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2. Stokes, G.G., On the effect of the internal friction on the motion of pendulums, Trans. Camb. Phil. Soc., 9 (1851), 8-106.

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Local and Global Problems: The Sheaf Approach

by

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0. The use of different approaches to a mathematical problem is always very profitable. Two strategies which have become of common use for geometers and analysts are the local approach to a problem and the global one. This has been the case for most of the existence questions which arise in analysis (existence of implicit ly defined functions, existence of solutions of differential equations), but the method of considering global problems from a local point of view is also of central importance for the construction of many objects when geometers try to extend classical results (a typical use of local definitions is represented by the notion of abstract manifold, which is defined as locally homeomorphic to $\mathbb{R}^n$, but which may be, from a global point of view, very different from it). One of the most useful tools in dealing with these situations is offered by the concept of sheaf, whose birth dates back to the late forties, and is due to J. Leray and H. Cartan (see [1], [5]).

This paper is intended as an introduction to sheaves, conceived as a bridge between local and global theories. The reader is assumed to have some familiarity with the basic notions of calculus, of topology and of algebra (mainly the definitions of group and of partially ordered set). Our examples are of varying degrees of difficulty, so that the more experienced reader can see how sheaf theory works in some more involved situations (e.g. the Weierstrass theorem on the existence of a holomorphic function with assigned zeroes, or the solvability of the inhomogeneous Cauchy-Riemann equations).

In section 1, we present in detail some typical situations where local study is useful to clarify the problem at hand, or where a global approach would be too restrictive (in particular we discuss the definition of orientability of surfaces and the problem of existence of implicit ly defined functions). Section 2 is devoted to the definition of sheaves of abelian groups and to the construction of a few concrete examples. In section 3 the problems exposed in section 1 are studied with the aid of sheaves, thus showing the versatility of this tool (this is probably the most technical section, and the less experienced reader may skip it without having any problem for the comprehension of the paper). In the second part of the paper (sections 4 and 5) we look at sheaves as the main example of the notion of sets variable with continuity (developed in the last twenty years by F.W. Lawvere and others, see [4]): this

notion illustrates how the "knowledge" obtained through sheaves moves, in some sense, in a continuous fashion, from the local to the global knowledge. In this sense, deeper than the one exposed in the first part of the paper, sheaves constitute the real link between local and global. More specifically we present sheaves as continuously variable sets and by means of a few examples, taken from elementary calculus, we study the basic notions concerning sets, such as equality, membership and the concept of function. The bibliography, while certainly non-exhaustive, points out some references where a very complete list can be found (see [2], [7]).

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1. We start by recalling some examples which should make clear the different approaches to a mathematical problem. The situations we will describe are very different from one another, but they all have one thing in common: they must be analyzed from at least two view-points: the "local" and the "global" one.

The first significant example concerns implicit functions. Consider the equation $F(x,y)=0$ where $F:\mathbb{R}^2 \to \mathbb{R}^2$ is a continuous function: does there exist a continuous function $f:\mathbb{R}\to\mathbb{R}$ such that $F(x,f(x)) \equiv 0$? If the answer is yes, we will say that the problem has a "global" solution. On the other hand, even if the problem has no global solutions, we can, nevertheless, look for "local" ones; by this we mean solutions defined only

on a small open set. We would like to remark that even if for each x_o the equation $F(x,y)=0$ admits a local solution $y=f_{x_o}(x)$ near x_o, together, these do not necessarily form a global one.

Another example which should clarify the "local-global dichotomy", comes out when comparing a cylinder Γ with a Moebius band B. Both of them can be obtained from a paper strip by pasting two opposite sides; if the joint is natural, the surface is a cylinder (see fig.1.a), if it is made after a 180°-twist, we obtain a Moebius band (see fig.1.b). The reader can easily convince himself that these two surfaces are "locally equal" (up to homeomorphism) (see fig.1.c), but "globally different". In fact it is impossible to deform continuously Γ to B as Γ is an orientable surface, while B is not. We shall come back to this example in the following section; in particular we will see how much sheaf terminology is able to explain orientability problems for surfaces.

Another, more technical, example, which we will refer to later, deals with a well known theorem of Weierstrass which solves the following problem: let $\{a_k\}_{k=1,2,...}$ be a discrete subset of $\mathbb{C}$ $(a_k \in \mathbb{C})$; the request is to construct a holomorphic function f vanishing at each a_k with assigned multiplicity m_k. (A slightly different problem considers meromorphic functions with given zeroes and poles). If we approach the problem from the local view-point, we can restrict our analysis to a "small" neighborhood U_k of a_k which contains

no other a_j $(j \neq k)$. Then a local solution of the problem is easily given in U_k by the function $f_k(z)=(z-a_k)^{m_k}$. Problems arise when we look for a global solution. Since the a_k's are infinitely many, the "very easy" answer

$$f(z) = \prod_{k=1}^{+\infty} (z-a_k)^{m_k}$$

may be meaningless, as, in general, such an infinite product does not converge. Indeed, Weierstrass' thoerem solved the problem, showing that if the a_k's do not accumulate in $\mathbf{C}$, it is possible to construct a function, globally defined on $\mathbf{C}$, with the assigned zeroes. To do this, Weierstrass defined

$$f(z) = \prod_{k=1}^{+\infty} f_k(z)\lambda_k(z),$$

where the λ_k's are holomorphic nowhere vanishing functions, introduced just to make the infinite product converge.

In section 3 we will describe a different approach to the problem based on sheaf theory. Not only will it allow us to generalize the theorem to the case of other surfaces, but also evidence the topological condition which makes this generalization possible.

2. First of all we describe two simple mathematical situations. Let $\mathbb{R}$ be the real axis, $\mathbb{Z}$ the abelian group of the integers, endowed with discrete topology, $\mathcal{Z}=\mathbb{R}\times\mathbb{Z}$ their product and $\pi : \mathcal{Z}\to\mathbb{R}$ the projection onto the first factor. The triplet $<\mathcal{Z},\mathbb{R},\pi>$ has the following properties:

1) $\mathcal{Z}$ and $\mathbb{R}$ are topological spaces;
2) for each z in $\mathcal{Z}$ ($z=(x,n)$; $x\in\mathbb{R}$; $n\in\mathbb{Z}$) there is an open neighborhood U of z homeomorphic to an open subset of $\mathbb{R}$ (see fig.2);
3) for each x_o in $\mathbb{R}$, the set $\mathcal{Z}_{x_o}=\pi^{-1}(x_o)=\{z\in\mathcal{Z}/\ \pi(z)=x_o\}$ is naturally endowed with a structure of abelian group (indeed $\mathcal{Z}_{x_o}\cong\mathbb{Z}$).

Another, less trivial, example is construed by using germs of continuous functions. First, consider two continuous functions $f,g\colon \mathbb{R}\to\mathbb{R}$ and say that, at x_o, f is equivalent to g if there is some open neighborhood U of x_o such that f and g agree, when restricted to U (see fig.3,a). An equivalence class $\{f\}_{x_o}$ of continuous functions at x_o is called a "germ" at x_o. Notice that functions belonging to a same germ at x_o must coincide not only at x_o, but also in some (not necessarily always the same) neighborhood of x_o (see figg.3,b and c). If now $\{f\}_{x_o}$ and $\{g\}_{x_o}$ are two germs at x_o, their sum is defined as the germ at x_o represented by the function $f+g$, in formula $\{f\}_{x_o}+\{g\}_{x_o}=\{f+g\}_{x_o}$

We can think of the germs at x_o as "seeds" constituting a "stalk" on x_o. The "sheaf" of all these stalks (as x_o varies

in $\mathbb{R}$) is usually denoted by $\mathscr{C}$. We can endow $\mathscr{C}$ with a standard topology (for this and for the proofs of statements of this section see [3]) and define a continuous map $\pi:\mathscr{C}\to\mathbb{R}$, by setting $\pi(\{f\}_{x_o}) = x_o$.

Like in the previous example, we can see that:

1) $\mathscr{C}$ and $\mathbb{R}$ are topological spaces;
2) for each germ in $\mathscr{C}$ there exists an open neighborhood in $\mathscr{C}$ containing it and homeomorphic to an open subset of $\mathbb{R}$;
3) for each x_o in $\mathbb{R}$, $\pi^{-1}(x_o)$, the set of all germs at x_o, is an abelian group.

The analogy between the above-mentioned cases should be clear; both of them are examples of a much more general mathematical concept which appears whenever a continuous map $\pi:\mathscr{S}\to X$ between topological spaces (e.g. $\mathscr{Y}$ and $\mathbb{R}$ or $\mathscr{C}$ and $\mathbb{R}$) "behaves well". So we are in a position to explain what a sheaf is.

We call sheaf (of abelian groups) (see [7]) a triplet $<\mathscr{S}, X, \pi>$ where:

1) $\mathscr{S}$ and X are topological spaces;
2) $\pi:\mathscr{S}\to X$ is a local homeomorphism (that is for each y in $\mathscr{S}$ there exists an open neighborhood U of y homeomorphic, via π, to an open subset of X);
3) for each x in X, the set $\mathscr{S}_x=\pi^{-1}(x)$ is an abelian group, and the group operations are continuous in the topology of $\mathscr{S}$.

The sets $\mathscr{S}_x$ ($x\in X$) are called stalks and $\mathscr{S}$ appears as the

"sheaf" of all the "stalks" $\mathscr{S}_x$ (see fig.4). The local nature of the definition should be clear on the grounds of 2). Notice that in 3), instead of "abelian group" we might require "group" or "set", defining sheaves of groups, sets, among others.

As already mentioned, sheaves are a good tool for deepening both local and global aspects of a problem. To explain this, let us define what is meant by "section" of a sheaf. Roughly speaking, a section is a continuous cut through stalks, so to say, parallel to X. More precisely, let $<\mathscr{S}, X, \pi>$ be a sheaf and U an open subset of X. A section of $\mathscr{S}$ on U is a continuous map $s: U \rightarrow \mathscr{S}$ such that $\pi(s(x))=x$ for each x in U (see fig.4); so that in case of $<\mathscr{Z}, \mathbb{R}, \pi>$ the sections are actually represented by segments parallel to $\mathbb{R}$ as shown in fig.2, while in the case of $<\mathscr{C}, \mathbb{R}, \pi>$ one can prove that a section on $U \subseteq \mathbb{R}$ is nothing but a real continuous functions defined on U.

The sections of a sheaf $\mathscr{S}$ on an open subset $U \subset X$, form an abelian group which we will denote by $\Gamma(U, \mathscr{S})$. In particular, taking U=X, we can consider the "global sections" of $\mathscr{S}$, i.e. the sections of $\mathscr{S}$ on X itself.

We conclude this section returning to the problem of orientability. If X is a surface in the euclidean space, consider a point x in X. To choose an orientation at x (hence on any disk D around x) means to decide which rotation, around x, should be considered positive: clockwise or anticlockwise. We

will use α_x (resp. β_x) to denote the anticlockwise (resp. clockwise) rotation around x, which defines an orientation α_D (resp. β_D) on D. If D_1 and D_2 are two disks in X, with non-empty intersection, we say that they are coherently oriented if they induce on $D_1 \cap D_2$ the same orientation (see figs.5, a and b). Finally, if X is a surface covered by n "disks" $D_1, \ldots, D_n$, we say that X is orientable if it is possible to assign an orientation to each D_i so that they be pairwise coherently oriented. It is now clear that while the cylinder is orientable, the Moebius band is not (see fig. 5, c).

It is not difficult to translate the construction explained above in terms of sheaves: on each point x in X the stalk $\mathcal{S}_x$ consists of two elements α_x and β_x; the space $\mathcal{S}$ is the disjoint union of the stalks $\mathcal{S}_x$ and the projection π maps every α_x (and β_x) into the point x. We can see that, on each disk D, the set $\Gamma(D,\mathcal{S})$ consists of the two possible orientations α_D and β_D described above (the phenomenon is analogous to the one of the sheaf $\langle \mathcal{Z}, \mathbb{R}, \pi \rangle$) (see fig. 5, d). Thus a global orientation is nothing but a global section of the sheaf (of sets) $\langle \mathcal{S}, \mathbb{R}, \pi \rangle$, usually called sheaf of orientations, and so X is orientable if and only if $\Gamma(X,\mathcal{S}) \neq \emptyset$.

3. Sheaf theory, besides giving excellent results in the study of geometrical objects (as curves, surfaces, etc.) has proved to be particularly useful in analysis, mainly when dealing with those problems which can be studied both from the local and the global point of view. In fact, as we have already pointed out, sheaves can often be used (with a standard technique, applicable to a wide range of different situations) to extend local results to global level.

The standard process we want to describe is the following: first one translates the property $\mathscr{P}$ to be extended (e.g. the solvability of a given differential equation) so that $\mathscr{P}$ holds locally exactly when a certain sheaf homomorphism $p:\mathscr{F}\to\mathscr{G}$ is surjective. However, this surjectivity does not imply surjectivity of the naturally induced map $P: \Gamma(X,\mathscr{F}) \to \Gamma(X,\mathscr{G})$ which would correspond to $\mathscr{P}$ being globally true. The second step consists in determining the sheaf $\mathscr{K}$, kernel of the homomorphism $p:\mathscr{F}\to\mathscr{G}$. At this point, either we already know the validity of $\mathscr{P}$ at a local level, or we prove it. Finally, the study of sheaves $\mathscr{K}$, $\mathscr{F}$ and $\mathscr{G}$, will enable us to decide, in a standard way (which uses the cohomology with coefficients in a sheaf), about the surjectivity of P, i.e. about the global validity of $\mathscr{P}$

We now wish to illustrate the process described above in two significant examples, which we have already mentioned.

Let us begin with the Weierstrass theorem or, better, with the extension of it to the case of meromorphic functions with zeroes and poles assigned (with multiplicity). As we have said, the problem consists in constructing two sheaves, and a suitable homomorphism between them, such that the surjectivity of it is equivalent to the local solvability of the Weierstrass' problem. With this in mind, let us construct, over $\mathbb{C}$, (similarly to what done in section 2 for the sheaf $\mathscr{C}$) the sheaf $\mathscr{M}^*$, whose sections are meromorphic functions which are not identically zero and the sheaf $\mathscr{O}^*$ whose sections are never vanishing holomorphic functions.

Since $\mathscr{O}^*$ is contained in $\mathscr{M}^*$, it is possible to construct in a natural way the quotient sheaf $\mathscr{D}=\mathscr{M}^*/\mathscr{O}^*$, whose stalk $\mathscr{D}_x$ on the point x in $\mathbb{C}$ is constituted by equivalence classes of germs of functions which are meromorphic in a neighborhood of x, where two germs are equivalent if their quotient is a germ in $\mathscr{O}^*_x$, i.e. if they have the same order in x (this, in particular, implies that the stalk $\mathscr{D}_x$ is isomorphic to the group $\mathbb{Z}$ of integers).

Hence we have a natural surjective homomorphism $p\colon \mathscr{M}^* \to \mathscr{D}$, whose kernel is the sheaf $\mathscr{O}^*$. We can now give a particularly simple description for the sheaf $\mathscr{D}$: if f_x belongs to $\mathscr{M}^*_x$, then $p(f_x) \in \mathscr{D}_x$ is the integer which expresses the order (of zero or of pole) of f_x; as a consequence, if $f \in \Gamma(U,\mathscr{M}^*)$, for U an open subset of $\mathbb{C}$, then $P(f) \in \Gamma(U,\mathscr{D})$ is what, in algebraic geometry, is called the divisor of f, and can be identified

with a discrete set of points x in U, the zeroes and the poles of f, to each of which a relative integer, expressing the order of f, is associated.

From this description, it is clear that the surjectivity of p is equivalent to the local solvability of the Weierstrass' problem, while the surjectivity of $P:\Gamma(\mathbb{C},\mathcal{M}^*) \to \Gamma(\mathbb{C},\mathcal{D})$ is equivalent to the global solvability of the Weierstrass' problem. With the techniques of cohomology theory, it can be proved that P is surjective.

If, more generally, we substitute $\mathbb{C}$ with a Riemann surface X, we can show that the problem of the extension of the theorem of Weierstrass, is not an analytical problem (how it could seem from the formulation given in section 1) but the purely topological problem of deciding whether $P:\Gamma(X,\mathcal{M}^*) \to \Gamma(X,\mathcal{D})$ is surjective (the fact that in case of Riemann surfaces the surjectivity of P is only a topological question, is not at all trivial: the interested reader is referred to [3] for details). In particular, it can be proved that if X is open, then the result of Weierstrass can be extended to it, while this is false for X compact. We wish to remark that, with this technique, we did not content ourselves with a new proof of a classical theorem, but we also came to an understanding of its hidden nature, thus opening the path to possible extensions of the theorem itself.

Let us proceed to the second example: the research of solutions (both local and global) of a differential equation. We will concentrate on a central example from complex analysis: the so called $\bar{\partial}$-problem. The operator $\bar{\partial}$ is a partial derivatives operator defined on infinitely differentiable complex functions of a single complex variable z=x+iy, by

$$\partial f/\partial \bar{z} = \frac{1}{2}(\partial f/\partial x + i\partial f/\partial y).$$

It is of some interest (the applications are manifold) to find out whether for every function u in C^{∞}, there is a C^{∞} function f such that

$$(*) \qquad \partial f/\partial \bar{z} = u.$$

For this purpose, we consider the sheaf $\mathscr{C}^{\infty}$ of germs of C^{∞} functions and we think of $\partial/\partial \bar{z}$ as an operator acting on those germs: in this way we obtain a sheaf homomorphism

$$\mathscr{C}^{\infty} \xrightarrow{\partial/\partial \bar{z}} \mathscr{C}^{\infty},$$

whose surjectivity is equivalent to the possibility of solving the equation (*), at least locally. We now proceed as we have indicated above: we look for the sheaf which is kernel of the homomorphism $\partial/\partial \bar{z}$, which in this case turns out to be the sheaf $\mathcal{O}$ of holomorphic functions: it can be shown, with techniques of the classical analysis, that (*) always has a local solution, and, finally, again by cohomological techniques, it can be proved that the local solvability of (*) can be extended to the level of sections, i.e. to the global level.

4. In this section we shall introduce the "presheaf" concept leading to the notion of complete presheaf which will provide an alternative description of sheaves and exhibit them as "continuously variable sets".

Let $\langle G, \leq \rangle$ be a partially ordered set (poset in the sequel), a presheaf of sets F on $\langle G, \leq \rangle$ is a law which assigns to each element a in G a set $F(a)$ and to each pair of elements a and b in G with $a \leq b$, a map $F_{ab}: F(b) \to F(a)$, satisfying the following "coherence" conditions:

(i) $\forall c \leq b \leq a;\ x \in F(a),\ F_{cb}(F_{ba}(x)) = F_{ca}(x)$;

(ii) $\forall a \in G,\ x \in F(a),\ F_{aa}(x) = x$.

The element $F_{ba}(x)$ is called the restriction of x to a. They are indeed restrictions in the following two examples. Let $\mathbb{R}$ be the collection of all open subsets of $\mathbb{R}$ ordered by inclusion. Let $C(U_\alpha)$ be the set of all real-valued continuous functions defined over U_α, that is

$$C(U_\alpha) = C_\alpha = \{f: U_\alpha \to \mathbb{R} \mid f \text{ continuous}\}.$$

If $U_\beta \subseteq U_\alpha$ and f is continuous over U_α, then the restriction $f_{|U_\beta}$ is continuous over U_β. This describes a law associating a set C_α to each open U_α on the real line, so that, whenever $U_\beta \subseteq U_\alpha$, we have a map from C_α to C_β: the restriction map.

We obtain a similar construction by replacing C_α with

$$B_\alpha = \{f: U_\alpha \to \mathbb{R} \mid f \text{ bounded}\}.$$

We shall refer to the last two examples as "presheaf C" of the continuous functions and "presheaf B" of the bounded functions.

Let's analyze presheaves from a different viewpoint. When G is a one-element set, $G = \{a\}$, a presheaf on $\langle G, \leq\rangle$ is given by just specifying a set F(a), thus identifying it with a "constant set". If G consists of two distinct elements, say t and a, with $a \leq t$, a presheaf is given by assigning two sets F(t) and F(a) and a map from F(t) into F(a). Thus a presheaf on this particular poset G is not only a pair of sets but a "variation from stage t to stage a". For an arbitrary poset G, a presheaf is a family of sets indexed by G and a collection of maps between them satisfying the coherence condition above. Again a presheaf can be thought of as a "discretely varying set" (see [4]).

We shall look for "elements" in these "variable sets". In the trivial case ($G=\{a\}$) the concept is the usual one. In the second case ($G=\{a,t\}$) an element of a presheaf is either just an element of F(a) or a pair constituted by an element of F(t) and its restriction to F(a). An element must be thought of either as an entity which, when born in F at stage t, still exists at stage a, or as an entity which does not appear at t and comes out at a. In the case of an arbitrary poset, an element is an entity of F at some stage, together with all its restrictions (see fig.6).

Let us look at an example in the presheaf C of continuous functions: consider the function $f(x) = 1/x$. Undefined in the origin, it is not continuous on all $\mathbb{R}$, however it is an element of C which is born at stage $\mathbb{R}\setminus\{0\}$ and exists from then on.

There is one fundamental difference between the presheaf B and the presheaf C. Take the following open covering of the real line:

$$\mathcal{U} = \{U_h = (h-1, h+1) \mid h \in \mathbb{Z}\},$$

and consider the function $f(x)=x$ and its restrictions $f_{|U_h}$. A function $f_{|U_h}$ is continuous and bounded in U_h; hence it is an element of both presheaves. Besides, $f_{|U_h}$ and $f_{|U_{h+1}}$ agree on $U_h \cap U_{h+1}$, so that they form a compatible family of functions; hence, intuitively, in both C and B, the $f_{|U_h}$'s should give a global element, i.e. f itself. But this happens only for C: the function $f(x)=x$ is continuous but unbounded in $\mathbb{R}$.

Complete presheaves will be defined as follows: let X be a topological space and τ the family of its open subsets ordered by inclusion. Let F be a presheaf of sets on $\langle\tau,\le\rangle$; F is a complete presheaf of sets if, for each covering $\{U_i\}_{i\in I}$ of an open U of X, the following compatibility condition holds:

- let $s_i \in F(U_i)$ for each $i \in I$; if the restrictions of s_k and s_j to $U_k \cap U_j$ coincide, then there exists one and only one $s \in F(U)$ whose restrictions to each U_i is s_i.

This formal property is enjoyed by C, but not by B.

A non- trivial theorem (see [3]) states that the concept of complete presheaf is equivalent to that of sheaf (see section 2), in the sense that to each complete presheaf F defined on the topology $\langle\tau,\le\rangle$ of a space X one can canonically associate a sheaf $\langle\mathcal{F}, X, \pi\rangle$ such that $\Gamma(U,\mathcal{F})=F(U)$, for any

$U \in \tau$; in particular the complete presheaf C will result exactly the sheaf $\langle \mathscr{C}, \mathbb{R}, \pi \rangle$. Below we shall not distinguish between a complete presheaf and its associated sheaf.

5. The concept of sheaf on a topological space we introduced above allows us to visualize the idea of a "continuously varying set", where the compatibility property links the variations of the set and the topology of the space: the variations are controlled, forced continuously by topology (see [4]). We spoke of "constant sets", "sets varying in two stages", "discretely variable sets" and "continuously variable sets", trying to justify the words we used. We shall analyze what sense the word set has, at least for sheaves (continuously variable sets, or c.v.sets).

Let X be a fixed topological space; first of all we must point out that any usual, common set A gives rise to a c.v. set on X: consider the discrete topology on A and form the space $A \times X$. Then $\langle A \times X = \mathscr{A}, X, \pi \rangle$ is a sheaf as described in section 2, where π is the projection from $A \times X$ to X. Then the sections of $\mathscr{A}$ generate a c.v. set and it is easy to check that $\Gamma(U, \mathscr{A})$ is the sum of as many copies of A as the number of connected components of U.

As before, we call element of a sheaf $\langle \mathscr{F}, X, \pi \rangle$ any object which is present at some stage, i.e. it belongs to $\Gamma(U, \mathscr{F})$ for some open U, together with all its restrictions. Let s and t be elements of a sheaf $\mathscr{F}$, set $S = \{W \subseteq X \mid s = t \text{ at stage } W\}$ and let

$V=\bigcup_{W\in S} W$; then by the compatibility property s=t at V. We can say therefore that, in a sheaf, local equality forces global equality.

It seems clear that a subset of a c.v. set F should be a continuously variable collection of elements of F, that is a collection of elements of F with the compatibility property. For instance, the sheaf $\mathscr{C}^\infty$ of infinitely differentiable real valued functions (given U open in $\mathbb{R}$, $C^\infty(U)$ is the set of infinitely differentiable real valued functions defined on U) satisfies the compatibility property; hence it is a sub-c.v. set of the c.v. set $\mathscr{C}$.

We describe now a sub-c.v. set G of a c.v. set F. Let s be a possible element of F existing at stage U. There are three possible cases: s belongs already to G; s will never be in G; s will belong to G from some stage W onward. Indeed, let

$$S = \{W \text{ open in } U \mid s\in G \text{ at stage } W\},$$

and let $V=\bigcup_{W\in S} W$; then, by the compatibility property, $s\in G$ at stage V. This means that an element in F cannot be included in G at different, <u>not comparable</u> stages (which instead may happen in a presheaf, a discretely variable set), but there is a precise time when the element begins to be in the sub-c.v. set. This translates as: in a sheaf local membership forces global membership. Therefore the chances of s being in G at stage U are as many as the open subsets of U.

Thanks to our study, we can now describe a function between c.v. sets, which acts as the characteristic function of a subset G of F. Classically, the characteristic function of a subset $X \subseteq Y$ divides the elements $x \in X$, assigning them the value 1, from the elements $y \in Y \setminus X$ which take value 0. Since in the case of a c.v. set the concept of membership relation is more sophisticated, in which an element x can be a member of a subset X at different stages, the range of the characteristic function must be a c.v. set: set Ω for the presheaf that to each open U of X assigns the set of open subsets of U. If $U \subseteq V$, the restriction Ω_{VU} is defined by

$$\Omega_{VU}(W) = U \cap W$$

for any open $W \subseteq V$. It is easy to verify that Ω is complete. The characteristic function $\chi_G : F \to \Omega$ takes an element s in F at the stage U to the stage $\chi_G(s) \subseteq U$ at which s will begin to belong to G.

We introduced and described in a very intuitive way the concepts of element, subset and characteristic function for c.v. sets in close analogy with the usual everyday sets. This comparison can be taken a good deal further; the authors do not intend to do so. We limit ourselves to refer the interested reader to the extensive literature on the subject (see [2]). Of course, the many problems mentioned in the last part of this paper are too profound, hard and still contested, to allow us to think we have satisfactorily exposed them in an essay of a few pages. We hope to have awakened the reader's interest in the matter.

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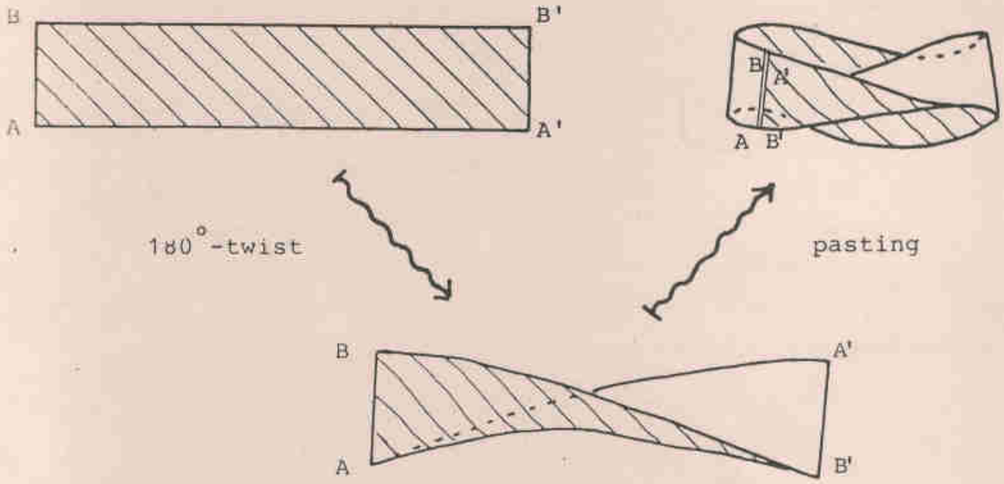

FIG.1,b. The 180°-twist, followed by the pasting, produces the Moebius band.

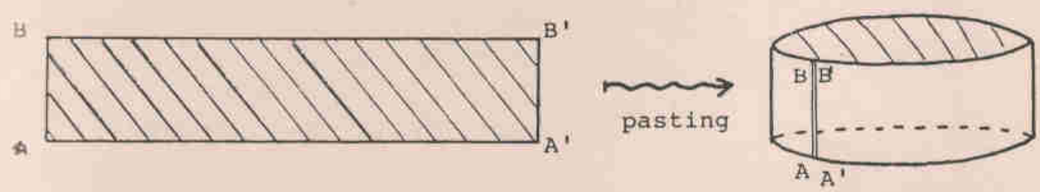

FIG.1,a. The pasting of two opposite sides of a rectangle as shown in figure, produces a cylinder.

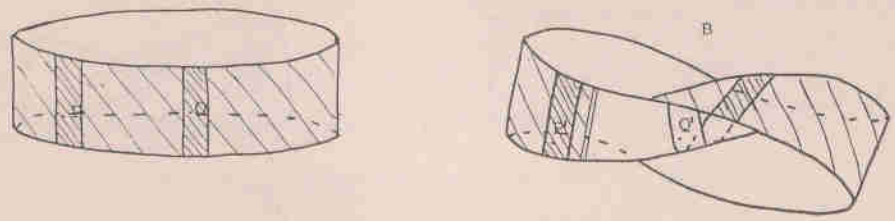

FIG.1.c. The cylinder and the Moebius band are "locally equal", since every point P' of B is contained in a strip which is homeomorphic to a strip in Γ, containing P.

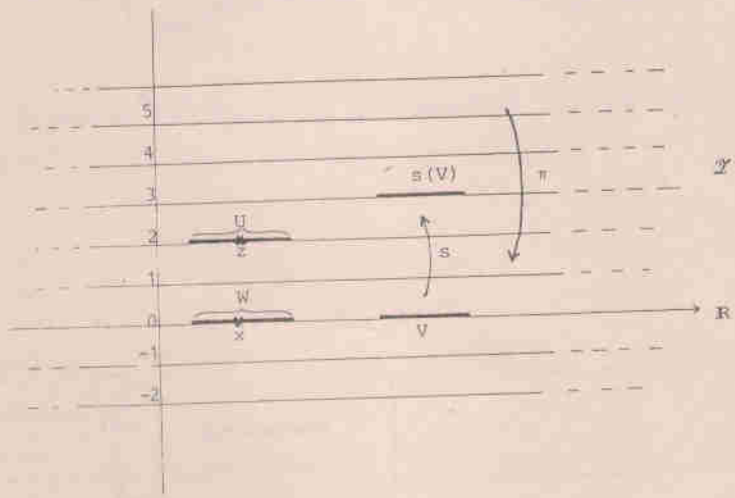

FIG.2. The topological spaces $\mathscr{Z}$ and $\mathbb{R}$ with the local homeomorphism a section $s \in \Gamma(V, \mathscr{Z})$.

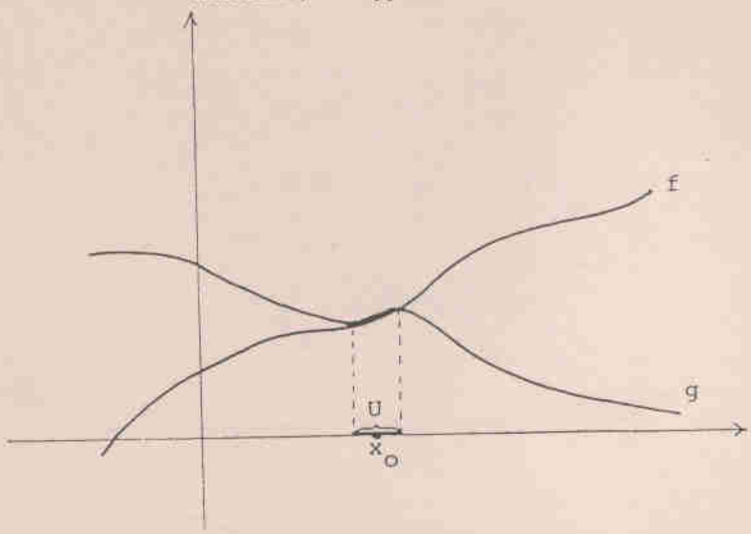

FIG.3,a. The functions f and g define the same germ at x_o, as they coincide in the neighborhood U of x_o.

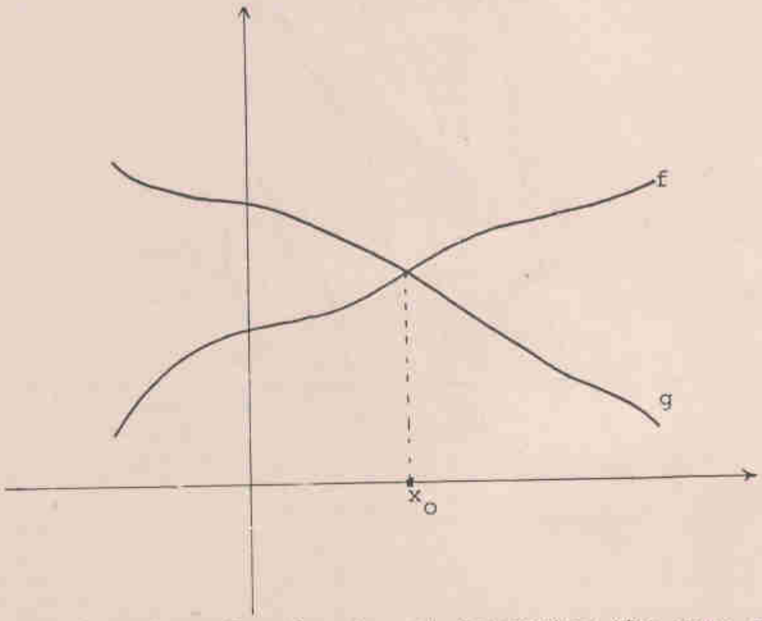

FIG.3,b. The functions f and g do not determine the same germ at the point x_o.

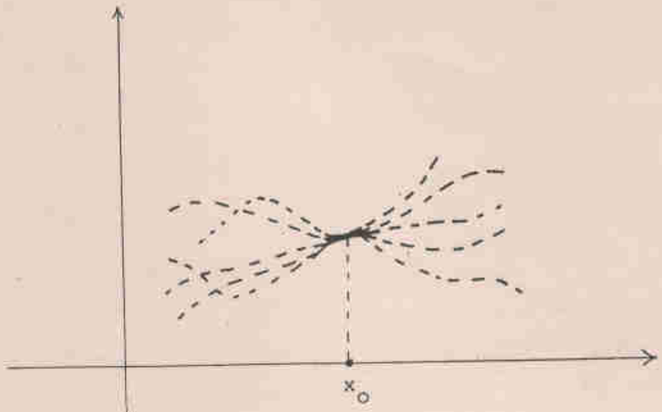

FIG.3.c. A germ at the point x_o.

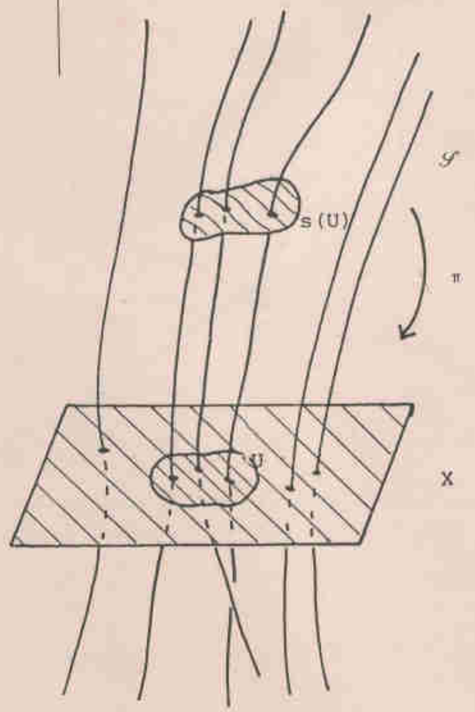

FIG. 4 . A section s on the open U of X.

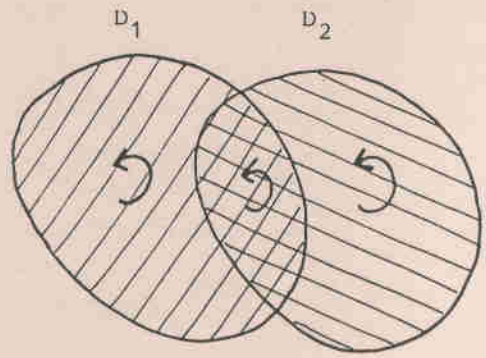

FIG. 5.a. D_1 and D_2 are coherently oriented.

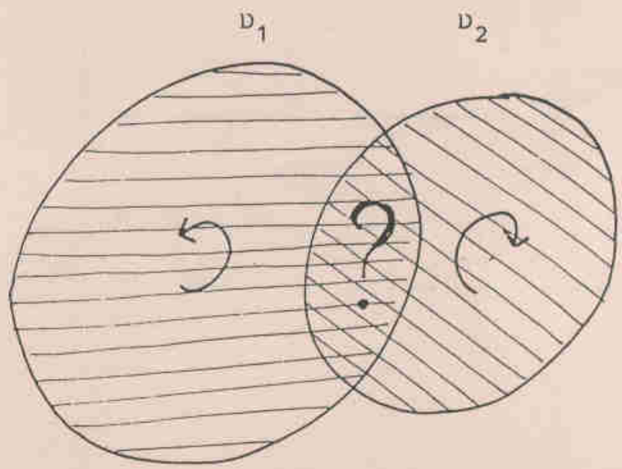

FIG. 5.b. D_1 and D_2 are not coherently oriented.

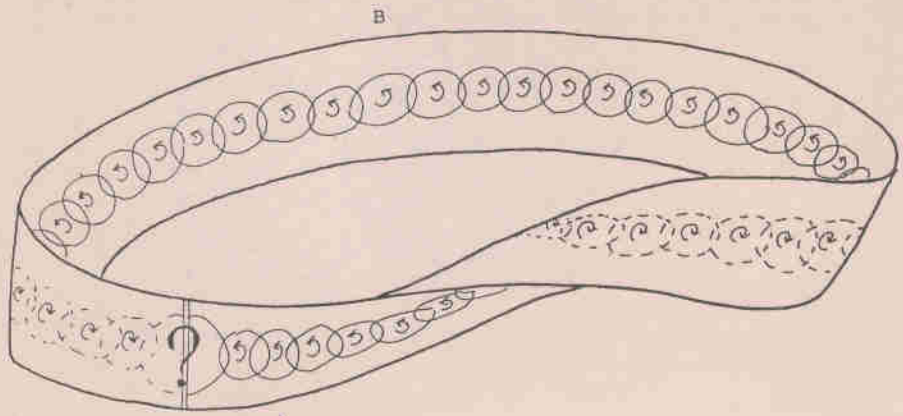

FIG. 5.c. The Moebius band B is not orientable, as it is impossible to give a coherent orientation to a chain of disks covering it.

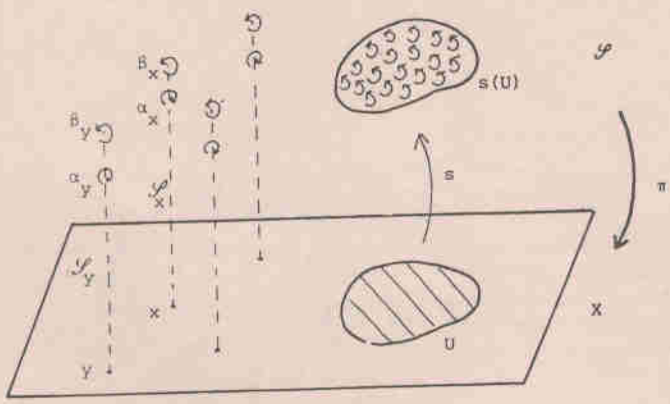

FIG 5.d. The sheaf $\langle \mathscr{S}, X, \pi \rangle$ of orientations on a surface X: stalks $\mathscr{S}_x$, $\mathscr{S}_y$, and a section s.

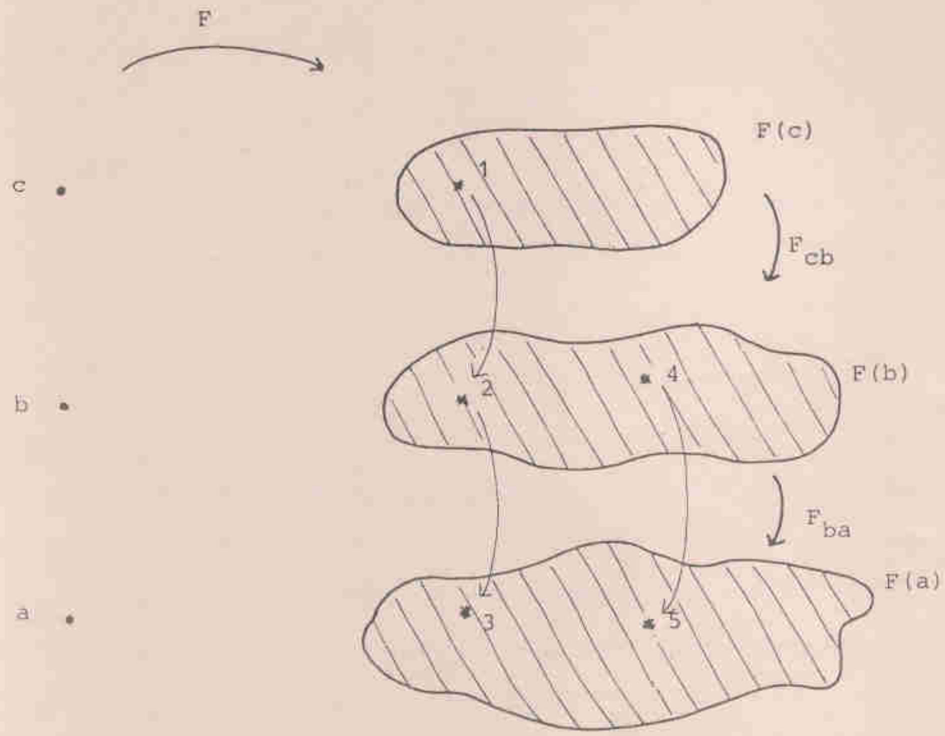

FIG. 6 . The figure shows an example of a presheaf F on the poset $G=\{a,b,c\}$ with $c \leq b \leq a$. In this example, the collections $\{1,2,3\}$, $\{2,3\}$, $\{3\}$, $\{4,5\}$, and $\{5\}$ are elements of the presheaf F, while, for example, $\{1,2\}$ is not an element, since $F_{ba}(2)=3$, which does not belong to $\{1,2\}$.

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An Indecomposibility Criterion for Rings

By

P. Singh and Gurprit Kaur

Abstract:

In this paper we have defined a ring R being an irredundant union of its ideals and established an indecomposibility criterion for R under certain conditions. For this purpose some other interesting results have also been proved.

Introduction:

Here we have considered rings which are irredundant unions of their proper ideals and established certain results concerning them. A ring R is said to be an irredundant union of the ideals S_i if none of the $S's$ is contained in the union of the remaining ones. Further basing on these results we have established an indecomposibility criterion for such rings.

Lemma 1:

Let a ring R be an irredundant union of the ideals S_i. Then for each i, S_i contains the intersection of all the remaining $S's$.

Proof:

Since R is an irredundant union of the ideals S_i, S_i (for any fixed i) cannot be contained in the union S of the remaining $S's$.

Let x be an element of S_i, which is not in S and Y be an element contained in the intersection of the remaining $S's$.

If $x + y \in S$;

then $x + y \in S_j$, for some j ($\neq i$).

But $y \in S_j$, implying that $x \in S_j$, which is a contradiction.

Therefore $x + y \notin S_j$. on the other hand, if $x + y \notin S$, then $x + y \in S_i$ and this implies that $y \in S_i$, which proves the Lemma.

Lemma 2

Let the ring R be an irredundant union of ideals S_i ($i = 1, \ldots, n > 2$). If $M = S_2 \cup S_3 \cup \cdots \cup S_n$, and x is not in M, then kx is in M forsome $k = 1, \ldots, n-1$.

Proof:

If $x \notin M$, then $x \in S_1$.

Let y be an element of S_2 such that $y \notin S_1$. Then $px + y$ is in M, for $p = 1, 2, \ldots, n+1$, since if $px + y$ is in S_1, then $y \in S_1$. Now if $px + y \in S_2$, for some $p = 1, \ldots, n-1$, then $px \in S_2 \subset M$, as desired.

If $px + y = qx + y$, for some $p, q = 1, \ldots, n-1$ with $p > q$, then $(p-q)x = 0$ is in M, as desired. Hence we may assume that the n-1 elements $x + y, 2x + y, \ldots, (n-1)x+y$ are distinct and in $S_3 \cup \cdots \cup S_n$.

It means that $px + y$, $qx + y$ are in S_m for some $m = 3, \ldots, n$ and some $p, q = 1, \ldots, n-1$ with $p > q$. Then $(p - q)x = (px+y) + [-(qx+y)]$ is $S_m \subset M$. Hence the Lemma.

Theorem 1.

If for every $x \in R$, $\frac{x}{k}$ is in R for every positive integer k less than a certain n, then R is not the irredundant union of n(or fewer) of its proper ideals.

Proof:

We will prove the theorem for a given n. The "Or fewer" part will follow obviously since n and any m smaller than n have analogous hypothesis.

Let R be the irredundant union of exactly n proper ideals S_i ($i = 1, \ldots, n > 2$). We set $M = S_2 \cup \cdots \cup S_n$. If $x \in M$, then $y = \frac{x}{p}$ is not in M for any positive integer p. Then $y = \frac{x}{(x-n)!}$ is not in M, but $y \in R$, by hypothesis. Therefore by Lemma 2, ky is in M for some k less than n. Since $x = (n-1)!$, $y = \frac{(n-1)!}{k}$ (ky), therefore x is in M, which is a contradiction. Hence R is not the irredundant union of n ideals.

If R is a finite ring with N elements, the hypothesis of theorem 1 is equivalent to the requirement that $(n-1)!$ be prime to N.

Therefore, we have the

Corollary:

Let R be a finite ring with N elements, p the smallest prime dividing N. Then R is not the union of p or fewer of its proper ideals.

Theorem 2.

Let R be a finite ring with N elements, p the smallest prime dividing N. Then if R is a union of exactly p + 1 proper ideals S_i then each ideal has index p and p^2 divides N.

Proof:

If n is the number of elements in an ideal S_i of R, then we will write: $O(S_i) = n$. If there is no ideal S_i ($i = 1, \cdots, p + 1$) with index p, then $S_i \; \forall i$ ($i = 1, \ldots, p + 1$) must have indices greater than p,

since p is the smallest prime divisor of N.

Hence $O(S_i) \leqslant \frac{N}{p+1}$ for all i.

Then we have $N \leqslant \sum O(S_i) < (p+1)\frac{N}{p+1} = N$, a contradiction.

Hence some S_j must have index p. Also $S_i + S_j$ $(i \neq j)$ is an ideal of R, which can not be proper bacause in that case R is a union of p ideals, namely $S_i + S_j$ and p-1 ideals S_k, $k = 1, 2, \ldots, p+1$, $k \neq i$, $k \neq j$, which is not possible.

Thus $S_i + S_j = R$, $i \neq j$.

We know that for ideals S , S , $i \neq j$

$$O(S_i + S_j).O(S_i \cap S_j) = O(S_i).O(S_j),$$

or that $O(R).O(S_i \cap S_j) = O(S_i).\frac{O(R)}{p}$,

or that $p.O(S_i \cap S_j) = O(S_i)$ for $i \neq j$.

Now we have to show that all S_i's have index p.

If q_i be the index of S_i, $i \neq j$, then $q_i \geqslant p$.

Let us suppose $q_i > p$ for some $i \neq j$.

Then

$$N = O(R) \geqslant O(S_j) + \sum_{i \neq j}\left[O(S_i) - O(S_i \cap S_j)\right]$$

$$= \frac{N}{p} + \sum_{i \neq j}\left[\frac{N}{q_i} - \frac{N}{pq_i}\right]$$

$$= \frac{N}{p} + \sum_{i \neq j}\frac{N}{q_i}(1 - \frac{1}{p})$$

$$= \frac{N}{p} + \sum_{i \neq j}(\frac{p-1}{p})\frac{N}{q_i}$$

$$< \frac{N}{p} + p.\frac{p-1}{p}.\frac{N}{p}$$

$$= \frac{N}{p} + N - \frac{N}{p} = N,$$

a contradiction.

Hence $q_i = p$ for all $i \neq j$.

Also $O(S_i \cap S_j) = \frac{N}{p^2}$, implying that p^2 divides N.

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On Tauberian Theorem for Lambert's Method

By

S. R. Sinha, S. K. Varma and Rakesh Maindwal

Abstract:

In the present note we have proved that the Tauberian condition for Lambert's method given by Peyerimhoff [1] is best possible. Our result is analogous to the result of Shapiro [2].

1. Definitions and notations: We denote the series

$$\sum_{n=1}^{\infty} a_n \tag{1.1}$$

by S and its value when it is convergent by s (so that, for example, $S=s$ means that Σa_n converges to s) By $S=s(A)$, $S=s(c)$ and $S=s(L)$, we mean that the series (1.1) is summable by Abel-method, Cesaro-method and Lambert-method to s respectively. We denote the hypothesis $S=s$, $S=s(A)$, $S=s(c)$ and $S=s(L)$ by K, K_A, K_C and K_L respectively. It is well established fact that if a series Σa_n is convergent then it is summable by any regular method of summability, but the converse is not true. If however we impose a suitable condition on the magnitude of a_n then the converse is true. Theorem of this type are called Tauberian theorems, after A. Tauber [3], who proved the first theorem of this type.

The most important Tauberian conditions with which we shall be concerned here are:

$$(o) \quad a_n = o(n^{-1})$$
$$(o') \quad a_n = o(n^{-1+\epsilon}),\ \epsilon > 0,$$

To avoid repetition we don't give the definition of Cesaro and Abel-method, but for the sake of completeness we give following definition of Lambert-method.

A series $\sum_{n=1}^{\infty} a_n$ is called summable to s by the Lambert-method (L) if

$$\sigma(x) = (1-x)\sum_{\nu=1}^{\infty} \frac{\nu a_\nu x^\nu}{1-x^\nu} \rightarrow s \quad \text{as} \quad x \rightarrow 1-0$$

This method L is regular (c.f. Peyerimhoff [1] p. 82-83)

2. Tauber's first theorem 'K_A and (o) imply K'

Corresponding to Tauber's first theorem Peyerimhoff [1] p. 83 proved following theorem for Lambert - method.

Theorem A 'K_L and (o) imply K'

Shapiro [2] proved that the Tauber's first theorem is best in the sense that (o) can not be replaced by (o').

So a very natural question arises;

Can we prove that the Theorem A is best possible in the sense of Shapiro [2] OR Can we prove that the Tauberian condition in Theorem A is best possible?

We have answered this in affirmative. In fact we shall prove the following theorem

Theorem 'K_L and (o') do not imply K'

3. PROOF OF THEOREM Let $x=1-\frac{1}{n}$ $(n=1,2,\ldots.)$ then

$K_L \Longrightarrow$

$$\sigma(x) = (1-x) \sum_{\nu=1}^{\infty} \frac{\nu a_\nu x^\nu}{1-x^\nu} \longrightarrow s \text{ as } x \to 1-0$$

$$\sigma(x) - s_n = (1-x) \sum_{\nu=n+1}^{\infty} \frac{\nu a_\nu x^\nu}{1-x} + (1-x) \sum_{\nu=1}^{n} a_\nu\left(\frac{\nu x^\nu}{1-x^\nu} - n\right)$$

(3.1) $$= I + II, \text{ Say}$$

$$I = O(1-x) \sum_{\nu=n+1}^{\infty} \frac{\nu^\epsilon x^\nu}{(1-x^\nu)} \quad \text{by } (O')$$

$$= O\left(\frac{1-x}{1-x^{n+1}} \cdot x^{n+1}\right) \left[x(n+1)^\epsilon + x^2(n+2)^\epsilon + \ldots \right]$$

(3.2) $\to \infty$ as $n \to \infty$ for $\epsilon > 0$ however small it may be

(This can be made $O(1)$ only when $\epsilon = 0$)

Now

$$n - \frac{\nu x^\nu}{1-x^\nu} = \nu + \frac{1}{1-x^\nu}\left(\frac{1-x^\nu}{1-x} - \nu\right) \leqslant \nu$$

and

$$n - \frac{\nu x^\nu}{1-x^\nu} = \frac{1}{1-x^\nu}\left(\frac{1-x^\nu}{1-x} - \nu x^\nu\right) \geqslant 0$$

Therefore

$$II = O(1-x) \sum_{\nu=1}^{n} |\nu a_\nu|$$

$$= O\left(1/n\right) \sum_{\nu=1}^{n} \nu^\epsilon \quad \text{by } (O')$$

(3.3) $\to \infty$ as $n \to \infty$ for $\epsilon > 0$ however small it may be,

This can be made $O(1)$ only if $\epsilon = 0$

So collection of (3.1), (3.2) and (3.3) completes the proof

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A Fixed Point Theorem in Compact Topological Spaces

By

K.P.R. Sastry and S.V.R. Naidu

In this paper, we prove a fixed point theorem for a family of self maps on a compact topological space which is a nontrivial extension of a result of Wong [2] in a compact metric space.

Notation: Suppose X is a nonempty set and $\mathcal{F}$ a nonempty family of self maps on X. A subset A of X is said to be $\mathcal{F}$- invariant if $f(A) \subset A$ for all f in $\mathcal{F}$.

For $x \in X$, the orbit $O(x)$ of x, with respect to $\mathcal{F}$, is defined as

$$O(x) = \left\{ y \;\middle|\; y = x \text{ or } y = gx \text{ where } g \text{ is the composite of a finite number of elements of } \mathcal{F} \right\}$$

(This idea of the orbit of a point with respect to a family of self maps is derived from Madhusudana Rao [1].)

For a subset A of a topological space, cl. A is the closure of A.

Theorem 1. Let $(X, \mathcal{T})$ be a compact topological space, $\mathcal{F}$ a nonempty family of continuous self maps on X and F a non-negative symmetric real valued function on XxX such that it is continuous in each variable, and $F(x,y) \neq 0$ whenever x, y are distinct elements of X.

Suppose that

$$(1.1) \quad F(fx,gy) \leq \max\left\{ \sup_{h\in\mathcal{F}} F(x,hy),\ \sup_{h\in\mathcal{F}}{}^{+} F(y,hx),\ \min\left\{ \sup_{z\in O(x)\cup O(y)} F(x,z),\ \sup_{z\in O(x)\cup O(y)} F(y,z) \right\} \right\}$$

for all $x, y \in X$ and $f, g \in \mathcal{F}$,

and

(1.2) whenever A is an $\mathcal{F}$-invariant closed subset of X, with $D(A) = \sup \left\{ F(u, v) \mid u, v \in A \right\}$ positive, there exist $y, w \in A$ such that

$$\sup_{z \in O(w)} F(y,z) < D(A).$$

Then the family $\mathcal{F}$ has a common fixed point in cl.O(x) for each x in X.

Proof. Fix $x \in X$. Since $O(x)$ is $\mathcal{F}$-invariant and every member of $\mathcal{F}$ is continuous on X, it follows that cl.O(x) is $\mathcal{F}$-invariant. By the compactness of X, there exists a nonempty minimal $\mathcal{F}$-invariant closed subset Y of cl.O(x).

Suppose that $D(Y) > 0$. Then, by (1.2), there exist $y, w \in Y$ such that

$$(1.3) \qquad r = \sup_{z \in O(w)} F(y,z) < D(Y).$$

Since cl.O(w) is an $\mathcal{F}$-invariant closed subset of Y, from the minimality of Y, it follows that

$$(1.4) \qquad Y = \text{cl. } O(w).$$

Let $S = \left\{ u \in Y \mid F(u,v) \leq r \text{ for all } v \text{ in } Y \right\}$.

From (1.3) and (1.4), we have $y \in S$. By the continuity of $F(.,v)$ on X for each $v \in X$, it follows that S is closed. Fix $u \in S$ and define $U = \left\{ v \in Y \mid F(fu, v) \leq r \text{ for all } f \text{ in } \mathcal{F} \right\}$. Then U is closed and $u \in U$ since $F(fu, u) = F(u, fu) \leq r$ for all f in $\mathcal{F}$ as Y is $\mathcal{F}$-invariant.

It follows from (1.1) that

$F(fu, gv) \leq r$ for all g in $\mathcal{F}$ and v in U so that U is $\mathcal{F}$-invariant.

Hence, by the minimality of Y we have $U = Y$ so that $fu \in S$ for all f

in $\mathcal{F}$. Hence S is $\mathcal{F}$-invariant so that again by the minimality of Y, $S = Y$. Hence $D(Y) = D(S) \leq r < D(Y)$ which is a contradiction. Hence $D(Y) = 0$, so that Y is a singleton.

The following is a special case of the above theorem with $F = d$.

Corollary 2. Let (X, d) be a compact metric space and $\mathcal{F}$ a nonempty family of continuous self maps on X. Suppose that

$$d(fx, gy) \leq \max \left\{ \sup_{h\in\mathcal{F}} d(x,hy), \sup_{h\in\mathcal{F}} d(y,hx), \min\left\{ \sup_{z\in O(x)\cup O(y)} d(x,z), \sup_{z\in O(x)\cup O(y)} d(y,z)\right\}\right\}$$

for all $f, g \in \mathcal{F}$ and $x, y \in X$

and whenever A is an $\mathcal{F}$-invariant closed subset of X with $\delta(A)$ (diameter of A) positive, there exist $y, w \in A$ such that

$$\sup_{z\in O(w)} d(y,z) < \delta(A).$$

Then the family $\mathcal{F}$ has a common fixed point in cl.O(x) for each x in X.

By taking $\mathcal{F}$ to be singleton, we get immediately the following, from corollary 2.

Corollary 3. Let (X,d) be a compact metric space, f a continuous self map on X. Suppose that

$$d(fx, fy) \leq \max \left\{ d(x,fy), d(y,fx), \min\left\{ \sup_{z\in O(x)\cup O(y)} d(x,z), \sup_{z\in O(x)\cup O(y)} d(y,z)\right\}\right\}$$

for all $s,y \in X$, and

whenever A is an f-invariant closed subset of X with $\delta(A)$ positive, there exist y, w in A such that

$$\sup_{n \geqslant 0} d(y, f^n(w)) < \delta(A).$$

Then, for each x in X, the sequence $\left\{f^n(x)\right\}$ of iterates of x, has a subsequence converging to a fixed point of f.

Now we state Wong's result which follows immediately from corollary 3.

<u>Corollary 4.</u> (Wong [3], Theorem 3). Let f be a continuous self map on a compact metric space (X,d). Suppose that there exist symmetric functions p, q, r on XxX into the closed interval [0, 1] such that, $p + q + r \leqslant 1$ and for all x, y in X,

$$d(fx,fy) \leqslant a\, d(x,y) + b\, d(x,fy) + c\, d(y,fx)$$

where $a = p(x,y)$, $b = q(x,y)$ and $c = r(x,y)$, and whenever A is an f-invariant closed subset of X with $\delta(A)$ positive, there exist $y, w \in A$ such that

$$\sup_{n \geqslant 0} d(y,f^n w) < \delta(A).$$

Then, for each x in A, the sequence $\left\{f^n(x)\right\}$ has a subsequence converging to a fixed point of f.

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