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2. Stokes, G.G., On the effect of the internal friction on the motion of pendulums, Trans. Camb. Phil. Soc., 9 (1851), 8-106.

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CONTENTS

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Coincidence Theorems and Fixed Point Theorems on 2-Metric Spaces and Applications

Virendra

Abstract:

Coincidence theorems for three maps on an arbitrary set with values in a 2-metric space and fixed point theorems for three maps on a 2-metric space are established. Our results extend and unify several known results in metric and 2-metric spaces. Some special cases are discussed.

S. Gahler [1] investigated the concept of 2-metric spaces in 1963 and published subsequently a series of papers [2] - [4] relevent to this topic.

A 2-metric space is a space Y with a real valued function d on $Y \times Y \times Y$ satisfying the following conditions:

i) for two distinct points a, b there is a point c such that $d(a, b, c) \neq 0$;

ii) $d(x, y, z) = 0$ if atleast two of the three points are equal;

iii) $d(x, y, z) = d(x, z, y) = d(y, z, x)$;

iv) $d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$

Iseki-Sharma [7,8] initiated the study of fixed points of contractive maps on 2-metric spaces. For an extensive bibliography concerning the mathematics on 2-metric spaces and fixed point theorems on 2-metric spaces, refer to [15], [16] and [17].

The study of coincidence theorems for a pair of mappings on an arbitary set with values in 2-metric spaces has recently been initiated in [18]. The coincidence theorems [18, Theorems 1,2,3] extend and improve the coincidence theorem of Goebel [6] and pave the way to establish improved versions of known fixed point theorems on 2-metric spaces and product (2-metric) spaces.

AMS (MOS) Subject Classification: 54 H 25

The aim of this note is to obtain coincidence theorems for three mappings on an arbitrary set with values in a 2-metric space. As a consequence we prove fixed point theorems for three mappings on a 2-metric space which extend and present improved versions of several known fixed point theorems. These fixed point theorems are applied to obtain new results what may be called fixed point theorems on product (2-metric) spaces. We give an example to show that Corollary 2.1 of Sharma [12] and Theorem 2 of Ganguly [5] are false in general. Theorem 3 and Corollary 2 of this paper present improved versions of Ganguly's results [5].

Throughout this note, let X be an arbitrary set, (Y, d) a 2-metric space, N the set of positive integers, ω the set of non-negative integers and R_+ the set of all non-negative real numbers. Further, let H be the family of all upper semi-continuous (u.s.c.) functions $h:R_+^5 \to R_+$ which are non-decreasing in each coordinate variable and

(A) $h(t, t, t, t, t,) < t$ for any $t > 0$.

Theorem 1. Let P, Q, T : $X \to Y$ be such that

(1.1) $d(Px, Qy, a) \leq h(d(Tx, Ty, a), d(Px, Tx, a),$
$d(Qy, Ty, a), d(Px, Ty, a), d(Qy, Tx, a))$

for all x, y in X, for all a in Y and for some h in H ;

(1.2) There exists a sequence $\{x_n\}_{n\in\omega}$ in X such that $Tx_n = Px_{n-1}$ if n is odd and $Tx_n = Qx_{n-1}$ if n is even ;

(1.3) $\sup \{d(Tx_i, Tx_j, a) : i, j \geq n, a \in Y$ and i, j are of different parities$\} = \sup \{d(Tx_i, Tx_j, a): i, j \geq n, a \in Y\} < \infty$,

for infinitely many n;

(1.4) the sequence $\{Tx_n\}$ has a subsequence converging to a point in $T(X)$;

Then P, Q and T have a coincidence point in X, that is, there exists a point z in X such that $Pz = Qz = Tz$.

Proof. Let $\delta_n = \sup \{d(Tx_i, Tx_j, a) : i, j \geq n, a \in Y\}$. Then δ_n is finite for each $n \in N$. Since $\delta_{n+1} \leq \delta_n$ for any $n \in N$, δ_n converges to some $\delta \geq 0$.

Let, if possible, $\delta > 0$ if $i \geqslant n+1$, $j \geqslant n+1$ and i odd and j even. Then by (1.1).

$$\begin{aligned} d(Tx_i, Tx_j, a) &= d(Px_{i-1}, Qx_{j-1}, a) \\ &\leq h(\, d(Tx_{i-1}, Tx_{j-1}, a), \\ &\quad d(Tx_i, Tx_{i-1}, a), d(Tx_j, Tx_{j-1}, a), \\ &\quad d(Tx_i, Tx_{j-1}, a), d(Tx_j, Tx_{i-1}, a)), \end{aligned}$$

that is

$$\delta_{n+1} \leqslant h(\delta_n, \delta_n, \delta_n, \delta_n, \delta_n) \text{ for all } 1 < n \in N.$$

So by the u.s. continuity of h,

$$\delta \leqslant h(\delta, \delta, \delta, \delta, \delta) < \delta,$$

proving $\delta = 0$. Therefore $\{Tx_n\}$ is a Cauchy sequence, and converges to some points $b \in T(X)$. Hence there exists a point z in X such that $Tz = b$. Now putting $x = z$ and $y = x_{2n+1}$ in (1.1) and passing to the limit we obtain

$d(Pz, Tz, a) \leqslant h(0, d(Pz, Tz, a), 0, d(Pz, Tz, a), 0)$. If for some $a \in Y$, $d(Pz, Tz, a) = t(\text{say}) \neq 0$, then this inequality gives

$$t \leqslant h(t, t, t, t, t) < t,$$

a contradiction. Thus $d(Pz, Tz, a) = 0$ for all $a \in Y$. So $Pz = Tz$. Similarly $Qz = Tz$. This completes the proof.

Theorem 2. Let $P, Q, T : X \to Y$ be such that

(2.1) $d(Px, Qy, a) \leqslant h(d(Tx, Ty, a), d(Px, Tx, a), d(Qy, Ty, a), t, t)$

for all x, y in X for all a in Y and for some h in H, where $t = [\, d(Px, Ty, a) + d(Qy, Tx, a)] / 2$;

(2.2) $P(X) \cup Q(X) \subseteq T(X)$;

(2.3) $T(X)$ is a complete subspace of Y. Then P, Q, and T have a coincidence point in X.

Proof. Let $\emptyset$ be a choice function for the family $\{T^{-1}y : y \in P(X) \cup Q(X)\}$, and $x_0 \in X$. Then there is a sequence $\{x_n\}_{n \in N}$

in X such that $x_n = \phi(T^{-1} Px_{n-1})$ if n is odd and $x_n = \phi(T^{-1}Qx_{n-1})$ if n is even. Following a standard technique, it can be shown that $\{Tx_n\}$ is a Cauchy sequence. Now the rest part of the proof may be completed following the proof of Theorem 1.

Remarks 1. The condition (2.2) guarantees (2.1) for any $x_0 \in X$.

2. The above theorems present improved versions of Theorems 1 and 2 of [18]. In fact, Theorem 2 [18] is obtained as a special case of Corollary 1 (below).

3. A slightly improved version of, perhaps, the most general fixed point theorem for a mapping on a 2-metric space due to Sharma [12, Th. 1] is obtained as a special case of Theorem 1 (above) or Theorem 3 (below) if X = Y, P = Q and Tx = x for every x in Y.

Corollary 1. Let P, Q, T : $X \to Y$ satisfy the conditions (1.2), (1.4) and the following :

$$(2.4)\quad d(Px, Qy, a) \leq q.\max\{d(Tx, Ty, a), d(Px, Tx, a), d(Qy, Ty, a), \tfrac{1}{2}[d(Px, Ty, a) + d(Qy, Tx, a)]\}$$

for all x, y in X, for all a in Y and some q in (0,1). Then P, Q and T have a coincidence point.

Proof By (2.4),

$$d(Tx_{2n+2}, Tx_{2n+1}, a) = d(Px_{2n}, Qx_{2n+1}, a) \leq q.\max\{d(Tx_{2n+1}, Tx_{2n}, a), \tfrac{1}{2}d(Tx_{2n+2}, Tx_{2n}, a)\}$$

and $d(Tx_{2n+1}, Tx_{2n+2}, Tx_{2n}) = 0$.

It can not be that

$$d(Tx_{2n+1}, Tx_{2n}, a) \leq \tfrac{1}{2}d(Tx_{2n+2}, Tx_{2n}, a)$$

for in such a situation

$$d(Tx_{2n+2}, Tx_{2n+1}, a) \leq \tfrac{1}{2} q\, d(Tx_{2n+2}, Tx_{2n}, a)$$

and so

$$\begin{aligned} d(Tx_{2n+2}, Tx_{2n}, a) &\leq d(Tx_{2n+1}, Tx_{2n}, a) \\ &\quad + d(Tx_{2n+2}, Tx_{2n+1}, a) \\ &\quad + d(Tx_{2n+2}, Tx_{2n}, Tx_{2n+1}) \\ &\leq \tfrac{1}{2} d(Tx_{2n+2}, Tx_{2n}, a) \\ &\quad + \tfrac{1}{2} q \cdot d(Tx_{2n+2}, Tx_{2n}, a) \end{aligned}$$

that is

$$d(Tx_{2n+2}, Tx_{2n}, a) \leq (1 + q)\, 2^{-1}\, d(Tx_{2n+2}, Tx_{2n}, a),$$

a contradiction, since $(1 + q)2^{-1} < 1$. Hence,

$$d(Tx_{2n+2}, Tx_{2n+1}, a) \leq q\, d(Tx_{2n+1}, Tx_{2n}, a)$$

Similarly $d(Tx_{2n+3}, Tx_{2n+2}, a) \leq q\, d(Tx_{2n+2}, Tx_{2n+1}, a)$.

Therefore $d(Tx_{n+1}, Tx_n, a) \leq q\, d(Tx_n, Tx_{n-1}, a)$.

In view of Lemma 1 [14], $\{Tx_n\}$ is a Cauchy sequence.
Now the rest part of the argument may be completed following the proof of Theorem 1.

Now we derive some fixed point theorems from the above results. First we give some definitions.

Definition 1 [12]. For a map $T : Y \to Y$ and a fixed $x_0 \in Y$, the space Y is said to be (T, x_0) - orbitally complete if every Cauchy sequence of the form $T^{n_i} x_0$ is convergent to some $u \in Y$, where $x_n = Tx_{n-1}$ defines the Piccard sequence of iterates of T at x_o.

Definition 2. If, for a point x_0 in Y, there exists a sequence x_n $(n \in \omega)$ in Y such that $Tx_n = Px_{n-1}$ if n is odd and $Tx_n = Qx_{n-1}$ if n is even, then $\{Tx_n : n \in \omega\}$ is said to be $(P, Q; T, x_0)$ - orbit or an orbit for P, Q, T at x_0.

Definition 3. The space Y is said to be $(P, Q; T, x_0)$ - orbitally complete for a fixed $x_0 \in Y$, if every Cauchy sequence of the form Tx_{n_i} is convergent to some $u \in Y$.

We remark that a complete 2-metric space Y is necessarily $(P, Q; T, x_0)$ - orbitally complete for an orbit for (P, Q, T) at x_0, while a

$(P, Q; T, x_0)$ - orbitally complete space need not be complete. Definitions 1-3 also hold for a metric (1-metric) space.

Theorem 3. Let $P, Q, T : Y \to Y$ satisfy the condition (1.1) with $X = Y$ and the following:

(3.1) T commutes with each P and Q;

(3.2) there exists a point x_0 in Y for which T(Y) is $(P, Q; T, x_0)$ - orbitally complete, and (1.3) holds. Then P, Q and T have a unique common fixed point and the sequence $\{Tx_n\}$ converges to the fixed point.

Proof In view of Theorem 1, there exist points z and b in Y such that $Tx_n \to b = Pz = Qz = Tz$.

By (3.1), $PPz = PTz = TPz = TTz = TQz = QTz = QQz$. So by (1.1),

$$\begin{aligned} d(PPz, Pz, a) &= d(PPz, Qz, a) \\ &\leqslant h(d(PPz, Pz, a), 0, 0, d(PPz, Pz, a), \\ &\quad d(PPz, Pz, a)) \\ &< d(PPz, Pz, a) \end{aligned}$$

a contradiction unless $d(PPz, Pz, a) = 0$. Since this is true for every $a \in Y$, $PPz = Pz$. Thus $Pz = b$ is a common fixed point of P, Q and T. The uniqueness of the common fixed point follows easily.

Remark 4. A multitude of fixed point theorems on 2-metric spaces may be obtained as special cases of Theorem 3. We mention a few. If $P = Q$ and T an identity map, then Theorem 1 of Sharma [12] is obtained as a special case (see also Remark 5), while Sharma's result generalizes several fixed point theorems for a self-map of a 2-metric space. If T be an identity map then Theorem 3 generalizes several fixed point theorems for two mappings including the result of Lal and Singh [9]. Theorem 3 extends the main result of Lal and Das [10].

Corollary 2. Let $P, Q, T : Y \to Y$ satisfy the conditions (3.1), (3.2) and the following:

$$(3.3)\quad d(Px, Qy, a) \leqslant q.\ \max \{ d(Tx, Ty, a), d(Px, Tx, a), d(Qy, Ty, a), d(Px, Ty, a), d(Qy, Tx, a) \}$$

for all x, y, a in Y and some q in (0, 1). Then P, Q, and T have a unique common fixed point and the sequence $\{ Tx_n \}$ converges to the fixed point.

Proof. If we define $h(t_1, t_2, t_3, t_4, t_5) = q.\ \max \{ t_1, t_2, t_3, t_4, t_5 \}$, the proof follows from Theorem 3.

Remark 5. If T be an identity map and P = Q in Corollary 2, then the condition (1.3) (cf. (3.2)) will not be needed, e.g. see [11]. In Theorem 4 (see below), the boundedness condition (1.3) used in Theorem 3 and Corollarly 2 is not required.

The following result is a metric (1-metric) analogue of Corollary 2.

Corollary 2' Let P, Q and T be maps from a metric space (M, d) to itself such that

(2'.1) T commutes with each of P and Q;

(2'.2) $d(Px, Qy) \leqslant q.\ \max \{ d(Tx, Ty), d(Px, Tx), d(Qy, Ty), d(Px, Ty), d(Qy, Tx) \}$

for all x, y in M and for some q in (0, 1);

(2'.3) there exists a point x_0 in M for which T(M) is $(P, Q; T, x_0)$ - orbitally complete;

(2'.4) $\sup \{ d(Tx_i, Tx_j) : i, j \geqslant n$ and i, j are of different parities$\} = \sup \{ d(Tx_i, Tx_j, a) : i, j \geqslant n \} < \infty$ for infinitely many n. Then P, Q and T have a unique common fixed point and $\{ Tx_n \}$ converges to the fixed point.

Corollary 2. 1 (with F_1 and F_2 single-valued maps) of Sharma [13, p. 410] is Corrollary 2' without the condition (2'.4). Corollary 2' without (2'.4) includes Theorem 2 of Ganguly [5]. The following example shows that Corollary 2' without (2'. 4) and with T = identity map is not true. Theorem 1 in [5] being a variant of Theorem 2 [5]

must also be false unless some further condition, such as (2'.4), is added to theorem 1/5/.

Example. Let $M = \{1, 2, 3, 4\}$ $d(1, 1) = d(2, 2) = d(3, 3) = d(4, 4) = 0$, $d(1, 3) = d(1, 4) = d(2, 3) = d(2, 4) = 2$, $d(1, 2) = d(3, 4) = 4$, $P(1) = P(3) = 2$, $P(2) = P(4) = 1$, $Q(1) = Q(4) = 3$, $Q(2) = Q(3) = 4$. Observe that $P(M) = \{1, 2\}$, $Q(M) = \{3, 4\}$. Consequently,

$$2 = d(Px, Qy) \leq 4 \quad q = q. \max \{d(x, y), d(x, Px), d(y, Qy), d(x, Qy), d(y, Px)\},$$

for all x, y in M and $q = 3/4$.

Remark 6 From the proof technique of Theorem 3, it is evident that $P, Q, T : Y \to Y$ satisfying all the hypothesis of Theorem 2 with $X = Y$ have a unique common fixed point provided T commutes with each P and Q.

Theorem 4. Let $P, Q, T : X \to Y$ satisfy the condition (3.1) and the following:

(3.3) there exists a point x_0 in Y for which T(Y) is $(P, Q; T, x_0)$ - orbitally complete;

(3.4) $$d(Px, Qy, a) \leqslant q.\max \{d(Tx, Ty, a), d(Px, Tx, a), d(Qy, Ty, a), \tfrac{1}{2}[d(Px, Ty, a) + d(Qy, Tx, a)]\}$$

for all x, y, a in Y and some $q \in (0, 1)$.

Then P, Q and T have a unique common fixed point and the sequence $\{Tx_n\}$ converges to the fixed point.

Proof. In view of Corollary 1, it may be completed following the proof of Theorem 3.

Now we give some applications of the above fixed point theorems.

Theorem 5. Let (Y, d) be a complete 2-metric space, and P, Q, T be maps of the product space Y x Y to Y such that

(5.1) $$d(P(x, y), Q(x', y'), a) \lesssim h(d(T(x, y), T(x', y'), a), d(P(x, y), T(x, y), a), d(Q(x', y'), T(x', y'), a), t, t)$$

for all x, y, x', y', a in Y and some $h \in H$, where

$t = [d(P(x, y), T(x', y'), a) + d(Q(x', y'), T(x, y), a)]/2$;

(5.2) $P(T(x, y), y) = T(P(x, y), y)$,

$Q(T(x, y), y) = T(Q(x, y), y)$ for all $x, y \in Y$;

(5.3) $P(Y\times\{y\}) \cup Q(Y\times\{y\}) \subseteq T(Y\times\{y\})$ for every $y \in Y$;

(5.4) $T(Y\times\{y\})$ is a complete subspace of Y for each $y \in Y$;

Then there exists exactly one point b such that

$P(b, y) = Q(b, y) = T(b, y) = b$ for all y in Y.

Proof. By (5.1),

$$\begin{aligned} d(P(x, y), Q(x', y), a) &\leq h(d(T(x, y), T(x', y), a),\\ &\quad d(P(x, y), T(x, y), a),\\ &\quad d(Q(x', y), T(x', y), a), p, p), \end{aligned}$$

for every $x, x', y, a \in Y$, where

$2p = d(P(x, y), T(x', y), a) + d(Q(x', y), T(x, y), a)$.

For a fixed $y \in Y$, this inequality corresponds to the condition (1.1) of Theorem 3. Further, for the fixed $y \in Y$, (5.2) corresponds to (3.1). Therefore in view of Remark 6, for each y in Y, there exists one and only one $x(y)$ in Y such that

$P(x(y), y) = Q(x(y), y) = T(x(y), y) = x(y)$.

For every $y, y' \in Y$, by (5.1), we have

$$\begin{aligned} d(x(y), x(y'), a) &= d(P(x(y), y), Q(x(y'), y', a))\\ &\leq h(d(x(y), x(y'), a), 0, 0, s, s)\\ &< d(x(y), x(y'), a), \end{aligned}$$

a contradiction unless $d(x(y), x(y'), a) = 0$ for all $a \in Y$,

where $2s = d(x(y), x(y'), a) + d(x(y'), x(y), a)$.

Consequently $x(y) = x(y')$. Hence $x(.)$ is some constant $b \in Y$, and so $P(b, y) = Q(b, y) = T(b, y) = b$ for all y in Y.

Theorem 6. Let (Y, d) be a complete 2-metric space, and P, Q, T be maps of the product space $Y \times Y$ to Y satisfying the conditions (5.2), (5.3) and the following:

$$\begin{aligned} d(P(x, y), Q(x', y'), a) &\leq q.\max\{d(T(x, y), T(x', y'), a),\\ &\quad d(P(x, y), T(x, y), a), \end{aligned}$$

$$d(Q(x', y'), T(x', y'), a),$$
$$\tfrac{1}{2}\,[d(P(x, y), T(x', y'), a) +$$
$$d(Q(x', y'), T(x, y), a)]\}$$

for all x, y, x', y', a in Y and some $q \in (0, 1)$; $T(Y \times \{y\})$ is a closed subspace of Y for each $y \in Y$. Then there exists exactly one point b such that $P(b, y) = Q(b, y) = T(b, y) = b$ for all $y \in Y$.

Proof. Using Corollary 2, it may be completed following the above proof.

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Department of Mathematics,
Gurukula Kangri University,
Hardwar 249 404,
India.

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Optimal Strategies for Absorbing Markov and Semi-Markov Processes with Rewards

H. B. Shrestha
Karnatak University,
Dharwad.

Abstract:

The present work discusses some optimality criterion for absorbing Markov and semi-Markov processes with and without discounting of rewards and costs. Such an analysis for the positively regular Markov Chain was introduced by Howard (1960). The expected total return for the process stopped at a certain state is obtained. The 'Markovian' optimality criterion to arrive at the stopping state which maximises the expected total return is also discussed.

1. Introduction:

Consider a time homogeneous absorbing Markov process $\{X_t, t \geqslant 0\}$ defined on the probability space $(\Omega, \mathfrak{F}, P)$ such that $X_t = i$ denotes that the system is in state i at time t. Let the state space of the process be $S=(T,T^C)$ where T denotes the state space of the s transient states and T^C denotes that of r absorbing states, s and r being any positive integers. We suppose that $P_i(A) > 0$ for all $i \in S$ so that the conditional probability of $A \in \mathfrak{F}$ given $X_0 = i$ is defined for all $i \in S$.

On S, real functions r and c are defined such that r_i denotes the reward earned by the process by staying at state i and c_i denotes the cost incurred by the process by staying at state i. We assume that both r_i and c_i are finite for all $i \in s$.

Transitions from one state to another are governed by the transition rates a_{ij}, i, j=1,2,...s and b_{il} , i=1,2.......s, l=1,2,....r. Then the probability that an individual in state $i \in T$ at time τ will

be in state $j \in T^c$ at time $\tau + d\tau$ is $a_{ij}\, d\tau$ and the probability that an individual in state $i \in T$ at time τ will be in state $l \in T^c$ at time $\tau + d\tau$ is $b_{il}\, d\tau$. The transition intensity matrix may then be written as

$$\bar{A} = \left(\begin{array}{c|c} A & B \\ \hline o & o \end{array}\right)$$

where the bar indicates that the matrix $\bar{A}$ is of order $s + r$. A is a $s \times s$ matrix which describes the transitions within the s transient states and B is a $s \times r$ matrix describing the transitions from transient to absorbing states. We assume that the absorption rate matrix B is not a zero matrix i.e there is at least one absorbing state in the system. Consider an interval $(o, t]$, $o \le t < \infty$. Define the transition probability matrices $((p_{il}(t)))$ and $((q_{il}(t)))$, $i, j = 1, 2, \ldots s$ and $l = 1, 2, \ldots, r$, where $p_{ij}(t)$ denotes the probability that an individual in state i at time o will be in state $j \in T$ at time t and $q_{il}(t)$ denotes the probability that an individual in state $i \in T$ at time o will be in state $l \in T^c$ at time t. The transition probability matrix for the system is then related to the transition intensity matrix as

$$\bar{P}(t) = \exp \left\{ t\, \bar{A} \right\}.$$

Define the duration of stay matrix as

$$\bar{D}(t) = \left(\begin{array}{c|c} D(t) & F(t) \\ \hline o & tI \end{array}\right)$$

where the elements $D_{ab}(t)$ are random variables measuring the time spent in state b within the interval $(\tau, \tau + t)$ given the initial state a. Let $d_{ab}(.)$ denote the density function of the duration of stay $D_{ab}(t)$ of the system. It is obvious that each row of $\bar{D}(t)$ sum up to t.

$$\text{Let } E(D_{ab}(t)) = \int_o^t u\, d_{ab}(u)\, du$$

$$= e_{ab}(t), \text{ if } b \in T$$

$$= \mathcal{E}_{ai}(t), \text{ if } i \in T^c \text{ and i is the state where}$$

the process is absorbed. The expected duration of stay matrix be

$$\bar{e}(t) = \left(\begin{array}{c|c} e(t) & \mathcal{E}(t) \\ \hline o & tI \end{array}\right).$$

The expected duration of stay matrix may also be obtained as

$\bar{e}(t) = \int_o^t \bar{P}(u)du = \sum_{k=o}^{\infty} [t^{k+1}\, \bar{A}^k / k+1!]$ which on pre-or post-multiplication by $\bar{A}$ results in

(1.1) $\bar{A}\,\bar{e}(t) = \bar{e}(t)\bar{A} = \bar{P}(t) - I.$

By definition, $\bar{A}$ is singular. So we consider the transient part A which, in an absorbing process, is diagonally dominant. Moreover A is a substochastic matrix so that A is non-singular and its inverse exists. Therefore, it follows from (1.1) that $e(t) = A^{-1}(P(t)-I) = (P(t) - I)A^{-1}$ and $e_{ab}(t)$ may be obtained as the (a,b)th element of e(t). The expected duration of stay matrix for absorbing states may be obtained as $\mathcal{E}(t) = \int_0^t Q(u)du = A^{-1}(e(t) - tI)\ B = A^{-1}(Q(t)-tB)$, and $\mathcal{E}_{ai}(t)$ may be obtained as the (a,i)-th element of $\mathcal{E}(t)$. It may be noted that

$$\mathcal{E}_{ai}(t) + \sum_{b \in T} e_{ab}(t) = t.$$

Alternative expressions for $e_{ab}(t)$ and $\mathcal{E}_{ai}(t)$ may be found in Chiang (1968).

In this paper, we develop a continuous time analogue, of an earlier paper (cf. Shrestha, 1984) on discrete time Markov and Semi-Markov process with rewards, both in the presence and absence of discounting. We consider the following situation.

Consider a rental agency which lets out flats on rental basis. The cost of keeping a flat idle in the ith state (e.g. ith day) be c_i and the rent that could be earned (by letting out the flat for that day) be r_i. Once rented out, it is assumed that the flat will be occupied till the end of the duration $(0,t]$. It is the concern of the agency to know how long or upto what state it can afford to keep a flat unoccupied without decreasing the expected total return during the given duration.

In section 2, we discuss the expected total return for the absorbing Markov process without discounting of rewards and costs. The expected total return for the positively regular Markov process has been discussed by Howard (1960). We also discuss the optimal strategy for such a process. In section 3, we discuss these for an absorbing Markov process with discounting of rewards and costs. In section 4, we discuss the absorbing semi-Markov process, the expected total return for such a process as well as the optimal strategy for arriving at the stopping state. In these discussions we consider the rewards and costs without

discounting. The case of discounted rewards and costs for the semi-Markov process will be taken up in section 5. All the strategies are expressed in terms of the actual rewards and costs rather than the values - as is the usual practice. The advantage of such a presentation will be the topic of section 6.

2. An Absorbing Markov Process without Discounting:

Let the process start at state a. The total cost incurred by the process staying in transient states prior to absorption into state i is given by $\sum_{b=a}^{i-1} D_{ab}(t)\, c_b$, where i-1 denotes the state preceeding state i, where from the process gets absorbed into state i in one step. The process will remain in state i for the remainder of $t - \sum_{b=a}^{i-1} D_{ab}(t)$ duration. The total reward earned by the process by staying in the absorbed state i for the rest of the duration is thus $\left\{ t - \sum_{b=a}^{i-1} D_{ab}(t) \right\} \times r_i$. The total return for the process started at state a and stopped at state i is then

$$(2.1) \quad R_i = \left\{ t - \sum_{b=a}^{i-1} D_{ab}(t) \right\} r_i - \sum_{b=a}^{i-1} D_{ab}(t)\, c_b .$$

The expected total return at state i is then

$$(2.2) \quad \begin{aligned} v_i &= \left\{ t - \sum_{b=a}^{i-1} e_{ab}(t) \right\} r_i - \sum_{b=a}^{i-1} e_{ab}(t)\, c_b \\ &= \mathcal{E}_{ai}(t)\, r_i - \sum_{b=a}^{i-1} e_{ab}(t)\, c_b . \end{aligned}$$

The existence of the expected total return v_i is guarenteed for all i since $\max_{i \in T^c} r_i < \infty$ and c_i is a non-negative function such that $\min_{i \in T} c_i > 0$. Also we note that, by assumption, r_i is a non-decreasing function in $i \in T^c$. The stopping state is defined as follows. The state i is a stopping state if for any $j > i$,

$$(2.3) \quad v_j - v_i \leq 0$$

for the first time, $i, j \in S$.

We now obtain the optimal strategy for the absorbing Markov process with undiscounted rewards and costs.

2.1 The Strategy:

Let j (> i) be any other state. The expected total return for state j is v_j. For the state i to be the stopping state, we need that

$$v_j - v_i = \mathcal{E}_{aj}(t)r_j - \mathcal{E}_{ai}(t)r_i - \sum_{b=i}^{j-1} e_{ab}(t)c_b \leq o$$

which yields on simplification

$$(2.4) \quad r_j \leq A_{ij}r_i + \sum_{b=i}^{j-1} B_{bj}\, c_b$$

where $A_{ij} = \mathcal{E}_{ai}(t) \,/\, \mathcal{E}_{aj}(t)$ and

$$B_{bj} = e_{ab}(t) \,/\, \mathcal{E}_{aj}(t).$$

The coefficient A_{ij} has the interpretation that it is the ratio of the duration of the system when the system is stopped at state i to remain there for the remaining duration to the duration of the system when the system is stopped at a further state j to remain there for the remainder of the duration. Clearly $A_{ij} \geq 1$ for all $i,j \in S$, the equality holding for i = j. Next, B_{bj} gives the proportion of the duration of the system at a particular state b, $i \leq b \leq j-1$ to that when the system is stopped at state j to remain in it for the rest of the duration. Obviously, inequality (2.4) is a "Markovian" one in that the decision as to whether the process should continue on after state i depends on the reward receivable at that state and the future costs the process would incur by leaving state i. Obviously, the stopped state will have the maximum expected total return.

3. The Markov Process with Discounting:

Usually the payments of rewards and costs stretches over a long passage of time so that a discounting of future returns become inevitable. Assume that the rewards as well as the costs are continuously discounted by a constant discount factor α, $o \leq \alpha \leq 1$. The process starts at state a at time o. The total discounted cost till it is absorbed in state i is given by

$$\sum_{k=a}^{i-1} c_k \quad \left(\int_{T_{k-1}}^{T_k} e^{-\alpha\tau} \, d\tau \right)$$

where $T_k = \sum_{b=a}^{k} D_{ab}(t)$. The total cost incurred by the process started at state a and stopped at i is thus

$$\frac{1}{\alpha} \sum_{k=a}^{i-1} e^{-\alpha T_{k-1}} \left(1 - e^{-\alpha(T_k - T_{k-1})} \right) c_k .$$

The reward earned by the process by remaining at state i for the remaining duration is

$$\frac{1}{\alpha} \left(e^{-\alpha T_{i-1}} - e^{-\alpha t} \right) r_i .$$

The total return for the discounted process started at state a and stopped at state i is then

$$(3.1) \quad R_i(\alpha) = \frac{1}{\alpha} \left\{ \left(e^{-\alpha T_{i-1}} - e^{-\alpha t}\right) r_i - \sum_{k=a}^{i-1} e^{-\alpha T_{k-1}} \left(1 - e^{-\alpha(T_k - T_{k-1})}\right) c_k \right\}.$$

Since the duration of stay at state b has the density function $d_{ab}(.)$, its probability generating function may be written as

$$E\left(e^{-\alpha D_{ab}(t)}\right) = \int_o^t e^{-\alpha D_{ab}(u)} d_{ab}(u) \, du = H_{ab}(t),$$

say. Then we may write

$$(3.2) \quad E\left(e^{-\alpha T_1}\right) = \prod_{b=a}^{1} E\left(e^{-\alpha D_{ab}(t)}\right) = \prod_{b=a}^{1} H_{ab}(t).$$

Taking the expectation of (3.1) and using the relation (3.2) we have

$$v_i(\alpha) = \frac{1}{\alpha} \left[\left(\prod_{b=a}^{i-1} H_{ab}(t) - e^{-\alpha t} \right) r_i - \sum_{k=a}^{i-1} \left\{ \prod_{b=a}^{k-1} H_{ab}(t) - \prod_{b=a}^{k} H_{ab}(t) \right\} c_k \right] .$$

If we let $H_n = \left(\prod_{b=a}^{n} H_{ab}(t) - e^{-\alpha t} \right) / \alpha$ we may rewrite the expected total return as

$$(3.3) \quad v_i(\alpha) = H_{i-1} r_i - \sum_{k=a}^{i-1} (H_{k-1} - H_k) \, c_k .$$

We now obtain the optimal strategy as in the case of Markov process without discounting.

3.1 The Optimality in Presence of Discounting:

For some state $j(>i)$, $v_j(\alpha)$ is similarly defined as in (3.3) We shall call state i as the stopping state if

(3.4) $v_j(\alpha) - v_i(\alpha) \leq o$.

Using (3.3) and simplifying we get

(3.5) $r_j \leq A^*_{i-1,j-1}\, r_i + \sum_{k=i}^{j-1} B^*_{k,j-1}\, c_k$

where

$$A^*_{i-1,j-1} = H_{i-1} / H_{j-1} \text{ and}$$

$$B^*_{k,j-1} = (H_{k-1} - H_k)/ H_{j-1},$$

which is again a "Markovian" inequality since the stopping state can be arrived at by considering the reward at state i and the costs involved in transition from state i to state j-1.

4. An Absorbing Semi-Markov Process without Discounting:

Consider a time homogenous absorbing semi-Markov process with s transient states and r absorbing states. The states occupied on successive transitions are governed by the transition probability matrix $\bar{P}(t)$ of the imbedded Markov process. When the process enters some states, it is held at that state for some time prior to making transitions to some other state. The holding times do not necessarily take integral values. Let $h_{kl}(.)$ denote the holding time density function of the holding time τ_{kl} of the process staying at state k before making a transition to state l. The mean holding time at state k is then $\bar{\tau}_{kl} = \int_0^t \tau h_{kl}(\tau)d\tau$ and the mean waiting time in state k is $\bar{\tau}_k = \sum_{l=a}^{s} p_{kl}\, \bar{\tau}_{kl}$. We assume that these means are finite. Let $\bar{\tau}$ denote the

diagonal matrix of mean waiting times for the s states. Then the expected duration of the process in state k given that the process started at state a is the (a,k)th element of the matrix $\bar{e}'(t)$ $= [P(t)-I]^{-1}\tilde{\tau}$. Let this element be denoted by $e'_{ak}(t)$. If the state k is an absorbing one then the expected duration of stay in state k is denoted by $\mathcal{E}'_{ak}(t)$. The expected duration of stay matrix $\bar{e}'(t)$ is thus obtained. It is easy to see that $\mathcal{E}'_{ak}(t) + \sum_{k \in T} e'_{ak}(t) = 1$.

Let the process start at state a. Let $D'_{ab}(t)$ denote the duration of stay in state b in the interval $(o,t]$. Let c_b be the cost per unit duration in state b and let g_b be the cost of transition into state b from some other state. Then arguing as in the case of the Markov process without discounting, the total cost of the process up to state i is

$$\sum_{b=a}^{i-1} (D'_{ab}(t)c_b + g_b)$$

and the total earnings after the absorption in state i is

$$\left\{ t - \sum_{b=a}^{i-1} D'_{ab}(t) \right\} r_i .$$

The total return for the process stopped at state i is thus

$$(4.1) \quad R_i^{\ *} = \left\{ t - \sum_{b=a}^{i-1} D'_{ab}(t) \right\} r_i - \sum_{b=a}^{i-1} (D'_{ab}(t)c_b + g_b)$$

and the expected total return is

$$v_i^{\ *} = \left\{ t - \sum_{b=a}^{i-1} e'_{ab}(t) \right\} r_i - \sum_{b=a}^{i-1} (e'_{ab}(t)c_b + g_b)$$

$$(4.2) \qquad = \mathcal{E}'_{ai}(t)\ r_i - \sum_{b=a}^{i-1} (e'_{ab}(t)c_b + g_b).$$

4.1 The Optimality Inequality:

Suppose that j ($>$i) is some other state. We wish to find out a stopping state i such that

$$(4.3) \qquad v_j^* - v_i^* \leq 0 \ .$$

Using (4.2) and simplifying (4.3), we get

$$(4.4) \quad r_j \leq E_{ij} r_i + \sum_{b=i}^{j-1} (F_{bj}\, c_b + G_j g_b)$$

where $E_{ij} = \mathcal{E}'_{ai}(t) \,/\, \mathcal{E}'_{aj}(t)$, $F_{bj} = e'_{ab}(t) \,/\, \mathcal{E}'_{aj}(t)$ and

$G_j = 1/\, \mathcal{E}'_{aj}(t)$. The inequality (4.4) has the same characteristic as (2.4) and the coefficients E_{ij} and F_{bj} have similar interpretations as A_{ij} and B_{bj} therein. The presence of G_j in the inequality indicates that the decision to proceed on to state j is also dependent on the 'entrance' cost of state j weighted by the reciprocal of the duration of the process after absorption in that state.

5. The Semi-Markov Process with Discounting:

When the rewards as well as the costs are continuously discounted, proceeding as in the case of the absorbing Markov process with discounting, it is easy to establish that the total return at state i for a process started at state a and absorbed at state i is

$$(5.1) \ R_i^*(\alpha) = \frac{1}{\alpha}\left[(e^{-\alpha T_{i-1}} - e^{-\alpha t})r_i - \sum_{k=a}^{i-1}\left\{e^{-\alpha T_{k-1}}(1-e^{-\alpha(T_k - T_{k-1})})c_k + e^{-\alpha T_k} g_k\right\}\right].$$

To obtain the expected total return, we first of all evaluate $E\left(e^{-} D_{ab}(t)\right)$ The exponential transform of a function $F(.)$ is defined as

$$F^e(\alpha) = \int_0^t e^{-\alpha\tau} F(\tau)\, d\tau \ .$$

The exponential transform of the holding time distribution for state b given the holding time density function $h_{ab}(.)$ for a process at state b given the initial state a is

$$h_{bb}^e(\alpha) = \int_0^t e^{-\alpha\tau} h_{bb}(\tau)\, d\tau \ .$$

The waiting time distribution for state b will then have an exponential transform

$$W_b^{\,e}(\alpha) = \sum_{l=a}^{s} p_{bl}\, h_{bl}^{e}\ (\alpha),$$

so that the exponential transform of the duration of stay $D_{ab}(t)$ for state b given the initial state a becomes (cf. Howard, 1971),

$$(5.2)\quad E\left(e^{-\alpha D'_{ab}(t)}\right) = d_b^e(\alpha) = \frac{W_b^{\,e}(\alpha) - p_{bb}h_{bb}^{\,e}(\alpha)}{1-p_{bb}h_{bb}^{e}\ (\alpha)} = H'_{ab}(t),$$

say. Using (5.2) and taking the expectation of (5.1) yields the expected total return for a discounted semi- Markov process as

$$v_i^{*}(\alpha) = \frac{1}{\alpha}\left[\left(\prod_{b=a}^{i-1} H'_{ab}(t) - e^{-\alpha t}\right) r_i - \sum_{k=a}^{i-1}\left\{\left(\prod_{b=a}^{k-1} H'_{ab}(t) - \prod_{b=a}^{k} H'_{ab}(t)\right)c_k + \prod_{b=a}^{k} H'_{ab}(t)\, g_k\right\}\right].$$

If we let $H_n^{*} = \left(\prod_{b=a}^{n} H_{ab}(t) - e^{-\alpha t}\right)/\alpha$ and

$G_n^{*} = \alpha \prod_{b=a}^{n} H'_{ab}(t)$ then we have,

$$(5.3)\quad v_i^{*}\ (\alpha) = H_{i-1}^{*}\, r_i - \sum_{k=a}^{i-1}\left\{\left(H_{k-1}^{*} - H_k^{*}\right)c_k + G_k^{*}\, g_k\right\}.$$

We now obtain the optimal strategy for the absorbing semi-Markov process with discounting.

5.1 The Optimality in Presence of Discounting:

The stopping state i is given by the condition

$$(5.4)\qquad v_j^{*}\ (\alpha) - v_i^{*}\ (\alpha) \leq 0$$

for all j ($>$i). Using (5.3) we get the optimality criterion as

$$(5.5)\quad r_j \leq E^{*}_{i-1,j-1}\, r_i + \sum_{k=i}^{j-1}\left(F^{*}_{k,j-1}\, c_k + G^{*}_{k,j-1}\, g_k\right)$$

where $E^{*}_{i-1,j-1} = H^{*}_{i-1}/\,H^{*}_{j-1}$, $F^{*}_{k,j-1} = (H^{*}_{k-1} - H^{*}_{k})\,/\,H^{*}_{j-1}$ and $G^{*}_{k,j-1} = G^{*}_{k}\,/\,H^{*}_{j-1}$. The coefficients of r_i and c_k have the interpretations as those in (4.4). The 'Markovian' characteristic is again retained.

6. Concluding Remarks:

It is seen that if the transition probability matrix, the cost factors and in case of discounted case, the discounting factor are known in advance, as we have assumed herein, the calculations of the coefficients of the costs factors in the Markovian optimality criterion pose no difficulty. Specifically if the holding time distributions and waiting time distributions in the semi-Markov process are known, the coefficients A,B,E,F and G's are easily derived. This means, given the cost factors, it is possible to pick-out the stopping state with little difficulty and it is not essential that the process be observed through its transitions during the given duration for the same.

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On a Structure Defined by a Tensor Field f ($\neq 0$) of Type (1,1) Satisfying $f^5 - a^2 f = 0$

Ram Nivas and C.S. Prasad

Summary:

Andreou [1] has studied the structure defined by a tensor field f ($\neq$ 0) of type (1,1) satisfying $f^5 + f = 0$. In the present paper, we have defined and studied f_a (5,1) - structure. We have also obtained a positive definite Riemannian metric with respect to which the complementary distributions are orthogonal.

1. Let M^n be an n-dimensional differentiable manifold of class C^∞ Suppose there exists on M^n , a (1,1) tensor field $f (\neq 0)$ satisfying

(1.1) $$f^5 - a^2 f = 0$$

where 'a' is a complex number not equal to zero. If a = i where $i = \sqrt{-1}$, our structure takes the form $f^5 + f = 0$ studied by Andreou [1].

Let us define on M^n, the operators l and m as follows:

(1.2) $l \overset{\text{def}}{=\!=\!=} (f^4/a^2)$ and $m \overset{\text{def}}{=\!=\!=} I - (f^4/a^2)$, I being unit tensor field.

In view of equations (1.1) and (1.2), we have

(1.3) $l^2 = l$, $m^2 = m$ and $l + m = I$. Thus we have

Theorem (1.1). For a tensor field f ($\neq$ 0) of type (1,1) satisfying (1.1) the operators l and m defined by (1.2), when applied to the tangent space of M^n at a point, are complementary projection operators.

Thus there exist complementary distributions L and M corresponding to projection operators l and m respectively. If the rank of f is constant every where and equal to r, the dimensions of L and M are r and (n-r) respectively [4]. Let us call such a structure as $f_a(5,1)$-structure of rank r.

Theorem (1.2). For a tensor field $f(\neq 0)$ of type (1,1) admitting $f_a(5,1)$-structure and for the projection operators l and m given by (1.2) we have

(1.4) $\quad fl = lf = f, \quad fm = mf = 0.$

Also

(1.5) $\quad f^2 l = lf^2 = f^2, \quad f^2 m = mf^2 = 0.$

Proof. The proof follows easily by virtue of equations (1.1) & (1.2).

Theorem (1.3). Let M^n be an n-dimensional differentiable manifold equipped with $f_a(5,1)$-structure. If f is of maximal rank, f^2 gives a GF-structure on M^n [2].

Proof. The rank of f is maximal if r = n. Thus dim L = n and dim M = 0. Consequently m = 0 and l = I. Hence in view of (1.2), we get

$$(f^4/a^2) = I$$

or

(1.5) $\quad f^4 = a^2 I.$

Thus f^2 gives a GF-structure on M^n.

Theorem (1.4). In the manifold M^n endowed with $f_a(5,1)$-structure, the (1,1) tensor field F given by $F \overset{\text{def}}{=\!=\!=} (2f^4/a^2) - I$ gives an almost product structure [3].

Proof. We have

$$F^2 = (4f^8/a^4) - (4f^4/a^2) + I$$

or

(1.6) $\quad F^2 = I$ by virtue of (1.1).

Hence F gives an almost product structure on M^n [3].

Theorem (1.5). Let p, q be tensors on M^n defined as follows:

(1.7) $\quad p = m + (f^2/a), \quad q = m - (f^2/a)$

Then we have

(1.8) $\quad pq = qp = m - l.$

Proof. The proof follows easily by virtue of equations (1.1) & (1.2).

2. Let us now introduce in the manifold M^n a local coordinate system and denote by f_i^h, l_i^h and m_i^h the local components of f, l and m respectively. We also introduce in M^n, a positive definite Riemannian metric by taking r mutually orthogonal unit vectors u_a^h in L (a,b,c, ... = 1,2,...,r) and (n-r) mutually orthogonal unit vectors u_B^h (A,B,C, ... = 1,2,..., n-r) in M. Thus we have [4]

$$l_i^h u_b^i = u_b^h , \quad l_i^h u_B^i = 0 ; \tag{2.1}$$

$$m_i^h u_b^i = 0 , \quad m_i^h u_B^i = u_B^h .$$

Let (v_i^a , v_i^A) be the matrix inverse to (u_b^h , u_B^h). Then v_i^a and v_i^A are components of linearly independent covariant vectors satisfying

$$v_i^a u_b^i = \delta_b^a , \quad v_i^a u_B^i = 0 ;$$

$$v_i^A u_b^i = 0 , \quad v_i^A u_B^i = \delta_B^A , \tag{2.2}$$

δ_j^i being Kronecker delta. Also

$$v_i^a u_a^h + v_i^A u_A^h = \delta_i^h . \tag{2.3}$$

In view of equations (2.1) and (2.2), we have

$$(l_i^h v_h^a) u_b^i = \delta_b^a , \quad (l_i^h v_h^a) u_B^i = 0 ;$$

$$(m_i^h v_h^A) u_b^i = 0 , \quad (m_i^h v_h^A) u_B^i = \delta_B^A . \tag{2.4}$$

Thus we have

$$l_i^h v_h^a = v_i^a , \quad l_i^h v_h^A = 0 ;$$

$$m_i^h v_h^a = 0 , \quad m_i^h v_h^A = v_i^A . \tag{2.5}$$

Since $fm = 0$, we have $f_i^h m_h^j = 0$. Hence contracting with v_j^A and making use of (2.5), we obtain

$$f_i^h v_h^A = 0 . \tag{2.6}$$

Further, since

$$l_j^h u_a^j = u_a^h , \text{ we have}$$

$$l_j^h u_a^j v_i^a = v_i^a u_a^h$$

or

$$l_j^h (\delta_i^j - v_i^A u_A^j) = v_i^a u_a^h$$

or

(2.7) $$l_i^h = v_i^a \, u_a^h \text{ by virtue of (2.1) and (2.3).}$$

Similarly

(2.8) $$m_i^h = v_i^A \, u_A^h \; .$$

Let us now put

(2.9) $$g_{ji} = v_j^a \, v_i^a + v_j^A \, v_i^A \; .$$

Then g_{ji} is globally defined positive definite Riemannian metric with respect to which (u_b^h, u_B^h) form an orthogonal frame and such that

(2.10) $$v_j^a = g_{ji} \, u_a^i \; , \quad v_j^A = g_{ji} \, u_A^i \; .$$

If we further put

(2.11) $$l_{ji} = l_j^t \, g_{ti} \quad \text{and} \quad m_{ji} = m_j^t \, g_{ti} \; ,$$

we have in view of (2.7) and (2.8)

(2.12) $$l_{ji} = v_j^a \, v_i^a \; , \quad m_{ji} = v_j^A \, v_i^A$$

and consequently

(2.13) $$l_{ji} + m_{ji} = g_{ji} \; .$$

The following equations can be proved easily

(2.14)
$$\text{(i)} \quad l_j^t \, l_i^s \, g_{ts} = l_{ji} \; ,$$
$$\text{(ii)} \quad l_j^t \, m_i^s \, g_{ts} = 0$$
and
$$\text{(iii)} \quad m_j^t \, m_i^s \, g_{ts} = m_{ji} \; .$$

If we now put

$$G_{ji} = \tfrac{1}{2} \left(g_{ji} + m_{ji} + f_t^s \, f_s^t \, g_{ij} \right) \; ,$$

then G_{ji} is again globally defined Riemannian metric satisfying

(2.16)
$$\text{(i)} \quad v_j^A = G_{ji} \, u_A^i$$
and
$$\text{(ii)} \quad m_{ji} = m_j^t \, G_{ti} .$$

Now

$$G(u_a, u_A) = \tfrac{1}{2} \left\{ g(u_a, u_A) + m(u_a, u_A) + f_t^s \, f_s^t \, g_{ij} \, u_a^i \, u_A^j \right\} = 0$$

by virtue of the fact that the distributions L and M are orthogonal with respect to Riemannian metric g. Thus L and M are orthogonal with respect to G also. Consequently, we have the following theorem.

Theorem (2.1). Let M^n be an n-dimensional differentiable manifold equipped with $f_a(5,1)$-structure of rank r. Then there exist complementary distributions L and M and a positive definite Riemannian metric G with respect to which the distributions are orthogonal.

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Department of Mathematics
and Astronomy,
Lucknow University,
Lucknow (India).

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On Semi-P2-Like and P -Finsler Spaces

U.P. Singh*

B.N. Gupta**

The (v)hv-torsion tensor P_{ijk} of a C2-like Finsler space leads to define a semi-P2-like Finsler space. A P_λ-Finsler space is a generalization of P* -Finsler space and semi-P2-like Finsler space. It has been shown that semi-P2-like and P_λ-Finsler spaces are C2-like Finsler spaces under certain conditions. The properties of hv-curvature tensors P_{hijk} of these spaces have been studied.

1. Introduction:

Let F_n $(n \geq 2)$ be an n-dimensional Finsler space equipped with the fundamental function L (x, y). The (h)hv-torsion tensor

C_{ijk} $\left(=\frac{1}{4}\frac{\partial^3 L^2}{\partial y^i \partial y^j \partial y^k}\right)$ of F_n satisfies the following identities

$C_{ojk} = C_{jok} = C_{jko} = o$

where o stands for transvection with respect to y^i. The hv-curvature tensor P_{hijk} of F_n is written in the form ([6])

$$(1.1)\quad P_{hijk} = \mathcal{U}_{(hi)} \left\{ C_{ijk|h} + C^r_{hj}\, C_{rik|o} \right\}$$

which is rewritten in terms of (v)hv-torsion tensor P_{ijk} as ([8])

$$(1.2)\quad P_{hijk} = \mathcal{U}_{(hi)} \quad P_{ijk|h} + P_{ikr}\, C^r_{jh}$$

where $C_{ijk|h}$ and $P_{ijk|h}$ stands for h-covariant differentiation with respect to x^h and v-covariant differentiation with respect to y^h respectively. $\mathcal{U}_{(hi)}$ denotes the interchange of indices h, i and subtraction.

* Department of Mathematics, University of Gorakhpur, Gorakhpur 273001

** Department of Mathematics, Post Graduate College, Ghazipur 233001

The (v)hv-torsion tensor P_{ijk} and v-curvature tensor S_{hijk} of F_n are respectively given by

(1.3) $P_{ijk} = P_{oijk} = C_{ijk|o}$

(1.4) $S_{hijk} = \mathfrak{U}_{(jk)} \left\{ C_{hkr} \ C^r_{ij} \right\}$

The curvature tensors P_{hijk} and S_{hijk} have the following properties:

(1.5) $P_{hijk} = - P_{ihjk}$

(1.6) $S_{hijk} = - S_{ihjk} = - S_{hikj} = S_{jkhi}$

(1.7) $S_{hijk} + S_{hjki} + S_{hkij} = o$

A Finsler space with $P_{ijk} = o$ is called a Landsberg space ([11]).

A Finsler space with P_{ijk} of the form

(1.8) $P_{ijk} = \lambda C_{ijk}$.

where λ is a scalar function, is called a P* -Finsler space ([2],[3]), while a Finsler space with $P_{hijk} = P_{hikj}$ is called a P-symmetric Finsler space ([8]). If the h-curvature tensor R_{hijk} of F_n is written in the form

(1.9) $R_{hijk} = R \left(g_{hj} \ g_{ik} - g_{hk} \ g_{ij} \right)$

where R is a non-zero scalar, then F_n is called an h-isotropic Finsler space ([1],[4]). We quote the following result which will be used in this paper.

Lemma 1 In an h-isotropic Finsler space $P_{hijk} = P_{hikj}$ and $S_{hijk} = o$ ([1]).

A Finsler space F_n $(n \geq 3)$ is called P2-like ([5]) if there exists a covariant vector field P_i such that the hv-curvature tensor of F_n is written in the form

(1.10) $P_{ijkl} = P_i \ C_{jkl} - P_j \ C_{ikl}$.

A Finsler space F_n $(n \geq 2)$ with $C^2 = C_i C^i$ $(\neq o)$ is called C2-like ([9]) if C_{ijk} is written in the form

(1.11) $C_{ijk} = \frac{1}{C^2} \ C_i \ C_j \ C_k$

where $C_i = g^{jk} \ C_{ijk}$.

From this definition it follows that a non-Riemannian Finsler space F_n is C2-like if and only if C_{ijk} is written in the form

(1.12) $C_{ijk} = L_i\, C_j\, C_k + L_j\, C_k\, C_i + L_k\, C_i C_j$

where $L_i\ (= \frac{1}{3C^2} C_i\ )$ are components of a covariant vector field.

3. Some Properties of C2-Like Finsler Spaces:

The h-covariant differentiation of relation (1.11) with respect to x^1 and its transvection with respect to y^1 gives

(2.1) $P_{ijk} = G_i C_j C_k + G_j C_k C_i + G_k C_i C_j - \emptyset\ C_i C_j C_k$

where we have put

$$G_i = \frac{1}{C^2}\ C_{i|o}\ , \quad \emptyset = \frac{2}{C^2}\ G_i C^i\ .$$

The right hand side of (2.1) equated to zero after contraction with respect to g^{jk} gives $C_{i|o} = o$. Again $C_{i|o} = o$ yields $P_{ijk} = o$.

Thus we have

Theorem 2.1 A C2-like Finsler space is a Landsberg space if and only if $C_{i|o} = o$.

In a P* - Finsler space we have in view of (1.8), $C_{i|o} = \lambda\, C_i$.

Conversely if $C_{i|o} = \lambda C_i$ and F_n is C2-like, then (2.1) gives $P_{ijk} = \lambda\, C_{ijk}$. Hence we have

Theorem 2.2 A C2-like Finsler space is a P* - Finsler space if and only if $C_{i|o} = \lambda\, C_i$.

Substituting from equations (1.11) and (2.1) into equation (1.1) we get the following form of hv-curvature tensor

(2.2) $P_{hijk} = C_j C_k E_{ih} + C_j M_{khi} + C_k M_{jhi}$

where we put

(2.3) $E_{ih} = \frac{1}{C^2}\ (\ C_{i|h} - C_{h|i} + C_h Q_i - C_i Q_h\ )$

(2.4) $Q_i = \frac{1}{C^2} \; (2C^1 \, C_{1|i} + C^4 \, G_i)$

(2.5) $M_{khi} = \frac{1}{C^2} \, (C_{k|h} \, C_i - C_{k|i} \, C_h)$.

Contracting equation (2.2) with respect to y^h and using equations (1.3), (2.3) to (2.5) we get the relation (2.1) which proves the following:

Theorem 2.3 If the hv-curvature tensor of a Finsler space is written in the form (2.2), then its (v)hv-torsion tensor is written in the form (2.1).

The following theorem can be easily deduced with the help of (2.2)

Theorem 2.4 The hv-curvature tensor of a C2-like Finsler space F_n satisfies the identities

(a) $P_{hijk} = P_{hikj}$

i.e. F_n is P-symmetric.

(b) $P_{hijk} + P_{ikjh} + P_{khji} = o$

In view of relations (1.11), (2.2) to (2.5) we have

(2.6) $P_{hijk} = K_h C_{ijk} - K_i C_{hjk} + M_{hijk}$

where $K_h = -Q_h$,

$$M_{hijk} = \frac{1}{C^2} \; (C_{i|h} - C_{h|i}) + C_j \, M_{khi} + C_k M_{jhi} .$$

Equation (2.6) yields the following:

Theorem 2.5 A C2-like Finsler space F_n $(n \geq 3)$ is P2-like if and only if there exists a covariant vector field $U_i (\neq - K_i)$ such that

$$M_{hijk} = U_h \, C_{ijk} - U_i \, C_{hjk}$$

3. Semi-P2-Like Finsler Spaces and its properties:

We consider an n-dimensional Finsler space F_n with the (v) hv-torsion tensor P_{ijk} of a special form

(3.1) $P_{ijk} = B_i C_j C_k + B_j C_k C_i + B_k C_i C_j$

where B_i is an indicatory vector field, positively homogeneous of degree two in y^i. From (3.1) it easily follows that the vector B_i is necessarily given by

(3.2) $B_i = \frac{1}{C^2} C_{i|o} - \frac{2}{3} \frac{C_{1|o} C^1}{C^4} C_i$.

This situation is similar to the case of C2-likeness. In two dimensional Finsler space we have ([10])

$$P_{ijk} = J_s m_i m_j m_k$$

$$C_{ijk} = \frac{J}{L} m_i m_j m_k$$

$$C_i = \frac{J}{L} m_i$$

where m_i is a unit vector normal to supporting element, J_s and J are scalar functions.

These relations show that in every two dimensional Finsler space the tensor P_{ijk} can be expressed in the form (3.1). Thus we introduce

Definition A non-Riemannian Finsler space F_n $(n \geq 2)$ is called semi-P2-like if the (v) hv-torsion tensor P_{ijk} of F_n is written in the form (3.1).

The right hand side of relation (3.1) equated to zero after contraction with respect to g^{jk} gives $B_i = o$. Conversely $B_i = o$ yields $P_{ijk} = o$. Thus we have

Theorem 3.1 A semi-P2-like Finsler space is a Landsberg space if and only if $B_i = o$.

We have concrete examples of non-Landsberg semi-P2-like Finsler spaces. If we put $G_i - \frac{\phi}{3} C_i = W_i$ in relation (2.1), then it takes the form (3.1). Thus we see that any C2-like Finsler space is semi-P2-like. We have certain examples of C2-like Finsler spaces. Some of them are given below, while other examples will be given in next article.

We suppose that a semi-P2-like Finsler space is a P* -Finsler space characterized by (1.8). In view of relations (1.8) and (3.1) we obtain

$$C_{ijk} = \frac{B_i}{\lambda} C_j C_k + \frac{B_j}{\lambda} C_k C_i + \frac{B_k}{\lambda} C_i C_j .$$

Consequently (1.12) gives that the space under consideration is C2-like provided $\lambda \neq o$, while $\lambda = o$ yields that the space is Landsberg. Hence we have the following:

<u>Theorem 3.2</u> A non-Landsberg P* -Finsler space is semi-P2-like if and only if it is a C2-like Finsler space.

Now we take a P2-like Finsler space characterized by the hv-curvature tensor of the form (1.10). The contraction of (1.10) with respect to y^i gives

$$P_{ijk} = P_o \, C_{ijk}$$

which shows that a P2-like Finsler space F_n $(n \geq 3)$ is a P* -Finsler space. Therefore in view of theorem (3.2) we get the following:

<u>Theorem 3.3</u> The necessary and sufficient condition that a non-Landsberg P2-like Finsler space F_n $(n \geq 3)$ be semi-P2-like is that it is C2-like.

By virtue of relations (3.1), (3.2) and $C_{i|o} = \tau C_i$ (suppose) we have

$$(3.3) \qquad P_{ijk} = \frac{\tau}{C^2} C_i C_j C_k \, .$$

Conversly the relation (3.3) gives $C_{i|o} = \tau C_i$, which yields the following:

<u>Theorem 3.4</u> The (v)hv-torsion tensor of a semi-P2-like Finsler space is given by (3.3) if and only if $C_{i|o} = \tau C_i$.

To find the form of hv-curvature tensor P_{hijk} of semi-P2-like Finsler space, we have from (1.2) and (3.1)

$$(3.4) \quad P_{hijk} = \mathcal{U}_{(hi)} \Big\{ B_{i|h} C_j C_k + C_j Q_{ikh} + C_k Q_{ijh} + C_i L_{jkh} + L_{ik} B_{jh} + C_i C_k F_{jh} \Big\}$$

where (a) $Q_{ikh} = B_i C_{k|h} + C_i B_{k|h}$

(b) $L_{jkh} = B_j C_{k|h} + B_k C_{j|h}$

(3.5) (c) $L_{ik} = C_i B_k + B_i C_k$

(d) $B_{jh} = C_r C^r_{jh}$

(e) $F_{jh} = B_r C^r_{jh}$.

A direct calculation with the help of (3.4) and (3.5) yields the following theorem:

Theorem 3.5 The hv-curvature tensor P_{hijk} of a semi-P2-like Finsler space satisfies the identity

$$P_{hijk} + P_{ikjh} + P_{khji} = o$$

Next we consider a P-symmetric semi-P2-like Finsler space. In view of (3.4) and $P_{hijk} = P_{hikj}$ we get

(3.6) $$\mathcal{U}_{(hi)}\left\{ L_{ik}B_{jh} - L_{ij}B_{kh} + C_iC_kF_{jh} - C_iC_jF_{kh} \right\} = o$$

which gives the following:

Theorem 3.6 A semi-P2-like Finsler space is P-symmetric if and only if (3.6) holds good.

Now let us suppose that a semi-P2-like Finsler space admits a concurrent vector field X_i ([7]). Then $X^i{}_{|j} = -\delta^i_j$ and $X^i{}_{|j} = o$, which gives

$$X^h P_{hijk} + C_{ijk} = o$$

Contracting this relation with respect to y^i and X^i respectively we get

(3.7) $P_{ijk} X^i = o$, $C_{ijk} X^i = o$

From second relation of (3.7) we get $C_iX^i = o$. Consequently (3.1) with the help of these relations yields

$$B_iX^iC_jC_k = o$$

Contraction of above relation with g^{jk} will give $B_iX^i = o$. Hence we have:

<u>Theorem 3.7</u> If a semi-P2-like Finsler space characterized by (3.1) admits a concurrent vector field X_i, then the vector B_i is orthogonal to X_i.

4. P_λ -Finsler Space and its Properties:

Let F_n be a Finsler space with (v)hv-torsion tensor P_{ijk} of the following form

$$(4.1)\quad P_{ijk} = \lambda C_{ijk} + a_i C_j C_k + a_j C_k C_i + a_k C_i C_j$$

where λ is a scalar function and a_i are components of a covariant vector field. λ is homogeneous of degree one in y^i while a_i is homogeneous of degree two in y^i. Also $P_{ijo} = o$ leads to $a_o = o$. It is to be noted that in any two dimensional Finsler space P_{ijk} can be expressed in the form (4.1). Thus we introduce

<u>Definition</u> A non-Riemannian Finsler space F_n $(n \geq 2)$ is called P_λ-Finsler space, if the (v)hv-torsion tensor of F_n is written in the form (4.1).

From (4.1) it follows that a_i is necessarily given by

$$a_i = \frac{1}{C^2}\left[C_{i|o} - \left(\frac{\lambda}{3} + \frac{2}{3C^2} C^l C_{l|o} \right) C_i \right]$$

The relation (4.1) shows that a P_λ -Finsler space reduces to a P* -Finsler space when a_i vanishes identically while it reduces to a semi-P2-like Finsler space when $\lambda = o$. Thus a P_λ -Finsler space is a generalization of P* -Finsler space and semi-P2-like Finsler space. From relations (1.11) and (2.1) we get that a C2-like Finsler space is a P_λ -Finsler space.

If the P_λ -Finsler space is a Landsberg space then from (4.1) it follows that

$$C_{ijk} = -\frac{1}{\lambda}\left(a_i C_j C_k + a_j C_k C_i + a_k C_i C_j\right)$$

and $$a_i = -\frac{\lambda}{3C^2} C_i$$

Hence we have the following:

Theorem 4.1 If a Finsler space is Landsberg space, then it is a P_λ-Finsler space if and only if it is C2-like.

Now we suppose that a P_λ-Finsler space is semi-P2-like. Then (3.1) and (4.1) give

$$\lambda C_{ijk} = (B_i - a_i)C_jC_k + (B_j - a_j)C_kC_i + (B_k - a_k)C_iC_j$$

This shows that if $B_i = a_i$, then $\lambda = o$. Otherwise the space under consideration is C2-like in view of (1.12). Hence we have:

Theorem 4.2 The necessary and sufficient condition for a P_λ-Finsler space to be semi-P2-like is that either $\lambda = o$ or the space is C2-like.

Now we suppose that a P_λ-Finsler space is a P* -Finsler space with scalar coefficient $\mathcal{T}$. Then (4.1) and (1.8) give

$$(\mathcal{T} - \lambda)C_{ijk} = a_iC_jC_k + a_jC_kC_i + a_kC_iC_j$$

which shows that the space is C2-like for $\mathcal{T} \neq \lambda$. When $\mathcal{T} = \lambda$ we have $a_i = o$, therefore (4.1) gives that the space is P* -Finsler space, which we have already supposed. Hence we get the following:

Theorem 4.3 A P* -Finsler space with scalar coefficient $\mathcal{T}$ is a P_λ-Finsler space if and only if it is a C2-like Finsler space, provided that $\mathcal{T} \neq \lambda$.

In previous article we have shown that every P2-like Finsler space is P* -Finsler space. Therefore in consequence of theorem (4.3) we have:

Theorem 4.4 A P2-like Finsler space characterized by (1.10) is a P_λ-Finsler space F_n $(n \geq 3)$ if and only if it is C2-like, provided that $P_o \neq \lambda$.

In view of relations (1.2), (1.4) and (4.1) we get the hv- curvature tensor P_{hijk} of a P_λ-Finsler space in the form

$$(4.2) \quad P_{hijk} = P_hC_{ijk} - P_iC_{hjk} - \lambda S_{hijk}$$

$$+ \mathcal{U}_{(hi)} \left\{ a_{i|h} C_j C_k + C_j Q'_{ikh} + C_k Q'_{ijh} + C_i L'_{jkh} + L'_{ik} B'_{jh} + C_i C_k F'_{jh} \right\}$$

where $P_h = \lambda_{|h}$ and the quantities with dashes have the same value as given in (3.5) except a_i replacing B_i. The form (4.2) of P_{hijk} shows that it is more general than the forms (1.10) and (3.4) of P_{hijk} in P2-like and semi-P2-like Finsler spaces respectively.

A direct calculation based on (4.2), (1.6) and (1.7) gives:

Theorem 4.5 The hv-curvature tensor of a P_λ-Finsler space satisfies the identity

$$P_{hijk} + P_{ikjh} + P_{khji} = o$$

Remark: It is to be noted that the above identity also holds in a semi-P2-like Finsler space.

Next we consider a P_λ-Finsler space and P-symmetric Finsler space. By virute of (1.6), (4.2) and $P_{hijk} = P_{hikj}$ we get

$$(4.3) \quad S_{hijk} = \frac{1}{2\lambda} \mathcal{U}_{(hi)} \left\{ L'_{ik} B'_{jh} - L'_{ij} B'_{kh} + C_i C_k F'_{jh} - C_i C_j F'_{kh} \right\}$$

which yields the following:

Theorem 4.6 A P_λ-Finsler space is P-symmetric if and only if its v-curvature tensor S_{hijk} is written in the form (4.3).

Now let us suppose that a P_λ-Finsler space is h-isotropic. Then by virtue of Lemma 1 it is P-symmetric and $S_{hijk} = o$. Hence from (4.3) we get

$$(4.4) \quad \mathcal{U}_{(hi)} \left\{ L'_{ik} B'_{jh} - L'_{ij} B'_{kh} + C_i C_k F'_{jh} - C_i C_j F'_{kh} \right\} = o$$

This gives the following:

Theorem 4.7 If a P_λ-Finsler space is h-isotropic, then the relation (4.4) holds good.

The relation (4.2) can be rewritten as

$$(4.5) \quad P_{hijk} = P_h C_{ijk} - P_i C_{hjk} + D_{hijk}$$

where

$$D_{hijk} = \mathfrak{A}_{(hi)}\Big\{ a_{i\,|\,h} C_j C_k + C_j Q'_{ikh} + C_k Q'_{ijh} + C_i L'_{jkh} + L'_{ik} B'_{jh} + C_i C_k F'_{jh} \Big\} - \lambda S_{hijk}$$

In consequence of (4.5) we have:

Theorem 4.8 A P_λ-Finsler space F_n $(n \geq 3)$ is P2-like if and only if there exists a covariant vector field $D_h (\neq -P_h)$ such that

$$D_{hijk} = D_h\, C_{ijk} - D_i\, C_{hjk}$$

Finally we suppose that a P_λ-Finsler space admits a concurrent vector field X_i. Then the relations (3.7) and (4.1) yield

$$a_i\, X^i\, C_j\, C_k = o$$

which after contraction with g^{jk} gives $a_i\, X^i = o$. Thus we have the following:

Theorem 4.9 If a P_λ-Finsler space admits a concurrent vector field X_i then the vector a_i is orthogonal to X_i.

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A Note on an Asymptotic Problem Concerning the Laplace Transform

R.M. Shreshtha*

1. Introduction:

The existence and asymptotic behaviour of a function $\phi(t)$ defined by

$$(1.1) \quad \int_0^\infty e^{-st}\, \phi(t)\, dt = \Phi\left(\left(\int_0^\infty e^{-st} f(t)\, dt\right)^m\right),$$

where $m \in \mathbb{Z}$, $\Phi(0) = 0$, $\Phi(z)$ is analytic for $|z| < R$ and $f \in L'(0, \infty)$ with $\int_0^\infty |f(t)|\, dt < R$, has been studied quite recently [1]. In the present note, we intend to sharpen the results in [1] by replacing the condition on the asymptotic behaviour of f with a weaker condition.

To begin with, we recall the following definitions [2,3].

Definition 1. A function $L(x)$, defined for all $x > 0$, will be called admissible whenever it is continuous and strictly positive for all $x > 0$, and satisfies the following two conditions

i) for all $h > 0$, $\lim\limits_{x \to \infty} \dfrac{L(x + h)}{L(x)} = 1$;

ii) there exists a constant $\lambda \geq 1$ such that

$$\max_{x \leq t \leq 2x} L(t) \leq \lambda\, L(2x).$$

The set of all admissible functions will be denoted by Λ.

Definition 2. If $f, g \in L'(0, \infty)$, then the convolution product of f and g, denoted by $f*g$, is defined by

$$(1.2) \quad (f*g)(x) := \int_0^x f(x - y)\, g(y)\, dy.$$

For convenience, we shall adopt the following notations for convolution of f with itself:

$f_1 := f$, and $f_k := f_{k-1} * f$, for all $k = 2,3,4,5\ldots$

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* This work was done with the support of Humboldt Fundation.

2. Existence Theorem

Theorem 1. If $\Phi(z)$ is analytic for $/z/ < R$, $(R > 0)$, $\Phi(0) = 0$ and $f \in L'(0, \infty)$ satisfies

$$/\int_0^\infty e^{-st} f(t)\, dt / \leq r < R$$

for all Re $s \geq 0$, then there is a $\phi \in L'(0, \infty)$ such that

$$(2.2) \qquad \int_0^\infty e^{-st} \phi(t)\, dt = \Phi((\int_0^\infty e^{-st} f(t)\, dt)^m),$$

$m \in \mathbb{Z}$, $\mathrm{Re}(s) \geq 0$.

Note: In [1], this theorem is proved with condition (2.1) replaced by

$$(2.3) \qquad \int_0^\infty /f(t)/\, dt < R$$

which obviously implies (2.1).

Proof. From (2.1), it readily follows that

$$(2.4) \qquad \Phi((\int_0^\infty e^{-st} f(t)\, dt)^m) = \sum_{n=0}^{\infty} \frac{\Phi^{(n)}(0)}{n!} (\int_0^\infty e^{-st} f(t)\, dt)^{mn}$$

$$= \sum_{n=1}^{\infty} \frac{\Phi^{(n)}(0)}{n!} \int_0^\infty e^{-st} f_{mn}(t)\, dt.$$

Now, since (Dym and Mokean, [4, Ex. 2.6.2]),

$$(2.5) \qquad \lim_{p \to \infty} \|f_p\|_1^{1/p} = \|\hat{f}\|_\infty,$$

where $\hat{f}$ denotes the Fourier transform of f, we have that

$$(2.6) \qquad \lim_{p \to \infty} \|f_p\|_1^{1/p} \leq r,$$

and, consequently,

$$(2.7) \qquad \phi(t) = \sum_{n=1}^{\infty} \frac{\Phi^{(n)}(0)}{n!} f_{mn}(t)$$

defines $\phi \in L'(0, \infty)$. So, we obtain

$$\sum_{n=1}^{\infty} \frac{\Phi^{(n)}(0)}{n!} \int_0^{\infty} e^{-st} f_{mn}(t)\, dt$$

$$= \int_0^{\infty} \left\{ \sum_{n=1}^{\infty} \frac{\Phi^{(n)}(0)}{n!} f_{mn}(t)\, dt \right\} e^{-st}\, dt.$$

From (2.4), then the theorem follows:

In particular, if we take m = 1, we arrive at the result due to Eggermont [5].

3. The Asymptotic Behaviour of the Original Function:

In order to discuss the asymptotic behaviour of the original function whose Laplace Transform is of the form

$$\Phi\left(\left(\int_0^{\infty} e^{-st} f(t)\, dt\right)^m\right),$$

we need the following lemma:

Lemma 3.1. If $f \in L'(0,\infty)$ and if $f(x) \sim \ell L(x)$, $x \to \infty$, for some $L \in \Delta$, and

$$(3.1) \qquad \left| \int_0^{\infty} e^{-st} f(t)\, dt \right| \leq r,$$

for all Re $s > 0$, then for all $k = 1,2,3,\ldots$,

$$(3.2) \qquad f_k(x) \sim k \ell L(x) \left(\int_0^{\infty} f(t)\, dt \right)^{k-1}, \quad x \to \infty$$

and for every $\varepsilon > 0$, there exists a constant λ and an $x_0 > 0$ such that for all $x > x_0$, and for all $k = 1,2,3,\ldots$,

$$(3.3) \qquad |f_k(x)| \leq k(|\ell| + 1)\, \lambda L(x)\, (r + \varepsilon)^{k-1}$$

The first conclusion (3.2) has been proved in [2] and formula (3.3) in [5]. Formula (3.3) as remarked by Eggermont sharpens the result in [2], where $(r + \varepsilon)^{k-1}$ is replaced by

$$(\cdot \int_0^\infty |f(t)|\, dt)^{k-1}$$

Theorem 3.1. If $\Phi(z)$ is analytic for $|z| < R$, $\Phi(0) = 0$ and $f \in L'(0, \infty)$ satisfies

$$(3.4) \qquad f(t) \sim \ell L(t), \qquad t \to \infty$$

for some $L(t) \in \Lambda$, and for all Re $s \geq 0$

$$(3.5) \qquad \left| \int_0^\infty e^{-st} f(t)\, dt \right| \leq r < R,$$

then $\emptyset(t)$, defined by

$$(3.6) \qquad \int_0^\infty e^{-st} \emptyset(t)\, dt = \Phi((\int_0^\infty e^{-st} f(t)\, dt)^m), \quad m \in \mathbb{Z},$$

satisfies

$$(3.7) \qquad \emptyset(t) \sim m \ell L(t) \left(\int_0^\infty f(t)\, dt \right)^{m-1} \Phi' \left((\int_0^\infty f(t)\, dt)^m \right).$$

as $t \to \infty$.

Proof. In view of (2.7), we have

$$(3.8) \qquad \emptyset(t) = \sum_{n=1}^\infty \frac{\Phi^{(n)}(0)}{n!} f_{mn}(t),$$

as the required function, whose asymptotic behaviour we intend to determine. Also, from Lemma (3.1), it follows that the series

$$(3.9) \qquad \sum_{n=1}^\infty \frac{\Phi^{(n)}(0)}{n!} \cdot \frac{f_{mn}(t)}{L(t)}$$

converges for large t. Hence

$$\lim_{t \to \infty} \frac{\emptyset(t)}{L(t)} = \sum_{n=1}^\infty \frac{\Phi^{(n)}(0)}{n!} \left(\lim_{t \to \infty} \frac{f_{mn}(t)}{L(t)} \right)$$

$$= \sum_{n=1}^\infty \frac{\Phi^{(n)}(0)}{n!}\, mn \ell \left(\int_0^\infty f(t)\, dt \right)^{mn-1}$$

$$= m\,\ell\,\left(\int_0^\infty f(t)\,dt \right)^{m-1} \Phi'\left(\left(\int_0^\infty f(t)\,dt \right)^m\right).$$

That is, $\phi(t) \sim m\,\ell\,L(t)\,\left(\int_0^\infty f(t)\,dt \right)^{m-1} \Phi'\left(\left(\int_0^\infty f(t)\,dt\right)^m\right)$.

This proves the theorem.

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Temporary Address
Mathematisches Institut
der Universitat
Am Hubland
8700 Wurzburg
West Germany.

Permanent Address
Mathematics Department
Tribhuvan University
Kirtipur, Kathmandu
Nepal.

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