THE NEPALI MATHEMATICAL SCIENCES REPORT

INSTITUTE OF SCIENCE
DEAN'S OFFICE
TRIBHUVAN UNIVERSITY
KIRTIPUR
NEPAL

VOLUME 2 NO. 2

JULY 1977

THE NEPALI

MATHEMATICAL SCIENCES REPORT

INSTITUTE OF SCIENCE
DEAN'S OFFICE

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KIRTIPUR

NEPAL

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On a Certain Class of Groups

Nep. Math. Sc. Rep.
1977, Vol.2, No. 2.

by D.R. Bajracharya

Let $\mathbb{K}$ be the class of all groups G such that every fully invariant sub-semigroup of G is a fully invariant sub-group, in other words, for an arbitrary element $x \in G$ there are endomorphisms $\delta_1, \delta_2, \ldots, \delta_n$ such that

$$x^{-1} = \delta_1(x)\ \delta_2(x) \ldots \delta_n(x).$$

It is evident that finite groups and Abelian groups are in $\mathbb{K}$. But it is not known whether the free group belongs to $\mathbb{K}$ (see Problem 1, [2]). In this note we shall prove the following:

Theorem: Any relatively free nilpotent group G of class ≤ 3 belongs to $\mathbb{K}$.

It is better to note, first of all, the following useful relations which can easily be deduced. All a's in the following relations belong to a nilpotent group of class 3.

1. $[[a_i,a_j],a_k]^{-1} = [a_k,[a_i,a_j]]$
2. $[[a_i,a_j],a_k^{-1}] = [[a_i,a_j],a_k]^{-1}$, $[[a_i,a_j]^{-1},a_k] = [[a_i,a_j],a_k]^{-1}$
3. $[[a_i,a_j],a_k a_l] = [[a_i,a_j],a_k] . [[a_i,a_j],al]$

   $[[a_i,a_j]\ [a_k,a_l],a_m] = [[a_i,a_j],a_m] . [[a_k,a_l],a_m]$
4. $[a_i,a_j]\ [a_k,a_l] = [a_k,a_l]\ [a_i,a_j]$
5. $[a_i^{-1},a_j^{-1}] = [a_i,a_j]\ [[a_i,a_j],a_i a_j]^{-1}$
6. $[[a_i^{-1},a_j^{-1}],a_k] = [[a_i,a_j],a_k]$
7. $[[a_i^{-1},a_j^{-1}],a_k^{-1}] = [[a_i,a_j],a_k]^{-1}$

Proof of the Theorem. Let G be a relatively free nilpotent group of class ≤ 3. (By a relatively free group, we mean a group which possesses a generating set such that every mapping of this set into the group can be extended to an endomorphism.)

(i) Let us first consider an element $x \in G$ which is of the form

(A) $$x = \prod_{i=1} a_{\mu_i}^{\alpha i} \cdot \prod_{1 \le j < i \le n} [a_{\mu_i}, a_{\mu_j}]^{\beta i,j}$$

where $a_{\mu_r} \in \{a_1, a_2, a_3, \ldots\}$, a set of relatively free generators of G.

Let ψ_i be the endomorphism of G such that

$$\psi_i(a_{\mu_i}) = a_{\mu_i}^{-1}, \qquad \psi_i(a_{\mu_j}) = 1 \text{ for } j \ne i.$$

Then $\psi_i(x) = a_{\mu_i}^{-\alpha_i}$, $(i = 1, 2, 3, \ldots\ldots)$.

Therefore, $\psi_n(x) \ldots \psi_1(x) . x = \prod_{1 \le j < i \le n} [a_{\mu_i}, a_{\mu_j}]^{\beta i,j}$

Now choose a factor $a_{\mu_k}^{\alpha_k}$ from (A) such that α_k is positive. Then

$$\{\psi_n(x) \ldots \psi_1(x) . x\}^{\alpha_k} = \prod_{1 \le j < i \le n} [a_{\mu_i}, a_{\mu_j}]^{\alpha_k \cdot \beta i,j}$$

Let $\delta_{i,j}$ be an endomorphism of G such that

$$\delta_{i,j}(a_{\mu_p}) = [a_{\mu_i}, a_{\mu_j}]^{-\beta i,j} \text{ for } p = k,$$
$$= 1 \qquad \text{for } p = k,$$

Then $\delta_{i,j}(x) = [a_{\mu_i}, a_{\mu_j}]^{-\alpha_k \beta i,j}$.

Therefore $\{\psi_n(x) \ldots \psi_1(x) . x\}^{\alpha_k} \prod_{1 \le j < i \le n} \delta_{i,j}(x) = 1$,
from which we can get the inverse of x.

In case all powers of a_{μ_r}'s in (A) are negative, choose any $a_{\mu_k}^{\alpha_k}$.
So $-\alpha_k$ is positive. Then proceed accordingly.

If all powers of a_{μ_r}'s are zero, i.e., if $x = \prod_{1 \le j < i \le n} [a_{\mu_i}, a_{\mu_j}]^{\beta i,j}$
then choose the endomorphism $\phi_{i,j}$ of G such that

(B) $$\phi_{i,j}(a_{\mu_i}) = a_{\mu_j}, \quad \phi_{i,j}(a_{\mu_j}) = a_{\mu_i}, \quad \phi_{i,j}(a_{\mu_k}) = 1 \text{ for } k \ne i \text{ or } j.$$

Then $\Phi_{i,j}(x) = [a_{\mu_i}, a_{\mu_j}]^{-\beta_{i,j}}$.

Therefore, $\prod_{1 \le j < i \le n} \Phi_{i,j}(x) = x^{-1}$.

(ii) Now consider an element $x \in G$ which is of the form

$$x = \prod_{i=1}^{n} a_{\mu_i}^{\alpha_i} . \prod_{1 \le j < i \le n} [a_{\mu_i}, a_{\mu_j}]^{\beta_{i,j}} . \prod_{\substack{1 \le j < i \le n \\ 1 \le k \le n}} [[a_{\mu_i}, a_{\mu_j}], a_{\mu_k}]^{\lambda_{i,j,k}}$$

Proceeding as in (i), we get

$$\Delta(x) = \{\Psi_n(x) \ldots \Psi_1(x).x\}^{\alpha_k} . \prod_{1 \le j < i \le n} \delta_{i,j}(x) = \prod_{\substack{1 \le j < i \le n \\ 1 \le k \le n}} [[a_{\mu_i}, a_{\mu_j}], a_{\mu_k}]$$

Now let w be the endomorphism of G such that

(C) $\quad w(a_{\mu_i}) = a_{\mu_i}^{-1} \quad$ for $i = 1, 2, \ldots\ldots$

Then $\quad w(\Delta(x)) = \prod_{\substack{1 \le j < i \le n \\ 1 \le k \le n}} [[a_{\mu_i}, a_{\mu_j}], a_{\mu_k}]^{-\gamma_{i,j,k}}$

Therefore, $\quad \Delta(x).w(\Delta(x)) = 1$,

from which we can get the inverse of x.

If

(D) $\quad x = \prod_{1 \le j < i \le n} [a_{\mu_i}, a_{\mu_j}]^{\beta_{i,j}} . \prod_{\substack{1 \le j < i \le n \\ 1 \le k \le n}} [[a_{\mu_i}, a_{\mu_j}], a_{\mu_k}]^{\lambda_{i,j,k}}$

then

$$x . \prod_{1 \le j < i \le n} \Phi_{i,j}(x) = \prod_{\substack{1 \le j \le i \le n \\ 1 \le k \le n}} [[a_{\mu_i}, a_{\mu_j}], a_{\mu_k}]^{\sigma_{i,j,k}}$$

where $\Phi_{i,j}$ is the endomorphism as defined in (B).

Now choose a factor $[a_{\mu_p}, a_{\mu_q}]^{\beta_{p,q}}$ from (D) such that $\beta_{p,q}$ is positive.

Then

(E) $\quad \{x . \prod_{1 \le j < i \le n} \Phi_{i,j}(x)\}^{\beta_{p,q}} = \prod_{\substack{1 \le j < i \le n \\ 1 \le k \le n}} [[a_{\mu_i}, a_{\mu_j}], a_{\mu_k}]^{\beta_{p,q}\sigma_{i,j,k}}$.

Choose the endomorphism $\vartheta_{i,j,k}$ of G such that

$$\vartheta_{i,j,k}(a_{\mu_p}) = [a_{\mu_i}, a_{\mu_j}]$$

$$\vartheta_{i,j,k}(a_{\mu_q}) = a_{\mu_k}^{-\sigma_{i,j,k}}$$

$$\vartheta_{i,j,k}(a_{\mu_r}) = 1 \quad \text{if } r \neq p \text{ or } q$$

Then $\vartheta_{i,j,k}(x) = [[a_{\mu_i}, a_{\mu_j}], a_{\mu_k}^{-\sigma_{i,j,k}}]^{\beta_{p,q}}$

$$= [[a_{\mu_i}, a_{\mu_j}], a_{\mu_k}]^{-\beta_{p,q}\sigma_{i,j,k}}$$

Therefore,

$$\text{(F)} \quad \prod_{\substack{1 \le j < i \le n \\ 1 \le k \le n}} \vartheta_{i,j,k}(x) = \prod_{\substack{1 \le j < i \le n \\ 1 \le k \le n}} [[a_{\mu_i}, a_{\mu_j}], a_{\mu_k}]^{-\beta_{p,q} \cdot \sigma_{i,j,k}}$$

Now from the result of multiplication of (E) and (F), which equals 1, we can get the inverse of x.

The other forms of x present no difficulties.||

Remark: There are also some solvable groups which belong to $\mathbb{K}$. One example is

$G = \{(a,b) \colon a \neq 0,\ b \in R$, the set of all rational numbers $\}$, where the multiplication is defined by

$$(a,b)\ (a',b') = (aa', ab'+b).$$

If the endomorphism Ψ_k and Φ are such that

$$\Psi_k(a,b) = (a,kb) \quad \text{and} \quad \Phi(a,b) = (a^{-1},0)$$

then $\Psi_{-a^{-1}}(a,b)\{\Phi(a,b)\}^2$ is the inverse of (a,b).

I express my deep gratitude to Dr. E. Plonka, Institute of Mathematics of the Polish Academy of Science, Wroclaw, for the suggestion of the problem and his valuable instruction.

References

1. M. Hall: The Theory of Groups, Macmillan, New York, 1959.

2. Hanna Neumann: Varieties of Groups, Springer-Verlag, 1967.

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Polynomials Generated by Exponential Differential Operators

Nep. Math. Sc. Rep.
1977, Vol.2, No. 2.

by R.M. Shrestha

1. Introduction

A number of formulae involving differential operators were furnished by Carlitz [1] quite recently. Two of them are

$$(DxD)^n = \sum_{s=0}^{n} \binom{n}{s} \frac{n!}{s!} x^s D^{n+s} \tag{1.1}$$

and

$$(D(xD)^m)^n = (xD+1)^m (xD+2)^m \ldots \ldots (xD+n)^m D^n \tag{1.2}$$

where $D = d/dx$.

The object of this note is to present, without the details of proofs, some properties of a set of polynomials generated by some exponential differential operators connected with the above operators. Our operators are in some sense generalisations of the exponential differential operator.

$$e^{-D^2}$$

used to define the Hermite polynomials by the relation

$$H_n(\tfrac{1}{2}x) = e^{-D^2} x^n. \tag{1.3}$$

Consider the differential operator defined by the relation

$$\exp[-(D(xD)^m)^p] = \sum_{n=0}^{\infty} \frac{[-D(xD)^m]^{pn}}{n!}, \tag{1.4}$$

where m and p are non-negative integers. When m = 0, it reduces to the exponential differential operator

$$\exp(-D^p)$$

used to define the Gould-Hopper polynomials [2].

2. The Polynomials $H_n(x;m)$

In this section, we restrict our discussion to the special case $p = 2$ in (1.4), and define a set of polynomials $H_n(x:m)$ by the formula

$$H_n(x:m) = e^{-(D(xD)^m)^2}\, x^n . \qquad (2.1)$$

A simple and direct computation yields the following closed form

$$H_n(x:m) = \sum_{k=o}^{[\frac{1}{2}n]} \frac{(-1)^k\,(n!)^{m+1}}{k!\,((n-2k)!)^{m+1}}\, x^{n-2k}, \qquad (2.2)$$

which is obviously a polynomial of degree in in x. It is easy to see that $H_n(x:m)$ is an even function of x if n is even and an odd function of x if is odd. This polynomial is clearly a direct generalisation of the Hermite polynomial defined by (1.3) to which it reduces when $m = o$.

Starting from (2.2), it is easy to arrive at the generating relation

$$\sum_{n=o}^{\infty} \frac{H_n(x:m)}{(n!)^{m+1}}\, t^n = e^{-t^2}\, {}_oF_m\,(-\,;\,1,1,\ldots,1;xt) \qquad (2.3)$$

which reduces to the well-known generating relation

$$e^{2xt\,-\,t^2} = \sum_{n=o}^{\infty} \frac{H_n(x)}{n!}\, t^n , \qquad (2.4)$$

if we put $m = o$ and replace x by 2x in (2.3).

We now give a hypergeometric representation obtainable as a direct consequence of the explicit representation (2.2). It is as follows:

$$H_n(x:m) = x^n\ {}_{2(m+1)}F_o\,(-\tfrac{1}{2}n,\ldots,-\tfrac{1}{2}n,-\tfrac{1}{2}(n+1),\ldots-\tfrac{1}{2}(n+1);\; -\,;\,-1/x^2) \qquad (2.5)$$

which reduces to the known hypergeometric form for the Hermite polynomials after usual reduction.

3. Generalised Gould-Hopper Polynomials

Application of the differential operator

$$\exp\left[-\left(D(xD)^m\right)^p\right]$$

to x^n and simple computation yield the following general polynomial

$$\begin{aligned} H_n(x:m:p) &= e^{-\left(D(xD)^m\right)^p}\; x^n \\ &= \sum_{k=o}^{[n/p]} \frac{(-1)^k\ (n!)^{m+1}}{((n-pk)!)^{m+1}}\, x^{n-pk}, \end{aligned} \tag{3.1}$$

which is a polynomial of precisely degree n in x. This set is obviously a generalisation of Gould-Hopper's generalisation [2] of the Hermite polynomial [3]. An immediate consequence of the definition (2.2) is the following interesting property

$$\left(D(xD)^m\right) H_n(x:m) = n^{m+1}\ H_{n-1}(x:m), \tag{3.2}$$

which corresponds to the property

$$D\ H_n(x) = n\ H_{n-1}(x),$$

the characteristic property of an Appell set. The corresponding property for the polynomial set defined by (3.1) is

$$\left(D(xD)^m\right) H_n(x:m:p) = n^{m+1} H_{n-1}(x:m:p). \tag{3.3}$$

Since this property corresponds to the property satisfied by an Appell set, we may call this polynomial set a generalised Appell set. Repeated applications of the differential operator

$$D(xD)^m$$

to the polynomial $H_n(x:m:p)$ yield the following general result

$$\left\{D(xD)^m\right\}^j H_n(x:m:p) = (j!)^{m+1} \binom{n}{j}^{m+1} H_{n-j}(x:m:p). \qquad (3.4)$$

The polynomial $H_n(x:m:p)$ being a direct generalisation of the Gould-Hopper polynomial it is believed that operational formulae analoguous to those obtained by Gould and Hopper may be developed for this set also.

References

1. Carlitz, L. : 1976: A theorem on differential operators, the Amer. Math. Monthly, 83(5), pp. 351-354.

2. Gould, H.W. and: Hopper, A.T. 1962: Operational formulas connected with two generalisations of Hermite polynomials, Duke Math. J. 29, pp. 51-63.

3. Rainville, E.D.: 1960: Special Functions, Macmillan and Co., N.Y.

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Some Generalisations of Stieltjes Transform

Nep. Math. Sc. Rep.
1977, Vol. 2, No.2.

S.R. Pant

1. Introduction

If

$$(1.1) \qquad f(s) = \int_0^\infty e^{-sx}\, \phi(x)\, dx \qquad (s > 0)$$

where

$$(1.2) \qquad \phi(x) = \int_0^\infty e^{-xt}\, \alpha(t)\, dt,$$

then we have

$$(1.3) \qquad f(s) = \int_0^\infty \frac{\alpha(t)}{s+t}\, dt.$$

Thus, the equation (1.3) which is known as the Stieltjes transform is obtained by the first iteration of the well-known Laplace transform (1.1). [7]. In this paper, we shall obtain some generalisations of (1.3) with the help of the following generalisations of (1.1):

$$(1.4) \qquad \int_0^\infty e^{-\frac{1}{2}sx}\, (sx)^{q-\frac{1}{2}}\, W_{p,q}(sx)\, \phi(x)\, dx,$$

$$(1.5) \qquad \int_0^\infty e^{-\lambda sx}\, (sx)^{-m-\frac{1}{2}}\, M_{k,m}(sx)\, \phi(x)\, dx. \qquad (\lambda > \tfrac{1}{2})$$

The generalisation (1.4) was given by Varma [6]. The second was studied by the author [4]. This transform, known as $M_{k,m}$- transform reduces to (1.1) on putting $\lambda = 3/2$ and $-k = m + \frac{1}{2}$. When $\lambda = \frac{1}{2}$, (1.5) is still true for $| t^{k-m-\frac{1}{2}}\, \phi(t) | \in L$

All parameters will be assumed to be real unless otherwise stated.

2. Some Generalisations

In this section, we obtain some generalisations of the Stieltjes transforms by considering the Laplace transform (1.1) and its generalisation (1.5).

Theorem 1. If

$$(2.1) \qquad f(s) = \int_0^\infty e^{-sx}\ \phi(x)\ dx \qquad (s > 0)$$

where

$$(2.2) \qquad \phi(x) = \int_0^\infty e^{-\lambda xt}\ (xt)^{-m-\frac{1}{2}} M_{k,m}(xt)\ \alpha(t)\ dt,$$

then

$$(2.3) \qquad f(s) = \int_0^\infty {}_2F_1\left[\tfrac{1}{2}+m-k, 1; 1+2m; \frac{t}{(\lambda+\frac{1}{2})t+s}\right]\left\{(\lambda+\tfrac{1}{2})t+s\right\}^{-1} \alpha(t)\ dt$$

provided that (i) 2m is not a negative integer

(ii) $\alpha(t) = O(t^{\mu})$ for small t, $\mu > 0$

$= O(e^{-\nu t})$ for large t, $\nu > 0$.

Proof. Substitution of (2.2) in (2.1) and simplification gives

$$f(s) = \int_0^\infty e^{-sx} \int_0^\infty e^{-\lambda xt}\ (xt)^{-m-\frac{1}{2}} M_{k,m}(xt)\ \alpha(t)\ dt\ dx$$

$$= \int_0^\infty \frac{\alpha(t)}{t^{m+\frac{1}{2}}} \left\{\int_0^\infty e^{-(s+\lambda t)x}\ x^{-m-\frac{1}{2}} M_{k,m}(xt)\ dx\right\} dt$$

$$= \int_0^\infty [(\lambda+\tfrac{1}{2})t+s]^{-1}\ {}_2F_1\ [\tfrac{1}{2}+m-k, 1;\ 1+2m;\ \frac{t}{(\lambda+\frac{1}{2})t+s}]\ \alpha(t)\ dt,$$

by using a result of Erdelyi [3, pp. 215]. The change of order of integration performed above is valid due to the absolute convergence of the t-integral resulting from the bounded behaviour of $\alpha(t)$ for large and small t, and for $\lambda > -\frac{1}{2}$, the x-integral is absolutely convergent when

$s > 0$. Also the resulting integral (2.3) is absolutely convergent for the same condition on $\alpha(t)$ and the asymptotic behaviour of ${}_2F_1$ [1].

Corollary. The integral (2.3) reduces to (1.3) when $-k = \frac{1}{2}+m$ and $\lambda = 3/2$, for the ${}_2F_1$ involved degenerates into the binomial rational expression

$$[(\lambda + \tfrac{1}{2})\, t + s][(\lambda - \tfrac{1}{2})\, t + s]^{-1}.$$

Theorem 2. If

$$(2.4) \qquad f(s) = \int_0^\infty e^{-\lambda sx}\,(sx)^{-m-\frac{1}{2}}\, M_{k,m}(sx)\, \phi(x)\, dx, \qquad (s > 0)$$

where

$$(2.5) \qquad \phi(x) = \int_0^\infty e^{-xt}\,\alpha(t)\, dt,$$

then

$$(2.6) \qquad f(s) = \int_0^\infty [(\lambda + \tfrac{1}{2})s + t]^{-1}\; {}_2F_1\left[\tfrac{1}{2}+m-k, 1;\, 1+2m;\, \frac{s}{(\lambda+\frac{1}{2})s+t}\right] \alpha(t)\,dt,$$

provided that

(i) 2m is not an integer

(ii) $\alpha(t) = o(t^{\mu}),\ t \to 0,\ \mu > 0$

$\qquad = O(e^{-\nu t}),\ t \to \infty, \nu > 0.$

Proof. The proof of this theorem is similar to that of the previous theorem.

3. In this section, we obtain the generalisation of (1 3) by the iteration of the $W_{k,m}$- and $M_{k,m}$ - transforms.

Theorem 3. If

$$(3.1) \qquad f(s) = \int_0^\infty e^{-\frac{1}{2}sx}\,(sx)^{q-\frac{1}{2}}\, W_{p,q}(sx)\;\phi(x)\, dx,$$

where

$$(3.2) \qquad \phi(x) = \int_0^\infty e^{-\lambda xt} (xt)^{-m-\frac{1}{2}} M_{k,m}(xt)\, \alpha(t)\, dt,$$

then

$$(3.3) \qquad f(s) = \frac{1}{s} \int_0^\infty \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{\Gamma(-2q)}{\Gamma(\frac{1}{2}-q-p)} \cdot \frac{(\frac{1}{2}+q-p)_n}{(1+2q)_n} \Gamma(1+2q+n) \left(1+(\lambda+\tfrac{1}{2})\frac{t}{s}\right)^{-1} \right.$$

$$ {}_2F_1 \left[\begin{matrix} \frac{1}{2}+m-k, 1+n+2q; \\ 1+2m; \end{matrix} \frac{t}{s+(\lambda+\frac{1}{2})t} \right] + \frac{\Gamma(2q)}{\Gamma(\frac{1}{2}+q-p)} \cdot \frac{(\frac{1}{2}-q-p)_n}{(1-2q)_n} \Gamma(n+1) \quad \times$$

$$\left\{1+(\lambda+\tfrac{1}{2})\frac{t}{s}\right\}^{-n-1} {}_2F_1 \left[\begin{matrix} \frac{1}{2}+m-k, 1+n; \\ 1+2m; \end{matrix} \frac{t}{s+(\lambda+\frac{1}{2})t} \right] \alpha(t)\, dt,$$

provided that

(i) $2m$ is not a negative integer

(ii) $q + \frac{1}{2} > 0$, $s > 0$

(iii) $\alpha(t) = o(t^{\mu})$, $t \to 0$, $\mu > 0$

$\qquad = O(e^{-\nu t})$, $t \to \infty$, $\nu > 0$.

Proof. To obtain the required result, use (3.2) in (3.1), change the order of integration, and then integrate term by term the result obtained by expressing the $W_{p,q}$ - function as the sum of two infinite series. [5, pp. 14]. It is easy to see that the change of order of integration and that of the summation and integration involved are justified under the given conditions since

$$\begin{aligned} W_{p,q}(x) &= o(x^{\pm q+\frac{1}{2}}) \text{ for small } x \\ &= O(e^{-\frac{1}{2}x} x^{p}) \text{ for large } x\ ; \end{aligned}$$

and

$$\begin{aligned} M_{k,m}(x) &= o(x^{m+\frac{1}{2}}) \text{ for small } x \\ &= O(e^{\frac{1}{2}x} x^{-k}) \text{ for large } x. \end{aligned}$$

Corollary 1. If we put $p+q = \frac{1}{2}$, we get

$$f(s) = \frac{1}{s}\int_0^\infty [1 + (\lambda+\tfrac{1}{2})\tfrac{t}{s}]^{-1} \; {}_2F_1\left[\begin{matrix}\tfrac{1}{2} + m - k, \; 1; \\ 1 + 2m;\end{matrix} \; \frac{t}{s + (\lambda+\frac{1}{2})t}\right] \alpha(t)\, dt,$$

a resu t already obtained in section 2.

Corollary 2. If we put $\lambda = 3/2$ and $-k = m+\tfrac{1}{2}$, in (3.3), we obtain after simplification, the following generalisation of the Stieltjes transform given by Varma [6]:

$$f(s) = \frac{\Gamma(1 + 2q)}{\Gamma(3/2 + q - p)}\,\frac{1}{s}\int_0^\infty {}_2F_1\left[\begin{matrix}2q + 1, \; 1 \; ; \\ 3/2 + q - p;\end{matrix} \; -t/s\right] \alpha(t)\, dt.$$

We may note here that a result similar to theorem (3.1) by interchanging the roles of the generalised Laplace transforms involved in this theorem.

References

1. Bromwich, T.J. : Introduction to the Theory of Infinite Series (1965), Macmillan.

2. Erdelyi, A : Higher Transcendental Functions, Vol. I. (1953), McGraw Hill, Book Co.

3. ----------- : Tables of Integral Transforms. Vol. I (1954), McGraw Hill, Book Co.

4. Pant S.R. : Study of a Generalised Laplace Integral and its Cesaro Mean, Ph.D. Thesis (1967), Ranchi University.

5. Slater, L.J. : Confluent Hypergeometric Functions (1960) Camb. Univ. Press.

6. Varma, R.S. : On a Generalised Laplace Integral, Proc. Nat. Acd. Sciences (India) Sect A. Vol.20 (1951).

7. Widder, D.V. : The Laplace Transform, (1963), Princeton.

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Sequential Life Testing on Exponential Distribution With Regular Interval Inspections

Nep. Math. Sc. Rep.
1977, Vol. 2, No.2.

by B.S. Rajbanshi

1. Introduction

A large literature exists on Life Testing, the length of life being supposed to follow the exponential distribution or the Weibull distribution [3], [4], [5]. Epstein and Sobel have made specific studies for the exponential distribution which include sequential distribution also. It may be pointed out that in their analysis, they have considered a continuous inspection plan [4] for n items taken at random from the population whose failure density function is given by

$$g(x,\theta) = \frac{1}{\theta} e^{-x/\theta}, \quad x > 0, \quad \theta > 0, \tag{1.1}$$

where θ may be interpreted as the average life time of the material under investigation, e.g. electric bulbs, radio tubes etc; because it is easily seen that

$$\theta = \int_0^\infty x g(x,\theta)\, dx. \tag{1.2}$$

In practice a continuous inspection plan is seldom enforced. In general, inspection takes place in regular intervals, say in every 4 or 6 hours or every day or every week at a fixed time. In the present paper, we intend to study the effects of the regular interval inspection plan for life testing on an exponential population given by (1.1). The test results in destruction of items (at least of those which fail).

Suppose that inspection takes place at the end of successive intervals whose duration is t each. And also suppose that x_i failures are observed at the ith inspection ($i = 1,2,\ldots,m$). The total number of units on test is n and at each inspection, the defective units are replaced by good ones. It is assumed that the ageing of the units while testing is negligible; in other words, the surviving old units are supposed to be as good as the new units.

2. The Sequential Probability Ratio

We adopt a sequential procedure for testing a simple hypothesis $H_0 : \theta = \theta_0$ against the alternative $H_1 : \theta = \theta_1$ ($\theta_1 < \theta_0$). It is given that the errors of the first kind and the second kind should not exceed small pre-assigned values α and β respectively.

If x_i failures occur at the ith inspection, the probability of x_i failures and $n - x_i$ survivals

$$= \binom{n}{x_i} p_k^{\,x_i} (1 - p_k)^{n - x_i} \quad \text{(k = 0, 1 corresponding to } H_o \text{ and } H_1 \text{ respectively)}$$

where

$$(2.1) \quad p_k = \frac{1}{\theta_k} \int_0^t e^{-x/\theta_k} \, dx = 1 - e^{-t/\theta_k} = 1 e^{-\lambda_k}, \text{ say}$$

and $1 - p_k = e^{-t/\theta_k} = e^{-\lambda_k}$, and θ_k is the true value of the parameter θ.

Let p_{1m} denote the probability of observing the sample $(x_1, x_2, \ldots, x_n)$ x_i failures occurring at the ith inspection, under hypothesis H_1 and p_{om} denote the same probability under hypothesis H_o. Then the sequential probability ratio upto and including the mth inspection is

$$(2.2) \quad \frac{p_{1m}}{p_{om}} = \prod_{i=1}^{m} \left(\frac{p_1}{p_o}\right)^{x_i} \left(\frac{1 - p_1}{1 - p_o}\right)^{n-x_i},$$

or

$$\log \frac{p_{1m}}{p_{om}} = \sum_{i=1}^{m} x_i \log \frac{p_1}{p_o} + (mn - \sum_{i=1}^{m} x_i) \log \frac{1 - p_1}{1 - p_o}.$$

Putting $\sum_{i=1}^{m} x_i = r$, the total number of failures in all the m inspections and substituting for p_k from (2.1), we get

$$(2.3) \quad \log \frac{p_{1m}}{p_{om}} = r \log \frac{1 - e^{\lambda_1}}{1 - e^{\lambda_o}} - mn\,(\lambda_1 - \lambda_o)$$

$$= r \log \Lambda - mn\,(\lambda_1 - \lambda_o), \text{ where } \Lambda = \frac{1 - e^{\lambda_1}}{1 - e^{\lambda_o}}$$

Inspection is continued if $\log B < \log \frac{p_{1m}}{p_{om}} < \log A$, where A and B are numbers determined by α and β.

(2.4) i.e. if $mn(\lambda_1 - \lambda_o) + \log B < r \log \Lambda < \log A + mn(\lambda_1 - \lambda_o)$

(2.5) H_1 is accepted if $r \log \Lambda \geq \log A + mn(\lambda_1 - \lambda_o)$,

(2.6) H_o is accepted if $r \log \Lambda \leq \log B + mn(\lambda_1 - \lambda_o)$.

There relations (2.4), (2.5) and (2.6) may be rewritten as:

Inspection is to be continued

(2.7) if $$\frac{\log B + mn(\lambda_1 - \lambda_o)}{\log \Lambda} < r < \frac{\log A + mn(\lambda_1 - \lambda_o)}{\log \Lambda},$$

(2.8) H_1 is accepted if the number of failures upto the mth inspection,

$$r = r_m \geq \frac{\log A + mn(\lambda_1 - \lambda_o)}{\log \Lambda},$$

Then r_m is called the rejection number.

(2.9) H_o is accepted if the number of failures upto the mth inspection,

$$r = a_m \leq \frac{\log B + mn(\lambda_1 - \lambda_o)}{\log}.$$

Then a_m is called the acceptance number.

As shown by Wald the test obtained by putting $A = (1 - \beta) / \alpha$ and $B = \beta/(1 - \alpha)$ is a satisfactory solution of the problem from the practical point of view, and the sampling plan prescribed by (2.7), (2.8) and (2.9) satisfies the requirements regarding the tolerated risks. The rejection and acceptance domains are characterised by two parallel lines as usual whose slopes and intercepts can be computed from (2.8) and (2.9).

We note that the case under study has a grouped binomial distribution each group consisting of n items under observation.

3. The Operation Characteristic Function

Wald's O.C. function in the parametric form will be used here.

The O.C. function

$$L(\theta) = \frac{A^{h(\theta)} - 1}{A^{h(\theta)} - B^{h(\theta)}}$$

$h(\theta) \neq 0$ is to be found out from the fundamental identity

$$\sum_{x=0}^{n} \left(\frac{f(x,\theta_1)}{f(x,\theta_o)}\right)^h \times f(x,\theta) = 1$$

$$\text{or} \quad \sum_{x=0}^{n} \binom{n}{x} \left[\left(\frac{p_1}{p_o}\right)^x \left(\frac{1-p_1}{1-p_o}\right)^{n-x}\right]^h p^x (1-p)^{n-x} = 1,$$

where $p = 1 - e^{-t/\theta} = 1 - e^{-\lambda}$, $1 - p = e^{-\lambda}$.

After simplification, we get

$$\left[e^{-\lambda - h(\lambda_1 - \lambda_o)} \left\{1 + \Lambda^h (e^{\lambda} - 1)\right\}\right]^n = 1$$

Using only the nth positive root of unity, we get

$$1 + \Lambda^h (e^{\lambda} - 1) = e^{\lambda + h(\lambda_1 - \lambda_o)}. \tag{3.1}$$

Further simplification yields

$$\theta = \frac{t}{\log\left[(1-\Lambda^h) / (e^{h(\lambda_1 - \lambda_o)} - \Lambda^h)\right]} \tag{3.2}$$

and

$$(3.3) \qquad L(\theta) = \frac{A^h - 1}{A^h - B^h} = \frac{\left(\frac{1-\beta}{\alpha}\right)^h - 1}{\left(\frac{1-\beta}{\alpha}\right)^h - \left(\frac{\beta}{1-\alpha}\right)^h} .$$

Relations (3.2) and (3.3) give the O.C. functions in the parametric form. For any arbitrary value of h, the point θ, $L(\theta)$ computed from (3.2) and (3.3) will be a point on the O.C. function. The O.C. curve can be drawn by plotting a sufficiently large number of points $[\theta, L(\theta)]$ corresponding to various values of h. This curve is described approximately in section 5.

4. The A.S.N. Function

Here A.S.N. will mean the average number of inspection (m) required to arrive at a decision. The approximate formula of Wald for A.S.N. E_θ (m) may be used if $E_\theta(z) \neq 0$, under the assumption that θ is the value of unknown parameter (average life-time):

$$(4.1) \qquad E_\theta(m) = \frac{\log A \{1 - L(\theta)\} + \log B\, L(\theta)}{E_\theta(z)}$$

where $z = \log \dfrac{f(x,\theta_1)}{f(x,\theta_o)}$, $\quad f(x,\theta_i) = \binom{n}{x} p_i^x (1 - p_i)^{n-x} \quad (i = o, 1)$

and $E_\theta(z) = \sum_{x=0}^{n} p(x).z$, where p(x) = probability of have x failures.

Thus

$$E_\theta(z) = \sum_{x=0}^{n} \binom{n}{x} p^x (1 - p)^{n-x} \log \frac{p_1^x (1 - p_1)^{n-x}}{p_o^x (1 - p_o)^{n-x}}$$

Substituting for p_1 and p_o and simplifying

$$E_\theta(z) = \sum_{x=0}^{n} \binom{n}{x} p^x (1 - p)^{n-x} \left[x \log \Lambda - n (\lambda_1 - \lambda_o)\right]$$

(4.2) $$= n\left[(1 - e^{-\lambda})\log \Lambda - (\lambda_1 - \lambda_0)\right]$$

Thus (4.1) gives the A.S.N. function as

(4.3) $$E_\theta(m) = \frac{\log \frac{1-\beta}{\alpha}\,[1 - L(\theta)] + L(\theta)\,\log \frac{\beta}{1-\alpha}}{n\,[(1 - e^{-\lambda})\log \Lambda - (\lambda_1 - \lambda_0)]}$$

5. <u>Plotting the O.C. Curve</u>

For the usual five values of $h = \pm\infty$, ± 1, 0, the values of θ and $L(\theta)$ are found out

(i) $h \to \infty$, $\theta \to \infty$, $L(\theta) = 1$,

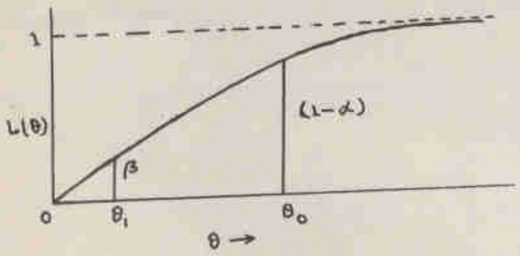
(ii) $h \to -\infty$, $\theta \to 0$, $L(\theta) = 0$,

(iii) $h = 1$, $\theta = \theta_0$, $L(\theta = 1 -$,

(iv) $h = -1$, $\theta = \theta_1$, $L(\theta) = B$,

(v) $h \to 0$, $\theta = -t/\log[1-(\lambda_1-\lambda_0)/\log \Lambda]$, $L(\theta) = \dfrac{\log A}{\log A + |\log B|}$

The O.C. curve will have the form shown below:



6. The Cost Function and Its Mini-max Solution

The cost function $f(n,t,\theta)$ consists of the following components:

(A) Cost of units destroyed in the test = $knp\ E_\theta(m)$, where k = cost of 1 unit,

(B) Cost of the inspection: Three possibilities may arise:

(i) It is $\xi E_\theta(m)$, where ξ = cost of one inspection over all the units,

(ii) If the cost is proportional to the number of items on inspection, it is $\zeta n\ E_\theta(m)$ where ζ = cost of inspection of one item,

(iii) Cost depends only on the number of defective units which alone are inspected and = $\zeta np\ E_\theta(m)$,

These three cases may be considered as exclusive or there may be a superposition of two or all of them.

(C) Since time is money (the sooner the test is finished, the better it is, because the products may then be sent to the market), assuming a linear relationship,

Cost of the time taken = $\mu t\ E_\theta(m)$, where μ = a constant, depending on the time t and average number of inspections $E_\theta(m)$,

(D) The cost of energy used for testing is also to be reckoned. It is approximately $\nu nt\ E_\theta(m)$ where ν = cost of energy consumed by unit item per unit time, neglecting the saving in energy due to the failures in each inspection.

So the cost function $f(n,t,\theta)$ may be set as equal to

(6.1) $E_\theta(m)\ [n(kp + \nu t) + \xi + \mu t]$

or $E_\theta(m)\ [knp + \zeta n + \mu t + \nu nt]$

or $E_\theta(m)\ [knp + \zeta np + \mu t + \nu nt]$

Corresponding to the conditions (i), (ii) or (iii) of (B) above.

The minimax solution is obtained theoretically as follows:

(6.2) $\dfrac{\partial f(n,t,\theta)}{\partial \theta} = 0$ should be solved for θ for f_{max} if it exists.
Then inserting this $\theta_{max} = \hat{\theta}$ (corresponding to f_{max}),

equate

(6.3) $$\frac{\partial f(n,t,\hat{\theta})}{\partial n} = 0$$

(6.4) $$\frac{\partial f(n,t,\hat{\theta})}{\partial t} = 0$$

The equations (6.3) & (6.4) should be solved to give a minimum for $f(n,t,\hat{\theta})$. Theoretically speaking the two equations are enough to give a set of solutions for n and t. But the exact mathematical expression have never been found out owing to complexity of differential coefficient of $L(\theta)$ in the parametric form. A numerical solution may be possible with the help of computers.

Nevertheless, a realistic solution is apparent from (6.3) as follows:

Taking the first form of (6.1) and using (4.3), we get

(6.5) $$\frac{\partial f}{\partial n} = \frac{\partial}{\partial n} \quad E_{\theta}(m) \left\{[n(kp + \nu t) \quad + \xi + \quad \mu t]\right\}$$

$$= -\frac{1}{n} E_{\theta}(m) \quad (\xi + \mu t) < 0,$$

Because all the quantities involved are positive,

$f(n,t,\theta)$ is a monotonically decreasing function of n.

Now we assume that the cost function has a simple linear relationship with n, say, of the form $k_1 n$, where k_1 = cost of the testing machinery per unit item.

Combining with (6.1), we get the new cost function f^*:

(6.7) $$f^* = k_1 n + f(n,t,\theta),$$

and solving
$$\frac{\partial f^*}{\partial n} = 0$$

we get

$$k_1 - \frac{1}{n} E_\theta(m) \ (\xi + \mu t) = 0$$

or

$$n^2 = \frac{(\xi + \mu t)}{k_1} C, \text{ where } C = \frac{L(\theta) \log B/A + \log A}{(1 - e^{-\lambda}) \log \Lambda - (\lambda_1 - \lambda_0)} \qquad (6.8)$$

For given values of t, θ, this gives an optimal solution for n.

References

1. A. Walk: : Sequential Analysis, J. Wiley & Sons, N.Y. (1947).
2. Columbia University SRG Report 255: Sequential Analysis of Statistical Data Application.
3. A.S. Qureishi : Discrimination between two Weibull processes, Technometrics, 21, Feb. 1964.
4. Epstein & Sobel: Sequential life tests in the exponential case; Amer. Math. Stat., 1955,26,1,pp.82-93.
5. ____________ : Life Testing; Journal Amer. Stat. Ass.., 1953, 48, pp.486.
6. M.A. Girshick : Contributions to the Theory of Sequential Analysis I; Amer. Math. Stat. (1946), V 17, p. 123.

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A Class of Polynomials

Nep. Math. Sc. Rep.
1977, Vol. 2, No.2.

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1. Introduction

A generating function for the classical Legendre polynomials $P_n(x)$ is known to have the form

$$(1 - xt)^{-c} \; {}_2F_1\left[\begin{matrix} \tfrac{1}{2}c, \; \tfrac{1}{2}c + \tfrac{1}{2}; \\ 1; \end{matrix} \; \frac{t^2(x^2 - 1)}{(1 - xt)^2}\right] = \sum_{n=0}^{\infty} \frac{(c)_n P_n(x) \, t^n}{n!}, \quad (1.1)$$

in which c may be any complex number. Starting with a generalisation of the hypergeometric function ${}_2F_1$ in the left side of (1.1), Shrestha [3], introduced a new generalised Legendre polynomial $P_n(x{:}m)$ by the relation

$$(1 - xt)^{-c} \; {}_{2m}F_{2m-1}\left[\begin{matrix} c/2m, \; (c+1)/2m, \ldots, (c+2m-1)/2m; \\ 1/m, \; 3m/2, \; 2/m, \ldots, \; (2m-1)/2m, 1; \end{matrix} \; \frac{t^{2m}(x^2-1)^m}{(1 - xt)^{2m}}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(c)_n \, P_n(x{:}m) t^n}{n!}, \quad (1.2)$$

which reduces to (1.1) when m = 1. He has derived some generating functions, explicit representation and hypergeometric form for $P_n(x{:}m)$. In the present and subsequent sections we intend to study various properties of a special case $P_n(x{:}2)$ of $P_n(x{:}m)$ in a greater detail. This set will be denoted by $R_n(x)$. Results deducible from those of $P_n(x{:}m)$ are as follows:

a) Generating functions

$$[(1 - xt)^4 - t^4(x^2 - 1)^2]^{-\frac{1}{4}} = \sum_{n=0}^{\infty} R_n(x) \, t^n, \quad (1.3)$$

$$(1 - xt)^{-1} \; {}_1F_0 \left[\tfrac{1}{4} ; - ; \frac{t^4(x^2 - 1)^2}{(1 - xt)^4} \right] = \sum_{n=0}^{\infty} R_n(x)\, t^n, \quad (1.4)$$

$$e^{xt} \; {}_0F_3 \left[- ; 1/2, 3/4, 1; \; \frac{t^4(x^2 - 1)^2}{16} \right] = \sum_{n=0}^{\infty} \frac{R_n(x)\, t^n}{n!}, \quad (1.5)$$

$$(1 - xt)^{-c} \; {}_4F_3 \left[\begin{matrix} \tfrac{1}{4}c, (c+1)/4, (c+2)/4, (c+3)/4; \\ 1/2, 3/4, 1; \end{matrix} \; \frac{t^4(x^2 - 1)^2}{(1 - xt)^4} \right]$$

$$= \sum_{n=0}^{\infty} \frac{(c)_n\, R_n(x)\, t^n}{n\,!} . \quad (1.6)$$

b) Explicit representation

$$R_n(x) = \sum_{k=0}^{[\frac{1}{4}n]} \frac{(\tfrac{1}{2})_k \; n! \; (x^2 - 1)^{2k} \, x^{n-4k}}{(n - 4k)! \; (4k)! \; k!} . \quad (1.7)$$

c) Hypergeometric form

$$R_n(x) = x^n \; {}_4F_3 \left[\begin{matrix} -\tfrac{1}{4}n, -\tfrac{1}{4}(n-1), -\tfrac{1}{4}(n-2), -\tfrac{1}{4}(n-3); \\ 1/2, \; 3/4, 1; \end{matrix} \; \frac{(x^2-1)^2}{x^4} \right] \quad (1.8)$$

2. Recurrence Relations

In this section, we proceed to establish some differential and pure recurrence relations for the polynomial $R_n(x)$.

Simplifying the generating relation (1.3), we arrive at

$$[1 - 4xt + 6\,x^2t^2 - 4x^3t^3 + (2x^2+1)t^4]^{-\frac{1}{4}} = \sum_{n=0}^{\infty} R_n(x)\; t^n. \quad (2.1)$$

Differentiating both sides with respect to x, and simplifying we get

$$R_n(x) - 3x\, R_{n-1}(x) + 3x^3\, R_{n-2}(x) - x\, R_{n-3}(x)$$

$$= R'_{n+1}(x) - 4x\, R'_n(x) + 6x^2\, R'_{n-2}(x) - 4x^3\, R'_{n-2}(x) + (2x^2 + 1)\, R'_{n-3}(x) \tag{2.2}$$

Again differentiating both sides of (2.1) with respect to t and equating the coefficients, we arrive at the pure recurrence relation

$$(4n-3)x\, R_{n-1}(x) + 3x^2(3-2x)\, R_{n-2}(x) + x^3(4n-9)\, R_{n-3}(x) - (2x^2+1)(n-3)\, R_{n-4}(x)$$
$$= n\, R_n(x). \quad (n \geqslant 4) \tag{2.3}$$

3. <u>Generating Function</u>

Differentiate (2.1) with respect to t, multiply the result by t, and add the product thus obtained to (2.1) to arrive at the following additional generating function for $R_n(x)$:

$$\frac{(1 - xt)^3}{[(1 - xt)^4 - t^4(x^2 - 1)^2]^{5/4}} = \sum_{n=0}^{\infty} (n+1)\, R_n(x) t^n. \tag{3.1}$$

4. <u>Expansions</u>

In this section, some expansions of $R_n(x)$ in series of Legendre, Hermite and Lagurre polynomials will be obtained from those of x^n in series of Legendre, Hermite and Lagurre polynomials [1]. We first quote some results for ready reference

a) $$x^n = \frac{n!}{2^n} \sum_{k=0}^{[\frac{1}{2}n]} \frac{(2n - 4k + 1)\, P_{n-2k}(x)}{k!\, (3/2)_{n-k}}, \tag{4.1}$$

b) $$x^n = n! \sum_{k=0}^{[\frac{1}{2}n]} \frac{H_{n-2k}(x)}{2^n\, s!\, (n - 2s)!} \tag{4.2}$$

c) $$x^n = n! \sum_{k=0}^{n} \frac{(-1)^k\, n!\, (1+\alpha)_n\, L_k^{\alpha}(x)}{(n-k)!\,(1+\alpha)_k} \quad (\alpha > -1) \tag{4.3}$$

Using the results (4.1) - (4.3) in the relation

$$\sum_{n=0}^{\infty} R_n(x)\, t^n = \sum_{n,k=0}^{\infty} \frac{(\tfrac{1}{4})_k\, (n+4k)!\, (x^2-1)^{2k} x^n t^{n+4k}}{n!\,(4k)!\,k!} \tag{4.4}$$

it is easy to obtain

$$R_n(x) = \frac{n!}{2^n} \sum_{s=0}^{[\frac{1}{2}n]} \frac{(2n-4s+1)\, P_{n-2s}}{s!\,(3/2)_{n-s}}\; {}_4F_3\left[\begin{matrix} -\frac{n}{2}, -\frac{n-1}{2}, \frac{2n-2s+1}{4}, \frac{2n-2s-1}{4}; \\ 1/2,\ 3/4, 1; \end{matrix}\ (x^2-1)^2\right],$$

$$R_n(x) = \frac{n!}{2^n} \sum_{s=0}^{[\frac{1}{2}n]} {}_6F_3\left[\begin{matrix} -\frac{n}{4}, -\frac{n-1}{4}, -\frac{n-2}{4}, -\frac{n-3}{4}, -\frac{s}{4}, -\frac{s-1}{2}; \\ 1/2,\ 3/4,\ 1; \end{matrix}\ 64(x^2-1)^2\right] \times$$

$$\times \frac{H_{n-2s}(x)}{s!\,(n-2s)!},$$

and

$$R_n(x) = (1+\alpha)_n \sum_{s=0}^{n} {}_4F_7\left[\begin{matrix} -\tfrac{1}{4}(n-k), -\tfrac{1}{4}(n-k-1), -\tfrac{1}{4}(n-k-2), -\tfrac{1}{4}(n-k-3); \\ -\tfrac{1}{4}(\alpha+n), -\tfrac{1}{4}(\alpha+n-1), -\tfrac{1}{4}(\alpha+n-2), -\tfrac{1}{4}(\alpha+n-3), \tfrac{1}{2}, 3/4, 1; \end{matrix}\ \frac{(x^2-1)^2}{256}\right]$$

$$\cdot \frac{(n-)_s\, L_s^{\alpha}(x)}{(1+\alpha)_s},$$

as the expansions in series of Legendre, Hermite and Laguerre polynomials respectively.

5. Miscellaneous Results Involving Products of Gegenbauer Polynomials

In this section, we give some results for the polynomial $R_n(x)$ involving products of Gegenbauer polynomials $C_n^{\nu}(x)$ defined by the relation

$$(1 - 2xt + t^2)^{-\nu} = \sum_{n=0}^{\infty} C_n^{\nu}(x)\ t^n. \tag{5.1}$$

Let us proceed with the generating relation

$$\left\{(1 - xt)^4 - t^4(x^2 - 1)^2\right\}^{-\frac{1}{4}} = \sum_{n=0}^{\infty} R_n(x)\ t^n.$$

This time we note that the left hand expression of the above relation equals

$$\left\{(1 - xt)^2 - t^2(x^2 - 1)\right\}^{-\frac{1}{4}} \left\{(1 - xt)^2 + t^2(x^2 - 1)\right\}^{-\frac{1}{4}} \tag{5.2}$$

whose first factor may be written as

$$\sum_{n=0}^{\infty} C_n^{\frac{1}{4}}(x)\ t^n.$$

The second factor, after suitable rearrangement, can be similarly expressed. Consequently, we get

$$\sum_{n=0}^{\infty} R_n(x)\ t^n = \sum_{n,k=0}^{\infty} C_n^{\frac{1}{4}}(x)\ t^n \sum_{r=0}^{k} \frac{(\frac{1}{4})_k\ (x)^{k-r}\ (-1)^r (2x-1)^r\ t^{k+r}}{r!\ (k-r)!}$$

from which we easily obtain

$$R_n(x) = \sum_{k=0}^{n} C_{n-k}^{\frac{1}{4}}(x)\ C_k^{\frac{1}{4}}\left(\frac{2x}{(2x^2-1)^{\frac{1}{2}}}\right).\ (2x^2 - 1)^{\frac{1}{2}k} \tag{5.3}$$

$$= \sum_{k=0}^{n} C_k^{\frac{1}{4}}(x)\ C_{n-k}^{\frac{1}{4}}\left(\frac{2x}{(2x^2-1)^{\frac{1}{2}}}\right).\ (2x^2 - 1)^{\frac{1}{2}(n-k)}$$

The results (1.8) and (5.3) provide us an interesting connection between the product of two Gegenbauer polynomials and a special case of ${}_4F_3$.

The result (5.3) may be used to obtain further properties of $R_n(x)$ from those of $C_n(x)$.

Acknowledgement

I am very much obliged to Dr. R.M. Shrestha, Mathematics Instruction Committee, Kirtipur Campus, T.U., for his helpful suggestions and guidance in the preparation of this note.

References

1. Rainville, E.D. : 1960 : Special Functions, Macmillan & Co., N.Y.

2. Shrestha, R.M. : 1970 : On generalised Legendre polynomials, T.U. Journal, 5 (1), pp. 82-86.

3. Shrestha, R.M. : 1976 : Some generating functions, T.U. Journal, 9 (1), pp. 59-63.

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On Generalised Hermite Polynomials

Nep. Math. Sc. Rep.
1977, Vol. 2, No. 2.

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1. Introduction

Hermite polynomial of degree n in x is defined by

$$(1.1) \qquad H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k \, n! \, (2x)^{n-2k}}{k! \, (n-2k)!}.$$

Various generalisations of this polynomial are known. In the present note we intend to investigate some properties of the following generalisation of (1.1)

$$(1.2) \qquad H^{(a,b)}_{n,2m,2v}(x) = \sum_{k=0}^{[n/2m]} \frac{(-1)^k \, n! \, (1+a)_k \, (2vx)^{n-2mk}}{k! \, (1+b)_k \, (n-2mk)!},$$

where m is a positive integer, n a non-negative integer, v, a and b are real parameters with $b > -1$.

This polynomial includes the polynomials considered by Lahari [1] when a = b, and Sister Celine [3], when b = 1 and m = 1.

2. Nature of the Generalised Hermite Polynomials

Replacing x by − x in (1.2), we have

$$(2.1) \qquad H^{(a,b)}_{n,2m,2v}(-x) = (-1)^n H^{(a,b)}_{n,2m,2v}(x),$$

showing that this set of polynomials is an even or odd function of x according as n is even or odd. Furthermore, the definition (1.2) yields

$$(2.2) \qquad H^{(a,b)}_{2mn,2m,2v}(0) = \frac{(-1)^n \, (2m)^{2mn} \prod_{r=1}^{2m-1} (r/2m)_n \, (1+a)_n}{(1+b)_n}$$

$$(2.3) \qquad H^{(a,b)}_{2mn+1,2m,2v}(0) = 0,$$

$$(2.4) \qquad DH^{(a,b)}_{n,2m,2v}(x) = \sum_{k=0}^{[(n-1)/2m]} \frac{(-1)^k\ n!\ (1+a)_k\ (2vx)^{n-2mk-1}}{k!\ (1+b)_k\ (n-2mk-1)!}$$

where $D = d/dx$,

The relation (2.4) again yields

$$(2.5) \qquad DH^{(a,b)}_{2mn,2m,2v}(0) = 0$$

and

$$(2.6) \qquad DH^{(a,b)}_{2mn+1,2m,2v}(0) = \frac{(-1)^n\ (2m)^{2mn} \prod_{r=1}^{2m} (\frac{1+r}{2m})_n\ (1+a)_n}{n!\ \ (1+b)_n}$$

The results (2.1) - (2.5) reduce to well-known results when the parameters are specialised.

3. Generating Functions

We shall obtain two generating functions for the generalised Hermite polynomials defined by (1.2).

Multiplying both sides of (1.2) by t^n and summing from 0 to ∞ , we find

$$(3.1) \sum_{n=0}^{\infty} \frac{H^{(a,b)}_{n,2m,2v}(x)\ t^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2m]} \frac{(-1)^k\ (1+a)_k\ (2vx)^{n-2mk}\ t^n}{k!\ (1+b)_k\ (n-2mk)!}$$

$$= \exp\ (2vxt)\ \ {}_1F_1\ (1+a;\ 1+b;\ -t^{2m}),$$

as desired.

To obtain the second generating function, we note that

$$\sum_{n=0}^{\infty} \frac{(c)_n\ H^{(a,b)}_{n,2m,2v}(x)\ t^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2m]} \frac{(-1)^k\ (c)_n\ (1+a)_k\ (2vx)^{n-2mk}\ t^n}{k!\ (1+b)_k\ (n-2mk)!}$$

(c a complex parameter)

$$= \sum_{k,n=0}^{\infty} \frac{(-1)^k (c)_{2mk} (1+a)_k t^{2mk}}{k!\ (1+b)_k} \cdot \frac{(c+2mk)_n (2vxt)^n}{n!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (2m)^{2mk} \prod_{j=1}^{2m}\left(\frac{c+j-1}{2m}\right)_k (1+a)_k t^{2mk}}{k!\ (1+b)_k (1-2vxt)^{c+2mk}}$$

From which, we obtain a generating relation in the hypergeometric form

$$(3.2) \quad (1-2vxt)^{-c} \ {}_{2m+1}F_1\left[\begin{matrix} c/2m,\ (c+1)/2m, \dots,\ (c+2m-1)/2m,\ 1+a; \\ 1+b; \end{matrix} \left(\frac{2mt}{1-2vxt}\right)^{2m}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(c)_n H_{n,2m,2v}^{(a,b)}(x)\ t^n}{n!}$$

The generating relations (3.1) and (3.2) reduce to those obtained by Lahari [2] and Sister Celine [3] on specialising the parameters.

4. Hypergeometric Form

The generalised Hermite polynomial $H_{n,2m,2v}^{(a,b)}(x)$ may be written in the following finite series form

$$(4.1) \quad H_{n,2m,2v}^{(a,b)}(x) = (2vx)^n \sum_{k=0}^{[n/2m]} \frac{(-1)^k (-n)_{2mk} (1+a)_k (2vx)^{-2mk}}{k!\ (1+b)_k}$$

Now, using the relation [3],

$$(a)_{kn} = k^{nk} \prod_{s=1}^{k} \left(\frac{a+s-1}{k}\right)_n ,$$

we get, from (4.1),

$$H^{(a,b)}_{n,2m,2v}(x) = (2vx)^n \sum_{k=0}^{[n/2m]} \frac{(-1)^k \prod_{s=1}^{2m} \left(\frac{-n+s-1}{2m}\right)_k}{k!\ (1+b)_k} (1+a)_k\ (vx/m)^{-2mk}$$

$$= (2vx)^n\ {}_{2m+1}F_1 \left[\begin{matrix} \frac{-n}{2m}, \frac{-n+1}{2m}, \ldots, \frac{-n+2m+1}{2m}, 1+a\ ; \\ 1+b\ ; \end{matrix}\ -(m/vx)^{2m} \right]$$

This representation is analogous to the known hypergeometric forms for Sister Celine's polynomial and the polynomial considered by Lahari.

5. Derivatives of $H^{(a,b)}_{n,2m,2v}(x)$

Differentiation of (1.2) with respect to x yields

(5.1) $$DH^{(a,b)}_{n,2m,2v}(x) = 2vn\ H^{(a,b)}_{n-1,2m,2v}(x).$$

If we repeat the differentiation s times we obtain

(5.2) $$D^s H^{(a,b)}_{n,2m,2v}(x) = \frac{(2v)^s\ n!\ H_{n-s,2m,2v}(x)}{(n-s)!}, \text{ where } D = d/dx.$$

The relation (5.1) shows that the Appell characteristic of the Hermite polynomials $H_n(x)$ is preserved in this case also.

6. Pure Recurrence Relation

Differentiating both sides of the generating relation (3.1) with respect to t and using the relation

(6.1) $$\exp(2vxt)\ {}_1F_1(2+a;2+b;\ -t^{2m}) = \sum_{n=0}^{\infty} \frac{H^{(1+a,1+b)}_{n,2m,2v}(x)\ t^n}{(n-2m)!},$$

we obtain

$$\sum_{n=0}^{\infty} \frac{n\, H^{(a,b)}_{n,2m,2v}(x)\; t^n}{n!} = -\frac{2m(1+a)}{(1+b)} \sum_{n=2m}^{\infty} \frac{H^{(1+a,\,1+b)}_{n-2m,2m,2v}(x)\; t^n}{(m-2m)!} +$$

$$+\ (2vx) \sum_{n=1}^{\infty} \frac{H^{(a,b)}_{n-1,2m,2v}(x)\; t^n}{(n-1)!}$$

Collecting the coefficients of t^n, we get the following relation

$$(6.2)\quad n\, H^{(a,b)}_{n,2m,2v}(x) - 2vnx\, H^{(a,b)}_{n-1,2m,2v}(x) + 2m\,\frac{(1+a)\; n!}{(1+b)\;(n-2m)!}\, H^{(1+a,1+b)}_{n-2m,2m,2v}(x)$$

7. Integral Form of $H^{(a,b)}_{n,2m,2v}(x)$

We first note that

$$\frac{(1+a)_k}{(1+b)_k} = \frac{\Gamma(1+b)}{\Gamma(1+a)\,\Gamma(b-a)} \cdot \frac{\Gamma(1+a+k)\;\Gamma(b-a)}{\Gamma(1+b+k)}$$

$$= \frac{\Gamma(1+b)}{\Gamma(1+a)\,\Gamma(b-a)}\; B\,(1+a+k,\; b-a)$$

$$= \int_0^1 t^{a+k}(1-t)^{b-a-1}\,dt,\quad \mathrm{Re}(1+b) > \mathrm{Re}(1+a) > 0$$

If we further assume that $\left(\frac{m}{vx}\right)^{2m} < 1$ and for $\mathrm{Re}(1+b) > \mathrm{Re}(1+a) > 0$, we get

$$H^{(a,b)}_{n,2m,2v}(x) = (2vx)^n\; \frac{\Gamma(1+b)}{\Gamma(1+a)\,\Gamma(b-a)} \sum_{k=0}^{[n/2m]} \frac{(-1)^k \prod_{j=1}^{2m} \left(\frac{-n+j-1}{2m}\right)_k}{k!} \times$$

$$\times \left(\frac{m}{vx}\right)^{2mk} \int_0^1 t^{a+k}(1-t)^{b-a-1}\, dt.$$

$$= (2vx)^n \frac{\Gamma(1+b)}{\Gamma(1+a)\,\Gamma(b-a)} \int_0^1 t^a (1-t)^{b-a-1}.$$

$${}_{2m}F_0 \left[\begin{matrix} \frac{-n}{2m}, \frac{-n+1}{2m}, \ldots, \frac{-n+2m-1}{2m} ; \\ - ; \end{matrix} \; -\left(\frac{m}{vx}\right)^{2m} t \right] dt,$$

as desired.

8. Integral Involving $H^{(a,b)}_{n,2m,2v}(x)$

Among many relations between $H_n(x)$ and $P_n(x)$, one interesting result is the following integral relation due to Curzon [3]:

$$(8.1) \qquad P_n(x) = \frac{2}{n!\sqrt{\pi}} \int_0^\infty \exp(-t^2)\, t^n H_n(xt)\, dt.$$

An analogue of the result (8.2) is

$$(8.2) \qquad P^{(a,b)}_{n,2m,2v}(x) = \frac{2}{n!\sqrt{\pi}} \int_0^\infty e^{-t^2} t^n H^{(a,b)}_{n,2m,2v}(xt)\, dt,$$

provided that $n - mk + \frac{1}{2} > 0$, and

$$(8.3) \qquad P^{(a,b)}_{n,2m,2v}(x) = \sum_{k=0}^{[n/2m]} \frac{(-1)^k (\frac{1}{2})_{n-mk} (1+a)_k (2vx)^{n-2mk}}{k!\,(1+b)_k\,(n-mk)!}$$

which is again an interesting generalisation of the Legendre polynomials $P_n(x)$. To establish (8.2), we use (2.1) and put $t^2 = sy$ with $x - 1/\sqrt{s}$ in the right side of (8.2) and arrive at

$$(8.4)\quad I = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{[n/2m]} \frac{(-1)^k\ (1+a)_k\ (2v)^{n-2mk}}{k!\ (1+b)_k\ (n-2mk)!}\ s^{\frac{1}{2}(n+1)} \int_0^\infty e^{-sy}\ y^{\ n-mk+\frac{1}{2}}\ dy$$

$$= \sum_{k=0}^{[n/2m]} \frac{(-1)^k (1+a)_k\ (2vx)^{n-2mk}\ (\tfrac{1}{2})_{n-mk}}{k!\ (1+b)_k\ (n-2mk)!},$$

as is required.

Acknowledgement

I would like to express my sincere thanks to Dr. R.M. Shrestha for his guidance in the preparation of this paper.

References

1. Lahari, M. : 1971 : On a generalisation of Hermite Polynomials, Proc. Amer. Math. Soc. 27, 117-121.

2. ________ : 1972 : An associated generalised Hermite Polynomials Indian J. Pure & Apl. Math. 3, No. 1, 92-95.

3. Rainville, E.D. : 1960 : Special Functions, Macmillan Co. N.Y.

Determination of Expected Entrant Population to an Academic Programme-A long Term Estimation

Nep. Math. Sc. Rep.
1977, Vol. 2, No.2.

by Hridaya Bahadur Shrestha

At the time of introduction of the National Education Systems Plan a number of directives and areas of responsibility for the development of education were introduced. Many of these were explicit in pointing the direction which education should take, as well as mentioning specific tasks necessary to reconstruct the educational system. Higher education did not escape criticisms of being theoretical and the skills developed within the students as being unrelated to the development of manpower needs of the Kingdom. One of the specific directives to the University was: The University will organise a separate admission test to determine eligibility for higher education (NESP; vii) so that the admission test could act as an aid to screening and placement procedure for entrance to educational programmes at various institutes and campuses of Tribhuvan University. It is five years that the admission tests have been practised. It is highly essential, therefore, to assess the entrant population to various institutes and thereby to the University, in relation with the School Leaving Certificate Examination (S.L.C.) clearing population as well as the relationship of the entrant population themselves for two consecutive years.

It is the object of this paper to propose statistical models of determining the relationship of the entrant populations themselves for two consecutive years. Later on the case for n years has been deduced through induction method.

This paper describes three methods of estimating the entrant population to an educational programme for an academic year, each illustrated with an example.

METHOD: A

Let the entrant population to the University at two successive points of time t and t+1 be E_t and E_{t+1}. Let $e_{t,i}$ and $e_{t+1,i}$ be the entrant population to ith educational programme at time t and t+1. Since the two consecutive observations on the entrant population refer to a small duration of time (one year), it is reasonable to assume a linear relationship between them as a first approximation. Therefore,

$$E_{t+1} = \alpha + \beta E_t \tag{1.1}$$

We can estimated α and β from observed values. In case of least square estimation method, the estimates are

$$\text{Est } \beta = \hat{\beta} = \frac{\sum_{i=1}^{n} e_{t,i}\, e_{t+1,i} - n\,\bar{e}_t\,\bar{e}_{t+1}}{\sum_{i=1}^{n} e_{t,i}^2 - n\,\bar{e}_t^{\,2}}$$

where $$\bar{e}_t = \frac{1}{n}\sum_{i=1}^{n} e_{t,i}$$

and, Est $\alpha = \hat{\alpha} = \bar{e}_{t+1} - \hat{\beta}\,\bar{e}_t$

Let the year of introduction of NESP to higher education be considered the base year. Then $E_{t,n}$ denotes the entrant population n years after the introduction of the NESP at the time period t.

(1.1) implies

$$E_{t+1,n+1} = \alpha + \beta E_{t,n} \tag{1.2}$$

Here we make one assumption that the pattern of growth of the entrant population remains on the average more or less the same as in year t and t+1.

Now, (1.2) is a difference equation of the first order. Its solution will be of the form

$$E_{t,n} = \lambda \beta^n + \frac{\alpha}{1-\beta} \tag{1.3}$$

where λ is a constant.

To estimate λ, the base year entrant population $E'_{t,s}$ may be written as

$$E'_{t,s} = \lambda + \frac{\alpha}{1-\beta} \quad [\because n=0]$$

$$\lambda = E'_{t,s} - \frac{\alpha}{1-\beta} \tag{1.4}$$

Therefore, the entrant population at the end of n years will be, using (1.4),

$$E_{t,n} = (E'_{t,s} - \frac{\alpha}{1-\beta})\ \beta^{n} + \frac{\alpha}{1-\beta}$$

$$= \beta^{n}.\ E'_{t,s} + (1-\beta^{n})\ \frac{\alpha}{1-\beta},\ \forall\, n \geqslant 1.$$

Example:

The entrant population to various institutes of T.U. for the academic sessions of 2030/31 and 2031/32 is summarized in the following table.

TABLE: 1 - Entrant Population to T.U. for academic sessions of 2030/31 and 2031/32.

Institute	Entrant Population for	
	2030/31	2031/32
Science	851	969
Humanities and Social Sciences	2811	3233
Business Administration, Commerce and Pblic Administration	1022	1202
Law	63	170
Education	927	668
Medicine	213	206
Applied Science and Technology	68	140
Engineering	319	187
Forestry	105	105
Agriculture	37	86
Total for the University	6416	6966

Source: Study Report on Examination and Admission (2032) by Pandey, K.R.; Risal, S. and Sarma, D.; Research Division Rector's Office, T.U.

By using the least squares estimation method,

$$\text{Est } \beta = \hat{\beta} = 1.128$$

and

$$\text{Est } \alpha = \hat{\alpha} = -27.125$$

From the table, we can read

$$E'_{t,s} = 6416$$

The expected entrant population for the academic year 2034/35 will be

$$E_{t,4} = 10{,}251.$$

METHOD: B

The entrant population to the University at any point of time is dependent on the number of students clearing the S.L.C. examination, hereafter referred to as S.L.C. clearing population. Let T_t represent the S.L.C. clearing population at the time t.

We shall proceed on to

i) determine the relationship between the entrant population E_t and the number of students clearing the S.L.C. examination, T_t.

ii) estimate the S.L.C. clearing population T_{t+1} for the next period assuming the relation determined in (i) to prevail for the second period, with the help of E_{t+1} and

iii) find a relationship between the calculated S.L.C. clearing populations T_t & T_{t+1}.

Relation (i)

Assume the relationship between the entrant population and the S.L.C. clearing population to follow one of the exponential curves. In our case assume that the above relationship is defined by the model

$$E_t = A.\ T_t^{\ B}$$

Taking logarithms on both sides

$$E_t^* = A^* + B\,T_t^* \tag{2.1}$$

where * over a quantity denotes the logarithm of the same and A* and B are estimates to be determined by the method of least squares estimation.

Relation (ii)

The estimate of the entrant population based on the S.L.C. clearing population in time t+1 will be

$$E_{t+1}^* = \hat{A}^* + \hat{B}\,T_{t+1}^* \tag{2.2}$$

where ^ over a constant denote their least square estimates. We deduce, therefore, that

$$E_{t,\,n}^* = \hat{A}^* + \hat{B}\ T_{t,\,n_1}^* \tag{2.3}$$

Relation (iii)

The relationship between the S.L.C. clearing population n years hence from the starting period of time and the S.L.C. clearing population at the starting period of time might be estimated by assuming the S.L.C. clearing population in successive years are linear related.

$$T_{t+1} = \nu + \delta T_t$$

and hence,

$$T_{t+1,\,n_1+1} = \nu + \delta\,T_{t,n_1} \tag{3.1}$$

where T_t, T_{t+1}, T_{t,n_1} & $T_{t+1,\,n_1+1}$ are the S.L.C. clearing population at the time period t & t+1 and n_1 & n_1+1 years hence from the starting point of time and ν & δ are the constant estimable by the method of least squares. (3.1) being a difference equation of first order we may deduce, as in (1.3), that

$$T_{t,n_1} = \xi\,\delta^{n_1} + \frac{\nu}{1-\delta} \tag{3.2}$$

where ξ is estimated by using the S.L.C. clearing population at the starting point of time $T'_{t,s}$ as the reference population. Clearly at the starting point of time

$$T'_{t,s} = \xi + \frac{\nu}{1-\delta}$$

$$\xi = T'_{t,s} - \frac{\nu}{1-\delta}$$

The total population of students clearing the S.L.C. examination on the $n_1{}^{th}$ year after the starting point is

$$T_{t,n_1} = \left(T'_{t,s} - \frac{\nu}{1-\delta}\right)\delta^{n_1} + \frac{\nu}{1-\delta}$$

$$\text{or} \quad T_{t,n_1} = \delta^{n_1} T'_{t,s} + \frac{\nu}{1-\delta}\left(1-\delta^{n_1}\right) \qquad (3.3)$$

Also
$$E^*_{t,n} = \hat{A}^* + \hat{B}\, T^*_{t,n_1}$$

Therefore,

$$E^*_{t,n} = \hat{A}^* + \hat{B} \log \left[\delta^{n_1} T'_{t,s} + \frac{\nu}{1-\delta}\left(1-\delta^{n_1}\right)\right] \qquad (3.4)$$

<u>Example</u>

The following table is available for the S.L.C. clearing population and the entrant population to T.U. for the academic sessions of 2028/29 to 2031/32.

TABLE. 2 S.L.C. clearing population and the entrant population for the academic sessions of 2028/29 to 2031/32

Year	S.L.C. clearing population	Entrant population
2028/29	5267	-
2029/30	5829	-
2030/31	7813	6416
2031/32	11454	6966

Source: As quoted in Table 1 above.

The least square estimate of A* & B were found to be

$$\hat{A}^{*} = 3.0447$$

$$\hat{B} = 0.196$$

The ν & δ for the S.L.C. clearing population was found to be

$$\delta = 1.394$$

$$\nu = -\ 421.382$$

For the academic session of 2034/35, estimate of T_{t,n_1} was

$$T_{t,n_1} = 31871$$

so that, $T^{*}_{t,n} = 3.9294$

Total entrant population in the year 2034/35 would be, therefore,

$$E_{t,4} = 9684$$

METHOD. C

The entrant population n years after the starting point of time may be estimated by

$$E_{t,n} = E'_{t,s} \quad (1 + \gamma_c)^n \tag{4.1}$$

where γ_c is the rate of growth of the entrant population assumed to remain the same for all subsequent years.

Example: The data on Table 2 enables us to compute

$$\gamma_c = 0.857$$

For the academic session of 2034/35, the expected entrant population would then be

$$E_{t,4} = 8,916$$

Summary

The paper describes statistical methods of determining the entrant population for any time period based on entrant population for earlier time periods. It also describes the method of estimating the entrant population based on the S.L.C. clearing population. The constant growth rate method of estimating the entrant population has been introduced here only for comparative purpose: The entrant population estimated by the above three methods are 10251, 9684 and 8916 respectively. These estimates are crude & based on very small number of observations. The estimates may be re-estimated, as time passes on, with additional data to ensure a more predictive measure of the entrant population. The academic session of 2030/31 would be, according to English Calendar the session of 1973-74.

References

Pandey, K.R., Risal S. and Sarma, S. 2032. Study Report on Examination and Admission, Research Division, Rector's Office, T.U.

Shrestha, H.B., 2031. Impact of qualitative development in Semester Examination System, Research Division, Rector's Office, T.U.

Richardson, C.H. 1954. Introduction to Calculus of Finite Differences, Van Norstrand. pp. 142.

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Abelian Theorems for the Error-Transform

Nep. Math. Sc. Rep.
1977, Vol. 2, No.2.

by L.K. Shrestha

1. Introduction

In this paper, we establish Abelian theorems for the Error-transform

$$f(s) = \int_0^\infty e^{-st} \text{ Erfc } (\sqrt{st}) \, d\,\alpha(t),$$

where ERfc(x) is the complementary Error function [1: pp.147] and is defined by

$$\text{Erfc}(x) = \int_x^\infty e^{-t^2} \, dt, \quad x > 0.$$

Here we assume s is real unless otherwise stated.

2. Abelian Theorems'

To establish Abelian theorems we need the following lemma.

Lemma 2.1 If r is positive real number, then

$$\frac{s^{r+1}}{H(r)} \int_0^\infty [e^{-st} \text{ Erfc}(\sqrt{st}) + \tfrac{1}{2} e^{-2st} (st)^{-\frac{1}{2}}] \; t^r \, dt = 1.$$

where

$$H(r) = \tfrac{1}{4} \frac{(r+3/2)}{(r+1) 2^{r+3/2}} \; {}_2F_1 \; [1, 3/2+r; \; r+2; \; \tfrac{1}{2}] + 2^{-r-3/2} \; (r + \tfrac{1}{2}).$$

The lemma follows easily by [1: pp. 138].

Theorem 2.1 If $\alpha(t)$ is of normalized bounded variation in the interval $o \leq t \leq R$ for every positive R and if the integral

$$(2.1) \qquad f(s) = \int_0^\infty e^{-st} \text{Erfc}(\sqrt{st}) \quad d\,\alpha(t)$$

converges for $s > 0$, then for any real number $r > 0$ and any constant A

$$\overline{\lim_{s\to\infty}} \left| s^r f(s) - A \right| \leq \overline{\lim_{t\to 0+}} \left| \frac{\alpha(t)H(r)}{t^r} - A \right| \tag{2.2}$$

$$\overline{\lim_{s\to 0+}} \left| s^r f(s) - A \right| \leq \overline{\lim_{t\to\infty}} \left| \frac{(t)H(r)}{t^r} - A \right| \tag{2.3}$$

Proof:- Since the integral (2.1) converges for $s > 0$, we have by theorem 2.4.1 [2: pp.20]

$$f(s) = s \int_0^\infty [e^{-st}\mathrm{Erfc}(\sqrt{st}) + \tfrac{1}{2}e^{-2st}(st)^{-\frac{1}{2}}]\, \alpha(t)\, dt.$$

Then

$$s^r f(s) - A = s^{r+1} \int_0^\infty [e^{-st}\mathrm{Erfc}(\sqrt{st}) + \tfrac{1}{2}e^{-2st}(st)^{-\frac{1}{2}}]\, [\alpha(t) - \frac{At^r}{H(r)}]\, dt.$$

If T is any positive number, then

$$\left| s^r f(s) - A \right| \leq s^{r+1} \int_0^T [e^{-st}\mathrm{Erfc}(\sqrt{st}) + \tfrac{1}{2}e^{-2st}(st)^{-\frac{1}{2}}] \left| \alpha(t) - \frac{At^r}{H(r)} \right| dt$$

$$+ s^{r+1} \int_T^\infty [e^{-st}\mathrm{Erfc}(\sqrt{st}) + \tfrac{1}{2}e^{-2st}(st)^{-\frac{1}{2}}] \left| \alpha(t) - \frac{At^r}{H(r)} \right| dt$$

$$= \frac{s^{r+1}}{H(r)} \int_0^T [e^{-st}\mathrm{Erfc}(\sqrt{st}) + \tfrac{1}{2}e^{-2st}(st)^{-\frac{1}{2}}]\, t^r \left| \frac{\alpha(t)H(r)}{t^r} - A \right| dt$$

$$+ s^{r+1} \int_T^\infty [e^{-st}\mathrm{Erfc}(\sqrt{st}) + \tfrac{1}{2}e^{-2st}(st)^{-\frac{1}{2}}] \left| \alpha(t) - \frac{At^r}{H(r)} \right| dt$$

$$= I_1 + I_2.$$

Using lemma 2.1 , we find that

$$I_1 \leq \underset{0 < t \leq T}{\mathrm{l.u.b.}} \left| \frac{\alpha(t)H(r)}{t^r} - A \right| ,$$

and for I_2 we note that because of the order properties of $\alpha(t)$, following from the convergence of the integral (2.1) [2], we can find a constant M for any positive ϵ such that

$$\left| \alpha(t) - \frac{At^r}{H(r)} \right| < M e^{2\epsilon t} \qquad (0 \leq t < \infty) .$$

Therefore

$$I_2 \leq Ms^{r+1} \int_T^\infty [e^{-(s-2\epsilon)t} \operatorname{Erfc}(\sqrt{st}) + \tfrac{1}{2} e^{-2t(s-\epsilon)} (st)^{-\frac{1}{2}}]\, dt$$

$$= Ms^{r+1} \int_0^\infty e^{-(s-2\epsilon)(T+y)} \operatorname{Erfc}(\sqrt{s(T+y)})\, dy +$$

$$+ \tfrac{1}{2} Ms^{r+1} \int_0^\infty e^{-2(s-\epsilon)(T+y)} (s(T+y))^{-\frac{1}{2}}\, dy$$

$$= Ms^{r+1} e^{-T(s-2\epsilon)} \int_0^\infty e^{-y(s-\epsilon)} \operatorname{Erfc}(\sqrt{s(T+y)})\, dy$$

$$+ \tfrac{1}{2} Ms^{r+1} e^{-2T(s-\epsilon)} \int_0^\infty e^{-2y(s-\epsilon)} (s(T+y))^{-\frac{1}{2}}\, dy .$$

Since the integrals on the right hand side are finite for $s > 2\epsilon$, we have

$$\lim_{s \to \infty} I_2 = \lim_{s \to \infty} [s^{r+1} e^{-T(s-2\epsilon)} + \tfrac{1}{2} s^{r+1} e^{-2T(s-\epsilon)}]$$

$$= 0 \qquad (s > 2\epsilon).$$

Thus

$$\lim_{s \to \infty} \left| s^r f(s) - A \right| \leq \operatorname*{l.u.b}_{0 \leq t \leq T} \left| \frac{\alpha(t)H(r)}{t^r} - A \right| .$$

Since the left hand side of the above inequality is independent of T, we allow it to approach zero and obtain

$$\overline{\lim_{s \to \infty}} \left| s^r f(s) - A \right| \leq \overline{\lim_{t \to 0}} \left| \frac{\alpha(t)H(r)}{t^r} - A \right| .$$

This establishes (2.2). On proceeding similarly (2.3) also follows.

Corollary 2.1. If for some non-negative number r and any constant A

$$\alpha(t) \sim \frac{At^r}{H(r)} \quad \begin{pmatrix} t \to \infty \\ t \to o+ \end{pmatrix}$$

then

$$f(s) \sim A\, s^{-r} \quad \begin{pmatrix} s \to o+ \\ s \to \infty \end{pmatrix}.$$

lemma 2.2. If $s > \sigma_c > 0$ and $r > 0$ then

$$s^{r+1} \int_0^{\infty} [e^{-st} \mathrm{Erfc}(\sqrt{st}) + \tfrac{1}{2} e^{-2st} (st)^{-\frac{1}{2}}]\, t^r e^{2\sigma t}\, dt = \frac{G(\sigma_c, s, r)}{H(r)} ,$$

where

$$G(\sigma_c, s, r) = H(r) \, [\tfrac{1}{2} \frac{\Gamma(r+3/2)}{(r+1)2^{r+3/2}} \cdot \frac{1}{(1-\sigma_c/s)^{r+3/2}}$$

$$ {}_2F_1 \, [1, r+3/2;\, r+2;\, \frac{s-2\sigma_c}{2s-2\sigma_c}] + \frac{\Gamma(r+\frac{1}{2})}{{}^{r+3/2}(1-\sigma_c/s)^{r+\frac{1}{2}}}]$$

and $H(r)$ is given in lemma 2.1.

Proof follows as in lemma 2.1.

Theorem 2.2. If the integral (2.1) converges for a $s > \sigma_c$ where σ_c is the abscissa of convergence of the integral, then for any $t > 0$ and any constant A

$$\lim_{s \to \sigma_c+} \left| \frac{s^{r+1} f(s)}{G(\sigma_c, s, r)} - \sigma_c A \right| \leq \lim_{t \to \infty} \sigma_c \left| \frac{\alpha(t)}{t^r e^{2\sigma_c t} H(r)} - A \right|$$

where $G(\sigma_c, s, r)$ is defined as in lemma 2.2.

Proceeding as before, we have

$$\left| \frac{s^{r+1}f(s)}{G(\sigma_c,s,r)} - A\sigma_c \right| \leq \frac{s^{r+1}\sigma_c}{G(\sigma_c,s,r)} \int_0^T [e^{-st}\mathrm{Erfc}(\sqrt{st})+\tfrac{1}{2}e^{-2st}(st)^{-\frac{1}{2}}] \times$$

$$\times \left| \alpha(t) - At^r e^{2\sigma_c t}H(r) \right| dt \quad + \frac{H(r)s^{r+1}\sigma_c}{G(\sigma_c,s,r)} \int_T^\infty [e^{-st}\mathrm{Erfc}(\sqrt{st})+\tfrac{1}{2}e^{-2st}(st)^{-\frac{1}{2}}]$$

$$\left| \frac{\alpha(t)}{t^r e^{2\sigma_c t}H(r)} - A \right| dt + \frac{s^{r+1}(s-\sigma_c)}{G(\sigma_c,s,r)} \int_0^\infty [e^{-st}\mathrm{Erfc}(\sqrt{st})+\tfrac{1}{2}e^{-2st}(st)^{-\frac{1}{2}}]\alpha(t)dt$$

$$= \quad I_1 + I_2 + I_3.$$

Since the integral (2.1) converges for $s > \sigma_c$, by using the order properties of $\alpha(t)$ [2: pp. 20] we can always find a constant K such that

$$\left| \alpha(t) - At^r e^{2\sigma_c t}H(r) \right| < K e^{2\sigma_c t} \quad (0 \leq t < \infty)$$

Hence

$$I_1 \leq \frac{s^{r+1}K\sigma_c}{G(\sigma_c,s,r)} \int_0^T [e^{-(s-2\sigma_c)t}\mathrm{Erfc}(\sqrt{st})+\tfrac{1}{2}e^{-(s-\sigma_c)2t}(st)^{-\frac{1}{2}}]\, dt.$$

The integral is convergent for $s > \sigma_c$. Hence

$$\lim_{s\to\sigma_c+} I_1 \leq \lim_{s\to\sigma_c+} \frac{s^{r+1}K\sigma_c}{G(\sigma_c,s,r)}$$

$$= \lim_{s\to\sigma_c} \frac{1}{\left\{ \frac{\frac{1}{2}\Gamma(r+3/2)}{(r+1)2^{r+\frac{1}{2}}} \frac{1}{(1-\sigma_c/s)^{r+3/2}}\, {}_2F_1\left[1,r+3/2;r+2;\frac{s/2-\sigma_c}{s-\sigma_c}\right] + \frac{\Gamma(r+\frac{1}{2})}{2^{r+3/2}(1-\sigma_c/s)^{r+\frac{1}{2}}} \right\}}.$$

Therefore

$$\lim_{s \to \sigma_c +} I_1 = 0.$$

Similarly

$$\lim_{s \to \sigma_c} I_3 = 0.$$

Finally

$$I_2 \leq \underset{T \leq t < \infty}{\text{l.u.b.}} \left| \frac{\alpha(t)}{t^r e^{2\sigma_c t} H(r)} - A \right| \cdot \frac{H(r)\, s^{r+1} \sigma_c}{G(\sigma_c, s, r)} \times$$

$$\times \int_T^{\infty} [e^{-st} \mathrm{Erfc}(\sqrt{st}) + \tfrac{1}{2} e^{-2st}(st)^{-\frac{1}{2}}]\, t^r e^{2\sigma_c t}\, dt$$

$$= \underset{T \leq t < \infty}{\text{l.u.b}}\ \sigma_c \left| \frac{\alpha(t)}{t^r e^{2\sigma_c t} (H(r))} - A \right| .$$

Since the left hand side is independent of T, we allow it to tend to infinity and thus obtain

$$\lim_{s \to \sigma_c +} \left| \frac{s^{r+1} f(s)}{G(\sigma_c, s, r)} - A\sigma_c \right| \leq \sigma_c \lim_{t \to \infty} \left| \frac{\alpha(t)}{t^r e^{2\sigma_c t} H(r)} - A \right|$$

which completes the proof.

I am thankful to Dr. R.P. Manandhar for his help and guidance in the preparation of this paper.

References

1. Erdelyi, A. ; Higher Transcendental Functions Vol. II, McGraw-Hill, 1954.

2. Shrestha, L.K.; Thesis "A Study of Error Transform" submitted to the Instruction Committee, Mathematics, 1976.

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Problem Section

The Section of the Report is devoted to problems and their solutions. You are invited to send your solutions to the problems to any member of the Coordinating Committee. One solution to each problem will be published in later issues of the Report with credit given to the solver. Other persons who submit correct solutions to the same problem will be acknowledged by name. You re also invited to send your favorite problems for inclusion in this Section to challenge others. A solution may be sent along with the problem if one is known.

Problems

1. You are given an obtuse triangle. Can you always cut it into acute triangles? If yes, what is the minimum number of triangles that will suffice in any case? (Hint: Yes! The minimum is 7.)

2. Let N be a natural number which is not divisible by 2 or by 5. Show that there is a multiple of N which consists entirely of ones.

3. Suppose that the function f whose domain is the real line is continuous at a point c and satisfies the functional equation $f(x+y) = f(x) + f(y)$ for all x and y. Prove that $f(x) = x\, f(1)$ for <u>all</u> x. (Thus functional additivity plus continuity at a point <u>implies</u> linearity.)

4. Let f and g be differentiable functions of the variable x. Students often feel that $(fg)'(x)$ should equal $f'(x)\, g'(x)$ and are disappointed when they learn that it is not generally so. Find functions f and g for which it is so; that is, for which $(fg)'(x) = f'(x)\, g'(x)$.