_1 \times \tau_2$, $\tau_o' = \tau_1' \times \tau_2'$. Then α_o : $(X_o, \tau_o) \to (Y_o, \tau_o')$ defined by $\alpha_o(x_1, x_2) = (\alpha_1(x_1), \alpha_2(x_2))$ for every $x_1 \in X_1$, $x_2 \in X_2$ is strongly irresolute.

<u>Proof</u>: Let $y = (y_1, y_2) \in Y$. Then $\alpha_o^{-1}(y) = \alpha_1^{-1}(y_1) \times \alpha_2^{-1}(y_2)$ is semi-open (by Theorem 11, Levine [3]) since $\alpha_i^{-1}(y_i)$ is semi-open for $i = 1, 2$. $\therefore$ α_o is strongly irresolute.

A set $P \subset X$ which cannot be expressed as the join of two sets $A, B \subset X$ such that $A \cap \underline{B} = \underline{A} \cap B = \phi$ is said to be semi-connected (Das [5]).

Theorem 7: Let α be strongly irresolute and let A be a semi-connected subset of X such that if O be semi-open or semi-closed in $(X, \mathcal{T})$, then $A \cap O$ is semi-open or semi-closed in $(A, \mathcal{T}_A)$. Then $\alpha(A)$ is a single point.

Proof: Let there exist more than one point in $\alpha(A)$. Let $p \in \alpha(A)$. Then $\alpha^{-1}(p)$ is a proper subset of A and since α is strongly irresolute $\alpha^{-1}(p)$ is both semi-open and semi-closed in $(X, \mathcal{T})$. $\therefore$ $\alpha^{-1}(p) \cap A$ is both semi-open and semi-closed in $(A, \mathcal{T}_A)$ which contradicts the fact that A is semi-connected. $\therefore$ $\alpha(A)$ is a single point.

Theorem 8: X is semi-connected iff every strongly irresolute mapping on X is constant.

Proof: Let every strongly irresolute mapping on X be constant. If X be not semi-connected, then there exists a proper subset A of X which is both semi-open and semi-closed. Let $Y = \{a,b\}$ $(a \neq b)$ and let $\mathcal{T}$ be any topology on Y. Let $\beta(A) = a$, $\beta(X-A) = b$. Then β is strongly irresolute but it is not a constant mapping. $\therefore$ X is semi-connected. The converse follows from Theorem 7.

In [1] Bohn and Lee defined semi-neighbourhood of a point $x \in X$ and in [3] Das defined semi-component in a manner analogous to neighbourhood and component.

Definition 2: X is said to be locally semi-connected if for every $x \in X$, every semi-open semi-neighbourhood of x contains a semi-connected semi-open semi-neighbourhood of x.

Every semi-component of a locally semi-connected space is a semi-open set.

Theorem 9: Let X be locally semi-connected and let γ be the cardinality of the family $\mathcal{C}$ of all semi-components of X. Then any space with cardinality $\leq \gamma$ is the image of X under some strongly irresolute mapping.

Proof: Let Y be any space whose cardinal number is $\beta \leq \gamma$. Let $\mathcal{C}'$ be a subfamily of $\mathcal{C}$ of cardinality β. Then there exists a bijective mapping $g: \mathcal{C}' \to Y$. $f: X \to Y$ is defined as follows: $f(x) = g(D_x)$ if

$x \in D_x \in \mathcal{C}'$ and $f(x) = g(D_o)$ where $x \in D \in \mathcal{C}$ but $D \notin \mathcal{C}'$, D_o being a fixed member of $\mathcal{C}'$. Since X is locally semi-connected, every member of $\mathcal{C}$ is both semi-open and semi-closed. $\therefore$ $f^{-1}(y)$ is both semi-open and semi-closed for every $y \in Y$. $\therefore$ f is strongly irresolute.

In [9] Arya and Gupta proved that a strongly continuous mapping $f:A \to Y$ can be extended strongly continuously to any locally connected space X which contains A as a closed and open set. But if $f:A \to Y$ be a strongly irresolute mapping and X, a locally semi-connected space containing A as a semi-closed and semi-open subset, then it may not be possible to extend f strongly irresolutely to X as shown by

Example 5: Let $A = \{a,b,c,d,e\}$,

$$\mathcal{T} = \{\phi, A, \{a,b\}, \{c\}, \{a,b,c\}\}.$$

$f: A \to A$ defined by

$$f(a) = f(b) = d, \quad f(c) = f(d) = f(e) = c$$

is strongly irresolute.

Let $X = \{a,b,c,d,e,f\}$,

$$\mathcal{T}' = \{\phi, X, \{a,b,c,d,e\}, \{f\}\}.$$

Then $(X, \mathcal{T}')$ is locally semi-connected and contains A as a semi-open and semi-closed subset. But it is not possible to extend f strongly irresolutely to X.

<u>Definition 3</u>: X is said to be weakly locally semi-neighbourhood if every point of X has a semi-connected semi-neighbourhood.

<u>Theorem 10</u>: A map $f:X \to Y$ from a weakly locally semi-connected space X to a space Y such that the image of every non-empty semi-connected subset of X is a singleton, is strongly irresolute.

<u>Proof</u>: Let $A \subset X$. Let $x \in A_{,}$. Since X is weakly locally semi-connected, there exists a semi-connected semi-neighbourhood N of x. Since $x \in A_{,}$, $N \cap A \neq \phi$. Now $f(x) \in f(N)$ and by our assumption $f(N)$ is a singleton. Since $f(A \cap N) \subset f(N)$, $f(x) = f(A \cap N) \subset f(A)$ $\therefore$ $f(x) \in f(A)$. $\therefore$ $f(A_{,}) \subset f(A)$. $\therefore$ f is strongly irresolute.

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A Fixed Point Theorem for Four Maps on a Metric Space

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Abstract

In this paper we obtain a fixed point theorem for four maps (in metric space set-up) generalizing and unifying certain results of Ciric, Das and Naik, and Fisher and also provide a good number of examples to give insight into the results discussed.

In this paper we investigate conditions under which four mappings from (a subset of) a metric space into the space admit a common fixed point.

In Section 1, we obtain a theorem generalizing and unifying the results of Ciric [1], Das and Naik [2] and Fisher [3], [4] and state a few important Corollaries. We also make a critical analysis of the necessity of the various conditions imposed in the results by means of several remarks.

In Section 2, we provide a number of examples to give insight into the results discussed in Section 1. The examples are often referred to in the remarks discussed in Section 1.

Sessa [6] introduced the concept of weak commutativity for two self-maps on a metric space. We slightly modify it as follows:

Definition: Let A,B,C be subsets of a metric space (X,d) with $C \subseteq A \cap B$. If $f: A \to X$ and $g: B \to X$ are such that

$$d(fgx,gfx) \leq d(gx,fx) \;\forall\; x \in C \ni gx \in A \text{ and } fx \in B$$

then we say that f,g are weakly commutative on C or f,g commute weakly on C or (f,g) is a weakly commuting pair (w.c.p.) on C. Throughout this paper, for any map h, R(h) denotes the range of h.

§ 1

Throughout this Section $\emptyset$ stands for an increasing map from R^+ to R^+ with $\emptyset(t+) < t \;\forall\; t > 0$ and $\lim\limits_{t \to \infty} [t-\emptyset(t)] = +\infty$.

Theorem 1: Let S,T be self-maps on a metric space (X,d) and $Y \subseteq X$. Let

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f,g be mappings from Y into X such that $d(fx,gy) \leq \phi\,(\max\{d(Sx,Ty), d(Sx,fx), d(Ty,gy), d(fx,Ty), d(Sx,gy)\})$ for all x,y in Y. ... (I)

Suppose that there are sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ in Y such that

$$\left.\begin{array}{l} fx_n = gy_n = Ty_{n+1} = Sx_{n+1} = z_{n+1} \quad (n = 0,1,2 \ldots) \\ \qquad Ty_o = Sx_o = z_o \end{array}\right\} \qquad \ldots \text{(II)}$$

Then $\{z_n\}$ is Cauchy. Suppose $\{z_n\}$ converges to some $z \in X$. Then the following statements hold:

1. If $z \in Y$, f is continuous at z and f,S commute weakly on Y then $fz = z$. If, further, $z \in T(Y)$ and g,T commute weakly on Y then $gz = z = Tz$. If, in addition, $z \in S(Y)$ then $Sz = z$.

2. If S is continuous at z and f,S commute weakly on Y then $Sz = z$. If, further, $z \in Y$ then $fz = z$ and f is continuous at z. If, in addition, $z \in T(Y)$ and g,T commute weakly on Y then $gz = z = Tz$.

3. If $fS = Sf$ on Y, $S(Y) \subseteq Y$ and S^m is continuous at z for some positive integer m then $S^m z = z$. If, further, $z \in Y$ then $Sz = z = fz$. If, in addition, $z \in T(Y)$ and g,T commute weakly on Y then $Tz = z = gz$.

4. Statements (1),(2),(3) remain valid if f,g,S,T are replaced by g,f,T,S respectively.

5. If $S = T$, S is continuous at z, $z \in Y$ and either f or g commutes weakly with S on Y then $fz = gz = Sz = z$.

Proof: Taking $x = x_{i-1}$ and $y = y_{j-1}$ in inequality (I) and using (II) we obtain

$$d(z_i,z_j) \leq \phi\,(\max\{d(z_{i-1},z_{j-1}), d(z_{i-1},z_i), d(z_{j-1},z_j), d(z_i,z_{j-1}), d(z_{i-1},z_j)\}) \qquad \ldots \text{(III)}$$

for all $i \geq 1$ and $j \geq 1$.

We have

$$d(z_o,z_j) \leq d(z_o,z_1) + d(z_1,z_j). \qquad \ldots \text{(IV)}$$

Let $\beta_n = \sup\{d(z_i,z_j)/0 \le i,\ j \le n\}$ $(n = 1,2,\ldots)$.
Then from (III) and (IV) it is clear that

$$\beta_n \le d(z_o,z_1) + \emptyset\,(\beta_n) \quad (n = 1,2,\ldots). \qquad \ldots \text{(V)}$$

$$\text{i.e. } \beta_n - \emptyset\,(\beta_n) \le d(z_o,z_1) \quad (n = 1,2,\ldots).$$

Since $\lim\limits_{t \to +\infty} [t-\emptyset(t)] = +\infty$, it follows that $\{\beta_n\}$ is bounded.

Hence $\sup\{d(z_i,z_j) \,/\, i \ge 0,\ j \ge 0\}$ is finite.

Let

$$\gamma_n = \sup\{d(z_i,z_j) \,/\, i \ge n,\ j \ge n\} \quad (n = 0,1,2,\ \ldots).$$

Then $\{\gamma_n\}$ is a decreasing sequence of non-negative real numbers and hence converges to a non-negative real number γ.

From inequality (III) we obtain

$$\gamma_n \le \emptyset\,(\gamma_{n-1}) \qquad (n = 1,2,\ \ldots). \qquad \ldots \text{(VI)}$$

Taking limits on both sides of the above inequality as $n \to \infty$, we obtain

$$\gamma < \emptyset\,(\gamma+).$$

Since $\emptyset\,(t+) < t \;\forall\; t > 0$, we must have $\gamma = 0$.

Hence $\{z_n\}$ is Cauchy.

I. Suppose $z \in Y$, f is continuous at z and f,S commute weakly on Y.
Consider

$$
\begin{aligned}
d(Sz_n,fz) &\le d(Sz_n,fz_{n-1}) + d(fz_{n-1},fz) \\
&= d(Sfx_{n-1},fSx_{n-1}) + d(fz_{n-1},fz) \\
&\le d(fx_{n-1},Sx_{n-1}) + d(fz_{n-1},fz) \\
&= d(z_n,z_{n-1}) + d(fz_{n-1},fz) \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}
$$

Hence $\{Sz_n\}$ converges to fz.

Taking $x = z_n$ and $y = y_n$ in inequality (I) and then taking limits on both sides as $n \to \infty$, we obtain

$$d(fz,z) \le \phi\,(d(fz,z)+)$$

so that $fz = z$.

Suppose $z \in T(Y)$.

Then $\exists\ w \in Y \ni Tw = z$.

Taking $x = x_n$ and $y = w$ in inequality (I) and then taking limits on both sides as $n \to \infty$, we obtain

$$d(z,gw) \le \phi\,(d(z,gw)+)$$

so that $gw = z$.

Suppose g, T commute weakly on Y.

Then $d(gz,Tz) = d(gTw,Tgw) \le d(Tw,gw) = 0$

so that $gz = Tz$.

Taking $x = x_n$ and $y = z$ in inequality (I) and then taking limits on both sides as $n \to \infty$, we obtain

$$d(z,gz) \le \phi\,(d(z,gz)+)$$

so that $gz = z$. Hence $Tz = z$.

Suppose $z \in S(Y)$.

Then $\exists\ u \in Y \ni Su = z$.

Taking $x = u$ and $y = z$ in inequality (I) we obtain

$$d(fu,z) \le \phi\,(d(fu,z))$$

so that $fu = z$.

Since f,S commute weakly on Y, we have

$$d(fz,Sz) = d(fSu,Sfu) \le d(Su,fu) = 0$$

so that $fz = Sz$.

Since $fz = z$ we must have $Sz = z$.

2. Suppose S is continuous at z and f,S commute weakly on Y.

Consider $d(fz_n,Sz) \le d(fz_n,Sz_{n+1}) + d(Sz_{n+1},Sz)$

$$\le d(fSx_n,Sfx_n) + d(Sz_{n+1},Sz)$$

$$\le d(Sx_n\ ,\ fx_n) + d(Sz_{n+1},Sz)$$

$$= d(z_n\ ,\ z_{n+1})+ d(Sz_{n+1},Sz)$$

$$\to\ 0 \text{ as } n \to \infty.$$

Hence $\{fz_n\}$ converges to Sz.

Taking $x = z_n$ and $y = y_n$ in inequality (I) and then taking limits on both sides as $n \to \infty$, we obtain

$$d(Sz,z) \le \emptyset\ (d(Sz,z)+)$$

so that $Sz = z$.

Suppose $z \in Y$.

Taking $x = z$ and $y = y_n$ in inequality (I) and then taking limits on both sides as $n \to \infty$, we obtain

$$d(fz,z) \le \emptyset\ (d(fz,z)+)$$

so that $fz = z$.

Let $\{u_n\}$ be a sequence in Y converging to z.

Taking $x = u_n$ and $y = y_i$ in inequality (I) and then taking limits on both sides as $i \to \infty$, we obtain

$$t_n \le \emptyset\ ((t_n+s_n)+), \quad (n = 1,2,\ \dots\ .) \qquad \dots\text{(VII)}$$

where $t_n = d(fu_n,z)$ and $s_n = d(Su_n,z)$.

Since S is continuous at z and $\{u_n\}$ converges to z, $\{s_n\}$ converges to zero. Hence there exists a positive real number M such that

$$s_n < M \ \forall\ n = 1,2,\ \dots\ .$$

Hence from inequality (VII) we have

$$t_n \le \emptyset\ (t_n+M) \ \forall\ n = 1,2,\ \dots\ .$$

$$(t_n+M) - \emptyset\ (t_n+M) \le M \quad n = 1,2,\ \dots\ .$$

Since $\lim_{t \to +\infty} [t - \emptyset(t)] = +\infty$, it follows that $\{t_n\}$ is bounded.

Suppose $\{t_{n_k}\}$ is a convergent subsequence of $\{t_n\}$ with limit t_o.

From inequality (VII) we have

$$t_{n_k} \leq \emptyset\ ((t_{n_k} + s_{n_k})+) \quad (k = 1,2, \ldots .)$$

Taking limits on both sides of the above inequality as $k \to \infty$, we obtain

$$t_o \leq \emptyset\ (t_o+)$$

so that $t_o = 0$.

Hence $\{t_n\}$ must converge to zero.

Hence f is continuous at z.

When $z \in T(Y)$ and g,T commute weakly on Y, it can be shown as in the proof of 1 that $gz = z = Tz$.

3. Suppose $fS = Sf$ on Y, $S(Y) \subseteq Y$ and S^m is continuous at z for some positive integer m.

Taking $x = S^m x_n$, $y = y_n$ in inequality (I) and using the commutativity of f and S on Y and equation (II) we obtain

$$d(S^m z_{n+1}, gy_n) \leq \emptyset\ (\max\ \{d(S^m z_n, Ty_n),\ d(S^m z_n, S^m z_{n+1}), d(Ty_n, gy_n),$$
$$d(S^m z_{n+1}, Ty_n),\ d(S^m z_n, gy_n)\}\ ).$$

Taking limits on both sides of the above inequality as $n \to \infty$, we obtain

$$d(S^m z, z) \leq \emptyset\ (d(S^m z, z)+)$$

so that $S^m z = z$.

Suppose $z \in Y$.

Then $S^{m-1} z \in Y$.

Taking $x = S^{m-1} z$ and $y = y_n$ in inequality (I) and then taking limits on both sides as $n \to \infty$, we obtain

$$d(fS^{m-1} z, z) \leq \emptyset\ (d(fS^{m-1} z, z)+)$$

so that $fS^{m-1} z = z$.

Hence $Sz = S(fS^{m-1}z) = f(S^m z) = fz$.

Taking $x = z$ and $y = y_n$ in inequality (I) and then taking limits on both sides as $n \to \infty$, we obtain

$$d(fz,z) \le \emptyset\ (d(fz,z)+)$$

so that $fz = z$. Hence $Sz = z$.

If $z \in T(Y)$ and g, T commute weakly on Y then it can be shown exactly as in the proof of statement (1) that $gz = z = Tz$.

4. This is evident from statements (1),(2),(3) since the initial hypothesis of the Theorem remains unaltered if f,g,S,T are replaced by g,f,T,S respectively.

5. Suppose $S = T$, S is continuous at z, $z \in Y$ and f,S commute weakly on Y.

Then from statement (2) we have $fz = z = Sz$.

Taking $x = y = z$ in inequality (I) and noting that $Tz = Sz = z$, we obtain

$$d(z,gz) \le \emptyset\ (d(z,gz)).$$

Hence $gz = z$.

Instead of assuming the weak commutativity of f and S, if we assume that of g and $T(=S)$, then from (4) we obtain $gz = z = Tz$. Now taking $x = y = z$ in inequality (I) it can be seen that $fz = z$.

Remark 1: When inequality (I) holds for all $x,y \in Y$, p is a common fixed point of f and S in Y and q is a common fixed point of g and T in Y then $p = q$ so that if $z \in Y$ is a common fixed point of f,g,S and T then z is the unique common fixed point of f and S and of g and T.

Remark 2: From inequalities (V) and (VI) it can be verified that $\gamma_o - \emptyset(\gamma_o) \le d(z_o,z_1)$ and $\gamma_n \le \emptyset^n\ (\gamma_o)$ so that in Theorem 1 one has the inequality

$$d(z_n,z) \le \emptyset^n\ (t_o),$$

where $t_o = \sup\ \{t \in R^+ \ /\ t-\emptyset(t) \le d(z_o,z_1)\}$.

<u>Remark 3</u>: Examples 2 and 13 show that in statement (1) of Theorem 1 one cannot drop the condition '$z \in S(Y)$' even when $Y = X$, X is complete, f,g,T are continuous on X, $gT = Tg$ and $\phi(t) = \alpha t$ on R^+, α being a constant in $[0,1/2]$. While in Example 2 the space X is compact, in Example 13 the mappings f and S are commutative.

<u>Remark 4</u>: If f commutes with S on Y, S is orbitally bounded on Y (i.e. $\{S^n x\}$ is bounded for each $x \in Y$) and the term $d(Sx,fx)$ is deleted from the right hand side of inequality (I) then the condition '$z \in S(Y)$' can be dropped from statement (1) of Theorem 1.

<u>Remark 5</u>: Example 3 shows that in statement (2) (and hence in statement (1)) of Theorem 1 one cannot drop the condition '$z \in T(Y)$' even when $Y = X$, X is compact, $fS = Sf$, $gT = Tg$ and $\phi(t) = \frac{t}{2}$ on R^+. In particular, it shows that the condition cannot be dropped from statement (3) of Theorem 1 even when $m = 1$. It also presents a situation where f,S are continuous on X but none of g,T has a fixed point.

Examples 1,4 and 12 also show that the condition '$z \in T(Y)$' cannot be dropped from statement (1) of Theorem 1. In all the three examples $Y = X$, X is complete and $\phi(t) = \alpha t$ on R^+ for some constant α in $[0,1/2]$. Examples 1 and 12 present a situation where f is continuous on X and none of g,S,T has z as a fixed point. In fact, in Example 1 none of g,S,T has a fixed point. While in Example 4, the space X is compact, in Example 12 the mappings f,S are commutative. Of course, in both the examples the mappings g,T are commutative. Example 4 illustrates a situation where both f and g are continuous on X but none of S,T has a fixed point.

<u>Remark 6</u>: Example 14 shows that in statements (1) and (2) of Theorem 1 one cannot drop the weak commutativity of f and S even when $Y = X$, X is finite and $gT = Tg$.

<u>Corollary 1</u>: Let f,g,S,T be self-maps on a complete metric space (X,d) such that (f,S), (g,T) are weakly commuting pairs and inequality (I) holds for all x,y in X. Suppose that atleast one of f,g,S,T is continuous $R(f) \subseteq R(T)$, $R(g) \subseteq R(S)$ and that there are sequences $\{x_n\}$, $\{y_n\}$ in X such that

$$fx_n = gy_n = Ty_{n+1} = Sx_{n+1} \quad (n = 0,1,2, \dots)$$

Then $\{fx_n\}$ is convergent and the limit of $\{fx_n\}$ is the unique common fixed point of f and S and of g and T.

Remark 7: Example 7 shows that Theorem 1 is a proper generalization of Corollary 1.

Remark 8: Examples 2 and 13 show that in Corollary 1 one cannot replace the condition '$R(f) \subseteq R(T)$ and $R(g) \subseteq R(S)$' by 'either $R(f) \subseteq R(T)$ or $R(g) \subseteq R(S)$' even when f,g,T are continuous on X, $gT \equiv Tg$ and $\phi(t) \equiv \alpha t$ on R^+, α being a constant in $[0,1/2]$. While in Example 2 the space X is compact, in Example 13 the mappings f,S are commutative.

Remark 9: Example 5 of Sastry and Naidu [5] shows that in Corollary 1 one cannot ensure the existence of a common fixed point for f,g,S,T (even when (X,d) is a finite metric space and S,T are identity maps on X) if the condition regarding the existence of the sequence $\{x_n\}$, $\{y_n\}$ is dropped.

Remark 10: Example 5 shows that in Corollary 1 on ecannot drop the condition on continuity even when X is compact, f commutes with S, g commutes with T and $\phi(t) = \frac{t}{4}$ on R^+. Example 6 also can be invoked here.

Remark 11: Example 15 shows that in Corollary 1 one cannot replace the condition '(f,S), (g,T) are weakly commuting pairs' by the condition 'either $fS \equiv Sf$ or $gT \equiv Tg$' even when X if finite and $f \equiv g$.

We can deduce the following two results from Corollary 1.

Corollary 2: (Theorem 1 of Fisher [4]):- Let f,S,T be self-maps on a complete metric space (X,d) such that $fS \equiv Sf$, $fT \equiv Tf$ and

$$d(fx,fy) \leq \alpha \max \{d(Sx,Ty),\ d(Sx,fx),\ d(Ty,fy),\ d(fx,Ty),\ d(Sx,fy)\}$$

for all x,y in X, where α is a constant in $[0,1)$. Suppose that for each x in X there exists y in X such that

$$fx \equiv Sy \equiv Ty$$

and at least one of f,S,T is continuous. Then f,S,T have a unique common fixed point.

Corollary 3: Let f,S be self-maps on a complete metric space (X,d) such that f,S commute weakly, $R(f) \subseteq R(S)$ and $d(fx,fy) \leq \phi$ (max $\{d(Sx,Sy)$, $d(Sx,fx)$, $d(Sy,fy)$, $d(fx,Sy)$, $d(Sx,fy)\}$) for all x,y in X. Suppose that

Remark 16: Corollary 5 is a generalization of Theorem 1 of Ciric [1]. Example 10 shows that it is a proper generalization.

Remark 17: Example 11 shows that in Corollary 5 one cannot ensure either the Cauchy nature of the sequence of iterates or the existence of a fixed point (even when (X,d) is complete) if the condition $\lim_{t\to+\infty} [t-\phi(t)] = +\infty$ is dropped.

§ 2. Examples

In Examples 1 to 6, X stands for the compact metric space $\{0,1,1/2,1/2^2,\ldots\}$ with the usual metric, whereas in Examples 7 to 10, it stands for the compact metric space $\{0,1,1/2,1/3,\ldots\}$ with the usual metric. Further in Examples 7 to 10, ϕ denotes the mapping from R^+ to R^+ defined by $\phi(t) = \frac{t}{1+t}$. It is clear that ϕ is increasing, continuous, $\phi(t) < t \;\forall t > 0$ and $\lim_{t\to+\infty} [t - \phi(t)] = +\infty$.

In all those examples where four self-maps f,g,S,T on X are considered, the equation

$$fx_n = gy_n = Ty_{n+1} = Sx_{n+1} = z_{n+1} \quad (n = 0,1,2,\ldots)$$

holds for a suitable choice of the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ in X and one such selection is specified in each case without explicitly stating the equation every time.

1. Define self-maps f,g,S,T on X by

$$f(0) = 0, \quad g(0) = 1/2, \quad S(0) = 1, \quad T(0) = 1$$

$$f(\tfrac{1}{2^n}) = \frac{1}{2^{n+3}}, \quad g(\tfrac{1}{2^n}) = \frac{1}{2^{n+5}}, \quad S(\tfrac{1}{2^n}) = \frac{1}{2^{n+2}}, \quad T(\tfrac{1}{2^n}) = \frac{1}{2^{n+4}}$$

$$(n = 0,1,2,\ldots).$$

Then f is continuous on X ; g,S,T are discontinuous at zero; f,S are weakly commutative but not commutative; g,T are commutative;

$$|fx-gy| \le \frac{1}{2}\max\{|Sx-Ty|, \ |Sx-fx|\}$$

$$|fx-gy| \le \frac{1}{2}\max\{|Sx-Ty|, \ |fx-Ty|\}$$

and

$$|fx-gy| \le \frac{1}{2}\max\{|fx-Ty|, \ |Sx-gy|\}$$

for all x,y in X.

For $x_n = \frac{1}{2^{n+2}}$ and $y_n = \frac{1}{2^n}$ we have $z_{n+1} = \frac{1}{2^{n+5}} \to 0$ as $n \to \infty$.

0 is the only fixed point of f.

None of g,S,T has a fixed point.

We note that $0 \notin R(T)$.

2. Let f, S be self-maps on X defined as in Example 1.
Define self-maps g,T on X by

$$g(0) = 0 \quad , \quad T(0) = 0$$
$$g(\frac{1}{2^n}) = \frac{1}{2^{n+2}} \, , \quad T(\frac{1}{2^n}) = \frac{1}{2^{n+1}} \quad (n = 0,1,2, \ldots).$$

Then f,g,T are continuous on X and any two of them are commutative;

$$R(f) \subset R(T) \quad ; \quad R(g) \not\subseteq R(S) \; ;$$

$$|fx-gy| \leq \frac{1}{2} |Sx-Ty| \quad \forall\, x,y \in X$$

and

$$|fx-gy| \leq \frac{1}{2} \max \{|fx-Ty| \, , \; |Sx-gy|\} \quad \forall\, x,y \in X.$$

For $x_n = \frac{1}{2^n}$ and $y_n = \frac{1}{2^{n+1}}$ we have $z_{n+1} = \frac{1}{2^{n+3}} \to 0$ as $n \to \infty$.

0 is the only fixed point of each of f,g and T.

S has no fixed point. In fact, $0 \notin R(S)$.

3. Let f,g,S,T be self-maps on X defined exactly as in Example 1 excepting for S(0) which we define here as 0.

Then f,S are continuous on X; g,T are discontinuous at 0;
$fS = Sf$, $gT = Tg$;

$$|fx-gy| \leq \frac{1}{2} \; |Sx-Ty| \quad \forall\, x, y \in X,$$

and

$$|fx-gy| \leq \frac{1}{2} \max \{|fx-Ty| \, , \; |Sx-gy|\} \;\forall\, x,y \in X.$$

For $\{x_n\}$, $\{y_n\}$ defined as in Example 1, we have
$z_{n+1} = \frac{1}{2^{n+5}} \to 0$ as $n \to \infty$.

0 is the unique fixed point of each of f and S.

But neither g nor T has a fixed point.

We note that $0 \notin R(T)$.

4. Let f,g,S,T be self-maps on X defined exactly as in Example 1 excepting for g(0) which we define here as 0.

Then f,g are continuous on X ; S,T are discontinuous at 0; g,T are weakly commutative but not commutative;

$$|fx-gy| \leq \frac{1}{2}\ |Sx-Ty| \quad \forall\, x,y \in X$$

and

$$|fx-gy| < \frac{1}{2} \max\ \{|fx-Ty|\ ,\ \ |Sx-gy|\} \ \forall\, x,y \in X.$$

For $\{x_n\}$, $\{y_n\}$ defined as in Example 1, we have

$$z_{n+1} = \frac{1}{2^{n+5}} \quad \to 0 \text{ as } n \to \infty .$$

0 is the unique fixed point of each of f and g.

But neither S nor T has a fixed point.

We note that $0 \notin R(S) \cup R(T)$.

5. Define self-maps f,g,S,T on X by

$$f(0) = \frac{1}{2^2}\ ,\ \ g(0) = \frac{1}{2^4}\ ,\ \ S(0) = 1\ ,\ \ T(0) = \frac{1}{2^2}$$

$$f(\frac{1}{2^n}) = \frac{1}{2^{n+5}}\ ,\ g(\frac{1}{2^n}) = \frac{1}{2^{n+6}}\ ,\ S(\frac{1}{2^n}) = \frac{1}{2^{n+3}}\ ,\ T(\frac{1}{2^n}) = \frac{1}{2^{n+4}}$$

$$(n = 0,1,2,\ \ldots\ ).$$

Then f,g,S,T are all discontinuous at zero ; $fS = Sf$, $gT = Tg$;

$R(f) \subset R(T)$, $R(g) \subset R(S)$;

$$|fx-gy| = \frac{1}{4}\ \ |Sx-Ty|$$

$$|fx-gy| < \frac{1}{3} \max\ \{|Sx-fx|\ ,\ |Ty-gy|\}$$

and

$$|fx-gy| < \frac{1}{4} \max\ \{|fx-Ty|\ ,\ |Sx-gy|\}$$

for all x,y in X.

For $x_n = \frac{1}{2^{2n+1}}$ and $y_n = \frac{1}{2^{2n}}$ we have $z_{n+1} = \frac{1}{2^{2n+6}} \to 0$ as $n \to \infty$.

None of f,g,S,T has a fixed point.

6. Define self-maps f,S on X by

$$f(0) = \frac{1}{2^3} \quad , \quad S(0) = 1$$

$$f(\tfrac{1}{2^n}) = \frac{1}{2^{n+6}} \quad , \quad S(\tfrac{1}{2^n}) = \frac{1}{2^{n+3}} \qquad (n = 0,1,2, \ldots\ .).$$

Then f,S are discontinuous at zero; $fS = Sf$; $R(f) \subset R(S)$;

$$|fx-fy| = \frac{1}{8}\ |Sx-Sy| ,$$

$$|fx-fy| < \frac{1}{7} \max\ \{|Sx-fx| \ , \ |Sy-fy|\}$$

and

$$|fx-fy| < \frac{1}{8} \max\ \{|fx-Sy| \ , \ |Sx-fy|\}$$

for all x,y in X.

For $x_n = \frac{1}{2^{3n}}$, $fx_n = Sx_{n+1} = \frac{1}{2^{3n+6}} \to 0$ as $n \to \infty$.

None of f and S has a fixed point.

7. Define self-maps f,g,S,T on X by

$$f(0) = g(0) = S(0) = T(0) = 0$$

$$f(\tfrac{1}{n}) = \begin{cases} \frac{1}{n+5} & \text{if n is even} \\ \frac{1}{n+12} & \text{if n is odd} \end{cases} \quad , \quad g(\tfrac{1}{n}) = \begin{cases} \frac{1}{7} & \text{if n is even} \\ \frac{1}{n+16} & \text{if n is odd} \end{cases}$$

$$S(\tfrac{1}{n}) = \begin{cases} \frac{1}{3} & \text{if n is even} \\ \frac{1}{n+4} & \text{if n is odd} \end{cases} \quad , \quad T(\tfrac{1}{n}) = \begin{cases} 1 & \text{if n is even} \\ \frac{1}{n+8} & \text{if n is odd.} \end{cases}$$

Then f is continuous on X; g,S,T are discontinuous at '0'; f,S are weakly commutative but not commutative; g,T are weakly commutative but not commutative; $R(g) \subset R(S)$; $R(f) \not\subset R(T)$;

$$|fx-gy| \le \phi\,(|Sx-Ty|) \ \forall\, x,y \in X\ ;$$

and

$$|fx-gy| \le \phi\,(\max\,\{|fx-Ty|\ ,\ |Sx-gy|\}\,)\ \forall\, x,y \in X.$$

But there is no constant α in $[0,1)$ such that

$|fx-gy| \le \alpha\ \max\ \{|Sx-Ty|\ ,\ |Sx-fx|\ ,\ |Ty-gy|\ ,\ |fx-Ty|\ ,\ |Sx-gy|\}$
for all x,y in X.

For $x_n = \frac{1}{8n+5}$ and $y_n = \frac{1}{8n+1}$ we have $z_{n+1} = \frac{1}{8n+7} \to 0$ as $n \to \infty$.

0 is the unique fixed point of each of f,g,S and T.

8. Let f,S be self-maps on X defined as in Example 7.
Define self-maps g,T on X by

$$g(0) = 0 \qquad , \qquad T(0) = 0$$

$$g(\tfrac{1}{n}) = \begin{cases} \frac{1}{7} \text{ if n is even} \\ \frac{1}{n+10} \text{ if n is odd} \end{cases} \quad , \quad T(\tfrac{1}{n}) = \begin{cases} 1 \text{ if n is even} \\ \frac{1}{n+2} \text{ if n is odd.} \end{cases}$$

Then f is continuous on X; g,S,T are discontinuous at zero; f, S are weakly commutative but not commutative; g,T are weakly commutative but not commutative; $R(f) \subset R(T)$, $R(g) \subset R(S)$;

$$|fx-gy| \le \phi\,(\max\ \{|Sx-Ty|\ ,\ |Sx-fx|\})\ \forall\, x,y \in X,$$

and

$$|fx-gy| \le \phi\,(\max\ \{|fx-Ty|\ ,\ |Sx-gy|\})\ \forall\, x,y \in X.$$

But there is no constant α in $[0,1)$ such that

$|fx-gy| \le \alpha\ \max\ \{|Sx-Ty|\ ,\ |Sx-fx|\ ,\ |Ty-gy|\ ,\ |fx-Ty|\ ,\ |Sx-gy|\}$
for all x,y in X.

For $x_n = \frac{1}{8n+1}$ and $y_n = \frac{1}{8n+3}$ we have $z_{n+1} = \frac{1}{8n+13} \to 0$ as $n \to \infty$.

0 is the unique fixed point of each of f,g,S and T.

9. Define self-maps f,S on X by

$$f(0) = 0 \qquad , \qquad S(0) = 0$$

$$f(\tfrac{1}{n}) = \frac{1}{n+2} \qquad , \quad S(\tfrac{1}{n}) = \frac{1}{n+1} \quad (n = 0,1,2,\ \dots\ .).$$

Then f,S are continuous on X; $fS = Sf$; $R(f) \subset R(S)$;

$$|fx-fy| \le \phi\,(|Sx-Sy|) \;\forall\, x,y \in X$$

and

$$|fx-fy| \le \phi\,(\max\{|fx-Sy|,\ |Sx-fy|\}) \;\forall\, x,y \in X.$$

But there is no constant α in $[0,1)$ such that

$|fx-fy| \le \alpha \max\{|Sx-Sy|,\ |Sx-fx|,\ |Sy-fy|,\ |fx-Sy|,\ |Sx-fy|\}$

for all x,y in X.

For $x_n = \frac{1}{n}$, $fx_n = Sx_{n+1} = \frac{1}{n+2} \to 0$ as $n \to \infty$.

0 is the unique fixed point of each of f and S.

10. Define $f : X \to X$ by $f(0) = 0$, $f(\frac{1}{n}) = \frac{1}{n+1}$ ($n = 1,2, \ldots$).
Then f is continuous on X;

$$|fx-fy| \le \phi\,(|x-y|) \;\forall\, x,y \in X$$

and

$$|fx-fy| \le \phi\,(\max\{|fx-y|,\ |x-fy|\}) \;\forall\, x,y \in X.$$

But there is no constant α in $[0,1)$ such that

$$|fx-fy| \le \alpha \max\{|x-y|,\ |x-fx|,\ |y-fy|,\ |fx-y|,\ |x-fy|\}$$

for all x, y in X.

0 is the unique fixed point of f.

11. Let $X = [1,\infty)$ with the usual metric.
Define $f : X \to X$ by $fx = 2x$,
and $\phi : R^+ \to R^+$ by $\phi(t) = \frac{2t^2}{1+2t}$.

Then X is a complete metric space ; f is continuous on X ; ϕ is an increasing continuous function on R^+, $\phi(t) < t \;\forall\, t > 0$,

$\lim_{t \to +\infty} [t-\phi(t)] = \frac{1}{2}$;

and

$$|fx-fy| \le \phi\,(\max\{|x-fy|,\ |y-fx|\})$$

for all x,y in X.

But f has no fixed point. In fact, for no x in X the sequence $\{f^n x\}$ is Cauchy.

12. Let $X = [0,\infty)$ with the usual metric.
Define self-maps f,g,S,T on X by

$$fx = x\ ,\quad gx = \begin{cases} \frac{1}{8} & \text{if } x = 0 \\ \frac{x}{8} & \text{if } x > 0 \end{cases}\ ,\quad Sx = \begin{cases} 1 \text{ if } x = 0 \\ 8\,x \text{ if } x > 0 \end{cases}\ ,\quad Tx = \begin{cases} 1 \text{ if } x = 0 \\ x \text{ if } x > 0\ . \end{cases}$$

Then X is a complete metric space ; f is continuous on X ; S,g,T are discontinuous at zero ; $fS = Sf$; $gT = Tg$;

$$|fx-gy| = \frac{1}{8}\max\ \{|Sx-Ty|\ ,\ |fx-Ty|\}$$
$$|fx-gy| \le \frac{1}{7}\max\ \{|Sx-Ty|\ ,\ |Ty-gy|\}$$
$$|fx-gy| \le \frac{1}{7}\max\ \{|Sx-fx|\ ,\ |Ty-gy|\}$$

and

$$|fx-gy| \le \frac{1}{8}\max\ \{|fx-Ty|\ ,\ |Sx-gy|\}$$

for all x,y in X.

For $x_n = \frac{1}{2^{3n+3}}$ and $y_n = \frac{1}{2^{3n}}$ we have $z_{n+1} = \frac{1}{2^{3n+3}} \to 0$ as $n \to \infty$.

0 is a fixed point of f only.
We note that $0 \notin R(T)$.

13. Let $X = [0,\infty)$ with the usual metric.
Define self-maps f,g,S,T on X by

$$fx = x\ ,\ gx = \frac{x}{16}\ ,\quad Tx = \frac{x}{2}\ ,\quad Sx = \begin{cases} 1 & \text{if } x = 0 \\ 8x & \text{if } x > 0. \end{cases}$$

Then X is a complete metric space; f,g,T are continuous on X, S is discontinuous at zero ; $fS = Sf$, $gT = Tg$; $R(f) = R(T)$, $R(g) \not\subseteq R(S)$,

$$|fx-gy| \le \frac{1}{7}\max\ \{|Sx-fx|\ ,\ |Ty-gy|\}\quad \forall\, x,y \in X$$

and

$$|fx-gy| \le \frac{1}{8}\max\ \{|fx-Ty|\ ,\ |Sx-gy|\}\quad \forall\, x,y \in X.$$

For $x_n = \frac{1}{2^{3n+4}}$ and $y_n = \frac{1}{2^{3n}}$ we have $z_{n+1} = \frac{1}{2^{3n+4}} \to 0$ as $n \to \infty$.

We note that '0' is the unique fixed point of each of g and T, 0 is a fixed point of f and $0 \notin R(S)$.

14. Let $X = \{1,2,3,4\}$ with $d(1,2) = d(1,3) = d(1,4) = d(2,4) = d(3,4) = 1$,

$$d(2,3) = \frac{1}{2}.$$

Define self-maps f,g,S,T on X by

$f1=2$, $f2=3$, $f3=f4=2$; $g1=g2=g3=g4=2$;

$S1=2$, $S2=4$, $S3=1$, $S4=3$; $T1=T2=2$, $T3=1$, $T4=3$.

Then f, S do not commute weakly ; $gT = Tg$; $R(f) \subset R(T)$, $R(g) \subset R(S)$;

$$d(fx,gy) \le \frac{1}{2}\, d(Sx,Ty) \quad \forall\, x,y \in X$$

$$d(fx,gy) \le \frac{1}{2}\, d(Sx,fx) \quad \forall\, x,y \in X$$

and

$$d(fx,gy) \le \frac{1}{2}\, d(Sx,gy) \quad \forall\, x,y \in X.$$

For $x_n = y_n = 1$, we have $fx_n = gy_n = Ty_{n+1} = Sx_{n+1} = 2$.

But none of f and S has a fixed point.

15. Let $X = \{1,2,3,4\}$ and d be a metric on X.

Let S,T be self-maps on X defined as in Example 14.

Define $f : X \to X$ by $fx = 2 \quad x \in X$.

Then f, S do not commute weakly ; $fT = Tf$; $R(f) \subset R(S) \cap R(T)$;

and $d(fx,fy) = 0 \ \forall\, x,y \in X$.

For $x_n = y_n = 1$, we have $fx_n = gy_n = Ty_{n+1} = Sx_{n+1} = 2$.

But S has no fixed point.

16. Let $X = \{1,2,3,4\}$ with $d(1,2) = d(1,3) = \frac{2}{3}$;

$$d(1,4) = d(2,4) = d(3,4) = 1;\ d(2,3) = \frac{1}{2}.$$

Let f,S be self-maps on X defined as in Example 14.

Then f,S do not commute weakly ; $R(f) \subset R(S)$;

$$d(fx,fy) \leq \frac{1}{2}\, d(Sx,Sy)$$

$$d(fx,fy) \leq \frac{1}{2} \max \{d(Sx,fx),\ d(Sy,fy)\}$$

and

$$d(fx,fy) < \frac{1}{2} \max \{d(fx,Sy),\ d(Sx,fy)\}$$

for all x,y in X.

Neither f nor S has a fixed point.

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Some Classes of P-valent Analytic Functions

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Abstract

Let $f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}$ $(p \in N = 1,2 \ldots)$ be analytic in the unit disk $E = \{z : |z| < 1\}$ and $D^{n+p-1}f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z)$ $(n > -p)$. Then $f(z)$ is said to be in the class $T_{n+p-1,\beta}$ if $\operatorname{Re}\left(\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}\right) > \beta$, $z \in E$, $0 \leq \beta \leq \frac{1}{2}$ and $f(z)$ is said to be in the class $T_{n+p-1,\beta}(\alpha)$, $\alpha \geq 0$, if $\operatorname{Re} J_{n+p-1}(f;\alpha) > \beta$, $z \in E$, $0 \leq \beta \leq \frac{1}{2}$ where $J_{n+p-1}(f;\alpha) = (1-\alpha)\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} + \alpha\frac{D^{n+p+1}f(z)}{D^{n+p}f(z)}$. In this paper we consider the classes $T_{n+p-1,\beta}$ and $T_{n+p-1,\beta}(\alpha)$. Class preserving integral operators for the Class $T_{n+p-1,\beta}$ are obtained. Our results are the generalizations of the earlier results obtained by Goel and Sohi, Ruscheweyh and Al-Amiri.

1. Introduction

Let A denote the class of functions of the form $f(z) = z + a_2 z^2 + \ldots$ which are analyfic in $E = \{z : |z| < 1\}$. Recently Ruscheweyh in [8] defined the classes K_n of functions $f(z) \in A$ and satisfying the condition

$$\operatorname{Re}\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right) > \frac{1}{2}, \quad z \in E, \text{ where } D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z),$$

$n \in N_0 = \{0,1,2,\ldots\}$ and * stands for Hadmard product. In [1] Al-Amiri defined the classes $K_n(\alpha)$. A function $f(z) \in A$ is said to belong to the class $K_n(\alpha)$ if $f(z) \cdot f'(z) \neq 0$ in $0 < |z| < 1$ and for $\alpha \geq 0$, $\operatorname{Re} J_n(f;\alpha) > \frac{1}{2}$ where

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$J_n(f\,;\alpha) = (1-\alpha)\dfrac{D^{n+1}f(z)}{D^n f(z)} + \alpha\dfrac{D^{n+2}f(z)}{D^{n+1}f(z)}$. In [4] Goel and Sohi defined the classes $T_{n,\beta}$ and $T_{n,\beta}(\alpha)$. A function $f(z) \in A$ is said to belong to the class $T_{n,\beta}$, if $\mathrm{Re}\left(\dfrac{D^{n+1}f(z)}{D^n f(z)}\right) > \beta$, $z \in E$ and $0 \leq \beta \leq \frac{1}{2}$. Similarly $f(z) \in A$ is said to belong to the class $T_{n,\beta}(\alpha)$, $\alpha \geq 0$ if $f(z).f'(z) \neq 0$ in $0 < |z| < 1$ and $\mathrm{Re}\, J_n(f\,;\alpha) > \beta$, $z \in E$, $0 \leq \beta \leq \frac{1}{2}$.

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \qquad (p \in N = 1,2,\ldots)$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$.

Let $D^{n+p-1}f(z) = \dfrac{z^p}{(1-z)^{n+p}} * f(z) \quad (n > -p)$ for $f(z) \in A(p)$.

The symbol $D^{n+p-1}f(z)$ was introduced by Goel and Sohi [3]. In this paper we define the classes $T_{n+p-1,\beta}$ and $T_{n+p-1,\beta}(\alpha)$. A function $f(z) \in A(p)$ is said to belong to the class $T_{n+p-1,\beta}$ if $\mathrm{Re}\dfrac{D^{n+p}f(z)}{D^{n+p-1}f(z)} > \beta$, $z \in E$, $0 \leq \beta \leq \frac{1}{2}$. Similarly $f(z) \in A(p)$ is said to belong to the class $T_{n+p-1,\beta}(\alpha)$, $\alpha \geq 0$ if $\mathrm{Re}\, J_{n+p-1}(f;\alpha)$, $z \in E$, $0 \leq \beta \leq \frac{1}{2}$,

where $J_{n+p-1}(f;\alpha) = (1-\alpha)\dfrac{D^{n+p}f(z)}{D^{n+p-1}f(z)} + \alpha\dfrac{D^{n+p+1}f(z)}{D^{n+p}f(z)}$.

Observe that

$$T_{n+o,\beta} = T_{n,\beta}\,, \quad T_{n+o,\beta}(\alpha) = T_{n,\beta}(\alpha)$$

$$T_{n,\frac{1}{2}} = k_n \qquad , \quad T_{n,\frac{1}{2}}(\;) = k_n(\;)\,.$$

Thus our results cover corresponding results of Goel and Sohi [4], Ruscheweyh [8] and Al-Amiri [1].

2. The classes $T_{n+p-1,\beta}$

Theorem 1. Let $n \in N_o$, $\mathrm{Re}\ c > (1-\beta)\,n - \beta p$.

If $f(z) \in T_{n+p-1,\beta}$ then

$$F(z) = \frac{p+c}{z^c}\int_0^z t^{c-1} f(t)\,dt \tag{1}$$

Also belongs to $T_{n+p-1,\beta}$.

Proof: It is easy to verify that the function $F(z)$ defined by (1) satisfies

$$z(D^{n+p-1}F(z))' = (c+p)D^{n+p-1}f(z) - c\,D^{n+p-1}F(z) \tag{2}$$

Define a regular function $w(z)$ in E by

$$\frac{D^{n+p}F(z)}{D^{n+p-1}F(z)} = \frac{1-(1-2\beta)w(z)}{1+w(z)} \tag{3}$$

obviously $w(o) = 0$, $w(z) \neq -1$ for $z \in E$.

Differentiating (3) logarithmically using (2), we get

$$\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} = \frac{1-(1-2\beta)w(z)}{1+w(z)} - \frac{2(1-\beta)zw'(z)}{(c+p)(1-(1-2\beta)w(z))(1+w(z))}\cdot\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}. \tag{4}$$

It is easy to check the following identity

$$z(D^{n+p-1}F(z))' = (n+p)D^{n+p}F(z) - nD^{n+p-1}F(z) \tag{5}$$

From (2), (3) and (5) after simple computation,

$$\frac{D^{n+p-1}f(z)}{D^{n+p-1}F(z)} = \frac{1+\left(1-\frac{2(1-\beta)(n+p)}{c+p}\right)w(z)}{1+w(z)} \tag{6}$$

(4) in conjunction with (6) gives

$$\frac{D^{n+p}F(z)}{D^{n+p-1}F(z)} = \frac{1-(1-2\beta)\,w(z)}{1+w(z)} -$$

$$\frac{2(1-\beta)}{c+p}\ \frac{zw'(z)}{\left(1+\frac{(c+p-2(1-\beta)(n+p))w(z)}{c+p}\right)(1+w(z))}$$

We claim that $|w(z)| < 1$, for otherwise by Jack's Lemma [5] there exists z_o, $|z_o| < 1$ such that $z_o w'(z_o) = kw(z_o)$, $|w(z_o)| = 1$ and $k \geq 1$. Then

$$\frac{D^{n+p}f(z_o)}{D^{n+p-1}f(z_o)} = \frac{1-(1-2\beta)w(z_o)}{1+w(z_o)} -$$

$$\frac{2(1-\beta)}{(c+p)}\ \frac{kw(z_o)}{\left(1+\frac{(c+p-2(1-\beta)(n+p))w(z_o)}{c+p}\right)(1+w(z_o))} . \tag{7}$$

Since $\operatorname{Re} \frac{1-(1-2\ )w(z_o)}{1+w(z_o)} = \beta$ and

$$\operatorname{Re}\left\{\frac{2(1-\beta)}{c+p}\ \frac{kw(z_o)}{\left(1+\frac{(c+p-2(1-\beta)(n+p))\,w(z_o))}{c+p}\right)(1+w(z_o))}\right\} > 0$$

for $\operatorname{Re} c \geq (1-\beta)\,n - p\beta$,

it follows from (7) that

$$\operatorname{Re}\left(\frac{D^{n+p}f(z_o)}{D^{n+p-1}f(z_o)}\right) \leq \beta \quad \text{for } \operatorname{Re} c > (1-\beta)\,n - \beta p.$$

This contradicts that $f \in T_{n+p-1,\beta}$.

Hence $|w(z)| < 1$ and by (3) $F \in T_{n+p-1,\beta}$.

Putting $p = 1$ in Theorem 1 we get the following result of Goel and Sohi [4].

<u>Corollary 1</u>. Let $n \in N_o$, $\operatorname{Re} c \geq (1-\beta)\,n - \beta$.

If $f(z) \in T_{n,\beta}$ then

$$F(z) = \frac{1+c}{z^c} \int_0^z t^{c-1} f(t)dt \text{ also belongs to } T_{n,\beta}.$$

Substituting $\beta = \frac{1}{2}$ in above Theorem we get the following result of Goel and Sohi [3].

Corollary 2. Let $n \in N_o$, Re $c > \frac{n-p}{2}$.

If $f(z) \in T_{n+p-1, \frac{1}{2}}$ then

$$F(z) = \frac{p+c}{z^c} \int_0^z t^{c-1} f(t)dt \text{ also belongs to } T_{n+p-1, \frac{1}{2}}.$$

Theorem 2. Let f A(p) and satisfies the condition

$$\text{Re } \frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} > \beta - \frac{1-\beta}{2\,(c-n+\beta(n+p))} \tag{8}$$

where c is any real number greater than $2(1-\beta)\,(n+p)-p$, then the function

$$F(z) \frac{p+c}{z^c} \int_0^z t^{c-1} f(t)dt \text{ belongs to } T_{n+p-1,\beta}.$$

Proof: Proceeding as in Theorem 1 we obtain

$$\frac{D^{n+p}f(z_o)}{D^{n+p-1}f(z_o)} = \frac{1-(1-2\beta)w(z_o)}{1+w(z_o)} - \frac{2(1-\beta)}{c+p} \frac{kw(z_o)}{\left(1+\frac{(c+p-2(1-\beta)(n+p))w(z_o)}{c+p}\right)(1+w(z_o))} \tag{9}$$

We have $\quad \text{Re } \frac{1-(1-2\beta)w(z_o)}{1+w(z_o)} = \beta$

and $\text{Re } \left\{ \frac{w(z_o)}{\left(1+\frac{(c+p-2(1-\beta)(n+p)w(z_o)}{(c+p)}\right)(1+w(z_o))} \right\} \geq$

$$\frac{c+p}{4(c-n+\beta(n+p))} \quad \text{for } c > 2(1-\beta)(n+p)-p.$$

The remaining part of the proof is similar to that of Theorem 1.

Corollary 3. Putting $n = \beta = 0$ and $p = c = 1$ in the above theorem it follows that if $f \in A$ and satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > -\frac{1}{2}, \quad \text{then } \mathrm{Re}\left(\frac{zF'(z)}{F(z)}\right) > 0 \text{ and hence}$$

$F(z)$ is starlike and univalent in E. This result was obtained earlier by R. Singh and S. Singh [10].

Corollary 4. By putting $n = 0$, $p = c = 1$ and $\beta \frac{\sqrt{17}-3}{4}$ in the above theorem we conclude that if $\mathrm{Re}\left(\frac{fz'(z)}{f(z)}\right) > 0$, then $\mathrm{Re}\left(\frac{zF'(z)}{F(z)}\right) > \frac{\sqrt{17}-3}{4}$. This result is due to Miller, Mocanu and Reade [7].

Corollary 5. Substituting $n = p = c = 1$ and $\beta = \frac{1}{2}$ it follows that if

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2} \quad \text{then}$$

$\mathrm{Re}\left(1 + \frac{zF''(z)}{F'(z)}\right) > 0$. This result is also due to R. Singh and S. Singh [10].

Remark 1. Results obtained in the above corollaries are stronger and extend the earlier results due to Libera [6] that is if $f(z)$ is member of S^* or k, then so also is $F(z) = \frac{2}{z}\int_0^z f(t)dt$, where S^* and k are the usual classes of starlike and convex functions respectively.

Theorem 3. Let $F \in T_{n+p-1,\beta}$ and $t = \mathrm{Re}\, c > n(1-\beta)-\beta p$, $n \in N_o$. Let f be defined as $F(z) = \frac{p+c}{z^c}\int_0^z \xi^{c-1} f(\xi)d\xi$, $\mathrm{Re}\, c > n(1-\beta) - \beta p$.

Then $f(z) \in T_{n+p-1,\beta}$ in $|z| < R_{n+p-1,c}$ where $R_{n+p-1,c}$ is the smallest positive root of

$[t - n - (1-2\beta)(n+p)]r^2 + 2[n+p-1-\beta(n+p)]r - t - p = 0$ and the result is sharp for c real.

Proof: Since $F \in T_{n+p-1,\beta}$. We have

$$\frac{D^{n+p}F(z)}{D^{n+p-1}F(z)} = \beta + (1-\beta)\, p(z) \tag{10}$$

where $p(0) = 1$ and $\operatorname{Re} p(z) > 0$ for $z \in E$.

Using (2), (5) and (10) we obtain

$$\begin{aligned}(c+p)D^{n+p}f(z) &= c\, D^{n+p}F(z) + z\,(D^{n+p}F(z))' \\ &= c\,(\beta + (1-\beta)\, p(z))\, D^{n+p-1}F(z) + \\ &\quad z(1-\beta)p'(z)D^{n+p-1}F(z) + (\beta+(1-\beta)p(z))\, z(D^{n+p-1}F(z))' \\ &= c\,(\beta+(1-\beta)p(z))D^{n+p-1}F(z) + \\ &\quad (1-\beta)\, zp'(z)\, D^{n+p-1}F(z) + \\ &\quad (\beta+(1-\beta)p(z))\, [(n+p)\, D^{n+p}F(z) - nD^{n+p-1}F(z)].\end{aligned}$$

Thus

$$\begin{aligned}(c+p)\, D^{n+p}f(z) = [&(c-n)(\beta+(1-\beta)p(z)) + (1-\beta)\, zp'(z) + \\ &(n+p)(\beta+(1-\beta)p(z))^2]\, D^{n+p-1}F(z)\end{aligned} \tag{11}$$

Also

$$\begin{aligned}(c+p)\, D^{n+p-1}f(z) &= c\, D^{n+p-1}F(z) + (n+p)\, D^{n+p}F(z) - n\, D^{n+p-1}F(z) \\ &= [(c-n) + (n+p)(\beta+(1-\beta)p(z))]\, D^{n+p-1}F(z)\end{aligned} \tag{12}$$

From (11) and (12) we have

$$\frac{\dfrac{D^{n+p}f(z)}{D^{n+p-1}f(z)} - \beta}{1-\beta} = p(z) + \frac{zp'(z)}{c-n+(n+p)(\beta+(1-\beta)p(z))} \tag{13}$$

Using the well known estimate

$$\frac{|zp'(z)|}{\operatorname{Re} p(z)} \le \frac{2r}{1-r^2} \quad \text{and} \quad \operatorname{Re} p(z) \ge \frac{1-r}{1+r},$$

$|z| = r$, we get from (13) that

$$\operatorname{Re}\left\{\frac{\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}-\beta}{1-\beta}\right\} \geq \operatorname{Re} p(z) \geq$$

$$\left\{1 - \frac{2r}{(1-r)\ [(t-n)(1+r)+(n+p)(1+r)\beta+(1-\beta)(1-r)(n+p)]}\right\} \tag{14}$$

The right hand side of (14) is positive provided $r < R_{n+p-1},c$.

The result is sharp for the function

$$f(z) = \frac{z^{1-c}}{p+c}\,(z^c F(z))'$$

where $F(z)$ is given by

$$\frac{D^{n+p}F(z)}{D^{n+p-1}F(z)} = \frac{1-(1-2\beta)z}{1+z}$$

Putting $p = 1$ in Theorem 3 we get the following result of Goel and Sohi [4].

Corollary 6. Let $F \in T_n$, and $t = \operatorname{Re} c > n(1-\beta) - \beta$, $n \in N_o$.
Let f be defined as

$$F(z) = \frac{1+c}{z^c}\int_o^z \xi^{c-1}f(\xi)d\xi \ , \ \operatorname{Re} c > (1-\beta)\,n - \beta \ .$$

Then $f(z) \in T_{n,\beta}$ in $|z| < R_{n,c}$ where $R_{n,c}$ is the smallest positive root of

$[t-n-(1(1-2\beta)\ (n+1)]\ r^2 + 2\ [n+z - \beta(n+1)]\ r - t - 1 = 0$,

and the result is sharp for c real.

Corollary 7. Substituting $n = \beta = 0$, $p = 1$ and $c > 0$, we obtain a generalization of Bernardi's result [2].

3. The classes $T_{n+p-1,\beta}(\alpha)$

Theorem 4. Let $f \in T_{n+p-1,\beta}$, $\alpha \geq 0$. Then

$$\operatorname{Re}\left(\frac{J_{n+p-1}(f;\alpha)-\beta}{1-\beta}\right) \geq \begin{cases} M_1(r) & \text{for } R_o \leq R_1 \\ M_2(r) & \text{for } R_o \geq R_1 \end{cases}$$

where

$$M_1(r) = \frac{\alpha}{2(n+p+1)(1-\beta)^2}\Big[\frac{2\alpha(1-2\beta)-2\beta(1-\beta)(n+p+1)}{\alpha} +$$

$$\left(\frac{2(n+p+1)(1-\beta)-\alpha(1-2\beta)}{\alpha}\right)\left(\frac{1-(1-2\beta)r}{1+r}\right) -$$

$$(1-2\beta)\frac{1+r}{1-(1-2\beta)r}\Big] ,$$

$$M_2(r) = \frac{\alpha}{(n+p+1)(1-\beta)^2}\Big[\frac{\alpha(1-2\beta)-\beta(1-\ )(n+p+1)}{\alpha} +$$

$$\frac{1}{1-r^2}\Big\{(2\beta)^{\frac{1}{2}}\left(\left(\frac{2(n+p+1)(1-\beta)+2\ \ \beta}{\alpha}\right)(1-2\beta r^2-(1-2\beta)r^4)\right)^{\frac{1}{2}} -$$

$$(1+(1-2\beta)r^2)\Big\}\Big] ,$$

$$R_o^2 = \frac{\alpha\beta(1+(1-2\beta)r^2)}{((n+p+1)(1-\beta)+\alpha\beta)(1-r^2)} \text{ and } R_1 = \frac{1-(1-2\beta)r}{1+r} .$$

The result is sharp.

We shall need the following Lemma due to Singh and Goel [9].

Lemma: If $w(z)$ is regular in E and satisfies the conditions $w(0) = 0$ and $|w(z)| < 1$, then

$$|zw'(z) - w(z)| \leq \frac{r^2 - |w|^2}{1 - r^2} , \quad |z| < 1 \tag{15}$$

Proof: Since $f \in T_{n+p-1,\beta}$. We can write

$$\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} = \frac{1-(1-2\beta)\,w(z)}{1+w(z)} \tag{16}$$

where $w(z)$ is regular in E, $w(0) = 0$ and $|w(z)| < 1$.

Differentiating (16) logarithmically and using (5) we get

$$\frac{D^{n+p+1}f(z)}{D^{n+p}f(z)} = \frac{1}{n+p+1}\left(1 + (n+p)\,\frac{1-(1-2\beta)w(z)}{1+w(z)} - \frac{2(1-\beta)\,zw'(z)}{(1-(1-2\beta)\,w(z))(1+w(z))}\right) \tag{17}$$

Substituting from (16) and (17) in $J_{n+p-1}(;\alpha)$, we get

$$J_{n+p-1}(f;\alpha) = \frac{1-(1-2\beta)\,w(z)}{1+w(z)} + \frac{2\alpha(1-\beta)}{n+p+1}\left[\frac{w(z)}{1+w(z)} - \frac{zw'(z)}{(1-(1-2\beta)w(z))(1+w(z))}\right] \tag{18}$$

This can be written as

$$\frac{J_{n+p-1}(f;\alpha)-\beta}{1-\beta} = \frac{1-w(z)}{1+w(z)} + \frac{2\alpha}{(n+p+1)}\,\frac{w(z)}{(1+w(z))} - - \frac{2\alpha}{(n+p+1)}\,\frac{zw'(z)}{(1-(1-2\beta)w(z))(1+w(z))}. \tag{19}$$

From (15) and (19) we get

$$\operatorname{Re}\left\{\frac{J_{n+p-1}(f;\alpha)-\beta}{1+\beta}\right\} \geq \operatorname{Re}\frac{1-w(z)}{1+w(z)} + \frac{2\alpha}{n+p+1}\left[\operatorname{Re}\left\{\frac{w(z)}{1+w(z)} - \frac{w(z)}{(1-(1-2\beta)w(z))(1+w(z))}\right\} - \frac{r^2-|w(z)|^2}{(1-r^2)\,|1-(1-2\beta)w(z)|\,|1+w(z)|}\right] \tag{20}$$

$$= \frac{\alpha}{2(n+p+1)\,(1-\beta)^2}\left[\frac{2\alpha(1-2\beta)-2\beta(1-\beta)(n+p+1)}{\alpha} + \frac{2(n+p+1)\,(1-\beta)-\alpha(1-2\ )}{\alpha}\operatorname{Re}p(z) - (1-2\beta)\operatorname{Re}\frac{1}{p(z)}\right.$$

$$- \frac{r^2 \, |p(z) + 1-2\beta|^2 - |1-p(z)|^2}{(1-r^2) \, |p(z)|} \Bigg] \tag{21}$$

where $p(z) = \frac{1-(1-2\beta)w(z)}{1+w(z)}$. It is easy to see that $p(z)$ maps the circle $|w(z)| < r$ onto the circle $|p(z)-a| \le d$,

$$a = \frac{1+(1-2\beta)r^2}{1-r^2}, \quad d = \frac{2(1-\beta)r}{1-r^2}, \quad r = |z| .$$

If we put $p(z) = a + u + iv$

$R^2 = |p(z)|^2 = (a+u)^2 + v^2$ and denote the right handside of (21) by $S(u,v)$ then we get

$$S(u,v) = \frac{\alpha}{2(n+p+1)(1-\beta)^2} \Big[\frac{2\alpha(1-2\beta) - 2\beta\,(1-\beta)(n+p+1)}{\alpha} +$$

$$\frac{2(n+p+1)\,(1-\beta) - \alpha(1-2\beta)}{\alpha}(a+u) -$$

$$(1-2\beta)(a+u)R^{-2} + (^2+c^2-d^2)R^{-1} \Big] \tag{22}$$

Now

$$\frac{\partial S}{\partial v} = \frac{\alpha}{2(n+p+1)(1-\beta)^2} vR^{-4}T(R) \text{ , where}$$

$$T(R) = 2(1-2\beta)\;(a+u) + 2R^3 + R(d^2 - u^2 - v^2) > 0 \text{ for } 0 \le \beta \le \tfrac{1}{2} .$$

The minimum of $S(u,v)$ inside the circle $|p(z) - a| \le d$ is attained on the diameter $v = 0$. On putting $v = 0$ in (22) we get

$$L(R) = S(u,o) = \frac{\alpha}{2(n+p+1)(1-\beta)^2} \Big[\frac{2\alpha(1-2\beta)-2\beta(1-\beta)(n+p+1)}{\alpha} +$$

$$\frac{(2(n+p+1)(1-\beta) + 2\alpha\beta)}{\alpha} R - \frac{(1-2\beta)}{R} + \frac{a^2-d^2}{R} - 2a \Big]$$

where $R = a+u$ and $a-d \le R \le a + d$. The absolute minimum of $L(R)$ in $(0,\infty)$ is attained at $R = R_o$ where

$$R_o = \left[\frac{\beta(1+(1-2\beta)r^2)}{((n+p+1)(1-\beta)+\alpha\beta)(1-r^2)}\right]^{\frac{1}{2}}$$

and equals to $M_2(r)$.

It is easy to see that $R_o < a + d$, but R_o is not always greater than $a - d$. In such case when $R_o \notin [a-d, a+d]$ the minimum of $L(R)$ on the segment is attained at $R_1 = a - d$ and equals to $M_1(r)$.

The equality signs for $M_1(r)$ and $M_2(r)$ are respectively attained for the functions $f_1(z)$ and $f_2(z)$ defined by

$$\frac{D^{n+p}f_1(z)}{D^{n+p-1}f_1(z)} = \frac{1-(1-2\beta)z}{1+z}$$

and

$$\frac{D^{n+p}f_2(z)}{D^{n+p-1}f_2(z)} = \frac{1-2\beta\cos\theta\, z+(2\beta-1)z^2}{1-2\cos\theta\, z+z^2}$$

where $\cos\theta$ is determined from

$$R_o = \frac{1-2\beta\cos\theta\, r+(2\beta-1)r^2}{1-2\cos\theta\, r+r^2}$$

Theorem 5. Let $f \in T_{n+p-1,\beta}$, $\alpha \geq 0$.

Then $f \in T_{n+p-1,\beta}(\alpha)$ for $|z| < r_o$ where r_o is the smallest positive root of the equation

$$[\beta^2(n+p+1)^2 + 4\alpha^2(1-2\beta)]\, r^4 + 2\beta[4\alpha^2 + 2\alpha(n+p+1)$$
$$- \beta(n+p+1)^2]\, r^2 + \beta[(n+p+1)^2\beta - 4\alpha(n+p+1)] = 0$$

if $R_o \geq R_1$ and

$$r_o = \frac{n+p+1}{(1-\beta)(n+p+1)+\sqrt{(1-\beta)^2(n+p+1)^2-(n+p+1)(n+p+1-2\alpha)(1-2\beta)}}$$

if $R_o \leq R_1$ where

$$R_o = \left[\frac{\alpha\beta(1 + (1-2\beta)r^2)}{((n+p+1)\ (1-\beta)\ +\alpha\beta)(1-r^2)}\right]^{\frac{1}{2}}$$

$$R_1 = \frac{1 - (1 - 2\beta)r}{1+r} \quad \text{and } r = |z| < 1.$$

The result is sharp.

The proof of this theorem follows easily from that of Theorem 4.

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Submanifolds of a Sasakian Manifold with Quarter Connections Symmetric Metric

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Summary

Various kinds of submanifolds of a Sasakian manifold [5] have been studied by Yano and Kon in a series of papers [7,8,9,10] and others. The purpose of the present paper is to study properties of quarter-symmetric metric connections defined by Golab [2] and studied by Rastogi [4], Mishra and Pandey [3] and Yano and Imai [11], in a submanifold of a Sasakian manifold. Some special cases of these results relating to invariant, anti-invariant, generic and contact CR-submanifolds have also been studied here.

1. Introduction

Let M^{2m+1} be a (2m+1)-dimensional differentiable manifold of class C^∞ covered by a system of coordinate neighbourhoods $\{U;x^h\}$, $h,i,j,\ldots = 1,2,\ldots,2m+1$. Let M^{2m+1} has a tensor field $F_i{}^h$, of type (1,1), a vector field f^h and a 1-form α_i satisfying

$$(1.1)\quad F_i{}^h F_j{}^i = -\delta_j^h + \alpha_j f^h,\ F_i{}^h f^i = 0,\ \alpha_i F_j{}^i = 0,\ \alpha_i f^i = 1.$$

Let $N_{ji}{}^h$ be the Nijenhuis tensor of $F_i{}^h$, then the almost contact structure (F,f,α) satisfying $N_{ji}{}^h + (\partial_j\alpha_i - \partial_i\alpha_j) f^h = 0$, is called normal almost contact structure. Let g_{ji} be the Riemannian metric of the manifold M^{2m+1} given by $g_{ts}F_j{}^t F_i{}^s = g_{ji} - \alpha_j\alpha_i, \alpha_i = g_{ji} f^j$. An almost contact structure satisfying $F_{ji} = \frac{1}{2}(\delta_j\alpha_i - \delta_i\alpha_j)$ is said to be contact. A manifold endowed with a normal contact metric structure is called a Sasakian manifold [5]. In a Sasakian manifold we have

$$(1.2)\quad F_i{}^h = \nabla_i f^h,\quad \nabla_j F_i{}^h = -g_{ji} f^h + \delta_j^h f_i,$$

where ∇_j denotes covariant differentiation with respect to Christoffel symbols $\{{}^{\,h}_{ji}\}$ formed with g_{ji} and $f_i = g_{ji} f^j$.

Let M^n be an n-dimensional C^∞, Riemannian manifold covered by a system of coordinate neighbourhoods $\{V;y^a\}$, $a,b,c,\ldots = 1,2,\ldots,n$. Let M^n be isometrically immersed in M^{2m+1}, then we have

$$(1.3)\quad g_{cb} = g_{ji} B^{ji}_{cb},\ B^{ji}_{cb} = B^j_c B^i_b,\ B^j_c = \partial_c x^j,\ \partial_c = \partial/\partial y^c.$$

Let $C_y{}^h$ be 2m+1-n mutually orthogonal unit vectors normal to M^n, where $x,y,z,\ldots = n+1,\ldots, 2m+1$, then we have [1]:

$$(1.4)\quad g_{zy} = g_{ji}\, C^{ji}_{zy},$$

$$(1.5)\quad \nabla_c B^h_b = h_{cb}{}^x\, C_x{}^h$$

and

$$(1.6)\quad \nabla_c C_y{}^h = - h^a_{cy}\, B^h_a,$$

where $C^{ji}_{zy} = C_z{}^j C_y{}^i$, g_{zy} is the metric tensor of the normal bundle, ∇_c is the operator of Vander Waerden-Bortolloti covariant derivative along M^n, $h_{cb}{}^x$ are second fundamental tensors of M^n and $h^a_{cy} = g_{zy}\, g^{ba}\, h_{cb}{}^z$.

In general, we have [10]

$$(1.7)\quad F_i{}^h B^i_b = B^h_a f^a_b - C_x{}^h f^x_b,$$

$$(1.8)\quad F_i{}^h C_y{}^i = B^h_a f^a_y + C_x{}^h f^x_y$$

and

$$(1.9)\quad f^h = B^h_a f^a + C_x{}^h f^x.$$

Since $F_{ji} = - F_{ij}$, $f_{cb} = - f_{bc}$, $f_{zy} = -f_{yz}$, where $f_{cb} = f^d_c\, g_{db}$, $f_{zy} = f^x_z\, g_{xy}$, therefore we have $f_{by} = f_{yb}$, where $f_{by} = f_b{}^z g_{zy}$, $f_{yb} = f^c_y\, g_{cb}$. Also we have [10]

$$(1.10)\quad f^a_c f^c_b - f^a_z f^z_b = - \delta^a_b + f_b f^a,\quad f^x_c f^c_b + f^x_z f^z_b = - f_b f^x,$$

$$f^a_c f^c_y + f^a_z f^z_y = f_y f^a,\quad f^x_c f^c_y - f^x_z f^z_y = \delta^x_y - f_y f^x,$$

$$f^a_c f^c + f^a_x f^x = 0,\quad f^x_c f^c - f^x_y f^y = 0,$$

and

$$(1.11)\quad \nabla_c f^a_b = - g_{cb} f^a + \delta^a_c f_b + f^a_x h_{cb}{}^x - h^a_{cx} f^x_b,$$

$$\nabla_c f^x_b = g_{cb} f^x - f^x_y h_{cb}{}^y + h_{ce}{}^x f^e_b,$$

$$\nabla_c f^a_y = \delta^a_c f_y + h^a_{cx}\, f^x_y - f^a_e\, h^e_{cy},$$

$$\nabla_c f^x_y = -\, h_{ce}{}^x\, f^e_y + f^x_e\, h^e_{cy},$$

$$\nabla_c f^a = f^a_c + h^a_{cx}\, f^x,\ \nabla_c f^x = -\, f^x_c - h_{ce}{}^x\, f^e.$$

2. Quarter-symmetric Metric Connections

The two quarter-symmetric metric connections in a Sasakian manifold M^{2m+1} can be expressed as

(2.1) $\Gamma_{ji}{}^h = \left\{{h \atop ji}\right\} - p_j\, F_i{}^h$

and

(2.2) $*\Gamma_{ji}{}^h = \left\{{h \atop ji}\right\} + \delta^h_j\, p_i - p_j F_i{}^h - g_{ji} p^h,$

where p_j is a vector field and $p^h = g^{hj} p_j$ is expressible as

(2.3) $p^h = p^a\, B^h_a + p^x\, C_x{}^h,$

such that

(2.4) $p^h\, B^b_h = p^b,\ p_h\, B^h_a = p_a,\ p^h\, C_h{}^y = p^y,\ p_h\, C_x{}^h = p_x.$

Let D_j be the operator of covariant differentiation based on $\Gamma_{ji}{}^h$ then we can obtain for a tensor T^h_l

(2.5) $D_j T^h_l = \nabla_j T^h_l - p_j\, F_i{}^h\, T^i_l + p_j\, F_l{}^i\, T_i{}^h,$

which leads to

$$D_j\, F^h_l = \nabla_j\, F_l{}^h,\ D_j p^h = \nabla_j p^h - p_j F_i{}^h p^i \text{ and } D_j f^h = \nabla_j f^h.$$

Let $*D_j$ be the operator of covariant differentiation based on $*\Gamma_{ji}{}^h$ then we can obtain

(2.6) $*D_j T^h_l = D_j T^h_l + T^i_l p_i\, \delta^h_j - T^h_j\, p_l - T_{lj} p^h + g_{jl} T^h_i,$

which leads to

(2.7) $*D_j F_l{}^h = \nabla_j F_l{}^h + F_l{}^i p_i \delta^h_j - F_j{}^h p_l - F_{lj} p^h + g_{jl} F_i{}^h p^i$

and

(2.8) $*D_j f^h = F_j{}^h + f^i p_i\, \delta^h_j - f_j p^h.$

From (2.7) and (2.8) we can obtain

$$(2.9)\quad {}^*D_j F_i^{\ j} = \nabla_j F_i^{\ j} + (2m-1)\, F_{ij} p^j$$

and

$$(2.10)\quad {}^*D_j f^j = 2m\, f^j p_j .$$

Let $\Gamma_{cb}^{\ \ a}$ and ${}^*\Gamma_{cb}^{\ \ a}$ be the induced connection parameters in M^n based on (2.1) and (2.2) respectively, then we can obtain

$$(2.11)\quad \Gamma_{cb}^{\ \ a} = \{^{a}_{cb}\} - p_c\, f^a_b$$

and

$$(2.12)\quad {}^*\Gamma_{cb}^{\ \ a} = \{^{a}_{cb}\} + p_b\, \delta^a_c - g_{cb} p^a - p_c f^a_b ,$$

whereas in the normal bundle both connections induce

$$(2.13)\quad \Gamma_{cy}^{\ \ x} = \{^{x}_{cy}\} - p_c\, f^x_y ,$$

where [1]

$$\{^{x}_{cy}\} = (\partial_c C_y^{\ h} + \{^{h}_{ji}\} B^j_c\, C_y^{\ i})\, C_h^{\ x}.$$

Hence we have

Theorem (2.1). A quarter-symmetric metric connection $\Gamma_{ji}^{\ \ h}$ $({}^*\Gamma_{ji}^{\ \ h})$ in a Sasakian manifold M^{2m+1} induces a quarter-symmetric metric connection $\Gamma_{cb}^{\ \ a}$ $({}^*\Gamma_{cb}^{\ \ a})$ in a submanifold M^n of a Sasakian manifold M^{2m+1}. In the normal bundle both quarter-symmetric connections induce the same connection (2.13).

If p^h is a vector normal to M^n, $p_c = 0$ and equations (2.11), (2.12) and (2.13) respectively reduce to

$$(2.14)\quad \Gamma_{cb}^{\ \ a} = \{^{a}_{cb}\},\ {}^*\Gamma_{cb}^{\ \ a} = \{^{a}_{cb}\},\ \Gamma_{cy}^{\ \ x} = \{^{x}_{cy}\}.$$

Hence we have

Theorem (2.2). For the vector p^h normal to M^n, both quarter-symmetric metric connections induce $\{^{a}_{cb}\}$ in M^n and $\{^{x}_{cy}\}$ in the normal bundle.

If the vector p^h is identical with f^h, equations (2.11), (2.12) and (2.13) respectively give

$$(2.15)\quad \Gamma_{cb}{}^{a} = \{{}^{a}_{cb}\} - f_c\, f^a_b\,,$$

$$*\Gamma_{cb}{}^{a} = \{{}^{a}_{cb}\} + f_b\,\delta^a_c - g_{cb}\,f^a - f_c\,f^a_b$$

$$\Gamma_{cy}{}^{x} = \{{}^{x}_{cy}\} - f_c\,f^x_y\,.$$

Now applying (1.11) to (2.15) and assuming that $\nabla_b f^a = 0$, we obtain

$$(2.16)\quad \Gamma_{cb}{}^{a} = \{{}^{a}_{cb}\} + f_c f^x\, h^a_{bx}$$

$$*\Gamma_{cb}{}^{a} = \{{}^{a}_{cb}\} + f_b\,\delta^a_c - g_{cb} f^a + f_c f^x\, h^a_{bx}.$$

These equations for a totally geodesic submanifold reduce to

$$(2.17)\quad \Gamma_{cb}{}^{a} = \{{}^{a}_{cb}\},\ *\Gamma_{cb}{}^{a} = \{{}^{a}_{cb}\} + f_b\,\delta^a_c - g_{cb}f^a,$$

which can also be obtained from (2.16) if we assume that f^h is a vector tangential to M^n. Hence we have

Theorem (2.3). <u>If the vector $p^h = f^h$, f^a is covariantly constant vector and either the submanifold is totally geodesic or f^h is tangential to M^n, the quarter-symmetric metric connections $\Gamma_{ji}{}^{h}$ $(*\Gamma_{ji}{}^{h})$ induce Christoffel symbols (a semi-symmetric metric connection Yano [6]) in M^n.</u>

If the submanifold M^n of a Sasakian manifold M^{2m+1} is anti-invariant [7], i.e., $f^a_b = 0$, equations (2.11) and (2.12) give

Theorem (2.4). <u>The quarter-symmetric metric connections $\Gamma_{ji}{}^{h}$ $(*\Gamma_{ji}{}^{h})$ induce $\{{}^{c}_{ba}\}$ (a semi-symmetric metric connection) in an anti-invariant submanifold M^n of a Sasakian manifold M^{2m+1}.</u>

If the submanifold M^n of a Sasakian manifold M^{2m+1} is generic [8], i.e., it satisfies

$$f^x_y = 0,$$

equation (2.13) gives

Theorem (2.5). <u>In a normal bundle of a generic submanifold of a Sasakian manifold M^{2m+1}, the two quarter symmetric metric connections induce $\{{}^{x}_{cy}\}$.</u>

3. <u>Mixed Covariant Derivative</u>

Let D_c and $*D_c$ respectively denote Vander Waerden-Bortolloti type of covariant derivatives based on $\Gamma_{ji}{}^{h}$ and $*\Gamma_{ji}{}^{h}$, then analogous to

(1.11) we can obtain

$$(3.1)\quad D_c f^a_b = \nabla_c f^a_b,\ D_c f^x_b = \nabla_c f^x_b + p_c\,(f^x_d\, f^d_b - f^y_b\, f^x_y),$$

$$D_c f^a_y = \nabla_c f^a_y - p_c\,(f^d_y\, f^a_d - f^a_x\, f^x_y),\ D_c f^x_y = \nabla_c f^x_y,$$

$$D_c f^a = \nabla_c f^a - p_c f^a_d f^d,\ D_c f^x = \nabla_c f^x - p_c f^x_y\, f^y$$

and

$$(3.2)\quad {}^*D_c f^a_b = D_c f^a_b - f^a_c p_b - f_{bc} p^a + f^d_a p_d\ \delta^a_c + f^a_d p^d g_{bc},$$

$${}^*D_c f^x_b = D_c f^x_b - p_b f^x_c + g_{cb} f^x_d p^d,$$

$${}^*D_c f^a_y = D_c f^a_y + f^d_y p_d\ \delta^a_c - f_{yc} p^a,\ {}^*D_c f^x = D_c f^x,$$

$${}^*D_c f^x_y = \nabla_c f^x_y,\ {}^*D_c f^a = D_c f^a + p_d\ \delta^a_c f^d - f_c p^a.$$

Since for an invariant submanifold of a Sasakian manifold we have $f^a_x = 0$, therefore equation (1.10) gives [10]

$$(3.3)\quad f^a_c f^c_b = -\delta^a_b + f_b f^a,\ f^x_z f^z_y = -\delta^x_y + f_y f^x,\ f^a_c f^c = 0,\ f^x_y f^y = 0.$$

Now using (3.3) in (3.1) and (3.2) we obtain

$$(3.4)\quad D_c f^a = \nabla_c f^a,\ {}^*D_c f^a = \nabla_c f^a + p_d f^d\ \delta^a_c - f_c p^a,\ {}^*D_c f^x = D_c f^x = \nabla_c f^x.$$

Hence we have

Theorem (3.1). <u>For an invariant submanifold of a Sasakian manifold the D and *D covariant derivatives satisfy (3.4).</u>

Since for an anti-invariant submanifold of a Sasakian manifold we have $f^a_b = 0$, therefore equations (1.10) reduce to [10]

$$(3.5)\quad f^a_z f^z_b = \delta^a_b - f_b f^a,\ f^x_z f^z_b = -f_b f^x,\ f^a_z f^z_y = f_y f^a,\ f^a_x f^x = 0.$$

Now using (3.5) in (3.1) and (3.2) we obtain

$$(3.6)\quad D_c f^x_b = \nabla_c f^x_b + p_c f^x f_b,\ {}^*D_c f^x_b = D_c f^x_b + g_{cb} f^x_d p^d - p_b f^x_c,$$

$$D_c f^a = \nabla_c f^a_y + p_c f^a f_y,\ {}^*D_c f^a_y = D_c f^a_y + f^d_y \delta^a_c p_d - p^a f_{yc},$$

$$D_c f^a = \nabla_c f^a,\ {}^*D_c f^a = \nabla_c f^a + p_d f^d\ \delta^a_c - f_c p^a.$$

Hence we have

Theorem (3.2). <u>For an anti-invariant submanifold of a Sasakian manifold the two covariant derivatives D and *D satisfy (3.6)</u>.

Since for a generic submanifold of a Sasakian manifold $f_y^x=0$, therefore (1.10) gives [10]

$$(3.7)\quad f_c^x f_b^c=-f_b f^x,\ f_c^a f_y^c=f_y f^a,\ f_c^x f^c=0,\ f_c^x f_y^c=\delta_y^x-f_y f^x.$$

Using (3.7) in (3.1) and (3.2) we obtain

$$(3.8)\quad D_c f_b^x=\nabla_c f_b^x-p_c f_b f^x,\ {*D_c} f_b^x=D_c f_b^x-p_b f_c^x+g_{cb} f_d^x p^d,$$

$$D_c f_y^a=\nabla_c f_y^a-p_c f_y f^a,\ {*D_c} f_y^a=D_c f_y^a+f_y^d p_d\ \delta_c^a-f_{yc} p^a,\ {*D_c} f^x=D_c f^x=\nabla_c f^x.$$

Hence we have

Theorem (3.3). <u>For a generic submanifold of a Sasakian manifold the two covariant derivatives satisfy (3.8)</u>.

Since for a contact CR-submanifold $f^x=0, f_a^x f_b^a=0$, therefore (1.10) gives [10]

$$(3.9)\quad f_z^x f_b^z=0,\ f_z^a f_y^z=0,\ f_c^x f_y^c-f_z^x f_y^z=\delta_y^x,\ f_c^a f^c=0,\ f_c^x f^c=0.$$

Using (3.9) in (3.1) and (3.2) we obtain

$$(3.10)\quad D_c f_b^x=\nabla_c f_b^x,\ {*D_c} f_b^x=\nabla_c f_b^x-p_b f_c^x+g_{cb} f_d^x p^d,\ D_c f_y^a=\nabla_c f_y^a,$$

$${*D_c} f_y^a=\nabla_c f_y^a+f_y^d p_d\ \delta_c^a-f_{yc} p^a,\ {*D_c} f^a=\nabla_c f^a+p_d f^d \delta_c^a-f_c p^a,$$

$$D_c f^a=\nabla_c f^a.$$

Hence we have

Theorem (3.4). <u>For a contact CR-submanifold of a Sasakian manifold two types of covariant derivatives satisfy (3.10)</u>.

4. Generalised Gauss, Weingarten Equtions

Differentiating B_b^h and C_x^h covariantly with respect to connection D, we obtain

$$(4.1)\quad D_c B_b^h = B_{cb}^h + \Gamma_{ji}{}^h B_{cb}^{ji} - \Gamma_{cb}{}^a B_a^h$$

and

$$(4.2)\quad D_c C_x{}^h = {}_c C_x{}^h + \Gamma_{ji}{}^h B_c^j C_x{}^i - \Gamma_{cx}{}^y C_y{}^h.$$

Substituting in (4.1) and (4.2) from (2.1), (2.11) and (2.13), we obtain on simplification by virtue of (1.5), (1.6), (1.7) and (1.8)

$$(4.3)\quad D_c B_b^h = {}'h_{cb}^{\ \ x}\, C_x^{\ h}$$

and

$$(4.4)\quad D_c C_x^{\ h} = -{}'h_{cx}^{a}\, B_a^h,$$

where

$$(4.5)\quad {}'h_{cb}^{\ \ x} \overset{\text{def.}}{\equiv} h_{cb}^{\ x} + p_c f_b^x,\quad {}'h_{cx}^{a} \overset{\text{def.}}{\equiv} h_{cx}^{a} + p_c f_x^a.$$

Equations (4.3) and (4.4) are the generalised Gauss and Weingarten equations respectively.

If in (4.5) either $p_c = 0$ or $f_b^x = 0$, we get ${}'h_{cb}^{\ x} = h_{cb}^{\ x}$, ${}'h_{cx}^{a} = h_{cx}^{a}$, therefore (4.3) and (4.4) reduce to (1.5) and (1.6) respectively. Hence we have

Theorem (4.1). <u>If either p^h is a vector normal to M^n or M^n is an invariant submanifold of a Sasakian manifold M^{2m+1}, the generalised Gauss and Weingarten equations reduce to Gauss and Weingarten equations (1.5) and (1.6) respectively.</u>

If the vector p^h is identical to f^h and f^h is a vector normal to M^n, we must have Yano and Kon [7] that M^n is a totally geodesic submanifold of a Sasakian manifold M^{2m+1}, therefore equation (4.5) gives ${}'h_{cb}^{\ x} = 0$ and ${}'h_{cx}^{a} = 0$. Hence we have

Theorem (4.2). <u>If the vector p^h is identically equal to f^h and f^h is a vector normal to M^n, the submanifold M^n of a Sasakian manifold has vanishing fundamental tensors ${}'h_{cb}^{\ x}$ and ${}'h_{cx}^{a}$.</u>

Similar to (4.3) and (4.4) corresponding to the connection $*D_c$ we can obtain

$$(4.6)\quad *D_c B_b^h = *h_{cb}^{\ \ x}\, C_x^{\ h}$$

and

$$(4.7)\quad *D_c C_x^{\ h} = -*h_{cx}^{a}\, B_a^h,$$

where

$$(4.8)\quad *h_{cb}^{\ \ x} \overset{\text{def.}}{\equiv} {}'h_{cb}^{\ x} - g_{cb}p^x,\quad *h_{cx}^{a} \overset{\text{def.}}{\equiv} {}'h_{cx}^{\ a} - \delta_c^a p_x.$$

From (4.8) we can see that if $p_x=0$, $*h_{cb}^{\ \ x}='h_{cb}^{\ \ x}$, $*h_{cx}^{a}='h_{cx}^{a}$ and conversely. Hence we have

Theorem (4.3). A necessary and sufficient condition for the generalised Gauss and Weingarten equations based on (2.1) and (2.2) to be identical is that p^h be a vector tangential to M^n.

If p^h is a vector normal to M^n or M^n is an invariant submanifold of M^{2m+1}, equations (4.8) reduce to

(4.9) $*h_{cb}^{\ \ x} = h_{cb}^{\ \ x}-g_{cb}p^x$, $*h_{cx}^{a} = h_{cx}^{a} - \delta_c^a p_x$.

Now if $*h_{cb}^{\ \ x} = 0$, $h_{cb}^{\ \ x} = g_{cb}p^x$, i.e., M^n is a totally umbilical submanifold. Conversely, if M^n is an umbilical submanifold satisfying $h_{cb}^{\ \ x} = g_{cb}p^x$, equation (4.9) gives $*h_{cb}^{\ \ x} = 0$. Hence we have

Theorem (4.4) If either p^h is a vector normal to M^n or M^n is an invariant submanifold of a Sasakian manifold M^{2m+1}, the necessary and sufficient condition for M^n to be an umbilical submanifold of a Sasakian manifold with the vector p^x is that $*h_{cb}^{\ \ x} = 0$.

Further if M^n satisfies $h_{cb}^{\ \ x}=0$, equation (4.9) gives $*h_{cb}^{\ \ x}=-g_{cb}p^x$ and conversly. Hence we have

Theorem (4.5). If either p^h is a vector normal to M^n or M^n is an invariant submanifold of a Sasakian manifold M^{2m+1}, the necessary and sufficient condition for M^n to be a totally geodesic submanifold of M^{2m+1} is that $*h_{cb}^{\ \ x}=-g_{cb}p^x$.

Remark: If $p^h=f^h$ and f^h is a vector normal to M^n, $h_{cb}^{\ \ x}=0$ and the condition $*h_{cb}^{\ \ x}=-g_{cb}p^x$ is identically satisfied.

5. Curvature Tensors

Let $R_{kji}^{\ \ \ h}$ and $*R_{kji}^{\ \ \ h}$ be the curvature tensors based on (2.1) and (2.2) then they are expressible as

(5.1) $R_{kji}^{\ \ \ h} = K_{khi}^{\ \ \ h} - (\nabla_k p_j - \nabla_j p_k)p_i^{\ h} + f^h(g_{ki}p_j - g_{ji}p_k) - (\delta_k^h p_j - \delta_j^h p_k)f_i$

and

(5.2) $*R_{kji}^{\ \ \ h} = R_{kji}^{\ \ \ h} - [\delta_k^h p_{ji} + p_k^h g_{ji} + p_j p^h F_{ki}$

$\qquad + \delta_k^h p_j p_t F_i^{\ t} + p_k p_i F_j^{\ h} + p_j p^t g_{ki} F_t^{\ h} - k|j]$,

where

$$p_{ji} = \nabla_j p_i - p_j p_i + \tfrac{1}{2} p_t p^t g_{ji}.$$

Let $R_{cba}{}^{d}$, $*R_{cba}{}^{d}$ and $R_{cbx}{}^{y}$ be the curvature tensors based on (2.11), (2.12) and (2.13) then we can define

$$(5.3)\quad R_{cba}{}^{d} = K_{cba}{}^{d} + f_a^d\,(\nabla_b p_c - \nabla_c p_b) - p_b\,\nabla_c f_a^d + p_c\,\nabla_b f_a^d,$$

$$(5.4)\quad *R_{cba}{}^{d} = R_{cba}{}^{d} - [\delta_c^d\, p_{ba} + p_c^d\, g_{ba} + p_b p^d f_{ca} + \delta_c^d\, p_b p_e f_a^e + p_c p_a f_b^d + p_h p^e f_e^d g_{ca} - c|b]$$

and

$$(5.5)\quad R_{cbx}{}^{y} = K_{cbx}{}^{y} + f_x^y\,(\nabla_b p_c - \nabla_c p_b) + p_c\,\nabla_b f_x^y - p_b\,\nabla_c f_x^y,$$

where

$$p_{ba} = \nabla_b p_a - p_b p_a + \tfrac{1}{2} p_c p^c g_{ba}.$$

If p^h is a vector normal to M^n, $p_c = 0$, therefore from (5.3), (5.4) and (5.5) we can obtain

$$(5.6)\quad *R_{cba}{}^{d} = R_{cba}{}^{d} = K_{cba}{}^{d},\quad R_{cbx}{}^{y} = K_{cbx}{}^{y}.$$

Hence we have

Theorem (5.1). <u>For a vector p^h normal to M^n, the curvature tensors of a submanifold M^n of a Sasakian manifold M^{2m+1} based on quarter-symmetric metric connections satisfy (5.6).</u>

For an anti-invariant submanifold M^n of a Sasakian manifold (5.3) and (5.4) give

$$(5.7)\quad R_{cba}{}^{d} = K_{cba}{}^{d}$$

and

$$(5.8)\quad *R_{cba}{}^{d} = K_{cba}{}^{d} - (\delta_c^d p_{ba} + p_c^d g_{ba} - c|b).$$

For a generic submanifold M^n of M^{2m+1}, (5.5) gives

$$(5.9)\quad R_{cbx}{}^{y} = K_{cbx}{}^{y}.$$

Hence we have

Theorem (5.2). <u>For an anti-invariant submanifold of a Sasakian manifold the curvature tensors satisfy (5.7) and (5.8), whereas for a generic submanifold of a Sasakian manifold the curvature tensor satisfies (5.9).</u>

Again multiplying (6.1) by $g_{hl} C_y^{\ l}$ and using (4.3), (4.4.) and

(6.4) $g_{hl} C_x^{\ l} D_c ('h_{ba}^{\ \ y} C_y^{\ h}) = D_c 'h_{bax}$, in (6.1) we obtain

(6.5) $(D_c 'h_{bax} - D_b 'h_{cax}) + 'h_{dax} S_{bc}^{\ \ d} = R_{kjih} B^{kji}_{cba} C_x^{\ h}$.

Equation (6.3) is the generalised Gauss curvature equation and (6.5) is the generalised Mainardi-Codazzi equation based on quarter-symmetric metric connection (2.1).

Similar to (6.1) from (4.2) we can obtain

(6.6) $D_c D_b C_x^{\ h} - D_b D_c C_x^{\ h} = R_{kji}^{\ \ \ h} B^{kj}_{cb} C_x^{\ i} - R_{cbx}^{\ \ \ y} C_y^{\ h} - S_{bc}^{\ \ a} D_a C_x^{\ h}$.

Also from (4.3) and (4.4) we can obtain

(6.7) $g_{hl} B^l_d (D_c D_b C_x^{\ h}) = - D_c 'h_{bdx}$

and

(6.8) $g_{hl} C_z^{\ l} (D_c D_b C_x^{\ h}) = - 'h^a_{bx} 'h_{caz}$.

Multiplying (6.6) by $g_{hl} B^l_d$ and using (6.7) we again obtain (6.5), whereas if we multiply (6.6) by $g_{hl} C_z^{\ l}$ and use (6.8) we obtain

(6.9) $R_{cbxz} = R_{kjil} B^{kj}_{cb} C_x^{\ i} C_z^{\ l} + 'h^a_{bx} 'h_{caz} - 'h^a_{cx} 'h_{baz}$,

which is the generalised Ricci equation based on (2.11) and (2.13).

If p^h is a vector normal to M^n, equations (6.3), (6.5) and (6.9) respectively reduce to

(6.10) $K_{cbad} = R_{kjih} B^{kjih}_{cbad} - (h_{bdx} h_{ca}^{\ \ x} - h_{cdx} h_{ba}^{\ \ x})$,

(6.11) $D_c h_{bax} - D_b h_{cax}) = R_{kjih} B^{kji}_{cba} C_x^{\ h}$

and

(6.12) $K_{cbxz} = R_{kjih} B^{kj}_{cb} C_x^{\ i} C_z^{\ h} + h^a_{bx} h_{caz} - h^a_{cx} h_{baz}$.

Similar to (6.3), (6.5) and (6.9), from (2.2), (2.12) and (2.13) we can obtain the following equations

(6.13) $*R_{cbad} = *R_{kjih} B^{kjih}_{cbad} - (*h_{bdx} *h_{ca}^{\ \ x} - *h_{cdx} *h_{ba}^{\ \ x})$,

(6.14) $(*D_c *h_{bax} - *D_b *h_{cax}) + *h_{dax} *S_{bc}^{\ \ d} = *R_{kjih} B^{kji}_{cba} C_x^{\ h}$

and

(6.15) $R_{cbxz} \doteq {}^*R_{kjih}\, B^{kj}_{cb}\, C_x{}^i\, C_z{}^h + {}^*h^a_{bx}\, {}^*h_{caz} - {}^*h^a_{cx}\, {}^*h_{baz}$,

where

$${}^*S_{bc}{}^d = {}^*\Gamma_{bc}{}^d - {}^*\Gamma_{cb}{}^d.$$

If the vector p^h is normal to M^n, equations (6.13), (6.14) and (6.15) reduce to

(6.16) $K_{cbad} \doteq {}^*R_{kjih}\, B^{kjih}_{cbad} - (h_{bdx}\, h_{ca}{}^x - h_{cdx}\, h_{ba}{}^x)$,

(6.17) ${}^*D_c\,(h_{bax} - g_{ba}\, p_x) - {}^*D_b\,(h_{cax} - g_{ca}\, p_x) \doteq {}^*R_{kjih}\, B^{kji}_{cba}\, C_x{}^h$

and

(6.18) $K_{cbxz} \doteq {}^*R_{kjih}\, B^{kj}_{cb}\, C_x{}^i\, C_z{}^h + h^a_{bx}\, h_{caz} - h^a_{cx}\, h_{baz}$.

From (6.10) and (6.16) we obtain

(6.19) $({}^*R_{kjih} - R_{kjih})\, B^{kjih}_{cbad} \doteq 0$, while from (6.12) and (6.18) we obtain

(6.20) $({}^*R_{kjih} - R_{kjih})\, B^{kj}_{cb}\, C_x{}^i\, C_z{}^h \doteq 0$.

If $p^h \doteq f^h$ and f^h is a vector normal to M^n, equations (6.10), (6.11) and (6.12) reduce respectively to

(6.21) $K_{cbad} \doteq R_{kjih}\, B^{kjih}_{cbad}$,

(6.22) $R_{kjih}\, B^{kji}_{cba}\, C_x{}^h \doteq 0$

and

(6.23) $K_{cbxz} \doteq R_{kjih}\, B^{kj}_{cb}\, C_x{}^i\, C_z{}^h$, while (6.16), (6.17) and (6.18) reduce to

(6.24) $K_{cbad} \doteq {}^*R_{kjih}\, B^{kjih}_{cbad}$,

(6.25) $g_{ca}\, {}^*D_b p_x - g_{ba}\, {}^*D_c p_x \doteq {}^*R_{kjih}\, B^{kji}_{cba}\, C_x{}^h$

and

(6.26) $K_{cbxz} \doteq {}^*R_{kjih}\, B^{kj}_{cb}\, C_x{}^i\, C_z{}^h$.

7. Some Applications

Multiplying equation (6.3) by $X^c X^a Y^b Y^d$ and using $X^i \doteq B^i_a\, X^a$, $Y^i \doteq B^i_a\, Y^a$, $({}'\rho_x)^{-1} \doteq {}'h_{ca}{}^x\, X^c X^a$ and $({}'\tau_x)^{-1} \doteq {}'h_{bd}{}^x\, Y^b Y^d$, we get on simplification

$$(7.1)\quad R_{cbad}\, X^c X^a Y^b Y^d = R_{kjih}\, X^k X^i Y^j Y^h$$
$$-[('\tau_x)^{-1}\, ('\rho_x)^{-1} - 'h_{cdx}\, 'h_{ba}^{\ x}\, X^c X^a Y^b Y^d].$$

Now dividing (7.1) by the respective sides of

$$(g_{ba}\, g_{cd} - g_{ca} g_{bd})\, X^c X^a Y^b Y^d = (g_{ji} g_{kh} - g_{ki} g_{jh})\, X^k X^i Y^j Y^h$$
$$= (1-(g_{ba} X^b Y^a)^2),$$

we get on simplification

$$(7.2)\quad r(y) = r(x) - [('\tau_x)^{-1}\, ('\rho_x)^{-1} - 'h_{cdx}\, 'h_{ba}^{\ x}\, X^c X^a Y^b Y^d].$$
$$.(1-(g_{ba} X^b Y^a)^2)^{-1},$$

where

$$(7.3)\quad r(x) \overset{\text{def.}}{=\!=\!=} (R_{kjih}\, X^k X^i Y^j Y^h).\, [(g_{ji} g_{kh} - g_{ki} g_{jh})\, X^k X^i Y^j Y^h]^{-1}$$

and

$$(7.4)\quad r(y) \overset{\text{def.}}{=\!=\!=} (R_{cbad}\, X^c X^a Y^b Y^d).\, [(g_{ba} g_{cd} - g_{ca} g_{bd})\, X^c X^a Y^b Y^d]^{-1}.$$

If p^h is a vector normal to M^n, equation (7.2) reduces to

$$(7.5)\quad k(y) = r(x) - (1-(g_{ba} X^b Y^a)^2)^{-1}.\, [(\tau_x)^{-1}\, (\rho_x)^{-1} - h_{cdx} h_{ba}^{\ x}\, .$$
$$.X^c X^a Y^b Y^d].$$

If $p^h = f^h$ and it is also normal to M^n, equation (7.5) reduces to

$$(7.6)\quad k(y) = r(x) = r(y).$$

Hence we have

Theorem (7.1). <u>If $p^h = f^h$ and it is a vector normal to M^n, the scalars $r(x)$ and $r(y)$ defined by (7.3) and (7.4) are identically equal to $k(y)$.</u>

If R_{kjih} is independent of directions X and Y equation (6.3) gives

$$(7.7)\quad R_{cbad} = r(x)\ (g_{ba} g_{cd} - g_{ca} g_{bd}) - ('h_{bdx}\, 'h_{ca}^{\ x} - 'h_{cdx}\, 'h_{ba}^{\ x}).$$

If $\breve{R}_{cbad}$ is also independent of directions equation (7.4) gives

$$(7.8)\quad \breve{R}_{cbad} = r(y)\ (g_{ba} g_{cd} - g_{ca} g_{bd}).$$

Thus if both R_{kjih} and R_{cbad} are independent of directions, from (7.7) and (7.8) we must have

$$(7.9)\quad (r(x) - r(y))\ (g_{ba} g_{cd} - g_{ca} g_{bd}) = ('h_{bdx}\, 'h_{ca}^{\ x} - 'h_{cdx}\, 'h_{ba}^{\ x}).$$

Hence we have

Theorem (7.2). A necessary and sufficient condition for the curvature tensors R_{kjih} and R_{cbad} to be independent of directions is given by (7.9).

Remark. If $p^h = f^h$ and f^h is normal to M^n, equation (7.9) is satisfied identically.

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Structure Defined by a Tensor Field F of Type (1,1) Satisfying $F^K - F = 0$

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Summary

Yano [3] has given the necessary and sufficient condition for an n-dimensional manifold to admit a tensor field $f \neq o$ of type (1,1) and of rank k such that $f^3 + f = o$, where k is even. Pokhariyal [2] has studied the structure defined on an n-dimensional manifold by a tensor field f of type (1,1) satisfying $f^5 - f = o$. In the present paper, we have considered the structure defined by a tensor field $F \neq o$ of type (1,1) satisfying $F^K - F = o$, where K is a positive integer ≥ 2. Here, we have defined the operators s, t as well as some tensors and established various results.

1. The operators s and t

Let us consider an n-dimensional differentiable manifold M^n of class C^∞ equipped with a non-null tensor field F of type (1,1) and of class C^∞ satisfying

$$(1.1) \qquad F^K - F = o ,$$

where K is a positive integer ≥ 2.

Let us put

$$(1.2) \qquad s \overset{\text{def}}{=} F^{K-1} , \quad t \overset{\text{def}}{=} I - F^{K-1},$$

I denoting the unit tensor field. Then we have

Theorem (1.1). For a tensor field $F \neq o$ satisfying (1.1) and the operators s, t defined by (1.2) and applied to the tangent space at each point of the manifold are complementary projection operators.

Proof. In consequence of (1.1) and (1.2), we have

$$(1.3) \qquad s + t = I ,$$

$$(1.4) \qquad s^2 = F^{2K-2} = F^K . F^{K-2}$$
$$= F.F^{K-2} = F^{K-1} = s,$$

$$(1.5) \qquad t^2 \equiv I + F^{2K-2} - 2F^{K-1}$$
$$\equiv I + F^{K-1} - 2F^{K-1} \equiv t\ ;$$

$$(1.6) \qquad s.t \equiv t.s \equiv F^{K-1} - F^{2K-2} \equiv o.$$

This proves the theorem.

Thus, if there is given a tensor field $F \not\equiv o$ of type (1,1) which satisfies (1.1); then there exist two complementary distributions S and T corresponding to the projection operators s and t respectively. Let the rank of F be constant and be equal to r everywhere, then the dimensions of S and T are r and n-r respectively. We call such a structure a 'F(K,-1) - structure of rank r', and the manifold M^n with this structure a 'F(K,-1) - manifold.'

If the rank of F is maximal, then $r \equiv n$. Thus $s \equiv I$ and $t \equiv o$. Hence F satisfies $F^{K-1} - I = o$. Consequently, $F^{1/2(K-1)}$ defines on M^n an almost product structure.

Theorem (1.2). For a tensor field $F \neq o$ satisfying (1.1) and the operators s, t defined by (1.2), we have

$$(1.7) \qquad Fs \equiv sF \equiv F\ , \quad Ft \equiv tF \equiv o\ ;$$

$$(1.8) \qquad F^2 s \equiv F^2\ , \quad F^2 t \equiv o\ .$$

Proof. The proof follows by virtue of the equations (1.1) and (1.2).

Theorem (1.3). For F satisfying (1.1) and s,t defined by (1.2), we have

$$(1.9) \qquad F^{K-2}\, s \equiv sF^{K-2} \equiv F^{K-2},\ F^{K-2} t \equiv tF^{K-2} \equiv o\ ;$$

$$(1.10) \qquad F^{K-1}\, s \equiv s \qquad ,\ F^{K-1}\, t \equiv o\ ;$$

that is, $F^{\frac{1}{2}(K-1)}$ acts on S as an almost product structure operator and on T as a null operator.

Proof. In consequence of (1.1) and (1.2), we have

$$F^{K-2} s \equiv F^{K-2}.F^{K-1} \equiv F^K.F^{K-3}$$
$$\equiv F.F.^{K-3} \equiv F^{K-2} \equiv sF^{K-2},$$

$$F^{K-1}s = F(F^{K-2}s) = F.F^{K-2} = F^{K-1} = s\ ,$$

$$F^{K-2}t = F^{K-2}(I-F^{K-1}) = F^{K-2} - F^{2K-3}$$

$$= F^{K-2} - F^{K}.F^{K-3} = F^{K-2} - F.F^{K-3}$$

$$= o = tF^{K-2},$$

$$F^{K-1}t = F(F^{K-2}t) = o\ .$$

Thus the theorem follows.

Theorem (1.4). For F satisfying (1.1) and s defined by (1.2), we have

$$(1.11)\qquad \{s + F^{\frac{1}{2}(K-1)}\}\{s - F^{\frac{1}{2}(K-1)}\} = o,\quad F^{K-1}s = F^{K-1}.$$

Proof. The proof follows by virtue of the equations (1.1) and (1.2).

Theorem (1.5). Suppose that there is given on M^n, a projection operator s and there exists a tensor field F of type (1,1) such that (1.11) is satisfied; then F satisfies (1.1).

Proof. From the first equation of (1.11), we have

$$s^2 - F^{K-1} = o\ ,$$

which in view of (1.4) gives

$$s - F^{K-1} = o.$$

Applying F to the above equation and using $Fs = F$, we obtain

$$F^K - F = o.$$

This proves the theorem.

2. Main Results

In this section, we shall define the tensors α,β; μ,υ; x,y ; p,q; u,v and establish various results in terms of these tensors and the operators s,t.

Theorem (2.1). For the tensors α and β defined by

$$(2.1)\qquad \alpha = t + F^{K-1}\ ,\quad \beta = t - F^{K-1}\ ,$$

We have

(2.2) $\alpha\beta = \beta,\ \ \alpha^2 = \alpha = \beta^2\ ,\ \ \ \alpha^3 = \alpha,\ \ \beta^3 = \beta\ ;$

(2.3) $\alpha t = t = \beta t\ ,$

$\alpha s = s = -\ \beta s\ .$

Proof. The proof follows easily by virtue of the equations (1.4), (1.5), (1.6), (1.10) and (2.1).

Theorem (2.2). Let μ and ν be tensors defined by

(2.4) $\mu = t + F^{K-2}\ ,\ \nu = t - F^{K-2};$

Then we have

(2.5) $\mu^2 = \nu^2,\ \ \mu^3 + \nu^3 = \mu + \nu\ ,$

(2.6) $\mu^2 + \mu\nu - \mu = \nu\ ,\ \ \nu^2 + \mu\nu - \nu = \mu\ ,$

(2.7) $\mu^4 + \mu^3\nu = \mu + \nu\ ,\ \ \nu^4 + \mu\nu^3 = \mu + \nu\ .$

Proof. In view of (1.1), (1.5), (1.7), (1.9) and (2.4), we have

$$\mu + \nu = 2t\ ,$$

$$\mu\nu = t^2 - F^{2K-4} = t - F^K.F^{K-4} = t - F^{K-3}\ ,$$

$$\mu^2 = t^2 + F^{2K-4} = t + F^K.F^{K-4} = t + F^{K-3}\ ,$$

$$\nu^2 = t^2 + F^{2K-4} = t + F^K.F^{K-4} = t + F^{K-3}\ ,$$

$$\mu^3 = (t+F^{K-2})\ (t+F^{K-3}) = t^2 + F^{2K-5}$$

$$= t + F^K.F^{K-5} = t + F^{K-4}\ ,$$

$$\nu^3 = (t-F^{K-2})\ (t+F^{K-3}) = t^2 - F^{2K-5}$$

$$= t - F^{K}.^{K-5} = t - F^{K-4}\ ,$$

$$\mu^4 = (t+F^{K-3})^2 = t^2 + F^{2K-6} = t + F^K.F^{K-6}$$

$$= t + F^{K-5} = \nu^4\ .$$

Therefore,

$$\mu^2 = \nu^2 \text{ and } \mu^3 + \nu^3 = 2t = \mu + \nu.$$

Also, μ^2

Similarly, ν^2

Further, μ^4

Similarly, i

ν^4

Theorem (2.3)

(2.8) $x =$

we have

(2.9) x^2

(2.10) x^2

Proof. In co

we have

$x -$

$xy = s^2$

$= s$

$x^2 = s^2 +$

$= s +$

Also,

$$\mu^2 + \mu\nu - \mu \equiv (t+F^{K-3}) + (t-F^{K-3}) - (t+F^{K-2})$$
$$\equiv t-F^{K-2} \equiv \nu \ .$$

Similarly,

$$\nu^2 + \mu\nu - \nu \equiv \mu \ .$$

Further,

$$\begin{aligned}\mu^4 + \mu^3\nu &\equiv t + F^{K-5} + (t+F^{K-4})\ (t-F^{K-2}) \\ &\equiv t + F^{K-5} + t^2 - F^{2K-6} \\ &\equiv t + F^{K-5} + t \ - F^K.F^{K-6} \\ &\equiv 2t + F^{K-5} - F.F^{K-6} \\ &\equiv 2t \equiv \mu + \nu \ .\end{aligned}$$

Similarly, it can be shown that

$$\nu^4 + \mu\nu^3 \equiv \mu + \nu \ .$$

Theorem (2.3). For the tensors x and y defined by

(2.8) $\quad x \equiv s + F^{K-2}\ ,\ y \equiv s - F^{K-2}\ ,$

we have

(2.9) $\quad x^2 + xy \equiv 2x\ ,\ y^2 + xy \equiv 2y\ ;$

(2.10) $\quad x^2 - y^2 \equiv 2(x-y).$

Proof. In consequence of (1.1), (1.4), (1.9) and (2.8), we have

$$x - y \equiv 2F^{K-2}\ ,$$
$$\begin{aligned}xy &\equiv s^2 - F^{2K-4} \equiv s - F^K F^{K-4} \\ &\equiv s \ - F.F^{K-4} \equiv s - F^{K-3}\ ,\end{aligned}$$
$$\begin{aligned}x^2 &\equiv s^2 + F^{2K-4} + 2sF^{K-2} \equiv s + F^K.F^{K-4} + 2F^{K-2} \\ &\equiv s + F^{K-3} + 2F^{K-2},\end{aligned}$$

$$y^2 = s^2 + F^{2K-4} - 2sF^{K-2} = s + F^K.F^{K-4} - 2F^{K-2}$$
$$= s + F^{K-3} - 2F^{K-2}.$$

Therefore,

$$x^2 + xy = 2(s + F^{K-2}) = 2x\ ,$$
$$y^2 + xy = 2(s - F^{K-2}) = 2y\ ,$$
$$x^2 - y^2 = 4F^{K-2} = 2(x-y).$$

Hence the result.

Theorem (2.4). If the tensors p and q are defined by

(2.11) $\quad p = t + F\ ,\quad q = t - F\ ;$

then we have

(2.12) $\quad ps = F\ ,\quad pt = t\ ,\quad qs = -F\ ,\quad qt = t\ ,$

(2.13) $\quad p^{K-1}\,s = s\ ,\ p^{K-1}\,t = t\ ,\ q^{K-1}\,s = (-1)^{K-1}\,s,\ q^{K-1}\,t = t\ .$

Proof. In view of (1.2), (1.5), (1.6), (1.7), (1.9) and (2.11), we obtain

$$ps = (t + F)s = ts + Fs = F\ ,$$
$$p^2 s = p(ps) = (t + F)F = F^2\ ,$$

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$$p^{K-1}s = p(p^{K-2}s) = (t + F)F^{K-2} = F^{K-1} = s\ .$$

Also,

$$pt = (t + F)t = t^2 + Ft = t\ ,$$
$$p^2 t = p(pt) = p(t) = t\ ,$$

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- - - - - - - - - -

$$p^{K-1}\,t = p(p^{K-2}\,t) = p(t) = t\ .$$

The remaining part of the theorem can be proved in a similar manner.

Theorem (2.5). For the tensors u and v defined by

(2.14) $u \equiv s + F\,, \quad v \equiv s - F\,,$

we have

(2.15) $us \equiv u,\ ut \equiv o,\ vs \equiv v\ ,\ vt \equiv o\ ;$

(2.16) $u^{K-1}s \equiv u^{K-1}\ ,\ u^{K-1}t \equiv o\ ,\ v^{K-1}s \equiv v^{K-1}\ ,\ v^{K-1}\ t \equiv o.$

Proof. The proof follows easily by virtue of the equations (1.4), (1.6). (1.7) and (2.14).

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