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G-Quasi-Interior Functions

Paul Long and Travis Thompson

Introduction. In this paper, a class of functions called G-quasi-interior functions are studied. These functions are investigated with a characterization given in terms of filterbases.

Definition 1. A function $f:X \to Y$ is said to be almost-continuous Singal-Singal) if for every x in X and each open set V in Y containing f(x), there exists an open set $U \subset X$ containing x such that $f(U) \subset \text{Int}(\text{Cl}(V))$.

Definition 2. A function $f:X \to Y$ is said to be weakly continuous if for each point x in X and each open set $V \subset Y$ containing f(x), there exists an open set $U \subset X$ containing x such that $f(U) \subset \text{Cl}(V)$.

Definition 3. A function $f:X \to Y$ is G-quasi-interior at a point $y \in Y$ provided, for each open $U \subset X$ such that a component of $f^{-1}(y)$ is contained in U, we have y in Int(f(U)). The function f is G-quasi-interior if it is G-quasi-interior at each y in Y. If a G-quasi-interior mapping is continuous, it is simply said to be a quasi-interior mapping.

Theorem 4. Let $f:X \to Y$ be a surjective almost-continuous function where X is compact and Y is Hausdorff. Let $y \in Y$ and let $U \subset X$ be open where $f^{-1}(y) \subset U$. Then $y \in \text{Int}(f(U))$.

Proof. Assume $y \notin \text{Int}(f(U))$. Then there exists a filterbase $F = \{B_a | a \in A\}$ on Y-f(U) converging to y and to no other point of Y. Then the filterbase $f^{-1}(F)$ lies in the compact X-U and hence accumulates to some $x_o \in X-U$. Thus, $f(x_o) \neq y$ so there exist open disjoint sets V_1 and V_2 in Y containing $f(x_o)$ and y, respectively. It follows that $\text{Int}(\text{cl}(V_1)) \cap V_2 = \emptyset$. Since F is almost-continuous, there exists an open W containing x_o such that $f(W) \subset \text{Int}(\text{Cl}(V_1))$. However, there exists some $B_b \in \{B_a\}$ such that $B_b \subset V_2$ and since $f^{-1}(B_b) \cap W \neq \emptyset$ for all $a \in A$, we conclude $f(W) \cap V_2 \neq \emptyset$. This contradicts $f(W) \subset \text{Int}(\text{Cl}(V_1))$ since $\text{Int}(\text{Cl}(V_1)) \cap V_2 = \emptyset$. Hence, our assumption $y \notin \text{Int}(f(U))$ is false and our theorem is proved.

The proof of Theorem 4 would hold if almost continuous were replaced with weakly continuous (with slight modification).

Theorem 5. Let $f:X \longrightarrow Y$ be a given surjection.

i) If f is open, then f is G-quasi-interior;

ii) If f is almost-continuous (weakly continuous) and point inverses are connected, then f is G-quasi-interior;

iii) If f is closed and point inverses are connected, then f is G-quasi-interior.

Proof. Part i) is obvious and ii) follows from Theorem 4. Part iii) follows from Theorem 11.2, page 86 of Topology by J. Dugundji.

If (Y, T) is a topological space, then let (Y, T_*) denote Y with the induced semi-regular topology. Let $f:X \longrightarrow Y$ be an almost-continuous surjection. Then if $f:X \to (Y, T_*)$ is G-quasi-interior, $f:X \longrightarrow (Y,T)$ is G-quasi-interior. However, if $f:X \longrightarrow (Y, T)$ is quasi-interior, $f:X \longrightarrow (Y, T_*)$ need not be G-quasi-interior as illustrated by Example 6.

Example 6. Let $I = [0, 1]$ have the topology T generated by the usual open sets together with $A = \{x \mid x \text{ rational and } 1/3 < x < 2/3\}$. Then $i: (I, T) \longrightarrow (I, T)$ is G-quasi-interior. However, $i:(I, T) \longrightarrow (I, T_*)$ (T_* is the usual topology) is not G-quasi-interior. Consider the point $x = 1/2$ in (I, T_*). Then $i^{-1}(X) = 1/2$ and A is an open set containing $i^{-1}(1/2) = 1/2$, but $1/2 \notin \text{Int}(f(A)) = \emptyset$.

Definition 7. Let A be a subset of a topological space X and $F = \{A_a \mid a \in A\}$ a filterbase on X. Then F accumulates to A if for each open U containing A, $U \cap A_a \neq \emptyset$ for all $a \in A$.

Theorem 8. Let $f:X \longrightarrow Y$ be any surjective function. Then f is G-quasi-interior at $y \in Y$ if and only if for each filterbase $F \longrightarrow y$, $f^{-1}(F)$ accumulates to each component of $f^{-1}(y)$.

Proof. Let f be a G-quasi-interior function at $y \in Y$ and let C be a component of $f^{-1}(y)$. Then for each open U containing C, $y \in \text{Int}(f(U))$. Since $F \longrightarrow y$, F also accumulates to y so that $A_a \cap \text{Int}(f(U)) \neq \emptyset$ for every $a \in A$. Consequently, $f^{-1}(F)$ accumulates to C.

Conversely, assume the condition is satisfied and U is an open set containing a component C of $f^{-1}(y)$. Assume $y \notin \mathrm{Int}(f(U))$. Then there exists a filterbase $F = \{A_a \mid a \in A\}$ on $Y - f(U)$ such that $F \longrightarrow y$. Therefore, $f^{-1}(F) \subset X{-}U$ and consequently cannot accumulate to C. This contradiction implies $y \in \mathrm{Int}(f(U))$.

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Barodiffusion in a Binary Mixture Near a Stagnation Point

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Abstract

The diffusion effect in a binary mixture of incompressible viscous fluids due to pressure gradient has been discussed when one of the components of the fluid is present in a small quantity. The flow has been discussed when a stream of such a mixture impinges on a wall perpendicular to it and flows away radially in all directions. It has been found that there is no separative effect when pressure gradient is ignored. The effect of the pressure gradient is to separate the two components of the mixture in a manner so that the heavier and more abundant component is deposited near the wall.

Introduction

We have discussed here a mixture of two components of fluids, the composition of which is described by the concentration C_1 defined as the ratio of mass of one component to the total mass of the fluid in a given volume element. In such a flow the diffusion of the individual species takes place by three mechanism namely concentration gradient, temperature gradient and pressure gradient. The diffusion flux density $\vec{i}$ is given by Landau and Lifshitz[1] as:

$$\vec{i} = - \rho D \,[\text{grade } C_1 + (K_T) \text{ grade } T + (K_p) \text{ grade } p], \tag{1}$$

where D is the diffusion coefficient or mass transfer coefficient, $K_T D$ is the thermal diffusion coefficient and $K_p D$ is the barodiffusion coefficient. In this paper we have considered the mixture to be isothermal so that the diffusion due to the temperature gradient has been neglected.

Sarma[3] has discussed the problem of barodiffusion in a binary mixture of viscous incompressible fluids when an infinite disk rotates with a constant angular velocity and there is a suction of mixture at the disk.

Srivastava[5] has discussed the barodiffusion in a binary mixture confined between two disks when one of the disks is rotating and the other is at rest. In this paper we discussed the barodiffusion in a binary mixture a stream of which impinges on a wall perpendicular to it and flows away radially in all directions. It has been shown that the equations admit an exact solution. Equations have been integrated by Karman-Pohlhausen method by taking Schmidt number as 1.0.

Mass Transfer Equations

We consider here the case when one of the components is present in a small quantity, hence the density and the viscosity of the mixture is independent of the distribution of the components. The flow problem of the binary mixture is identical to that of a single fluid but the velocity is understood as the mass average velocity $\vec{V} = (\rho_1\vec{V_1} + \rho_2\vec{V_2})/\rho$ and the density $\rho = \rho_1 + \rho_2$ where the subscripts 1 and 2 respectively denote the rarer and more abundant component in the binary mixture. The equations of motion and continuity for a steady motion are:

$$\rho(\vec{V}\cdot\nabla)\vec{V} = -\nabla p + \mu\nabla^2\vec{V}, \qquad (2)$$

$$\nabla\cdot\vec{V} = 0, \qquad (3)$$

where μ is the coefficient of viscosity.

The additional equation for the species conservation is given by

$$\rho(\vec{V}\cdot\nabla)\, C_1 = -\nabla\cdot\vec{i}\,. \qquad (4)$$

Substituting $\vec{i}$ from the equation (1), we get

$$\rho(\vec{V}\cdot\nabla)\, C_1 = \rho D\,[\nabla^2 C_1 + \nabla\cdot(K_p\nabla p)]. \qquad (5)$$

The explicit expression for the barodiffusion coefficient K_p has been given by Landau and Lifshitz[1] as:

$$K_p = (m_2 - m_1)\left(\frac{C_1}{m_1} + \frac{C_2}{m_2}\right)\frac{C_1C_2}{p_\infty} \qquad (6)$$

where p_∞ denotes the pressure in the working medium, and m_1 and m_2 are masses of two kinds of particles, we have the following relation between C_1 and C_2

$$C_1 + C_2 = 1 \tag{7}$$

since $C_1 = \rho_1/\rho$, $C_2 = \rho_2/\rho$. In general we take K_p as proportional to the product C_1C_2. Assuming that the square of C_1 is negligible, we get

$$K_p = \frac{(m_2 - m_1)C_1}{m_2\ p_\infty} \tag{8}$$

Boundary Conditions

The boundary conditions on (5) are different in different cases. At the solid surface of the body insolvable in the fluid the mass flux of this rarer component of the mixture normal to the surface is zero. This can be written as

$$\rho V_n C_1 - \rho\ D \left[\frac{\partial C_1}{\partial n} + K_p \frac{\partial p}{\partial n}\right] = 0 \text{ at the surface,} \tag{9}$$

where V_n is the fluid velocity normal to the surface and $(\partial/\partial n)$ denotes the derivative in the direction normal to the surface. The first part represents the convective flux and the second part in the parenthesis denotes the diffusion flux. If, however, there is diffusion from a body which dissolves in the fluid, the equilibrium is rapidly established near its surface, and the concentration in the fluid adjoining the body is the saturation constant C_o. The boundary condition at such surface is, therefore,

$$C_1 = C_o. \tag{10}$$

Flow Near a Stagnation Point

In this section we discuss the flow and diffusion of a stream of a binary mixture of incompressible viscous fluids when a stream of such a mixture impinges on a wall $z = o$ and flows away radially in all directions. We take here cylindrical polar coordinates with stagnation point

as the origin and the flow direction as negative z-axis. We denote the radial and axial components of the velocity in the frictionless flow by U and W respectively where as those in viscous flow are denoted by $u = u\ (r,\ z)$ and $w = w\ (r,\ z)$. The boundary conditions on velocity field are $u = o$, $w = o$ at $z = o$ and $u \longrightarrow U$ at $z \longrightarrow \infty$. For frictionless case we have [See Schlichiting [4]]

$$U = ar,\ W = -2az, \tag{11}$$

where a is a constant. We take the following form for u, w and p in the boundary layer region:

$$u = (a\nu)^{1/2}\ x\ \phi'(y),\ w = -2(a\nu)^{1/2}\ \phi'(y), \tag{12}$$

$$p - p_o = \rho a \nu\ P(x,\ y), \tag{13}$$

where $Y = (a/\nu)^{1/2}\ z,\ x = (a/\nu)^{1/2}\ r,\ \nu = \mu/\rho$

and p_o is the pressure at stagnation point.

Substituting expressions for u and w from (12), p from (13) and K_p from (8), the diffusion equation (5) becomes

$$S(x\ \phi'\ \frac{\partial C_1}{\partial x}) - 2S\ (\phi\frac{\partial C_1}{\partial y}) = \frac{\partial^2 C_1}{\partial y2}$$

$$+\beta\left(\frac{\partial}{x\,\partial x}\ (x\ C_1\frac{\partial P}{\partial x}) + \frac{\partial}{\partial y}\ (C_1\frac{\partial P}{\partial y})\right), \tag{14}$$

where $S = \nu/D$ the Schmidt number and β is a barodiffusion number given by

$$\beta = \frac{m_2 - m_1}{m_2\ P_\infty} \cdot \frac{\rho a^2}{2}.$$

The expression for $P(x,\ y)$ is given by Srivastava[5] as

$$P(x,\ y) = -\frac{1}{2}\left\{x^2 + 4\ (\phi^2 - \phi' - 1)\right\}. \tag{15}$$

The boundary conditions (9) and (10) give

$$\frac{\partial C_1}{\partial y} + \beta C_1 \frac{\partial P}{\partial y} = 0 \text{ at } y = 0 \tag{16}$$

and $C_1 \longrightarrow C_o$ as $y \longrightarrow \infty$. (17)

<u>Solution of Equation</u>

We assume that the barodiffusion number β is small and write

$$C_1 = C_o \left\{ f(x, y) + \beta g(x, y) \right\} . \tag{18}$$

Substituting (18) in (14) and equating terms independent of β and coefficient of β on both sides of the equation, we get

$$S\left(x\, \phi' \frac{\partial f}{\partial x} - 2\, \phi \frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}, \tag{19}$$

$$S\left(x\, \phi' \frac{\partial g}{\partial x} - 2\, \phi \frac{\partial g}{\partial y}\right) = \frac{\partial^2 g}{\partial y^2} + \left[\frac{\partial}{x\partial x}\left(x\, f \frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial y}\left(f \frac{\partial P}{\partial y}\right)\right]. \tag{20}$$

Substituting (18), boundary conditions (16) and (17) yield following conditions of f and g :

$$\frac{\partial f}{\partial y} = 0 \quad \text{at } y = 0,$$

$$f \longrightarrow 1 \quad \text{as } y \longrightarrow \infty, \tag{21}$$

$$\frac{\partial g}{\partial y} + f \frac{\partial P}{\partial y} = 0 \quad \text{at } y = 0,$$

$$g \rightarrow 0 \quad \text{as } y \longrightarrow \infty. \tag{22}$$

We further assume that

$$f(x, y) = f_1(y) + x^2 f_2(y)$$

Substituting this in (19) and equating terms independent of x and co-efficient of x^2, we get following equations:

$$- 2S \phi f_1'' = f_1'', \tag{23}$$

$$2S (\phi' f_2 - \phi f_2') \quad f_1'' f_2''. \tag{24}$$

Boundary conditions on f_1 and f_2 are given by (21) as

$$\begin{aligned} & f_1' = 0, \ f_2' = 0 \quad \text{at } y = 0 , \\ & f_1 \rightarrow 1, \ f_2 \rightarrow 0 \text{ as } y \rightarrow \infty. \end{aligned} \tag{25}$$

The solution of equation (24) is given as

$$f_1 = 1, \ f_2 = 0 , \tag{26}$$

which gives $f = 1$.

Again we write

$$g (x, y) = g_1 (y) + x^2 g_2 (y) .$$

Substituting this form of $g(x, y)$, $f = 1$ and expression for $P(x, y)$ for (15) in (20) and equating terms independent of x and coefficient of x^2, we get the following equations for g_1 and g_2 :

$$g_1'' + 2 S g_1' = 2 + 4 \phi \phi'' + 4 \phi'^2 - 2 \phi''' , \tag{27}$$

$$2S (\phi g_2 - \phi g_2') = g_2''. \tag{28}$$

The boundary conditions on g_1 and g_2 are given by equation (22) after substituting P from (15) and $f = 1$ as

$$g_1\ (0) = -2\ \emptyset''\ (0),\ g_2'\ (0) = 0\ ,$$

$$g_1 \to o,\ g_2 \to o \text{ as } y \to \infty\ . \tag{29}$$

The solution of equations (27), (28) and (29) is

$$g_1 = A\int e^{-2S\int \emptyset\, dy}\, dy + 2\int e^{-2S\int \emptyset\, dy}\left[\int (1 + 2\ \emptyset'^2 + 2\ \emptyset\ \emptyset'' - \emptyset''')\ e^{2S\int \emptyset\, dy}\, dy\right] dy + B\ , \tag{30}$$

$$g_2 = 0\ , \tag{31}$$

where A and B are constants.

It is not possible to get analytical expression for $\emptyset$, hence we solve (27) by Karman-Pohlhausen method. We assume that the diffusion boundary layer is of the same order as the viscous one i.e., S = 1. The expression for $\emptyset$ is given by Ratna and Rajeshwari[2]

$$\emptyset(y) = D^*(1{\cdot}656\ y^2 - 1{\cdot}0832\ y^3 - 0{\cdot}0314 + 0{\cdot}3501\ y^5 - 0{\cdot}1146\ y^6), \tag{32}$$

where D* = 2.5494 is the non-dimensional boundary layer thickness. Assuming fifth order polynomial for $g_1(y)$ and calculating the constants by Karman-Pohlhansen method, we get

$$g_1(y) = D^*\ (-2{\cdot}1327 + 1{\cdot}0195\ y + 0{\cdot}5000\ y^2 - 0{\cdot}0095\ y^3 + 1{\cdot}4936\ y^4 - 0{\cdot}8710\ y^5). \tag{33}$$

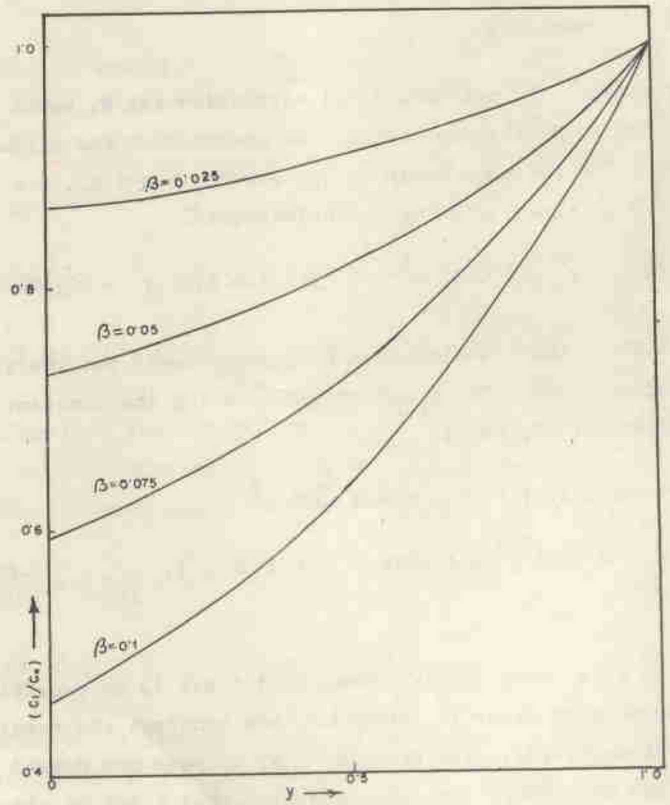
Discussion

The function f (x, y) = 1 which shows that there is no separative effect when pressure gradient is ignored. This confirms the results of Sarma[3] and Srivastava[5]. The function g(x, y) does not depend on x and is a function of y only. Calculations give that $g_1(y)$ is always negative within the boundary layer region. The function C_1/C_o is given

by $(C_1/C_o) = 1 + \beta(-5{\cdot}4371 + 2{\cdot}5991\ y + 1{\cdot}2747\ y^2 -$

$$- 0{\cdot}0242\ y^3 + 3{\cdot}8078\ y^4 - 2{\cdot}2205\ y^5). \tag{34}$$

The function (C_1/C_o) has been plotted for $\beta = 0.025, 0.050, 0.075$ and 0.100 in Fig. 1. The graph reveals that the value of (C_1/C_o) is always less than 1 near the boundary layer region and its values at the wall are 0.8641, 0.7282, 0.5922 and 0.4562 respectively for the above mentioned values of β. This means that the concentration of the rarer component is much less near the wall than that maintained at a large distance from it. But $C_1 + C_2 = 1$ which shows that the heavier component gets deposited more near the wall.



The Graph of (C_1/C_o) against y

Figure 1

Conclusion

When a stream of binary mixture, in which the lighter component is present in a small quantity, impinges on a wall perpendicular to it separation takes place near the wall due to pressure gradient. The ratio of the lighter component is reduced near the wall, as a result of which the heavier component gets more concentrated near the wall.

References

[1] Landau, L.D. and Lifshitz, E.M. (1959), 'Fluid Mechanics', Pergamon Press, London, English Ed. Page 217.

[2] Ratna, S.L. and Rajeshwari, "The Flow of a Second-order Fluid near a stagnation point", Z. Angew. Math. Phys., 14, (1962) page 43.

[3] Sarma, G.S.R., "Barodiffusion in a binary mixture at a rotating disk", Z. Angew. Math. Phys., 24, (1973) page 789-800.

[4] Schlichting, H., "Boundary Layer Theory" (Translated by J. Kestin), Mc. Graw-Hill Book Company, Inc. New York (1962), Page 78.

[5] Srivastava, A.C., "Barodiffusion in a binary mixture confined between two disks", Math. Forum, Vol. 2, No. 2, (1979), page 16-21.

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Stochastic Production Inventory Model with no randomness in the Evolution of the Inventory

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Introduction

A stochastic version of the production planning and inventory model with an inventory of items was introduced by Sethi and Thompson (1980). They developed a model to determine the production levels over time to minimise a discounted quadratic loss function, which consisted of the cost of control and the cost of terminal deviation. Shrestha (1982) extended their model by introducing a deterioration factor in that the items deteriorate in the inventory after some time from their arrival in the inventory system. The deterioration was assumed to take place according to some instantaneous decay rate which could be constant or follow some known decay distribution e.g., Wiebull distribution. In both of these works, the change in the inventory was assumed to be given by Ito's differential stock flow equation consisting of a deterministic component and a stochastic component. The stochastic component was assumed to take care of the randomness arising in the production and the inventory levels, and it was considered to be a White noise process with zero mean and unit variance.

For many production inventory models, the randomness arising in the model may be solely due to the randomness arising in the production mechanism only. For instance, if the demand of the items are constant over time the change in the inventory level will still be described by the Ito's differential stock flow equation, but since the randomness in the evolution of the inventory is due to the randomness of the production levels, the variance of the White noise process may be made to depend on the control variable (production level) and as such the variance of the White noise process may be described as a function of the control variable. This is what we have done in this paper in evolving a stochastic production inventory model with no randomness in the evolution of the

inventory. Such an assumption reflects reality in many applications.

In what follows, we present the model and the assumptions under which it is concieved. The notations and the general assumptions needed for developing the model are very much the same as in Shrestha (1982). We develop an undiscounted model with no deteriorating items and obtain the optimal production rates for the cases of both finite and infinite planning horizon. Further we develop an undiscounted model with deteriorating items on the lines of study of Shrestha (1982). Finally we make some general observations and conclude the paper.

The model

Consider a production inventory mechanism involving the production and stocking of a homogeneous good. We define the following quantities:

$x(t)$ = inventory level at time t (state variable)
$u(t)$ = production level at time t (control variable)
S = constant demand rate at time t; $S > 0$
T = length of the planning horizon
u_1 = factory optimal production level
x_1 = factory optimal inventory level
x_0 = initial inventory level
h = inventory holding cost coefficient and
B = salvage value per terminal inventory.

Let the stock flow equation be given by Ito's differential equation

$$dx = (u - s)dt + d\,\xi(t)$$

where $d\,\xi(t)$ is a stochastic process with, for all t,

$$E\,(d\,\xi(t)) = 0$$

$$E\,(d\,\xi(t))^2 = \delta^2\,u^2 dt$$

$$E\,(d\,\xi(t))^n = o\,(\triangle t),\ n > 2$$

and δ^2 is a constant. Since the cost of control is now reflected in the uncertainty attendant upon the use of the control, the criterion function may be simply written as

$$E\left\{\int_0^T h(x - x_1)^2 dt - Bx(T)\right\}$$

and we aim to minimize this function. Here the first term explains the expected cost of control and the other term explains the expected loss due to terminal inventory. The quadratic cost function has been assumed in order to accommodate the possibly negative inventory i.e., backlogging of demand.

We wish to choose the production rate u such that the objective function is minimum. To that extent, we define the value function from time t to the planning horizon T as

$$V(t, x) = \max_u E\left\{-\int_t^T h(x - x_1)^2 dt + Bx(T)\right\}$$

with boundary condition $V(T, x) = Bx(T)$ and the control variable reflected in the stock flow equation.

Following Sethi and Thompson (1980) and using the principle of optimality, we get the Hamilton-Jacobi-Bellman equation as

$$V(t, x) = \max_u E\left\{-h(x - x_1)^2 dt + V(t + dt, x + dx)\right\}.$$

Expanding $V(t + dt, x + dx)$ by using Taylor's series and taking the expectations gives with $x_1 = 0$, for simplicity,

$$0 = \max_u \left\{-hx^2 dt + (V_t + (u - s)V_x) dt + \tfrac{1}{2}u^2\sigma^2 V_{xx} dt\right\}$$

where V_t, V_x, V_{xx} are the first order partial derivative of V with respect to t and x and the second order partial derivative of x respectively. We presume that all these derivatives exist. Dividing by dt we get

$$0 = \max_u \left\{-hx^2 + V_t + (u - s)V_x + \tfrac{1}{2}u^2\sigma^2 V_{xx}\right\}.$$

To maximize the expression within the bracket with respect to u, we take its derivative with respect to u and equate it to zero. This results in the optimal production rate as

$$u = -V_x / \sigma^2 V_{xx},$$

Substituting this into the Hamilton-Jacobi-Bellman equation we get the second order partial differential equation

$$0 = -hx^2 + V_t - SV_x - \tfrac{1}{2} \frac{V_x^2}{\sigma^2 V_{xx}},$$

which we need to solve to obtain the optimal production rate.

Assume that $V(t, x)$ has the seperable form

$$V(t, x) = Q(t)x^2 + R(t)x + M(t)$$

where $M(t)$ reflects the cost of randomness. We may now obtain V_t, V_x and V_{xx} and proceed as in Shrestha (1982) to derive

$$Q = h\sigma^2 \left\{ \exp\left((t - T)/\sigma^2\right) - 1 \right\} = h\sigma^2 (y - 1)$$

where $y = \exp\left((t - T)/\sigma^2\right)$ and,

$$R = 2Sh\sigma^4 + (B - 2Sh\sigma^2 (T - t) - 2Sh\sigma^4)y.$$

The optimal rate is

$$u^* = S - x/\sigma^2 + \left\{ \frac{2h\sigma^2 (T - t) S - B}{2h\sigma^4} \right\} \frac{y}{y - 1}$$

Thus the optimal rate equals the demand rate corrected by terms depending on the level of inventory, variance of the production mechanism and the distance from the planning horizon.

Remarks

1. Since $\sigma^2 > 0$, greater the variance of the production rates, lesser should be the optimal production rate. This is because greater variance in the production rates imply greater chances of the demands not being fulfilled or items pilling up in the inventory.

2. If σ^2 is infinitely large but the planning horizon is finite, then $u^* \longrightarrow S$ and the optimal policy would be to produce just as much as the current demand and have no inventory at all.

3. If δ^2 is very low as compared to the inventory level there is a chance that the production level will be negative. In such a case, disposal of items from the inventory might be advocated.

4. If the planning horizon is too large then $y \to 0$ exponentially faster than $T \to \infty$ and $u^*_\infty = S - x/\delta^2$. In such a case the production rate should be just x/δ^2 more than the demand rate.

The model with deteriorating items

Assume now that the items deteriorate according to some instantaneous decay rate. The items start deteriorating in the inventory after some time Y (>0) of their arrival in the inventory. At time t', a fraction $\phi(t')$ of the on-hand inventory deteriorates per unit time ($t' \geq Y$). We assume $\phi(t')$ to be integrable. Let $\psi(t') = 1 - \phi(t')$ denote the fraction of the on-hand inventory which did not deteriorate by time t'.

Let $x'(t')$ and $u'(t')$ be the inventory and production levels at time t', $Y \leq t' \leq T'$ and x_1' and u_1' be those at the beginning of the deterioration epoch respectively. Let S' be the demand rate at time t'. We observe the process for the duration $(Y, T']$.

The rate of change of inventory may be described by the stock flow equation

$$dx' = (u' - S')\,dt' + d\,\xi(t')$$

with the initial condition $x'(Y) = x_1$. Here $\xi(dt')$ is a stochastic process with, for all t',

$$E(\xi(dt')) = 0$$

$$E(\xi^2(dt')) = u'^2 \delta^2 dt'$$

$$E(\xi^n(dt')) = o(dt'),\ n > 2.$$

We wish to control $u'(t')$ so as to guide the production process in such a way as to adjust the inventory level initially at x_1' towards S'. The objective function may be written as

$$E\left\{\int_Y^{T'} h\psi^2(x' - x_1')dt' - \int_Y^{T'} B_1\phi(x' - x_1')dt' - Bx'(T' - Y)\right\}$$

where B_1 is the salvage value per deteriorating item, and we wish to minimize this function with respect to $u'(t')$. The second term in this expression has been introduced to take into account the expected loss due to deterioration of items. For ease in presentation, we make a time translation from t', $Y \le t' \le T'$ to t, $0 \le t \le T$ and rewrite the objective function as

$$E\left\{\int_0^T h\psi^2 x^2 dt - \int_0^T B_1\phi x dt - Bx(T)\right\}.$$

Define a value function from time t to the planning horizon T as

$$V(t, x) = \max_u E\left\{-\int_t^T h\psi^2 x^2 dt + \int_t^T B_1\phi x dt + Bx(T)\right\}$$

with boundary condition $V(T, x) = Bx(T)$. The Hamilton-Jacobi-Bellman equation now becomes

$$0 = \max_u\left\{-h\psi^2 x^2 + B_1\phi x + V_t + (u - s)V_x + \tfrac{1}{2}u^2\sigma^2 V_{xx}\right\}$$

and the optimal rate

$$u_\psi = -V_x / \sigma^2 V_{xx}.$$

The second order partial differential equation, the solution to which yields the optimal production rate is then,

$$0 = -h\psi^2 x^2 + B_1\phi x + V_t - \tfrac{1}{2}\frac{V_x^2}{\sigma^2 V_{xx}} - SV_x .$$

Let $V(t, x)$ have the separable form

$$V(t, x) = Q(t)\ (\psi x)^2 + R(t)\psi x + M(t)$$

then we may derive

$$Q = -hy\eta/\psi^2$$

where $\eta = \eta(t)$, $\eta(s) = \int_s^T \psi^2(\mu)\exp(-\mu - T)/\sigma^2)d\mu$, $y = y(t)$ and $y(s) = \exp((s - T)/\sigma^2)$,

$$R = (1/\psi)\,(By - B_1\sigma^2\,(y - 1) - \int_t^T \nu(s)ds)$$

where $\nu(s) = B_1 y\,\psi(s)y^{-1}(s) - 2Shy\,\eta(s)$, and

$$M = -\int_t^T (SR\psi + R^2/4\sigma^2 Q)ds.$$

The optimal production rate in presence of deterioration is thus

$$u^*_\psi = S - x/\sigma^2 + (2h\sigma^2 y\,\eta)^{-1}\,(By + B_1\sigma^2(1 - y)) - \int_t^T \nu_1(s)ds$$

where $\nu_1(s) = (2h\sigma^2 y\,\eta)^{-1}\,\nu(s) + S/(T - t)$.

The optimal production rate is then the combined effect of the on-hand inventory level, holding and salvage cost coefficients, the distance from the planning horizon and the deterioration rate.

Remarks

1. The net effect on the optimal production rate due to deterioration may be simply obtained as $\Delta u_\psi = u^*_\psi - u^*$. Thus Δu_ψ denotes the additional increase in the production rate in presence of a decay factor, so as to fulfill a constant demand S^*.

2. Note that the additional production needed due to presence of a deterioration factor does not depend upon the inventory level at all.

Particular cases

1. When no deterioration takes place, $\phi(t) = 0$ and $\psi(t) = 1$. Also $B_1 = 0$. The expression for the optimal production rate reduces to that derived earlier.

2. When the deterioration rate is constant over time, i.e., $\phi(t) = \phi$, $\psi(t) = \psi$ then the optimal production rate is given by

$$u^*_{\psi} = S - x/\delta^2 + (y/y - 1)\left\{\frac{2h\delta^2\psi^2 S(T - t) - B}{2h\psi^2\delta^4} + \frac{B_1\phi\delta^2(y - 1)}{2h\psi^2\delta^4}\right\}.$$

The change in the optimal production rate when a constant deterioration of items is present is seen to be

$$\frac{y}{1 - y}\left\{\frac{B(1 - \psi^2)}{2h\,\psi^2\delta^4}\right\} + \frac{B_1\phi}{2h\,\psi^2\delta^2}\cdot y \geq 0.$$

Greater the deterioration rate, larger should be the production rate and greater the variance of the production process, lesser should be the production rate. Further greater the planning horizon lesser should be the increase in the optimal production rate.

If the planning horizon is too large i.e., $T \longrightarrow \infty$ then $y \longrightarrow 0$ and $u^*_{\psi,\infty} \longrightarrow u^*_{\infty}$ so that the effect of deterioration is nullified.

Concluding remarks

We have discussed in this paper a stochastic production inventory model where the variation in the inventory level is due to the randomness of the production level. We observe that the optimal production rate depends on the demand rate, the point of time from where deterioration starts, the on-hand inventory level, the deterioration rate, the distance from the planning horizon and the variance of the production rates. The other observations follow as in Shrestha (1982) except that in the present model, the additional production needed due to the presence of the deterioration factor need not depend upon the inventory level at all.

References

[1] Sethi, S.P. and Thompson, G.L. (1980). "Simple Models in Stochastic Production Planning" in Applied Stochastic Control in Econometrics and Management Sciences, Eds. Bensoussan, A., Kleindorfer, P. and Tapiero, C.S., North Holland Pub. Co., Amsterdam.

[2] Shrestha, H.B. (1982). Stochastic Production - Inventory Model for Deteriorating Items, The Nepali Mathematical Sciences Report, 7, 87-94.

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Integrability Conditions of a $\xi(p, -(p-q))$-Structure Satisfying $\xi^{p} - \xi^{p-q} = 0$ ($\xi \neq 0$; p, q odd ; I)

M.D. Upadhyay and Ashwani Garg

Summary

Yano, Houh and Chen [1] have studied the structures defined by a tensor field ϕ of the type (1,1) satisfying $\phi^4 \pm \phi^2 = 0$. Gadea and Cordero [4] have obtained its integrability conditions. Upadhyay and Gupta [3] have obtained some integrability conditions of $\xi(K,-(K-2))$-structure, satisfying $\xi^{K} - \xi^{K-2} = 0$, where ξ is a tensor field of the type (1,1). Upadhyay and Garg [2] have obtained some integrability conditions of $\xi(p,-(p-q))$-structure satisfying $\xi^{p} - \xi^{p-q} = 0$ ($\xi \neq 0$, p even, I), where ξ is a tensor of the type (1,1). The purpose of this paper is to obtain some integrability conditions of $\xi(p,-(p-q))$-structure satisfying $\xi^{p} - \xi^{p-q} = 0$ ($\xi \neq 0$; p, q odd ; I) where ξ is a tensor of the same type.

1. Preliminaries

Let M^n be n-differentiable manifold of class C^{∞}, equipped with a (1,1) tensor field ξ ($\xi \neq 0$, I) and of class C^{∞} satisfying

(1.1) $\quad \xi^{p} - \xi^{p-q} = 0 \quad (2 \text{ rank } \xi - \text{rank } \xi^{p-q}) = \dim M^n$,

operators s and t have been defined as follows:

(1.2) $\quad s = \xi^{p-q} ; \qquad t = I - \xi^{p-q}$.

I denoting identity operator and $p > q$ and q is any odd integral number.

2. Some Results

Now we will prove the following theorems:

Theorem (2.1). For a tensor field ξ ($\xi \neq 0$) satisfying (1.1), the operator s and t defined by (1.2) and applied to the tangent space at a point of the manifold are complementary projection operators.

Proof. In consequence of (1.1) and (1.2), we have

$$(2.1) \qquad s + t = I$$

$$(2.2) \qquad s^2 = \xi^{2p-2q} = \xi^p \, \xi^{p-2q}$$

$$= \xi^{p-q} \, \xi^{p-2q} = \xi^p \, \xi^{p-3q}$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \xi^p \, \xi^{p-kq} \, ,$$

where k is some integral value such that $kq = p$, i.e.

$$s^2 = \xi^p \, \xi^{p-kq} = \xi^p \, \xi^{p-p}$$

$$= \xi^p = \xi^{p-q}$$

$$= s \, .$$

$$(2.3) \qquad t^2 = I + \xi^{2p-2q} - 2\, \xi^{p-q} = I + \xi^p \, \xi^{p-2q} - 2\, \xi^{p-q}$$

$$= I + \xi^{p-q} \, \xi^{p-2q} - 2\, \xi^{p-q} = I + \xi^{2p-3q} - 2\, \xi^{p-2q}$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= I + \xi^p \, \xi^{p-kq} - 2\, \xi^{p-q} \, ,$$

where k is some integral value such that $kq = p$, i.e.

$$t^2 = I + \xi^p \, \xi^{p-kq} - 2\, \xi^{p-q} = I + \xi^p \, \xi^{p-p} - 2\, \xi^{p-q}$$

$$= I + \xi^{p-q} - 2\, \xi^{p-q} = I - \xi^{p-q}$$

$$= t \, .$$

$$(2.4)\qquad st = ts = \xi^{p-q} - \xi^{2p-2q}$$
$$= \xi^{p-q} - \xi^{p-q} = 0 .$$

Let S and T be complementary distributions corresponding to the projection operators s and t respectively. Let the rank of ξ be constant and be equal to r, then from (1.1) we have

$$\text{dis } S = (2r - n) \quad \text{and} \quad \dim T = (2n - 2r)$$

Here dimension T is even but dim S is not necessarily even. Obviously $n \leq 2r \leq 2n$. Such a structure has been called a generalised $\xi(p, -(p-q))$-structure or rank r and the manifold M^n with this structure a ' $\xi(p, -(p-q))$-manifold'.

Theorem (2.2). For a tensor field ξ ($\xi \neq 0$, I) satisfying (1.1) and the operators s and t defined by (1.2), we have

$$(2.5)\qquad \xi^{p-q}\, s = s\, \xi^{p-q} = \xi^{p-q} ;$$

$$(2.6)\qquad \xi^{p-q}\, t = t\, \xi^{p-q} = 0 ;$$

$$(2.7)\qquad \xi^{p-q+1}\, s = s\, \xi^{p-q+1} = \xi^{p-q+1} ;$$

$$(2.8)\qquad \xi^{p-q+1}\, t = t\, \xi^{p-q+1} = 0 .$$

Proof. The proof is obvious from (1.1) and (1.2).

Theorem (2.3). For a tensor field ξ($\xi \neq 0$, I) satisfying (1.1) and the operators s and t defined by (1.2), we have

$$(2.9)\qquad \xi s = s \xi = \xi^{p-q+1} , \quad \xi t = t \xi = \xi - \xi^{p-q+1};$$

$$(2.10)\qquad \xi^{q}\, s = \xi^{p-q} , \qquad \xi^{q}\, t = \xi^{q}\, \xi^{p-q} .$$

Proof. The proof is obvious from (1.1) and (1.2).

Corollary (2.1). The $\xi(p, -(p-q))$-structure of maximal rank is an almost product structure at $q = 2$.

Proof. The rank of ξ is maximal, $r = n$. Then $t = 0$. Thus satisfies:

(2.11) $\quad I - \xi^{p-q} = 0$,

multiplying equation (2.11) by ξ^q, we obtain

(2.12) $\quad \xi^q - \xi^p = 0$,

which in view of (2.11) implies that

(2.13) $\quad \xi^q - I = 0$,

for $q = 2$ in (2.13) we have

(2.14) $\quad \xi^2 - I = 0$.

Hence the result.

Corollary (2.2). The $\xi(p, -(p-q))$-structure of minimal rank is a $\xi(p-q)$-structure.

Proof. If the rank of ξ is minimal, $2r=n$. Then $s = 0$. Thus ξ satisfies:

(2.15) $\quad \xi^{p-q} = 0$.

Following the general nomenclature, we cal such a structure a $\xi(p-q)$-structure.

3. Nijenhuis Tensor of $\xi(p, -(p-q))$-structure

Let ξ be a $\xi(p, -(p-q))$-structure of rank r. Then the Nijenhuis tensor $N(X,Y)$ of ξ is

(3.1) $\quad N(X,Y) = [\xi X, \xi Y] - \xi[\xi X,Y] - \xi[X, \xi Y] + \xi^2 [X,Y]$.

Equation (3.1) in consequence of (2.1) and (2.9) can also be expressed in the form :

(3.2) $\quad N(X,Y) = N(sX,sY) + N(sX,tY) + N(tX,sY) + N(tX,tY)$

If the distribution S is integrable, $N(sX,sY)$ becomes the Nijenhuis tensor of $\xi/s \overset{\text{def}}{===} \xi_s$. If the distribution T is integrable, $N(tX,tY)$ also becomes the Nijenhuis tensor of $\xi/T \overset{\text{def}}{===} \xi_T$.

Let $\mathcal{L}_Y \xi$ be Lie derivative of the tensor field ξ with respect to a vector Y. Then we have the definition :

$$(3.3) \qquad (\mathcal{L}_Y \xi)X = \xi[X,Y] - [\xi X, Y] \, ,$$

where $\mathcal{L}_Y \xi$ is a tensor field of the same type as ξ . Now by virtue of (3.1) and (3.3), we obtain

$$(3.4) \qquad N(sX,tY) = \xi(\mathcal{L}_{tY}\xi)sX - (\mathcal{L}_{\xi tY}\xi)sX \, ,$$

and

$$(3.5) \qquad N(tX,sY) = \xi(\mathcal{L}_{sY}\xi)tX - (\mathcal{L}_{\xi sY}\xi)tX \, .$$

4. Integrability conditions

In this section, we shall obtain the partial integrability conditions of the $\xi(p, -(p-q))$-structure.

Theorem (4.1). For any two vector fields X and Y, the following hold:

(i) The distribution S is integrable if and only if $sN(sX,sY) = 0$.

(ii) The distribution T is integrable if and only if $sN(tX,tY) = 0$.

Proof. It is well known that for any two vector fields X and Y, the distribution S and T are integrable if and only if $t[sX,sY] = 0$ and $s[tX,tY] = 0$, respectively. Thus by virtue of (2.4), (2.8), (2.9) and (3.1), the theorem follows.

Theorem (4.2). For any two vector fields X and Y, the distributions S and T are both integrable if and only if

$$(4.1) \qquad N(X,Y) = s.N(sX,sY) + N(sX,tY) + N(tX,sY) + t.N(tX,tY)$$

Proof. In consequence of (2.1), equation (3.3) can be expressed as

$$(4.2) \qquad N(X,Y) = s.N(sX,sY) + t.N(sX,sY) + N(sX,tY)$$
$$+ N(tX,sY) + sN(tX,tY) + t.N(tX,tY) \, .$$

Now the result follows by virtue of the equation (4.2) and Theorem (4.1).

Theorem (4.3). If the distribution S is integrable, a necessary and sufficient condition for the almost product structure defined by $\xi/s = \xi_s$ on each integral manifold of S to be integrable is that, for any vector

field X and Y,

(4.3) $N(sX,tY) = 0$,

which is equivalent to :

(4.4) $s.N(sX,sY) = 0$.

Proof. Suppose that the distribution S is integrable, then ξ induces on each integral manifold of S an almost product structure. The induced structure is integrable if and only if its Nijenhuis tensor vanishes identically. Thus theorem follows.

Definition (4.1). We say that the $\xi(p, -(p-q))$-structure is 's_p-partially integrable' if the distribution S is integrable and the almost product structure induced from ξ on each integral manifold of S is also integrable.

Theorem (4.4). For any vector field X and Y, a necessary and sufficient condition for the $\xi(p, -(p-q))$-structure to be s_p-integrable is that

(4.5) $N(sX,sY) = 0$.

Proof. The proof of the Theorem follows from the Theorem (4.1)(i) and (4.3).

Theorem (4.5). If the distribution T is integrable, a necessary and sufficient condition for the $\xi(p-q)$ structure defined by $\xi/_T = \xi_T$ on each integral manifold of T to be integrable is that, for any vector field X and Y

(4.6) $N(tX,tY) = 0$,

which is equivalent to :

(4.7) $t.N(tX,tY) = 0$.

Proof. The proof of the theorem follows from a pattern of the proof of Theorem (4.3).

Definition (4.2). We say that the $\xi(p, -(p-q))$-structure is 't_p-partially integrable' if the distribution T is integrable and the $\xi(p-q)$-

structure ξ_T induced from ξ on each integral manifold of T is also integrable.

Theorem (4.6). For any vector fields X and Y, a necessary and sufficient condition for the $\xi(p, -(p-q))$-structure to be t_p-partially integrable is that

(4.8) $\quad N(tX, tY) = 0$.

Proof. The proof of the Theorem follows from the Theorem (4.1)(ii) and (4.5).

Definition (4.3). We say that a $\xi(p, -(p-q))$-structure is 'partially integrable' if and only if it is s_p-partially integrable and t_p-partially integrable simultaneously.

Theorem (4.7). For any vector fields X and Y, a necessary and sufficient condition for the $\xi(p, -(p-q))$-structure to be partially integrable is that

(4.3) $\quad N(X,Y) = N(sX, tY) + N(tX, sY)$.

References

/1/ K.Yano, C. Houh and B. Chen. Structure defined by a tensor field ϕ of type (1,1) satisfying $\phi_4 + \phi_2 = 0$.
Tensor (N.S.) Vol. 23(1972), 81-87.

[2] M.D. Upadhyay and V.C. Gupta. Integrability conditions of a $\xi(K, -(K-2))$-structure satisfying $\xi^K - \xi^{K-2} = 0$ ($\xi \neq 0$, I).
Rev. Univ. Nac. Tucuman, Vol. 20, No. 1-2, (1976), 31-44.

[3] M.D. Upadhyay and Ashwani Garg. Integrability conditions of a $\xi(p, -(p-q))$-structure satisfying $\xi^p - \xi^{p-q} = 0$ ($\xi \neq 0$, p, even, I).

[4] P.M. Godes and A. Luis Cordero. Integrability conditions of a structure ϕ satisfying $\phi_4 \pm \phi_2 = 0$. Tensor (N.S.) Vol. 28 (1974), 78-82.

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Some Results On Isotopes Of Groups

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A quasigroup G is defined to be a system of three compositions viz. product ('.'), right division ('/') and left division ('\') such that for every $a,b,c \in G$, $ab = c \Leftrightarrow c/b = a \Leftrightarrow a\backslash c = b$.

Functions are understood to operate on the right.

An isotopy of a groupoid G_1 onto a groupoid G_2 with products $\emptyset_1, \emptyset_2$ is defined to be a triple $h = (\alpha, \beta, \nu)$ where $\alpha, \beta, \nu : G_1 \longrightarrow G_2$ satisfy $(\alpha x \beta)\, \emptyset_2 = \emptyset_1 \nu$. An isotopy of the form (α, β, I_G) of a groupoid $(G, \emptyset_1)$ onto a groupoid $(G, \emptyset_2)$ is said to be a principal isotopy.

A quasigroup G is said to satisfy the left (resp. right) property if there exists a bijective mapping λ (resp. μ): $G \longrightarrow G$ such that $Lx^{-1} = Lx\lambda$ (resp. $Rx^{-1} = Rx\mu$) where Lx, Rx are the left and right translations by x in G. λ (resp. μ) is said to be the left (resp. right) inverse operator in G.

<u>Theorem 1</u>: Let $(G, \cdot)$ be a group and let (G, o) be obtained from $(G, \cdot)$ via a principal isotopy (α, β, IG): $(G, o) \longrightarrow (G, \cdot)$. Then (G, o) possesses a left (resp. right) identity and satisfies the left (resp. right) inverse property having ρ as the left (resp. right) inverse operator if and only if there exists an element $a \in G$ such that $\beta = La^{-1}$ (resp. $\alpha = Ra^{-1}$) and $\rho\alpha = \alpha i\, La\, Ra$ (resp. $\rho\beta = \beta\, i\, La\, Ra$) where La, Ra are the left and right translations by a and 'i' is the inverse operator in $(G, \cdot)$.

<u>Proof</u>: Let (G, o) possess a left identity e and satisfy the left inverse property having ρ as the left inverse operator.

We have, $x \text{ o } y = x\alpha .\, y\beta$ for every $x,y \in G$ ... (1)
$x = e$ in (1) gives $y = y\beta La$ for every $y \in G$ where $a = e\alpha$.
$\therefore\ \beta = La^{-1}$ and then $x \text{ o } y = x\alpha . a^{-1}y$ for every $x, y \in G$.

Again for every $x, y \in G$, $y = x\rho \circ (x \circ y) = x\rho\alpha \,.\, a^{-1} (x\alpha \,.\, a^{-1}y)$.

$\therefore \quad x\rho\alpha \,.\, a^{-1} x\alpha a^{-1} = 1$ where 1 is the identity in $(G,.)$.

$\therefore \quad x\rho\alpha = (a^{-1} x \alpha a^{-1})^{-1} = x\alpha i L_a R_a$ for every $x \in G$

$\therefore \quad \rho\alpha = \alpha i L_a R_a$.

Conversely let there exist an element $a \in G$ such that $\beta = L_a^{-1}$ and $\rho\alpha = \alpha i L_a R_a$.

Let $a\alpha^{-1} = e$. Then $e \circ y = e\alpha \,.\, a^{-1} y = a.a^{-1}y = y$ for every $y \in G$. $\therefore$ e is a left identity in $(G.o)$.

Also $x\rho \circ (x \circ y) = x\rho\alpha \,.\, a^{-1} x\alpha a^{-1} u = a (x\alpha)^{-1} a.\ a^{-1} x\alpha a^{-1} y$ ($\because \rho\alpha = \alpha i L_a R_a$) $= y$ for every $x, y \in G$.

$\therefore$ (G,o) satisfies the left inverse property having ρ as the left inverse operator.

The other case may be similarly disposed of.

<u>Theorem 2</u>: Let $(G,.)$ be a group and let (G, o) be a principal isotope of $(G,.)$ under a principal isotopy $(\alpha, \beta, I_G): (G, o) \longrightarrow (G,.)$. Then (G, o) possesses an element a such that $R_a = L_a = i$ if, and only if there exist $b, c \in G$ such that $bc = a$ and $\alpha = i S_b$, $\beta = i T_c$ where R_a, L_a (resp. S_a, T_a) are the right, left translations by a in (G,o) (resp. $(G,.)$) and i is the inverse operator in $(G,.)$.

<u>Proof.</u> Let there exist an element a in G such that $R_a = L_a = i$. Let $b = a\beta i$, $c = a\alpha i$.

Since $R_a = i$, $x^{-1} = x \circ a = x\alpha \,.\, a\beta = x\alpha \,.\, b^{-1}$ and then $x\alpha = x^{-1}b = x\, i\, S_b$ for every $x \in G$. $\therefore$ $\alpha = i\, S_b$.

Similarly $\beta = i\, T_c$.

Again $a \circ a = a^{-1}$ gives $a^{-1} = a^{-1}bc\, a^{-1}$ and then $bc = a$.

Conversely if $b, c \in G$ be such that $\alpha = iS_b$, $\beta = iT_c$ then $bc = a \in G$ is such that $R_a = L_a = i$.

<u>Definition 1</u>: A groupoid $(G,.)$ is said to be an iso-group if there exists a group (G, o) such that '.' and 'o' are connected by the relation $xy = a^{-}\rho \circ y\rho$ where ρ is the inverse operator in (G, o).

Theorem 3: A groupoid $(G,.)$ is an iso-group if, and only if there exists an element e in G satisfying the following conditions

(i) $x(yx) = (xy)x = y$,

(ii) $e(ex.ye) = (ex.\ ye)e = yx$,

(iii) $ex.zy = yx.ze$

for every $x,y,z \in G$.

Proof: Let there exist an element e in G satisfying (i), (ii), (iii).

'.' is cancallative. For $ba = ca \Longrightarrow b = a(ba)$ (by (i)) $= a(ca)=c$ (by (i)). Similarly $ab = ac \Longrightarrow b = c$.

(i) and the cancellativity of '.' show that the unique solutions of the equations $xz = y$ and $zx = y$ are yx and xy respectively.

$\therefore$ $(G,.)$ is a quasigroup.

$ex = xe$ for every $x \in G$. For if $x = x_1x_2$, then replacing x by x_1e and y by ex_2 in (ii), one gets that $e(e(x_1e).\ (ex_2)e)$ $= (e(x_1e).\ (e(x_2)e)e$ ie, $e\ (x_1x_2) = (x_1x_2)e$ (by (i)) ie,

$ex = xe$ ... ... (1)

Again replacing x,y by ee in (ii) and using (i) one gets that $e(ee) = ee.\ ee$. $\therefore$ $ee = e$ ... ... (2)

Let $x \circ y = ex.ye = xLe.\ yRe$ for every $x,y \in G$ where Le, Re are the left and right translations by c in $(G,.)$.

Then (Le, Re, IG) is a principal isotopy of (G,o) onto $(G,.)$. Since $(G,.)$ is a group, (G,o) is at least a quasigroups.

Since for every $x, y, z \in G$, $(xoy)\ oz = e\ (ex.ye).ze = yx.ze$ (by (ii)) $= ex.\ zy$ (by (iii)) $= ex.\ (ey.ze)e = x \circ (yoz)$, (G,o) is group.

e is the identity element in (G,o). For, $e \circ x = ee.xe =$ $e(xe)$ (by (2)) $= x$ (by (i)).

Since $x \circ ex = ex.\ (ex)e = ex.\ e(ex)$ (by(1)) $= e$ (by (i)), $Le = \rho$. Since $Le = Re$, $Re = \rho$.

$\therefore$ $(G,.)$ is an iso-group.

Conversely let (G,.) be an iso-group. Then there exists a group (G,o) such that x.y = xpoyp. Then the identity element c is (G,o) is an element satisfying (i), (ii), (iii).

<u>Definition 2</u>: A groupoid G is said to be totally symmetric if it is commutative and $x(xy) = y$ for every $x,y \in G$. G is said to be abelian if for every $a,b,c,d \in G$, $(ab)(cd) = (ac)(bd)$.

<u>Theorem 4</u>: A groupoid (G,.) is abelian and totally symmetric if, and only if it can be obtained from a commutative group (G,o) via an isotopy $(i,i,Ta):(G,.) \longrightarrow (G,o)$ where i is the inverse operator and Ta the right translation by a in (G,o). Moreover, (G,.) processes an idmpotent element if, and only if Ta = IG.

<u>Proof</u>: Let (G,.) be abelian and totally symmetric. '.' is cancellative. For, $ab = ac \Longrightarrow b = a(ab) = a(ac) = c$. The unique solution of the equation ax = b is ab..'. Since (G,.) is commutative, it is a quasigroup. Let $p \in G$. Then there exists an isotopy $(\phi, \psi, Ta): (G,.) \longrightarrow (G,o)$ where (G,o) is a commutative group having p as the identity element, ϕ, ψ defined by the relations $(x)\phi = ((xp)p)/(((pp)p)\setminus p)$, $(x)\psi = (p/p)x$ for every $x \in G$ are commuting automorphisms of (G,o) and $a = (pp)/(((pp)p)\setminus p)$ (Das [2]).

Since (G,.) is abelian and totally symmetric, $(x)\phi = (x)\psi = (pp)x$ and $a = (pp)(pp)$. Also $\phi^2 = IG$.

$\phi = \psi = i$. For, for every $x \in G$, $x\phi ox = x[(pp)x]oa = ppoa$ ($\because$ (G,.) is totally symmetric) = p op ($\because \phi$ is an automorphism of (G,o)) = p.

$\therefore$ (i,i, Ta) is an isotopy of (G,.) onto a commutative group (G,o).

If (G,.) possesses an idempotent element, then if we take p as that idempotent element, then Ta becomes IG.

Conversely if (G,.) be obtained from a commutative group (G,o) via an isotopy $(i,i,Ta) : (G,.) \longrightarrow (G,o)$, then it can be readily verified that (G,.) is abelian and totally symmetric. Moreover, if Ta = IG, then the identity element in (G,o) is an idempotent element in (G,.).

Corollary 1: Isotopic totally symmetric abelian groupoids possessing idempotent elements are isomorphic.

Proof: Let (G, ϕ) and (H, ψ) be two isotopic totally symmetric abelian groupoids possessing idempotent elements. Then by Theorem 4 there exist commutative groups (G, ϕ') and (H, ψ') such that $\phi = (i \times i)\phi'$, $\psi = (i' \times i')\psi'$ where i, i' are inverse operators in (G, ϕ'), (H, ψ') respectively. Since the relation of being isotopic is an equivalance relation, the groups (G, ϕ'), (H, ψ') are isotopic and therefore they are isomorphic. Let α be an isomorphism of (G, ϕ') onto (H, ψ'). Then $\phi\alpha = (i \times i)\ \phi'\alpha = (i\alpha \times i\alpha)\psi' = (\alpha i' \times \alpha i')\psi'$.
$(\because\ i\alpha = \alpha i') = (\alpha \times \alpha)(i' \times i')\psi' = (\alpha \times \alpha)\psi$.
$\therefore$ α is an isomorphism of (G, ϕ) onto (H, ψ).

Theorem 5: A necessary and sufficient condition that a groupoid $(G,.)$ may be an abelian right (resp. left) loop of exponent 2 is that $(G,.)$ can be obtained from a commutative group (G,o) via an isotopy (IG, i, IG) (resp. (i, IG, IG)):
$(G,.) \longrightarrow (G,o)$ where i is the inverse operator in (G,o).

Proof: The sufficiency of the condition is obvious.

Next let $(G,.)$ be an abelian right loop of exponent 2 with e as the right identity. If $x \circ y = x.\ yLe^{-1}$, then (G,o) is a commutative group with e as the identity element (Das [2]). Since $x \circ ex = x.\ exLe^{-1} = x.x = e$, $Le = i$. $\therefore$ (IG,i,IG) is an isotopy of $(G,.)$ onto a commutative group (G,o).

The case of a left loop may be similarly disposed of.

Corollary 2: Isotopic abelian right (resp. left) loops of exponent 2 are isomorphic.

The proof can be constructed as in Corollary 1 by using Theorem 5.

Theorem 6: A group isotopic to an abelian quasigroup is commutative.

Proof: Let G be a group which is isotopic to an abelian quasigroup $(H,.)$. Since $(H,.)$ is an abelian quasigroup, it is isotopic to a commutative group (H,o) (Das [2]). Since the relation of being isotopic is an equivalence relation G and (H,o) are isotopic and therefore they

are isomorphic. Since (H,o) is commutative, so also is G.

Theorem 6: Every principal isotope of a commutative group (G, +) is abelian if, and only if for any two bijective mappings α, β of G onto itself, the mappings ϕ_α, ϕ_β : G $\longrightarrow$ G defined by the relations $(x)\phi_\alpha = (x)\alpha - (o)\alpha$, $(x)\phi_\beta = (x)\beta - (o)\beta$ for every $x \in G$ where 0 is the null element in (G, +) are two commuting automorphisms of (G, +).

The result follows from Lemmas 1 and 2, Das [2].

Corollary 3: If there exist two bijective mappings of a commutative group (G, +) which leave the null element of (G, +) fixed and which do not commute, then at least one of the principal isotopes of (G, +) is not an abelian quasigroup.

Theorem 7: Every principal isotope of a group of order ≤ 3 is an abelian quasigroup.

Proof: The result is obvious for a group of order 1.

Next let G = $\{a, b\}$ be a group of order 2. The addition table in G, may, without any loss of generality be taken as:

	a	b
a	a	b
b	b	a

The only bijective mappings of G onto itself are α_1 = IG and $\alpha_2 = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$. Then $\phi_{\alpha_1} = \phi_{\alpha_2}$ = IG is an automorphism of G and ϕ_{α_1}, ϕ_{α_2} commute. $\therefore$ By Theorem 6, every principal isotope of G is an abelian quasigroup.

Finally let G = $\{a, b, c\}$ be a group of order 3. We can assume, without any loss of generality that the addition table of G is

	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

The only bijective mappings of G onto itself are

$\alpha_1 = IG$, $\alpha_2 = \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}$,

$\alpha_3 = \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}$; $\alpha_4 = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}$,

$\alpha_5 = \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix}$, $\alpha_6 = \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}$.

Then $\phi\alpha_i = IG$ for $i = 1, 4, 5$ and $\phi\alpha_i = \alpha_2$ for $i = 2, 3, 6$. IG and α_2 are automorphisms of G and they commute. $\therefore$ By Theorem 6 every principal isotope of G is an abelian quasigroup.

Theorem 8: If $(G, +)$ be a group of order > 3, then at least one of the principal isstopes of $(G, +)$ is not an abelian quasigroup.

Proof: Let a, b, c be any three distinct elements in G none of which is the null element in $(G, +)$. $\alpha, \beta : G \longrightarrow G$ are defined as follows:

$(a)\alpha = b$, $(b)\alpha = a$, $(x)\alpha = x$ for every $x\ (\neq a, b)$ in G

and $(b)\beta = c$, $(c)\beta = b$, $(x)\beta = x$ for every $x\ (\neq b, c)$ in G.

Then α, β are bijective, leave the null element of $(G, +)$ fixed but $\alpha\beta \neq \beta\alpha$ since $(a)\alpha\beta = c \neq b = (a)\beta\alpha$. $\therefore$ By Corollary 3, at least one of the principal isotopes of $(G, +)$ is not an abelian quasigroup.

References

1. R.H. Bruck, A survey of binary systems, Springer-Verlag, Berlin and New York, 1966.

2. Phullendu Das, Isotopy of abelian quasigroups, Proc. Amer. Math. Soc. Vol. 63, No. 2, (1977), 317-323.

3. D.C. Murdoch. Quasigroups which satisfy certain generalised associative laws, Amer. J. Math. 61 (1939), 509-522.

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Extended Decompositions of Topological Vector Spaces

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1. Introduction

The concept of extended Schauder decomposition of a topological vector space (TVS) was firstly given by Fleming and Ruckle in [1]. In fact, they generalised the concept of Schauder decomposition of TVS by replacing a sequence of subspaces by a net or family of subspaces and obtained certain results on extended Schauder decompositions of TVS which are already known for Schauder decompositions and vector sequence spaces. This study was further continued in locally convex spaces by Webb [6].

Our aim in this paper is to obtain certain more results on extended Schauder decompositions of TVS. A necessary and sufficient condition has been obtained for a family of subspaces of TVS to become an extended Schauder decomposition.

2. Notations and Terminology

Let Λ be a directed set and Σ be the set of all nonempty finite subsets of Λ. We denote the members of Σ by $\sigma, \tau, \sigma_0, \tau_0, \ldots$. In general, we use $\{\ \}$ to denote sets, $[\ ]$ to denote the closed linear spans of the indicated sets, and ( ) to denote the nets or families.

Let X be a TVS. A family $(M_\alpha)_{\alpha \in \Lambda}$ of subspaces of X is called an extended decomposition of X if for each $x \in X$, there exists a unique net $(x_\alpha)_{\alpha \in \Lambda}$, $x_\alpha \in M_\alpha$, $\alpha \in \Lambda$ such that $x = \sum_{\alpha \in \Lambda} x_\alpha$, the convergence being in the topology of X. The uniqueness implies the existence of (not necessarily continuous) associated projections P_α of X onto M_α such that $P_\alpha P_\beta = \delta_{\alpha\beta} P_\beta$, where $\delta_{\alpha\beta} = 0$ for $\alpha \neq \beta$ and $\delta_{\alpha\beta} = 1$ for $\alpha = \beta$. If we write $Q_\sigma = \sum_{\alpha \in \sigma} P_\alpha$, $\sigma \in \Sigma$, then $Q_\sigma Q_\tau = Q_{\sigma \cap \tau}$ for all $\sigma, \tau \in \Sigma$. If each P_α is continuous (equi-continuous), the extended decomposition is called an extended Schauder

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decomposition (equi-extended Schauder decomposition) and we denote it by $(M_\alpha, P_\alpha)_{\alpha \in \Lambda}$. Thus the family $(M_\alpha)_{\alpha \in \Lambda}$ of closed subspaces of X is an extended Schauder decomposition of X if $\lim_\sigma \sum_{\alpha \in \sigma} P_\alpha(x) = x$ or $\lim_\sigma Q_\sigma(x) = x$, for each $x \in X$.

Now, we state some results in the form of the lemmas which we shall be using in our work.

Lemma 2.1. Let X and Y be two topological vector spaces. Let $(p_\lambda)_{\lambda \in D}$ and $(q_\mu)_{\mu \in D'}$ be the families of pseudonorms generating the topologies on X and Y, respectively; D, D' being the directed sets. A linear map $f: X \to Y$ is continuous if and only if for every $\mu \in D'$, there exists $\lambda \in D$ and a number $M > 0$ depending on μ and independent of $x \in X$ such that

$$q_\mu(f(x)) \leq M\, p_\lambda(x), \quad \text{for all } x \in X.$$

Lemma 2.2. Let X and Y be two topological vector spaces whose topologies are generated by the families of pseudonorms $(p_\lambda)_{\lambda \in D}$ and $(q_\mu)_{\mu \in D'}$, respectively; D, D' being directed sets. A family of linear maps from X into Y is equi-continuous if and only if for each $\mu \in D'$, there exists a $\lambda \in D$ and a number $M > 0$ such that

$$q_\mu(f(x)) \leq M\, p_\lambda(x)$$

for all $x \in X$ and $f \in \mathcal{C}$.

Lemma 2.3. (i) A complete subset of a Hausdorff topological vector space is closed.

(ii) Every closed subset of a complete topological vector space is complete.

Lemma 2.4. Let A be a subset of a topological vector space X, and Y be a complete Hausdorff topological vector space. If f is a continuous linear map from A into Y, then there exists a unique continuous linear map $\bar{f}$ from $\bar{A}$ into Y such that $\bar{f}(x) = f(x)$ for all $x \in \bar{A}$.

3. Main Results

Theorem 3.1. Let X be a Hausdorff topological vector space where the topology on X is generated by the family D of pseudonorms p,q,r,... and $(M_\alpha, P_\alpha)_{\alpha \in \Lambda}$ be an equi-extended Schauder decomposition. Then, for each $p \in D$ there exists a $q \in D$ and a constant $K > 0$ depending on p such that

$$p\left(\sum_{\alpha \in \sigma} x_\alpha\right) \leq Kq\left(\sum_{\alpha \in \tau} x_\alpha\right)$$

for all $\sigma, \tau \in \Sigma$ with $\sigma \subseteq \tau$ and for any net $(x_\alpha)_{\alpha \in \Lambda}$ of X where $x_\alpha \in M_\alpha$, $\alpha \in \Lambda$.

Proof Since $(M_\alpha, P_\alpha)_{\alpha \in \Lambda}$ is an equi-extended Schauder decomposition, the projections Q_σ's are equicontinuous. By lemma 2.2 for each $p \in D$, there exists a $q \in D$ and a constant $K > 0$ depending on p such that

$$p(Q_\sigma(x)) \leq Kq(x), \text{ for each } x \in X.$$

In particular, let $x = \sum_{\alpha \in \tau} x_\alpha$, $x_\alpha \in M_\alpha$, $\alpha \in \tau$. Then $x = \sum_{\alpha \in \tau} x_\alpha = \sum_{\alpha \in \tau} P_\alpha(x) = Q_\tau(x)$. For $\sigma \subseteq \tau$, we have

$$p(Q_\sigma(x)) \leq K\, q(x)$$

$$\Rightarrow\ p(Q_\sigma(Q_\tau(x))) \leq K\, q(Q_\tau(x))$$

$$\Rightarrow\ p\left(\sum_{\alpha \in \sigma} x_\alpha\right) \leq K\, q\left(\sum_{\alpha \in \tau} x_\alpha\right).$$

Theorem 3.2. Let X be a complete Hausdorff topological vector space and $(M_\alpha)_{\alpha \in \Lambda}$ be a family of nontrivial closed subspaces of X such that $[\bigcup_{\alpha \in \Lambda} M_\alpha] = X$. Then $(M_\alpha)_{\alpha \in \Lambda}$ is an extended Schauder decomposition of X if for each $p \in D$ there exists a $q \in D$ and a constant $K > 0$ depending on p such that

$$p\left(\sum_{\alpha \in \sigma} x_\alpha\right) \leq K\, q\left(\sum_{\alpha \in \tau} x_\alpha\right) \tag{3.1}$$

for all $\sigma, \tau \in \Sigma$ with $\sigma \subseteq \tau$ and for all nets $(x_\alpha)_{\alpha \in \Lambda}$ of X where $x_\alpha \in M_\alpha$, $\alpha \in \Lambda$.

<u>Proof</u>: Let $E_\sigma = [\bigcup_{\alpha\in\sigma} M_\alpha]$. We shall prove by induction that $E_\sigma = \bigoplus_{\alpha\in\sigma} M_\alpha$ for all $\sigma \in \Sigma$. If σ has only one element, then clearly $E_\sigma = \bigoplus_{\alpha\in\sigma} M_\alpha = M_\alpha$. Now, let us assume that the result be true for σ containing n elements. Then, we will prove it for a set $\tau\in\Sigma$ containing n+1 elements. Let $x \in E_\tau$. Then there is a net $(x_\lambda)_{\lambda\in D'}$ in $[\bigcup_{\alpha\in\tau} M_\alpha]$ such that $\lim_\lambda x_\lambda = x$, where D' is a directed set. Now $x_\lambda = \sum_{\alpha\in\tau} x_{\lambda_\alpha}$, $x_{\lambda_\alpha} \in M_\alpha$, $\alpha \in \tau$. From (3.1), we have

$$(3.2)\quad p(x_{\lambda_\alpha} - x_{\mu_\alpha}) \leq Kq(\sum_{\alpha\in\tau}(x_{\lambda_\alpha} - x_{\mu_\alpha})) = Kq(x_\lambda - x_\mu).$$

Now $(x_\lambda)_{\lambda\in D'}$ is a Cauchy net. Hence by (3.2) (x_{λ_α}) is a Cauchy net in M_α for each $\alpha\in\tau$. But M_α is closed, therefore by lemma 2.3, it is complete and hence $\lim_\lambda x_{\lambda_\alpha} = x_\alpha$ for some $x_\alpha \in M_\alpha$. Also by continuity of addition in TVS, we have

$$\lim_\lambda x_\lambda = \lim_\lambda (\sum_{\alpha\in\tau} x_{\lambda_\alpha}) = \sum_{\alpha\in\tau} (\lim_\lambda x_{\lambda_\alpha}) = \sum_{\alpha\in\tau} x_\alpha.$$

This implies that $x = \sum_{\alpha\in\tau} x_\alpha$, since X is Hausdorff. Thus any $x \in E_\tau$ can be written in the form $x = \sum_{\alpha\in\tau} x_\alpha$, $x_\alpha \in M_\alpha$, $\alpha \in \tau$. Also, $E_\sigma \cap M_{\tau\sim\sigma} = \{0\}$. Suppose $E_\sigma \cap M_{\tau\sim\sigma} \neq \{0\}$. Then there exists some $x \in E_\sigma \cap M_{\tau\sim\sigma} \Rightarrow x \in E_\sigma$ and $x \in M_{\tau\sim\sigma}$. Let $p \in D$ be be arbitrary, then from (3.1), we get

$$p(x) = p(\sum_{\alpha\in\sigma} x_\alpha) \leq Kq(\sum_{\alpha\in\tau} x_\alpha),$$

where we choose $x_\delta = -x$, $\delta \in \tau\sim\sigma$. Since X is Hausdorff and $p \in D$ is arbitrary, $p(x) = 0 \Rightarrow x = 0$. Thus $E_\tau = \bigoplus_{\alpha\in\tau} M_\alpha$.

Now, we define $Q_{\sigma\tau} : E_\tau \to E_\sigma$ $(\sigma \subset \tau)$ by $Q_{\sigma\tau}(z) = x$, where $z \in E_\tau$, $z = x+y$, $x \in E_\sigma$ and $y \in \bigoplus_{\sigma\in\tau\sim\sigma} M_\alpha$. Clearly, $Q_{\sigma\tau}$ is a well defined projection of E_τ onto E_σ and

$$p(Q_{\sigma\tau}(z)) = p(x) \leq Kq(x+y) = Kq(z).$$

Thus by lemma 2.1, it follows that $Q_{\sigma\tau}$'s are continuous for $\sigma, \tau \in \Sigma$. Now, let us assume that $F = \{x \in X: x \in E_\tau$ for some $\tau \in \Sigma\}$. Then $\bigcup_{\alpha \in \Lambda} M_\alpha \subset F$. Hence by hypothesis $\bar{F} = X$. Further, for $\sigma \in \Sigma$, we define $Q'_\sigma : F \to E_\sigma$ by $Q'_\sigma(y) = Q_{\sigma\tau}(y)$, where $y \in E_\tau$, $\tau \in \Sigma$. Clearly, Q'_σ is continuous projection since each $Q_{\sigma\tau}$ is continuous and

$$Q'^2_\sigma(y) = Q'_\sigma(Q'_\sigma(y)) = Q_{\sigma\sigma}(Q'_\sigma(y)) = Q'_\sigma(y).$$

Each E_σ is closed, hence by lemma 2.3, it is complete. Thus by lemma 2.4, Q'_σ has a unique extension Q_σ on $\bar{F} = X$ whose range is E_σ.

Now we shall prove that (Q_τ) is equi-continuous family. Consider a net $(x^{(\lambda)})_{\lambda \in W}$ (where W is a directed set) in F. Then $\lim_\lambda x^{(\lambda)} = x$ for some $x \in X$. By continuity of pseudonorms at 0, we have $\lim_\lambda p(x^{(\lambda)} - x) = 0$ for each $p \in D$. Also, $Q_\sigma(x^{(\lambda)}) \to Q_\sigma(x)$, since Q_σ is continuous. Now for $p \in D$ and $\sigma \in \Sigma$ (arbitrary but fixed), we have

$$\begin{aligned} p(Q_\sigma(x)) &= p(Q_\sigma(x - x^{(\lambda)}) + Q_\sigma(x^{(\lambda)})) \\ &\leq r(Q_\sigma(x - x^{(\lambda)})) + r(Q_\sigma(x^{(\lambda)})) \text{ for some } r \in D \\ &\leq r(Q_\sigma(x - x^{(\lambda)})) + Kr_1(x^{(\lambda)}) \text{ for some } r_1 \in D \text{ and } K > 0 \\ &\leq r(Q_\sigma(x - x^{(\lambda)})) + Kq(x^{(\lambda)} - x) + Kq(x) \end{aligned}$$

where $q \in D$. But $Q_\sigma(x^{(\lambda)} - x) \to 0$ (σ fixed), hence $r(Q_\sigma(x - x^{(\lambda)})) \to 0$ for all $r \in D$. Therefore

$$(3.3) \quad p(Q_\sigma(x)) \leq Kq(x)$$

for all $x \in X$. By Lemma 2.2 this shows that (Q_τ) is an equicontinuous family. Further, we shall prove that $x = \lim_\tau Q_\tau(x)$ for each $x \in X$. For this, let $x \in X$, $p \in D$ be arbitrary but fixed and $\varepsilon > 0$ be given. Then for any $y \in X$ we have $s \in D$ such that

$$p(Q_\tau(x) - x) \leq s(Q_\tau(x) - y) + s(x-y).$$

Also by (3.3) there exists $s_1 \in D$ such that

$$s(Q_\tau(x)) \leq K\, s_1(x),$$

where s_1 is independent of τ and K is independent of $x \in X$. Further, we can find $r \in D$ with $r(x) \geq \max\{s(x), s_1(x)\}$ for all $x \in X$. Since $\bar{F} = X$, there exists some $y \in E_\sigma$, $\sigma \in \Sigma$ and $r \in D$ such that $r(x-y) < \varepsilon$, $\varepsilon > 0$. For $y \in E_\sigma$, $x \in X$ and $\sigma \subseteq \tau$, we have

$$\begin{aligned} p(Q_\tau(x) - x) &\leq s(Q_\tau(x) - y) + s(x-y) \\ &= s(Q_\tau(x-y)) + s(x-y) \\ &\leq K\, s_1(x-y) + s(x-y) \\ &\leq (K+1)\, r(x-y) \\ &< (K+1)\varepsilon . \end{aligned}$$

But $p \in D$ is arbitrary, therefore $\lim_\tau Q_\tau(x) = x$. Also $Q_\sigma Q_\tau = Q_{\sigma \cap \tau}$ and $Q_\sigma(X) = M_\alpha$ if σ contains only one element α. For $\sigma \subseteq \tau$, where σ contains n elements and τ contains n+1 elements, we have $Q_\tau - Q_\sigma = P_\alpha$, $\alpha \in \tau \sim \sigma$. Therefore, $(Q_\tau - Q_\sigma)(X) = P_\alpha(X) = M_\alpha$ for $\alpha \in \Lambda$. Hence $(M_\alpha, P_\alpha)_{\alpha \in \Lambda}$ is an extended Schauder decomposition of X.

Remark. Theorem 3.2 gives sufficient condition for the family $(M_\alpha)_{\alpha \in \Lambda}$ of nontrivial closed subspaces of X to be an extended Schauder decomposition of X. In fact, it is equi-extended Schauder decomposition of X.

Theorems 3.1 and 3.2 can be combined as follows:

Theorem 3.3. Let X be a complete Hausdorff topological vector space and $(M_\alpha)_{\alpha \in \Lambda}$ be the family of nontrivial closed subspaces of X such that $[\bigcup_{\alpha \in \Lambda} M_\alpha] = X$. Then $(M_\alpha)_{\alpha \in \Lambda}$ is an equiextended Schauder decomposition of X if and only if for each $p \in D$ there exists a $q \in D$ and a constant $K > 0$ depending on p such that

$$p\left(\sum_{\alpha \in \sigma} x_\alpha\right) \leq Kq\left(\sum_{\alpha \in \tau} x_\alpha\right)$$

for all $\sigma, \tau \in \Sigma$ with $\sigma \subset \tau$ and for all nets $(x_\alpha)_{\alpha \in \Lambda}$, $x_\alpha \in M_\alpha$, $\alpha \in \Lambda$.

REFERENCES

[1] Fleming, D.J. and Ruckle, W., Continuous decompositions, Rev. Roum. Math. Pures Appl. 15 (1970), 1153-1170.

[2] Kelley, J.L. and Namioka, I., Linear topological spaces, Van Nostrand, Princeton, N.J. (1963).

[3] Köthe, G., Topological vector spaces I, Springer-Verlag, Berlin (1970).

[4] Ray, S.K., Decompositions of topological vector spaces, Doctoral dissertation, I.I.T. Kanpur (India), (1974).

[5] Schaeffr, H.H., Topological vector spaces, Springer-Verlag, Berlin (1966).

[6] Webb, J.H., Extended Schauder Decompositions of locally convex spaces, Glasgow Math. J. 15 (1974), 166-171.

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