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Elementary Euclidean Geometry and Algebra - A Dualist Point of View

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The duality in the title first bothered the ancient Greeks. To put it in our XXth century language, they noted both that the coordinate field of Euclidean geometry is and that the field of rational numbers isn't Pythagorean (discovery of incommensurable magnitudes (cf. [16])).

The first who realized the duality was René Descartes, a philosopher, celebrated by Hegel as the true founder of modern philosophy. His "Géometrie" (1637), an appendix to his famous "Discours de la méthode", contained implicitly the idea that - what we today call - a 'Cartesian plane' may serve as a model for Euclidean geometry. It took more than 250 years to prove the converse, i.e. that any model of Euclidean geometry, based on a complete axiom - system (AS), is isomorphic to the Cartesian plane constructed over the field of real numbers $\underline{\underline{R}}$. And this was done by the most influential mathematician of the early XXth century, David Hilbert in 1899 ([15]). In between (around 1797 cf.[8]), C.F. Gauss, the "princeps mathematicorum", discovered the geometric interpretation of comples numbers, of what we shall call a 'Gauss plane'.

The works of Descartes, Gauss and Hilbert were startingpoints. It is the purpose of this paper to convince the reader how fruitful a field of research they opened. Now, there are two essentially different ways of thinking of a duality between geometry and algebra. First, think of a faithful correspondence between the points of a line and the elements of an algebraic structure; an operation called at the empirical level measuring. This first way originated with Descartes. The second way originated with Felix Klein's Erlangen Programme and consists of thinking of a group of transformations as an algebraic synonym for a given geometry. This way was followed by Bachmann [2] and will not be

discussed here. The present paper is aimed to answer the following question: "Which algebraic structures are, geometrically speaking, 'coordinate lines'?".

We shall be concerned only with theories formalized within first - order logic. A set $\underline{T}$ of first - order sentences is called a theory whenever $Cn(\underline{T}) = \underline{T}$, where by $Cn(\underline{T})$ we understand the set of all logical consequences of $\underline{T}$. A theory $\underline{T}$ is said to be complete iff for any sentence φ we have either $\varphi \in \underline{T}$ or $\neg\varphi \in \underline{T}$. We say that $\underline{T}$ is decidable if one can decide algorithmically whether for a given sentence φ $\varphi \in \underline{T}$ or $\neg(\varphi \in \underline{T})$. If $\underline{T}$ is a first - order theory in a language L_{BD}, containing (besides logical constants) only the ternary predicate B and the quaternary predicate D, then a model of T is a structure $\underline{M} = (u(\underline{M}), B_{\underline{M}}, D_{\underline{M}})$, where $u(\underline{M})$ is a set, called the universe of the model and $B_{\underline{M}} \subset u(\underline{M})^3$, $D_{\underline{M}} \subset u(\underline{M})^4$, such that any sentence of $\underline{T}$ is satisfied in $\underline{M}$ (the notions of model and satisfaction were introduced in [40]). A model $\underline{M}$ is said to have cardinality x iff $u(\underline{M})$ has cardinality x. The class of models of $\underline{T}$ will be denoted by $Mod(\underline{T})$. First - order logic, although very convenient from the model - theoretic viewpoint (one can often describe all the models of a first - order theory), has the curious property called

<u>The Löwenheim - Skolem Theorem</u>.

<u>If a theory $\underline{T}$ has an infinite model, then it has a model of any given infinite cardinality</u>.

Therefore, Hilbert's result, by which 'Euclidean geometry' has (up to isomorphism) a single model cannot be obtained within first - order logic. His 'Euclidean geometry' is based on a second - order AS. The task of expressing Euclidean geometry in first - order logic was successfully solved by Alfred Tarski (an influential philosopher, too (cf. [18])). With the simplifications due to his graduate student, H.N. Gupta [11] it reads (in L_{BD}, B and D having the following intuitive

meanings: 'B(abc)' iff' b lies between a and c' and 'D(abcd)' iff 'the segment ab is congruent to the segment cd'):

A 1. (Inner transitivity property of Betweenness)

$$\forall xyzu \;\; B(xyu) \wedge B(yzu) \longrightarrow B(syz)$$

A 2. (Reflexivity property of Equidistance)

$$\forall xy \;\; D(xyyx)$$

A 3. (Identity property for Equidistance)

$$\forall xyz \;\; D(xyzz) \longrightarrow (x=y)$$

A 4. (Transitivity property for Equidistance)

$$\forall xyzuvw \;\; D(xyzu) \wedge D(vwzu) \longrightarrow D(xyvw)$$

A 5. (Pasch's axiom)

$$\forall txyzu \; \exists \, v \;\; B(xtu) \wedge B(yuz) \longrightarrow B(xvy) \wedge B(ztv)$$

A 6. (Euclid's axiom)

$$\forall txyzu \; \exists \, vw \;\; B(sut) \wedge B(yuz) \wedge \neg(x=u) \longrightarrow B(xzv) \wedge$$
$$\wedge B(xyw) \wedge B(vtw)$$

A 7. (Five - segment axiom)

$$\forall xx'yy'zz'uu' \;\; D(xyx'y') \wedge D(yzy'z') \wedge D(xux'u') \wedge$$
$$\wedge D(yuy'u') \wedge B(xyz) \wedge B(x'y'z') \wedge \neg(x=y) \longrightarrow D(zuz'u')$$

A 8. (Zxiom of segment construction)

$$\forall xyuv \; \exists \, z \;\; B(xyz) \wedge D(yzuv)$$

A 9. (Lower - dimension axiom)

$$\exists \, xyz \;\; \neg B(xyz) \wedge \neg B(yzx) \wedge \neg B(zxy)$$

A 10. (Upper - dimension axiom)

$$\forall xyzuv \;\; D(xuxv) \wedge D(yuyv) \wedge D(zuzy) \wedge \neg(u=v) \longrightarrow$$
$$\longrightarrow B(xyz) \vee B(yzx) \vee B(zxy)$$

A 11. (Elementary continuity axiom of Cantor - Dedekind)

All sentences of the form

$$\forall vw\ldots \; \Big\{ [\exists \, z \; \forall xy \;\; \varphi(x) \wedge \psi(y) \longrightarrow B(zxy)] \longrightarrow$$
$$\longrightarrow [\exists \, u \; \forall xy \;\; \varphi(x) \wedge \psi(y) \longrightarrow B(xuy) \vee (u=x) \vee (u=y)] \Big\}$$

where $\varphi(x)$ stands for any formula in which the variables x,v,w,... but neither y nor z nor u, occur free, and similarly for $\psi(y)$, with x and y interchanged.

Each of its axioms is independent of all the remaining ones, i.e. $A\,i \notin Cn(A\,1,\ldots,\widehat{A\,i},\ldots,A\,11)$, for $i = 1,\ldots,11$, where $\hat{A}$ means 'A omitted'.

Let $\ddot{\underline{E}}_2 = Cn(A\,1 - A\,11)$. $\ddot{\underline{E}}_2$ is not finitely axiomatizable (cf. [42]). The following representation theorem holds for $\ddot{\underline{E}}_2$ (cf. [42], [29], [6]):

Any model of $\ddot{\underline{E}}_2$ is isomorphic to a Cartesian plane constructed over a real closed field.

A Cartesian space constructed over an ordered field F is the following structure

$$\underline{\underline{C}}_n(F) = (F^n, D_F, B_F)$$

with $D_F = \left\{(a,b,c,d) \in (F^n)^4 \mid \|a - b\| = \|c - d\|\right\}$ and $B_F = \left\{(a,b,c) \in (F^n)^3 \mid \exists\ t \in F,\ 0 \leqslant t \leqslant 1 \text{ and } (a - b) = t(a - c)\right\}$, where $\|(a_1,\ldots,a_n)\| \sum_{i=1}^{n} a_i^2$ and $a - b = (a_1 - b_1,\ldots,a_n - b_n)$, for $a = (a_1,\ldots,a_n)$, $b = (b_1,\ldots,b_n)$, with $a_i, b_i \in F$. We shall also use the notations $\underline{\underline{D}}_n(F)$ and $\underline{\underline{B}}_n(F)$ for (F^n, D_F) and (F^n, B_F) respectively. $\underline{\underline{C}}_2(F)$ is called a Cartesian plane. With these notations, the above theorem reads

$$\underline{\underline{M}} \in \mathrm{Mod}(\ddot{\underline{E}}_2) \ \text{ iff } \ \underline{\underline{M}} \simeq \underline{\underline{C}}_2(F),$$

where F is a real closed field.

A real closed field is an ordered field for which any positive element has a square root and any polynomial of odd degree has a root therein. They were introduced and studied by Artin and Schreier [1] (cf. also [43]).

Tarski [41] also proved that $\ddot{\underline{E}}_2$ is decidable and complete (cf. also [28]). This is of interest to us because it proves that

$$\ddot{\underline{E}}_2 = \mathrm{Th}_{L_{BD}}(\underline{\underline{C}}_2(\underline{\underline{R}}))$$

$\mathrm{Th}_{L_{BD}}(\underline{\underline{C}}_2(\underline{\underline{R}}))$ being the theory containing all L_{BD} - sentences true in $\underline{\underline{C}}_2(\underline{\underline{R}})$. To put it more simply

$\ddot{\underline{E}}_2$ says anything one can first - order - say in Hilbert's Euclidean geometry.

An AS for $\underline{E}_n''$ (n - dimensional) is obtained by replacing the dimension axioms A 9 and A 10 by (e.g.):

$$A\ 9_n.\quad \exists x_0 x_1 \dots x_n \;\; \perp(x_1 x_0 x_2) \wedge \dots \wedge \perp(x_1 x_0 x_n) \wedge$$
$$\wedge \perp(x_2 x_0 x_3) \wedge \dots \wedge \perp(x_2 x_0 x_n) \wedge$$
$$\wedge \perp(x_{n-1} x_0 x_n)$$

and A 10_n = $\neg$A 9_{n+1}, where $\perp(xyz)$ is an abbreviation of the formula

$\neg(x=y) \wedge \neg(z=y) \wedge \exists u\ (B(zyu) \wedge D(xuxz) \wedge D(yuyz))$ ($\perp(xyz)$ 'means' : the points x,y,z form the vertices of a right - angled triangle with the right angle at y).

A 9_n means intuitively that there exist n mutually orthogonal lines concurrent at a point.

An AS for $\underset{\sim}{BE}_2'' = \underset{\sim}{E}_2'' \cap L_B$ (i.e. the theory of all L_B - sentences in $\underline{E}_2''$) was given in [33] and we have $\underline{\underline{M}} \in \mathrm{Mod}(\underset{\sim}{BE}_2'')$ iff $\underline{\underline{M}} \simeq \underline{\underline{B}}_2(F)$, where F is a real closed field.

Let $\underline{E}_2$ = Cn (A 1 - A 10). Its corresponding algebraic structure is given by the following

<u>Representation theorem</u> ([42])

$\underline{\underline{M}} \in \mathrm{Mod}(\underset{\sim}{E}_2)$ <u>iff</u> $\underline{\underline{M}} \simeq \underline{\underline{C}}_2(F)$, <u>where</u> F <u>is a Pythagorean ordered field.</u>

(An ordered field is called <u>Pythagorean</u> if

(1) $\forall xy \exists z \quad x^2 + y^2 = z^2$)

A first question concerns the geometric equivalent of arbitnary ordered fields. The answer was given by H.N. Gupta (and, independently, for plane geometry by Z. Piesyk) in 1965 (cf. [10], [11], [12]). Let $\bar{\underset{\sim}{E}}_n$ = Cn(B 1, B 2, A 1, A 3, A 4', A 5 - A 7, A 8_1, A 8_2, A 8_3, A $9_n'$, A 10), where

B 1. $\forall xy \;\; B(xyx) \longrightarrow x=y$

B 2. $\forall xyzu \wedge B(xyz) \wedge B(xyu) \wedge \neg(x=v) \longrightarrow B(xzu) \vee B(xuz)$

A 4'. $\forall xyzuvw \;\; D(xyzu) \wedge D(xyvw) \longrightarrow D(zuvw)$

A 8_1. (Existence of the reflection in a line)

$\forall xyz \;\; \exists uv \;\; B(zuv) \wedge D(vxxz) \wedge D(vyyz) \wedge (B(xyu) \vee$
$\vee B(yux) \vee B(uxy))$

A 8_2. (Existence of the reflection in a point)

$\forall xy\ \exists z\ B(xyz) \wedge D(yzxy)$

A 8_3. (For lines, on which there exist eongruent segments, one can transport further congruent segments from one to the other)

$$\forall xyzx'y'\ \exists z'\ B(xyz) \wedge D(xyx'y') \longrightarrow B(x'y'z') \wedge$$
$$\wedge D(yzy'z')$$

A $9'_n$ is A $9_n \wedge D(x_0x_1x_2) \wedge \ldots \wedge D(x_0x_1x_0x_n)$. The axiom A 8_i (for i = 1,2,3) are special cases of A 8. We have the following

<u>Representation theorem</u> ([11])

$\underline{\underline{M}} \in \mathrm{Mod}(\bar{\underline{E}}_n)$ <u>iff</u> $\underline{\underline{M}} \simeq \underline{\underline{C}}_n(F)$, <u>where</u> F <u>is an ordered field.</u>

L. W. Szczerba [30] gave a finite AS for $\underset{\sim}{B\bar{E}_2} = \bar{\underline{E}}_2 \cap L_B$ and proved that

$\underline{\underline{M}} \in \mathrm{Mod}(\underset{\sim}{B\bar{E}_2})$ <u>iff</u> $\underline{\underline{M}} \simeq \underline{\underline{B}}_2(F)$, <u>where</u> F <u>is an ordered field.</u>

An important step towards realizing the connection between the geometric 'Betweenness' relation and the order relation of a field was the proof of the independence of Pasch's axiom (for the original AS of Tarski [41] it is still an open question whether Pasch's axiom is independent or not).

Let $\underset{\sim}{SOE_2}$ = Cn(B 3, A 1 - A 3, A 4', A 6', A 7 - A 10, M),

where

B 3. $\forall abc\ B(abc) \longrightarrow B(cba)$

A 6'. (another form of Euclid's axiom)

$$\forall abc\ \exists p\ \neg(B(abc) \vee B(bca) \vee B(bac)) \longrightarrow$$
$$\longrightarrow D(apbp) \wedge D(bpcp)$$

M. (xistence of the midpoint)

$\forall ac\ \exists b\ B(abc) \wedge D(abbc)$

Let F be a Pythagorean <u>semi - ordered</u> field, i.e. the order relation satisfies

(2) $\forall x \quad x \geqslant 0 \quad -x \geqslant 0$

(3) $x \quad (x \geqslant 0 \quad -x \geqslant 0) \longrightarrow x=0$

(4) $xy \quad (x \geqslant o \quad y \geqslant o) \longrightarrow x + y \geqslant o$

(5) $1 > 0$

For F, the Cartesian plane is defined as $\underline{\underline{C}}_2(F) = (F^2, B_F, D_F)$, where

$B_F(abc) \longleftrightarrow \|a - b\| + \|b - c\| = \|a - c\|$

$D_F(abcd) \longleftrightarrow \|a - b\| = \|c - d\|$

with $\|x\| = \left|\sqrt{x_1^2 + x_2^2}\right|$ and $|x| = \begin{cases} x \ , \ x \geqslant 0 \\ -x \ , \ -x \geqslant 0 \end{cases}$

The representation theorem for $\underset{\sim}{SOE_2}$ is (cf. [32]):

$\underline{\underline{M}} \in \mathrm{Mod}(\underset{\sim}{SOE_2})$ iff $\underline{\underline{M}} \simeq \underline{\underline{C}}_2(F)$, where F is a semi - ordered Pythagorean field.

A first example of a semi-ordered real closed (and thus in particular Pythagorean) field which does not satisfy

(6) $\forall xy \ (x \geqslant 0 \wedge y \geqslant 0) \longrightarrow xy \geqslant 0$

was given by Szczerba [30]. He defined a semi - order on $\underline{\underline{R}}$. Since $\{1, \sqrt{2}, \sqrt[4]{2}\}$ is a linearly independent set in $\underline{\underline{R}}$ (regarded as a vector space over $\underline{\underline{Q}}$), we can extend it to a vector basis $\{h_\alpha\}_{\alpha \in \underline{\underline{R}}}$ of $\underline{\underline{R}}$ over $\underline{\underline{Q}}$ (by Zorn's lemma) with $h_o = 1$, $h_1 = \sqrt{2}$, $h_2 = \sqrt[4]{2}$. Any $x \in \underline{\underline{R}}$ can be written as

$$x = \sum_\alpha \omega_\alpha(x).h_\alpha \text{ with } \omega_\alpha(x) \in \underline{\underline{Q}} \text{ uniquely determined by } x,$$

and such that $\omega_\alpha(x) = o$ with the possible exception of a finite number of indices α. Let $f : \underline{\underline{R}} \longrightarrow \underline{\underline{R}}$, $f(x) = 2\,w_o(x) - x$ and $f(\{x \in \underline{\underline{R}} \mid x \geqslant o\}) = P$. Let

$$x \geqslant_s 0 \text{ iff } x \in P.$$

$\geqslant_s$ is a semi - order on $\underline{\underline{R}}$, not satisfying (6). Constructions of semi - orders were thoroughly studied by Prestel [22]. Together with Gupta he also proved the Following

Representation theorem ([13], [14])

$\underline{\underline{M}} \in \mathrm{Mod}(\underset{\sim}{SOE_2} \cup GP)$ iff $\underline{\underline{M}} \simeq \underline{\underline{C}}_2(F)$, where F is a Pythagorean quadratically semi - ordered field,

GP being the following axiom

GP. $\forall abcd \ \perp(abc) \wedge \perp(adb) \wedge (B(adc) \vee B(dca) \vee B(cad)) \longrightarrow B(adc)$

A semi - order is called quadratic if it satisfies

(7) $\forall xy \quad x \geqslant 0 \longrightarrow xy^2 \geqslant 0.$

Quadratic semi - orders were thoroughly investigated in [5], [21]. Consider now the following "circle axiom" (stating intuitively that the circle constructed with center in 0 and radius larger than the distance from 0 to a given line, intersects that line):

C. $\forall abcp\ \exists c'\ \ B(abc) \longrightarrow B(pbc') \wedge D(ac'ac)$ and denote $\underset{\sim}{E}_2' = Cn(\underset{\sim}{SOE}_2 \cup \{C'\})$. Wanda Szmielew [34] proved that

$\underline{\underline{M}} \in Mod(\underset{\sim}{E}_2')$ iff $\underline{\underline{M}} \simeq \underline{\underline{C}}_2(F)$, <u>where</u> F <u>is an Euclidean ordered field</u>. An ordered field is called <u>Euclidean</u> if it satisfies

(8) $\forall x\ \exists y\ \ x \geqslant 0 \longrightarrow x = y^2$.

Real closed fields are in particular Euclidean ordered and these are in particular Pythagorean ordered, therefore $\underset{\sim}{E}_2'' \supset \underset{\sim}{E}_2' \supset \underset{\sim}{E}_2$. The first inclusion is strict because $\underset{\sim}{E}_2'$ has a finite AS, whereas $\underset{\sim}{E}_2''$ is does not admit any finite AS. To prove that the second inclusion is strict, we give an example of a Pythagorean ordered, not Euclidean ordered field. Early examples of such fields were provided by Royden [25] and Frasnay [7, pp. 36 - 40]. A thorough study of both Euclidean and Pythagorean ordered fields may be found in [4]. Royden's example is:

Let F be the smallest field containing all algebraic numbers, an indeterminate w, and closed under the operation of taking the square root of a sum of squares. Thus each element of F is an algebraic function f(w) with algebraic coefficients. We make F into two distinct ordered fields F_1 and F_2 by taking two different transcendental numbers w_1 and w_2 and setting

in F_1 $\qquad f(w) >_1 0 \qquad$ iff $\qquad f(w_1) > 0$

in F_2 $\qquad f(w) >_2 0 \qquad$ iff $\qquad f(w_1) > 0$.

Since an Euclidean field admits a unique order, F is not Euclidean, but $(F, >_1)$ is a Pythagorean ordered field. In 1972, Szmielew found the geometric analog of Euclid's field property (8). Consider the following axiom (a statement on the intersection of two circles):

$$C_2. \quad \forall abca'b'\ \exists q\ \ (B(abc) \vee B(bca) \vee B(cab)) \wedge B(ab'c) \wedge$$
$$\wedge B(ba'c) \wedge D(ab'b'c) \wedge D(ba'a'c) \longrightarrow$$
$$\longrightarrow D(aqqb) \wedge (D(ab'b'q) \vee D(ba'a'q))$$

<u>Representation theorem</u> ([35])

$\underline{\underline{M}} \in \mathrm{Mod}(\underset{\sim}{SOE_2} \cup \{C_2\})$ <u>iff</u> $\underline{\underline{M}} \simeq \underline{\underline{C}}_2(F)$, <u>where</u> F <u>is a formally real Euclidean semi - ordered field</u>; i.e. F is semi - ordered and satisfies (1),

(9) $\forall xy \ (x^2 + y^2 = 0) \longrightarrow (x=0)$ and

(10) $\forall x \ \exists y \ \ -x = y^2 \vee x = y^2$.

Since $(\underline{\underline{R}}, > s)$ is a formally real Euclidean semi - ordered field not satisfying (7), we have $A\ 5 \notin \ \notin \mathrm{Cn}(\underset{\sim}{SOE_2} \cup \{C_2\})$. We also have

$SOE_2 \vdash (C \longleftrightarrow A\ 5 \wedge C_2)$ and $\underset{\sim}{SOE_2} \vdash (C \longleftrightarrow GP \wedge C_2)$.

The algebraic structures appearing so far win the representation theorems were fields with additional properties and with an additional relation of - at least - semi - order. The question is now: are there geometric equivalents for a wider class of fieldsX The positive answer to this question is contained in [2], but Bachmann's treatment is not in the first - order axiomatic spirit of this paper. Rudolf Schnabel [26] gave in 1981 a first - order axiomatization of geometric structures, which were previously group-theoretically defined in [2]. Consider the following axioms (written in L_D) :

S 1. $\forall abcd \ D(abcd) \longrightarrow D(cdab)$

S 2. $\forall abcdef \ D(abcd) \wedge D(cdef) \longrightarrow D(abef)$

S 3. $\forall ab \ D(abba)$

S 4. $\forall abca'b'c'xm \ \exists x' \ \forall y \ \neg(a=b) \wedge \neg(c=m) \wedge D(acam) \wedge$
$\wedge D(bcbm) \wedge D(a'b'ab) \wedge D(a'c'ac) \wedge D9b'c'bc) \longrightarrow$
$\longrightarrow D(a'x'ax) \wedge D(b'x'bx) \wedge D(c'x'cx) \wedge (D(a'yax) \wedge$
$\wedge D(b'ybx) \wedge D(c'ycx) \longrightarrow y=x')$

S 5. $\forall abcd \ \exists m \ \neg(a=b) \wedge \neg(c=d) \wedge D(acad) \wedge D(bcbd) \longrightarrow$
$\longrightarrow D(amcm) \wedge D(bmcm) \wedge \neg(m=a) \wedge \neg(m=b) \wedge \neg(m=c)$

S 6. $\exists abcd \ \neg(a=b) \wedge \neg(c=d) \wedge D(acad) \wedge D(bcbd)$

S 4 is called 'rigidity axiom', S 5 says the same thing as A 6' and S 6 is a lower dimension axiom. Let $\underset{\sim}{T} = \mathrm{Cn}(S\ 1 - S\ 6)$. A different AS for T can be found in [9]. We have the following

<u>Representation theorem</u> ([26])

$\underline{\underline{M}} \in \mathrm{Mod}(\underline{T})$ <u>iff</u> $\underline{\underline{M}} \simeq \underline{\underline{G}}(K, \sigma)$,

K being a field and σ an involution, i.e. $\sigma : K \longrightarrow K$, an automorphism and $\sigma \circ \sigma = id_K$. $\underline{\underline{G}}(K, \sigma) = (K, D_K)$, where $D_K(abcd)$ iff $\|a - b\| = \|c - d\|$, for all $a,b,c,d \in K$, with $\|x\| = x . \sigma(x)$ for all $x \in K$. $\underline{\underline{G}}(K, \sigma)$ is called the Gauss plane over (K, σ). It is a generalization of $\underline{\underline{G}}(\underline{\underline{C}}, \sigma)$, with $\sigma(z) = \bar{z}$ for all $z \in \underline{\underline{C}}$, $\bar{z}$ being the complex conjugate of z. The characteristic of $\underline{\underline{G}}(K, \sigma)$ is the characteristic of the field K.

The minimal model of $\underset{\sim}{T}$ contains four points and has characteristic 2. The next model contains nine points and has characteristic 3. If we strengthen the axiom S 6 to

$$S\ 6_1.\quad \exists abcdm\ \forall m'\ \neg(a=b) \wedge \neg(c=d) \wedge D(acad) \wedge D(bcbd) \wedge$$
$$\wedge D(ambm) \wedge (D(amam') \wedge D(bmbm') \longrightarrow m=m')$$

then all models of characteristic 2 are excluded; if to

$$S\ 6_2.\quad \exists abcm\ \forall b'\ \neg(a=b) \wedge \neg(c=m) \wedge D(cacb) \wedge D(mamb) \wedge$$
$$\wedge \neg D(cmbm) \wedge (D(abab') \wedge D(mbmb') \longrightarrow b=b')$$

then the one with nine points is excluded too (cf. [26]). Let $\underset{\sim}{T}_2$ = Cn (S 1 – S 5, S 6_2). Let $F = \mathrm{Fix}\,\sigma = \{x \in K \mid \sigma(x) = \}$; F is a subfield of K of index 2. If the characteristic is $\neq 2$, then there is an $i \in K \setminus F$ with $\sigma(i) = -i$; $i^2 = f \in F$, but $f \notin F^2 = \{f^2 \mid f \in F\}$, (therefore F is not quadratically closed), such that all $x \in K$ have a unique representation

$$x = a + bi\ ,\ \text{with } a,b \in F.$$

We further have $\|x\| = x.\sigma(x) = (a + bi)(a = bi) = a^2 - fb^2$. We now define a predicate S by:

$$S(abcd) \longleftrightarrow \forall m\ \exists s\quad D(cmdm) \longrightarrow D(abcs) \wedge D(cmsm)$$ having the intuitive meaning 'ab $\leqslant$ cd'. Consider the following two axioms

$$S\ 7_1.\quad \exists abcd\quad S(abcd) \wedge \neg(a=b) \wedge \neg D(abcd)$$

$$S\ 7_2.\quad \forall abcd\quad S(abcd) \vee S(cdab)$$

(which, written without the abbreviation S, are

$$S\ 7_1.\quad \exists abcd\ \forall m\ \exists s\quad (D(cmdm) \longrightarrow D(abcs) \wedge D(cmsm)) \wedge$$
$$\wedge \neg(a=b) \wedge \neg D(abcd)$$

$$S\ 7_2.\quad \forall abcdmn\ \exists s\quad (D(cmdm) \longrightarrow D(abcs) \wedge D(cmsm)) \vee$$
$$\vee (D(anbn) \longrightarrow D(cdas) \wedge D(ansn)).)$$

<u>Representation theorem</u> ([26])

(i) $\underline{\underline{M}} \in \mathrm{Mod}(\underline{T}_2 \cup \{S\ 7_1\})$ <u>iff</u> $\underline{\underline{M}} \simeq \underline{\underline{G}}(K, \sigma)$, <u>where</u> Fix$\sigma$ <u>is a Pythagorean field.</u>

(ii) $\underline{\underline{M}} \in \mathrm{Mod}(\underline{T}_2 \cup \{S\ 7_2\})$ <u>iff</u> $\underline{\underline{M}} \simeq \underline{\underline{G}}(K, \sigma)$, <u>where</u> Fix$\sigma$ <u>is an Euclidean field.</u>

There are obvious similarities between Gauss planes and Cartesian planes; the difference is that there is no betweenness relation in a Gauss plane and that its norm looks rather unusual (for $f \neq -1$). It is not always possible to choose $f = -1$ if Fixσ is only Pythagorean (take $F =$ Fixσ Pythagorean, non Euclidean and $f \notin F$ but $f \in F^2$ and $-f \notin F^2$). In order to obtain the standard Euclidean norm, we should add the axiom (cf. [19], [27] and also [2], 13.2)

S 8. abcde $\exists(a=b)\ \neg(a=c) \wedge \neg(a=d) \wedge \neg(b=c) \wedge$
$\wedge \neg(b=d) \wedge \neg(c=d) \wedge D(abbc) \wedge D(bccd) \wedge$
$\wedge D(cdda) \wedge D(aebe) \wedge D(cebe) \wedge D(cede) \wedge$
$\wedge \neg D(aeac)$

which states that there is a square and excludes planes of characteristic 2 and the nine - points plane. Let S 8' be S 8 without $\wedge \neg D(aeac)$. Put $\underset{\sim}{T}_2' = \mathrm{Cn}(S\ 1 - S\ 5,\ S\ 8')$. Then

$\underline{\underline{M}} \in \mathrm{Mod}(\underline{T}_2')$ <u>iff</u> $\underline{\underline{M}} \simeq \underline{\underline{D}}_2(F)$, <u>where</u> F <u>is a field which is not quadratically closed and has characteristic</u> $\neq 2$.

A field is said to be quadratically closed if $\forall a\ \exists b\ \ b^2 = a$.

We also have $\mathrm{Cn}(S\ 1 - S\ 5,\ S\ 7_1,\ S\ 8) = \underset{\sim}{DE}_2$ (note that we do not know any AS for $\underset{\sim}{DE}_n$ $(n \geqslant 3)$). W. Szmielew ([37], [38]) proved that

$E_2 = \mathrm{Cn}(\underset{\sim}{DE}_2,\ B\ 3,\ A\ 1,\ A\ 10',\ WP)$

where

A 10'. $\forall abc\ \exists uv\ \neg(u=v) \wedge D(auav) \wedge D(bubv) \wedge D(cucv) \longleftrightarrow$
$\longleftrightarrow B(abc) \vee B(bca) \vee B(cab)$

WP. (Weak Pasch axiom)

$\forall abcdp\ \exists q\ \ B(apd) \wedge B(bdc) \longrightarrow (B(bpq) \vee B(pqb) \vee$
$\vee B(qbp)) \wedge (B(aqc) \vee B(qca) \vee B(caq))$

Therefore $\{S\ 1 - S\ 5,\ S\ 7_1,\ S\ 8,\ B\ 3,\ A\ 1,\ A\ 10',\ WP\}$ represents an AS for $\underset{\sim}{E}_2$. Moreover, S 8 may be replaced by S 8'. Each of its

axioms is independent of all the remaining ones (cf. [27]).

Finally we have $\underline{E}_2' = Cn(\underline{T}_2 \cup \{S\ 7_2'\})$ in $\underline{E}_2'$, B is definable by D (cf. [24]), thus any sentence of E_2' containing B may be rewritten as a sentence in L_D.)

With the following form of the continuity axiom (cf. [25])

$$\text{Co.}\ \forall vw\ldots\ \{[\exists z \forall xy\ \ \varphi(x) \wedge \psi(y) \longrightarrow S(zxzy)] \text{ —}$$
$$\longrightarrow [\exists u\ \ xy\ \ \varphi(x) \wedge \psi(y) \longrightarrow S(zxzu) \wedge S(zuzy)]\}$$

– where φ and ψ are formulae of L_D with the same requirements as in A 11 – we have $Cn(\underline{T}_2 \cup \{S\ 7_2, Co\}) = \underline{E}_2''$ (with the same remark concerning B as above).

It is interesting to mention that, except $\underline{E}_2''$ and $\underline{BE}_2''$, none of the above theories is decidable (cf. [44]).

And so, the story of the duality between elementary Euclidean geometry and the elementary theory of fields has come to an end.

Can fields be replaced by weaker algebraic structures? The answer depends on what one is willing to accept under the heading 'Geometry'. In a recently published book ([39]), Wanda Szmielew gave ASs for plane affine geometry over ternary fields (a rather complicated structure, which is neither necessarily commutative nor associative!), strong left quasi - fields, skew fields and fields. Affine geometry over ternary fields is quite 'natural', since it represents the starting-point to Euclidean geometry which allows coordonatization and thus a representation theorem (cf. also [20]).

Which is the weakest algebraic structure, one could suspect of a reasonable geometric meaning? Since a line should look algebraically like

$$\{(x,y)\ |\ ax + by + c = 0\},$$

we must have two operations + and $\cdot$, so that the natural answer should be 'rings'.

Geometries over rings (in their full generality) were first considered by D. Barbilian [3] (also a very original Romanian poet), but his treatment is not in our first-order axiomatic manner. First - order axiomatizations for affine Barbilian - planes over Z - rings

(unitary rings with ab = 1 iff ba = 1) were given by Leissner [17] and for affine Barbilian-structures over arbitrary unitary rings by Rado [23]. Because of the curious deviations from ordinary geometry, Barbilian himself restricted the class of rings to Z – rings. Even with this restriction, there are couples of points for which the joining line is not unique. Therefore the interest for geometries over rings came from the <u>algebrist</u> Barbilian. A traditional geometer would hardly call them 'geometries', to say nothing of 'Euclidean'.

A synthes's of the present paper is given below. (Index n means 'n – dimensional with $n \geqslant 2$!)

GEOMETRY	ALGEBRA
$\underset{\sim}{E}''_n$, $\underset{\sim}{BE}''_n$	real closed fields
$\underset{\sim}{E}'_n$	Euclidean ordered fields
$\underset{\sim}{E}_n$	Pythagorean ordered fields
$\underset{\sim}{\bar{E}}_n$, $\underset{\sim}{B\bar{E}}_2$, ordered Pappian midpoint planes [39, p. 156 ff]	ordered fields
$Cn(\underset{\sim}{SOE}_n \cup \{GP\})$	quadratically semi – ordered Pythagorean fields
$Cn(\underset{\sim}{SOE}_n \cup \{C_2\})$	formally real Euclidean semi – ordered fields
$Cn(\underset{\sim}{SOE}_n)$	Pythagorean semi – ordered fields
$Cn(\underset{\sim}{T}_2 \underset{\underset{\sim}{T}_2}{\cup} \{S\ 7_1\})$	Pythagorean fields fields of characteristic $\neq 2$, not quadratically closed
Pappian affine planes [39, p. 43]	fields
ordered Desarguean midpoint planes [39, p. 156 ff]	ordered skew fields

GEOMETRY	ALGEBRA
ordered midpoint planes [39, p. 156 ff]	ordered strong left quasi - fields
Desarguean affine planes [39, p. 43]	skew fields
weakly Desarguean affine planes [39, p. 43]	strong left quasi - fields
parallelity planes [39, p. 19]	ternary fields
affine Barbilian - planes	Z - rings
affine Barbilian - structures	unitary rings

References

[1] E. Artin, O. Schreier, Algebraische Konstruktion reeller Korper, Abh. math. Sem. Univ. Hamburg 5 (1926), 83 - 115

[2] F. Bachmann, Aufbau der Geometrie aus dem Spiegelungs-begriff, 2. Auflage, Springer - Verlag, Berlin, Heidelberg, New York, 1973 (1. Auflage 1959)

[3] D. Barbilian, Zur Axiomatik der projektiven ebenen Ringgeometrien I, II, Jber. Deutsche Math.-Verein. 50 (1940), 179 - 229, 51 (1941), 34 - 76

[4] E. Becker, Euklidische Korper und euklidische Hullen von Korpern, J. reine angew. Math. 268/269 (1974), 41 - 52

[5] E. Becker, E. Kopping, Reduzierte quadratische Formen und Semi - ordnungen reeller Korper, Abh. math. Sem. Univ. Hamburg 46 (1975), 143 - 177

[6] K. Borsuk, W. Szmielew, Foundations of geometry, Studies in Logic and the Foundations of Mathematics, North - Holland, Amsterdam, 1960

[7] C. Frasnay, Un development de la geometrie elementaire, Alger - Mathematiques 6 (1959), 5 - 43

[8] C. F. Gauss, Theoria residuorum biquadraticorum, commentatio secunda, Gottinger gelehrte Anzeigen, 1831 April 23, in: C.F. Gauss Werke, Gottingen 1876, II. Band, 169 - 178 (especially 174 - 178), and C. F. Gauss Werke, X. Bandes 2. Abt., Gottingen, 1933, III. d. (L. Schlesinger), 53 - 55

[9] M. Grochowska, Euclidean two - dimensional equidistance theory, to appear in Demonstratio Math.

[10] H.N. Gupta, An axiomatization of finite - dimensional Cartesian spaces over arbitrary ordered fields, Bull. Acad. Polon. Sci., Ser. Sci. Math., Astron., Phys. 13 (1965), 551 - 552

[11] H.N. Gupta, Contributions to the axiomatic foundations of Euclidean geometry, Ph.D. Thesis, University of California at Berkeley, 1965

[12] H.N. Gupta, Z. Piesyk, An axiomatization of two-dimensional Cartesian spaces over arbitrary ordered fields, Bull. Acad. Sci., Ser. Sci. Math., Astron., Phys. 13 (1965), 549 - 550

[13] H.N. Gupta, A. Prestel, On a class of Pasch - free Euclidean planes, Bull. Acad. Polon. Sci., Ser. Sci. Math., Astron., Phys. 20 (1972), 17 - 23

[14] H.N. Gupta, A. Prestel, Triangle and Schwarz inequality in Pasch - free geometry, Bull. Acad. Polon. Sci., Ser. Sci. Math., Astron., Phys. 20 (1972), 999 - 1003

[15] D. Hilbert, P. Bernays, Grundlagen der Geometrie, 12. Auflage, Teubner, Stuttgart, 1977 (L. Auflage 1899)

[16] W.R. Knorr, The evolution of the Euclidean Elements. A study of the theory of incommensurable magnitudes and its significance for early Greek geometry, D. Reidel, Dordrecht, Holland, 1975

[17] W. Leissner, Affine Barbilian - Ebenen, I, II, J. of Geometry 6 (1975), 31 - 57; 105 - 129

[18] A. Mostowski, Tarski Alfred, in: The Encyclopedia of Philosophy, vol. 8, ed. Paul Edwards, MacMillan and Free Press, New York 1967, 77 - 81

[19] V. Pambuccian, Das Problem der axiomatischen Grundlegung der euklidischen Geometrie, Dissertation, Bukarest, 1982

[20] V. Pambuccian, Review of /39/, to appear in The Amer. Math. Monthly

[21] A. Prestel, Quadratische Semi - Ordnungen und quadratische Formen, Math. Z. 133 (1973), 319 - 342

[22] A. Prestel, Euklidische Geometrie ohne das Axiom von Pasch, Abh. Math. Sem. Vniv. Hamburg 46 (1979), 143 - 177

[23] F. Rado, Affine Barbilian structures, J. of Geometry, 14 (1980), 75 - 102

[24] R.M. Robinson, Binary relations as primitive notions in elementary geometry, in: The Axiomatic Method, Proc. Symp. at Berkeley 1957/1958, North - Holland, Amsterdam, 1959, 68 - 85

[25] H.L. Royden, Remarks on primitive notions for elementary Euclidean and non - Euclidean plane geometry, in: The Axiomatic Method, Proc. Symp. at Berkeley 1957/1958, North - Holland, Amsterdam, 1959, 86 - 96

[26] R. Schnabel, Euklidische Geometrie, Habilitations-schrift, Kiel, 1981

[27] R. Schnabel, V. Pambuccian, Die metrisch - euklidische Geometrie als Ausgangspunkt fur die geordneteuklidische Geometrie, to appear in Expo. Math.

[28] W. Schwabhauser, Uber die Vollstandigkeit der elementaren euklidischen Geometrie, Z. f. math. Logik Grundl. Math. 2 (1956), 137 - 165

[29] W. Schwabhauser, Metamathematische Methoden in der Geometrie, to appear

[30] L. W. Szczerba, Independence of Pasch's axiom, Bull, Acad. Polon. Sci., Ser. Sci. Math., Astron., Phys. 18 (1970), 491 - 498

[31] L.W. Szczerba, Weak general affine geometry, Bull. Acad. Polon. Sci., Ser. Sci. Math., Astron., Phys. 20 (1972), 717 - 803

[32] L.W. Szczerba, W. Szmielew, On the Euclidean geometry without the Pasch axiom, Bull, Acad. Sci., Ser. Sci. Math., Astron., Phys. 18 (1970), 659 - 666

[33] L. W. Szczerba, A. Tarski, Metamathematical properties of some affine geometries, Proc. of the 1964 Internat. Congr. of Logic, Methodol. and Philos. of Sci., Jerusalem, North - Holland, Amsterdam, 1964, 166 - 178

[34] W. Szmielew, The Pasch axiom as a consequence of the circle axiom, Bull. Acad. Polon. Sci., Ser. Sci. Math., Astron., Phys. 18 (1970), 751 - 758

[35] W. Szmielew, A statement on two circles as the geometric analog of Euclid's field property, Bull. Acad. Polon. Sci., Ser. Sci. Math., Astron., Phys. 18 (1970), 759 - 764

[36] W. Szmielew, The role of the Pasch axiom in the foundations of Euclidean geometry, in: The Tarski Symp., Berkeley 1971; Proc. Symp. Pure Math. Vol. 25, AMS, Providence, Rhode Island, 1974, 123 - 132

[37] W. Szmielew, The order and the semi - order of n - dimensional Euclidean space in the axiomatic and model - theoretic aspects, Grundlagen der Geometrie und algebraische Methoden, Potsdamer Forschungen, Reihe B, Heft 3, 1974, 69 - 80

[38] W. Szmielew, Concerning the order and semi - order of n - dimensional Euclidean space, Fund. Math. 107 (1980), 47 - 56

[39] W. Szmielew, From Affine to Euclidean Geometry, D. Reidel, Dordrecht and PWN Warsaw, 1983

[40] A. Tarski, Der Wahrheitsbegriff in den formalisierten Sprachen, Studia Philosophica 1 (1935), 261 - 405

[41] A. Tarski, A decision method for elementary algebra and geometry, 2nd, ed., University of California Press, Berkeley and Los Angeles, 1951 (1st ed. 1948)

[42] A. Tarski, What is elementary geometry?, in: The Axiomatic Method, Proc. Symp. at Berkeley 1957/1958, North - Holland, Amsterdam, 1959, 16 - 29

[43] B.L. van der Waerden, Algebra I, 8. Auflage, Springer - Verlag, Berlin, Heidelberg, New York, 1971

[44] M. Ziegler, Einige unentscheidbare Korpertheorien, L' enseignement mathematique 28 (1982), 269 - 280

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Common Fixed Points Under Asymptotic Regulatiry Condition

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Unique common fixed point theorems for four self maps on a complete metric space under asymptotic regularity condition are obtained. One of these generalizes the results of Fisher [1] and Ray [2]. Incidentally two results of Singh and Kasahara [5] are shown to be incorrect even under the condition, suggested in corrigendum [6] and reasonable modifications are proved.

Recently, Singh and Kasahara [5] proved the following Theorem 1: ([5], Th.1). Let f,g,S and T be self maps on a complete metric space (X,d) satisfying

(1.1) $d(fx,gy)\ \phi(d(Sx,Ty),d(Sx,gy),d(Ty,fx),d(Sx,fx),d(Ty,gy))$ for all $x,y \in X$, where $\phi: \mathbb{R}_+^5 \longrightarrow \mathbb{R}_+$ is upper semi continuous (u.s.c.), non decreasing in each variable and $\phi(t,t,t,t,t) < t$ for all $t > 0$. Further, suppose that

(1.2) $fS = Sf$, $fT = Tf$, $gS = Sg$, $gT = Tg$ and $ST = TS$;

(1.3) there exists a sequence $\{x_n\}$ in X such that $fTx_{2n} = TSx_{2n+1}$,

$gSx_{2n+1} = TSx_{2n+2}$ for $n = 0,1,\ldots$ with

(1.3)' $\operatorname{Sup}\{d(TSx_i,TSx_j) \mid i,j \in N\} < \infty$ and

(1.4) S and T are continuous.

Then f,g,S and T have a unique common fixed point $z \in X$ and $\{TSx_n\}$ converges to z.

Theorem 2: ([5], Th.2). Let f,g,S and T be self mpas on a metric space (X,d) satisfying (1.1), (1.2), (1.3), (1.3)' and the following

(2.1) the sequence $\{TSx_n\}$ has subsequences converging to $z \in X$

(2.2) S and T are continuous at z.

Then z is the unique common fixed point of f,g,S and T.

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The following example due to Fisher [1], shows that theorem 1 is false even when (X,d) is a bounded metric space, S and T are identity maps on X and
$\phi(t_1,t_2,t_3,t_4,t_5) = \frac{1}{2}\,\mathrm{Max}\,\{t_1,t_2,t_3,t_4,t_5\}$ for all $(t_1,t_2,t_3,t_4,t_5) \in \mathbb{R}^5_+$.

Example 3: ([1]). Let $X = \{0,1,2,\ldots,n,\ldots\}$ with a metric d defined by $d(m,m) = 0$ for all $m \in X$ and for $m \neq n$,

$$d(m,n) = \begin{cases} 1, & \text{if } m+n \text{ is odd} \\ 2, & \text{if } m+n \text{ is even.} \end{cases}$$

Define $f,g: X \longrightarrow X$ by $f(2n) = f(2n+1) = 2n + 2$ and $g(2n) = 2n+1$, $g(2n+1) = 2n + 3$ for $n = 0,1,2,\ldots$.

The following example due to Sastry and Naidu [3] also shows that both the theorems are false even when (X,d) is a finite metric space if we select ϕ, S and T as in the above.

Example 4: ([3], Ex5). Let $X = \{1,2,3,4\}$, $d(1,2) = d(3,4) = 2$, $d(1,3) = d(1,4) = d(2,3) = d(2,4) = 1$. Define $f,g:X \longrightarrow X$ by $f1 = f4 = 2$, $f2 = f3 = 1$ and $g1 = g3 = 4$, $g2 = g4 = 3$.

S.L. Singh in his corrigendum [6], mentioned that theorems 1 and 2 of [5] are valid only if (1.3)' is replaced by

$$(1.3)'' \quad \mathrm{Sup}\,\{d(TSx_i,TSx_j) \mid i,j \in N,\ i,j \text{ are not of the same parity}\} = \mathrm{Sup}\,\{d(TSx_i,TSx_j) \mid i,j \in N\} < \infty .$$

We find that even the above alteration does not ensure the existence of a common fixed point for f,g,S and T in view of the following two examples. In these examples, we take ϕ,S,T as in the above examples.

Example 5: Let $X = \{-3, -2, -1, 0,1,2,\ldots\}$ with distance d defined as follows.

$d(x,x) = 0$ for all $x \in X$, $d(-3,-2) = 4$, $d(-3,x) = 2$ if $x \neq -3,-2$,

$d(-2,x) = 2$ if $x \neq -3,-2$, $d(-1,x) = \begin{cases} 1, & \text{if } x \text{ is even and positive or zero} \\ 2, & \text{if } x \text{ is odd and positive.} \end{cases}$

For $x \geqslant 0$, $y \geqslant 0$, $x \neq y$

$$d(x,y) = \begin{cases} 1, & \text{if } x+y \text{ is odd} \\ 2, & \text{if } x+y \text{ is even.} \end{cases}$$

Define f,g on X as follows

$f(-3) = -2$, $f(-2) = f(-1) = 0$, $f(2n+1) = f(2n) = 2n+2$, for $n = 0,1,2,\ldots.$

$g(-3) = f(-1) = 1$, $g(-2) = -1$, $g(2n) = 2n+1$, $g(2n+1) = 2n+3$, for $n=0,1,2,\ldots.$

Then (1.3)" is satisfied when $x_o = -3$.
But neither f nor g has a fixed point.

The following example also shows that his modified condition (1.3)" does not ensure the common fixed point even when the space is finite.

Example 6: Let $X = \{-3,-2,-1,1,2,3,4\}$. Define d as follows
$d(x,x) = 0$ for all $x \in X$, $d(-3,-2) = 4$, $d(-3,x) = 2$ for $x \neq -3,-2$.
$d(-2,x) = 2$ for $x \neq -3,-2$.

$$d(-1,x) = \begin{cases} 2 \text{ for } x = 3 \text{ or } 4 \\ 1 \text{ for } x = 1 \text{ or } 2 \end{cases}$$

$d(1,2) = d(3,4) = 2$, $d(1,3) = d(1,4) = d(2,3) = d(2,4) = 1$.

Define f,g on X as follows

$f1 = f4 = 2$, $f2 = f3 = f(-2) = f(-1) = 1$, $f(-3) = -2$,

$g1 = g3 = 4$, $g2 = g4 = g(-3) = g(-1) = 3$, $g(-2) = -1$.

For $x_o = -3$, condition (1.3)" is satisfied. But neither f nor g has a fixed point.

However, Theorems 1 and 2 of [5] hold good with a slight modification of (1.3)" namely

$$\text{Sup}\left\{ d(TSx_i,TSx_j) \mid i,j \geqslant n,\ i,j \text{ are not of the same parity}\right\}$$
$$= \text{Sup}\left\{ d(TSx_i,TSx_j) \mid i,j \quad n \right\} < \infty$$

for infinitely many n.

Now, we suggest modifications to Theorems 1 and 2 of [5], under asymptotic regularity condition.

Definition: Let f,g,S and T be self maps on a metric space (X,d). We say that the pair (f,g) is asymptotically regular (a.r) with respect

to the pair (T,S) at $x_o \in X$ if there exists a sequence $\{x_n\}_{n=o}^{\infty}$ in X such that $fTx_{2n} = TSx_{2n+1}$, $gSx_{2n+1} = TSx_{2n+2}$ and $d(TSx_n, TSx_{n+1}) \longrightarrow 0$ as $n \longrightarrow \infty$. If $S = T = I$ (Identity map) then we simply say that (f,g) is a.r at x_o.

Theorem 7: Let f,g,S and T be self maps on a metric space (X,d). Suppose that

(7.1) $ST = TS$,

(7.2) (f,g) is a.r. with respect to (T,S) at $x_o \in X$ and

(7.3) $d(fx,gy) \leq \phi(\text{Max } \{d(Sx,Ty), d(Sx,fx), d(Ty,gy), d(Sx,gy), d(Ty,fx)\})$

for all $x,y \in X$, where ϕ: $\mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is continuous, increasing and $\phi(t) < t$ for all $t > 0$.

Then $\{TSx_n\}$ is Cauchy. Further, if either (i) (X,d) is complete, S,T are continuous at the limit z of $\{TSx_n\}$ and $fS = Sf, gT = Tg$ or (ii) $\{TSx_n\}$ has a subsequence converging to z;S,T are continuous at z and $fS = Sf$, $gT = Tg$, then z is the only common fixed point of f,g,S and T.

Proof: For i odd, j even and $j \neq 0$, we have

$$d(TSx_i, TSx_j) = d(fTx_{i-1}, gSx_{j-1})$$
$$\leq \phi(\text{Max } \{d(TSx_{i-1}, TSx_{j-1}),\ d(TSx_{i-1}, TSx_i),\ d(TSx_{j-1}, TSx_j)$$
$$d(TSx_{i-1}, TSx_j),\ d(TSx_{j-1}, TSx_i)\}).$$

When i and j are positive integers and i+j is odd, we have

$$d(TSx_i, TSx_j) \leq \phi(\text{Max } \{d(TSx_{i-1}, TSx_{j-1}),\ d(TSx_{i-1}, TSx_i),\ d(TSx_{j-1}, TSx_j),$$
$$d(TSx_{i-1}, TSx_j), d(TSx_{j-1}, TSx_i)\}).$$

Suppose $\{TSx_n\}$ is not Cauchy.

Then there exists an $\varepsilon > 0$ and strictly increasing sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $m_k < n_k$ with $d(TSx_{m_k}, TSx_{n_k}) \geq \varepsilon$ and $d(TSx_{m_k}, TSx_{n_k - 1}) < \varepsilon$ for all $k = 1,2,\ldots$.

Then from (7.2), it follows that

$$d(TSx_{m_k}, TSx_{n_k}) \longrightarrow \varepsilon \text{ as } k \longrightarrow \infty .$$

Let $B_1 = \{k \mid m_k$ is even and n_k is odd$\}$, $B_2 = \{k \mid m_k$ is even and n_k is even$\}$, $B_3 = \{k \mid m_k$ is odd and n_k is even$\}$ and $B_4 = \{k \mid m_k$ is odd and n_k is odd$\}$. Clearly at least one of B_1, B_2, B_3 and B_4 is infinite. Suppose B_1 is infinite. Then for all $k \in B_1$, we have

$$d(TSx_{m_k+1}, TSx_{n_k+1}) \leq \phi(\text{Max}\{d(TSx_{m_k}), d(TSx_{m_k}, TSx_{m_k+1}), d(TSx_{n_k}, TSx_{n_k+1}), d(TSx_{n_k}, TSx_{m_k}+1), d(TSx_{m_k}, TSx_{n_k+1})\}).$$

By taking limit as $k \longrightarrow \infty$ in B_1, we get that $\varepsilon \leq \phi(\varepsilon) < \varepsilon$ which is a contradiction. Suppose B_2 is infinite. Then for all $k \in B_2$, we have

$$d(TSx_{m_k+1}, TSx_{n_k}) \leq \phi(\text{Max}\{d(TSx_{m_k}, TSx_{n_k-1}), d(TSx_{m_k}, TSx_{m_k+1}), D(TSx_{n_k}, TSx_{n_k-1}), d(TSx_{m_k}, TSx_{n_k}), d(TSx_{m_k+1}, TSx_{n_k-1})\}).$$

Letting $k \longrightarrow \infty$ in B_2, we get that $\varepsilon \leq \phi(\varepsilon) < \varepsilon$ which is a contradiction. Similar is the situation in the remaining cases.
Hence $\{TSx_n\}$ is Cauchy. Suppose (i) holds.
Then there exists a $z \in X$ such that $TSx_n \longrightarrow z$ as $n \longrightarrow \infty$.

Now applying (7.3) to $d(fz, gSx_{2n+1})$, $d(fTx_{2n}, gz)$, $d(fTSx_{2n}, gTSx_{2n+1})$ and $d(fTSx_{2n}, gSx_{2n+1})$ and taking limit as $n \longrightarrow \infty$, we get that z is a common fixed point of f, g, S and T. Uniqueness of common fixed point follows easily from (7.3). Suppose (ii) holds. Since $\{TSx_n\}$ is Cauchy follows that $TSx_n \longrightarrow z$ as $n \longrightarrow \infty$. Now the proof follows similarly.

Corollary 8: Let f and g be self maps on a metric space (X,d). Suppose that

(8.1) (f,g) is a.r. at $x_o \in X$ and

(8.2) $d(fx,gy) \leq \phi(\text{Max}\{d(x,y), d(x,fx), d(y,gy), d(x,gy), d(y,fx)\})$

for all $x, y \in X$ where $\emptyset: \mathbb{R}_+ \dashrightarrow \mathbb{R}_+$ is as in theorem 7.
Then $\{x_n\}$ is Cauchy.
If (X,d) is complete, then the limit of $\{x_n\}$ is the only common fixed point of f and g.

Proof: Take $S = T = I$ in the above theorem.

The following example shows that the above corollary fails, even when $f = g$, if the condition (8.1) is dropped.

Example 9: ([3], Ex.3). Let $X = [1, \infty)$ with the usual metric and $f: X \dashrightarrow X$ be given by $fx = 2x$.
Define $\emptyset: \mathbb{R}_+ \dashrightarrow \mathbb{R}_+$ as $\emptyset(t) = \dfrac{2t^2}{1 + 2t}$.

Example 3 also shows that the above corollary fails even in a bounded metric space when the condition (8.1) is dropped.

Corollary 10: Let f and g be commuting self maps on a complete, bounded metric space (X,d) satisfying (8.2). Then f and g have a unique common fixed point.

Proof: Let d be bounded by a positive real number M.
It can be easily seen that
$d(fg)^n x, (fg)^n y) \leqslant \emptyset^n(M)$ for all $x, y \in X$ and for all $n = 0,1,2,\ldots$.
By the properties of $\emptyset$, we can show that $\emptyset^n(M) \dashrightarrow 0$ as $n \dashrightarrow \infty$.
Let $x_o \in X$. Define $x_{2n+1} = fx_{2n}$, $x_{2n+2} = gx_{2n+1}$, $n = 0,1,2,\ldots$.
Then
$d(x_{2n}, s_{2n+1}) = d((fg)^n x_o, (fg)^n fx_o) \dashrightarrow 0$ as $n \dashrightarrow \infty$ and
$d(x_{2n+1}, x_{2n+2}) = d((fg)^n fx_o, (fg)^n fgx_o) \dashrightarrow 0$ as $n \dashrightarrow \infty$.
Therefore $d(x_n, x_{n+1}) \dashrightarrow 0$ as $n \dashrightarrow \infty$; Hence (f,g) is a.r. at x_o.
Now, by corollary 8 follows the result.

Theorem 4 of Fisher [1] follows from corollary 10 by taking $\emptyset(t) = \alpha t$ where $\alpha \in [0,1)$.

Corollary 11: (B.K. Ray [2], Th.4). Let f and g be commuting self maps on a complete, bounded metric space (X,d) satisfying

(11.1) $[d(fx, g^2y)]^2 \leqslant \alpha \operatorname{Max} \{d(x,fx) d(gy, g^2y) d(x, g^2y)\ d(fx, gy)\}$ for all $x, y \in X$ where $\alpha \in [0,1)$.

Suppose that g is continuous.

Then f and g have a unique common fixed point.

Proof: We have

$$d(fx,gy) \leqslant \sqrt{\alpha}\,\mathrm{Max}\left\{ [d(x,fx)d(y,gy)]^{\frac{1}{2}},\ [d(x,gy)d(fx,y)]^{\frac{1}{2}} \right\}$$

for all $x,y \in g(X)$.

Using the inequality $\sqrt{ab} \leq \frac{a+b}{2} \leq \mathrm{Max}\{a,b\}$

for non-negative reals a and b, we get

$$d(fx,gy) \leq \sqrt{\alpha}\,\mathrm{Max}\left\{ d(x,fx), d(y,gy), d(x,gy), d(y,fx) \right\} \text{ for all } x,y \in g(X).$$

Now, from corollary 10 (by taking $\phi(t) = \sqrt{\alpha t}$) it follows that for any $x_0 \in g(X)$, the sequence $\{x_n\}$ defined by $x_{2n+1} = fx_{2n}$, $x_{2n+2} = gx_{2n+1}$, $n=0,1,2,\ldots$

is Cauchy. Hence, there exists a $z \in X$ such that $x_n \longrightarrow z$ as $n \longrightarrow \infty$.

$$z = \underset{n\to\infty}{\mathrm{Lt}}\ x_{2n+2} = \underset{n\to\infty}{\mathrm{Lt}}\ gx_{2n+1} = g\ \underset{n\to\infty}{\mathrm{Lt}}\ x_{2n+1} = gz.$$

From (11.1) follows that $fz = z$ and z is the only common fixed point of f and g.

In corollary 10 (by taking $\phi(t) = \alpha t$ where $\alpha \in [0,1)$) incidentally we proved that, in the presence of (8.2), boundedness of the space (X,d) and commutativity of f and g together imply the asymptotic regularity of (f,g) at any point, while example 3 shows that boundedness of X alone does not imply the asymptotic regularity of (f,g) at any $x_0 \in X$. Hence we have the following natural Problem: In the presence of (8.2) with $\phi(t) = \alpha t$, what additional condition on f and g guarantee the asymptotic regularity of (f,g) at some point $x_0 \in X$ when the space is bounded?

Theorem 12: Let f,g,S and T be self maps on a metric space (X,d) satisfying (7.3) and the following

(12.1) $fg = gf$;

(12.2) there exists a sequence $\{x_n\}$ in X such that $Sgx_{2n+1} = gfx_{2n}$, $Tfx_{2n+2} = gfx_{2n+1}$ for $n = 0,1,2,\ldots$ with $d(gfx_n, gfx_{n+1}) \longrightarrow 0$ as $n \longrightarrow \infty$. Then $\{gfx_n\}$ is Cauchy. If (X,d) is complete, S,T are continuous at the limit z of $\{gfx_n\}$ and $fS = Sf$, $gT = Tg$ then z is the only common fixed point of f,g,S and T.

Proof: Similar to that of theorem 7.

Theorem 13: Let f,g,S and T be self maps on a metric space (X,d) satisfying the following

(13.1) $ST = TS$,

(13.2) $d(fs,gy) \leq \phi\text{Max}\left\{ d(Sx,Ty), d(Sx,fx), d(Ty,gy), \tfrac{1}{2}[d(Sx,gy) + d(Ty,fx)] \right\})$

for all $s,y \in X$ where $\phi: \mathbb{R}_+ \dashrightarrow \mathbb{R}_+$ is continuous, increasing and $\phi(t) < t$ for all $t > 0$.

(13.3) there exists a sequence $\{x_n\}$ in X such that $fTx_{2n} = TSx_{2n+1}$, $gSx_{2n+1} = TSx_{2n+2}$, $n = 0,1,2,\ldots$.

Then $\{Tsx_n\}$ is Cauchy. If $fS = Sf$, $gT = Tg$, (X,d) is complete and S,T are continuous at the limit z of $\{TSx_n\}$, then z is the only common fixed point of f,g,S and T.

Proof: It can easily be seen from (13.1), (13.2) and (13.3) that $d(TSx_n, TSx_{n+1}) \dashrightarrow 0$ as $n \longrightarrow \infty$.

Now the result follows from theorem 7.

The following theorem can also be proved on similar lines, using theorem 12.

Theorem 14: Let f,g,S and T be self maps on a metric space (X,d) satisfying (13.2) and the following

(14.1) $fg = gf$;

(14.2) there exists a sequence $\{x_n\}$ in X such that $Sgx_{2n+1} = gfx_{2n}$, $Tfx_{2n+2} = gfx_{2n+1}$, $n = 0,1,2,\ldots$.

Then $\{gfx_n\}$ is Cauchy. If (X,d) is complete, $fS = Sf$, $gT = Tg$ and S,T are continuous at the limit z of $\{gfx_n\}$, then z is the only common fixed point of f,g,S and T.

Example 9 also shows that in (13.2) of theorem 13, $\tfrac{1}{2}[d(Sx,gy) + d(fx,Ty)]$ can not be replaced by $\text{Max}\{d(Sx,gy), d(fx,Ty)\}$ even when $f = g$ and $S = T = I$.

The conditions imposed on ϕ in (7.3) are not really stringent compared to the apparently weaker conditions, namely

(i) ϕ is upper semi continuous with $\phi(t) < t$ for all $t > 0$ and

(ii) ϕ is upper semi continuous from the right, increasing and $\phi(t) < t$ for all $t > 0$

which, many authors are usually tempted to assume; in fact, it is shown in [4] that all these are equivalent.

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References

[1] B. Fisher: Results on common fixed points on bounded metric spaces, Math. Sem. Notes, 7 (1979), 73-80.

[2] B.K. Ray: On a theorem of B. Fisher, Indian. J. Pure Appl. Math. 10 (1979), 629-632.

[3] K.P.R. Sastry and S.V.R. Naidu: Fixed point theorems for generalized contraction mappings, Yokohama Math. J. 28 (1980), 15-29.

[4] K.P.R. Sastry, S.V.R. Naidu, I.H.N. Rao, K.P.R. Rao: Fixed points under semi group of self maps, Nat. Acad. Sci. Letters (to appear).

[5] S.L. Singh and S. Kasahara: On some recent results on common fixed points. Indian. J. Pure-Appl. Math. 13 (1982), 757-761.

[6] S.L. Singh: Corrigendum on some recent results on common fixed points (I.J.P. Am, 13(7), 757-61). I.J.PAM. 1983 (1975).

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On Some Insignificant Regression Coefficients

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Abstract

While dealing with a multiple regression model often an analyst suffers with an apparent contradiction to the conception or to the theory when one or more of the regression coefficients appear with reverse sign or with insignificant value. This type of problem will embarass the analyst as it is very difficult to interprete such coefficients. In the present paper six different sources that can cause the problem are discussed explicitly with statistical justifications. The suggestions to overcome the problem when these causes are in effect are also mentioned.

1. Introduction

Today, regression analysis has become one of the most widely used statistical techniques for analysing multifactor data. Its broad appeal results from a conceptually simple process of using a single equation model to express the relationship between a set of variables. Due to mathematical approach regression analysis has become very interesting theoretically too and can be regarded as the most widely used statistical method with its applications occuring in almost every field including the physical sciences, the biological sciences, and the social sciences.

By regression analysis we mean that statistical device which reveals the average functional relationship between two or more variables. Models that are considered to establish such relationships between the variables are regression models. A regression model that includes more than one regressor is a multiple regression model. The general form of a multiple linear regression model with n regressors is,

$$Y = \beta_o + \beta_1 X_1 + \beta_2 X_2 + \ldots\ldots\ldots\ldots\ldots + \beta_n X_n + \varepsilon \quad (1)$$

The parameters β_i's are the regression coefficients. If the error term ε is ignored, this equation describes a hyperplane in the n-dimensional space. The regression coefficient β_i represents the expected change in the response Y per unit change in X_i.

Regression models can be used for several purposes. (a) Data description, (b) estimation of the parameters, (c) prediction or the estimation of the response, and (d) the control on the response are the most common purposes of a regression model. A set of data can be summarised or described in a regression model which may be much more convenient and useful than a tabular or even a graphic form, if the model is adequate and complete. Likewise, parameter estimation problems and estimation and prediction of the response can be solved by the method of regression analysis. Regression models are helpful for control purposes too. For example, a metallurgist can use regression analysis to develop a model relating the malleability of a certain metal to the quality of ore and the ingredients to be added while melting. This model can then be used to control the malleability strength of the metal to a desired level by regulating on the ore quality and on the additional ingredients. But when a regression analysis is used for control purpose, it is important that the variables should be in a cause and effect relationship which may not be necessary for the former three purposes.

2. The problem

To estimate the regression coefficients β_i's in (1) the method of least squares which has got many sophisticated statistical justifications, is widely applied. Using such models as in (1), occassionally, an analyst experiences an apparent contradiction to the intuition or to the theory when one or more of the regression coefficient seems to have a reverse sign or insignificant value. That is, the theory or the problem situation may imply that a particular regression coefficient should be positive or significant while the actual regression coefficient turns out to be negative or insignificant, or vice versa. For example,

while estimating the production function for a particular crop the regressors like croping area, supply of labour, amount of fertilizers used, use of high yielding variety seeds, use of modern machines & tools, use of pesticides, proper rainfall, technical guidelines, soil & climatic conditions, government budget on agricultural sector, availability of credit facility, irrigation facility and market facility all should have the estimated coefficients with significant positive values. But it is likely that one or more of the regression coefficients may be obtained with the insignificant and reverse sign (negative say). This type of problem will be embarassing, as it is very difficult to justify a negative estimate of a parameter in the model used when the user believes that the parameter should be positive.

3. Causes of the problem

There may be several sources that can cause the regression coefficients either with the problem of sign reversal or insignificancy in them. The following causes are relatively more important and are to be pondered while developing a multiple regression model. That is, the problem may arise, if,

(i). the regressors have values in a very small range,
(ii). the important regressors have not been included in the model,
(iii). the problem of multicollinearity exists,
(iv). the problem of autocorrelation exists,
(v). the computational errors have been made,
(vi). the data are biased.

Each of the above points is crucial in the sense that any one of them can create the problem. So it will be noteworthy to discuss them one by one.

(i). Let the regressors (X's) have values in a very small range. Consider the simple linear regression model

$$Y_i = \beta_o + \beta_1 X_i + \varepsilon_i \tag{2}$$

In the above model the variance of the estimated regression coeffcient $\hat{\beta}_1$ is,

$$(3) \quad \text{Var.}(\hat{\beta}_1) = E(\hat{\beta}_1 - \beta_1)^2 = E\left[\frac{\Sigma xiyi}{\Sigma xi^2} - \beta_1\right]^2,$$

Where, $x_i = X_i - \bar{X}$ and $y_i = Y_i - \bar{Y}$

For the above model, let it be assumed that the error terms εi's are independently and normally distributed around zero mean and with a constant variance σ^2, i.e. $E(\varepsilon i) = 0$

$$E(\varepsilon i\, \varepsilon j) = 0, \text{ for all } i \neq j \text{ and}$$
$$= \sigma^2 \text{ for all } i = j = 1, 2, \ldots, n.$$

so that,

$$(4) \quad y_i = Y_i - \bar{Y} = \beta_1 x_i + \varepsilon i$$

Therefore, from (3),

$$(5) \quad \hat{\text{Var}}.(\hat{\beta}_1) = E\left[\frac{\Sigma xi(\beta_1 xi + \varepsilon i)}{\Sigma xi^2} - \beta_1\right]^2 = E\left[\frac{\Sigma xi\,\varepsilon i}{\Sigma xi^2}\right]^2$$

$$= E\left[\frac{\Sigma xi\,\varepsilon i}{\Sigma xi^2}\right]^2 = \frac{\sigma^2}{\Sigma xi^2} = \frac{\sigma^2}{\Sigma(Xi-\bar{x})^2}$$

Note that the variance of $\hat{\beta}_1$ is inversely proportional to the range or scatteredness of the regressor. Now if the range of X is too small, variance of the estimated coefficient of X will be relatively large, which in turn results in insignificancy of the estimated coefficient. It is also supported by the Student's t statistic,

$$t = \frac{\hat{\beta}}{\sqrt{\hat{\text{Var}}.(\hat{\beta})}} \sim t_{n-k},$$

from which, it is obvious that lesser the value of the calculated t, greater will be the chance that the coefficient is insignificant.

(ii). If important regressors are not included in the model then also the estimates of the regression coefficients will have the problem of sign reversal. It is due to the partial nature of the regression

coefficients. For example, consider the following set of data,

Y	X_1	X_2
2	3	1
7	5	1
3	4	2
9	6	3
5	8	3
3	10	4
11	12	4
7	14	5
4	13	6
14	15	7

If we fit a simple regression model of the form, $Y = \beta_o + \beta_1 X_1 + \varepsilon$, for the response Y on X_1 only, the estimated model turns out to be, $\hat{Y} = 2.1557 + 0.4827 X_1$, where $\hat{\beta}_1 = 0.4827$ is the positive regression coefficient for the variable X_1 showing the total effect of X_1 on Y and ignoring the informations contained in X_2. But, if both X_1 and X_2 are included, simultaneously, in a similar linear regression model, $Y = \beta_o + \beta_1 X_1 + \beta_2 X_2 + \varepsilon$, then the estimated model becomes, $\hat{Y} = -4.2295 - 0.2389X_1 + 3.2201 X_2$, where $\hat{\beta}_1$ has turned out to be negative. Thus, when we ignore the variable X_2, a positive relationship between X_1 and Y is indicated but when X_2 is also included alongwith X_1 in the model, then a real negative relationship between X_1 and Y is indicated.

In the second model, $\hat{\beta}_1$ measures the effect of X_1 given that X_2 is also included. Thus, a reverse sign may indicate that some important regressors are missing in the model. In practice such case happens when we observe Y as production of a particular crop and X_1 and X_2 as the amount of local and chemical fertilizers used respectively.

(iii). Multicollinearity among the regressors also causes the problem of insignificancy to the estimated regression coefficients. Severe multicollinearity effectively inflates the variances of the regression coefficients which in turn causes the hazardness in one or more of the regression coefficients. This is illustrated in the following example.

If the correlation between the regressors is perfect then not only the variances of the estimated coefficients will be infinite but

also the parameters will be indeterminate. For example, let X_1 and X_2 be the two regressors for the model,

$$Y = \beta_1 X_1 + \beta_2 X_2 + \varepsilon . \tag{6}$$

The method of least squares will yield the estimated $\hat{\beta}_j$'s $(j = 1,2)$ as,

$$\underset{\sim}{\hat{\beta}} = (X'X)^{-1} X'Y$$

$$\text{i.e., } (X'X)\underset{\sim}{\hat{\beta}} = X'Y$$

But X'X can be defined as ([5]),

$$(X'X) = \begin{pmatrix} 1 & r_{12} \\ r_{12} & 1 \end{pmatrix} \text{ and } X'Y = \begin{pmatrix} r_{1y} \\ r_{2y} \end{pmatrix}$$

where r_{12} is the simple correlation between X_1 and X_2 and r_{jy} is the simple correlation coefficient between $X_j, (j = 1,2)$ and Y.
Since,

$$C = (X'X)^{-1} = \frac{1}{1 - r_{12}^2} \begin{pmatrix} 1 & -r_{12} \\ -r_{12} & 1 \end{pmatrix}, \text{ the variance of } \underset{\sim}{\hat{\beta}} \text{ is}$$

$$\text{Var.}(\underset{\sim}{\hat{\beta}}) = \frac{\sigma^2}{1 - r_{12}^2}, \quad (\because \text{Var.}(\underset{\sim}{\hat{\beta}}) = \sigma^2 c_{ii}, \text{ where } c_{ii} \text{ is the diagonal element of the matrix C}), \tag{7}$$

and the estimates of the regression coefficients are,

$$\hat{\beta}_1 = \frac{r_{1y} - r_{12} r_{2y}}{1 - r_{12}^2} \quad \text{and,} \tag{8}$$

$$\hat{\beta}_2 = \frac{r_{2y} - r_{12} r_{1y}}{1 - r_{12}^2} \tag{9}$$

If there exists a strong relationship between X_1 and X_2 then r_{12} will tend to 1 and from (7) it is clear that the variance of $\hat{\beta}_j$ will tend to infinity. Again from (8) and (9) we obtain,

$$\hat{\beta}_1 = \left[\frac{(\Sigma x_1 y)(\Sigma x_2^2) - (\Sigma x_1 x_2)(\Sigma x_2 y)}{(\Sigma x_1^2)(\Sigma x_2^2) - (\Sigma x_1 x_2)^2}\right] \sqrt{\frac{\Sigma x_1^2}{\Sigma y^2}} \quad \text{and,} \tag{10}$$

$$\hat{\beta}_2 = \frac{(\Sigma x_2 y)(\Sigma x_1^2) - (\Sigma x_1 x_2)(\Sigma x_1 y)}{(\Sigma x_2^2)(\Sigma x_1^2) - (\Sigma x_1 x_2)^2} \sqrt{\frac{\Sigma x_2^2}{\Sigma y^2}}, \tag{11}$$

where we have subtituted,

$$r_{1y} = \frac{\Sigma x_1 y}{\sqrt{\Sigma x_1^2 \ \Sigma y^2}}$$

$$r_{2y} = \frac{\Sigma x_{2y}}{\sqrt{\Sigma x_2^2 \Sigma y^2}} \qquad \text{with } x_1 = X_1 - \bar{X}_1$$

$$x_2 = X_2 - \bar{X}_2 \text{ and}$$

and

$$r_{12} = \frac{\Sigma x_1 x_2}{\sqrt{\Sigma x_1^2 \Sigma x_2^2}} \qquad y = Y - \bar{Y}$$

Now, if X_1 and X_2 are related by some exact relation as $X_1 = K.\ X_2$ then we find the parameters β_1 and β_2 to be indeterminate and as such the separate values of $\hat{\beta}_1$ and $\hat{\beta}_2$ cannot be obtained. This is due to the fact that if X_1 is perfectly related to X_2 the matrix $(X'X)$ becomes singular so that $|X'X| = 0$ and consequently $(X'X)$ cannot be inverted.

In practice, more often the situations will not be of 'perfect' multicollinearity and $(X'X)$ will not be exactly singular but may be close to it. The nearer the value of $|X'X|$ is to zero the larger will be the variance of the estimated regression coefficients. Sometimes, this variance will be so large that it renders to flash out an insignificant coefficient (as discussed in (i)).

(iv). Variance of the estimated response in an autocorrelated model will be considerably larger than when there is no autocorrelation in the model. This variance depends upon the variance of the regression coefficients and that of the error terms. In an autocorrelated model, the least square regression coefficients, though unbiased, are no longer minimum variance estimates. For example, consider a two-variable linear regression model,

(12) $$Y_t = \beta_o + \beta_1 X_t + \varepsilon_t$$

Let the error term ε_t be serially correlated by the relationship ([5]),

(13) $$\varepsilon_t = \rho \varepsilon_{t-1} + U_t$$

which is a first-order auto-regressive relationship between the distrubance terms, and where ρ is a simple autocorrelation coefficient and

U_t is a non-autocorrelated error term for which,

$$E(U_t) = 0$$

$$E(U_t\, U_{t+i}) = \sigma_U^2, \text{ if } i = 0$$
$$= 0, \text{ if } i \neq 0 \text{ and for all } t.$$

Now (13) may be written as,

$$(14) \qquad \varepsilon_t = \rho_{t-1} + U_t = \rho(\rho\varepsilon_{t-2} + U_{t-1}) + U_t$$
$$= U_t + \rho U_{t-1} + \rho^2 U_{t-2} + \text{----------} = \sum_{\gamma=0}^{\infty} \rho^{\gamma} U_{t-\gamma}$$

So that,

$$(15) \qquad E(\varepsilon_t) = \sum_{\gamma=0}^{\infty} \rho^{\gamma} E(U_{t-\gamma}) = 0, \text{ and}$$

$$\mathrm{Var}(\varepsilon_t) = E[\varepsilon_t - E(\varepsilon_t^2)] = E[\varepsilon_t]^2 \quad [\because \text{ from } (15)]$$
$$= E[U_t + \rho U_{t-1} + \rho^2 U_{t-2} + \ldots]^2$$
$$= E[U_t^2 + \rho^2 U_{t-1}^2 + \rho^4 U_{t-2}^2 + \ldots + \text{the product term}]$$
$$= \sigma_U^2[1 + \rho^2 + \rho^4 + \ldots\ldots] = \frac{\sigma_U^2}{1-\rho^2}, \text{ for all } t.$$

If the variance of ε_t be denoted by σ_ε^2 then,

$$(16) \qquad \sigma_\varepsilon^2 = \frac{\sigma_U^2}{1-\rho^2}, \ -1 \leq \rho \leq 1$$

Here, σ_ε^2 is the variance of the autocorrelated error terms and σ_u^2 is the variance of the non-autocorrelated error terms. If the autocorrelatation coefficient $\rho = 0$, then $\sigma_\varepsilon^2 = \sigma_u^2$ so that ε_t becomes U_t in the model. But since ρ^2 is always positive (unless $\rho = 0$), from (16) it is clear that as ρ^2 increses σ_ε^2 also increases but rapidly. So, larger the value of the autocorrelation coefficient ρ, greater will be the variance of the error terms, which in turn causes larger standard errors of the coefficients and severity leads to the problem of insignificancy in them. This is so because, from (5) we have the variance of

the least square estimate of the regression coefficient β_1 as,

$$\text{Var.}(\hat{\beta}_1) = \frac{1}{(\Sigma xi^2)^2} E[x_1^2 \varepsilon_1^2 + x_2^2 \varepsilon_2^2 + ---- + x_n^2 \varepsilon_n^2 + 2x_1 x_2 \varepsilon_1 \varepsilon_2 + ---$$

$$------ + 2x_{n-1}x_n \varepsilon_{n-1} \varepsilon_n]$$

Since for the first order auto-regressive scheme,

$$\text{Var.}(\varepsilon_t) = E(\varepsilon_t^2) = \sigma_\varepsilon^2 \text{, for all } t = 1,2,\ldots\ldots,n$$

(17)

$$\text{and } E(\varepsilon_t \varepsilon_{t-r}) = \rho^r \sigma_\varepsilon^2 \text{, for all } t \neq r,$$

(18) We have, $\text{Var.}(\hat{\beta}_1) = \frac{\sigma_\varepsilon^2}{\Sigma x_i^2} + 2\sigma_\varepsilon^2 \Sigma \frac{xi\ xj}{(\Sigma xi^2)^2} \rho^r$, for all

$i \neq j = 1,2,\ldots., n.$

Thus from (18), it is clear that, if there is no autocorrelation the second term on the right hand side of (18) will vanish, i.e., if $\rho = 0$, var. $(\hat{\beta}_1) = \frac{\sigma_\varepsilon^2}{\Sigma xi^2}$, which is as described in (5). But, if there is a positive autocorrelation ($\rho > o$), then the variance of $\hat{\beta}_1$ will be relatively larger than when there is no autocorrelation in the model.

(v). While estimating the regression coefficients of a multiple regression model, the computational errors also causes the problem of insignificant coefficients. The carryover after a considerable place of decimal point also affects the coefficients to a greater extent. Moreover, different computer programs are run with different way of carrying over or the rounding off the digits. In case of perfect multicollinearity too, the process of rounding off may yield the inverse of the matrix which will be absurd because of its singularity.

(vi). If the data are biased then also the reverse signs or insignificant coefficients appear in the regression analysis. There is a well known saying, 'Garbage in Garbage out' (GIGO) in the computer system. To avoid this type of problem, first of all checks on the quality of data should be carried out critically and it is advisable to collect fresh (primary) data as far as possible. The secondary data

may display their own errors due to the biasedness or due to the different type of interest of the former investigator.

4. Conclusion

The above points are extremely important, though they may be insufficient in dealing with the problem of insignificant coefficients of a multiple regression model. It will be an advantage to know which of the causes has played the predominant role. But in general, each of the above cause can create the hazard, so the effect of each of them should be kept as minimum as possible. To do so, the following suggestions might be helpful.

(i) The regressors should not have values in a very small range, and as such one should try to collect a large amount of data: whether time series or cross-sectional.

(ii) Caution should be exercised to see that the important regressors are not excluded from the model.

(iii) To overcome the problem of multicollinearity there are several methods including the ridge regression method, a sophisticated one originally proposed by Hoerl and Kennard [4]. But the following methods are relatively simple and widely applicable.

(a) Farrar and Glauber [3] have suggested that the size of the sample should be increased by collecting additional data in order to reduce the effect of multicollinearity in a multiple regression model. But this method will be more effective only if multicollinearity is due to the errors in measurement and happens to exist only in the sample and not in the population.

(b) Use of different sets of data (e.g. time series and cross-sectional) also helps to solve the problem of multicollinearity. It is so because the correlation among the variables of the two sets will definitely be minimum.

(c) Theoretical considerations will be helpful to decide on dropping off one of the collinear variables.

(iv). As autocorrelation causes serious problem in the multiple regression analysis, it should be eliminated as far as possible. A widely applied method of detecting the presence of autocorrelation is the wellknown Durbin and Watson [2] test statistic and for a large sample, lagged variable model the test is due to Durbin [1].

Since (a) exclusion of important regressors, (b) mis-specification of the mathematical form of the model, (c) mis-specification of the true random error term, (d) unnecessary smoothening of data, and (e) presence of multicollinearity in the regressors are the causes of autocorrelation in the multiple regression model, they should be avoided as far as possible. Re-specification of the model as suggested by Durbin [1] can also be used to eliminate the effect of autocorrelation.

(v). To avoid the problem of computational errors and the bias in the data, an investigation for the accuracy of computer codes or the rechecking of the calculating mechanism and a rechecking in the quality of data might be helpful.

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References

1. Durbin, J. (1970). 'Testing for serial correlation in least squares regression when some of the regressors are lagged dependent variables', Econometrica, Vol. 38,410-421.
2. Durbin, J and Watson, G.S. (1951). 'Testing for serial correlation in least squares regression', Biometrika, Vol. 37, 409-428 and Vol. 38, 159-178.
3. Farrar, D.E. and Glauber R.R. (1967). 'Multicollinearity in regression analysis: the problem revisited,' Review of Economics and Statistics, Vol 49, 92-107.

4. Hoerl, A.E. and Kennard R.W. (1970). 'Ridge regression: Applications to nonorthogonal problems', Technometrics, Vol. 12, 69-82.

5. Johnston J. (1972). Econometric Methods, 2nd Edition, McGraw-Hill Kogakusha Ltd., Tokyo.

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On a Submanifold of a Riemannian Manifold with Semi-symmetric Metric Connection

Ram Nivas

Summary

Hypersurfaces of a Riemannian manifold with semisymmetric metric connection have been studied by Tyuiti Imai [1] and others. In the present paper, I have considered a submanifold of codimensions 2 of the above Riemannian manifold with semisymmetric metric connection and studied some of its properties.

1. Preliminaries.

Let M^{n+1} be an (n+1)-dimensional differentiable manifold of differentiability class C and M^{n-1}, and (n-1)-dimensional manifold immersed differentiability in M^{n+1} by the immersion $\tau : M^{n-1} \to M^{n+1}$. Let us denote the differential $d\tau$ of the immersion τ by B, so that the vector field X in the tangent space of M^{n-1} corresponds to a vector field BX in that of M^{n+1}. Suppose that M^{n+1} is a Riemannian manifold with metric tensor $\tilde{g}$. Then the submanifold M^{n-1} is also Riemannian with metrio tensor $\tilde{g}$ such that $\tilde{g}(BX, BY) = g(X,Y)$ for arbitrary vector fields X,Y in M^{n-1} [1].

If the Riemannian manifolds M^{n-1} and M^{n+1} are both orientable, we can choose mutually orthogonal unit normals N_1 and N_2 defined along M^{n-1} such that $\tilde{g}(BX,N_1) = \tilde{g}(BX,N_2) = \tilde{g}(N_1,N_2) = 0$ and $\tilde{g}(N_1,N_1) = \tilde{g}(N_2,N_2) = 1$ for arbitrary vector field X in M^{n-1} [2].

We now suppose that the enveloping manifold M^{n+1} admits a semi-symmetric metric connection given by ([1],[3])

$$(1.1) \qquad \tilde{\nabla}_{\tilde{X}} \tilde{Y} = \overset{\circ}{\tilde{\nabla}}_{\tilde{X}} \tilde{Y} + \tilde{\pi}(\tilde{Y})\tilde{X} - \tilde{g}(\tilde{X},\tilde{Y})\tilde{P},$$

for arbitrary vector fields $\tilde{X}$, $\tilde{Y}$ in M^{n+1} where $\overset{\circ}{\tilde{\nabla}}$ denotes the Levi-Civita connection with respect to the Riemannian metric $\tilde{g}$, $\tilde{\pi}$ is a

1-form and $\tilde{P}$, the vector field devined by $\tilde{g}(\tilde{P},\tilde{X}) = \tilde{\pi}(\tilde{X})$, for arbitrary vector field $\tilde{X}$ of M^{n+1}. Let us now put

(1.2) $\tilde{P} = BP + \lambda N_1 + \mu N_2$, P being a vector field in the tangent space of M^{n-1} and λ, μ being functions on M^{n-1}. We have the following theorem:

<u>Theorem</u> (1.1). The connection induced on the submanifold M^{n-1} of codimensions 2 of the Riemannian manifold M^{n+1} with semisymmetric metric connection is also semisymmetric one.

<u>Proof</u>. Let $\overset{\circ}{\nabla}$ be the connection induced on the submanifold M^{n-1} from the connection ∇ on the enveloping manifold M^{n+1}, with respect to unit normals N_1 and N_2. Then we have [2].

$$\overset{\tilde{\circ}}{\nabla}_{BX} BY = B(\overset{\circ}{\nabla}_X Y) + h(X,Y)N_1 + k(X,Y)N_2 \tag{1.3}$$

for arbitrary vector fields X, Y of M^{n-1} where h and k are second fundamental tensors of M^{n-1}. Similarly, if ∇ be connection induced on M^{n-1} from the semi-symmetric metric connection $\tilde{\nabla}$ on M^{n+1}, we have

$$\tilde{\nabla}_{BX} BY = B(\nabla_X Y) + m(X,Y)N_1 + n(X,Y)N_2 , \tag{1.4}$$

m and n being tensor fields of type (0,2) of the submanifold M^{n-1}. We also have, in view of (1.1)

$$\tilde{\nabla}_{BX} BY = \overset{\tilde{\circ}}{\nabla}_{BX} BY + \tilde{\pi}(BY)BX - \tilde{g}(BX,BY)\tilde{P}$$

which in view of (1.2), (1.3) and (1.4) becomes,

$$\begin{aligned} & B(\nabla_X Y) + m(X,Y)N_1 + n(X,Y)N_2 \\ &= B(\overset{\circ}{\nabla}_X Y) + h(X,Y)N_1 + k(X,Y)N_2 \\ &\quad + \pi(Y)BX - g(X,Y)\{BP + \lambda N_1 + \mu N_2\} \end{aligned} \tag{.15}$$

where $\tilde{\pi}(BX) = \pi(X)$ is a 1-form in M^{n-1}. Thus, we have

$$\nabla_X Y = \overset{\circ}{\nabla}_X Y + \pi(Y)X - g(X,Y)P, \tag{1.6}$$

where λ and μ are chosen such that

$$(1.7)\quad \begin{aligned} &(a)\quad m(X,Y) = h(X,Y) - \lambda g(X,Y), \quad \text{and} \\ &(b)\quad n(X,Y) = k(X,Y) - \mu g(X,Y). \end{aligned}$$

Thus,

$$\nabla_X Y - \nabla_Y X = \overset{\circ}{\nabla}_X Y - \overset{\circ}{\nabla}_Y X + \pi(Y)X - \pi(X)Y$$

or

$$(1.8)\quad \nabla_X Y - \nabla_Y X - [X,Y] = \pi(Y)X - \pi(X)Y.$$

Hence the connection ∇ induced on M^{n-1} is semisymmetric one [3].

2. Totally Geodesic and Totally Umbilical Submanifolds

Let $X_1, X_2, \ldots, X_{n-1}$ be $(n-1)$ or the normal vector fields in the submanifold M^{n-1}. Then the function

$$\frac{1}{2(n-1)} \sum_{i=1}^{n-1} \left\{ h(X_i,X_i) + k(X_i,X_i) \right\}$$

is the mean curvature of M^{n-1} with respect to the Riemannian connection $\overset{\circ}{\nabla}$ and $\frac{1}{2(n-1)} \sum_{i=1}^{n-1} \left\{ m(X_i,X_i) + n(X_i,X_i) \right\}$ is the mean curvature of M^{n-1} with respect to the semi-symmetric connection ∇ [1].

From this we have the following definitions [1].

Definition (2.1) : If h and k vanish separately the submanifold M^{n-1} is called totally geodesic with respect to the Riemannian connection $\overset{\circ}{\nabla}$.

Definition (2.2) : The submanifold M^{n-1} is called totally umbilical with respect to the connection $\overset{\circ}{\nabla}$ if h and k are proportional to the metric tensor g.

Definition (2.3): M^{n-1} is called totally geodesic and totally umbilical with respect to the semi-symmetric connection ∇ according as the functions m and n vanish separately and are proportional to the metric tensor g respectively.

Now we have the follwing results:

Theorem (2.1). In order that the mean curvature of M^{n-1} with respect to the connection ∇ may coincide with that of M^{n-1} with respect to the

connection ∇, it is necessary and sufficient that $\widetilde{P}$ is in the tangent space of M^{n+1}.

Proof. In view of (1.7), we have

$$m(X_i, X_i) + n(X_i, X_i) = h(X_i, X_i) + k(X_i, X_i) - (\lambda + \mu) g(X_i, X_i).$$

Summing up for $i = 1,2,\ldots,(n-1)$ and dividing by $2(n-1)$, we obtain,

$$\frac{1}{2(n-1)} \sum_{i=1}^{n-1} \left\{ m(X_i,X_i) + n(X_i, X_i) \right\} = \frac{1}{2(n-1)} \sum_{i=1}^{n-1} \left\{ h(X_i,X_i) + k(X_i,X_i) \right\}$$

if and only if, $\lambda = \mu = 0$. Hence from (1.2), it follows that $P = BP$. Thus the vector field $\widetilde{P}$ is in the tangent space of M^{n+1}.

Theorem (2.2). The submanifold M^{n-1} is totally umbilical with respect to the Riemannian connection $\overset{\circ}{\nabla}$ if and only if it is totally umbilical with tespect to the semi-symmetric connection ∇.

Proof. The proof follows from equations (1.7) (a) and (1.7) (b).

3. The Curvature Tensor and Weigarten Equations.

Now, we shall obtain the Weingarten equations with respect to the semi-symmetric metric connection $\widetilde{\nabla}$. For the Riemannian connection $\overset{\circ}{\nabla}$, these equations are given by [2].

$$\text{(3.1)} \quad \begin{aligned} &\text{(a)} \quad \overset{\circ}{\widetilde{\nabla}}_{BX} N_1 = - BHX + l(X)N_2 \text{, and} \\ &\text{(b)} \quad \overset{\circ}{\widetilde{\nabla}}_{BX} N_2 = - BKX - l(X)N_1 \end{aligned}$$

where H and K are tensor fields of type (1.1) such that

$$\text{(3.2)} \quad \begin{aligned} &\text{(a)} \quad g(HX,Y) = h(X,Y) \\ &\text{and} \\ &\text{(b)} \quad g(KX,Y) = k(X,Y). \end{aligned}$$

Also, making use of (1.1) and (3.1) (a), we get

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$$\tilde{\nabla}_{BX} N_1 = - BHX + l(X)N_2 + \tilde{\Pi}(N_1)BX - g(BX,N_1)\tilde{P}$$

Since $\tilde{\Pi}(N_1) = \tilde{g}(\tilde{P},N_1) = \lambda$ and $\tilde{g}(BX,N_1) = 0$, we have

$$\tilde{\nabla}_{BX} N_1 = - BHX + l(X)N_2 + \lambda BX,$$

or

(3.3) $\tilde{\nabla}_{BX} N_1 = - B(H - \lambda I)X + l(X)N_2$, I denotes the identity tensor field.

Similarly, from (1.1) and (3.1) (b), we get

(3.4) $\tilde{\nabla}_{BX} N_2 = - B(K - \mu I)X - l(X)N_1$.

Putting $H - \lambda I = M_1$ and $K - \mu I = M_2$ in (3.3) and (3.4) respectively, we get,

(3.5)
(a) $\tilde{\nabla}_{BX} N_1 = - BM_1X + l(X)N_2$,
and
(b) $\tilde{\nabla}_{BX} N_2 = - BM_2X - l(X)N_1$

(3.5)(a),(b) are the equations of Weingarten with respect to the semi-symmetric metric connection $\tilde{\nabla}$.

The Riemann curvature tensor for the semi-symmetric metric connection can be obtained as follows.

Let $\tilde{R}(\tilde{X},\tilde{Y})\tilde{Z}$ be the Riemann curvature tensor of the enveloping manifold M^{n+1} with respect to the semi-symmetric metric connection $\tilde{\nabla}$. Then,

(3.6) $\tilde{R}(\tilde{X},\tilde{Y})\tilde{Z} = \tilde{\nabla}_{\tilde{X}} \tilde{\nabla}_{\tilde{Y}} \tilde{Z} - \tilde{\nabla}_{\tilde{Y}} \tilde{\nabla}_{\tilde{X}} \tilde{Z} - \tilde{\nabla}_{[\tilde{X},\tilde{Y}]} \tilde{Z}$,

Now, replacing $\tilde{X}$ by BX, $\tilde{Y}$ by BY and $\tilde{Z}$ by BZ, we get

$$\tilde{R}(BX,BY)BZ = \tilde{\nabla}_{BX} \tilde{\nabla}_{BY} BZ - \tilde{\nabla}_{BY} \tilde{\nabla}_{BX} BZ - \tilde{\nabla}_{[BX,BY]} BZ$$

which in view of (1.4) becomes

$$\begin{aligned}\tilde{R}(BX,BY)BZ = {} & \tilde{\nabla}_{BX} \{B(\nabla_Y Z) + m(Y,Z)N_1 + n(Y,Z)N_2\} - \\ & - \tilde{\nabla}_{BY} \{B(\nabla_X Z) + m(X,Z)N_1 + n(X,Z)N_2\} - \\ & - \{B(\nabla_{[X,Y]} Z) + m([X,Y],Z)N_1 + n([X,Y],Z)N_2\}\end{aligned}$$

which, again, by virtue of (1.4), Weingarten equations (3.5)(a),(b) and the condition (1.8), becomes,

$$\begin{aligned}\widetilde{R}(BX,BY)BZ = {} & BR(X,Y)Z + m(\Pi(Y)X - \Pi(X)Y,Z)N_1 + n(\Pi(Y)X - \\ & - \Pi(X)Y,Z)N_2 + \{(\nabla_X m)(Y,Z) - (\nabla_Y m)(X,Z)\}N_1 + \\ & + \{(\nabla_X n)(Y,Z) - (\nabla_Y n)(X,Z)\}N_2 + B\{m(X,Z)M_1 Y - \\ & - m(Y,Z)M_1 X + n(X,Z)M_2 Y - n(Y,Z)M_2 X\} + l(X)\{m(Y,Z)N_2 - \\ & - n(Y,Z)N_1\} - l(Y)\{m(X,Z)N_2 - n(X,Z)N_1\},\end{aligned}$$

$R(X,Y)Z$ being the Riemann curvature tensor of the submanifold with respect to the semi-symmetric connection ∇.

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References

1. Imai, Tyiuti. Hypersurfaces of a Riemannian manifold with semi-symmetric metric connection. Tensor, N.S., Vol. 23, 1972, pp. 300-306.
2. Mishra, R.S. Almost complex and Almost contact submanifolds. Tensor, N.S., Vol. 25, 1972, pp. 419-433.
3. Imai, Tyiuti Notes on semisymmetric metric connection. Tensor, N.S., Vol. 24, 1972, pp. 293-296.

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On an Asymptotic Problem Concerning the Laplace Transform

R.M. Shreshtha*

Abstract

The asymptotic behaviour of a function $\emptyset(t)$ at infinity is determined interms of the asymptotic behaviour of a function f at infinity and the function Φ, where $\emptyset$, f, and Φ are connected by the equation

$$\int_0^\infty e^{-st}\,\emptyset(t)\,dt = \Phi\left(\left(\int_0^\infty e^{-st} f(t)\,dt\right)^m\right),$$

where $m \in \mathbf{Z}$, $\Phi(0) = 0$ and $\Phi(z)$ is analytic in $/z/ < R$.

1. Introduction

The Laplace Transform Φ of a function $\emptyset$ is defined by [1]

(1.1) $$\Phi(s) := \int_0^\infty e^{-st}\,\emptyset(t)\,dt$$

for $\mathrm{Re}(s) > 0$. In the present note, we shall be concerned with the problem of determining the asymptotic behaviour of $\emptyset$ at infinity in terms of the asymptotic behaviour of a function f at infinity and the function Φ, where $\emptyset$, f, and Φ are connected by the equation

(1.2) $$\int_0^\infty e^{-st}\,\emptyset(t)dt = \Phi\left(\left(\int_0^\infty e^{-st} f(t)\,dt\right)^m\right), \quad (m \in \mathbf{Z})$$

with $\Phi(0) = 0$ and $\Phi(z)$ analytic in $/z/ < R$.

2. Luxemburg's Theorem on Convolution Product

In order to consider the asymptotic behaviour of the convolution product of two integrable functions, Luxemburg [2] has recently introduced a class of admissible functions. His definition of the class of admissible functions and the standard definition of the convolution

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product of two integrable functions are as follows:

Definition 2.1. A function $L(x)$, defined for all $x > 0$, will be called admissible whenever it is continuous and strictly positive for all $x > 0$, and satisfies the following two conditions:

i) For every $h > 0$, $\lim_{x \to \infty} L(x+h)/L(x) = 1$, i.e., $L(\exp(x))$ is slowly oscillating in the sense of Karamata,

ii) There exists a constant $\lambda \geqslant 1$, depending on L, such that for all $x > 0$, $\max(\, L(t) : x \leqslant t \leqslant 2x) \leqslant \lambda L(2x)$.

The set of admissible functions is denoted by Λ.

Definition 2.2. If $f, g, \in L^1(0, \infty)$, then the convolution product of f and g, denoted by $f*g$, is defined by [3].

$$(2.1)\qquad (f*g)(x) := \int_0^x f(x-y)\, g(y)\, dy\,, \qquad (x > 0).$$

For convenience, we shall adopt the following notations for the convolutions of a function with itself:

$f_1 := f$, and $f_k := f_{k-1} * f$, for all $k = 2,3,4,\ldots$.

With these definitions at our disposal, we are now in a position to state Luxemburg's theorem and corollary on the asymptotic behaviour of the convolution of two integrable functions.

Luxemburg's Theorem. If $f, g \in L^1(0, \infty)$ are real or complex valued functions integrable over $x > 0$ such that $f(x) \sim \ell L(x)$ $(x \to \infty)$, and $g(x) \sim m\, M(x)$ $(x \to \infty)$, where $L, M \in \Lambda$ are admissible and ℓ, m are constants, then

$$(2.2)\qquad (f*g)(x) \sim \ell\Big(\int_0^\infty g(t)\, dt\Big) L(x) + m\Big(\int_0^\infty g(t)\, dt\Big) M(x) \text{ as } x \to \infty.$$

Furthermore, there exists a positive constant x_o, depending only on f and g such that for all $x \geqslant x_o$, we have the following estimate

$$(2.3)\qquad |(f*g)(x)| \leqslant (|\ell|+1)\Big(\lambda \int_0^\infty |g(t)|\, dt\Big) L(x) + (|m|+1)\Big(\mu \int_0^\infty |f(t)|\, dt\Big) M(x)$$

where λ and μ are the constants specified by (ii) of definition 2.1 for L and M respectively.

As a corollary of the forgoing theorem, the following result is obtained.

Lemma. If $f \in L^1(0, \infty)$ and if $f(x) \sim \ell L(x)$ $(x \to \infty)$, where $L \in \Lambda$, then for all $k = 1,2,3,4,\ldots$, we have

(2.4) $$f_k(x) \sim \ell k \left(\int_0^\infty f(t)\, dt \right)^{k-1} L(x),$$

and there exists a constant λ and an $x_o > 0$ such that for all $x > x_o$ and for all $k \geqslant 2$, we have

(2.5) $$/f_k(x)/ \leq k(/\ell/+1) \left(\lambda \int_0^\infty /f(t)/dt \right)^{k-1} L(x).$$

3. Representation of the Original Function

Theorem 3.1. If $\Phi(z)$ is analytic for $\mathrm{Re}(z) > 0$, $\Phi(0) = 0$ and $f \in L^1(0, \infty)$ satisfies the condition

(3.1) $$\int_0^\infty /f(t)/dt < R,$$

then, there exists an integrable function $\emptyset$ defined by

(3.2) $$\emptyset(t) := \sum_{n=1}^{\infty} \frac{\Phi^{(n)}(0)}{n!} f_{mn}(t), \quad t > 0,\ m = 0,1,2,\ldots$$

whose Laplace Transform is

$$\Phi\left(\left(\int_0^\infty \exp(-st)\, f(t)\, dt \right)^m \right).$$

Proof. Using (3.1) it readily follows that

(3.3) $$\Phi\left(\left(\int_0^\infty \exp(-st)\, f(t)\, dt \right)^m \right) = \sum_{n=1}^{\infty} \frac{\Phi^{(n)}(0)}{n!} \left(\int_0^\infty \exp(-st)\, f(t) dt \right)^{mn}$$

$$= \sum_{n=1}^{\infty} \frac{\Phi^{(n)}(0)}{n!} \left(\int_0^\infty \exp(-st)\, f_{mn}(t) \right) dt$$

and, consequently,

(3.4) $$\emptyset(t) = \sum_{n=1}^{\infty} \frac{\Phi^{(n)}(0)}{n!} f_{mn}(t)$$

defines $\emptyset \in L^1(0, \infty)$.

This completes the proof of the theorem.

In particular, if we take $m = 1$, we arrive at the result due to Luxemburg [2].

4. The Asymptotic Behaviour of the Original Function

We shall now state and prove a theorem which provides us with the asymptotic behaviour of the original function ϕ at infinity.

Theorem 4.1. If $\Phi(z)$ is analytic for $\mathrm{Re}(z) > -R$, $R > 0$, with $\Phi(0) = 0$ and $f \in L^1(0, \infty)$ satisfies

$$(4.1) \qquad f(t) \sim \ell L(t), \; t \to \infty, \text{ for some } L \in \Lambda,$$

and

$$(4.2) \qquad \int_0^\infty /f(t)/ \, dt < R/\lambda \;,$$

then $\phi(t)$, defined by (1.2), satisfies

$$(4.3) \qquad \Phi(t) \sim m\ell L(t)\, \Phi' \left(\int_0^\infty f(t)dt \right), \; t \to \infty.$$

Proof. In view of the theorem (3.1).

$$(4.4) \qquad \phi(t) = \sum_{n=1}^{\infty} \frac{\Phi^{(n)}(0)}{n!} f_{mn}(t)$$

is the required function, whose asymptotic behaviour we intend to determine. Also, from the lemma in section two, it follows that

$$\sum_{n=1}^{\infty} \frac{\Phi^{(n)}(0)}{n!} \frac{f_{mn}(t)}{L(t)}$$

converges uniformly in t (for large t). Hence

$$\lim_{t \to \infty} \frac{\phi(t)}{L(t)} = \sum_{n=1}^{\infty} \frac{\Phi^{(n)}(0)}{n!} \left(\lim_{t \to \infty} \frac{f_{mn}(t)}{L(t)} \right)$$

$$= \sum_{n=1}^{\infty} \frac{\Phi^{(n)}(0)}{n!} mn\ell \left(\int_0^\infty f(t) \, dt \right)^{mn-1}$$

$$= m\ell \left(\int_0^\infty f(t) \, dt \right)^{m-1} \Phi' \left(\left(\int_0^\infty f(t) \, dt \right)^m \right)$$

and this completes the proof.

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References:

1. D.V. Widder; The Laplace Transform, Princeton, 1946. [2] W.A.J. Luxemburg; Applicable Analysis, Vol.(2). No. 1, (1978), 61. [3] I.I. Hirschman and D.V. Widder; The Convolution Transform, Princeton, 1955.

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