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Contents

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Closure Semi-Continuity

Phullendu Das

Abstract

Closure semicontinuity is defined and some results of closure semicontinuity analogous to those for closure continuity are obtained.

Let (X, τ) and (Y, τ') be any two topological spaces.

A set $A \subset X$ is said to be a semi-open set if there exists an open set 0 such that $0 \subset A \subset \overline{0}$ where $\overline{0}$ is the closure of 0 (Levine [6]).

S.O. (τ) will denote the class of all semi-open sets of (X, τ). If $x \varepsilon X$, $0(\tau, x)$ and S.O. (τ, x) will denote respectively the class of all open and semi-open sets of (X, τ) containing x.

In [5] Das defined semi limit point and in [3] Crossley and Hildebrand defined semi-closure $\underline{A}$ of a set $A \subset X$ in a manner analogous to limit point and closure.

Unless otherwise mentioned α will denote a mapping of (X, τ) into (Y, τ').

α is said to be semi-continuous if $U \varepsilon \tau' \Longrightarrow (U)\alpha^{-1} \varepsilon$ S.O.(τ) (Levine [6]).

α is said to be closure continuous at a point $x \varepsilon X$ iff for every $V \varepsilon 0(\tau', (x)\alpha)$ there exists a $U \varepsilon 0(\tau, x)$ such that $(\overline{U})\alpha \subset \overline{V}$. If α be closure continuous at every $x \varepsilon X$, then α is said to be closure continuous on X (Andrew and Whittlesy [1]).

Definition 1: α is said to be closure semi-continuous at a point $x \varepsilon X$ iff for every $V \varepsilon 0(\tau', (x)\alpha)$, there exists a $U \varepsilon$ S.O. (τ, x) such that $(\underline{U})\alpha \subset \overline{V}$. If α be closure semi-continuous at every $x \varepsilon X$, then α is said to be closure semi-continuous on X.

Theorem 1: If α be semi-continuous, then α is closure semi-continuous on X.

Proof: Let α be semi-continuous. Let $x \in X$ and let $V \in O(\tau', (x)\alpha)$. Since α is semi-continuous, there exists a $U \in S.O.(\tau, x)$ such that $(U)\alpha \subset V$ and then $(\underline{U})\alpha \subset (\overline{U})\alpha$ (by Theorem 4, Biswas [2]) $\subset \overline{V}$. α is closure semi-continuous at x. Since x is any point of X, α is closure semi-continuous on X.

Note 1: The converse of Theorem 1 is not true as shown by

Example 1: Let $X = \{a, b, c, d\}$,

$$= \{\phi, X, \{a\}, \{b, c\}, \{a, b, c\}\}.$$

Then $S.O.(\tau) = \tau \cup \{\{a, d\}, \{b, c, d\}\}$.

$\alpha : X \to X$ is defined by

$$(a)\alpha = b, \ (b)\alpha = a, \ (c)\alpha = d, \ (d)\alpha = c.$$

α is closure semi-continuous on X. Since $\{a\} \in \tau$ but $(\{a\})\alpha^{-1} = \{b\} \notin S.O.(\tau)$, α is not semi-continuous.

Note 2 : Closure continuity $\Longrightarrow$ Closure semi-continuity but not conversely since α (defined in Example 1) is not closure continuous at d. For, $\{b, c\} \in O(\tau, (d)\alpha)$ but as the only open set containing d is X and $(\underline{X})\alpha = X \not\subset \overline{\{b, c\}} = \{b, c, d\}$.

Theorem 2 : Let (Y, τ') be a T_3 - space and let α be closure semi-continuous at a point $x \in X$. Then α is semi-continuous at x.

Proof: Let $V \in O(\tau', (x)\alpha)$. Since (Y, τ') is a T_3-space, there exists a $V_1 \in O(\tau', (x)\alpha)$ such that $\overline{V}_1 \subset V$. Since α is closure semi-continuous at x, there exists a $U \in S.O.(\tau, x)$ such that $(\underline{U})\alpha \subset \overline{V}_1$. Then $(\underline{U})\alpha \subset V$. Hence α is semi-continuous at x.

Corollary 1 : If (Y, τ') be a T_3-space, then α is closure semi-continuous on X $\Longrightarrow$ α is semi-continuous.

Note 3 : Semi-continuity at a point does not imply closure semi-continuity at that point as shown by

Example 2 : Let $X = \{a, b, c\}$,

$$\tau = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\},$$

$$\tau' = \{\phi, X, \{b\}, \{a, c\}\}.$$

Then τ = S.O. (τ), τ' = S.O. (τ')

The identity mapping $I_X : (X,\tau) \to (X,\tau')$ is semi-continuous at c. Since $\{a, c\} \varepsilon\ 0(\tau',c)$ but for every U ε S.O. (τ, c), $(\underline{U})\ I_X = X \not\subset \overline{\{a,c\}} = \{a,c\}$, α is not closure semi-continuous at C.

In [1] Andrew and Whittlesy proved that if α be continuous at a point $x \varepsilon X$ at which α is not closure continuous, then every U ε $0(\tau, x)$ has a limit point at which α is not continuous.

But if α be semi-continuous at a point $x \varepsilon X$ at which α is not closure semi-continuous, then every U ε S.O. (τ, x) may not possess a semi-limit point at which α is not semi-continuous as shown by

<u>Example 3</u> : Let $X = \{a,b,c\}$,

$$\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}.$$

Then S.O. (τ) $= \tau \cup \{\{a, c\}, \{b, c\}\}$.

$\alpha : X \to X$ is defined by

$$(a)\alpha = b,\ (b)\alpha = a,\ (c)\alpha = b.$$

is semi-continuous at C. α is not closure semi-continuous at C. For, $\{b\} \varepsilon\ 0\ (\tau, (c)\alpha)$ but for every U ε S.O (τ, c), $(\underline{U})\alpha = (X)\alpha = \{a, b\} \not\subset \{b, c\} = \overline{\{b\}}$.

Now $\{a, c\}\ \varepsilon$ S.O. (τ, c). But it has no semi-limit point.

However we have the following result

<u>Theorem 3</u> : If $x \varepsilon X$ be such that (i) α is semi-continuous at x, (ii) α is not closure semi-continuous at x and (iii) S.O. (τ, x) is closed w.r. to intersection, then every U ε S.O. (τ, x) has a semi-limit point y at which α is not semi-continuous.

<u>Proof</u> : By (ii), there exists a $V \varepsilon\ 0(\tau', (x)\alpha)$ such that $(\underline{U})\alpha \not\subset \overline{V}$ for every U ε S.O.(τ, x). By (i), there exists a $U_1 \varepsilon$ S.O.(τ, x) such that $(U_1)\alpha \subset V$. Let U ε S.O.(τ, x). Let $U_2 = U \cap U_1$. Then by (iii) $U_2\ \varepsilon$ SO.(τ, x). Hence $(\underline{U}_2)\alpha \subset \overline{V}$ but $(U_2)\alpha \subset V$. Hence there exists a $y \varepsilon X$ such that y is a semi-limit point of U_2 and therefore of U but $(y)\alpha \notin \overline{V}$. Let $V_1 = y - \overline{V}$. Then $V_1\ \varepsilon\ 0(\tau', (y)\alpha)$. Let $U_3\ \varepsilon$ S.O. (τ, y). Since y is a semi-limit point of U_2,

$U_2 \cap (U_3 - \{y\}) \neq \phi$. Let $Z \in U_2 \cap (U_3 - \{y\})$. Then $(Z)\alpha \in V$. Hence $(U_3)\alpha \not\subset V_1$ for every $U_3 \in S.O(\tau, y)$. $\therefore$ is not semi-continuous at y .

Definition 2 : (X,τ) is said to be a ST_3 - space if for every semi-closed set F of X and every $p \notin F$, there exist $U, V \in S.O.(\tau)$ such that $F \subset U$, $p \in V$ and $U \cap V = \phi$ (Das [4]).

(X,τ) is a ST_3 - space iff for every $x \in X$ and for every $U \in S.O.(\tau,x)$ there exists a $V \in S.O.(\tau, x)$ such that $\underline{V} \subset U$ (Das [4]).

Theorem 5 : If (i) (X,τ) be a ST_3 - space and (ii) α be semi-continuous at a point $x \in X$, then α is closure semi-continuous at x.

Proof : Let $V \in O(\tau', (x)\alpha)$. By (ii) there exists a $U_1 \in S.O.(\tau,x)$ such that $(U_1)\alpha \subset V$. By (i) there exists a $U \in S.O.(\tau, x)$ such that $\underline{U} \subset U_1$. Hence $(\underline{U})\alpha \subset (U_1)\alpha \subset V \subset \overline{V}$. Hence α is closure semi-continuous at x .

Note 4 : The converse of Theorem 4 is not true as shown by

Example 4 : Consider the topological spaces (X,τ), (X,τ') defined in Example 2. (X,τ') is a ST_3 - space. $I_x: (X,\tau') \to (X,\tau)$ is closure semi-continuous at a. But it is not semi-continuous at a.

It follows from Theorem 1, 2 and 4 that

Theorem 5 : If (X,τ) be a ST_3 - space and (Y,τ') a T_3-space, then α is semi-continuous at $x \in X$ (resp. on X) $\Longleftrightarrow$ α is closure semi-continuous at x (resp. on X).

Theorem 6: Let $A \subset X$. If α be closure continuous, then $\alpha' = \alpha \mid A: (A,\tau_A) \to (Y,\tau')$ is also closure continuous.

Proof: Let $x \in A$. Let $V \in O(\tau', (x)\alpha')$. Since α is closure continuous, there exists a $U \in O(\tau,x)$ such that $(\overline{U})\alpha \subset \overline{V}$. Then $U_1 = U \cap A \in O(\tau_A, x)$ and $(\bar{U}_{1\tau A})\, \alpha' \subset (\overline{U})\alpha \subset V$. $\therefore$ α' is closure continuous.

Note 5: Restriction of a closure semi-continuous function is not always closure semi-continuous as shown by

Example 5: Consider the closure semi-continuous mapping α defined in Example 1.

Let $A = \{a, b, d\}$.

Then $\tau_A = \{\phi, A, \{a\}, \{b\}, \{a, b\}\}$

and $S.O.(\tau_A) = \tau_A \cup \{\{a, d\} , \{b, d\}\}$.

$\alpha' = \alpha | A$ is not closure semi-continuous at d. For $\{b, c\} \in O(\tau, (d)\alpha')$ but there does not exist any $U \in S.O.(\tau_A, d)$ such that $(\underline{U}\tau_A)\alpha' \subset \overline{\{b,c\}} = \{b, c, d\}$.

The product of two closure continuous mappings is closure continuous. But the product of two closure semi-continuous mappings is not always closure semi-continuous as shown by

Example 6: Let $X = \{x, y, z\}$,

$\tau_1 = \{\phi, X, \{x\} , \{y, z\}\}$,

$\tau_2 = \{\phi, X, \{x\} , \{y\}, \{x, y\}\}$,

τ_3 = discrete topology in X.

$I_x : (X,\tau_1) \to (X,\tau_2)$ is closure semi-continuous.

$\alpha : (X,\tau_2) \to (X,\tau_3)$ defined by

$(x)\alpha = x$, $(y)\alpha = y$, $(z)\alpha = x$

is also closure semi-continuous

But $I_x\alpha : (X,\tau_1) \to (X,\tau_3)$ is not closure semi-continuous at z. For, $\{x\} \in O(\tau_3, (z)\alpha)$ but there does not exist any $U \in S.O.(\tau_1, z)$ such that $(\underline{U})\alpha \subset \overline{\{x\}} = \{x\}$.

Theorem 7: Let $\alpha_i : (X_i,\tau_i) \to (Y_i,\tau_i')$ be closure semi-continuous for $i = 1,2$. Let $X = X_1 x X_2$, $Y = Y_1 x Y_2$, $\tau = \tau_1 x \tau_2$, $\tau' = \tau_1' x \tau_2'$, $\alpha = \alpha_1 x \alpha_2$. Then $\alpha : (X,\tau) \to (Y,\tau')$ is closure semi-continuous.

Proof: Let $x = (x_1, x_2) \in X$ and $W \in O(\tau', (x)\alpha)$. Then there exist $U_i \in O(\tau_i', (x_i)\alpha_i)$ $(i = 1, 2)$ such that $U_1 x U_2 \subset W$.

Since α_i is closure semi-continuous, there exist $V_i \in$ S.O.(τ_i, x_i) such that $(\underline{V}_i)\alpha \subset \overline{U}_i$ $(i = 1, 2)$. Then $V = V_1 \times V_2 \in$ S.O. (τ, x) and $(\underline{V})\alpha = (\underline{V}_1 \times \underline{V}_2)\alpha = (\underline{V}_1)\alpha_1 \times (\underline{V}_2)\alpha_2 \subset \bar{U}_1 \times \bar{U}_2 \subset \bar{W}$. $\therefore$ α is closure semi-continuous.

REFERENCES

[1] Andrew, D.R. and Whittlesy, E.K. (1966). Closure continuity, Amer. Math. Monthly, 73, 758-759.

[2] Biswas, N. (1970). On characterizations of semi-continuous functions, Atti. Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Math. Natur., (8):48, 399-402.

[3] Crossley, S.G. and Hildebrand, S.K. (1971). Semi-closure, Texas J. Sc., 22 (2 + 3), 99-112.

[4] Das, Phullendu (1970). Note on ST_i - spaces. Proc. 57th session of Indian Science Congress.

[5] Das, Phullendu (1973). Note on some applications of semi-open sets. Progress of Mathematics, Vol. 7, No. 1, 33-44.

[6] Levine, N. (1965). Semi-open sets and semi-continuity in topological spaces. Amer. Math. Monthly, 70, 36-41.

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Solving Integral Equation of n-Variables by $\mathcal{L}$ and $\mathcal{L}^{-1}$ Operators

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1. Introduction

Recently in the year 1971, Fox [4] has solved a large variety of integral equations of one variable by using Laplace transforms and their inverses. In this connection he proved the following property in the form of a theorem as -

If (i) $\alpha > 0$, $\frac{1}{2}\alpha + \beta > 0$, $t > 0$ (ii) $s = \sigma + i\mu$, σ and μ both real; $F(s) \in L\left(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty\right)$,

then

$$\mathcal{L}^{-1}\left\{\frac{1}{2\pi i}\int_C \Gamma(\alpha s+\beta)\, F(s) t^{-(\alpha s+\beta)} ds\right\} = \frac{1}{2\pi i}\int_C F(s) x^{\alpha s+\beta-1}\, ds \tag{1.1}$$

where, for both the integrals the contour C may be the line $\sigma = \frac{1}{2}$, a line parallel to the imaginary axis in the complex s-plane.

An attempt has been made in the present paper, to generalize the above result (1.1) to the case of n-variables and then apply this result in solving an integral equation of n-variables.

2. Laplace and Mellin Transforms of n-variables

Ahuja [2] has defined the Laplace transform of n-variables in the same fashion as in the case of two variables as

$$\mathcal{L}\{f(x_n)\} = \phi(p_n) = (n)\int_0^\infty f(x_n) \prod_{r=1}^{n} \{\exp(-p_r x_r)\,(dx_r)\}. \tag{2.1}$$

The inverse Laplace transform of $\phi(p_n)$ is $f(x_n)$, and is denoted by

$$f(x_n) = \mathcal{L}^{-1}\{\phi(p_n)\} = \frac{1}{(2\pi i)^n}\,(n)\int_{\sigma_n - i\infty}^{\sigma_n + i\infty} \phi(p_n) \prod_{r=1}^{n} \{\exp(p_r x_r)\,(dx_r)\} \tag{2.2}$$

where $(n)\int_0^\infty$ denotes the product of integrals $\int_0^\infty \int_0^\infty \ldots$ n times. Also equations (2.1) and (2.2) implies that

$$\mathcal{L}^{-1}\,[\{\phi(p_n)\}] = \phi(p_n) \tag{2.3}$$

In 1974, Aggarwal and Goyal [1] have defined the Mellin transform and its inverse for n-variables under similar suitable conditions as due to Reed [5] as:

If $$M\,\{f(x_n)\} = F(s_n) = (n)\int_0^\infty f(x_n) \prod_{i=1}^{n} \{x_i^{s_i-1}\,(ds_i)\} \tag{2.4}$$

then

$$M^{-1}\,\{F(s_n)\} = f(x_n) = \frac{1}{(2\pi i)^n}\,(n)\int_{-i\infty}^{i\infty} F(s_n) \prod_{i=1}^{n} x_i^{-s_i}\,(ds_i)\} . \tag{2.5}$$

and also restated the Parseval's theorem for n variables identical to the one by Fox [3], for one variable, in the form as:

If $M\,\{h(u_n)\} = H(s_n)$ and $M\,\{f(x_n u_n)\} = F(s_n) \prod_{i=1}^{n}\{x_i^{-s_i}\}$

where $M\,\{f(u_n)\} = F(s_n)$,

then

$$(n)\int_0^\infty h(x_n u_n)\,f(u_n) \prod_{i=1}^{n} \{(du_i)\} = \frac{1}{(2\pi i)^n}\,(n)\int_{-i\infty}^{i\infty} H(s_n)F(1-s_n) \prod_{i=1}^{n}\{x_i^{-s_i}(ds_i)\}. \tag{2.6}$$

3. In this section we shall prove a theorem which is a generalization of (1.1).

<u>Theorem 1</u> - If

(i) $\alpha_r > 0$, $\frac{1}{2}\alpha_r + \beta_r > 0$, $t_r > 0$ and

(ii) $s_r = \sigma_r + i\mu_r$, σ_r and μ_r both real; $F(s_n) \in L((\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ $(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty), \ldots$ n times), then

$$\mathcal{L}^{-1}\left[\frac{1}{(2\pi i)^n}\,(n)\int_{C_n} F(s_n) \prod_{r=1}^{n} \{\Gamma(\alpha_r s_r + \beta_r)\,t_r^{-\alpha_r s_r - \beta_r}\,(ds_r)\}\right]$$

$$= \frac{1}{(2\pi i)^n}\,(n)\int_{C_n} F(s_n) \prod_{r=1}^{n} \{x_r^{\alpha_r s_r + \beta_r - 1}\,(ds_r)\}, \tag{3.1}$$

where for (n) integrals the contour C_n may be the line $\sigma_r = \frac{1}{2}$, $r = 1, 2, \ldots, n$; (n) lines parallel respectively to the imaginary axes in the complex $s_1, s_2, \ldots, s_n$ spaces.

<u>Proof</u> - Let

$$R_n = \frac{1}{(2\pi i)^n} (n) \int_{C_n} F(s_n) \prod_{r=1}^{n} \{x_r^{\alpha_r s_r + \beta_r - 1} (ds_r)\}, \tag{3.2}$$

where C_n are the n-lines $\sigma_r = \frac{1}{2}$, $r = 1, 2, \ldots, n$. From the conditions $\alpha_r > 0$ and $F(s_n) \in L\,((\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)X(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)X\ldots \text{ n times})$, we see that the n integrals in (3.2) with the factor $\prod_{r=1}^{n} (x_r^{\beta r - 1})$ excluded, is absolutely convergent for all x_r. Now, let us consider,

$$\mathcal{L}\{R_n\} = (n) \int_0^\infty \prod_{r=1}^{n} (e^{-t_r x_r}) \{ \frac{1}{(2\pi i)^n} (n) \int_{C_n} F(s_n) \prod_{r=1}^{n} \{(x_r^{\alpha_r s_r + \beta_r - 1} (ds_r)\}(dx_r). \tag{3.3}$$

where C_n are the (n) lines $\sigma_r = \frac{1}{2}$, $r = 1, 2 \ldots, n$.

The integrand of the integral on the right hand side of the equation (3.3) is

$$F(s_n) \prod_{r=1}^{n} \{(e^{-t_r x_r}) x_r^{\alpha_r s_r + \beta_r - 1}\}.$$

Modules of this integrand is

$$| F(s_n) | \prod_{r=1}^{n} \{(e^{-t_r x_r}) (x^{\frac{1}{2}\alpha_r + \beta_r - 1}\}.$$

Hence from the conditions $\frac{1}{2}\alpha_r + \beta_r > 0$, $r = 1, 2\ldots, n$ and $F(s_n) \in L\,((\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)X(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)X\ldots \text{ n times})$,

we see that

$$\frac{1}{(2\pi i)^n} (n) \int_0^\infty (n) \int_{C_n}^\infty |F(s_n)| \prod_{r=1}^{n} \{|e^{-t_r x_r} x_r^{\alpha_r s_r + \beta_r - 1}| (ds_r) (dx_r)\}$$

is convergent. Therefore the right hand side of (3.3) is an absolutely convergent integral. So we may change the order of integration and integrate first with respect to x. Thus, we get

$$\mathcal{L}\{R_n\} = \frac{1}{(2\pi i)^n}\,(n)\int_{C_n} F(s_n)\Big\{(n)\int_0^{\infty} \prod_{r=1}^{n} \{(e^{-t_r x_r})\, x_r^{\alpha_r s_r + \beta_r - 1}\,(dx_r)\,(ds_r)\}$$

$$= \frac{1}{(2\pi i)^n}\,(n)\int_{C_n} F(s_n) \prod_{r=1}^{n} \{t_r^{-\alpha_r s_r - \beta_r}\,\Gamma(\alpha_r s_r + \beta_r)\,(ds_r)\} \,. \quad (3.4)$$

On applying the inverse operator $\mathcal{L}^{-1}$ to both the sides of (3.4) and then on using equation (2.3), we get the required result (3.1).

4. In this section, with the help of Theorem 1, we shall obtain the solution of the integral equation

$$g(t_n) = (n)\int_0^{\infty} f(u_n) \prod_{r=1}^{n} \{(u_r t_r)^{\nu_r} K_{\nu_r}(u_r t_r)\,(du_r)\} \,, \quad (4.1)$$

where $K_{\nu_r}(x_r)$ for $r = 1, 2, \ldots, n$, is the associated Bessel function defined by Watson [7, p. 78 (6)], $g(t_n)$ is a known function and $f(x_n)$ is to be found.

Proof - By [6, p. 197; eq^n (7.9.12)],

$$M\Big[\prod_{r=1}^{n} \{u^{\nu_r} K_{\nu_r}(u_r)\}\Big] = \prod_{r=1}^{n} \{2^{s_r + \nu_r - 2}\,\Gamma(\tfrac{s_r}{2})\,\Gamma(\tfrac{s_r}{2} + \nu_r)\} \,. \quad (4.2)$$

On applying Parseval's theorem of n-variables given by (2.6), to the right hand side of equation (4.1), we get

$$g(t_n) = \frac{1}{(2\pi i)^n}\,(n)\int_{-i\infty}^{i\infty} F(1-s_n) \prod_{r=1}^{n} \{2^{s_r + \nu_r - 2}\,\Gamma(\tfrac{s_r}{2})\,\Gamma(\tfrac{s_r}{2} + \nu_r)\cdot t_r^{-s_r}\,(ds_r)\} \,. \quad (4.3)$$

Now, applying the inverse Laplace transform $\mathcal{L}^{-1}$ to both the sides of equation (4.3) and then using equation (3.1), we obtain

$$\mathcal{L}^{-1}\{g(t_n^{\frac{1}{2}})\} = \frac{1}{(2\pi i)^n}\,(n)\int_{-i\infty}^{i\infty} F(1-s_n)\prod_{r=1}^{n}\{2^{\frac{s_r}{2}+\nu_r-2}\, x_r^{\frac{s_r}{2}-1}\,\Gamma(\frac{s_r}{2}+\nu_r)(ds_r)\}$$

writing $x_n = \frac{1}{t_n}$ and multiplying both the sides by $t_n^{-\nu_n-1}$, we get

$$t_n^{-\nu_n-1}\,\mathcal{L}^{-1}\{[g(t_n^{1/2})]_{x_n=\frac{1}{t_n}}\} = \frac{1}{(2\pi i)^n}\,(n)\int_{-i\infty}^{i\infty} F(1-s_n)\prod_{r=1}^{n}\{t^{-\frac{s_r}{2}-\nu_r}$$

$$2^{\frac{s_r}{2}+\nu_r-2}\,\Gamma(\frac{s_r}{2}+\nu_r)\,(ds_r)\}\ .$$

Again using the inverse Laplace transform $\mathcal{L}^{-1}$ to both the sides of equation, we obtain

$$\mathcal{L}^{-1}[t_n^{\nu_n-1}\{[\mathcal{L}^{-1}\,g(t_n^{1/2})]_{x_n=\frac{1}{t_n}}\}]$$

$$= \frac{1}{(2\pi i)^n}\,(n)\int_{-i\infty}^{i\infty} F(1-s_n)\prod_{r=1}^{n}\{2^{s_r+\nu_r-2}\, x_r^{\frac{s_r}{2}+\nu_r-1}\,(ds_r)\}$$

$$= \frac{1}{(2\pi i)^n}\,(n)\int_{i\infty}^{i\infty} F(1-s_n)\prod_{r=1}^{n}\{2^{\nu_r-1}\, x^{\nu_r-1/2}\,(2x_r^{1/2})^{s_r-1}\,(ds_r)\}$$

$$= \frac{1}{(2\pi i)^n}\int_{r=1}^{n}\{2^{\nu_r-1}\, x^{\nu_r-1}\, f(2x_r^{-1/2})\}\ ,$$

which is the required solution of our equation (4.1).

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REFERENCES

[1] Aggarwal, I. and Goyal, A.N. (1974). On a class of Dual Integral Equations Involving $G(x_n)$. Ind. J. Pure Appl. Math., Vol. 5, pp. 572-584.

[2] Ahuja, G. (1978): "Study of Certain Integral Transforms and Integral Equations." Thesis submitted to L.U., Lucknow.

[3] Fox, C. (1965): "A Formal Solution of Certain Dual Integral Equations." Trans. Amer. Math. Sco., 119, 389-95.

[4] Fox, C. (1971): "Solving Integral Equations by L and L^{-1} Operators." Proc. Amer. Math. Soc., Vol. 39, No. 2, pp. 299-306.

[5] Reed, I.S. (1944): "The Mellin Type of Double Integral." Duke Math. J., pp. 565.

[6] Titchmarsh, E.C. (1977): "Theory of Fourier Integrals." Clarendon Press, Oxford.

[7] Watson, G.N. (1922): "Theory of Bessel Functions." Cambridge Univ. Press, New York.

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Infinistesimal Variation of Hypersurfaces of an Almost r-Contact Hyperbolic Structure Manifold

C.B. Singh & Jaya Pant

Summary

The infinitesimal variation of the structure tensors of an almost contact metric structure induced on the hypersurface of a Kahlerian manifold under various conditions has been studied by Yano. In this paper we have studied the infinitesimal variation of the structure tensors of an almost r-contact hyperbolic structure induced on the hypersurface of a differentiable manifold equipped with an almost r-contact hyperbolic structure.

1. Introduction

Let M^{n+r} be an (n+r) dimensional differentiable manifold of differentiability class C^∞. Let there exist on M^{n+r} a C^∞ vector valued linear function F, r-C^∞ linearly independent and non zero contravariant vector fields $T^1, T^2, \ldots, T^r$ such that

$$(1.1) \qquad F^2X = X + \sum_{\ell=1}^{r} A_\ell(X)T^\ell$$

for arbitrary vector field X on M^{n+r}. Also

$$(1.2) \qquad F(X) \overset{\text{def}}{===} \overline{X}.$$

In view of (1.1) let M^{n+r} be endowed with the Riemannian metric tensor G such that it satisfies the following condition

$$(1.3) \qquad G(\overline{X},\overline{Y}) + G(X,Y) + \sum_{\ell=1}^{r} A_\ell(X)A_\ell(Y) = 0.$$

Thus M^{n+r} satisfying the conditions (1.1) and (1.3) will be called an almost r-contact hyperbolic structure manifold [2].

In M^{n+r}, the following results hold

$$\text{(1.4)}\quad \begin{aligned} &(a)\quad \overline{T}^{\ell} = 0 , \\ &(b)\quad A_{\ell}(\overline{X}) = 0 , \text{ for arbitrary vector field } X \\ &(c)\quad A_{\ell}(T^{m}) + \delta_{\ell}^{m} = 0 , \end{aligned}$$

where δ_{ℓ}^{m} is Kronecker delta and ℓ , m take the values 1,2, ..., r.

Let us imbed a hypersurface M^{n+r-1} into M^{n+r} by the isometric immersion $b : M^{n+r-1} \to M^{n+r}$. Corresponding to this we have the Jacobian b^{*} of b denoted by B which carries $T_{q}(M^{n+r-1})$ into $T_{b(q)}(M^{n+r})$ injectively. Since the immersion is isometric, we have

$$\text{(1.5)}\qquad G(BX,BY) \circ b = g(X,Y) ,$$

g being the metric induced on the hypersurface and X,Y denote arbitrary vector fields. We have

$$\text{(1.6)}\qquad G(BX,N) = 0 ,$$

$$\text{(1.7)}\qquad G(N,N) = 1 .$$

The transformation equations are

$$\text{(1.8)}\qquad FBX = BfX + \alpha(X)N ,$$

$$\text{(1.9)}\qquad FN = BP + \eta N ,$$

where f is a tensor field of type (1.1) and α is a 1-form on M^{n+r-1} from equation (1.8) and the relations

$$\text{(1.10)}\quad \begin{aligned} &(a)\quad T^{\ell} = Bt^{\ell} + P_{\ell} N , \\ &(b)\quad A_{\ell}(BX) \circ b = a_{\ell}(X) , \\ &(c)\quad \alpha(X)P = 0 . \end{aligned}$$

we get

$$\text{(1.11)}\qquad f^{2}X = X + \sum_{\ell=1}^{r} a_{\ell}(X)t^{\ell} .$$

The metric g in (1.5) is found to satisfy

$$\text{(1.12)}\qquad g(fX,fY) + g(X,Y) + \sum_{\ell=1}^{r} a_{\ell}(X)a_{\ell}(Y) = 0 ,$$

Consequently an almost r-contact hyperbolic structure gets induced on M^{n+r-1}.

Let D be the Riemannian connexion induced on M^{n+r-1} by g. Then we have the Gauss and Weingartan equation [1]

$$(1.13)\qquad E_{BX}BY = BD_XY + H(X,Y)N\ ,$$

$$(1.14)\qquad E_{BX}N = -\,B'HX\ ,$$

where H is the 2^{nd} fundamental form of M^{n+r-1} and 'H is a tensor field of type (1,1) associated with H. Let K and $\tilde{K}$ stand for the curvature tensors of the hypersurface and the enveloping manifold. Then we have Gauss and Codazzi equations

$$(1.15)\qquad \tilde{K}(BX,BY,BZ,BU) = {}'K(X,Y,Z,U) - H(Y,Z)H(X,U) + H(X,Z)\ H(Y,Y)$$

and

$$(1.16)\qquad \tilde{K}(BX,BY,BZ,N) = (D_XH)(Y,Z) - (D_YH)(X,Z)$$

where 'K and $'\tilde{K}$ are the associate covariant curvature tensors of M^{n+r-1} and M^{n+r}.

Now let us differentiate equation (1.8) along the hypersurface and use $(E_{\tilde{X}}F) = 0$, hence

$$E_{BX}BfY = F((E_{BX}BY) - \{(D_XA)Y + A(D_XY)\}\ N - A(Y)E_{BX}N$$

In view of (1.9), (1.13) and (1.14) we get

$$(1.17)\qquad (D_Xf)Y = H(X,Y)P + \alpha(Y)\ 'HX\ ,$$

$$(1.18)\qquad (D_X\alpha)Y = H(X,Y)\eta - H(X,fY).$$

Covariant differentiation of (1.9) along M^{n+r-1} yields

$$(1.19)\qquad D_XP = a'HX - {}'HfX\ .$$

<u>Definition 1.1</u>: An almost r-contact hyperbolic structure is said to be normal if

$$(1.20)\qquad S(X,Y) = N(X,Y) + \sum_{\ell=1}^{r} \{(D_X\alpha_\ell)Y - (D_Y\alpha_\ell)X\}\ t^\ell = 0$$

where

$$N(X,Y) = (D_{fX}f)Y - (D_{fY}f)X + f(D_Y f)X - f(D_X f)Y + \sum_{\ell=1}^{r} a_\ell [X,Y]\ t^\ell ,$$

so that the normality condition (1.20) takes the form

$$S(X,Y) = (D_{fX}f)Y - (D_{fY}f)X + f(D_Y f)X - f(D_X f)Y$$
$$+ \sum_{\ell=1}^{r} a_\ell [X,Y]\ t^\ell + \sum_{\ell=1}^{r} \{(D_X\alpha_\ell)Y - (D_Y\alpha_\ell)X\}\ t^\ell = 0 .$$

If almost r-contact hyperbolic structure induces on M^{n+r} be normal, from the last equation and from (1.17) and (1.18), we obtain

$$\alpha(X)\ \{'Hf - f'H\}\ Y - \alpha(Y)\ \{'Hf - f'H\}\ X = 0 .$$

(1.21) $\quad 'Hf = f'H$.

Therefore, it follows that [1]

(1.22) $\quad H(P,P) = {}'HP$

showing that $H(P,P)$ is an eigen value of $'H$ and the corresponding eigen vector is P. Let us denote $H(P,P)$ by τ.

<u>Definition 1.2</u>: An almost r-contact hyperbolic structure is called r-hyperbolic Sasakian if

(1.23) $\quad \sum_{\ell=1}^{r} \{(D_X\alpha_\ell)Y - (D_Y\alpha_\ell)X\} = r'f(X,Y)$

we have $'f(X,Y) = g(fX,Y)$.

More generally in a normal r-contact hyperbolic structure hypersurface of M^{n+r} we assume that [3]

(1.24) $\quad \sum_{\ell=1}^{r} \{(D_X\alpha_\ell)Y - (D_Y\alpha_\ell)X\} = r\ \beta\ 'f(X,Y)$.

Applying (1.18) to the above equation we have

(1.25) $\quad 'H\eta = {}'Hf = -\ r'\ \beta\ f.$

Thus we obtain

(1.26) $\quad 'HX = -\ r'\ \beta\ X + (\tau + r'\ \beta)\ \alpha(X)P.$

Equations (1.17), (1.18), (1.19) then transform as

(1.27) $\quad (D_X f)Y = - r'\beta \{g(X,Y)P + \alpha(Y)X\} + 2(\tau + r'\beta)\,\alpha(X)\,\alpha(Y)$

(1.28) $\quad (D_X \alpha)Y = r'\,\beta\, f(X,Y)$,

(1.29) $\quad D_X P = - (\eta - f)\, r'\,\beta\, X.$

Let β be a constant so that from (1.27) and (1.29) we obtain

$$K(X,Y,P) = - r'^2\ \beta^2 \eta\ \ (\alpha(Y)X - \alpha(X)Y)\ ,$$

which shows that for a normal r-contact hyperbolic structure hypersurface satisfying (1.24) and involving constant $r'\ \beta$, the sectional curvature with respect to a plane section containing P is $r'^2\ \beta^2$.

Let us call such a structure a normal r-contact hyperbolic structure with f sectional curvature $r'^2\ \beta^2$.

2. Infinitestimal Variation of a Hypersurface of an Almost r-Contact Hyperbolic Structure Manifold

Let us take the restriction of an almost decomposable killing vector field U on the enveloping manifold of the hypersurface. According the variation of the differential of imbedding is given by [4]

(2.1) $$(\delta B)X = \varepsilon E_{BX} U$$

where ε is infinitesimally small number spliting U into its tangential and normal parts as

(2.2) $$U = BV + \lambda N$$

and from (1.13), (1.14) we express (2.1) as

(2.3) $\quad (\delta B)(X) = \varepsilon\{B(D_X V - \lambda' HX) + (X\lambda + H(X,V)N\}$.

Infinitesimal variation of N is given by [5]

$$\delta N = \varepsilon L_U N = \varepsilon BW\ .$$

The Lie derivative of N (i.e. $L_U N$) being orthogonal to N, infinitesimal variation of equation (1.6) yields

$$G(BD_{\bar{X}}V + H(X,V)N + (X\lambda)N - \lambda B'HX, N) = -G(BX.BW)$$

which implies that $W = -('HV + \Lambda)$ where Λ stands for the vector field associate to the gradient of λ. Thus we have

$$\delta N = -B('HV + \Lambda)$$

Now varying equation (1.8) infinitesimally, we get

$$(\delta B)(fX) + B(\delta f)X = F((\delta B)X) - (\delta N)\alpha(X) - \delta\alpha(X)N$$

Making use of (1.8), (2.3) and (2.4) in it, we find

$$B(\delta f)X + (\delta\alpha)(X)N = \varepsilon\Big[\{Bf(D_{\bar{X}}V - \lambda'HX) + \alpha(D_XV - \lambda'HX)N$$

$$+ (X\lambda + H(X,V)(BP + \eta N) + \alpha(X)B('VH + \Lambda)$$

$$- \{B(D_{fX}V - \lambda'HfX) + (fX\lambda + H(fX,V)N\}\Big]$$

Comparing the tangential and normal components, we have

$$(2.5) \qquad (\delta f)X = \varepsilon\{f(D_XV - \lambda'HX) + (H(X,V) + \lambda)P$$

$$+\alpha(X)('HV + \Lambda) - D_{fX}V + \lambda'HfX\}$$

and

$$(2.6) \qquad (\delta\alpha)X = \varepsilon\{(D_XV - \lambda'HX) + \eta(X\lambda + H(X,V) - fX\lambda - H(fX,V)\}$$

Since the derivative of f along V is given by

$$(L_Vf)X = L_V(fX) - f(L_VX) = D_V(fX) - D_{fX}V - f(D_VX - D_XV).$$

Therefore equation (2.5) assumes the following form

$$(\delta f)X = \varepsilon\{(L_Vf)X + \lambda('Hf - f'H)X + X\lambda\ P + \alpha(X)\Lambda + 2H(X,V)P\}$$

Applying equation (1.18) and the definition

$$(L_V\alpha)\ X \overset{\text{def}}{=\!=\!=} (D_V\alpha)X + \alpha(D_XV)$$

$$(2.8) \qquad (\delta\alpha)X = \Big[\{(L_V\alpha)X - \alpha\lambda'HX - (fX)\lambda\}$$

$$+ 2H(X,V)\eta + 2H(V,fX)\Big].$$

Next varying equation (1.9) infinitesimally, we get

$$- \varepsilon FB('HV + \Lambda) = [B(\delta P) + \varepsilon\{ B(D_P V - \lambda' HP) + P\lambda + H(P,V)N\}$$
$$- \varepsilon\eta B('HV + \Lambda)] ,$$

which by virtue of (1.8) and (2.3) yields

$$B\delta P + \varepsilon\{B(D_P V - \lambda' HP) + (P\lambda + H(P,V)N\} - \varepsilon\eta B('HV + \Lambda)$$
$$= - \varepsilon[Bf('HV + \Lambda) + \alpha('HV + \Lambda) N],$$

whose tangential part reduces in virtue of (1.19) to the form

$$(2.9) \qquad \delta P = \varepsilon[\lambda' HP + L_U P + \Lambda(\eta - f)] .$$

Again varying equation (1.5) infinitesimally, we get

$$(\delta g)(X,Y) = G((\delta B)X, BY) + G(BX, (\delta B)Y) ,$$

which in virtue of (2.3) reduces to

$$(2.10) \qquad (\delta g)(X,Y) = \varepsilon\{(L_V g)(X,Y) - 2\lambda\, H(X,Y)\} .$$

Thus we establish the following theorem.

<u>Theorem 2.1</u>: When a hypersurface of an almost r-contact hyperbolic structure manifold varied infinitesimally by means of a vector field $U = BV + \lambda N$ the structure tensors of almost r-contact hyperbolic structure hypersurface vary according to equations (2.7), (2.8), (2.9) and (2.10).

<u>Corollary 2.1</u>: When a hypersurface of an almost r-contact hyperbolic structure manifold is given infinitesimally tangential variation by means of BV, the variation of the induced almost r-contact hyperbolic structure tensors on the hypersurface are given by their Lie derivatives along V.

<u>Corollary 2.2</u>: When a hypersurface of an almost r-contact hyperbolic structure manifold is given infinitesimal normal variation by means of λN, the variation of the induced almost r-contact hyperbolic structure tensors on the hypersurface are given by

$$
(2.11)\quad
\begin{aligned}
&(a)\quad (\delta f)X = \varepsilon[\lambda('Hf - f'H)X + X\lambda P + \alpha(X)\Lambda + 2H(X,V)P],\\
&(b)\quad (\delta\alpha)X = \varepsilon[-\alpha\lambda'HX - fX\lambda + 2H(X,V)\eta + 2H(V,fX)],\\
&(c)\quad \delta P = \varepsilon[\lambda'HP + \Lambda(\eta - f)],\\
&(d)\quad (\delta g)(X,Y) = -2\varepsilon\lambda H(X,Y).
\end{aligned}
$$

The infinitesimal variation is said to be parallel when BX and $\bar{B}X$ are both parallel equivalently and when $(\delta B)X$ is tangential to the original hypersurface. Since

$$(\delta B)X = \varepsilon\{B(D_X V - \lambda'HX) + (X\lambda + H(X,V)N\},$$

therefore for an infinitesimal parallel variation it is necessary and sufficient that

$$(2.12)\qquad X\lambda + H(X,V) = 0,$$

<u>Corollary 2.3</u>: When hypersurface of an almost r-contact hyperbolic structure manifold is given infinitesimal parallel variation the hypersurface varies as

$$
(2.13)\quad
\begin{aligned}
&(a)\quad (\delta f)X = \varepsilon[\lambda('Hf - f'H) + \alpha(X)\Lambda],\\
&(b)\quad (\delta\alpha)X = \varepsilon[-\alpha\lambda'HX],\\
&(c)\quad \delta P = \varepsilon\lambda'HP,\\
&(d)\quad (\delta g)(X,Y) = -2\varepsilon\lambda H(X,Y).
\end{aligned}
$$

<u>Corollary 2.4:</u> Let the structure induced on a hypersurface of an almost r-contact hyperbolic structure manifold be a normal r-contact hyperbolic structure with f-sectional curvature $-r'^2\beta^2$ then the infinitesimal normal parallel variation of the hypersurface makes the structure tensor vary as

$$
(2.14)\quad
\begin{aligned}
&(\delta f)X = (X)\Lambda,\\
&(\delta\alpha)X = -\varepsilon\lambda\tau P,\\
&\delta P = \varepsilon\lambda\tau P,\\
&(\delta g)(X,Y) = -2\varepsilon\lambda\{-r'\beta g(X,Y) + (\tau + r'\beta)\alpha(X)\alpha(Y)\}.
\end{aligned}
$$

3. Variation of r-Hyperbolic Sasakian Hypersurface with f-Sectional Curvature $r'^2\ \beta^2$.

We now assume that an almost r-contact hyperbolic structure induced on the hypersurface is a r-hyperbolic Sasakian structure with f-sectional curvature $r'^2\ \beta^2$, we have [1]

$$H(X,'HY) = r'^2\ \beta^2\ g(X,Y) + (\tau^2 + r'^2\ \beta^2)\ \alpha(X)\ \alpha(Y) \tag{3.1}$$

and

$$H(X,Y) = -\ r'\beta\ g(X,Y) - r'\beta\ (\delta g)(X,Y) + (\tau + r'\beta)\ \alpha(X)\ \alpha(Y)\ . \tag{3.2}$$

The variation in the connexions and the second fundamental form are given by [1]

$$(\delta D)(X,Y) = \varepsilon(L_U D)(X.Y) - D_Y\ \lambda'H)X - (D_X\lambda'H)Y + H(X,Y) + \lambda H^*(X,Y) \tag{3.3}$$

where $\quad gH^*(X,Y)Z = (D_Z H)(X,Y)$

and

$$(\delta H))X,Y) = \varepsilon\{(L_V H)(X,Y) - \lambda H(X,'HY) + XY\lambda - (D_X Y)\lambda + \lambda'\tilde{K}(N,\ BX,BY,N)\}\ . \tag{3.4}$$

If the infinitesimal variation of the hypersurface are normal the variation of D and H would be given by [1]

$$(\delta D)(X,Y) = \varepsilon\{XY\lambda - (D_X Y)\lambda + '\tilde{K}(N,BX,BY,N) - \lambda H(X,'HY)\}. \tag{3.5}$$

Varying equation (3.2) infinitesimally, we have

$$(\delta H)(X,Y) = -\ (\delta r'\beta)g(X,Y) - r'\beta\ (\delta g)(X,Y) + \delta(\tau + r'\beta)\ \alpha(X)\ \alpha(Y) + (\tau + r'\beta)\ \}\ (\delta\alpha)(X)\ \alpha(Y) + \alpha(X)(\delta\ \alpha)(Y)\}\ , \tag{3.6}$$

which with the help of equations (2.8), (2.9), (2.10), (3.5) and

$$(3.7)\qquad (L_V H)(X,Y) = -r'\beta\,((L_V g)(X,Y) + \{(L_V H)(P,P) + 2H(L_V P,P)\,\alpha(X)\,\alpha(Y) + (\tau + r'\beta)\,(L_V \alpha)(X)\,\alpha(Y) + \alpha(X)\,(L_V \alpha)(Y)\}$$

becomes

$$\varepsilon\{XY\lambda - (D_X Y)\lambda + \lambda'\tilde{K}(N,BX,BY,N) - \lambda H(X,'HY)\}$$
$$= -r'\beta\varepsilon\lambda\ H(X,Y) + \varepsilon\{PP\lambda - (D_P P)\lambda$$
$$(3.8)\qquad - \lambda H(P,HP) - 2H(P,\ \lambda HP - \Lambda(\eta - f)) + \frac{\delta r'\beta}{\varepsilon}\}\alpha(X)\,\alpha(Y)$$
$$+ \varepsilon(\tau + r'\beta)\ \{-\alpha\lambda'HX - fX\lambda + 2H(X,V)\eta + 2H(V,fX)\}\ \alpha(Y) + \{-\alpha\lambda'HY + fY\lambda + 2H(Y,V) + 2H(V,fY)\}\,\alpha(X).$$

Conversely if λ satisfies the differential equation (3.8) then by retreating the steps we get (3.3).

Hence we have the following theorem :

Theorem 3.1: In order that for an infinitesimal variation (2.1) may have a r-hyperbolic Sasakian hypersurface with f-sectional curvature $-r'^2\beta^2$ in a r-hyperbolic Sasakian with f-sectional curvature $-r'^2\beta^2 - \delta r'^2\beta^2$, it is necessary and sufficient that the function λ satisfies the relation

$$\varepsilon\,[XY\lambda - (D_X Y)\lambda + \lambda\{'K(N,BX,BY,N) + r'^2\beta^2\,(g(X,Y) - \alpha(X)\,\alpha(Y))\}$$
$$+ (PP\lambda - (D_P P)\lambda)\ \alpha(X)\,\alpha(Y) + (\tau + r'\beta)\ \{- fX\lambda\,\alpha(Y) - fY\lambda\,\alpha(X)\} + \{2H(X,V)\eta + 2H(V,fX)\}\,\alpha(Y)$$
$$+ \{2H(Y,V) + 2H(V,fY)\}\,\alpha(X)\,]$$
$$= \delta r'\beta\ (\alpha(X)\,\alpha(Y) - g(X,Y))\ .$$

Corollary 3.1: The infinitesimal normal parallel variation carries a normal r-hyperbolic Sasakian hypersurface with f-sectional curvature $-r'^2\beta^2$ to a normal r-hyperbolic Sasakian hypersurface with f-sectional curvature $-r'^2\beta^2 - \delta r'^2\beta^2$ if and only if

$$(3.9) \qquad \lambda\varepsilon\{'K(N,BX,BY,N) + r'^2\ \beta^2\ (g(X,Y) - \alpha(X)\ \alpha(Y))$$

$$= \{\alpha(X)\ \alpha(Y) - g(X,Y)\}\ \ \delta r'\beta$$

Corollary 3.2: If the enveloping manifold of corollary (3.1) be flat the condition reduces to $\delta r'\beta\ = -\ \lambda\varepsilon r'^2\ \beta^2$

Hence the proof is obvious.

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REFERENCES

[1] B.B. Sinha and Ramesh Sharma (1979): Infinitesimal variations of hypersurfaces of an almost product and almost decomparable manifold. Ind. J. of Pure and Appl. Math., Vol. 10/8, 1009-1019.

[2] K.K. Dube and Ram Nivas (1978): Almost r-contact hyperbolic structure in a product manifold. Demonstratio Mathematica, Vol. XI, No. 4.

[3] Sato, I. (1977): On a structure similar to almost contact structure II. Tensor (NS) 31, 199-205.

[4] Yano, K. (1977): Infinitesimal variation of hypersurface of a Kahlerian manifold. J. Math. Soc., Japan, 29, 287-301.

[5] Yano, K. (1957): The theory of Lie-derivative and its application. North Holland Publishing Co. Amsterdam.

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Common Fixed Points for Self Maps

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In this paper, we obtain sufficient conditions (i) for a family of continuous self maps on a compact topological space and (ii) for an equicontinuous family of self maps on a metric space, to admit common fixed points. From these, the fixed point theorems of Kirk [1] follow as corollaries.

Definition 1: For a non-empty family $\mathcal{F}$ of self-maps on a non-empty set X and $x \in X$, the orbit $O(\mathcal{F}, x)$ of x with respect to $\mathcal{F}$ is defined as

$$O(\mathcal{F}, x) = \{y \mid y = x \text{ or } y \text{ is the image of } x \text{ under the composite of a finite number of elements of } \mathcal{F}\}.$$

(This idea of the orbit of a point with respect to a family of self maps is due to Madhusudhana Rao [2]).

We write $O(x)$ for $O(\mathcal{F}, x)$ when $\mathcal{F}$ is understood. For a subset A of a topological space, cl. A is the closure of A.

Theorem 2: Let $(X, \mathcal{J})$ be a compact topological space, $\mathcal{F}$ a non-empty family of continuous self maps on X and F a non-negative real valued function on X x X such that F is continuous in each variable and $F(x,y) > 0$ whenever x,y are distinct elements of X. Suppose that

$$\inf_{y \in O(x)} \; \sup_{u,v \in O(y)} F(u,v) < \sup_{u,v \in O(x)} F(u,v) \tag{2.1}$$

whenever $x \in X$ is such that the right hand side is positive. Then the family $\mathcal{F}$ has a common fixed point in cl. $O(x)$ for each x in X.

Proof: Fix $x \in X$. Since $O(x)$ is $\mathcal{F}$-invariant (that is $f(O(x)) \subset O(x)$ for all f in $\mathcal{F}$) and every member of $\mathcal{F}$ is continuous, it follows that cl. $O(x)$ is $\mathcal{F}$-invariant. By the compactness of $(X, \mathcal{J})$, there exists a minimal non-empty $\mathcal{F}$-invariant closed subset Y of cl. $O(x)$. By the

minimality of Y, it follows that $Y = \text{cl. } 0(y)$ for each y in Y. Since F is continuous in each variable, it follows that

$$\begin{aligned} D(y) &= \sup \{ F(u,v) \mid u,v \in 0(y) \} \\ &= \sup \{ F(u,v) \mid u,v \in \text{cl. } 0(y) \} \\ &= \sup \{ F(u,v) \mid u,v \in Y \} \end{aligned}$$

for each y in Y, so that D is constant on Y. Fix $z \in Y$. Then $D(y) = D(z)$ for all $y \in 0(z)$ so that

$$\inf \{ D(y) \mid y \in 0(z) \} = D(z).$$

Hence, from (2.1), $D(z) = 0$, consequently z is a common fixed point for the family $\mathfrak{F}$.

Corollary 3. Let (X,d) be a compact metric space and $\mathfrak{F}$ a non-empty family of continuous self maps on X. Suppose that

(3.1) $\quad \inf \{ \delta(0(y)) \mid y \in 0(x) \} < \delta(0(x))$

whenever $x \in X$ with the diameter $\delta(0(x)) > 0$. Then $\mathfrak{F}$ has a common fixed point in cl. 0(x) for each x in X.

Corollary 4. (Kirk [1], Theorem 2). Let (X,d) be a compact metric space and f a continuous self-map on X with diminishing orbital diameters on X. Then, for each x in X, the sequence $\{f^n(x)\}$ has a subsequence converging to a fixed point of f.

Now we prove a theorem on the existence of common fixed points for a family of self maps on a metric space, not necessarily compact. From this follows another result of Kirk [1].

Theorem 5. Let (X,d) be a metric space, $\mathfrak{F}$ a non-empty family of self maps on X and

$\mathcal{C} = \{ f \mid$ either f is the identity map on X or f is the composite of a finite number of elements of $\mathfrak{F} \}$.

Suppose that $\mathcal{C}$ is equicontinuous on X, (3.1) holds and $x \in X$. If $z \in \bigcap \{ \text{cl. } 0(y) \mid y \in 0(x) \}$, then z is a common fixed point of $\mathfrak{F}$ and

$$r(x) = \inf \{ \delta(0(y)) \mid y \in 0(x) \} = 0 .$$

Proof. Since every member of $\mathcal{C}$ is continuous, we have $O(z) \subset \text{cl. } O(y)$ for all $y \in O(x)$. Hence

(5.1) $\quad \delta(O(z)) \leq r(x)$.

Let $\varepsilon > 0$. Since $\mathcal{C}$ is equicontinuous on X, there exists $\eta > 0$ such that $d(fu, fv) < \varepsilon$ for all $f \in \mathcal{C}$ whenever $d(u,v) < \eta$. Fix $w \in O(z)$ and $y \in O(x)$. Then $w \in \text{cl. } O(y)$ so that there exists $k \in \mathcal{C}$ such that $d(ky,w) < \eta$. Let $g,h \in \mathcal{C}$. Then

$$\begin{aligned} d(gky, hky) &\leq d(gky, gw) + d(gw, hw) + d(hw, hky) \\ &< \varepsilon + \delta(O(w)) + \varepsilon . \end{aligned}$$

Hence $\delta(O(ky)) < \delta(O(w)) + 2\varepsilon$. We have $r(x) \leq \delta(O(ky))$ since $y \in O(y)$. Hence $r(x) < \delta(O(w)) + 2\varepsilon$. This is true for all $\varepsilon > 0$. Hence $r(x) \leq \delta(O(w))$. This is true for all $w \in O(z)$. Hence

(5.2) $\quad r(x) \leq r(z)$.

From (5.1) and (5.2) we have $r(x) = r(z) = \delta(O(z))$. Hence $(o(z)) = 0$, so that z is a common fixed point of $\mathcal{F}$ and $r(x) = 0$.

Corollary 6. (Kirk [1], Theorem 3). Let (X,d) be a metric space and $f: X \to X$ be such that

(6.1) $\quad$ f has diminishing orbital diameters and

(6.2) $\quad$ there exists a constant α such that for each positive integer k and for each $x, y \in X$,

$$d(f^k x, f^k y) \leq \alpha\, d(x,y).$$

If for some $x \in X$, a subsequence of the sequence $\{f^n x\}$ has limit z, then $\lim f^n x = z$ and $fz = z$.

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REFERENCES

[1] Kirk, W.A.: On mappings with diminishing orbital diameters, J. Lond. Math. Soc., 44(1969), 107-111.

[2] Madhusudana Rao, J.: On some theorems of Iseki, Indian J. Pure and Appl. Math, 12(5) (1981), 580-584.

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Certain Expansions of Generalized Basic Hypergeometric Functions of Two Variables

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1. Introduction

In this note, we shall derive some expansions involving generalized basic hypergeometric series of two variables. These include, as special cases, many of the known expansions of generalized basic hypergeometric functions of one variable which have been derived by Aruna Verma [10].

2. Notation

The following usual notation will be used throughout this note.

$$[a;r] = [q^a;r] = (1-q^a)\,(1-q^{a+1})\,\ldots\,(1-q^{a+r-1})$$

$$[a;0] = [q^a;0] = 1 .$$

The generalized basic hypergeometric series of two variables is defined as

$$\phi\left[\begin{array}{c|c} x & \begin{array}{c}(a)\\(b)\;;\;(c)\\(d)\\(e)\;;\;(f)\end{array} \\ y & \end{array}\right] = \sum_{m\geq o}\sum_{n\geq o}\frac{[(a);m+n][(b);m][(c);n]\,x^m\,y^n}{[(d);m+n][(e);m][(f);n][1;m][1;n]}$$

3. Main Results

In this section we shall establish the following results:

$$(3.1)\qquad \sum_{r=o}^{\infty}\frac{[(a);2r][b-e;r][b-c;r][e;r][e+c-b;r]\,x^r y^r q^{u(r)}}{[1;r][b+r-1;r][(d);2r][b;2r]}\,x\,.$$

$$\phi\left[\begin{array}{c|c} xq^{e+c-b} & \begin{array}{c}(a)+2r\\ b-e+r,b-c+r\;;\;b-e+r,\;e+c-b+r\\(d)+2r\\ b+2r\;\;;\;\;b+2r\end{array} \\ y & \end{array}\right]$$

$$=\phi\left[\begin{array}{c|c} xq^{e+c-b} & \begin{array}{c}(a),b-e\\ a-c;\;\;e+c-b\\(d),\;b\end{array} \\ y & \end{array}\right],\text{ where } u(r) = r(c+r-1)$$

(3.2)
$$\sum_{r=0}^{\infty} \frac{[(a);2r][e;r][b;r][\delta-2b;r][b;r][c-e;r]\; x^r y^r q^{u(r)}}{[1;r][c+r-1;r][(d);2r][\delta;r][c;2r]} \times$$

$$\phi\left[\begin{array}{c|c} & (a)+2r \\ xq^{b} & e+r,b+r,\delta-2b+r;b+r,e+r \\ y & (d)+2r \\ & \delta+r,\ c+2r\ ;\ c+2r \end{array}\right]$$

$$= \phi\left[\begin{array}{c|c} & (a),e \\ xq^{b} & b,\delta-2b;b \\ y & (d),c \\ & \delta\ \ ;- \end{array}\right],$$

where $u(r) = r(b+c+r-1)$.

(3.3)
$$\sum_{r=0}^{\infty} \frac{[(a);2r][\alpha;r][e-b;r][\beta;r][\alpha;r][\beta+1;r]\; x^{2r} q^{u(r)}}{[(d);2r][e+r-1;r][1;r][\alpha+\beta-\frac{1}{2};r][e;2r][\alpha+\beta-\frac{1}{2};r]} \times$$

$$\phi\left[\begin{array}{c|c} & (a)+2r \\ xq^{-\frac{1}{2}} & \alpha+r,b+r,\beta+r;\alpha+r,b+r,\beta-1+r \\ x & (d)+2r \\ & \alpha+\beta-\frac{1}{2}+r,e+2r;\alpha+\beta-\frac{1}{2}+r,e+2r \end{array}\right]$$

$$= \phi\left[\begin{array}{c|c} & (a),b \\ xq^{-\frac{1}{2}} & \alpha,\beta\ ;\ \alpha,\beta-1 \\ x & (d),e \\ & \alpha+\beta-\frac{1}{2};\ \alpha+\beta-\frac{1}{2} \end{array}\right]$$

where, $u(r) = r(b+r-1)$

(3.4)
$$\sum_{r=0}^{\infty} \frac{[(a);2r][e;r][b;r][\beta;r]\; x^r y^r q^{u(r)}}{[(d);2r][1;r][c;r][\alpha;r]} \times$$

$$\phi\left[\begin{array}{c|c} & (a)+2r \\ x & e+r,\ b+r\ ;\ e+r,\ \beta+r \\ y & (d)+2r \\ & c+r\ \ ;\ \ c+r \end{array}\right]$$

$$= \phi\left[\begin{array}{c|c} & (a),e \\ x & b\ \ ;\ \ \beta \\ & (d) \\ & c\ \ ;\ \ \alpha \end{array}\right],$$

where $u(r) = r(e+r-1)$.

4. Proofs of (3.1) to (3.4)

We shall prove the above results:

To prove (3.1), we write the left hand side as

$$\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[(a);2r+m+n][b-e;r+m][b-c;r+m][e;r][b-e;r+n]}{[(d);2r+m+n][1;r][b+r-1;r][b-e;r][b;2r+m]}\times$$

$$\times\frac{[e+c-b,r+m]\,x^{r+m}\,y^{r+n}}{[b;2r+n][1;m][1;n]}\,q^{r(r+1)+r(c-2)+m(e+c-b)}$$

$$=\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{r=0}^{\infty}\frac{[(a);s+t][b;2r][e;r][b-e;s][b-c;s]}{[(d);s+t][1;r][b+r-1;r][b-e;r][b;r+s]}\times$$

$$\times\frac{[b-e;t][e+c-b;t]\,x\,y^{t}\,q^{r(r+1)+r(c-2)+(s-r)(e+c-b)}}{[b;r+t][1;s-r][1;t-r]}$$

(By taking $r+m = s$ and $r+n = t$)

$$=\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{[(a);s+t][b-c;s][b-e;s][b-e;t][e+c-b;t]}{[(d);s+t][b;s][b;t][1;s][1;t]}\,.$$

$$x^{s}y^{t}q^{s(e+c-b)}\;.\;{}_6\phi_5\left[\begin{matrix}b-1,\ \frac{b+1}{2},\ -\frac{b+1}{2},\ e,\ -s,\ -t;\\ \frac{b-1}{2},\ -\frac{b-1}{2},\ b-e,\ b+s,b+t;\end{matrix}\ Q\right]$$

where $Q = q^{b-e+s+t}$.

Now summing well poised series ${}_6\phi_5$ with the help of known result [Slater (IV-9), p. 247],

we get the right hand side of (3.1).

Similarly, (3.2) and (3.3) may be proved. To prove (3.4), we replace the left hand side by

$$\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[(a);2r+m+n][e;r+m][b;r+m][e;r+n][\beta;r+n]\,x^{r+m}y^{r+n}q^{r(e+r-1)}}{[(d);2r+m+n][1;r][c;r+m][\alpha;r+n][e;r][1;m][1;n]}$$

$$=\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{r=0}^{\infty}\frac{[(a);s+t][e;s][b;s][e;t][\beta;t]\,x^{s}y^{t}q^{r(e+r-1)}}{[1;r][(d);s+t][c;s][\alpha;t][e;r][1;s-r][1;t-r]}.$$

(By taking $r + m = s$ and $r + n = t$)

$$= \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{[(a);s+t][e;s][b;s][e;t][\beta;t]\, x^s y^t}{[(d);s+t][c;s][\alpha;t][1;s][1;t]} \times$$

$$_2\phi_1 \left[\begin{matrix} -s, -t; & q^{e+s+t} \\ e & \end{matrix}\right]$$

Summing $_2\phi_1$ by Gauss's theorem [Slater (IV. 1), p. 247], we get the right hand side of (3.4).

5. Special Cases

In this section, we shall deduce some known results as special cases:

(i) Putting $A = D = 0$, $y = x$ in (3.1), we get

$$(5.1) \qquad \sum_{r=0}^{\infty} \frac{[e;r][b-e;r][b-c;r][e+c-b;r]\, x^{2r} q^{r(c+r-1)}}{[1;r][b+r-1;r][b;2r]} \times$$

$$_2\phi_1 \left[\begin{matrix} b-e+r, b-c+r; & \\ & xq^{e+c-b} \\ b+2r & \end{matrix}\right] \times {}_2\phi_1 \left[\begin{matrix} b-e+r, e+c-b+r; & \\ & x \\ b+2r; & \end{matrix}\right]$$

$$= \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{[b-e;s+t][b-c;s][e+c-b;t]}{[b;s+t][1;s][1;t]}$$

Now setting $s+t = v$ in the right hand side of (5.1) and changing the order of summation and summing the inner $_2\phi_1$ by the basic analogue of Gauss Theorem [Slater (IV-3), p. 247], we have,

$$\sum_{r=0}^{\infty} \frac{[e;r][b-e;r][b-c;r][e+c-b;r]\, x^{2r} . q^{r(c+r-1)}}{[1;r][b+r-1;r][b;2r]} \times$$

$$_2\phi_1 \left[\begin{matrix} b-e+r, b-c+r; & \\ & xq^{e+c-b} \\ b+2r & \end{matrix}\right] \times {}_2\phi_1 \left[\begin{matrix} b-e+r, e+c-b+r; & \\ & x \\ b+2r & \end{matrix}\right]$$

$$= {}_2\phi_1 \left[\begin{matrix} e, b-e; & \\ & x \\ b & \end{matrix}\right]$$

which is equivalent to a result due to F.H. Jackson [9, § 55].

(ii) Letting $A = D = 0$ and $y = x$ in (3.2), we get,

$$(5.2) \quad \sum_{r=0}^{\infty} \frac{[e;r][c-e;r][\delta-2b;r][b;r][b;r]\, x^{2r} q^{r(b+c+r-1)}}{[1;r][c+r-1;r][\delta;r][c;2r]} \times$$

$$\times\ {}_3\phi_2\left[\begin{matrix} e+r, b+r, \delta-2b+r; \\ \delta+r,\ c+2r \end{matrix}\ xq^{b}\right]\ {}_2\phi_1\left[\begin{matrix} b+r,\ e+r; \\ c+2r \end{matrix}\ x\right]$$

$$= \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{[e;s+t][b;s][\delta-2b;s][b;t]\, x^{s+t}\, q^{b-s}}{[c;s+t][\delta;s][1;s][1;t]}$$

Now, putting $s+t = u$ in the right hand side of (5.2) and summing the inner ${}_3\phi_2$ by Sear's [13, § 3.1], we have,

$$\sum_{r=0}^{\infty} \frac{[e;r][c-e;r][\delta-2b;r][b;r][b;r]\, x^{2r} q^{r(b+c+r-1)}}{[1;r][c+r-1;r][\delta;r][c;2r]} \times$$

$${}_3\phi_2\left[\begin{matrix} e+r, b+r,\ \delta-2b+r; \\ \delta+r, c+2r \end{matrix}\ xq^{b}\right] \times\ {}_2\phi_1\left[\begin{matrix} b+r,\ e+r; \\ c+2r \end{matrix}\ x\right],$$

$$= {}_3\phi_2\left[\begin{matrix} 2b, e, \delta-b; \\ c, \delta \end{matrix}\ x\right]$$

(iii) Taking $A = D = 0$ in (3.3), we have,

$$(5.3) \quad \sum_{r=0}^{\infty} \frac{[\alpha;r][\alpha;r][\beta;r][\beta-1;r][e-b;r]\, x^{2r} q^{r(b+r-1)}}{[e+r-1;r][1;r][\alpha+\beta-\frac{1}{2};r][\alpha+\beta-\frac{1}{2};r][e,2r]} \times$$

$${}_3\phi_2\left[\begin{matrix} \alpha+r, b+r, \beta+r; xq^{-\frac{1}{2}} \\ \alpha+\beta-\frac{1}{2}+r,\ e+2r \end{matrix}\right] \times\ {}_3\phi_2\left[\begin{matrix} \alpha+r, b+r, \beta+r; x \\ \alpha+\beta-\frac{1}{2}+r, e+2r \end{matrix}\right]$$

$$= \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{[b;s+t][\alpha;s][\beta;s][\alpha;t][\beta-1;t]\, x^{s+t} q^{-s/2}}{[e;s+t][\alpha+\beta-\frac{1}{2};s][\alpha+\beta-\frac{1}{2};t][1;s][1;t]}$$

Now, putting $s+t = u$ in the right hand side of (5.3); we get after some simplification,

$$\sum_{u=0}^{\infty} \frac{[\alpha;u][\beta-1;u][b;u]x^{u}}{[\alpha+\beta-\frac{1}{2};u][e;u][1;u]} \times\ {}_4\phi_3\left[\begin{matrix} \alpha, \beta,\ \frac{3}{2}-\alpha-\beta-u; q \\ \alpha+\beta-\frac{1}{2},\ 1-\alpha-u,\ 2-\beta-u \end{matrix}\right]$$

Further changing the order of summation and summing the inner ${}_4\phi_3$, we get a result due to N. Agrawal [2; (3.1)]

(iv) Lastly, putting y = x and A = D = 0 in (3.4), we get

$$(5.4) \qquad \sum_{r=0}^{\infty} \frac{[e;r][b;r][\beta;t]\, x^{2r}\, q^{r(e+r-1)}}{[1;r][c;r][d;r]} \times {}_2\phi_1\left[\begin{matrix} e+r, b+r; x \\ \\ c+r \end{matrix}\right] \times {}_2\phi_1\left[\begin{matrix} e+r, \beta+r; & \\ & x \\ \alpha+r & \end{matrix}\right]$$

$$= \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{[b;s][\beta;t][e;s+t]\, x^{s+t}}{[c;s][\alpha;t][1;s][1;t]}$$

Now, setting s+t = u in the right hand side of (5.4) we get, after some simplification, a result due to F.H. Jackson [9, 35].

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References

[1] N. Agarwal: Certain basic hypergeometric identities of the Cayley-Orr type, Journal of the London Mathematical Society, Vol. 34 (1959), pp. 37-46.

[2] N. Agarwal: On certain basic hypergeometric identities, Rivista di Matematica della Universita di Parma, to appear.

[3] R.P. Agarwal: Some basic hypergeometric identities, Annales de la societe Scientifique de Bruxelles, Vol. 67 (1953), pp. 1-21.

[4] W.N. Bailey: Cambridge Tract on 'Generalized Hypergeometric Series', 1935.

[5] J.L. Burchnall and T.W. Chaundy: Expansions of Appell's double hypergeometric function, Quarterly Journal of Mathematics (Oxford), Vol. 11 (1940), p. 249.

[6] F.H. Jackson: On basic double hypergeometric functions, Quarterly Journal of Mathematics (Oxford), Vol. 13 (1942), pp. 69-82.

[7] F.H. Jackson: On basic double hypergeometric functions, II, Quarterly Journal of Mathematics (Oxford), Vol. 15 (1944), pp. 49-61.

[8] D.B. Sears: Transformations of basic hypergeometric series of the special type, Proceedings of the London Mathematical Society (27, Vol. 52) (1951), pp. 467-483.

[9] T.W. Chaundy: Expansions of hypergeometric functions, Quarterly Journal of Mathematics (Oxford), Vol. 13 (1942), pp. 159-171.

[10] Arun Verma: Certain expansions of generalized Basic Hypergeometric functions, Duke Math. Journal, Vol. 31 (1964).

[11] R.Y. Denis: A general expansion theorem for product of generalized hypergeometric series, Proc. Nat. Inst. Sc. (India), 35 (1969), 70-76.

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Non-linear Two-point Boundary Value Problems and Their Applications to Population Dynamics

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Abstract

In this paper existence and uniqueness of solutions to two-point boundary value problems associated with a system of first order non-linear matrix differential operators satisfying general boundary conditions are derived. These results are applied to obtain approximate analytical solutions to Lotka-Volterra equations which arise in population dynamics.

1. Introduction

Non-linear boundary value problems play an important role in almost all branches of Science and Engineering. In finding solutions to two and three-point boundary value problems for non-homogeneous as well as non-linear problems the construction of Green's function is vital. In Section 3, we present criteria for the existence and uniqueness of solutions to two-point boundary value problems associated with a system of first order matrix differential operators.

$$Ly = p(t)y' + Q(t)y = f(t,y) \tag{1.1}$$

satisfying the general linear boundary conditions

$$My(a) + Ny(b) = \alpha \tag{1.2}$$

where $p(t) \varepsilon C^2 [a,b]$ and $Q(t) \varepsilon C^1 [a,b]$ are square matrices of order n, p(t) being non singular, M and N being square matrices of order n and f, α are column vectors of order n and f being non-linear. We assume that the related homogeneous boundary value problem has only the trivial solution. In Section 4, we apply the theory developed in Section 3, to

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an interesting problem that arises in population dynamics and fairly good approximate analytical solutions are obtained.

2. Preliminaries

The equation

$$P(t)\, y' + Q(t)\, y = 0 \tag{2.1}$$

is equivalent to

$$y' = A(t)y \tag{2.2}$$

where $A = -(P^{-1}Q)$

Definition 2.1: Any set of n linearly independent solutions of (2.2) is a fundamental set of solutions. The matrix with these elements as columns is a fundamental matrix for the given equation.

Definition 2.2: If $Y(t)$ is a fundamental matrix for the system (2.2) then the matrix D defined by

$$D = MY(a) + NY(b)$$

is called a characteristic matrix for the boundary value problem:

$$y' = A(t)y$$

$$My(a) + Ny(b) = 0$$

Lemma 2.1: Any solution of the matrix differential equation (1.1) is also a solution of the integral equation

$$y(t) = Y(t) \int_a^t Y^{-1}(s)P^{-1}(s)f(s,y(s))ds + Y(t)C$$

where C is a constant matrix.

Proof. The proof is similar to the proof of theorem 3.3.1 and 3.3.2 of Chapter 3[2].

Lemma 2.2: If $Y(t)$ is a fundamental matrix of (2.2) then the matrix $\emptyset(t)$ defined by $\emptyset(t) = Y(t)\, D^{-1}C$ is also a fundamental matrix for (2.2).

Lemma 2.3: Let $\emptyset$ be a fundamental matrix of $y' = (-P^{-1}Q)y$. Then, the boundary value problem (1.1) satisfying (1.2) has a unique solution $y = y_1 + \emptyset$ if and only if y_1 is the only solution of

$$\begin{aligned} Py' + Qy &= f(t, y + \emptyset) \\ My(a) + Ny(b) &= 0 \end{aligned} \tag{2.3}$$

Proof. Suppose y_1 is the only solution of (2.3).

Define $y = y_1 + \emptyset$. Then

$$\begin{aligned} Py' + Qy &= Py_1' + Qy_1 + P\emptyset' + Q\emptyset \\ &= f(t, y_1 + \emptyset) + 0 \\ &= f(t,y) \end{aligned}$$

Also

$$\begin{aligned} My(a) + Ny(b) &= [My_1(a) + Ny_1(b)] + [M\emptyset(a) + N\emptyset(b)] \\ &= 0 + [My(a) + Ny(b)]\, D^{-1}\alpha \\ &= \alpha . \end{aligned}$$

Conversely, suppose $y = y_1 + \emptyset$ is the only solution of (1.1) satisfying (1.2), then, it can be easily seen that

$$Py_1' + Qy_1 = f(t, y_1 + \emptyset)$$

and

$$My_1(a) + Ny_1(b) = 0 .$$

3. Two-point Boundary Value Problems

In this section we establish existence and uniqueness of solutions to two-point boundary value problems associated with a system of first order non-linear differential equations (1.1) satisfying (1.2). We assume that the function $f(t,y)$ satisfies a Lipschitz condition in the second variable i.e., there exists a non-negative constant k such that

$$||f(t,y) - f(t,z)|| \leq k||y-z|| \;\forall\; (t,y),(t,z)\in[a,b]\times R^n \tag{3.1}$$

Theorem 3.1: Suppose the homogeneous boundary value problem is incompatible and f satisfies a Lipschitz condition (3.1) on $[a,b]\times R^n$ and suppose that

$$k \max_{[a,b]} \int_a^b ||G(t,s)||ds < 1 , \tag{3.2}$$

then there exists a unique solution to the boundary value problem (1.1) satisfying (1.2), where G is the Green's function for the corresponding homogeneous boundary value problem. From Lemma 2.1 any solution of (1.1) satisfying (1.2) is a solution of the integral equation

$$y(t) = Y(t) \int_a^t Y^{-1}(s)P^{-1}(s)f(s,y)(s))ds + Y(t)C$$

where $Y(t)$ is a fundamental matrix for $y' = (-P^{-1}Q)y$ and C is a constant matrix and will be determined uniquely from the boundary conditions (1.2).

Substituting the general form of $y(t)$ in the boundary conditions, we get

$$[MY(a) + NY(b)]C + NY(b) \int_a^b Y^{-1}(s)P^{-1}(s)f(s,y(s))ds = \alpha$$

$$\text{Thus } C = D^{-1}[\alpha - NY(b) \int_a^b Y^{-1}(s)P^{-1}(s)f(s,y(s))ds]$$

where D is the characteristic matrix for the boundary value problem.

Thus $\quad y(t) = Y(t)\, D^{-1}\alpha + \int_a^b G(t,s)f(s,y(s))ds$

where

$$G(t,s) = \begin{cases} Y(t)Y^{-1}(s)P^{-1}(s) - Y(t)D^{-1}NY(b)Y^{-1}(S)P^{-1}(s) & (a < s < t \le b) \\ -Y(t)D^{-1}NY(b)Y^{-1}(s)P^{-1}(s) & (a \le t < s < b \end{cases}$$

Let $Y(t)\, D^{-1}\alpha = \emptyset(t)$; Note this analysis corresponds to the homogeneous boundary value problem. To obtain unique solution to the homogeneous boundary value problem, we consider the linear space J of functions $u(t) \in C^n\, [a,b]$ fitted with norm

$$||u(t)|| = \max_{t\varepsilon[a,b]} |u(t)|.$$

It is clear that J is a Banach space.

Let T: $J \to J$ defined by

$$(Tu)(t) = \int_a^b G(t,s)\; f(s,u(s))ds\ .$$

Then for any $u,v \in J$

$$||Tu - Tv|| \leq K \int_a^b ||G(t,s)|| [\,||u(s)-v(s)||\,] ds$$

$$\leq K \,||u-v||\, \max_{[a,b]} \int_a^b ||G(t,s)|| ds$$

$$\leq \alpha \,||\, u-v \,||$$

where $\alpha = K \max_{[a,b]} \int_a^b ||G(t,s)|| ds < 1$.

Thus T is a contraction mapping and hence by a Banach fixed point theorem the homogeneous boundary value problem has a unique solution.

Now applying the above procedure to the boundary value problem

$$Py' + Qy = f(t, y + \emptyset)$$

$$My(a) + Ny(b) = 0$$

a unique solution y_1 is constructed. Hence by Lemmas 2.2 and 2.3 ,

$y = y_1 + \emptyset$ is the unique solution of the boundary value problem (1.1) satisfying (1.2).

4. Application to Lotka-Volterra Equations

In 1977, Verma [3] has obtained exact solutions of the Lotka-Volterra equations:

$$\begin{aligned} n_1 &= \alpha_1 n_1 - \beta_1 n_1 n_2 \\ n_2 &= \alpha_2 n_2 + \beta_2 n_1 n_2 \end{aligned} \tag{4.1}$$

under the assumption $\alpha_1 = \alpha_2$. In 1980, Wilson [4] gave the form of exact solutions of (1.3) when α_1, α_2, β_1, β_2 are functions of time with the assumption $\alpha_1(t) = \alpha_2(t)$ and $\beta_1(t) = K\, \beta_2(t)$ where K is a constant. In 1982, Burnside [1] gave the form of exact solutions with a different assumption namely,

$$(\alpha_1 - \alpha_2)\, \beta_1 \beta_2 = \alpha_2 \alpha_1 - \alpha_1 \beta_2. \tag{4.2}$$

In the real world, the assumptions of $\alpha_1 = \alpha_2$ or the more complicated assumptions of (4.2) may not be satisfied. Thus it is of interest to determine exact or atleast approximate analytical solutions of Lotka-Volterra equations (4.1) without making any of the above mentioned assumptions.

(4.1) can be written in the form

$$\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}' = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} + \begin{pmatrix} -\beta_1 n_1 n_2 \\ \beta_2 n_1 n_2 \end{pmatrix} \tag{4.3}$$

$$\begin{pmatrix} n_1(o) \\ n_2(o) \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{4.2}$$

The boundary conditions have a physical significance i.e., both pre and predator population do not vanish at time $t = 0$.

$$n' = An + f(t,n)$$

where $A = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$, $f(t,y) = \begin{pmatrix} -\beta_1 n_1 n_2 \\ \beta_2 n_1 n_2 \end{pmatrix}$

Fundamental matrix for (4.3) is given by

$$Y(t) = \begin{pmatrix} e^{\alpha_1 t} & 0 \\ 0 & e^{\alpha_2 t} \end{pmatrix}$$

The characteristic matrix for the boundary value problem is

$$D = MY(a) + NY(b) .$$

Here $M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $Y(a) = \begin{pmatrix} n_1(o) & 0 \\ 0 & n_2(o) \end{pmatrix}$, $N = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$D = MY(a) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} n_1(o) & 0 \\ 0 & n_2(o) \end{pmatrix} = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$$

$$G(t,s) = \begin{cases} \begin{pmatrix} e^{a_1(t-s)} & 0 \\ 0 & e^{a_2(t-s)} \end{pmatrix} & a < s < t \leq b \\ 0 & a \leq t < s < b \end{cases}$$

By Banach fixed point theorem whenever

$$K \max_{[a,b]} \int_a^b ||G(t,s)||ds < 1 ,$$

a unique solution to the boundary value problem exists.

REFERENCES

[1] Burnside, R.R.: 'A note on exact solutions on prey predator equation', Bull. Math. Biology, 44(1982), 893-897.

[2] Cole, R.H.: 'Theory of ordinary differential equations' Appleton-Century Crafts, New York (1968).

[3] Varma, V.S.: 'Exact solutions for a special prey-predator on competing species system', Bull. Math. Biology, 39 (1977), 619-622.

[4] Wilson, A.J.: 'On Varma's prey-predator problem', Bull. Math. Biol. 42 (1980), 599-600.

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Meromorphic Multivalent Functions with Positive Coefficients

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Abstract

Let $f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_{p+n} z^{p+n} (a_{p+n} \geq 0)$ be analytic and p-valent in $0 < |z| < 1$ and $Q^*[P,A,B]$ where $-1 \leq A < B \leq 1$ and $A + B \geq 0$ be the class of functions $f(z)$ which satisfy

$$\left| \frac{\frac{zf'(z)}{f(z)} + p}{B\frac{zf'(z)}{f(z)} + Ap} \right| < 1 \quad \text{for } |z| < 1.$$

Coefficient estimates, representation formula, distortion theorem, radius of convexity and closure theorem are obtained for the class $Q^*[P,A,B]$.

1. Introduction

Let $f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_{p+n} z^{p+n} (a_{p+n} \geq 0)$ be analytic and p-valent in the punctured disk $0 < |z| < 1$. For A,B fixed $-1 \leq A < B \leq 1$ and $A + B \geq 0$, we say that $f(z) \in Q^*[p,A,B]$ if and only if

$$\left| \frac{\frac{zf'(z)}{f(z)} + p}{B\frac{zf'(z)}{f(z)} + Ap} \right| < 1, \quad z \in E = \{z : |z| < 1\}.$$

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In this paper we obtain coefficient estimates, representation formula, distortion theorem and radius of convexity for the class $Q^*[p,A,B]$. Further it is shown that this class is closed under convex linear combinations.

For suitable choices of p,A and B, we obtain the earlier results of the authors [5] as special cases of the results obtained here.

Goel and Sohi [1], Sarangi and Uralegaddi [2], Shukla and Dashrath [3], and Silverman,H. [4] studied certain subclasses of analytic functions with negative coefficient.

2. Coefficient Inequalities

Theorem 1. A function $f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_{p+n} z^{p+n}$ $(a_{p+n} \geq 0)$ is in $Q^*[p,A,B]$ if and only if

$$\sum_{n=0}^{\infty} ((1+B)n + (2+B+A)\,p)\, a_{p+n} \leq p(B-A) \tag{1}$$

Proof: Suppose (1) holds

we have

$$\left| z\frac{f'(z)}{f(z)} + p \right| - \left| Bz\frac{f'(z)}{f(z)} + Ap \right| < 0$$

provided

$$\left| \sum_{n=0}^{\infty} (2p+n)\, a_{p+n} z^{2p+n} \right| - \left| p(B-A) - \sum_{n=0}^{\infty} ((B+A)\,p + Bn)\, a_{p+n} z^{2p+n} \right| < 0 . \tag{2}$$

For $|z| = r < 1$ the left side of (2) is bounded above by

$$\sum_{n=0}^{\infty} (2p+n)\, a_{p+n} r^{2p+n} - p\,(B-A)$$

$$+ \sum_{n=0}^{\infty} ((B+A)\ p + Bn)\ a_{p+n} r^{2p+n}$$

$$= \sum_{n=0}^{\infty} ((1+B)\ n + (2 + B + A)\ p)\ a_{p+n} r^{2p+n} - p\ (B-A)$$

$$< \sum_{n=0}^{\infty} ((1+B)\ n + (2 + B + A)\ p) a_{p+n} - p\ (B-A)$$

$$\leq 0$$

Hence $f(z) \in Q^* [p,A,B]$.

Conversely, suppose

$$\left| \frac{\frac{zf'(z)}{f(z)} + p}{Bz \frac{f'(z)}{f(z)} + Ap} \right| = \left| \frac{\sum_{n=0}^{\infty} (2p+n)\ a_{p+n} z^{2p+n}}{p(B-A) - \sum_{n=0}^{\infty} (Bn + (B+A)\ p) a_{p+n} z^{2p+n}} \right|$$

$$< 1 \text{ for } z \in E.$$

Since $|Re(z)| \leq |z|$ for all z, we have

$$\text{(3)} \quad Re \left\{ \frac{\sum_{n=0}^{\infty} (2p+n)\ a_{p+n} z^{2p+n}}{p(B-A) - \sum_{n=0}^{\infty} (Bn+(B+A)p)\ a_{p+n} z^{2p+n}} \right\} < 1.$$

Choose values of z on the real axis so that $z \frac{f'(z)}{f(z)}$ is real. Upon clearing the denominator in (3) and letting $z \to 1$ through real values we obtain (1).

Corollary: If $f(z) \in Q^* [p,A,B]$ then

$$a_{p+n} \leq \frac{p(B-A)}{(1+B)\ n + (2+B+A)\ p} \text{ for } n = 0,1,2, \ldots$$

with equality for $f(z) = \frac{1}{z^p} + \frac{p(B-A)\ z^{p+n}}{(1+B)n + (2+B+A)\ p}$

3. Representation Formula

Theorem 2. A function $f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_{p+n} z^{p+n}$

$(a_{p+n} \geq 0)$ is in $Q^* [p,A,B]$ if and only if

(4) $$f(z) = z^{-p} \exp \{ - p(B-A) \int_0^z \frac{t^{2p-1} \phi(t)}{1 - Bt^{2p} \phi(t)} dt \}, \quad 0 < |z| < 1$$

where $\phi(z)$ is analytic and satisfies $|\phi(z)| \leq 1$ for $z \in E$.

Proof: Let $f(z)$ be in $Q^* [p,A,B]$. Then

$$\left| \frac{\frac{zf'(z)}{f(z)} + p}{Bz \frac{f'(z)}{f(z)} + Ap} \right| < 1, \quad z \in E .$$

Since the absolute value vanishes at $z = 0$,

we have

(5) $$\left(\frac{z \frac{f'(z)}{f(z)} + p}{Bz \frac{f'(z)}{f(z)} + Ap} \right) = z^{2p} \phi(z)$$

where $\phi(z)$ is analytic and $|\phi(z)| \leq 1$ for $z \in E$. Integrating (5) we obtain (4). The other part follows by differentiating (4).

4. Distortion Theorem

Theorem 3. If $f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_{p+n} z^{p+n}$

$(a_{p+n} \geq 0)$ is in $Q^* [p,A,B]$, then

$$\frac{1}{r^p} - \frac{B - A}{2+B+A} r^p \leq |f(z)| \leq \frac{1}{r^p} + \frac{B - A}{2+B+A} r^p$$

and

$$\frac{p}{r^{p+1}} - \frac{p(B-A)}{1+B} r^{p-1} \leq |f'(z)| \leq \frac{p}{r^{p+1}} + \frac{p(B-A)}{1+B} r^{p-1}$$

Proof: Since from theorem 1.

$$(2+B+A) \quad p\sum_{n=0}^{\infty} a_{p+n} \le \sum_{n=0}^{\infty} ((1+B)n + (2+B+A)p)\, a_{p+n}$$

$$\le p\,(B-A)$$

we have

$$\sum_{n=0}^{\infty} a_{p+n} < \frac{B-A}{2+B+A}$$

$$|f(z)| \le \frac{1}{r^p} + \sum_{n=0}^{\infty} a_{p+n} r^{p+n}$$

$$\le \frac{1}{r^p} + r^p \sum_{n=0}^{\infty} a_{p+n}$$

$$\le \frac{1}{r^p} + \frac{B-A}{2+B+A} r^p$$

and

$$|f(z)| \ge \frac{1}{r^p} - \sum_{n=0}^{\infty} a_{p+n} r^{p+n}$$

$$\ge \frac{1}{r^p} - r^p \sum_{n=0}^{\infty} a_{p+n}$$

$$\ge \frac{1}{r^p} - \frac{B-A}{2+B+A} r^p$$

The bounds for $|f(z)|$ are sharp and are attained for the function $f(z) = \frac{1}{z^p} + \frac{B-A}{2+B+A} z^p$ at $z = r,\ re^{i\frac{\pi}{2p}}$.

Further we have

$$|f'(z)| \le \frac{p}{r^{p+1}} + \sum_{n=0}^{\infty} (p+n)\, a_{p+n} r^{p+n-1}$$

$$\le \frac{p}{r^{p+1}} + r^{p-1} \sum_{n=0}^{\infty} (p+n)\, a_{p+n}$$

and

$$|f'(z)| \ge \frac{p}{r^{p+1}} - \sum_{n=0}^{\infty} (p+n)\, a_{p+n} r^{p+n-1}$$

$$\ge \frac{p}{r^{p+1}} - r^{p-1} \sum_{n=0}^{\infty} (p+n)\, a_{p+n}$$

Since

$$(1+B)\ (p+n) \le (1+B)\ n + (2+B+A)\ p,\ n = 0,1,2,\ldots$$

we have

$$(1+B) \sum_{n=0}^{\infty} (p+n)\ a_{p+n} \le \sum_{n=0}^{\infty} (1+B)\ n + (2+B+A)\ p)\ a_{p+n}$$

$$\le p\ (B-A)$$

and the bounds for $|f'(z)|$ follow.

5. Radius of Convexity

Theorem 4. If $f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_{p+n} z^{p+n}\ (a_{p+n} \ge 0)$

is in $Q^* [p,A,B]$ then $f(z)$ is p-valently convex in

$$0 < |z| < R_p = \operatorname*{Inf}_n \left(\frac{(1+B]n+(2+B+A)p}{(B-A)(n+p)^2}\ p \right)^{\frac{1}{2p+n}},\ n=0,\ 1,2,.$$

The estimate is sharp for the function

$$f(z) = \frac{1}{z^p} + \frac{p(B-A)}{(1+B)n + (2+B+A)p}\ z^{p+n} \text{ for some } n\ .$$

Proof: It suffices to show that

$$\left| \frac{1+z\frac{f''(z)}{f'(z)} + p}{1+z\frac{f''(z)}{f'(z)} - p} \right| < 1 \text{ for } 0 < |z| < R_p\ .$$

We have

$$\left| \frac{1+z\frac{f''(z)}{f'(z)} + p}{1+z\frac{f''(z)}{f'(z)} - p} \right| = \left| \frac{\sum_{n=0}^{\infty} (2p+n)(p+n)\ a_{p+n} z^{2p+n}}{2p^2 + \sum_{n=0}^{\infty} n(p+n)a_{p+n} z^{2p+n}} \right|$$

$$\le \frac{\sum_{n=0}^{\infty} (2p+n)(p+n)\ a_{p+n}\ |z|^{2p+n}}{2p^2 - \sum_{n=0}^{\infty} n(p+n)\ a_{p+n} |z|^{2p+n}}$$

The last expression is bounded above by 1 if

$$(6) \qquad \sum_{n=0}^{\infty} \left(\frac{p+n}{p}\right)^2 a_{p+n} |z|^{2p+n} \leq 1 .$$

From theorem 1, we have

$$\sum_{n=0}^{\infty} \frac{(a+B)\ n+(2+B+A)p}{p(B-A)} a_{p+n} \leq 1 .$$

Hence (6) is satisfied if

$$(7) \qquad \left(\frac{p+n}{p}\right)^2 |z|^{2p+n} \leq \frac{(1+B)n+(2+B+A)p}{p(B-A)}, \quad n = 0,1,2, \ldots$$

Solving (7) for $|z|$ we obtain

$$(8) \qquad |z| \leq \left(\frac{(1+B)\ n + (2+B+A)\ p}{(B-A)\ (n+p)^2}\ p\right)^{\frac{1}{2p+n}}, \quad n = 0,1,2, \ldots$$

Putting $|z| = R_p$ in (8) the result follows.

6. CLOSURE THEOREM

Theorem 5. Let $f_{p-1}(z) = \frac{1}{z^p}$ and

$$f_{p+n}(z) = \frac{1}{z^p} + \frac{p(B-A)\ z^{p+n}}{(1+B)n + (2+B+A)p} \quad \text{for } n = 0,1,2, \ldots$$

Then $f(z) \in Q^* [p,A,B]$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=-1}^{\infty} \lambda_{p+n} f_{p+n}(z) \text{ where } \lambda_{p+n} \geq 0 \text{ and } \sum_{n=-1}^{\infty} \lambda_{p+n} = 1$$

Proof: Suppose $f(z) = \sum_{n=1}^{\infty} \lambda_{p+n} f_{p+n}(z)$

$$= \frac{1}{z^p} + \sum_{n=0}^{\infty} \lambda_{p+n} \frac{p(B-A)\ z^{p+n}}{(1+B)\ n + (2+B+A)p}$$

Then $$\sum_{n=0}^{\infty} \lambda_{p+n} \frac{p\,(B-A)}{(1+B)\,n + (2+B+A)p} \quad \frac{(1+B)\,n + (2+B+A)p}{p(B-A)}$$

$$= \sum_{n=0}^{\infty} \lambda_{p+n} = 1 - \lambda_{p-1} \le 1 .$$

Hence $f(z) \in Q^* [p,A,B]$.

Conversely, suppose $f(z) \in Q^* [p,A,B]$. Then

$$a_{p+n} \le \frac{p(B-A)}{(1+B)n + (2+B+A)p} \quad \text{for } n = 0,1,2, \ldots$$

Setting $$\lambda_{p+n} = \frac{(1+B)n + (2+B+A)p}{p(B-A)} \, a_{p+n} , \; n = 0,1,2, \ldots$$

and $$\lambda_{p-1} = 1 - \sum_{n=0}^{\infty} \lambda_{p+n}$$

we obtain $f(z) = \sum_{n=-1}^{\infty} \lambda_{p+n} f_{p+n}(z)$

Remarks: Putting $p = 1$, $A = (2\alpha - 1)\beta$ and $B = \beta$ where $0 \le \alpha < 1$ and $0 < \beta \le 1$, in the above theorems we obtain the results in [5].

REFERENCES

[1] R.M. Goel and N.S. Sohi, "Multivalent function with Negative coefficients". Indian J. Pure Appl. Math., 12 (1981), 844-853.

[2] S.M. Sarangi and B.A. Uralegaddi, "The radius of convexity and starlikeness for certain classes of analytic functions with negative coefficients I", Rendiconti Accademia Nazionale dei Lincei, LXV (1978), 38-42.

[3] S.L. Shukla and Dashrath, "On certain classes of multivalent functions with negative coefficients", Soochow J. Math., 8 (1982), 179-188.

[4] H. Silverman, "Univalent functions with negative coefficients", Pro. Amer. Math. Soc., 51 (1975), 109-116.

[5] B.A. Uralegaddi and M.D. Ganigi, "Certain Class of meromorphically starlike functions with positive coefficients." (Submitted).

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