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On Univalence of Certain Analytic Functions Associated With Starlike Functions--1

M.I. Rizvi*

1. Introduction

Let S be the class of functions $f(Z)=Z+\sum_{n=2}^{\infty} a_n Z^n$ which are regular and univalent in the unit disc D $\{|Z| < 1\}$, where S* denotes the class of functions in S, which maps D onto a starlike region with respect to the origin. An equivalent analytic characterization for functions of S* is well known [1]. S_β^* denotes the class of functions f(Z) in S* having the additional property:

$$\text{Re}\left\{\frac{Zf'(Z)}{fZ}\right\} \geq \beta, \quad Z\varepsilon D; \; 0 \leq \beta \leq 1 \tag{1.1}$$

Here β is referred as the order of Starlike functions f(Z) and we identify $S_o^* \equiv S^*$

In this paper we are mainly concerned with the radius of starlikeness of the function F(Z). Incidently the results of Padmanabhan [2], Bajpai and Srivastava [3], Bernadi [6] and Libra [7] follow from ours.

2. Theorem 1: If $f(Z) = Z + \sum_{n=2}^{\infty} a_n Z^n \;\varepsilon\; S_\beta^*$ and

$$f(Z) = \frac{(C+\alpha)}{Z^{C-\alpha+1}} \int_o^Z t^{C-\alpha} [f(t)]^\alpha \, dt \, , \tag{2.1}$$

then, $\quad F(Z) \;\varepsilon\; S_{\alpha\beta}^*$.

<u>Proof</u>: Let $J(Z) = \int_o^Z t^{C-\alpha} [f(t)]^\alpha \, dt$

then, $\quad J'(Z)\, Z^{-C+\alpha} = [F(Z)]^\alpha$

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and

$$(2.2) \qquad F(Z) = (C+\alpha)\, Z^{-C+\alpha-1}\, J(Z).$$

Now,

$$F'(Z) = (C+\alpha)\, [(-C+\alpha-1)\, Z^{-C+\alpha-2}\, J(Z) + Z^{-C+\alpha-1}\, J'(Z)].$$

Hence,

$$Z^{C-\alpha+2}\, F'(Z) = (C+\alpha)[(-C+\alpha-1)\, J(Z) + ZJ'(Z)].$$

$$= (C+\alpha)\, [(-C+\alpha-1)J(Z) + Z^{C-\alpha+1}(f(Z))^{\alpha}]$$

Now,

$$[Z^{C-\alpha+2}\, F'(Z)]' = (C+\alpha)[(-C+\alpha-1)J'(Z) + (C-\alpha+1)Z^{C-\alpha}\{f(Z)\}^{\alpha} + \alpha Z^{C-\alpha+1}\{f(Z)\}^{\alpha-1}\, f'(Z)]$$

$$= (C+\alpha)\ \alpha\ Z^{C-\alpha+1}\, \{f(Z)\}^{\alpha-1}\, f'(Z)$$

Hence,

$$\frac{[Z^{C-\alpha+2}F'(Z)]'}{J'(Z)} = (C+\alpha)\alpha\ \frac{Zf'(Z)}{f(Z)}.$$

Therefore using (1.1), we get

$$(2.3) \qquad \mathrm{Re}\, \left\{\frac{(Z^{C-\alpha+2}\, F'(Z))'}{J'(Z)}\right\} = (C+\alpha)\alpha\, \left\{\mathrm{Re}\ \frac{Zf'(Z)}{f(Z)}\right\}$$

$$\geq (C+\alpha)\alpha\beta.$$

Therefore using ([4], Lemma, p. 430) in (2.3), we get

$$(2.4) \qquad \mathrm{Re}\, \left\{\frac{Z^{C-\alpha+2}\, F'(Z)}{J(Z)}\right\} \geq (C+\alpha)\alpha\beta$$

or

$$\mathrm{Re}\, \left\{\frac{Z^{C-\alpha+2}\, F'(Z)}{F(Z)Z^{C-\alpha+1}/(C+\alpha)}\right\} \geq (C+\alpha)\alpha\beta$$

or

$$(2.5) \qquad \mathrm{Re}\, \left\{\frac{ZF'(Z)}{F(Z)}\right\} > \alpha\beta$$

Using (1.1), (2.5) gives $F(Z) \in S^*_{\alpha\beta}$.

Corollary 1. Theorem C of Bernadi [4] follows from the above theorem by taking $\alpha = 1$ and $C = 1, 2, 3$.

Corollary 2. Theorem A of Libra [7] follows from the above theorem by taking $\alpha = 1$, $\beta = 0$ and $C = 1$.

Theorem 2 : If $f(Z) = Z + \sum_{n=2}^{\infty} a_n Z^n$ and

$$F(Z) = \frac{(C+\alpha)}{Z^{C-\alpha+1}} \int_0^Z t^{C-\alpha} [f(t)]^\alpha \, dt \in S^*_{\alpha\beta}$$

where $0 \le \alpha \le 1$ and $0 \le \beta \le 1$

Then,

$f(Z) \in S^*_\beta$ in the region,

$$Z < r_o = \frac{-(2-\alpha\beta)+\{3+\alpha^2\beta^2+(C-\alpha+1)^2+2(C-\alpha+1)\alpha\beta-2\alpha\beta\}^{1/2}}{(C - \alpha + 2\alpha\beta)}, \text{ if } C=2, 3, \ldots$$

$$= \frac{1}{2} \qquad \text{if } C=1 \text{ and } \beta=o$$

$$= - \frac{(2-\alpha\beta)+(4-\alpha^2\beta^2)^{1/2}}{2\alpha\beta} \qquad \text{if } C=1 \text{ and } o<\beta<1$$

Proof: By the hypothesis of the above theorem, we have

$$(3.1) \qquad \frac{ZF'(Z)}{F(Z)} = \frac{ZJ'(Z) - (C-\alpha+1)J(Z)}{J(Z)}$$

Further since $F(Z)$ is a starlike function of order $\alpha\beta$, so there exist a function $w(Z)$, which is regular in the unit disc and satisfies the conditions of Schwartz Lemma, such that

$$(3.2) \qquad \frac{ZF'(Z)}{F(Z)} = \frac{1-(1-2\alpha\beta)w(Z)}{1+w(Z)}$$

From (3. 1) and (3.2), it follows that

$$(3.3) \qquad \{f(Z)\}^\alpha = \frac{[(C-\alpha+2)+\{(C-\alpha+1)+(2\alpha\beta-1)\}\, w(Z)]J(Z)}{[1+w(Z)]Z^{C-\alpha+1}}$$

Differentiating eq. (3.3) logarithmically and simplifying, finally we get

$$(3.4)\qquad \frac{\alpha Zf'(Z)}{f(Z)} - \alpha\beta = (1-\alpha\beta)\left[\frac{1-w(Z)}{1+w(Z)} - \frac{2Zw'(Z)}{[1+w(Z)][(C-\alpha+2)+w(Z)(C-\alpha+1+2\alpha\beta-1)]}\right]$$

But

$$(3.5)\qquad \mathrm{Re}\left\{\frac{1-w(Z)}{1+w(Z)}\right\} = \frac{1-|W(Z)|^2}{|1+W(Z)|}$$

and

$$(3.6)\qquad \mathrm{Re}\left\{\frac{2Zw'(Z)}{[1+w(Z)][(C-\alpha+2)+(C-\alpha+1+2\alpha\beta-1)w(Z)]}\right\}$$

$$\leq \frac{2|z|(1-|w(Z)|^2)}{(1-|z|^2)\,|1+w(Z)|\,|(C-\alpha+2)+(C-\alpha+1+2\alpha\beta-1)w(Z)|},$$

The last last inequality is obtained by using the well known inequality ([5], p. 168),

$$(3.7)\qquad |w'(Z)| \leq (1-|w(Z)|^2) \,/\, (1-|z|^2)\,.$$

From (3.4), (3.5) and (3.6), it follows that $f(Z)$ is starlike function of order β, if

$$\frac{2|z|(1-|w(Z)|^2)}{|1+w(Z)|(1-|z|^2)|(C-\alpha+2)(C-\alpha+2\alpha\beta)w(Z)|} \leq \frac{1-|w(Z)|^2}{|1+w(Z)|^2}$$

or

$$(3.8)\qquad \frac{2|z|}{1-|z|^2} < (C-\alpha+2)\left|1+\frac{C-\alpha+2\alpha\beta}{C-\alpha+2}\,w(Z)\right| \,/\, |1+w(Z)|$$

Since

$$|w(Z)| \leq |Z| \text{ and } (C-\alpha+2\alpha\beta)/(C-\alpha+2) \leq 1,$$

we have

$$(3.9)\qquad 1+\frac{(C-\alpha+2\alpha\beta)}{(C-\alpha+2)}\,|z| \,/\, (1+|z|) \leq \left|1+\frac{(C-\alpha+2\alpha\beta)}{(C-\alpha+2)}\,w(Z)\right| \,/\, |1+w(Z)|.$$

From (3.8) and (3.9), we obtain that $f(Z) \in S^*_\beta$ if

$$2|z| \leq \{(C-\alpha+2) + (C-\alpha+2\alpha\beta)\,|z|\}\,(1-|z|),$$

i.e. if

$$(C-\alpha+2)-2(2-\alpha\beta)|Z|-(C-\alpha+2\alpha\beta)|Z|^2 > 0.$$

Let $P(|Z|) = P(r) = (C-\alpha+2)-2(2-\alpha\beta)r-(C-\alpha+2\alpha\beta)r^2$,

Since $P(o) = (C-\alpha+2)$ and $P'(r) < 0$, the positive root r_o for which $P(r) > 0$ must be less than the root of the polynomial $P(r) = 0$, that gives the required value of r_o and the proof of theorem 2 is complete.

Corollary 1. Theorem G of Bernadi [6] follows from the above theorem by taking $\alpha = 1$, $\beta=0$ and $C=1, 2, 3, \ldots$

Corollary 2. If $\alpha=1$, $C=1$ and $0 \leqslant \beta \leqslant 1/2$ then the theorem of Padmanabhan [2] follows.

Corollary 3. If $\alpha=1$, Theorem 1 of Srivastava [3] follows from the above theorem.

4. Theorem 3. If $f(Z)=Z+\sum_{n=2}^{\infty} a_n Z^n \in S^*_{\beta_1}$ and

$$g(Z)=Z+\sum_{n=2}^{\infty} b_n Z^n \in S^*_{\beta_2}$$

and

$$F(Z) = \frac{(C+p+q)}{Z^{C-p-q+1}} \int_o^Z t^{C-p-q} \{f(t)\}^p \{g(t)\}^q \, dt, \tag{4.1}$$

$$F(Z) \in S^*_{p\beta_1+q\beta_2}$$

Proof: Let $J(Z) = \int_o^Z t^{C-p-q} \{f(t)\}^p \{g(t)\}^q \, dt$,

then,

$$J'(Z) = Z^{C-p-q} \{f(Z)\}^p \{g(Z)\}^q,$$

and

$$F(Z) = (C+p+q)\, Z^{-C+p+q-1} J(Z). \tag{4.2}$$

Now,

$$F'(Z) = (C+p+q)[(-C+p+q-1)Z^{-C+p+q-2}J(Z)+Z^{-C+p+q-1}J'(Z)]$$

$$= (C+p+q)[(-C+p+q-1)Z^{-C+p+q-2}J(Z)+Z^{-1}\{f(Z)\}^p\{g(Z)\}^q]$$

Hence,

$$[Z^{C-p-q+2}F'(Z)] = (C+p+q)[(-C+p+q-1)J(Z) + Z^{C-p-q+1}\{f(Z)\}^{p}\{g(Z)\}^{q}].$$

Now,

$$[Z^{C-p-q+2}F'(Z)]' = (C+p+q)\,[-C+p+q-1)J'(Z) + (C-p-Q+1)Z^{C-p-q}\{f(Z)\}^{p}\{g(Z)\}^{q} + Z^{C-p-q+1}p\,\{f(Z)\}^{p-1}f'(Z) + Z^{C-p-q+1}q\{g(Z)\}^{q-1}g'(Z)]$$

$$= (C+p+q)Z^{C-p-q+1}[p\{f(Z)\}^{p-1}f'(Z) + q\{g(Z)\}^{q-1}g'(Z)]$$

Hence,

$$\frac{[Z^{C-p-q+2}F'(Z)]'}{J'(Z)} = (C+p+q)\,[p\frac{Zf'(Z)}{f(Z)} + q\,\frac{Zg'(Z)}{g(Z)}] \tag{4.3}$$

Hence using (1.1) in (4.3) we get

$$\mathrm{Re}\,[\frac{\{Z^{C-p-q+2}F'(Z)\}'}{J'(Z)}] = (C+p+q)[p\mathrm{Re}\,\{\frac{Zf'(Z)}{f(Z)}\} + q\mathrm{Re}\,\{\frac{Zg'(Z)}{f(Z)}\}] \tag{4.4}$$

$$\geq (C+p+q)\,[p\beta_1 + q\beta_2].$$

Therefore using ([4], Lemma p. 430) in (4.4), we get

$$\mathrm{Re}\,\{\frac{Z^{C-p-q+2}F'(Z)}{J(Z)}\} \geq (C+p+q)\,[p\beta_1 + q\beta_2]. \tag{4.5}$$

or

$$\mathrm{Re}\,\{\frac{Z^{C-p-q+2}F'(Z)}{F(Z)Z^{C-p-q+1}/(C+p+q)}\} \geq (C+p+q)\,[p\beta_1 + q\beta_2].$$

or

$$\mathrm{Re}\,\{\frac{ZF'(Z)}{F(Z)}\} \geq (p\beta_1 + q\beta_2). \tag{4.6}$$

Therefore using (1.1) in (4.6), we get

$$F(Z) \in S^{*}_{p\beta_1 + q\beta_2}.$$

<u>Corollary 1</u>. If $\beta_2 = 0$, $p = \alpha$ and $\beta_1 = \beta$, the theorem 1 of our paper follows.

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On a Special Tensor C_{hijk} of a Finsler Space

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Abstract

In the present paper we have defined a special tensor in a Finsler space and discussed its properties. This tensor is not conformally invariant but it is defined with the help of Cartan's third curvature tensor in the same way as the conformal curvature tensor is defined in Riemannian space. The forms of the tensor Chijin in Finsler space of scalar curvature and of constant curvature have been also discussed in this paper.

1. Introduction

The properties of conformal curvature tensor in a Riemannian space have been studied in [2], [5] and [6]. Rund ([8] page 226-227) has pointed out the existence of a conformal invariant which is not a tensor. In the present paper we have defined a special tensor $C_{hijk}(x,\dot{x})$ in a Finsler space F_n and discussed its properties. This tensor is not conformally invariant but it is defined with the help of Cartan's third curvature tensor in the same way as the conformal curvature tensor is defined in Riemannian space. The forms of the tensor C_{hijk} in Finsler spaces of scalar curvature and of constant curvature have been also discussed in this paper.

We shall use the following identities involving the curvature tensor R_{hijk} ([8] pages 105, 107, 111).

(1.1) (a) $R_{hijk} = - R_{ihjk}$

(b) $R_{hijk} = - R_{hikj}$ and

(c) $R_{hj} = R_{hijk}\, g^{ik}$,

(1.2) $$R_{ijhk} + R_{ikjh} + R_{ihkj} + (C_{ijl}\, K^{l}_{rhk} + C_{ikl}\, K^{l}_{rjh} + C_{ihl}\, K^{l}_{rkj})\dot{x}^{r}=0$$

and

$$(1.3)\quad R^i_{hjk|m} + R^i_{hmj|k} + R^i_{hkm|j} + R^l_{km} P^i_{hjl} + R^l_{jk} P^i_{hml} + R^l_{mj} P^i_{hkl} = 0,$$

where $R^i_{hjk} = g^{ri} R_{hrjk}$,

$$(1.4)\quad R^l_{hk} = R^l_{rhk} \dot{x}^r = K^l_{rhk} \dot{x}^r ,$$

$$(1.5)\quad P_{ijkl} \ (= g_{jh} P^h_{ikl}) = C_{jkl|i} - C_{ikl|j} + C_{ikh} C^h_{jl|o} - C_{jkh} C^h_{il|o},$$

the index 'o' stands for contraction with respect to $\dot{x}^i$ and the tensor K^l_{rkj} has been defined in [8] (page 97). From the equations (1.1) a), (1.2) and (1.4) we find

$$(1.6)\quad R^i_{jhk} + R^i_{kjh} + R^i_{hkj} = C^i_{jl} R^l_{hk} + C^i_{kl} R^l_{jh} + C^i_{hl} R^l_{kj} .$$

2. The Special Tensor C_{hijk}

An R_3-like Finsler space F_n $(n \geq 3)$ is characterised by the relation ([4]):

$$(2.1)\quad R_{hijk} = L_{hj} g_{ik} + L_{ik} g_{hj} - g_{hk} L_{ij} - g_{ij} L_{hk},$$

where L_{ij} are components of a covariant tensor of second order. In a three-dimensional Finsler space R_{hijk} can always be written in the form (2.1)

Contracting the equation (2.1) with g^{ik} and using the relation (1.1)c) we get

$$(2.2)\quad R_{hj} = (n-2) L_{kj} + L g_{hj} ,$$

where $L \overset{def}{===} L_{ij} g^{ij}$

Contracting the equation (2.2) with g^{hj} we get

$$(2.3)\quad R = 2 (n-1) L,$$

where

(2.4) $R \overset{\text{def}}{=\!=\!=} g^{ij} R_{ij}$.

Substituting the value of L from the equation (2.3) in the equation (2.2) we get

$$(2.5)\quad L_{hj} = \frac{1}{(n-2)} R_{hj} - \frac{R}{2(n-1)(n-2)} g_{hj} .$$

Putting the value of L_{hj} from the equation (2.5) in the equation (2.1) we get

$$(2.6)\quad R_{hijk} = \frac{1}{(n-2)} (R_{hj}\, g_{ik} + R_{ik}\, g_{hj} - R_{ij}\, g_{hk} - R_{hk}\, g_{ij}) - - \frac{R}{(n-1)\ (n-2)} (g_{hj}\, g_{ik} - g_{ij}\, g_{hk}) .$$

It is to be noted that the Ricci tensor R_{ij} is not symmetric tensor in general.

Definition (2.1). We define a special tensor $C_{hijk}\ (x,\dot{x})$ in F_n by the relation.

$$(2.7)\quad C_{hijk} = R_{hijk} - \frac{1}{(n-2)} (R_{hj}\, g_{ik} + R_{ik}\, g_{hj} - g_{hk}\, R_{ij} - R_{hk}\, g_{ij}) + + \frac{R}{(n-1)(n-2)} (g_{hj}\, g_{ik} - g_{ij}\, g_{hk}) .$$

The tensor C_{hijk} is of the same form as the conformal curvature tensor of a Riemannian space. However, tensor is not conformally invariant.

The equation (2.6) proves the following lemma:

Lemma (2.1). In an R_3-like Finsler space C_{hijk} vanishes identically.

As a particular case of this lemma we have

Lemma (2.2). In a three-dimensional Finsler space C_{hijk} vanishes identically.

From the equation (2.7) we find

$$(2.8)\quad C^i_{hjk} = R^i_{hjk} - \frac{1}{(n-2)} (\delta^i_k\, R_{hj} + R^i_k\, g_{hj} - R^i_j\, g_{hk} - R_{hk}\, \delta^i_j\ ) +$$

$$+ \frac{R}{(n-1)(n-2)} (g_{hj}\, \delta^i_k - g_{hk}\, \delta^i_j),$$

where $C_{hrjk}\, g^{ri} = C^i_{hjk}$, $R^i_k = R_{rk}\, g^{ri}$ and $g_{ij}\, g^{jk} = \delta^k_i$.

<u>Theorem (2.1).</u> The tensor C_{hijk} satisfy the following identities:

(2.9) a) $C^i_{ijk} = 0$ (b) $C^i_{hik} = 0$

c) $C^i_{hji} = 0$ and (d) $C_{hijk} = - C_{ihjk}$

<u>Proof</u>. The proof follows immediately from the equations (2.7) and (2.8).

From the equation (2.7) we find

$$(2.10) \quad C_{hijk} - C_{jkhi} = \frac{1}{(n-2)} [g_{ik}(R_{jh} - R_{hj}) + g_{jh}(R_{ki} - R_{ik}) - g_{ij}(R_{kh} - R_{hk}) - g_{hk}(R_{ji} - R_{ij})].$$

If $C_{hijk} = C_{jkhi}$, then the right hand side of the equation (2.10) vanishes. Contracting the right hand side of the equation (2.10) with g^{ik} we get

$$(2.11) \quad R_{jh} = R_{hj}$$

Conversely, if the equation (2.11) holds then from the equation (2.10) we get

$$C_{hijk} = C_{jkhi} .$$

Hence we have

<u>Theorem (2.2).</u> The tensor C_{hijk} is symmetric in pairs of indices (hi) and (jk) if and only if the Ricci tensor R_{ij} is a symmetric tensor. We shall use the following lemma ([9]):

<u>Lemma (2.3)</u>. The Ricci tensor R_{ij} of a Finsler space of scalar curvature is a symmetric tensor.

From the lemma (2.3) and theorem (2.2) we have the following theorem:

<u>Theorem (2.3)</u>. If F_n is a Finsler space of scalar curvature then C_{hijk} is symmetric in pairs of indices (hi) and (jk).

From the equations (2.8) and (1.6) we have

$$
\begin{aligned}
(2.12)\qquad C^i_{hjk} + C^i_{jkh} + C^i_{khj} = {} & -(C^i_{jl}\, R^i_{hk} + C^i_{kl}\, R^l_{jh} + \\
& + C^i_{hl}\, R^l_{kj}) - \frac{1}{n-2}\,(\delta^i_k\, \hat{R}_{hj} + \\
& + \delta^i_j\, \hat{R}_{kh} + \hat{R}_{jk}\, \delta^i_h),
\end{aligned}
$$

where $\hat{R}_{hj} = R_{hj} - R_{jh}$.

Taking the Cartan's covariant derivative of the equation (2.8) and using the relation $g_{ij|h} = o$ we get

$$
\begin{aligned}
(2.13)\qquad C^i_{hjk|m} = {} & R^i_{hjk|m} - \frac{1}{n-2}\,(\delta^i_k\, R_{hj|m} + R^i_{k|m}\, g_{hj} - \\
& - R^i_{j|m}\, g_{hk} - R_{hk|m}\, \delta^i_j) + \frac{R_{|m}}{(n-1)(n-2)}\,(g_{hj}\, \delta^i_k - \delta^i_j\, g_{hk}).
\end{aligned}
$$

After using the Bianchi identity (1.3), the equation (2.13) become

$$
\begin{aligned}
(2.14)\qquad C^i_{hjk|m} + C^i_{hkm|j} + C^i_{hmj|k} = {} & -(R^l_{km}\, P^i_{hjl} + R^l_{jk}\, P^i_{hml} + \\
& + R^l_{mj}\, P^i_{hkl}) - \frac{1}{n-2}\,[\delta^i_m\,(R_{hk|j} - \\
& - R_{hj|k}) + \delta^i_k\,(R_{hj|m} - R_{hm|j}) + \\
& + \delta^i_j\,(R_{hm|k} - R_{hk|m}) + (R^i_{k|m} - \\
& - R^i_{m|k})\, g_{hj} + g_{hk}\,(R^i_{m|j} - R^i_{j|m}) + \\
& + g_{hm}\,(R^i_{j|k} - R^i_{k|j})] + \frac{1}{(n-1)(n-2)}\,[
\end{aligned}
$$

With the help of the Bianchi identity (1.3) and relations (3.2)a); (3.2)c), (3.4) and (1.1)a) we get

$$\text{(3.6)(a)} \qquad R^{i}_{hjk|m}\, X^{m} = 2\cdot R^{i}_{hjk},$$

$$\text{(b)} \qquad R_{hj|m}\, X^{m} = 2\, R_{hj}$$

and

$$\text{(c)} \qquad R_{hj|m}\, X^{m}\, g^{hj} = R_{|m}\, X^{m} = 2R.$$

Assuming that X^i is a concurrent vector field of F_n, we define a modified Finsler space $\overset{*}{F}_n$ whose metric function $\overset{*}{F}(x,\dot{x})$ is given by

$$\text{(3.7)} \qquad F^{*2} = F^{2} + (X_i(x)\, \dot{x}^{i})^{2}.$$

It has been shown in [7] that the metric tensors of the spaces F_n and $\overset{*}{F}_n$ satisfy the relation

$$\text{(3.8)} \qquad \overset{*}{g}_{ij} = g_{ij} + X_i X_j .$$

Also we have the relation ([7]):

$$\text{(3.9)} \qquad \overset{*}{R}_{ijkl} = R_{ijkl} + \frac{g_{ik}\, g_{jl} - g_{il}\, g_{jk}}{1 + X^{2}}$$

$$\text{(3.10)} \qquad \overset{*}{R}_{ij} = R_{ij} + \frac{g_{ij}\{(n-2)\, X^{2} + (n-1)\}}{(1 + X^{2})^{2}} + \frac{X_i\, X_j}{(1 + X^{2})^{2}}$$

and

$$\text{(3.11)} \qquad \overset{*}{R} = R + (n-1)\, \frac{\{(n-2)\, X^{2} + n\}}{(1 + X^{2})^{2}},$$

where $X^2 = g_{ij}\ (x,\dot{x})\ X^i X^j$.

The special tensor $\overset{*}{C}_{hijk}$ in $\overset{*}{F}_n$ is defined as

$$\text{(3.12)} \qquad \overset{*}{C}_{hijk} = \overset{*}{R}_{hijk} - \frac{1}{n-2}\, (\overset{*}{g}_{ik}\, \overset{*}{R}_{hj} + \overset{*}{R}_{ik}\, \overset{*}{g}_{hj} - \overset{*}{R}_{ij}\, \overset{*}{g}_{hk} - \overset{*}{g}_{ij}\, \overset{*}{R}_{hk}) + \frac{\overset{*}{R}}{(n-1)(n-2)}\, (\overset{*}{g}_{ik}\, \overset{*}{g}_{hj} - \overset{*}{g}_{ij}\, \overset{*}{g}_{hk}).$$

With the help of the equations (2.7), (3.8), (3.9), (3.10), (3.11) and (3.12) we get

$$(3.13)\quad \overset{*}{C}_{hijk} = C_{hijk} - \frac{1}{n-2}\,(X_i X_k R_{hj} + R_{ik} X_h X_j - R_{ij} X_h X_k - X_i X_j R_{hk}) + $$

$$+ \frac{R}{(n-1)(n-2)}\,(g_{ik}\,X_h X_j + X_i X_k g_{hj} - $$

$$- g_{hj}\,X_i X_j - X_h X_k\,g_{ij})\,.$$

<u>Theorem (3.1)</u>. If a Finsler space F_n admits a concurrent vector field X^i then the necessary and sufficient condition that the special tensor C_{hijk} is invariant under the transformation (3.7) is

$$(3.14)\quad R_{ij} = \frac{R}{n-1}\,(g_{ij} - \frac{X_i X_j}{X^2})\,.$$

<u>Proof</u>. In order to prove the necessary part we assume that $\overset{*}{C}_{hijk} = C_{hijk}$ is given. Then from the equation (3.13) we find

$$X_i X_k R_{hj} + R_{ik}\,X_h X_j - X_h X_j R_{ik} - R_{hk}\,X_i X_j = \frac{R}{n-1}\,(g_{ik}\,X_h\,X_j + $$

$$+ X_i X_k g_{hj} - g_{hk}\,X_i X_j - g_{ij}\,X_h X_k).$$

Multiplying the above equation by X^h, X^j using equation (3.3) and relations $g_{ij}\,X^j = X_i$, $X_i X^i = X^2$ we get the equation (3.14).

In order to prove the sufficient part we substitute from (3.14) the expression of the components R_{hj}, R_{ik} R_{ij}, R_{hk} in (3.13). This substitution gives $\overset{*}{C}_{hijk} = C_{hijk}$.

4. <u>Finsler Space of Scalar Curvature</u>

The tensor R_{hijk} in a Finsler space F_n of scalar curvature, K is given by ([9])

$$(4.1)\quad R_{hijk} = \frac{1}{2}\,\{(h_{hj}\,N_{ik} + h_{ik}\,N_{hj}) - (N_{hk}\,h_{ij} + N_{ij}\,h_{kh})\} - Q_{hijk},$$

where h_{ij} is the angular metric tensor defined by

$$h_{ij} = g_{ij} - l_i l_j ,$$

l_i is unit vector in the direction of the element of support, N_{ij} is a symmetric tensor given by ([9])

$$N_{ij} = Kg_{ij} + \frac{F^2}{3} \frac{\partial^2 K}{\partial \dot{x}^i \, \partial \dot{x}^j} + \frac{\partial K}{\partial \dot{x}^i} l_j F + \frac{\partial K}{\partial \dot{x}^j} l_i F + l_i l_j F \tag{4.2}$$

and

$$Q_{hijk} = P_{hrj} P^r_{ik} - P_{hrk} P^r_{ij} , \tag{4.3}$$

P_{ijk} being defined by the equation (1.5).

The scalar K occurring above is called as the scalar curvature. It is positively homogeneous function of degree zero with respect to the element of support $\dot{x}$.

If the scalar K is constant then F_n is called a Finsler space of constant curvature. In a space of scalar curvature K the relation (1.1)c, (2.4) and (4.1) yield

$$R_{hj} = (n-1)Kg_{hj} + [(n-3)F^2 \frac{\partial^2 K}{\partial \dot{x}^h \, \partial \dot{x}^j} + (3n-7)F \frac{\partial K}{\partial \dot{x}^j} l_h + \frac{\partial K}{\partial \dot{x}^h} l_j) + F^2 \frac{\partial^2 K}{\partial \dot{x}^s \, \partial \dot{x}^r} g^{sr} h_{hj}] /6 - Q_{hj} \tag{4.4}$$

and

$$R = (n-1)\, n\, K + \frac{n-2}{3} F^2 \frac{\partial^2 K}{\partial \dot{x}^s \, \partial \dot{x}^r} g^{sr} - Q, \tag{4.5}$$

where $Q_{hj} \overset{\text{def}}{=} Q_{hijk}\, g^{ik} = Q_{jh}$ and $Q_{hj}\, g^{hj} = Q$.

With the help of the equations (4.1), (4.2), (4.4), (4.5) and (2.7) we get

$$C_{hijk} = \frac{F^2}{6(n-2)} (g_{hj} \frac{\partial^2 K}{\partial \dot{x}^i \, \partial \dot{x}^k} + g_{ik} \frac{\partial^2 K}{\partial \dot{x}^h \, \partial \dot{x}^j} - \frac{\partial^2 K}{\partial \dot{x}^i \, \partial \dot{x}^j} g_{hk} - \tag{4.6}$$

$$- g_{ij} \frac{\partial^2 K}{\partial \dot{x}^h \partial \dot{x}^k}) + \frac{F}{6(n-2)} [g_{hj} (\frac{\partial K}{\partial \dot{x}^i} l_k + \frac{\partial K}{\partial \dot{x}^k} l_i) +$$

$$+ g_{ik} (\frac{\partial K}{\partial \dot{x}^h} l_j + \frac{\partial K}{\partial \dot{x}^j} l_h) - g_{hk} (\frac{\partial K}{\partial \dot{x}^i} l_j + \frac{\partial K}{\partial \dot{x}^j} l_i) -$$

$$- g_{ij} (\frac{\partial K}{\partial \dot{x}^h} l_k + \frac{\partial K}{\partial \dot{x}^k} l_h)] - \frac{F^2}{6} (\frac{\partial^2 K}{\partial \dot{x}^i \partial \dot{x}^h} l_h l_j +$$

$$- l_i l_k \frac{\partial^2 K}{\partial \dot{x}^h \partial \dot{x}^j} - l_h l_k \frac{\partial^2 K}{\partial \dot{x}^i \partial \dot{x}^j} - l_i l_j \frac{\partial^2 K}{\partial \dot{x}^h \partial \dot{x}^k}) -$$

$$- \frac{F^2}{3(n-2)} \frac{\partial^2 K}{\partial \dot{x}^s \partial \dot{x}^r} g^{sr} [\frac{1}{n-1} (g_{hj} g_{ik} - g_{ij} g_{hk}) +$$

$$+ \frac{1}{2} (g_{ik} l_h l_j + g_{hj} l_i l_k - l_i l_j g_{hk} - g_{ij} l_h l_j)] -$$

$$- L_{hijk} ,$$

where

$$L_{hijk} = Q_{hijk} - \frac{1}{n-2} (g_{ik} Q_{hj} + g_{hj} Q_{ik} - g_{hk} Q_{ij} - Q_{hk} g_{ij}) + \frac{Q}{(n-1)(n-2)} (g_{hj} g_{ik} - g_{ij} g_{hk}). \tag{4.7}$$

We have thus proved the following theorem.

<u>Theorem (4.1)</u>. In a Finsler space of scalar curvature K the tensor C_{hijk} is given by the equation (4.6).

In a space of constant curvature $\frac{\partial K}{\partial \dot{x}^i} = o$. Therefore the equation (4.6) gives:

<u>Theorem (4.2)</u>. In Finsler space of constant curvature K the special tensor C_{hijk} is given by

$$C_{hijk} = - Q_{hijk} + \frac{1}{n-2} (g_{ik} Q_{hj} + g_{hj} Q_{ik} - g_{hk} Q_{ij} - Q_{hk} g_{ij}) - \frac{Q}{(n-1)(n-2)} (g_{ik} g_{hj} - g_{ij} g_{hk}). \tag{4.8}$$

In a Landsberg space $P_{hij} = o$ ([7]). Therefore the equation (4.3) yields $Q_{hijk} = o$ which implies $Q_{hj} = o$ and $Q = o$.

Substituting these values in (4.8) we get $C_{hijk} = o$.

It has been shown in [3] that an R_3-like Finsler space is characterised by the vanishing of the tensor C_{hijk}.

Therefore we have the following theorem:

Theorem (4.3). A Landsberg space of constant curvature is R_3-like.

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Fixed Point Theorems, Compact Metric Spaces and Nearly Densifying Maps

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In this paper we first obtain a fixed point theorem of Jungck type [10] for nearly densifying self-maps on a complete metric space, generalizing some of the results of Chattopadhyay [2] and Iseki [9]. Later we prove some fixed point theorems for a family of self-maps on a compact metric space and obtain their analogues for a family of nearly densifying self-maps on a complete metric space. One of these results (Theorem 2) is a generalization of some of those of Bailey [1], Edelstein [3], Hardy and Rogers [8], and Fisher [4], [5], [6] and [7].

Throughout this paper:

(X,d) is a metric space ; A,B are subsets of X ;
δ (A) is the diameter of A : $\bar{A}$ is the closure of A ;
f,g,S,T are self-maps on X ; I is the identity map on X ; and
$\mathcal{F}$ is a non-empty family of self-maps on X.

Definition 1 (Kuratowski [11]):- If A is bounded then $\alpha(A)$, the measure of non-compactness of A, is defined as inf $\{ \varepsilon > 0 \mid$ A admits a finite cover consisting of subsets of X with diameter less than $\varepsilon \}$.

If A,B are bounded subsets of X then

$$\alpha(A \cup B) = \max \{ \alpha(A), \alpha(B) \} .$$

In a complete metric space, the measure of non-compactness of a bounded set is zero if and only if the closure of the set is compact.

Definition 2 (Sastry and Naidu [27]): f is said to be nearly densifying if $\alpha(fA) < \alpha(A)$ for every f-invariant bounded subset A of X with $\alpha(A) > 0$.

Definition 3 (Sessa [13]): f,g are said to be weakly commutative on X if $d(fgx, gfx) \leq d(gx, fx)$ for all $x \in X$.

Theorem 1: Suppose that (X,d) is complete; f,S are commutative, continuous and nearly densifying; and for $x,y \in X$

$$(1.1) \qquad d(fx,fy) < \max \{ d(Sx,Sy),\ d(Sx,fx),\ d(Sy,fy),\ \tfrac{1}{2}[d(Sx,fy) + d(fx,Sy)] \}$$

provided $fx \neq fy$ and $Sx \neq Sy$.

Suppose also that there is an $x_o \in X$ such that $A = \{ f^i S^j x_o \mid i$ and j are non-negative integers $\}$ is bounded. Then f and S have a unique common fixed point.

Proof: Clearly $f(A) \subseteq A$.
From the commutativity of f and S we have $S(A) \subseteq A$.
From the continuity of f and S we now have $f(\bar{A}) \subseteq \bar{A}$ and $S(\bar{A}) \subseteq \bar{A}$.
Since f and S are nearly densifying, A is bounded and $A = \{x_o\} \cup f(A) \cup S(A)$, it is clear that $\alpha(A) = 0$ so that, from the completeness of X, $\bar{A}$ is compact.

Since fS is continuous, $(fS)^n(\bar{A})$ is compact for each positive integer n.

Since the sequence $\{(fS)^n(\bar{A})\}$ is a decreasing sequence of non-empty compact sets, the set $H = \bigcap_{n=1}^{\infty} (fS)^n(\bar{A})$ is a non-empty compact set.

From the commutativity of f and S it is clear that $f(H) \subseteq H$ and $S(H) \subseteq H$.

Let $x \in H$. Then $x \in (fS)^{n+1}(\bar{A})$ $(n=0,1,2,\ldots)$ so that we can find a sequence $\{x_n\}$ such that $x_n \in (fS)^n(\bar{A})$ and $fSx_n = x$ $(n=1,2,\ldots)$. The sequence $\{x_n\}$, being a sequence in the compact set $\bar{A}$, admits a convergent subsequence with limit p for some p in $\bar{A}$. It is clear that $p \in H$ and $fSp = x$. Hence H is a subset of $f(H)$ as well as $S(H)$.

Thus we have $f(H) = H$ and $S(H) = H$.

From the continuity of f and S and the compactness of H we can find a $u \in H$ such that $d(Su,fu) = \inf \{ d(Sx,fx) \mid x \in H \}$.

We can choose $w \in H$ such that $Sw = u$.

Suppose now that $fSw \neq ffw$ and $SSw \neq Sfw$.

Then taking $x = Sw$ and $y = fw$ in inequality (1.1) and making use of the commutativity of f and S and the fact that $Sw = u$ we obtain $d(Sfw,ffw) < \max \{d(Su,fu), d(Sfw,ffw), \frac{1}{2} d(Su,ffw)\}$ so that on applying triangle inequality we get

$$d(Sfw,ffw) < d(Su,fu)$$

which is a contradiction to the selection of u. Hence we have $fz = Sz$ where $z = fw$ or Sw. Also $S^2z = Sfz = fSz$. Hence if $S^2z \neq Sz$ then one can apply inequality (1.1) for $x = Sz$ and $y = z$ and arrive at a contradiction.

Thus Sz is a common fixed point of f and S.

From inequality (1.1) it is evident that f and S cannot have two distinct common fixed points.

Hence the theorem.

Remark 1. Theorem 2 of Chattopadhyay [2] which is a generalization of the theorems of Iseki [9] is a special case of Theorem 1 with $S = I$.

Remark 2. Example 1 shows that in Theorem 1 the commutativity condition on f and S cannot be replaced by weak commutativity.

Example 1. Let $X = \{ 1,2,3 \}$ with the usual metric.

Define $f : X \longrightarrow X$ as $f1 = 2$, $f2 = 1$, $f3 = 2$

and $S : X \longrightarrow X$ as $S1 = 3$, $S2 = 3$, $S3 = 1$.

Then f, S are weakly commutative and

$|fx - fy| < \max \{|Sx - fx|, |Sy - fy|\}$ for all x,y in X.

f, S being self-maps on a finite metric space are continuous and densifying.

But neither f nor S has a fixed point.

Problem. Does Theorem 1 remain valid if inequality (1.1) is altered by replacing the average of $d(fx,Sy)$ and $d(Sx,fy)$ with their maximum?

Theorem 2. Suppose (X,d) is compact, $(fg)^p$ is continuous on X for some positive integer p ; f,g commute with each other and also with every member of $\mathfrak{F}$; and for $x,y \in X$

(2.1) $d(fx,gy) < \delta\,(\mathfrak{S}(x) \cup \mathfrak{S}(y))$
provided $fx \neq gy$, where $\mathfrak{S}(x) = \{Sx \mid S \in \tau\}$, τ being the semi-group of self-maps on X generated by $\mathfrak{F} \cup \{f,g,I\}$. Then the family $\mathfrak{F} \cup \{f,g\}$ has a unique common fixed point.

Proof: Let $H = \bigcap_{n=1}^{\infty} (fg)^{pn}(X)$.

From the compactness of X, the continuity of $(fg)^p$ and the commutativity of f and g it can be shown that H is a non-empty compact set, $f(H) = H$ and $g(H) = H$.

Since f and g commute with every member of $\mathfrak{F}$, we have $SH \subseteq H$ for all $S \in \mathfrak{F}$.

From the compactness of H there exist $z_1,z_2 \in H$ such that $\delta(H) = d(z_1,z_2)$.

There exist $x_1,x_2 \in H$ such that $fx_1 = z_1$ and $gx_2 = z_2$.

Suppose $z_1 \neq z_2$.

Then from inequality (2.1) we obtain

$$\delta(H) = d(fx_1,gx_2) < \delta(\mathfrak{S}(x_1) \cup \mathfrak{S}(x_2)) \leq \delta(H)$$

which is a contradiction.

Hence $H = \{ z \}$ for some z in X.

Since every member of $\mathfrak{F} \cup \{f,g\}$ is H-invariant, it is clear that z is a common fixed point of $\mathfrak{F} \cup \{f,g\}$. From inequality (2.1) it is evident that the family $\mathfrak{F} \cup \{f,g\}$ cannot have two distinct common fixed points.

Remark 3. Theorem 2 is a generalization of some of the theorems of Bailey [1], Edelstein [3], Fisher [4] and Hardy and Rogers [8]. It is also a generalization of Theorem 4 of Fisher [5] and Theorem 2 of Fisher [7]

Corollary 1. Suppose that (X,d) is compact, f is continuous on X, $fS = Sf$, $fT = Tf$ and for $x,y \in X$

$$d(fx,fy) < \max \{ d(Sx,Ty),\ d(Sx,fx),\ d(Ty,fy),\ d(fx,Ty),\ d(Sx,fy)\}$$

provided the right hand side of the inequality is positive. Then f,S and T have a unique common fixed point.

Proof: When the right hand side of the above inequality is zero then $fx = fy$. Hence from the hypothesis it is evident that the above inequality holds when $fx \neq fy$. Now the Corollary follows from Theorem 2 on taking $g = f$ and $\mathcal{F} = \{ S,T \}$.

Remark 4. Theorem 2 of Fisher [7] (which in turn is an improvement over Theorem 5 of Fisher [6]) is nothing but Corollary 1 with the following additional restrictions on the hypothesis:

(1) S,T are continuous on X and

(2) For each x in X, there exists y in X such that $fx = Sy = Ty$.

Theorem 3. Suppose that (X,d) is complete, $\mathcal{F}$ is finite, every member of $\mathcal{F} \cup \{f,g\}$ is continuous and nearly densifying, f,g commute with each other and also with every member of $\mathcal{F}$, and inequality (2.1) holds when $fx \neq gy$, where $\mathfrak{G}(x)$ is as defined in Theorem 2.

Suppose also that $\mathfrak{G}(x_o)$ is bounded for some $x_o \in X$.

Then the family $\mathcal{F} \cup \{f,g\}$ has a unique common fixed point in X.

Proof: Write $A = \mathfrak{G}(x_o)$ and $\mathcal{S} = \mathcal{F} \cup \{f,g\}$.

Clearly $S(A) \subseteq A$ for all $S \in \mathcal{S}$.

Since every member of $\mathcal{S}$ is continuous on X, we have $S(\bar{A}) \subseteq \bar{A}$ for all $S \in \mathcal{S}$.

Clearly $A = \{x_o\} \cup \left(\bigcup_{S \in \mathcal{S}} S(A) \right)$.

Since (X,d) is complete, A is bounded, $\mathcal{S}$ is finite and every member of $\mathcal{S}$ is nearly densifying it follows that $\bar{A}$ is compact. Now from Theorem 2 it follows that $\mathcal{S}$ has a common fixed point z in the compact space $\bar{A}$.

Theorem 4. Suppose (X,d) is compact, $\mathcal{F}$ is finite, every member of $\mathcal{F} \cup \{f,g\}$ is continuous on X and any two members of it are commutative and inequality (2.1) holds for all those x,y in X for which

$$\sum_{S \in \mathcal{F}} \left[d(Sx,Sy) + \inf_{u \in \mathfrak{S}(x) \cup \mathfrak{S}(y)} d(fu,Su) + \inf_{u \in \mathfrak{S}(x) \cup \mathfrak{S}(y)} d(gu,Su) \right] \neq 0$$

where $\mathfrak{S}(x)$ is as defined in Theorem 2. Then the family $\mathcal{F} \cup \{f,g\}$ has a unique common fixed point.

Proof: Let $K = \bigcap_{n=1}^{\infty} (fg)^n(X)$ and $\mathcal{S} = \mathcal{F} \cup \{f,g\}$.

Then K is a non-empty compact set, $f(K) = K = g(K)$ and $S(K) \subseteq K$ for all $S \in \mathcal{F}$.

There exist $z_1, z_2, w_1, w_2 \in K$ such that $\delta(K) = d(z_1,z_2)$, $fw_1 = z_1$ and $gw_2 = z_2$.

Let $\mathcal{F} = \{S_1, S_2, \ldots, S_n\}$.

If $\inf_{u \in \mathfrak{S}(w_1) \cup \mathfrak{S}(w_2)} d(fu, S_1u) > 0$ then from inequality (2.1) we have

$$\delta(K) = d(fw_1, gw_2) < \delta(\mathfrak{S}(w_1) \cup \mathfrak{S}(w_2)) \leq \delta(K),$$ which is a contradiction.

Hence $\inf_{u \in \mathfrak{S}(w_1) \cup \mathfrak{S}(w_2)} d(fu, S_1u) = 0$.

Now from the continuity of f and S_1 and the compactness of X it follows that there exists $u_1 \in \overline{\mathfrak{S}(w_1) \cup \mathfrak{S}(w_2)}$ such that $fu_1 = S_1u_1$.

Since any two members of $\mathcal{S}$ are commutative it now follows that $f = S_1$ on $\mathfrak{S}(u_1)$.

Hence from the continuity of f and S_1 we have $f = S_1$ on $\overline{\mathfrak{S}(u_1)}$. Clearly $S(\mathfrak{S}(u_1)) \subseteq \mathfrak{S}(u_1)$ for all $S \in \mathcal{S}$.

Since every member of $\mathcal{S}$ is continuous we must have

$S(\overline{\mathfrak{S}(u_1)}) \subseteq \overline{\mathfrak{S}(u_1)}$ for al $S \in$.

Taking $\overline{\mathfrak{S}(u_1)}$ in the place of X in the definition of K and proceeeding almost along the same lines as above, we can show the existence of $u_2 \in \overline{\mathfrak{S}(u_1)}$ such that $f = S_2$ on $\overline{\mathfrak{S}(u_2)}$.

Further, $\overline{\mathfrak{S}(u_2)}$ is S-invariant for every $S \in \mathcal{S}$.

Now we have $f = S_1 = S_2$ on $\overline{\mathfrak{S}(u_2)}$.

Continuing like this, it can be shown in (n+1) steps that there exists $v \in X$ such that $f = S_1 = S_2 = \ldots = g$ on $\overline{\mathfrak{S}(v)}$.

Further, $\overline{\mathfrak{S}(v)}$ is S-invariant for every $S \in \mathcal{S}$.

Let $H = \bigcap_{n=1}^{\infty} (fg)^n(\overline{\mathfrak{S}(v)})$.

Then H is a non-empty compact set, $fH = H$,

$f = S_1 = S_2 = \ldots = S_n = g$ on H.

There exist $v_1, v_2, v_3, v_4 \in H$ such that $\delta(H) = d(v_1, v_2)$, $fv_3 = v_1$ and $gv_4 = v_2$.

If $d(Sv_3, Sv_4) > 0$ for some $S \in \mathcal{S}$ then from inequality (2.1) we get
$\delta(H) = d(fv_3, gv_4) < \delta(\mathfrak{S}(v_3) \cup \mathfrak{S}(v_4)) \leq \delta(H)$,
which is a contradiction.

Hence $Sv_3 = Sv_4$ for all $S \in \mathcal{S}$.

Hence $v_1 = fv_3 = S_1v_3 = S_1v_4 = gv_4 = v_2$.

Hence $H = \{z\}$ for some z in X.

Since H is S-invariant for every S in $\mathcal{S}$ it follows that z is a common fixed point of $\mathcal{S}$.

From inequality (2.1) it follows that z is the only common fixed point of $\mathcal{S}$.

Theorem 5. Theorem 3 with 'f,g commute with each other and also with every member of $\mathcal{F}$' and 'inequality (2.1) holds when $fx \neq gy$' being read as 'any two members of $\mathcal{F} \cup \{f,g\}$ are commutative' and 'inequality (2.1) holds when

$$\sum_{S\in\mathcal{F}} \left[d(Sx,Sy) + \inf_{u\in (x)\cup(y)} d(fu,Su) + \inf_{u\in (x)\cup(y)} d(gu,Su)\right] \neq 0 \text{ '}$$

respectively.

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A Subclass of Univalent Functions With Negative Coefficients

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Abstract

Let $f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n$ $(a_n \geq 0,\ a_1 > 0)$ be analytic in a unit disc E.

Let $Q(\alpha)$ be a class of functions $f(z)$ satisfying $\operatorname{Re} \frac{f'(z)}{a_1} > \alpha$ $(0 \leq \alpha < 1)$ for $z \in E$ and T a class of functions $f(z)$ that satisfy $f'(z_o) = 1$ $(0 < z_o < 1)$. The subclass $R(\alpha_1 z_o) = Q(\alpha) \cap T$ is considered and coefficient inequalities, distortion theorems, radius of convexity and closure theorems are obtained for this class.

1. Introduction

Let A denote the class of functions $f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n$ which are analytic in the unit disc $E = \{z, |z| < 1\}$ where $a_1 > 0$ and $a_n \geq 0$ for $n = 2,3,\ldots$

Let $Q(\alpha)$ be a subclass of A, consisting of functions $f(z)$ satisfying

$$\operatorname{Re} \frac{f'(z)}{a_1} > \alpha \ (0 \leq \alpha < 1) \text{ for } z \in E \text{ and T a class of functions } f(z)$$

that satisfy $f'(z_o) = 1$ $(0 < z_o < 1)$.

For given α and z_o fixed, let

$$R(\alpha, z_o) = Q(\alpha) \cap T.$$

In this paper we obtain a few sharp results concerning coefficient inequalities, distortion theorems and radius of convexity for the class $R(\alpha, z_o)$. Further, we show that the class $R(\alpha, z_o)$ is closed under Arithmetic means and convex linear combinations.

H. Silverman's [1] techniques are used for establishing the theoreims.

2. Coefficient Inequalities

Theorem 1. A function $f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n$ where $a_1 > 0$ and $a_n \geq 0$ $(n=2,3,\ldots)$, is in $Q(\alpha)$ if and only $\sum_{n=2}^{\infty} n\, a_n < a_1\,(1-\alpha)$. (1)

The proof is obtained in [2].

Theorem 2. A function $f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n$ is in $R(\alpha, z_o)$ if and only if $\sum_{n=2}^{\infty} n\,\{1-(1-\alpha)\,z_o^{\,n-1}\}\,a_n < 1-\alpha$

Proof: Since $f'(z_o) = 1 = a_1 - \sum_{n=2}^{\infty} n a_n z_o^{\,n-1}$, the result follows by substituting

$$a_1 = 1 + \sum_{n=2}^{\infty} n a_n z_o^{\,n-1} \tag{2}$$

in the statement of theorem 1.

Corollary. If $f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n$ is in $R(\alpha, z_o)$ then $a_n \leq \dfrac{1-\alpha}{n\{1-(1-\alpha)z_o^{\,n-1}\}}$ $(n=2,3,\ldots)$ with equality for $f(z) = \dfrac{nz-(1-\alpha)\,z^n}{n\{1-(1-\alpha)z_o^{\,n-1}\}}$.

S

3. Distortion The

Theorem 3:- If f(z)

then

$|f(z)| \leq$ 2(1-

and

$|f'(z)| \leq$

Proof: From Theore

$\sum_{n=2}^{\infty} a_n$

Therefore,

$|f(z)| \leq a_1 r$ +

That is

$|f(z)| \leq$ 2(1-

Hence (3) follows.

Further,

$|f'(z)| \leq a_1$ +

Also from theorem 2,

$\sum_{n=2}^{\infty} na_n \leq$

3. Distortion Theorem

Theorem 3:- If $f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n$ is in $R(\alpha, z_o)$

then

$$|f(z)| \le \frac{2}{2\{1-(1-\alpha)z_o\}} [2 + r(1-\alpha)] \quad (|z| = r) \ldots \tag{3}$$

and

$$|f'(z)| \le \frac{1}{1-(1-\alpha)z_o} [1 + r(1-\alpha)] \quad (|z| = r) \ldots \tag{4}$$

Proof: From Theorem 2, we have

$$\sum_{n=2}^{\infty} a_n \le \frac{1-\alpha}{2\{1-(1-\alpha)z_o\}}$$

Therefore,

$$|f(z)| \le a_1 r + r^2 \sum_{n=2}^{\infty} a_n \le \frac{r}{1-(1-\alpha)z_o} + \frac{r^2(1-\alpha)}{2\{1-(1-\alpha)z_o\}}$$

That is

$$|f(z)| \le \frac{r}{2\{1-(1-\alpha)z_o\}} [2 + r(1-\alpha)].$$

Hence (3) follows.

Further,

$$|f'(z)| \le a_1 + r \sum_{n=2}^{\infty} na_n$$

Also from theorem 2, we have

$$\sum_{n=2}^{\infty} na_n \le \frac{1-\alpha}{1-(1-\alpha)z_o}$$ and hence (4) follows.

Note: The bounds are sharp since the equalities are obtained for the function

$$f(z) = \frac{z}{1-(1-\alpha)z_o} - \frac{(1-\alpha)z^2}{2\{1-(1-\alpha)z_o\}}$$

4. Radius of Convexity

Theorem 4: If $f(z) \in R(\alpha, z_o)$, then $f(z)$ is convex in the disk

$|z| < r = r(\alpha) = \inf_n \left[\frac{1}{n(1-\alpha)}\right]^{\frac{1}{n-1}}$. The result is sharp with the extremal function

$$f_n(z) = \frac{nz - (1-\alpha)z^n}{n\{1-(1-\alpha)z_o^{\,n-1}\}} \quad (n=2,\ 3,\ \ldots..)$$

Proof: It is enough if we show that

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \quad \text{for} \quad |z| \leq r\,(\alpha)$$

we know,

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{\sum_{n=2}^{\infty} n(n-1)a_n|z|^{n-1}}{a_1 - \sum_{n=2}^{\infty} na_n|z|^{n-1}}$$

Hence

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \text{ , if}$$

$$\sum_{n=2}^{\infty} n(n-1)a_n|z|^{n-1} \leq 1 + \sum_{n=2}^{\infty} na_n z_o^{\,n-1} - \sum_{n=2}^{\infty} na_n|z|^{n-1}$$

or

$$\sum_{n=2}^{\infty} (n^2|z|^{n-1} - n\,z_o^{\,n-1})\,a_n \leq 1 \tag{5}$$

From theorem 2, we have

$$\sum_{n=2}^{\infty} \left(\frac{n}{1-\alpha} - n\, z_o^{\,n-1}\right) a_n \le 1$$

therefore, (5) holds good if

$$\sum_{n=2}^{\infty} (n^2|z|^{n-1} - nz_o^{\,n-1})a_n \le \sum_{n=2}^{\infty} \left(\frac{n}{1-\alpha} - nz_o^{\,n-1}\right) a_n$$

or

$$n^2|z|^{n-1} - n\, z_o^{\,n-1} \le \frac{n}{1-\alpha} - nz_o^{\,n-1} \quad (n = 2, 3, \ldots\ldots) \tag{6}$$

Solving (6) for $|z|$ we have

$$|z| \le \left[\frac{1}{n\,(1-\alpha)}\right]^{\frac{1}{n-1}} \quad (n = 2, 3, \ldots\ldots)$$

writing $|z| = r(\alpha)$, the result follows.

5. Closure Theorems

Theorem 5:- if $f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n$ and

$$g(z) = b_1 z - \sum_{n=2}^{\infty} b_n z^n \text{ are in } R(\alpha,\, z_o)$$

then

$$h(z) = c_1 z - \frac{1}{2} \sum_{n=2}^{\infty} (a_n + b_n) z^n, \quad \left(c_1 = \frac{a_1 + b_1}{2}\right)$$

is also in $R(\alpha,\, z_o)$.

Proof: In view of theorem 2, since $f(z)$ and $g(z)$ belong to $R(\alpha,\, z_o)$, we have

$$\sum_{n=2}^{\infty} \left(\frac{n}{1-\alpha} - nz_o^{\,n-1}\right) a_n \le 1 \tag{7}$$

and

$$\sum_{n=2}^{\infty} \left(\frac{n}{1-\alpha} - n\, z_o^{\,n-1}\right) b_n \le 1 \qquad (8)$$

For $h(z)$ to be in $R(\alpha, z_o)$ it is sufficient to show that

$\frac{1}{2} \sum_{n=2}^{\infty} \left(\frac{n}{1-\alpha} - n\, z_o^{\,n-1}\right) (a_n + b_n) \le 1$ which follows immediately from (7) and (8).

Theorem 6: If $f_1(z) = z$, $f_n(z) = \dfrac{nz - (1-\alpha)z^n}{n\{1-(1-\alpha)z_o^{\,n-1}\}}$ $(n=2,3,\ .....)$

then $f(z) \in R(\alpha, z_o)$ if and only if it can be expressed in the form

$f(z) = \sum_{n=2}^{\infty} \lambda_n f_n(z)$, where

$$\lambda_n \ge 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \lambda_n = 1 .$$

Proof: Let $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$ where $\lambda_n \ge 0$ and $\sum_{n=1}^{\infty} \lambda_n = 1$

then

$$f(z) = \left[\lambda_1 + \sum_{n=2}^{\infty} \frac{\lambda_n}{1-(1-\alpha)z_o^{\,n-1}}\right] z - \sum_{n=2}^{\infty} \frac{(1-\alpha)\,\lambda_n\, z^n}{n\{1-(1-\alpha)z_o^{\,n-1}\}}$$

Note that $f'(z_o) = \sum_{n=1}^{\infty} \lambda_n f'_n(z_o) = \sum_{n=1}^{\infty} \lambda_n = 1$

we also have

$$\sum_{n=2}^{\infty} \frac{\lambda_n (1-\alpha)}{n\{1-(1-\alpha)z_o^{\,n-1}\}} \left[\frac{n\{1-(1-\alpha)z_o^{\,n-1}\}}{(1-\alpha)}\right] = \sum_{n=2}^{\infty} \lambda_n = 1 - \lambda_1 \le 1$$

Hence from theorem 2, $f(z) \in R(\alpha, z_o)$.

Conversely, suppose that $f(z) \in R(\alpha, z_o)$.

Since

$$a_n \leq \frac{1-\alpha}{n\{1-(1-\alpha)z_o^{n-1}\}} \quad (n = 2, 3, \ldots\ldots)$$

We may set

$$\lambda_n = \left[\frac{n\{1-(1-\alpha)z_o^{n-1}\}}{(1-\alpha)}\right] a_n \; (n = 2, 3, \ldots\ldots)$$

and

$$\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n$$

Then $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$

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On Quasi-Regular and Jacobson Radicals

Z.K. Warsi & Prahlad Singh

Abstract

Here in the paper we have introduced an operation 0_α in the Γ-ring and established certain new results concerning the Γ-ring.

1. Introduction

Let $M = \{a, b, c, \ldots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \ldots\}$ be additive abelian groups. If there exists a mapping $M \times \Gamma \times M \rightarrow M$ satisfying the conditions:

1. $(a + b)\alpha c = a\alpha c + b\alpha c$
   $a(\alpha + \beta) c = a\alpha c + a\beta c$
   $a\alpha (b + c) = a\alpha b + a\alpha c;$

2. $(a\alpha b)\beta c = a\alpha(b\beta c)$;

3. $a\alpha b = 0$ and if $a, b \neq 0$ then $\alpha = 0$;

then M is said to be Γ-ring (c.f. Barnes 1966). Also if there is a mapping $\Gamma \times M \times \Gamma \rightarrow \Gamma$ satisfying the conditions:

1a. Same as 1 above

2a. $(a\alpha b)\beta c = a(\alpha b \beta)c = a\alpha(b\beta c)$;

3a. $a\alpha b = 0$, if $\alpha \neq 0$ then either $a = 0$ or $b = 0$;

then M is said to be a Γ-ring in the sense of Nobusawa (1964). A Nobusawa Γ-ring M is said to be commutative if $a\alpha b = b\alpha a$ and $\alpha a\beta = \beta a\alpha$ hold for every $a, b \in M$ and $\alpha, \beta \in \Gamma$.

The notion of Γ-ring was introduced by N. Nobusawa (1964) and he used it to generalize the Wedderburn theorem. Later on, it was discussed by several authors (J. Luch 1968, 1969; W.E. Barnes 1966; W.E. Coppage and Luh 1971, and Z.K. Warsi 1978).

Throughout the paper we have assumed that Γ-ring is in the sense of Nobusawa. The following basic notions and definitions are due to Barnes (1966), Coppage and Luh (1971), and Warsi (1978).

A subset A of the Γ-ring M is a right ideal if A is an additive subgroup of M and $a\,\Gamma\,M = \{a \alpha c : a \in A, \alpha \in \Gamma, c \in M\} \subseteq A$. Similarly a left ideal.

If A is both a left and a right ideal, then A is a two-sided ideal or simply an ideal of M.

If A and B are two ideals of M, then their sum defined as $A + B = \{a + b : a \in A, b \in B\}$ is an ideal.

Let M be a Γ-ring, $a \in M$ then a principal ideal generated by a is represented as $(a) = Za + a\Gamma M + M\Gamma a + M\Gamma a\Gamma M$, where Z is the set of all integers.

If A and B are subsets of M, then $A\Gamma B$ is the set of all finite sums of the form $\sum a_i \alpha_i b_i$ where $a_i \in A$, $b_i \in B$, $\alpha_i \in \Gamma$.

If A is a two-sided ideal of M, then $M/A = \{x + A : x \in M\}$, the set of cosets of A is again a Γ-ring with respect to the operations:

$$(x + A) + (y + A) = (x + y) + A$$

$$(x + A)\,\alpha\,(y + A) = (x \alpha y) + A, \text{ for } x, y \in A \text{ and } \alpha \in \Gamma.$$

An ideal A of the Γ-ring M is a nil ideal if for each $a \in A$ and $\alpha \in \Gamma$

$$(a \alpha)^n a = 0 \text{ for some integer } n.$$

2. Our Definitions:

We start with an operation 0_α on the Γ-ring M, which is defined as follows:

$$a\, 0_\alpha\, b = a + b - a \alpha b$$

for every $a, b \in M$ and $\alpha \in \Gamma$.

It is obvious that the operation is closed. Further we have the following:

$$(a\ 0_\alpha\ b)\ 0_\alpha\ c = (a + b - a\alpha b)\ 0_\alpha\ c$$
$$= a + b - a\alpha b + c - (a + b - a\alpha b)\alpha c$$
$$= a + b - a\alpha b + c - a\alpha c - b\alpha c + a\alpha b\alpha c.$$

Also, we have

$$a\ 0_\alpha\ (b\ 0_\alpha\ c) = a\ 0_\alpha\ (b + c - b\alpha c)$$
$$= a + b + c - b\alpha c - a\alpha(b + c - b\alpha c)$$
$$= a + b + c - b\alpha c - a\alpha b - a\alpha c + a\alpha b\alpha c$$

$$\Longrightarrow (a\ 0_\alpha\ b)\ 0_\alpha\ c = a\ 0_\alpha\ (b\ 0_\alpha\ c).$$

Hence the operation 0_α is associative.

Let $a \in M$ and $\alpha \in \Gamma$, then we have 0 in M such that

$$a\ 0_\alpha\ 0 = a + 0 - a\alpha 0 = a$$

$\Longrightarrow$ 0 is an additive identify for the operation 0_α. Thus the operation 0_α is closed, associative and there exists an identity for 0_α.

<u>Definition:</u> Let $a \in \Gamma$-ring M and if there exists an element b in the Γ-ring M such that $a\ 0_\alpha\ b = 0$ then we say that a is right quasi-regular (r.q.r.) and b is said to be right quasi-inverse (r.q.i.).

<u>Definition:</u> Let $a \in \Gamma$-ring M and if there exists an element c in M such that $c\ 0_\alpha\ a = 0$ then we say that a is left quasi-regular and c is said to be left quasi-inverse.

<u>Definition:</u> An element $a \in M$ is said to be quasi-regular if it is left quasi-regular as well as right quasi-regular.

<u>Definition:</u> An element $d \in M$ is said to quasi-inverse if it is a left as well as a right quasi-inverse.

<u>Example:</u> Let $M = \{ \begin{bmatrix} x & y \\ o & z \end{bmatrix}$: x, y, z are integars}

and $\Gamma = \{ \begin{bmatrix} o & x \\ o & o \end{bmatrix}$: is an integar}.

Let $a = \begin{bmatrix} 2 & 1 \\ o & 1 \end{bmatrix} \in M$, $\alpha = \begin{bmatrix} o & 1 \\ o & o \end{bmatrix} \in \Gamma$. We choose $b = -\begin{bmatrix} 2 & 3 \\ o & 1 \end{bmatrix} \in M$

so that

$$\begin{aligned} a\, 0_{\alpha}\, b &= a + b - a \alpha b \\ &= \begin{bmatrix} 2 & 1 \\ o & 1 \end{bmatrix} + \left(- \begin{bmatrix} 2 & 3 \\ o & 1 \end{bmatrix}\right) - \left\{ - \begin{bmatrix} 2 & 1 \\ o & 1 \end{bmatrix} \begin{bmatrix} o & 1 \\ o & o \end{bmatrix} \begin{bmatrix} 2 & 3 \\ o & 1 \end{bmatrix} \right\} \\ &= \begin{bmatrix} o & -2 \\ o & o \end{bmatrix} + \begin{bmatrix} o & 2 \\ o & o \end{bmatrix} \\ &= \begin{bmatrix} o & o \\ o & o \end{bmatrix}. \end{aligned}$$

Hence $\begin{bmatrix} 2 & 1 \\ o & 1 \end{bmatrix}$ is right quasi-regular and $-\begin{bmatrix} 2 & 3 \\ o & 1 \end{bmatrix}$ is a right quasi-inverse.

3. Some Theorems:

Theorem 1 : The set of right quasi-regular elements in the Γ-ring M is a group under the opeation 0_{α}.

Proof : It has been proved that the operation 0_{α} is closed, associative and also there exists an element $0 \in M$ such that $a\, 0_{\alpha}\, 0 = 0$.

This element is an identity element for the operation 0_{α}. Let a be right quasi-regular. Then there exists an element b in M such that $a\, 0_{\alpha}\, b = 0$. Then we say that be is a right inverse of a and vice-versa. Thus the inverse of each element of the set of right quasi-regular elements exists. Hence the set of all right quasi-regular elements forms a group with respect to the operation 0_{α}.

Theorem 2 : Every nilpotent element of the Γ-ring M is quasi-regular.

Proof : Let a $\varepsilon\Gamma$-ring M and let it be nilpotent.

Then $(a\alpha)^n a = 0$. We set an element $b = -\sum_{k=0}^{n-1} (a\alpha)^k a$, then

$$a \circ_\alpha b = a + b - a\alpha b$$

$$= a + (-\sum_{k=o}^{n-1} (a\alpha)^k a) - a\alpha(-\sum_{k=o}^{n-1} (a\alpha)^k a)$$

$$= -\sum_{k=1}^{n-1} (a\alpha)^k a + \sum_{k=1}^{n-1} (a\alpha)^k a + (a\alpha)^n a$$

$= 0$, Hence a is r.q.r.

Theorem 3 : An element of a Γ-ring M is right quasi-regular if and only if the right ideal $A = \{a\alpha x - x : x \in M\}$ coincides with the Γ-ring M.

Proof: Let $A = \{a\alpha x - x : x \in M\}$ be an ideal in M. Obviously $A \subseteq M$. Let $a \in M$ be r.q.r. Then $a\, 0_\alpha\, y = 0$ for some $y \in M$.

$$\Longrightarrow a + y - a\alpha y = 0$$

$$\Longrightarrow a = a\alpha y - y \in A$$

$$\Longrightarrow a \in A$$

$$\Longrightarrow M \leq A$$

$$\Longrightarrow M = A.$$

To prove the converse, let $A = \Gamma$-ring M, $a \in M \Longrightarrow a \in A$, then for some $y \in M$

$$a = a\alpha y - y \in A$$

$$\Longrightarrow a + y - a\alpha y = 0$$

$\Longrightarrow$ a is right quasi-regular.

Theorem 4 : Let a, b $\in$ Γ-ring M such that $a\alpha b$ is right quasi-regular. Then $b\alpha a$ is also right quasi-regular.

Proof: Let $a\alpha b$ be right quasi-regular. Then there exists an element c in the Γ-ring M such that

$(a\alpha b)0_\alpha\; c = 0$ i.e. $a\alpha b + c - a\alpha b\alpha c = 0$

Now $(b\alpha a)\; 0_\alpha\; (-b\alpha a + (b\alpha c)\alpha a)$

$$= (b\alpha a) - b\alpha a + (b\alpha c)\alpha a - [(b\alpha a)\alpha(-b\alpha a+(b\alpha c)\alpha a)]$$

$$= b\alpha a - b\alpha a + b\alpha c\alpha a + (b\alpha a)\alpha(b\alpha a)$$

$$- (b\alpha a)\alpha(b\alpha c)\alpha a$$

$$= b\alpha[c\alpha a + a\alpha b\alpha a - a\alpha(b\alpha c)\alpha a]$$

$$= b\alpha[c + a\alpha b - a\alpha(b\alpha c)]\alpha a$$

$$= b\alpha[a\alpha b \; 0_\alpha \; c]\alpha a$$

$$= 0.$$

Thus b is also r.q.r.

<u>Theorem 5</u> : Let M be a commutative Γ-ring, and A, a right quasi-regular ideal. Then A is quasi-regular.

Proof : Let an ideal A of the Γ-ring M be a right quasi-regular ideal, then for $a \in A$ we have an ideal $A' = \{a\alpha x - x : x \in M\}$ which coincides with the Γ-ring M (Th. 3).

i.e. A' = Γ-ring M.

As $a \in A$, there exists $b \in$ Γ-ring M such that $a\, 0_\alpha\, b = 0$, and $b \in A'$ ($\because A' = M$)

$\Longrightarrow b = a\alpha b - a$

$\Longrightarrow b + a - b\alpha a = 0 \quad \because a\alpha b = b\alpha a$

$\Longrightarrow b\, 0_\alpha\, a = 0$

$\Longrightarrow$ a is left quasi-regular.

Thus A is left quasi-regular.

<u>Definision</u> : Let M be a Γ-ring, then the Jacobson Radical $J_\alpha(M)$ of the Γ-ring M is defined as

$$J_\alpha(M) = \{a : a\alpha M \text{ is r.q.r. for } a \in M\}.$$

<u>Theorem 6</u> : Let M be a commutative Γ-ring, then $J_\alpha(M)$ is a quasi-regular ideal.

Proof: Firstly we prove that $J_\alpha(M)$ is an ideal in the Γ-ring M. If $a \in J_\alpha(M)$ so that $a\alpha M$ is r.q.r., then for each $x \in M$ we have $(a\alpha x)\alpha M \subseteq a\alpha M$. So $(a\alpha x)\alpha M$ is right quasi-regular and $a\alpha x \in J_\alpha(M)$. For each $x, y \in \Gamma$-ring M, it follows that $a\alpha y\alpha x$ is r.q.r. implying $x\alpha a\alpha y$ is r.q.r. (Th. 4).

Hence $(x\alpha a)\alpha M$ is r.q.r. as $(a\alpha x)\alpha M$ is right quasi-regular.

Let a' be right quasi-inverse of $a\alpha x$ implying $(a\alpha x)\, 0_\alpha\, a' = 0$. As $b \in J_\alpha(M)$, $[b\alpha(-x + x\alpha a)]$ is r.q.r. and hence there exists an element w' in M such that $[b\alpha(-x + x\alpha a')]0_\alpha w' = 0$.

Now consider

$$[(a - b)\alpha x]\, 0_\alpha\, (a'\, 0\, w') = ([(a - b)\alpha x]\, 0_\alpha\, a')\, 0_\alpha w'$$

$$= [(a - b)\alpha x + a' - (a - b)\alpha x\alpha a']\, 0_\alpha\, w'$$

$$= [a\alpha x - b\alpha x + a' - a\alpha x\alpha a' + b\alpha x\alpha a']\, 0_\alpha\, w'$$

$$= [a\alpha x + a' - a\alpha x\alpha a' - b\alpha x + b\alpha x\alpha a']\, 0_\alpha\, w'$$

$$= [- b\alpha x + b\alpha x\alpha a']\, 0_\alpha\, w'$$

$$= b\alpha(- x + x\alpha a')\, 0_\alpha\, w'$$

$$= 0.$$

Thus $(a - b)\alpha M$ is r.q.r. and $(a - b) \in J_\alpha(M)$.

Therefore $J_\alpha(M)$ is an ideal.

If $a \in J_\alpha(M)$, then $a\alpha M$ is r.q.r. in particular $a\alpha a$ is r.q.r.

Then for some c in the Γ-ring M, $(a\alpha a)\, 0_\alpha\, c = 0$.

Let

$$a\, 0_\alpha\, [- a\, 0_\alpha\, c] = [a\, 0_\alpha\, (-a)]\, 0_\alpha\, c$$

$$= [a-a-a\alpha(-a)]\, 0_\alpha\, c$$

$$= (a\alpha a)\, 0_\alpha\, c$$

$$= 0.$$

Hence $-a\, 0\, c$ is r.q.r. and a is r.q.r., so that $J_\alpha(M)$ is a quasi-regular ideal.

Theorem 7 : If J_Γ (M) is the jacobson radical of the Γ-ring M, then J_Γ (M / J_Γ (M) = 0.

Proof : Let a + J_Γ (M) ε J_Γ (M / J_Γ (M). Then for each $x \varepsilon M$, $(a + J_\Gamma (M)) \alpha (x + J_\Gamma (M)$ is right quasi-regular in M/J_Γ (M). Thus there is an element $s + J_\Gamma$ (M) of M/J_Γ (M) such that

$$[(a + J_\Gamma (M)) \alpha (x + J_\Gamma (M))] 0_\alpha [s + J_\Gamma (M)] = J_\Gamma (M)$$

$$\Longrightarrow [a \alpha x + J_\Gamma (M)] 0_\alpha [s + J_\Gamma (M)] = J_\Gamma (M)$$

$$\Longrightarrow a \alpha x 0_\alpha s \in J_\Gamma (M).$$

Therefore, there exists y in M sich that $[a \alpha x 0_\alpha s] 0_\alpha y = 0$

$\Longrightarrow$ $a\alpha x$ has $s\, 0_\alpha\, y$ as r.q.i. in the Γ-ring

$\Longrightarrow$ $a\alpha M$ is r.q.r. and hence $a \in J_\Gamma (M)$.

This shows that a + J_Γ (M) is the zero of the Γ-ring M.

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