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A Note on the Consistency of Criticality Parameter of a Supercritical Multitype Galton Watson Branching Process with Random Environments

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Sumary

The multitype analogues of the criticality parameter estimators of Becker and Nanthi are defined and the strong consistency of these estimators established.

Introduction

The branching process with random environments was first introduced by Smith (1968) and Smith and Wilkinson (1969) and later on generalised by Athreya and Karlin (1972). Such a branching process with random environments (BPRE) is particularly well adapted to describe the population growth. Becker (1977) used it to model the growth of an epidemic and proposed a strongly consistent estimator for the criticality parameter of a single type BPRE. Dion and Esty (1979) suggested estimators for two of the main parameters of the BPRE, viz., the reproduction average per individual and the mean of the logarithm of the environmental mean. The later parameter is known as the criticality parameter indicating the certain extinction or possible explosion of the process. The consistency and asymptotic normality of the estimators were also proved.

Recently, Nanthi (1978, 1979) gave an alternative estimator to estimate the criticality parameter of a supercritical BPRE. Nanthi's estimator uses more information than Becker's estimator and is also a strongly consistent and asymptotic normal estimator of the criticality parameter.

In the present work, we define the multitype analogues of the criticality parameter estimators of Becker and Nanthi, and examine the strong consistency of these estimators.

Preliminaries and Some Useful Results:

Consider a collection of particles of r different types. Let $X_n(j)$ denote the number of individuals of type j, $1 \leqslant j \leqslant r$, in the nth. generation. Then $X_n = (X_n(1), X_n(2), \ldots, X_n(r))$ denotes the population vector of

the nth. generation and $\|X_n\| = \sum_{j=1}^{r} X_n(j)$ denotes the total number of particles of all types in the nth. generation. $\underset{\sim}{X} = \{X_n, n \geqslant 0\}$ denotes a supercritical r-type ($r \geqslant 2$) Galton Watson process with $\underset{\sim}{X}_0 = \underset{\sim}{e}_i$ defined on the probability space $(\Omega, \mathcal{F}, P)$, where e_i denotes a r-dimensional vector with 1 in the ith. position and zero elsewhere. To get from the nth. generation to the (n+1)th. generation, each particle of type i splits according to the offspring distribution $\xi_n(i)$ yielding a progeny vector. $\bar{\xi}_n = (\xi_n(1), \xi_n(2), \ldots, \xi_n(r))$ is called the environment of the nth. generation and $\ddot{\xi} = (\bar{\xi}_0, \bar{\xi}_1, \ldots)$ is called the environmental sequence. For a given $\ddot{\xi}$, the process generated in this manner is a non-homogeneous multitype branching process or equivalently, a multitype branching process in a random environment (MBPRE) conditional on the environment $\bar{\xi}$. Throughout, the stochastic process $\bar{\xi}$ is assumed to be stationary and ergodic unless otherwise stated.

To each vector $\bar{\xi}_k = (\xi_k(1), \xi_k(2), \ldots, \xi_k(r))$ there corresponds an r-variate probability generating function (pgf) $\Phi_{\bar{\xi}_k}(\bar{s}) = (\phi^{(1)}_{\bar{\xi}_k}(\bar{s}), \ldots, \phi^{(r)}_{\bar{\xi}_k}(\bar{s}))$ where $\phi^{(j)}_{\bar{\xi}_k}(\bar{s})$ is the pgf of $\xi_k(j)$. $\Phi_{\bar{\xi}_k}(\bar{s})$ is called the associated pgf corresponding to the environment $\bar{\xi}_k$.

Let $p_{\alpha_1 \alpha_2 \ldots \alpha_r}(\xi_k(j))$ denote the probability that a j-type particle with offspring distribution $\xi_k(j)$ gives birth to α_1 particles of type 1, α_2 particles of type 2, ..., α_r particles of type r. Denote by $p_{ij}^{(1)}(\bar{\xi}_k)$ the probability that a j-type particle produces 1 i-type particles under the environment $\bar{\xi}_k$. Then

$$\phi^{(j)}_{\bar{\xi}_k}(s) = \sum_{\bar{\alpha} \in I} p_{\bar{\alpha}}(\xi_k(j)) \bar{s}^{\bar{\alpha}}$$

where $\bar{s}^{\bar{\alpha}} = s_1^{\alpha_1} s_2^{\alpha_2} \ldots s_r^{\alpha_r}$ and $\bar{\alpha} = \alpha_1 \alpha_2 \ldots \alpha_r$. Corresponding to the mean of the offspring distribution defined for BPRE, the mean matrix $M_k = M(\bar{\xi}_k)$ corresponding to the environment $\bar{\xi}_k$ is defined by

$$m_{ij} = m_{ij}(\bar{\xi}_k) = \text{expected number of offsprings of type i produced by a particle of type j under the environment } \bar{\xi}_k$$

$$= \sum_{l=0}^{\infty} l p_{ij}^{(1)}(\bar{\xi}_k) = \frac{\partial \phi^{(j)}_{\bar{\xi}_k}(\bar{1})}{\partial s_i}$$

where $\bar{1}$ is the r-dimensional vector with components identically equal to 1.

Let $\mathcal{F}_n(\bar{\xi})$ be the σ-field generated by $X_0, X_1, \ldots, X_n$ and $\bar{\xi}$. Then the process $\{X_n; n \geq 0\}$ is an MBPRE if

$$E\left\{ \bar{s}^{\tilde{X}_n} \mid \mathcal{F}_{n-1}(\bar{\xi}) \right\} = \left\{ \phi_{\bar{\xi}_{n-1}}(\bar{s}) \right\}^{X_{n-1}}$$

Athreya and Karlin (1972) have shown that the following are true with probability 1:

$$0 < m_{ij} < \infty \qquad i,j = 1,2,\ldots,r$$

$$0 < \frac{\partial^2 \phi^{(j)}_{\bar{\xi}_k}(\bar{1})}{\partial s_i \, \partial s_l} < \infty \quad i,j,l, = 1,2,\ldots r$$

$$E\left[-\log \langle \bar{1} - \bar{\phi}_{\bar{\xi}_k}(0), \bar{1} \rangle \right] < \infty$$

where $\langle \bar{x}, \bar{y} \rangle$ designates the innerproduct of the indicated vectors. Further we assume that $E \left| \log \| M_k \| \right| < \infty$ where $\| \cdot \|$ is defined as $\| \bar{A} \| = \max_i \sum_{j=1}^{r} | a_{ij} |$ and M_k is a strictly positive matrix with probability 1.

It was proved by Furstenburg and Kesten (1960) that

(1) $$\lim_{n\to\infty} n^{-1} \log \| M_n . M_{n-1} \cdots M_0 \| = \pi$$

exists with probability 1 and that π is finite and also that

(2) $$\lim_{n\to\infty} n^{-1} E \log \| M_n . M_{n-1} \cdots M_0 \| = \pi .$$

Defined $\lambda_n(\bar{\xi})$ as the spectral radius of the matrix

$$\Gamma_n(\bar{\xi}) = M_{n-1} M_{n-2} \cdots M_0$$

then the limit relations (1) and (2) can be expressed equivalently in the form

(3) $$\lim_{n\to\infty} n^{-1} \log \lambda_n(\bar{\xi}) = \lim_{n\to\infty} n^{-1} E \log \lambda_n(\bar{\xi}) = \pi .$$

Regularity Condition

The following condition is the random environment analogue of the usual regularity condition imposed on the non-extinction probabilities for the multitype Galton Watson process.

(C) $$P(\bar{q}(\bar{\xi}) << \bar{1}) = 1$$

where $\bar{q}(\bar{\xi}) = (q_1(\bar{\xi}),\ldots,q_r(\bar{\xi}))$ is called the extinction probability vector and

$$\bar{q}(\bar{\xi}) = \lim_{n \to \infty} \bar{\Phi}_0(\bar{\Phi}_{\xi_1}(\bar{\Phi}_{\xi_2}(\ldots \bar{\Phi}_{\xi_n}(\sigma))\ldots)).$$

We shall also need the following Classification Theorem due to Tanny (1981).

Let $\{\underset{\sim}{X}_n; n \geqslant 0\}$ be a MBPRE satisfying condition (C) and let

$$\limsup_{n \to \infty} n^{-1} \log \| M_{n-1}M_{n-2}\cdots M_0 \| = \pi$$

then $\pi < 0$ implies that $P(\bar{q}(\bar{\xi}) = 1) = 1$

(4) $\pi > 0$ implies that, if $\{\underset{\sim}{X}_n; n \geqslant 0\}$ is stable, then

$$\lim_{n \to \infty} n^{-1} \log \| X_n \| = \pi \text{ a.e. on } \{\omega: \| X_n \| \not\to 0 \text{ as } n \to \infty\}.$$

When $\pi > 0$, the process $\{\underset{\sim}{X}_n; n \geqslant 0\}$ is said to be a supercritical MBPRE. Under the regularity assumption made above, for a supercritical MBPRE, drawing analogy with that of a supercritical multitype Galton Watson branching process (see Karlin and Taylor, 1975), we have

(5) $$\left\{ \prod_{k=0}^{n-1} \lambda_k(\bar{\xi}) \right\}^{-1} \underset{\sim}{X}_n \to W\underset{\sim}{V},$$

where W is a nondegenerate random variable such that $P(W > 0) > 0$, and $\underset{\sim}{V}$ denotes the left eigenvector associated with the maximal eigenvector $\underset{\sim}{\lambda}(\bar{\xi}) = \{\lambda_n(\bar{\xi}); n \geqslant 0\}$ of the offspring mean matrix.

In this note, we are concerned with the questions of consistency of Becker's and Nanthi's estimators for the supercritical MBPRE when $\bar{\xi}_k$'s are independently and identically distributed random variables. In such a case, it is easy to see that $\{\Phi^{(j)}_{\bar{\xi}_k}(\bar{s}); 1 \leqslant j \leqslant r\}$ will be a family of independent and identically distributed random variables.

Consistency of Becker's Estimator

For a single type Galton Watson branching process, it was observed by Becker (1977) and Dion and Esty (1979) that the estimator

$$m_n^* = x_n^{1/n} \quad \text{if } x_n > 0$$
$$= 1 \qquad x_n = 0$$

is strongly consistent for e^{π}. For the multitype MBPRE, we define the multitype analogue of the Becker estimator as

$$\theta_n^* = \| X_n \|^{1/n} \quad \text{if } \| X_n \| > 0 \tag{6}$$
$$= 1 \qquad \| X_n \| = 0$$

Given the environmental sequence $\bar{\xi}$ assume that $\{ X_n ; n \geq 0 \}$ is an increasing sequence bounded by some finite constant c then the Classification Theorem implies that, if $\pi > 0$ then

$$\lim_{n \to \infty} n^{-1} \log \| X_n \| = \pi \text{ a.e.}, \tag{7}$$

on $\{ \omega : \| X_1 \| \not\to 0 \text{ as } n \to \infty \}$. This establishes the strong consistency of Becker's estimator for an MBPRE. Formally we may state the above result as follows.

Proposition 1:

Under the regularity assumptions stated above, θ_n^* is a strongly consistent estimator for e^{π}.

Consistency of Nanthi's Estimator

For the simple BPRE, Nanthi (1978, 1979) showed that

$$\bar{\theta}_n = (x_1 x_2 \ldots x_n)^{2/n^2} \quad \text{if } x_n > 0$$
$$= 1 \qquad x_n = 0$$

is a strongly consistent estimator of e^{π}. The multitype analogue of such an estimator may be written as

$$\bar{\theta}_n^* = \left[\prod_{k=1}^{n} \| X_k \| \right]^{2/n^2} \quad \text{when } \| X_k \| > 0 \tag{8}$$
$$= 1 \qquad \| X_k \| = 0.$$

For the jth. type offspring, (5) may be written as

$$\left\{ \prod_{k=0}^{n-1} \lambda_k(\bar{\xi}) \right\}^{-1} X_n(j) \to W\, v(j). \tag{9}$$

Summing over the r-types, we have

$$\left\{ \prod_{k=0}^{n-1} \lambda_k(\bar{\xi}) \right\}^{-1} \| X_n \| \to W \sum_{j=1}^{r} v(j). \tag{10}$$

Let N_1 be the exceptional set relating to (10) and N_1^c be its complimentary set. Conditional on the non-extinction set $A = \{W > 0\}$, we have for any $\omega \in N_1^c \cap [W > 0] = S_1$,

i) $\| X_n \| > 0$ for $n \geqslant 0$

ii) $\log \| X_n \| - \sum_{k=0}^{n-1} \log \lambda_k(\bar{\xi}) \to \log \left[W \sum_{j=1}^{r} v(j) \right]$.

Setting $U_k = \log \lambda_k(\bar{\xi})$, $k \geqslant 0$ and following Nanthi (1979), we get from (ii) above,

$$2n^{-2} \left[\sum_{k=1}^{n} \log \| X_k \| - \sum_{k=0}^{n-1} (n-k)u_k \right] \to 0$$

which, in turn, may be written as

$$\log \bar{\theta}_n^* - \pi - \frac{\pi}{n} - \frac{2}{n} \sum_{k=0}^{n-1} (u_k - \pi) - \frac{2}{n^2} \sum_{k=0}^{n-1} k(u_k - \pi) \to 0 . \tag{11}$$

An application of the results in (3) into (11) yields

(12)

i) $n^{-1} \sum_{k=0}^{n-1} (u_k - \pi) \to 0$ a.s.(P)

ii) $n^{-1} \sum_{k=0}^{n-1} k(u_k - \pi) \to 0$ a.s.(P).

Set $P_1(A) = P(A \mid W > 0)$ for $A \in \mathcal{F}$. Arguing as in Nanthi (1979), the above results are seen to be true a.e.(P). Consequently, on S_1, we have the following.

Proposition 2:

For a supercritical MBPRE, $\log \bar{\theta}_n^*$ is a strongly consistent estimator of the offspring mean π under P_1.

Further, set,

$$\bar{\theta}_n^*(j) = \left[\prod_{k=1}^{n} X_k(j) \right]^{2/n^2} \quad \text{if } X_k(j) > 0$$

$$= 1 \qquad X_k(j) = 0$$

for $j = 1,2,\ldots,r$. Let N_2 be the exceptional set relating to (9) and N_2^c be its complimentary set. Conditional on the nonextinction set A, we have for any $\omega \in N_2^c \cap [W > 0] = S_2$,

i) $X_n(j) > 0$ for $n \geqslant 0$

ii) $\log X_n(j) - \sum_{k=1}^{n-1} \log \lambda_k(\bar{\xi}) \to \log [W\, v(j)]$.

Set $u_l = \log \lambda_l(\bar{\xi})$, $l \geqslant 0$ and

$$Y_n(j) = \left[\prod_{k=1}^{n} \left\{ \left(\prod_{l=0}^{k-1} \lambda_l(\bar{\xi}) \right) X_k(j) \right\} \right]^{2/n^2}$$

Taking logarithm and on simplification we get,

$$\log Y_n(j) = 2n^{-2} \sum_{k=1}^{n} \log X_k(j) - 2n^{-2} \sum_{k=1}^{n} (n-k)u_k. \tag{14}$$

Condition (9) above yields that, on S_2,

$$\log Y_n(j) \to 0 \quad \text{a.s.}$$

Consequently, on S_2, an application of the results in (3) into (14) yields

$$\bar{\theta}_n^*(j) = \left[\prod_{k=1}^{n} X_k(j) \right]^{2/n^2} \to e^{\pi} \quad \text{a.s., } j = 1,2,\ldots,r.$$

We have just proved the following.

Proposition 3:

For a supercritical MBPRE, $\log \bar{\theta}_n^*(j)$, $j = 1,2,\ldots,r$, is a strongly consistent estimator of π under P_1.

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Closure Semi-Continuity

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Abstract

Closure semi-continuity is defined and some results of closure semi-continuity analogous to those for closure continuity are obtained.

Let (X, τ) and (Y, τ') be any two topological spaces.

A set $A \subset X$ is said to be a semi-open set if there exists an open set 0 such that $0 \subset A \subset \bar{0}$ where $\bar{0}$ is the closure of 0 (Levine [6]).

S.O. (τ) will denote the class of all semi-open sets of (X, τ). If $x \in X$, $0(\tau, x)$ and S.O. (τ, x) will denote respectively the class of all open and semi-open sets of (X, τ) containing x.

In [5] Das defined semi limit point and in [3] Crossley and Hildebrend defined semi-closure $\underline{A}$ of a set $A \subset X$ in a manner analogous to limit point and closure.

Unless otherwise mentioned α will denote a mapping of (X, τ) into (Y, τ').

α is said to be semi-continuous if $U \in \tau' \Longrightarrow (U)\alpha^{-1} \in$ S.O.(τ) (Levine [6]).

α is said to be closure continuous at a point $x \in X$ iff for every $V \in 0(\tau', (x)\alpha)$ there exists a $U \in 0(\tau, x)$ such that $(\bar{U})\alpha \subset \bar{V}$. If α be closure continuous at every $x \in X$, then α is said to be closure continuous on X (Andrew and Whittlesy [1]).

Definition 1: α is said to be closure semi-continuous at a point $x \in X$ iff for every $V \in 0(\tau', (x)\alpha)$ there exists a $U \in$ S.O.(τ, x) such that $(\underline{U})\alpha \subset \bar{V}$. If α be closure semi-continuous at every $x \in X$, then α is said to be closure semi-continuous on X.

Theorem 1: If α be semi-continuous, then α is closure semi-continuous on X.

Proof: Let α be semi-continuous. Let $x \in X$ and let $V \in 0(\tau', (x)\alpha)$. Since α is semi-continuous, there exists a $U \in S.0(\tau, x)$ such that $(U)\alpha \subset V$ and then $(\underline{U})\alpha \subset \overline{(U)\alpha}$ (by Theorem 4, Biswas [2]) $\subset \bar{V}$. $\therefore$ α is closure semi-continuous at x. Since x is any point of X, α is closure semi-continuous on X.

Note 1: The converse of Theorem 1 is not true as shown by

Example 1: Let $X = \{a,b,c,d\}$,

$$\tau = \{\phi, X, \{a\}, \{b,c\}, \{a,b,c\}\}.$$

Then $S.0.(\tau) = \tau \cup \{\{a,d\}, \{b,c,d\}\}$.

$\alpha : X \to X$ is defined by

$$(a)\alpha = b, \quad (b)\alpha = a, \quad (c)\alpha = d, \quad (d)\alpha = c.$$

α is closure semi-continuous on X. Since $\{a\} \in \tau$ but $(\{a\})\alpha^{-1} = \{b\} \notin S.0.(\tau)$, α is not semi-continuous.

Note 2: Closure continuity $\Longrightarrow$ Closure semi-continuity but not conversely since α (defined in Example 1) is not closure continuous at d. For, $\{b,c\} \in 0(\tau, (d)\alpha)$, but as the only open set containing d is X and $(\underline{X})\alpha = X \not\subset \{\overline{b,c}\} = \{b,c,d\}$.

Theorem 2: Let (Y, τ') be a T_3 - space and let α be closure semi-continuous at a point $x \in X$. Then α is semi-continuous at x.

Proof: Let $V \in 0(\tau', (x)\alpha)$. Since (Y, τ') is a T_3-space, there exists a $V_1 \in 0(\tau', (x)\alpha)$ such that $\bar{V}_1 \subset V$. Since α is closure semi-continuous at x, there exists a $U \in S.0.(\tau, x)$ such that $(\underline{U})\alpha \subset \bar{V}_1$. Then $(\underline{U})\alpha \subset V$. Hence α is semi-continuous at x.

Corollary 1: If (Y, τ') be a T_3-space, then α is closure semi-continuous on X $\Longrightarrow$ α is semi-continuous.

Note 3: Semi-continuity at a point does not imply closure semi-continuity at that point as shown by

Example 2: Let $X = \{a,b,c\}$,

$$\tau = \{\phi, X, \{a\}, \{a,b\}, \{a,c\}\},$$

$$\tau' = \{\phi, X, \{b\}, \{a,c\}\}.$$

Then $\tau = S.0.(\tau)$, $\tau' = S.0.(\tau')$.

The identity mapping $I_X:(X, \tau) \longrightarrow (X, \tau')$ is semi-continuous at c. Since $\{a,c\} \in 0(\tau',c)$ but for every $U \in S.0.(\tau,c)$, $(\underline{U}) I_X = X \not\subset \overline{\{a,c\}} = \{a,c\}$, α is not closure semi-continuous at c.

In [1] Andrew and Whittlesy proved that if α be continuous at a point $x \in X$ at which α is not closure continuous, then every $U \in 0(\tau,x)$ has a limit point at which α is not continuous.

But if α be semi-continuous at a point $x \in X$ at which α is not closure semi-continuous, then every $U \in S.0.(\tau, x)$ may not possess a semi-limit point at which α is not semi-continuous as shown by

<u>Example 3</u>: Let $X = \{a,b,c\}$,

$$\tau = \{\phi, X, \{a\}, \{b\}, \{a,b\}\}.$$

Then $S.0.(\tau) = \tau \cup \{\{a,c\}, \{b,c\}\}$.

$\alpha : X \longrightarrow X$ is defined by

$(a)\alpha = b$, $(b)\alpha = a$, $(c)\alpha = b$.

α is semi-continuous at c. α is not closure semi-continuous at c. For, $\{b\} \in 0(\tau, (c)\alpha)$ but for every $U \in S.0.(\tau, c)$, $(\underline{U})\alpha = (X)\alpha = \{a,b\} \not\subset \overline{\{b\}} = \{b,c\}$.

Now $\{a,c\} \in S.0.(\tau, c)$. But it has no semi-limit point.

However we have the following result

<u>Theorem 3</u>: If $x \in X$ be such that (i) α is semi-continuous at x, (ii) α is not closure semi-continuous at x and (iii) $S.0.(\tau, x)$ is closed w.r. to intersection, then every $U \in S.0.(\tau, x)$ has a semi-limit point y at which α is not semi-continuous.

<u>Proof</u>: By (ii), there exists a $V \in 0(\tau', (x)\alpha)$ such that $(\underline{U})\alpha \not\subset \overline{V}$ for every $U \in S.0.(\tau, x)$. By (i), there exist a $U_1 \in S.0(\tau,x)$ such that $(U_1)\alpha \subset V$. Let $U \in S.0.(\tau, x)$. Let $U_2 = U \cap U_1$. Then by (iii) $U_2 \in S.0.(\tau, x)$. Hence $(\underline{U_2})\alpha \not\subset \overline{V}$ but $(U_2)\alpha \subset V$. Hence there exists a $y \in X$ such that y is a semi-limit point of U_2 and therefore of U but $(y)\alpha \notin \overline{V}$. Let $V_1 = Y - \overline{V}$. Then $V_1 \in 0(\tau', (y)\alpha)$. Let $U_3 \in S.0.(\tau, y)$. Since y is a semi-limit point of U_2, $U_2 \cap (U_3 - \{y\}) \neq \phi$. Let $z \in U_2 \cap (U_3 - \{y\})$. Then $(z)\alpha \in V$. Hence $(U_3)\alpha \not\subset V_1$ for every $U_3 \in S.0.(\tau,y)$. $\therefore$ α is not semi-continuous at y.

Definition 2: (X, τ) is said to be a ST_3- space if for every semi-closed set F of X and every $p \notin F$, there exist $U, V \in S.O.(\tau)$ such that $F \subset U$, $p \in V$ and $U \cap V = \phi$ (Das [4]).

(X, τ) is a ST_3 - space iff for every $x \in X$ and for every $U \in S.O.(\tau, x)$, there exists a $V \in S.O.(\tau, x)$ such that $\underline{V} \subset U$ (Das /4/).

Theorem 5: If (i) (X, τ) be a ST_3 - space and (ii) α be semi-continuous at a point $x \in X$, then α is closure semi-continuous at x.

Proof: Let $V \in 0(\tau', (x)\alpha)$. By (ii) there exists a $U_1 \in S.O.(\tau, x)$ such that $(U_1)\alpha \subset V$. By (i) there exists a $U \in S.O.(\tau, x)$ such that $\underline{U} \subset U_1$. Hence $(\underline{U})\alpha \subset (U_1)\alpha \subset V \subset \bar{V}$. Hence α is closure semi-continuous at x.

Note 4: The converse of Theorem 4 is not true as shown by

Example 4: Consider the topological spaces (X, τ), (X, τ') defined in Example 2. (X, τ') is a ST_3 - space. $I_X: (X, \tau') \dashrightarrow (X, \tau)$ is closure semi-continuous at a. But it is not semi-continuous at a.

It follows from Theorems 1, 2 and 4 that

Theorem 5: If (X, τ) be a ST_3 - space and (Y, τ') a T_3 - space, then α is semi-continuous at $x \in X$ (resp. on X) $\Longleftrightarrow$ α is closure semi-continuous at x (resp. on X).

Theorem 6: Let $A \subset X$. If α be closure continuous, then $\alpha' = \alpha | A$: $(A, \tau_A) \rightarrow (Y, \tau')$ is also closure continuous.

Proof: Let $x \in A$. Let $V \in 0(\tau', (x)\alpha')$. Since α is closure continuous, there exists a $U \in 0(\tau, x)$ such that $(\bar{U})\alpha \subset \bar{V}$. Then $U_1 = U \cap A \in 0(\tau_A, x)$ and $(\bar{U}_{1\tau_A})\alpha' \subset (\bar{U})\alpha \subset \bar{V}$. $\therefore$ α' is closure continuous.

Note 5: Restriction of a closure semi-continuous function is not always closure semi-continuous as shown by

Example 5: Consider the closure semi-continuous mapping α defined in Example 1.

Let $A = \{a, b, d\}$.
Then $\tau_A = \{\phi, A, \{a\}, \{b\}, \{a,b\}\}$.
and $S.O.(\tau_A) = \tau_A \cup \{\{a,d\}, \{b,d\}\}$.

$\alpha' = \alpha|A$ is not closure semi-continuous at d. For $\{b,c\} \in O(\tau, (d)\alpha)$ but there does not exist any $U \in S.O(\tau_A, d)$ such that

$$(\underline{U})_{\tau_A}\alpha \subset \overline{\{b,c\}} = \{b,c,d\}.$$

The product of two closure continuous mappings is closure continuous. But the product of two closure semi-continuous mappings is not always closure semi-continuous as shown by

<u>Example 6</u>: Let $X = \{x, y, z\}$,

$\tau_1 = \{\phi, X, \{x\}, \{y,z\}\}$,
$\tau_2 = \{\phi, X, \{x\}, \{y\}, \{x, y\}\}$,
τ_3 = discrete topology in X.
$I_X : (X, \tau_1) \longrightarrow (X, \tau_2)$ is closure semi-continuous.

$\alpha : (X, \tau_2) \longrightarrow (X, \tau_3)$ defined by

$$(x)\alpha = x,\ (y)\alpha = y,\ (z)\alpha = x$$

is also closure semi-continuous.

But $I_X\alpha = \alpha : (X, \tau_1) \longrightarrow (X, \tau_3)$ is not closure semi-continuous at z. For, $\{x\} \in O(\tau_3, (z)\alpha)$ but there does not exist any $U \in S.O(\tau_1, z)$ such that $(\underline{U})\alpha \subset \{\bar{x}\} = \{x\}$.

<u>Theorem 7</u>: Let $\alpha_i : (X_i, \tau_i) \longrightarrow (Y_i, \tau'_i)$ be closure semi-continuous for $i = 1,2$. Let $X = X_1 \times X_2$, $Y = Y_1 \times Y_2$, $\tau = \tau_1 \times \tau_2$, $\tau' = \tau'_1 \times \tau'_2$, $\alpha = \alpha_1 \times \alpha_2$. Then $\alpha : (X, \tau) \longrightarrow (Y, \tau')$ is closure semi-continuous.

<u>Proof</u>: Let $x = (x_1, x_2) \in X$ and $W \in O(\tau', (x)\alpha)$. Then there exist $U_i \in O(\tau'_i, (x_i)\alpha_i)$ $(i=1,2)$ such that $U_1 \times U_2 \subset W$. Since α_i is closure semi-continuous, there exist $V_i \in S.O(\tau_i, x_i)$ such that $(\underline{V}_i)\alpha_i \subset \bar{U}_i$ $(i=1,2)$. Then $V = V_1 \times V_2 \in S.O.(\tau, x)$ and $(\underline{V})\alpha = (\underline{V}_1 \times \underline{V}_2)\alpha = (\underline{V}_1)\alpha_1 \times (\underline{V}_2)\alpha_2 \subset \bar{U}_1 \times \bar{U}_2 \subset \bar{W}$. $\therefore$ α is closure semi-continuous.

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Oscillations of a Spherical Bubble in Walters Liquid B'

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Abstract

In this paper damped non-linear oscillations of a spherical bubble filled with a gas obeying Boyle's law in Walters liquid B' have been investigated by employing a perturbation scheme in terms of ϵ which measures the departure of the cavity pressure from the pressure at infinity. It is found that for $\epsilon > 0$ the effect of elasticity is to increase the frequency.

Introduction

The growth of spherical bubble in a non-newtonian liquid by introducing second order terms in stress strain velocity relations of classical hydrodynamics has been studied by Jain [1]. Here we propose to study the oscillations of a spherical bubble or cavity in an incompressible homogeneous Walters liquid B' subjected to a uniform pressure at infinity and filled with a gas obeying Boyle's law.

The liquid pressure at infinity is taken to be Π and the pressure inside the bubble when of radius R_0 is supposed to be $m^3\Pi$. The perturbation theory is adopted to discuss the non-linear oscillations for $m \sim 1$.

General Theory

The constitutive equation[3] for Walters liquid B' with short memries is

$$p^{ik} = - pg_{ik} + 2\eta_o e^{ik} - 2k_o \frac{\delta}{\delta t} e^{ik}$$

where $e^i_k = \frac{1}{2}\left\{v^i_{,k} + v^k_{,i}\right\}$ and η_o is the limiting viscosity at small rates of shear, K_o the elasticity of the fluid, v^i the velocity tensor, $\frac{\delta}{\delta t}$ denotes the convected differential of a tensor quantity for any contravarient tensor b^{ik} and is given by

$$\frac{\delta b^{ik}}{\delta t} = \frac{\delta b^{ik}}{\delta t} + v^m \frac{\delta b^{ik}}{\delta x^m} - b^{im} \frac{\partial v^m}{\delta x^m} - b^{mk} \frac{\partial v^i}{\delta x^m}$$

In spherical polar coordinates (r, θ, ϕ) with centre of bubble as pole, the velocity components (u,v,w) for symmetrical motion of a spherical bubble can be taken as

(1) $u = F(t_1)/ r^2, \; v = o, \; w = o$

where $F(t_1) = R_1^2 R_1'$, R_1 being the radius of cavity and accent representing differentiation with respect to t_1.

Now the equation of motion can be written as

$$(2) \quad \frac{\delta p_{rr}}{\delta r} + \frac{1}{r}\frac{\delta p_{r\theta}}{\delta \theta} + \frac{1}{r\sin\theta}\frac{\delta p_{r\theta}}{\delta \phi} + \frac{1}{r}\left[2p_{rr} - p_{\theta\theta} - p_{\phi\phi} + p_{r\theta}\cos\theta\right] + \rho\left(f_r - \frac{\delta u}{\delta t}\right) = o$$

where the constitutive equations of Walters liquid B' in the symmetric case provides

$$p_{rr} = 2\eta_o \frac{\delta u}{\delta r} - 2K_o\left[\frac{\delta^2 u}{\delta t \delta r} + u\frac{\delta^2 u}{\delta r^2} - 2\left(\frac{\delta u}{\delta r}\right)^2\right],$$

$$(3) \quad p_{\theta\theta} = 2\eta_o \frac{u}{r} - 2K_o\left[\frac{1}{r}\frac{\delta u}{\delta t} + \frac{u}{r}\frac{\delta u}{\delta r} - \frac{u^2}{r^2}\right],$$

$$p_{\phi\phi} = 2\eta_o \frac{u}{r} - 2K_o\left[\frac{1}{r}\frac{\delta u}{\delta t} + \frac{u}{r}\frac{\delta u}{\delta r} - \frac{u^2}{r^2}\right],$$

$$p_{r\theta} = p_{r\phi} = p_{\theta\phi}$$

Substituting (3) in (2) and using (1), we obtain, after integration following equation

$$(4) \quad \rho\left[-\frac{F'(t_1)}{r} + \frac{1}{2}u^2\right] = -p + \frac{14}{3}K_o\frac{F^2(t_1)}{r^6} + \Pi$$

where the condition $p = \Pi$ as $r \longrightarrow \infty$ has been used.

Now, from Boyle's Law the pressure inside the bubble, when it is of radius R_1 is $\frac{m^3 \Pi R_o}{R_1{}^3}$, and considering the equilibrium at the surface of the bubble, we get,

$$(5) \quad \frac{m^3 \Pi R_o{}^3}{R_1{}^3} = -p - 4\eta_o\frac{R_1'}{R_1} + 4K_o\left[\frac{3R_1'^2}{R_1^2} + \frac{R_1''}{R_1}\right]$$

Eliminating p from (4) and (5), we get,

$$(6)\quad R'' + \frac{3}{2}\,\frac{(R^2-\frac{11}{9}K^2)}{R(R^2-K^2)}\,R'^2 + \frac{2\lambda R'}{R^2-K^2} = \frac{(m^3/R^3)-1}{R^2(R^2-K^2)}$$

where $t = \sqrt{\rho}\;\frac{R_o t_1}{\Pi}$, $\lambda = \frac{2\eta_o}{R_o\rho}\,\frac{\sqrt{\rho}}{\Pi}$, $K^2 = \frac{4K_o}{\rho R_o^2}$ and $R = \frac{R_1}{R_o}$,

is the non-dimensional radius of the bubble.

Thus equation (6) shows that R=m is the critical point being the point of static equilibrium. To discuss the motion near this point we let $R = 1 - \epsilon(1-x)$ in (6), $\epsilon = 1-m$ being a small quantity.

Thus expressing (6) in terms of x, we get,

$$(7)\quad x'' + 2\gamma x' + \nu^2 x = -\frac{3}{2}\epsilon\,\frac{(1-\frac{11}{9}K^2)}{1-K^2}\,x'^2 - 4\nu\epsilon\,\frac{(1-x)}{(1-K^2)^2}\,x' - 3\epsilon\,\frac{x(2+4K^2-xK^2-3x)}{(1-K^2)^2} + o(\epsilon^2).$$

where $\gamma = \lambda/(1-K^2)$ and $\nu^2 = 3/(1-K^2)$

Above equation is similar to the equation representing damped non-linear oscillations investigated earlier by Mendelson [2]. Thus, following his procedure, we seek a solution by periodic in ψ as $x = x(a,\psi)$, where $\frac{da}{dt} = \xi(a)$ and $\frac{dw}{dt} = w(a)$ and set

$$(8)\quad \begin{aligned} x &= x_o + \epsilon x_1 + \dots \\ w &= w_o + \epsilon w_1 + \dots \\ \xi &= -\gamma a + \epsilon\xi + \dots \end{aligned}$$

when $w_o^2 = \nu^2 - \gamma^2$. It should be noted that for oscillatory solution $w_o^2 > o$ which gives $K^2 < 1 - \frac{\lambda^2}{3}$.

The zeroth order solution is same as given by Mendelson and can be easily obtained as

$$(9)\quad x_o = a\cos\psi$$

Using the value of x_o, the equation denoting the first order approximation, x_1, can be obtained by substituting (8) in (7) and equating terms of order ϵ. Thus, we have,

$$w_o^2 \frac{\partial^2 x_1}{\partial \psi^2} + \gamma^2 a^2 \frac{\delta^2 x_1}{\delta a^2} - 2w_o \gamma a \frac{\delta^2 x_1}{\delta a \delta \psi} + 2\gamma w_o \frac{\delta x_1}{\delta \psi} - a\gamma^2 \frac{\delta x_1}{\delta a} + \nu^2 x_1$$

$$= -\frac{3}{4} a^2 \left[\frac{(1-\frac{11}{9}K^2)}{1-K^2} (w_o^2 + \gamma^2) - \frac{2(1+K^2)(3-K^2)}{(1-K^2)} + \frac{4}{3} \frac{\gamma^2}{1-K^2} \right] +$$

$$(10) \quad + \cos\psi \left[2 w_o w_1 a + \gamma \left(a \frac{\delta \xi_1}{\delta a} - \xi_1\right) - \frac{6a(1+2K^2)}{(1-K^2)^2} + \frac{2\gamma^2 a}{1-K^2} \right] +$$

$$+ \sin 1\psi \left[2w_o \xi_1 - \gamma a \frac{\delta w_1}{\delta a} + \frac{2 w_o a}{1-K^2} \right] + a^2 \cos 2\psi \left[-\frac{3}{4} \frac{(1-\frac{11}{9}K^2)(\gamma^2 - w_o^2)}{1-K^2} + \right.$$

$$\left. + \frac{3}{2} \frac{(1+K^2)(3-K^2)}{(1-K^2)} - \frac{\gamma^2}{1-K^2} \right] + a^2 \sin 2\psi \left[-\frac{3}{2} \frac{(1-\frac{11}{9}K^2)}{1-K^2} w_o \gamma - \frac{\gamma w_o^2}{1-K^2} \right]$$

To prevent the occurrence of secular terms, we get from equation (10),

$$(11) \quad \begin{aligned} & 2w_o w_1 a + \gamma \left(a \frac{\delta \xi_1}{\delta a} - \xi_1\right) = \frac{6a(1+2K^2)}{(1-K^2)^2} - \frac{2\gamma^2 a}{1-K^2}, \\ & 2w_o \xi_1 - \gamma a^2 \frac{\delta w_1}{\delta a} = 2 \frac{\gamma w_o a}{1-K^2}. \end{aligned}$$

Now, it is easy to see that equations (11) are satisfied provided

$$(12) \quad \begin{aligned} \xi_1 &= -a \frac{\gamma}{1-K^2}, \\ w_1 &= \frac{1}{w_o} \left[\frac{3(1+2K^2) - \gamma^2(1-K^2)}{(1-K^2)^2} \right]. \end{aligned}$$

To determine x_1, we now let

$$(13) \quad x_1(a, \psi) = \frac{1}{2} a^2 A_o + a^2 (A_2 \cos 2\psi + B_2 \sin 2\psi)$$

Substituting above in (10), we get

$$(14)\quad A_o = -\left[\frac{3(1-\frac{11}{9}K^2)(w_o^2+\gamma^2) - 3(1+K^2)(3-K^2) + \gamma^2}{(1-K^2)(w_o^2-\gamma^2)}\right],$$

and

$$A_2 = \left[\frac{\gamma w_o\left\{12\ (1+K^2)(3-K^2) - 3(1-\frac{11}{9}K^2)(3\gamma^2-5w_o^2) - 2\ (5\gamma^2+3w_o^2)\right\}}{2\ (1-K^2)\left\{(\gamma^2-3w_o^2)^2 + 16\gamma^2 w_o^2\right\}}\right]$$

(15)

$$B_2 = \left[\frac{\gamma w_o\left\{3\ (\gamma^2+w_o^2)\ (3-\frac{11}{9}K^2) - 12\ (1+K^2)\ (3-K^2)\right\}.}{2\ (1-K^2)\ \left\{(\gamma^2-3\,w_o^2)^2 + 16\gamma^2 w_o^2\right\}}\right]$$

Now using the value of ξ_1 and w_1 as given (12), we have

$$\frac{da}{dt} = -\gamma a + \epsilon\,\xi_1 = -\gamma a\left\{1 + \epsilon/(1-\ ^2)\right\}$$

(16)

$$\frac{d\psi}{dt} = w_o + \epsilon\ w_1 = w_o + \frac{\epsilon}{w_o}\left\{\frac{3(1+2K^2) - \gamma^2(1-K^2)}{(1-K^2)^2}\right\}.$$

The equations (16) are to be integrated under the condition $a = a_o$ for $t = 0$, and $\psi = \psi_o$ for $t = t_o$, we get,

$$a = a_o\,\epsilon^{-\gamma}(1 + \frac{\epsilon}{1-K^2})\ t\ ,$$

(17)

$$= \psi_o + \frac{\sqrt{3(1-K^2) - \lambda^2}}{1-K^2}\left[1 + \frac{\epsilon\left\{3(1+2K^2)(1-K^2) - \lambda^2\right\}}{(1-K^2)\left\{3(1-K^2) - \lambda^2\right\}}\right]t\ .$$

Finally, correct upto order ϵ, we have

$$x = a\cos\psi + \epsilon a^2\left[-\frac{1}{2}\,\frac{3(1-\frac{11}{9}K^2)(w_o^2+\gamma^2) - 3(1+K^2)(3-K^2) + \gamma^2}{2(1-K^2)\left\{(\gamma^2-3w_o^2)^2 + 16\gamma^2 w_o^2\right\}}\right. +$$

$$(18)\quad + \frac{\gamma w_o\left\{12(1+K^2)(3-K^2) - 3(1-\frac{11}{9}K^2)(3\gamma^2-5w_o^2) - 2(5\gamma^2+3w_o^2)\right\}}{2(1-K^2)\left\{(\gamma^2-3w_o^2)^2 + 16\gamma^2 w_o^2\right\}}\cos 2\psi$$

$$\left. + \frac{\gamma w_o\left\{3(\gamma^2+w_o^2)(3-\frac{11}{9}K^2) - 12(1-K^2)(3-K^2)\right\}}{2(1-K^2)\left\{(\gamma^2-3w_o^2)^2 + 16\gamma^2 w_o^2\right\}}\sin 2\psi\right].$$

Some idea about the effect of elastic parameter K^2 can be ascertained from the first term in equation (18) which under the conditions $x = 1$, $\frac{dx}{dt} = o$ at $t = o$ can be represented as

$$x_1 = f_1(\lambda, K)\, e^{f_2(\lambda, K)t} \cos\left[\tan^{-1}\left\{f_3(\lambda, K) + f_4(\lambda, K)t\right\}\right]$$

where

$$f_1(\lambda, K) = \left[1 + \frac{\lambda^2 (1-K^2+\epsilon)^2 \left\{3(1-K^2) - \lambda^2\right\}^2}{\left[(1-K^2)\left\{3(1-K^2) - \lambda^2\right\} + \epsilon\left\{3(1+2K^2)(1-K^2) - \lambda^2\right\}\right]^2}\right],$$

$$f_2(\lambda, K) = -\frac{\lambda(1-K^2+\epsilon)}{(1-K^2)^2},$$

$$f_3(\lambda, K) = -\frac{\lambda(1-K^2+\epsilon)\sqrt{3(1-K^2) - \lambda^2}}{(1-K^2)\left\{3(1-K^2) - \lambda^2\right\} + \epsilon\left\{3(1+2K^2) - \lambda^2\right\}};$$

$$f_4(\lambda, K) = \frac{\sqrt{3(1-K^2) - \lambda^2}}{(1-K^2)}\left\{1 + \epsilon\frac{3(1+K^2)(1-K^2) - \lambda^2}{(1-K^2)\left\{3(1-K^2) - \lambda^2\right\}}\right\}.$$

It should be noted that positive and negative values of ϵ (= 1-m) correspond to the situations when the initial external pressure is more or less than the initial internal pressure $m^3\Pi$. Curves drawn for the damping of with time for various values of parameters in figures 1 and 2, lead to the following conclusions.

(i) For $\epsilon > 0$, the effect of elasticity is to increase the frequency of oscillations while reverse is the case for $\epsilon < 0$;

(ii) In general, for both positive and negative values of ϵ, the damping is faster with increase in value of elastic parameter except for the initial stages when $\epsilon > 0$.

(iii) For the same numerical difference of pressure the frequency is greater when the interior pressure is exceeded by the internal pressure than when it exceeds Π.

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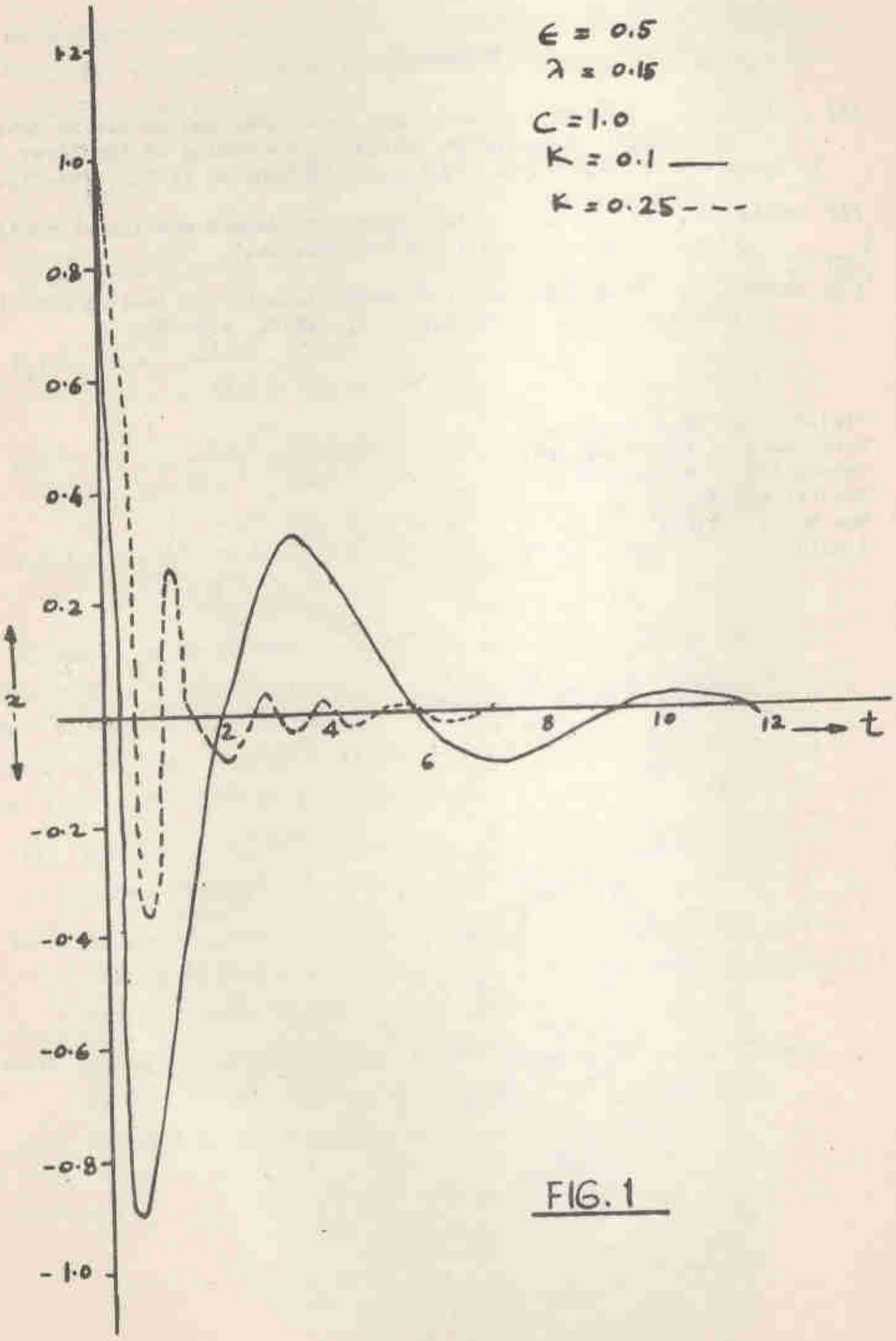

$\epsilon = 0.5$
$\lambda = 0.15$
$C = 1.0$
$K = 0.1$ ——
$K = 0.25$ - - -
1.2
1.0
0.8
0.6
0.4
0.2
z
-0.2
-0.4
-0.6
-0.8
-1.0
2
4
6
8
10
12
t

FIG. 1

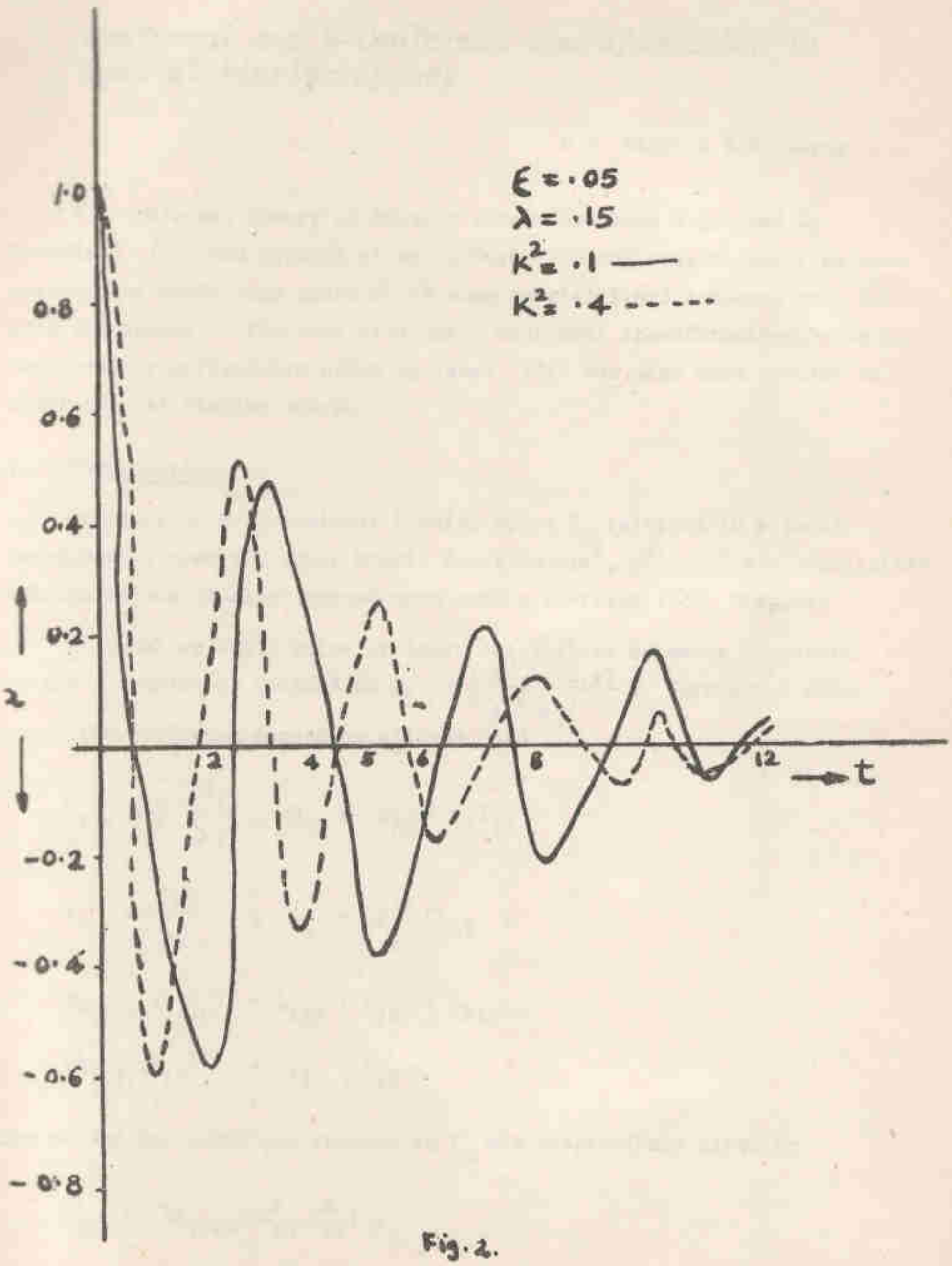

Fig. 2.

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Conformal and h-Conformal Transformation in Special Finsler Spaces

U.P. Singh & B.N. Gupta

The conformal theory of Finsler spaces has been discussed by Knebelman [3] and several other authors. In the present paper we have established conditions under which some special Finsler spaces have become the spaces of the same kind under conformal transformation. The h-conformal transformation given by Izumi [2] has also been studied in these special Finsler spaces.

1. Introduction

Consider an n-dimensional Finsler space F_n referred to a local coordinate system x^i, whose metric function $L(x^i, y^i)$, $(y^i = \dot{x}^i)$ satisfies the conditions usually imposed upon such a function ([8], Chap. 1).

As usual we shall raise or lower the indices by means of metric tensor g_{ij} which is defined as $g_{ij} = \frac{1}{2}\frac{\partial^2 L^2(x,y)}{\partial y^i \partial y^j}$. Throughout this paper the following notations will be used

$$C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}, \quad h_{ij} = g_{ij} - l_i l_j,$$

$$l_i = \frac{\partial L}{\partial y^i}, \quad C_i = g^{jk} C_{ijk},$$

$$\mathfrak{S}_{(ijk)}(L_{ijk}) = L_{ijk} + L_{jki} + L_{kij},$$

$$\mathcal{U}_{(ij)}(L_{ij}) = L_{ij} - L_{ji}.$$

The v- and hv- curvature tensors of F_n are respectively given by

$$S^i_{hjk} = \mathcal{U}_{(jk)}(C^i_{mj} C^m_{hk}),$$

$$P_{hijk} = \mathcal{U}_{(hi)} (C_{ijk|h} + C^m_{hj} P_{mik}) ,$$

where the symbol $|$ denotes the h-covariant derivative and P_{mik} is the (v) hv-torsion tensor defined by

$$P_{mik} = y^h P_{hmik} = C_{imk|j} y^j .$$

A Finsler space with $S^i_{hjk} = o$ is called a flat Finsler space.

In terms of P_{ijk}, we define $P_i = g^{jk} P_{ijk}$.

Definition (1.1). Two metric spaces are said to be conformal if their metric tensors are proportional to each other ([8]).

Consider two Finsler spaces F_n and $\bar{F}_n$ represented by the same co-ordinate system. It is assumed that the spaces F_n and $\bar{F}_n$ are conformal. Thus we have the relation

$$(1.1) \qquad \bar{g}_{ij} = e^{2\sigma} g_{ij} ,$$

where $\sigma = \sigma(x)$ is a scalar function.

If σ is constant, then transformation is called homothetic. A quantity without bar will be defined in F_n and a quantity with bar will stand for the corresponding quantity in the space $\bar{F}_n$. If the Finsler spaces F_n and $\bar{F}_n$ are conformal then the following relations are always satisfied ([2])

$$\text{(a) } \bar{g}^{ij} = e^{-2\sigma} g^{ij} , \quad \text{(b) } \bar{h}_{ij} = e^{2\sigma} h_{ij} ,$$

$$\text{(c) } \bar{C}_{ijk} = e^{2\sigma} C_{ijk} , \quad \bar{C}^i_{jk} = C^i_{jk} ,$$

$$(1.2) \qquad \bar{C}_i = C_i , \quad \bar{C}^i = e^{-2\sigma} C^i ,$$

$$\text{(d) } \bar{P}_{jlk} = e^{2\sigma} (P_{jlk} + V^h_{jlk} \sigma_h) ,$$

$$\text{(e) } \bar{P}^i_{hjk} - P^i_{hjk} = a^i_{hj|k} + a^i_{hm} C^m_{jk} - S^i_{hmk} b^m_j ,$$

where

$$V^h_{jlk} = y_j \, C^h_{lk} + y_k \, C^h_{lj} + L^2 \, g_{il} \frac{\partial C^{ih}_j}{\partial y^k} + y_l \, C^h_{jk} + y^h \, C_{jlk}$$

$$+ L^2 \, (C^m_{jl} \, C^h_{mk} + C^m_{kl} \, C^h_{mj} - C^h_{ml} \, C^m_{jk}) \, ,$$

$$\sigma_h = \frac{\partial \sigma}{\partial x^h} , \; y_i = g_{ij} \, y^j \, , \; C^{ih}_j = g^{im} \, C^h_{jm} \, ,$$

$$a^i_{jk} = (\delta^i_j \, \delta^h_k + \delta^i_k \, \delta^h_j - g^{ih} \, g_{jk} + y_j \, C^{ih}_k - y^i \, C^h_{jk} - L^2 \, g^{hm} S^i_{jkm}) \sigma_h .$$

$$b^i_j = (\delta^i_j \, y^h + \delta^h_j \, y^i - g^{ih} \, y_j + L^2 \, C^{ih}_j) \, \sigma_h$$

and the symbol $|_i$ stands for v-covariant differentiation with respect to y^i.

If a conformal vector σ_i satisfies the h-condition defined by ([2]):

$$\text{(1.3)} \qquad L \, C^h_{ij} \, \sigma_h = \sigma_1 \, h_{ij} \, ,$$

where $\sigma_1 = L \, C^i \, \sigma_i \, /(n-1)$,

then the transformation is called h-conformal transformation ([2]). It has been shown by Izumi ([2]) that σ_1 is independent of y^i. We now introduce a new vector $\rho_i \, (x,y)$ defined by

$$\text{(1.4)} \qquad \rho_i = \sigma_i + \sigma_1 \, l_i \, ,$$

which is rewritten as ([2])

$$\text{(1.5a)} \qquad \rho_i = (\delta^j_i + L \, C^j \, l_i/(n-1)) \, \rho_j \, .$$

In view of (1.4) and (1.5a), we have

$$\text{(1.5b)} \qquad \sigma_i = (\delta^j_k - L \, C^j \, l_i/(n-1)) \, \rho_j \, .$$

We see that if one of two vectors σ_i, ρ_i vanishes the other also vanishes. We shall use the following lemma introduced by Izumi ([2]).

Lemma (1.1). If a Finsler space $F_n (n \geq 3)$ is C-reducible ([5]), that is it satisfies

$$C_{ijk} = \frac{1}{(n+1)} (h_{ij} C_k + h_{jk} C_i + h_{ki} C_j)$$

$(C_{ijk} \neq 0)$, then the h-conformal transformation is reduced to a homothetic one, that is $\sigma_i = o$ and $\rho_i = o$.

Under h-conformal transformation, the following relations are satisfied ([2])

$$\text{(a)} \quad \bar{P}^i_{jk} - P^i_{jk} = \rho C^i_{jk}, \text{ where } \rho = \rho_i y^i, \quad \rho_i = \frac{\partial \rho}{\partial y^i},$$

$$\text{(b)} \quad \bar{P}_j - P_j = \rho C_j,$$

$$(1.6) \qquad \text{(c)} \quad \bar{P}^i_{jkl} - P^i_{jkl} = \rho_j C^i_{kl} - \rho^i C_{jkl} - \rho S^i_{jkl},$$

$$\text{(d)} \quad \bar{\lambda} = \lambda + \rho, \text{ where } \lambda = P_j C^j / C^2, \; C^2 = C_i C^i$$

2. Conformal Transformation in Some Special Finsler Spaces

Definition (2.1). A Finsler space is called a Landsberg space if $P_{ijk} = o$.

From (1.2 d), we get the following

Theorem (2.1). Under a conformal transformation a Landsberg space is transformed into Landsberg space if and only if $V^h_{jlk} \sigma_h$ vanishes.

If the conformal transformation is homothetic then $\sigma_h = o$ and we have

Corollary (2.1). Under a homothetic transformation a Landsberg space is transformed into a Landsberg space.

Definition (2.2). A Finsler space is said to be a P^*-Finsler space (/1/) if the torsion tensor P_{ijk} has the form

$$(2.1) \qquad P_{ijk} = \lambda\, C_{ijk} ,$$

where λ is a scalar function given by

$$(2.2) \qquad \lambda = P_i\, C^i / C^2$$

In view of relations (1.2 c), (1.2 d) and (2.2), we get

$$(2.3) \qquad \bar{\lambda} = \lambda + \frac{1}{C^2} V^h_j\, C^j \sigma_h ,$$

where

$$V^h_j = V^h_{jlk}\, g^{lk} = y_j\, C^h + y^h\, C_j + L^2 \left(\frac{\partial C^{ih}_j}{\partial y^i} + C^i\, C^h_{ij}\right).$$

From (1.2 c), (1. 2 d) and (2.3), we have

$$\bar{P}_{ijk} - \bar{\lambda}\, \bar{C}_{ijk} = e^{2\sigma} \left[P_{ijk} - \lambda\, C_{ijk} + \left(V^h_{ijk} - \frac{1}{C^2} V^h_m\, C^m\, C_{ijk}\right) \sigma_h \right] ,$$

which proves the following

Theorem (2.2). Under a conformal transformation a P^*-Finsler space will be P^*-Finsler space if and only if

$$V^h_{ijk}\, \sigma_h = \frac{1}{C^2} V^h_m\, C^m\, C_{ijk}\, \sigma_h .$$

Corollary (2.2). A P^*-Finsler space is transformed into a P^*-Finsler space, provided that the transformation is homothetic.

Definition (2.3). A Finsler space F_n $(n \geq 3)$ is called P-reducible (/6/) if the (v) hv-torsion tensor P_{ijk} of F_n is written in the form

$$(2.4) \qquad P_{ijk} = G_i\, h_{jk} + G_j\, h_{ki} + G_k\, h_{ij} ,$$

where $G_i = \frac{1}{(n+1)} P_i$.

By virtue of (1.2 d), we get

$$(2.5) \quad \bar{G}_i = G_i + \frac{1}{(n+1)} V_i^h \sigma_h .$$

The relations (1.2 b), (1.2 d) and (2.5) give

$$\bar{P}_{ijk} - \mathfrak{S}_{(ijk)} (\bar{G}_i \bar{h}_{jk}) = e^{2\sigma} \Big[P_{ijk} - \mathfrak{S}_{(ijk)} (G_i h_{jk}) + \Big\{ V^h_{ijk} - \frac{1}{(n+1)} \mathfrak{S}_{(ijk)} (V_i^h h_{jk}) \Big\} \sigma_h \Big] ,$$

which shows the following

Theorem (2.3). Under a conformal transformation a P-reducible Finsler space is transformed into P-reducible space if and only if

$$V^h_{ijk} \sigma_h = \frac{1}{(n+1)} \mathfrak{S}_{(ijk)} (V_i^h h_{jk}) \sigma_h .$$

Corollary (2.3). Under a homothetic transformation a P-reducible Finsler space is transformed into a P-reducible Finsler space.

Definition (2.4). A non-Riemannian Finsler space F_n is called semi-P 2-like ([9]) if the (v) hv-torsion tensor of F_n is written in the form

$$P_{ijk} = B_i C_j C_k + B_j C_k C_i + B_k C_i C_j ,$$

where

$$(2.6) \quad B_i = \frac{1}{C^2} \Big[P_i - \frac{2}{3C^2} (P_j C^j) C_i \Big] .$$

A direct calculation based on equations (1.2 a), (1.2 c) and (1.2 d) gives

$$(2.7) \quad \bar{B}_i = e^{2\sigma} \Big[B_i + \frac{1}{C^2} (V_i^h - \frac{2}{3C^2} V_j^h C^j C_i) \sigma_h \Big] .$$

From (1.2 c), (1.2 d) and (2.7), we have

$$\bar{P}_{ijk} - \mathfrak{G}_{(ijk)}\,(\bar{B}_i \bar{C}_j \bar{C}_k) = e^{2\sigma}\Big[P_{ijk} - \mathfrak{G}_{(ijk)}\,(B_i C_j C_k) + \Big\{V^h_{ijk} - \mathfrak{G}_{(ijk)}\,(L^h_i C_j C_k)\Big\}\sigma_h\Big], \tag{2.8}$$

where

$$L^h_i = \frac{1}{C^2}\,(V^h_i - \frac{2}{3C^2}\,V^h_j C^j C_i).$$

The relation (2.8) yields the following

Theorem (2.4). Under a conformal transformation a semi-P_2-like Finsler space is transformed into a semi-P2-like Finsler space if and only if

$$V^h_{ijk}\,\sigma_h = \mathfrak{G}_{(ijk)}\,(L^h_i C_j C_k)\,\sigma_h .$$

Corollary (2.4). A semi-P2-like Finsler space is semi-P2-like under a homothetic transformation.

Definition (2.5). A Finsler space is called P-symmetric ([6]) if hv-curvature tensor P_{hijk} of space satisfies $P^i_{hjk} = P^i_{hkj}$.

In view of (1.2 e), we have

$$\bar{P}^i_{hjk} - \bar{P}^i_{hkj} = P^i_{hjk} - P^i_{hkj} + \mathfrak{U}_{(jk)}\,(a^i_{hj}|_k + S^i_{hmj}\, b^m_k),$$

which proves that

Theorem (2.5). Under a conformal transformation a P-symmetric Finsler space remains P-symmetric if and only if

$$\mathfrak{U}_{(jk)}\,(a^i_{hj}|_k + S^i_{hmj}\, b^m_k) = o.$$

Theorems (2.1) to (2.4) give the following

Theorem (2.6). If a Finsler space F_n $(n \geqslant 3)$ satisfies the condition $V^h_{ijk} = o$, then under a conformal transformation the following properties hold good:

(I) A Landsberg space is transformed into a Landsberg space.

(II) A P^*- Finsler space is transformed into a P^* -Finsler space.

(III) A P-reducible space is transformed into a P-reducible space.

(IV) A semi-P2-like Finsler space is transformed into a semi-P2-like Finsler space.

3. h-Conformal Transformation in Some Finsler Spaces

In this section we suppose that the two Finsler spaces F_n and $\bar{F}_n$ are conformal and the conformal vector σ_i satisfies the h-condition given by (1.3). The transformation in this case will be called h-conformal.

Now we shall discuss the h-conformal transformation in special Finsler spaces introduced in previous section.

The condition $\rho = o$ gives $\rho_i = o$ and therefore (1.5 b) shows $\sigma_i = o$, that is the transformation is homothetic. Conversely, under a homothetic transformation, (1.5 a) gives $\rho_i = o$, which implies $\rho = o$. These facts in view of equation (1. 6 a) and definition (2.1) yield

Theorem (3.1). Under h-conformal transformation a non-Riemannian Landsberg space is transformed into a Landsberg space if and only if the transformation is homothetic.

By virtue of relations (1.1), (1.2 c), (1.6 a) and (1.6 d), we have

$$\bar{P}_{ijk} - \bar{\lambda}\, \bar{C}_{ijk} = e^{2\sigma} (P_{ijk} - \lambda\, C_{ijk}) .$$

This relation and definition (2.2) prove

Theorem (3.2). Under h-conformal transformation a P^* -Finsler space is transformed into a P^*-Finsler space.

Further, the relation (1. 6 b) gives

$$\bar{G}_i = G_i + \frac{1}{(n+1)} \rho\, C_i .$$

Therefore we have

$$\bar{P}_{ijk} - \mathfrak{G}_{(ijk)} (\bar{G}_i \bar{h}_{jk}) = e^{2\sigma} \Big[P_{ijk} - \mathfrak{G}_{(ijk)} (G_i h_{jk}) + \rho \Big\{ C_{ijk} - \frac{1}{(n+1)} \mathfrak{G}_{(ijk)} (C_i h_{jk}) \Big\} \Big] ,$$

which in view of lemma (1.1) and definition (2.3) proves the following

Theorem (3.3). Under h-conformal transformation a P-reducible Finsler space is transformed into a space of same kind if and only if the transformation is homothetic.

Next, under h-conformal transformation, making use of (1.2 d) and (1.6 b), the vector B_i is transformed as

$$\text{(3.1)} \qquad \bar{B}_i = e^{2\sigma} \left(B_i + \frac{1}{3\, C^2} \rho\, C_i \right) .$$

By virtue of (1.1), (1.2 d), (1.6 a) and (3.1), we get

$$\text{(3.2)} \qquad \bar{P}_{ijk} - \mathfrak{G}_{(ijk)} (\bar{B}_i \bar{C}_j \bar{C}_k) = e^{2\sigma} \Big[P_{ijk} - \mathfrak{G}_{(ijk)} (B_i C_j C_k) + \rho \Big(C_{ijk} - \frac{1}{C^2} C_i C_j C_k \Big) \Big] .$$

A non-Riemannian Finsler space having its (h) hv-torsion tensor C_{ijk} in the form

$$C_{ijk} = \frac{1}{C^2} C_i C_j C_k , \quad C^2 \neq o$$

is called C2-like ([6]).

In view of relation (3.2) and definition (2.4), we have

Theorem (3.4). Under h-conformal transformation a semi-P2-like Finsler space is transformed into a semi-P2-like space if and only if either the transformation is homothetic or the space is C2-like.

From (1.6 c), we have

$$\bar{P}^i_{jkl} - \bar{P}^i_{jlk} = P^i_{jkl} - P^i_{jlk} - 2\rho \; S^i_{jkl} \, .$$

This relation and definition (2.5) give

Theorem (3.5). Under h-conformal transformation a P-symmetric Finsler space remains P-symmetric if and only if either the transformation is homothetic or the space is flat.

Definition (3.1). If the hv-curvature tensor P_{ijkl} of a Finsler space F_n $(n \geqq 3)$ satisfies

$$P_{ijkl} = \lambda_i \; C_{jkl} - \lambda_j \; C_{ikl}$$

then the space is called P2-like ([4]).

It has been shown by Matsumoto ([4]) that in a P2-like Finsler space either $P_{ijkl} = o$ or $S_{ijkl} = o$.

In view of (1.1), (1.2 c) and (1.6 c), we get

$$\bar{P}_{ijkl} - (\bar{\lambda}_i \, \bar{C}_{jkl} - \bar{\lambda}_j \, C_{ikl}) = e^{2\sigma} \Big[P_{ijkl} - (\lambda_i \, C_{jkl} - \bar{\lambda}_j \, C_{ikl}) + \rho_i \, C_{jkl} - \rho_j \, C_{ikl} - \rho S_{ijkl} \Big] \, .$$

This yields the following

Theorem (3.6). Under h-conformal transformation a P2-like Finsler space $(P_{hijk} \neq o)$ is transformed into a P2-like space, provided that

$$\bar{\lambda}_j = \lambda_j + \rho_j \, .$$

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On a Dual to Ratio Estimator for Estimating Finite Population Variance

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Summary

In this paper we have proposed three new product-type estimators for estimating finite population variance $\sigma_y^2 = \sum_{i=1}^{N} (y_i - \bar{Y})^2/N$ of the study variable y in the situations where population mean $\bar{X}$ or variance σ_x^2 or coefficient of variation C_x $(= \sigma/\bar{X})$ of an auxiliary character x is known. Under large sample approximation we have studied their properties and compared their efficiences with the conventional estimators. Here we have confined ourselves to sampling scheme SRSWOR.

1. Introduction

Let $U = (1,2,\ldots\ldots N)$ be a finite population of N units and y be a real valued function defined on U taking the value y_i for the unit i in U $(1 \leq i \leq N)$. Let $\bar{Y} = \sum_{i=1}^{N} y_i/N, \sigma_y^2 = \sum_{i=1}^{N} (y_i - \bar{Y})^2/N, \sigma_x = \sum_{i=1}^{N} (x_i - \bar{X})^2/N$ denote the population means and variances of the study character y and an auxiliary character x. When the auxiliary information on a variate x is available, Das and Tripathi (1978) utilized this information and suggested a number of estimators for the estimation of σ_y^2.

In this paper we propose the estimators for σ_y^2 in situations where (i) the population mean $\bar{X}$, (ii) the population variance σ_x^2, and (iii) the population coefficient of variation C_x $(= \sigma_x/\bar{X})$, of x is known.

In what follows we shall use the following notations:

$$s_x^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2/(n-1),\ \mu_r(x) = \frac{1}{N}\sum_{i=1}^{N} (x_i - \bar{X})^r,\ r = 3,4,$$

$$\mu_{rs}(y,x) = \frac{1}{N}\sum_{i=1}^{N} (y_i - \bar{Y})^r (x_i - \bar{X})^s,\ r,s = 1,2,\ \beta_2(x) = \frac{\mu_4(x)}{\sigma_x^4}$$

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Similar notations for $\mu_r(y)$ and $\beta_2(y)$ can also be established.

2. Estimators and Their Properties

According to the situation in which mean or variance or coefficient of variation of the auxiliary character x is known, we consider the sampling strategies for σ_y^2 as (i) $S_1 = (E, e_1^*)$, (ii) $S_2 = (E, e_2^*)$ and (iii) $S_3 = (E, e_3^*)$ where E is the SRSWOR and e_1^*, e_2^* and e_3^* are the estimators defined as

$$e_1^* = s_y^2 \cdot (\bar{x}^*/\bar{X}) \tag{1}$$

$$e_2^* = s_y^2 \cdot (s_x^{*2}/\sigma_x^2) \tag{2}$$

$$e_3^* = s_y^2 \cdot (C_x^{*2}/C_x^2) \tag{3}$$

where $\bar{x}^* = (N\bar{X} - n\bar{x})/(N-n)$, $s_x^{*2} = (N\sigma_x^2 - ns_x^2)/(N-n)$ and $C_x^{*2} = s_x^{*2}/\bar{x}^{*2}$.

Assuming $\left|\dfrac{\bar{x} - \bar{X}}{\bar{X}}\right| < 1$ and $\left|\dfrac{s_x^2 - \sigma_x^2}{\sigma_x^2}\right| < 1$,

we obtain the relative bias (RB) and MSE's of the estimators e_1^*, e_2^* and e_3^*, to the terms of order $0(n^{-1})$, as

$$\text{RB}\ (e_1^*) = \left|\frac{K}{N}\right| \tag{4}$$

$$\text{RB}\ (e_2^*) = \left|\frac{(h-1)}{N}\right| \tag{5}$$

$$\text{RB}\ (e_3^*) = \left|\frac{[g(C_x^2 - 2m + \beta_2(x) - \frac{n-3}{n-1}) + (h - 2k - 1)]}{N}\right| \tag{6}$$

$$\text{MSE}\ (e_1^*) = (\sigma_y^4/n)(1-f)\ [g(g\ C_x^2 - 2K) + (\beta_2(y) - \tfrac{n-3}{n-1})] \tag{7}$$

$$\text{MSE}\ (e_2^*) = (\sigma_y^4/n)(1-f)\ [(\beta_2(y) - \tfrac{n-3}{n-1}) + g\{g(\beta_2(x) - \tfrac{n-3}{n-1}) - 2(h-1)\}] \tag{8}$$

$$\mathrm{MSE}\ (e_3^*) = (\sigma_y^4/n)\ (1-f)\ \Big[\beta_2(y) - \frac{n-3}{n-1} + g^2 \Big\{4(C_x^2 - m)$$

$$+ \beta_2(x) - \frac{n-3}{n-1}\Big\} - 2\ g(h-2k-1)\Big] \qquad (9)$$

where $g = n/(N-n)$, $f = n/N$, $h = \mu_{22}\ (y,x)/\ (\sigma_x^2 \cdot \sigma_y^2)$,

$$m = \mu_3(x)\ /\ (\bar{X}\sigma_x^2) \text{ and } K = \mu_{21}(y,x)/(\bar{X}\sigma_y^2)\ .$$

3. Theoretical Comparisons

We estimate the variance σ_y^2 by $e_1 = s_y^2 \cdot (\bar{X}/\bar{x})$, $e_2 = s_y^2 \cdot (\sigma_x^2 / s_x^2)$, and $e_3 = s_{y_2}^2 \cdot \{ C_x^2 / (s_x^2 / x^2)\}$ according as the population mean $\bar{X}$, variance σ_x^2 and coefficient of variation C_x is known, with the MSE's as

$$\mathrm{MSE}\ (s_y^2) = (\sigma_y^4/n)\ (1-f)\ \Big[\beta_2(y) - \frac{n-3}{n-1}\Big] \qquad (10)$$

$$\mathrm{MSE}\ (e_1) = (\sigma_y^4/n)\ (1-f)\ \Big[(\beta_2(y) - \frac{n-3}{n-1}) + (C_x^2 - 2K)\Big] \qquad (11)$$

$$\mathrm{MSE}\ (e_2) = (\sigma_y^4/n)\ (1-f)\ \Big[\beta_2(x) + \beta_2(y) - 2(h-1) - 2\ \frac{n-3}{n-1}\Big] \qquad (12)$$

$$\mathrm{MSE}\ (e_3) = (\sigma_y^4/n)\ (1-f)\ \Big[\beta_2(x) + \beta_2(y) + 4(C_x^2 - m) -$$

$$- 2(h-2K-1) - 2\ \frac{n-3}{n-1}\Big] \qquad (13)$$

When the population mean $\bar{X}$ is known, the proposed estimator e_1^* is more efficient than s_y^2 or e_1 according as

$$\frac{K}{C_x^2} > (g/2), \qquad (14)$$

$$\left.\begin{array}{ll} \text{either} & \dfrac{K}{C_x^2} > (1+g)/2;\ (g > 1) \\ \text{or} & \dfrac{K}{C_x^2} < (1+g)/2;\ (g < 1)\ , \end{array}\right\} \qquad (15)$$

When the population variance σ_{x}^{2} is assumed to be known the proposed estimator e_2^* is more efficient than s_y^2 or e_2 according as

$$\frac{(h-1)}{\beta_2(x) - \frac{n-3}{n-1}} > \frac{g}{2}\,, \tag{16}$$

$$\left.\begin{array}{ll} \text{either} & \dfrac{(h-1)}{\beta_2(x) - \frac{n-3}{n-1}} > \dfrac{(1+g)}{2}\,;\ (g > 1) \\[2ex] \text{or} & \dfrac{(h-1)}{\beta_2(x) - \frac{n-3}{n-1}} < \dfrac{(1+g)}{2}\,;\ (g < 1) \end{array}\right\} \tag{17}$$

Further, when the population coefficient of variation C_x is known the proposed estimator e_3^* is more efficient than s_y^2 or e_3 according as

$$\frac{(h - 2K - 1)}{4(C_x^2 - m) + \beta_2(x) - \frac{n-3}{n-1}} > \frac{g}{2}\,, \tag{18}$$

$$\left.\begin{array}{ll} \text{either} & \dfrac{(h - 2K - 1)}{4(C_x^2 - m) + \beta_2(x) - \frac{n-3}{n-1}} > \dfrac{(1 + g)}{2}\,;\ (g > 1) \\[2ex] \text{or} & \dfrac{(h - 2K - 1)}{4(C_x^2 - m) + \beta_2(x) - \frac{n-3}{n-1}} < \dfrac{(1 + g)}{2}\,;\ (g < 1) \end{array}\right\}. \tag{19}$$

In a similar manner we have proposed three new ratio type estimators for the variance σ_y^2 and derived the conditions when these will be efficient than the conventional estimators under the three stated situations.

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On the Geometric Means of an Entire Dirichlet Series of Order(R) Zero

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1. Consider the Dirichlet series

$$(1.1)\quad f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n} \quad (s = \sigma + it,\ \lambda_1 \geq 0,\ \lambda_n < \lambda_{n+1} \to \infty \text{ with } n),$$

which we assume to be absolutely convergent for all finite S.

The geometric means of $|f(s)|$ for $\mathrm{Re}(s) = \sigma$ are defined as ([1], p. 359)

$$(1.2)\quad G(\sigma) = \exp\left\{ \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \log |f(\sigma + it)|\, dt \right\}$$

and

$$(1.3)\quad g_K(\sigma) = \exp\left\{ \frac{K}{e^{K\sigma}} \int_{o}^{\sigma} e^{Kx} \log G(x)dx \right\}$$

where K is a positive number.

We have defined the logarithmic geometric mean $g_K^*(\sigma)$ of $|f(s)|$ for an entire Dirichlet series of order (R) zero as [2]

$$(1.4)\quad g_K^*(\sigma) = \exp\left\{ \frac{K+1}{\sigma^{K+1}} \int_{o}^{\sigma} x^K \log G(x)dx \right\}$$

where K is a positive number.

Since $\log G(\sigma)$ is a convex function of σ ([1], p. 359) which can be represent as

$$(1.5)\quad \log G(\sigma) = \log G(\sigma_o) + \int_{\sigma_o}^{\sigma} n(x)dx, \quad (\sigma_o \geq 0)$$

where $n(x)$ is a non-decreasing function of x tending to ∞ with x and has only ennumerable discontinuties on the left, we have ([3], p. 71).

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$$(1.6) \quad \lim_{\sigma \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log n(\sigma)}{\sigma} = \begin{matrix} \rho_1 \\ \lambda_1 \end{matrix} \quad (0 \le \lambda_1 \le \rho_1 < \infty) .$$

For a class of entire Dirichlet series of order(R) zero let us define

$$(1.7) \quad \lim_{\sigma \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log n(\sigma)}{\log \sigma} = \begin{matrix} \rho_1^* \\ \lambda_1^* \end{matrix}, \quad (0 \le \lambda_1^* \le \rho_1^* < \infty)$$

Also we have ([4]).

$$(1.8) \qquad N(\sigma) = \int_0^{\sigma} n(x)dx$$

Without loss of generality we may assume that $|f(it)| = 1$, for all t, so that $G(0) = 1$.

In this paper we have obtained some inequalities and growth properties regarding $G(\sigma)$ and $g_K^*(\sigma)$. The results are given in the form of theorems.

2. Theorem 1:- Let the series (1.1) be absolutely convergent for all finite s. If $0 < \sigma_1 < \sigma_2$

$$\left(\frac{\sigma_2^{K+2} - \sigma_1^{K+2}}{K + 2}\right) n(\sigma_1) \le \sigma_2^{K+1} \log \left\{\frac{G(\sigma_2)}{g_K^*(\sigma_2)}\right\} - \sigma_1^{K+1} \log \left\{\frac{G(\sigma_1)}{g_K^*(\sigma_1)}\right\}$$

$$\le \frac{\sigma_2^{K+2} - \sigma_1^{K+2}}{K + 2} \; n(\sigma_2)$$

Proof: Since $n(\sigma)$ is a non-decreasing function of σ, therefore, we have from Lemma 2[2]

$$\sigma_2^{K+1} \log\left\{\frac{G(\sigma_2)}{g_K^*(\sigma_2)}\right\} - \sigma_1^{K+1} \log\left\{\frac{G(\sigma_1)}{g_K^*(\sigma_1)}\right\} = \int_{\sigma_1}^{\sigma_2} x^{K+1} n(x)dx$$

$$\le \left(\frac{\sigma_2^{K+2} - \sigma_1^{K+2}}{K+2}\right) n(\sigma_2)$$

and

$$\sigma_2^{K+1} \log \frac{G(\sigma_2)}{g_K^*(\sigma_2)} - \sigma_1^{K+1} \log \frac{G(\sigma_1)}{g_K^*(\sigma_1)} = \int_{\sigma_1}^{\sigma_2} x^{K+1} \; n(x)dx$$

$$\ge \frac{\sigma_2^{K+2} - \sigma_1^{K+2}}{K + 2)} \cdot n(\sigma_1)$$

Corollary:- If β $(0 < \beta < 1)$ is a constant, then

$$\lim_{\sigma \to \infty} \frac{\left\{ \dfrac{G(\beta\sigma)}{g_K^*(\beta\sigma)} \right\}^{\beta^{K+1}}}{\left\{ \dfrac{G(\sigma)}{g_K^*(\sigma)} \right\}} = 0$$

Putting $\sigma_1 = \beta\sigma$ and $\sigma_2 = \sigma$ in (2.1) and taking the limits the result follows.

3. Let us set, for $0 < \rho_1^* < \infty$

$$(3.1) \quad \lim_{\sigma \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \left\{ \dfrac{G(\sigma)}{g_K^*(\sigma)} \right\}^{1/\sigma}}{\sigma^{\rho_1^*} \phi(\sigma)} = \begin{matrix} a \\ b \end{matrix} \quad (0 < b \leq a < \infty);$$

$$(3.2) \quad \lim_{\sigma \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{n(\sigma)}{\sigma^{\rho_1^*} \phi(\sigma)} = \begin{matrix} c \\ d \end{matrix} \quad (0 < d \leq c < \infty),$$

where $\phi(\sigma)$ satisfies the following two conditions:

(i) $\phi(\sigma) > 0$ is continuous for $x > \sigma_0$.

(ii) $\phi(L\sigma) \sim \phi(\sigma)$ as $x \to \infty$, for every constant $L > 0$.

Theorem 2. Let the series (1.1) be absolutely convergent for all finite s and be of order (R) zero. Then

$$(3.3) \quad \frac{d}{\rho_1^* + K+2} \leq b \leq a \leq \frac{c}{\rho_1^* + K+2}$$

$$(3.4) \quad a \geq \frac{c^{1 + \frac{\rho_1^*}{(K+2)}} \, \rho_1^{*\,\frac{\rho_1^*}{(K+2)}}}{(\rho_1^* + K+2) \left\{ c(\rho_1^* + K+2) - d(K+2) \right\}^{\rho_1^*/(K+2)}}$$

and

$$b \leq \frac{d}{(K+2)} \left\{ 1 - \frac{\rho_1^*}{(\rho_1^* + K+2)} \left(\frac{d}{c}\right)^{\left(\frac{K+2}{\rho_1^*}\right)} \right\}.$$

Proof:- We have from Lemma 2 [2]

$$(3.6) \quad \log\left\{\frac{G(\sigma)}{g_K^*(\sigma)}\right\}^{1/\sigma} = \frac{1}{\sigma^{K+2}} \int_0^{\sigma} x^{K+1} n(x)\,dx .$$

For $h > 1$ and for any $\varepsilon > 0$ and $\sigma \geqslant \sigma_0$,

$$\log\left\{\frac{G(\sigma h^{1/\rho_1^*})}{g_K^*(\sigma h^{1/\rho_1^*})}\right\}^{1/\sigma h^{1/\rho_1^*}} = \frac{1}{(\sigma h^{1/\rho_1^*})^{K+2}} \int_0^{\sigma h^{1/\rho_1^*}} x^{K+1} n(x)\,dx$$

$$= \frac{1}{(\sigma h^{1/\rho_1^*})^{K+2}} \left\{ \int_0^{\sigma_0} + \int_{\sigma_0}^{\sigma} + \int_{\sigma}^{\sigma h^{1/\rho_1^*}} x^{K+1} n(x)\,dx \right\}$$

$$> \frac{A}{\sigma^{K+2}} + \frac{(d-\varepsilon)}{(\sigma h^{1/\rho_1^*})^{K+1}} \int_{\sigma_0}^{\sigma} x^{\rho_1^*+K+2}\, \phi(x)\,dx +$$

$$+ \frac{n(\sigma)}{(\sigma h^{1/\rho_1^*})^{K+2}} \int_{\sigma}^{\sigma h^{1/\rho_1^*}} x^{K+1}\,dx .$$

Now, by Lemma 5([5], p. 54)

$$\int_{\sigma_0}^{\sigma} u^{\beta-1} \phi(u)\,du \approx \frac{\sigma^{\beta}\phi(\sigma)}{\beta}$$

for every positive β , so we get

$$\log\left\{\frac{G(\sigma h^{1/\rho_1^*})}{g_K^*(\sigma h^{1/\rho_1^*})}\right\}^{1/\sigma h^{1/\rho_1^*}} > \frac{A}{\sigma^{K+2}} + \frac{(d-\varepsilon)\phi(\sigma)\,\sigma^{\rho_1^*+K+2}}{(\sigma h^{1/\rho_1^*})^{K+2}(\rho_1^*+K+2)} +$$

$$+ \frac{n(\sigma)}{(K+2)} \; \frac{(\sigma h^{1/\rho_1^*})^{K+2} - \sigma^{K+2}}{(\sigma h^{1/\rho_1^*})^{K+2}}$$

Therefore,

$$\frac{\log\left\{\frac{G(\sigma h^{1/\rho_1^*})}{g_K^*(\sigma h^{1/\rho_1^*})}\right\}^{1/\sigma h^{1/\rho_1^*}}}{(\sigma h^{1/\rho_1^*})^{\rho_1^*}\,\phi(\sigma h^{1/\rho_1^*})} > \frac{1}{h}\left[\left[\frac{A}{\sigma^{\rho_1^*+K+2}} + \frac{(d-\varepsilon)}{h^{(K+2)/\rho_1^*}(\rho_1^*+K+2)} + \right.\right.$$

$$+ \frac{n(\sigma)}{\sigma^{\rho_1^*}\phi(\sigma)} \left\{ \frac{h^{(K+2)/\rho_1^*} - 1}{(K+2)\, h^{(K+2)/\rho_1^*}} \right\} \Bigg]$$

Hence, taking limits and using (3.1) and (3.2), we get

$$(3.7) \quad a \geqslant \frac{1}{h} \left\{ \frac{d}{(\rho_1^*+K+2)\, h^{(K+2)/\rho_1^*}} + \frac{c}{(K+2)} \left(\frac{h^{(K+2)/\rho_1^*} - 1}{h^{(K+2)/\rho_1^*}} \right) \right\}$$

and

$$(3.8) \quad b \geqslant \frac{1}{h} \left\{ \frac{d}{(\rho_1^*+K+2)\, h^{(K+2)/\rho_1^*}} + \frac{d}{(K+2)} \left(\frac{h^{(K+2)/\rho_1^*} - 1}{h^{(K+2)/\rho_1^*}} \right) \right\}.$$

It can be seen that the maxima of the right hand side expressions in (3.7) and (3.8) occur at

$$(3.9) \quad h = \left\{ \frac{c(\rho_1^*+K+2) - d(K+2)}{\rho_1^* c} \right\}^{\rho_1^*/(K+2)},$$

and

$$(3.10) \quad h = 1$$

respectively. Substituting these values of h from (3.9) and (3.10) in (3.7) and (3.8) respectively, we get (3.4) and the left-hand inequality of (3.3).

Again,

$$\log \left\{ \frac{G(\sigma h^{1/\rho_1^*})}{g_K^*(\sigma h^{1/\rho_1^*})} \right\}^{1/\sigma h^{1/\rho_1^*}} < \frac{A}{\sigma^{K+2}}$$

$$+ \frac{(c+\varepsilon)}{(h^{1/\rho_1^*})^{K+2}} \int_{\sigma_0}^{\sigma} x^{\rho_1^*+K+1} \phi(x)\, dx$$

$$+ \frac{n(h^{1/\rho_1^*})}{(h^{1/\rho_1^*})^{K+2}} \int_{\sigma}^{\sigma h^{1/\rho_1^*}} x^{K+1}\, dx$$

$$\simeq \frac{A}{\sigma^{K+2}} + \frac{(c+\varepsilon)\,\phi(\sigma)}{h^{(K+2)/\rho_1^*}} \frac{\sigma^{\rho_1^*}}{(\rho_1^*+K+2)} + \frac{n(\sigma h^{1/\rho_1^*})}{(K+2)\, h^{(K+2)/\rho_1^*}} \left\{ h^{(K+2)/\rho_1^*} - 1 \right\}$$

and so we have

$$(3.11)\quad a \le \frac{1}{h}\left\{\frac{c}{(\rho_1^*+K+2)h^{(K+2)/\rho_1^*}} + \frac{c}{(K+2)}\,\frac{h\left(h^{\frac{(K+2)}{\rho_1^*}}-1\right)}{h^{(K+2)/\rho_1^*}}\right\},$$

and

$$(3.12)\quad b \le \frac{1}{h}\left\{\frac{c}{(\rho_1^*+K+2)h^{(K+2)/\rho_1^*}} + \frac{d}{(K+2)}\,\frac{h\left(h^{(K+2)/\rho_1^*}-1\right)}{h^{(K+2)/\rho_1^*}}\right\}.$$

It can also be seen that the minima of the right hand side expressions in (3.11) and (3.12) occur at

$$(3.13)\qquad h = 1$$

and

$$(3.14)\qquad h = \frac{c}{d},$$

respectively. Substituting these values of h from (3.13) and (3.14) in (3.11) and (3.12) respectively, we get the right-hand inequality of (3.3) and (3.5). This completes the proof.

Theorem 3. If $\log\left\{\frac{G(\sigma)}{g_K^*(\sigma)}\right\}^{1/\sigma} \sim a\,\sigma^{\rho_1^*}\phi(\sigma)$, then

$$n(\sigma) \sim (\rho_1^*+K+2)\,a\,\sigma^{\rho_1^*}\,\phi(\sigma)$$

and conversely.

Proof: From (3.3), if c = d, then $a = b = \frac{c}{(\rho_1^*+K+2)}$. Suppose now a = b, we shall show that c = d. If $0 < \eta < 1$, we have from (3.6)

$$\eta\,\sigma^{K+2}\,\eta(\sigma) < \int_\sigma^{(1+\eta)\sigma} x^{K+1}\,n(x)\,dx$$

$$= \int_0^{(1+\eta)\sigma} x^{K+1}\,n(x)\,dx - \int_0^{\sigma} x^{K+1}\,n(x)\,dx$$

$$= \{(1+\eta)\sigma\}^{K+2}\log\left[\frac{G\{\sigma(1+\eta)\}}{g_K^*\{\sigma(1+\eta)\}}\right]^{1/\sigma(1+\eta)} - \sigma^{K+2}\log\left\{\frac{G(\sigma)}{g_K^*(\sigma)}\right\}^{1/\sigma}.$$

Therefore,

$$\eta\, n(\sigma) < \frac{a\,\sigma^{\rho_1^*}(1+\eta)^{\rho_1^*+K+2}\,\phi\{(1+\eta)\sigma\}}{} - a\sigma^{\rho_1^*}\,\phi(\sigma) + O(\sigma^{\rho_1^*}\phi(\sigma))$$

Hence,

$$\limsup_{\sigma\to\infty} \frac{n(\sigma)}{\sigma^{\rho_1^*}\phi(\sigma)} \leq a\left\{(\rho_1^*+K+2) + H\eta\right\}$$

where H is constant. Since η is arbitrary, we get

$$\limsup_{\sigma\to\infty}\left\{\frac{n(\sigma)}{\sigma^{\rho_1^*}\phi(\sigma)}\right\} \leq (\rho_1^*+K+2)\ a\ .$$

Considering

$$\sigma^{K+2}\log\left\{\frac{G(\sigma)}{g_K^*(\sigma)}\right\}^{1/\sigma} - \left\{\sigma(1-n)\right\}^{K+2}\log\left\{\frac{G\left\{\sigma(1+\eta)\right\}}{g_K^*\left\{\sigma(1+\eta)\right\}}\right\}^{1/\sigma(1+\eta)}$$

and proceeding as above, we get

$$\liminf_{\sigma\to\infty}\frac{n(\sigma)}{\sigma^{\rho_1^*}\phi(\sigma)} \geq (\rho_1^*+K+2)a$$

and hence

$$n(\sigma) \sim (\rho_1^*+K+2)a\ \sigma^{\rho_1^*}\phi(\sigma)\ .$$

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