so that

$$(\alpha-\delta)^{-1} H_n(P:Q) = (\alpha-\delta)^{-1} \log \left(\sum_{i=1}^{n} p_i^{\alpha+\beta_i-1} q_i^{\gamma-\alpha} / \sum_{i=1}^{n} p_i^{\beta_i}\right)$$

$$= G(P:Q)^{(\alpha,\bar{\beta},\gamma,\delta)}$$

This establishes the theorem.

3. Particular Cases

In this section we investigate the particular cases of 'Generalized Information Measure' (1.1).

(i) For $\gamma=\alpha$, $\delta=2\alpha-1$, (1.1) reduces to Rathie's (1970) generalized entropy of order α and type β_i defined as

$$H_{\alpha}^{\left\{p_i\right\}}_{\left\{\beta_i\right\}} = (1-\alpha)^{-1} \log \left(\sum_{i=1}^{n} p_i^{\alpha+\beta_i-1} / \sum_{i=1}^{n} p_i^{\beta_i}\right), \alpha \neq 1$$

(ii) For $\beta_i=\beta$, $i=1,2,\ldots,n$, $\gamma=\alpha$, $\delta=2\alpha-1$, (1.1) gives Kapur's (1967) generalized entropy of order α and type β defined as

$$H_{\alpha}^{\left\{p_i\right\}}_{\beta} = (1-\alpha)^{-1} \log \left(\sum_{i=1}^{n} p_i^{\alpha+\beta-1} / \sum_{i=1}^{n} p_i^{\beta}\right), \alpha \neq 1$$

(iii) Putting $\beta_i=1$, $i=1,2,\ldots,n$, $\gamma=\alpha$, $\delta=2\alpha-1$, we see that (1.1) reduces to

$$\hat{H}_{n,\alpha}(_1,p_2,\ldots,p_n) = (1-\alpha)^{-1} \log\left(\sum_{i=1}^{n} p_i^{\alpha} / \sum_{i=1}^{n} p_i\right), \alpha \neq 1$$

(iv) Also, when the distribution P is complete, that is $\sum_{i=1}^{n} p_i=1$, $\hat{H}_{n,\alpha}(p_1,p_2,\ldots,p_n)$ becomes Renyi's (1961) additive entropy of order α defined as

$$\hat{H}_{n,\alpha}(p_1,p_2,\ldots,p_n) = (1-\alpha)^{-1} \log \left(\sum_{i=1}^{n} p_i^{\alpha}\right), \alpha \neq 1.$$

(v) For $\alpha=1$, $\beta_i=1$, $i=1,2,\ldots,n$ and $\delta=\gamma$, (1.1) becomes

$$\hat{H}_{n,\gamma}(p_1,p_2,\ldots,p_n:q_1,q_2,\ldots,q_n) = (1-\gamma)^{-1} \log\left(\sum_{i=1}^{n} p_i q_i^{\gamma-1} / \sum_{i=1}^{n} p_i\right), \gamma \neq 1$$

which reduces to the additive inaccuracy of order γ , viz.

$$\hat{H}_{n,\gamma}(p_1,p_2,\ldots,p_n;q_1,q_2,\ldots,q_n) = (1-\gamma)^{-1}\log\left(\sum_{i=1}^{n} p_i q_i^{\gamma-1}\right),\ \gamma\neq 1$$

when the distribution P is complete.

(vi) Substitution of $\gamma=1$, $\delta=1$, in (1.1) gives information of order α and type β_i defined by Rathie (1970) as

$$I_{\alpha}\left[\begin{Bmatrix}p_i\\ \beta_i\end{Bmatrix}/Q\right] = (\alpha-1)^{-1}\log\left(\sum_{i=1}^{n} p_i^{\alpha+\beta_i-1}\, q_i^{1-\alpha}\Big/\sum_{i=1}^{n} p_i^{\beta_i}\right),\ \alpha\neq 1.$$

(vii) For $\beta_i=\beta$, $i=1,2,\ldots,n$, $\gamma=1$, $\delta=1$, (1.1) reduces to Kapur's (1968) information of order and type β viz.

$$I_{\alpha}\left[\begin{Bmatrix}p_i\\ \beta\end{Bmatrix}/Q\right] = (\alpha-1)^{-1}\log\left(\sum_{i=1}^{n} p_1^{\alpha+\beta-1}\, q_i^{1-\alpha}\Big/\sum_{i=1}^{n} p_i^{\beta}\right),\ \alpha\neq 1.$$

(viii) For $\beta_i=1$, $i=1,2,\ldots,n$, $\gamma=1$, $\delta=1$, (1.1) reduces to

$$I_{\alpha}(P/Q) = (\alpha-1)^{-1}\log\left(\sum_{i=1}^{n} p_i^{\alpha} q_i^{1-\alpha}\Big/\sum_{i=1}^{n} p_i\right),\ \alpha\neq 1$$

which gives Renyi's (1961) information of order α , viz.

$$I_{\alpha}(P/Q) = (\alpha-1)^{-1}\log\left(\sum_{i=1}^{n} p_i^{\alpha} q_i^{1-\alpha}\right),\ \alpha\neq 1.$$

when the distributions P and Q are complete.

(ix) For $\alpha=1$, $\delta=\gamma$, (1.1) gives Rathie's (1970a) generalized inaccuracy measure defined as

$$H_{\gamma}\left[\begin{Bmatrix}p_i\\ \beta_i\end{Bmatrix}/Q\right] = (1-\gamma)^{-1}\log\left(\sum_{i=1}^{n} p_i^{\beta_i} q_i^{\gamma-1}\Big/\sum_{i=1}^{n} p_i^{\beta_i}\right),\ \gamma\neq 1.$$

(x) For $\beta_i=\beta$ for all $i=1,2,\ldots,n$, $\alpha=1$, $\delta=\gamma$, (1.1) gives the generalized inaccuracy defined as

$$H_{\gamma}\left[\begin{Bmatrix}p_i\\ \beta\end{Bmatrix}/Q\right] = (1-\gamma)^{-1}\log\left(\sum_{i=1}^{n} p_i^{\beta} q_j^{\gamma-1}\Big/\sum_{i=1}^{n} p_i^{\beta_i}\right),\ \gamma\neq 1.$$

4. Remarks

From 'Generalized Information Measure' different measures stated above can be deduced by putting different values of α, β_i, γ, δ and consequently their characterization theorems can be obtained by putting different values of these parameters.

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On Com-Solvable Groups

— P. Singh and Gurprit Kaur

Abstract

In the paper "On Com-solvable Groups", we have defined a special class of groups with the help of the notion of commutator. These groups have been called com-solvable. We have established a necessary and sufffi-cient condition for the com-solvability of a group. In this connection we have also considered direct products of groups. Further, we have also defined centre-solvability of a group and established a relationship be-tween the notions of com-solvability and centre-solvability.

1. Introduction

This paper is devoted to a study of a special class of solvable groups. We designate these special groups as com-solvable groups. A solvable group is defined to be a group G for which there exists a chain of sub-groups $G_o = G \supseteq G_1 \supseteq G_2 \supseteq G_n = \{e\}$, where e is the identity of G, G_{i+1} is normal in G_i and $\frac{G_i}{G_{i+1}}$ is abelian, i = 0, 1, ..., n-1. If there exists an abelian subgroup H of G such that the commutator subgroup G' of G is contained in H, then evidently G will be solvable. In this event we have the following chain of subgroups serving the purpose: $G_o = G \supseteq H \supseteq \{e\}$. We call such groups G, Com-solvable.

2. Definition

A group G is called com-solvable if, and only if there exists an abelian subgroup H of G such that the commutator subgroup G' of G is contained in H.

Theorem 1

Every com-solvable group is solvable.

Proof

Let G be a com-solvable group, then the commutator subgroup G' of G is contained in some abelian subgroup H of G. Hence H is normal in G and G/H is abelian. Now $G \supseteq H \supseteq \{e\}$ is a chain of subgroups of G, making G solvable.

Example 1

Every abelian group is com-solvable.

For, if G be any abelian group then it is an abelian subgroup of itself and it contains $G' = \{e\}$.

Example 2

Let us consider the symmetric group $G = s_3$, the elements are

$$f_1 = e,\ f_2 = \begin{pmatrix} 123 \\ 213 \end{pmatrix},\ f_3 = \begin{pmatrix} 123 \\ 132 \end{pmatrix},\ f_4 = \begin{pmatrix} 123 \\ 321 \end{pmatrix},\ f_5 = \begin{pmatrix} 123 \\ 231 \end{pmatrix},$$

$f_6 = \begin{pmatrix} 123 \\ 312 \end{pmatrix}$. Then $G' = \{e, f_5, f_6\}$ which is abelian. $G \supseteq G' \supseteq e$ is a chain of subgroups of G, satisfying our need for com-solvability of S_3.

Example 3

We know that the symmetric group S_4 is solvable. The commutator subgroup of S_4 is the alternating subgroup A_4 of S_4, which is non-abelian and as such S_4' is not contained in any abelian subgroup. Hence S_4 cannot be com-solvable.

3. Definition

A group G is called centre-solvable if, and only if, there exists a subgroup H of G in the centre, $Z(G)$, of G such that G/H is abelian.

Theorem 2

Every centre-solvable group is com-solvable, but not conversely.

Proof:

Let G be a centre-solvable group. Then there exists a subgroup H of G in the centre $Z(G)$ such that G/H is abelian. Then $G' \subseteq H$. Also H is abelian. Thus G is com-solvable.

That the converse of the theorem is not true, follows from the fact that the summetric group S_3 is com-solvable, but it is not centre-solvable.

Theorem 3

Every subgroup of a com-solvable group is com-solvable.

Proof

Let G be a com-solvable group and let H be an abelian subgroup of G such that $G' \subseteq H$. If K be any subgroup of G, then $K' \subseteq G' \subseteq H$. Also $K' \subseteq K$. Hence $K' \subseteq H \cap K$, which is abelian, proving the com-solvability of K.

Theorem 4

Let H be a normal subgroup of a group G.

Then G/H is com-solvable if G is com-solvable.

Proof

Let G be com-solvable. Then there exists an abelian subgroup K of G such that $G' \subseteq K$. Therefore, $\frac{G}{K}$ is abelian. Since H and K are normal subgroups of G, HK is a normal subgroup of G containing H and K. Therefore $\frac{HK}{H}$ is a normal subgroup of G/H.

Now

$$\frac{G/H}{HK/H} \cong \frac{G}{HK} \cong \frac{G/K}{HK/K}.$$

Also $\frac{G/K}{HK/K}$, being a quotient group of the abelian group G/K, is abelian.

Now,

$$\text{Since } \frac{G/H}{HK/H} \text{ is abelian, } \left(\frac{G}{H}\right)' \subseteq \frac{HK}{H}.$$

We claim that $\frac{HK}{H}$ is abelian.

Let $a, b \in \frac{HK}{H}$. Then there exist $h_1, h_2 \in H$ and $k_1, k_2 \in K$ such that $a = H(h_1 k_1)$ and $b = H(h_2 k_2)$.

$a = H(h_1 k_1) = (Hh_1)k_1 = H k_1$; similarly $b = H k_2$.

$$\begin{aligned} \text{Now, } \quad a\,b &= (H k_1)(H k_2) \\ &= H (k_1 k_2) \\ &= H (k_2 k_1) \quad (K \text{ is abelian}) \\ &= (H k_2)(H k_1) \\ &= b\,a\,. \end{aligned}$$

Hence G/H is com-solvable.

The converse of the above theorem is not true as can be seen from the following example:

We take $G = S_4 . H = \{e, (12)(34), (13)(24), (14)(23)\}$ is a normal subgroup of S_4 and $\frac{S_4}{H} \cong S_3$. By ex. 2, S_3 is com-solvable and so is $\frac{S_4}{H}$. But S_4 is not com-solvable, by ex. 3.

Theorem 5

If $\frac{G}{H}$ and $\frac{G}{K}$ are com-solvable groups, then the group $\frac{G}{H \cap K}$ is com-solvable.

Proof

Since $\frac{G}{H}$ and $\frac{G}{K}$ are com-solvable, there exist A and B satisfying $(\frac{G}{H})' \subseteq \frac{A}{H}$ and $(\frac{G}{K})' \subseteq \frac{B}{K}$ and such that $\frac{A}{H}$ and $\frac{B}{K}$ are abelian.

Now, $\frac{G/H}{A/H}$ is abelian and $\frac{G/H}{A/G} \cong \frac{G}{A}$. Hence $\frac{G}{A}$ is abelian and therefore $G' \subseteq A$. Similarly $G' \subseteq B$ and so $G' \subseteq A \cap B$, implying that $\frac{G}{A \cap B}$ is abelian.

Now since $\frac{A}{H}$ is abelian, we have $Ha_1 \, Ha_2 = Ha_2 \, Ha_1$ for all $a_1, a_2 \in A$,

Or $Ha_1a_2 = Ha_2a_1$.

Similarly $Kb_1 \, b_2 = Kb_2 \, b_1$, for all $b_1, b_2 \in B$.

Since $Hxy = Hyx$ and $Kxy = Kyx$ for all $x, y \in A \cap B$,

We have $Hxy \cap Kxy = Hyx \cap Kyx$,

or $(H \cap K)xy = (H \cap K)yx$,

or $(H \cap K)x \, (H \cap K)y = (H \cap K)y \, (H \cap K)x$.

Hence $\frac{A \cap B}{H \cap K}$ is abelian.

Now, $\frac{G/H \cap K}{A \cap B/H \cap K} \cong \frac{G}{A \cap B}$ which is abelian.

Hence $\left(\frac{G}{H \cap K}\right)' \subseteq \frac{A \cap B}{H \cap K}$ Hence the theorem.

It is known that if G_1 and G_2 are two groups, the composition in each being denoted multiplicatively, then $G_1 \times G_2 = \{(g_1, g_2) : g_1 \in G_1, g_2 \in G_2\}$ under the binary operation, defined by

$(g_1, g_2)(h_1, h_2) = (g_1h_1, g_2h_2)$, where $g_1, h_1 \in G_1$ and $g_2, h_2 \in G_2$, is a group called the (external) direct product of G_1 and G_2.

Theorem 6

Let G_1 and G_2 be com-solvable groups. Then the direct product $G_1 \times G_2$ is com-solvable. Conversely, if $G_1 \times G_2$ is com-solvable, G_1 and G_2 are com-solvable.

Proof

Since G_1 is com-solvable, there exists an abelian subgroup H_1 of G_1 such that $G_1' \subseteq H_1$. Thus $\frac{G_1}{H_1}$ is abelian. Similarly there exists an abelian subgroup H_2 of G_2 such that $\frac{G_2}{H_2}$ is abelian.

Since H_1 and H_2 are abelian normal subgroups of G_1 and G_2 respectively, it follows that $H_1 \times H_2$ is an abelian normal subgroup of $G_1 \times G_2$.

We shall now prove that $\frac{G_1 \times G_2}{H_1 \times H_2}$ is abelian.

or that $\left[(H_1 \times H_2)(g_1, g_2)\right]\left[(H_1 \times H_2)(g_1', g_2')\right]$

$= \left[(H_1 \times H_2)(g_1', g_2')\right]\left[(H_1 \times H_2)(g_1, g_2)\right]$

$g_1, g_1' \in G_1$ and $G_2, g_2' \in G_2$.

Or that $(H_1 \times G_2)\left[(g_1,g_2)(g_1',g_2')\right] = (H_1 \times H_2)\left[(g_1',g_2')(g_1,g_2)\right]$,

Or that $(H_1 \times H_2)(g_1g_1', g_2g_2') = (H_1 \times H_2)(g_1' g_1, g_2' g_2)$.

Let $(h_1,h_2)\ (g_1g_1',\ g_2g_2') \in (H_1 \times H_2)\ (g_1g_1',\ g_2g_2')$, $h_1 \in H_1, h_2 \in H_2$.

Now $(h_1,h_2)\ (g_1g_1',\ g_2g_2') = (h_1(g_1g_1'),\ h_2(g_2g_2'))$.

Since, $\frac{G_1}{H_1}$ is abelian, $H_1\ (g_1g_1') = H_1(g_1'g_1)$.

Thus $h_1(g_1g_1') \in H_1(g_1g_1') \Longrightarrow h_1(g_1g_1') = h_1'\ (g_1'g_1)$ for some $h_1' \in H_1$.

Similarly, $h_2(g_2g_2') = h_2'\ (g_2'g_2)$ for some $h_2' \in H_2$.

Hence $(h_1,h_2)\ (g_1g_1',\ g_2g_2') = (h_1'\ (g_1'\ g_1),\ h_2'\ (g_2'\ g_2))$

$$= (h_1',\ h_2')\ (g_1'\ g_1,\ g_2'\ g_2)$$

$$\in (H_1 \times H_2)\ (g_1'\ g_1,\ g_2'\ g_2).$$

Therefore $(H_1 \times H_2)\ (g_1g_1',\ g_2g_2') \subseteq (H_1 \times H_2)\ (g_1'\ g_1,\ g_2'\ g_2)$

Similarly $(H_1 \times H_2)\ (g_1'g_1,\ g_2'g_2) \subseteq (H_1 \times H_2)\ (g_1\ g_1',\ g_2\ g_2')$.

Therefore $(H_1 \times H_2)\ (g_1g_1',\ g_2g_2') = (H_1 \times H_2)\ (g_1'\ g_1,\ g_2'\ g_2)$.

The converse follows from theorem 3.

By G'', we denote $(G')'$. Thus G'' is normal subgroup of G' and as such of G.

Theorem 7

A group G is com-solvable if, and only if $G'' = \{e\}$.

Proof

First we suppose that G is com-solvable. Then there exists an abelian subgroup H of G such that $G' \subseteq H$, showing that G' is abelian and hence $G'' = \{e\}$.

Conversely, let $G'' = \{e\}$. Then G' is an abelian subgroup of G. Since $G' \subseteq G'$, it follows that G is com-solvable.

4. Definition

$G = G^{1'}$, $G^2 = [G, G]$ = the commutator subgroup of G. $G^n = [G^{n-1}, G]$ = the subgroup of G generated by all elements of the type, $a^{-1} b^{-1} ab$, where $a \in G^{n-1}$, $b \in G$.

The group G is nilpotent if for some non-negative integer n, $G^{n+1} = \{e\}$, G is nilpotent of class n if n is the least integer so that $G^{n+1} = \{e\}$.

Theorem 8

If a group G is nilpotent of class at most 2, then G is com-solvable.

Proof

If G is nilpotent of class 2, then $G^3 = \{e\}$.

Obviously, $[a, b] \in [G^2, G] = \{e\}$. Therefore $a^{-1} b^{-1} ab = e$, that is $ab = ba$ for all $a \in G^2$, $b \in G$. Thus $a \in Z(G)$, the centre of G. Accordingly, $G^2 \subseteq Z(G)$ which is an abelian subgroup. Hence G is com-solvable.

Now, if G is nilpotent of class 1, then $G^2 = [G, G] = \{e\}$.

Therefore G is an abelian group and hence G is com-solvable.

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"On the Exponent of Covergence and Lower Order of the Zeros of an Integral Function"

— M.I. Rizvi

1. Let $f(z)$ be an Integral function of finite non zero order ρ . Also let $n(r)$ denote the number of zeros of $f(z)$ for $|z| \leq r$ and ρ_1, λ_1 denote the exponent of Convergence and lower order of zeros of $f(z)$ respectively. Further, let Δ denote the upper density of the set of zeros of $f(z)$. If $f(z)$ has at least one zero in $|z| \leq r$, then it is known ([1], pp. 15, 17)

$$(1.1) \qquad \limsup_{r\to\infty} \inf \frac{\log n(r)}{\log r} = \begin{matrix}\rho_1 \\ \lambda_1\end{matrix} .$$

It is also known ([2], p. 10)

$$(1.2) \qquad \limsup_{r\to\infty} \frac{n(r)}{r^{\rho_1}} = \Delta .$$

In this paper, we have derived relations between the exponents of convergence and lower orders of zeros of two or more Integral functions. The results have been given in the form of theorems.

2. Theorem 1:

Let $n_1(r)$, $n_2(r)$, $n(r)$ denote respectively the number of zeros of integral function $f_1(z)$, $f_2(z)$, $f(z)$, each having at least one zero in $|z| \leq r$. Also, let $\rho_1^{(1)}$, $\rho_1^{(2)}$, ρ_1 and $\lambda_1^{(1)}$, $\lambda_2^{(2)}$, λ_1 denote the exponents of convergence and lower orders of zeros of $f_1(z)$, $f_2(z)$, $f(z)$ respectively. Then if,

$$\log n(r) \sim \frac{m_1 \log n_1(r) + m_2 \log n_2(r)}{m_1 + m_2} \quad \text{for } r \to \infty$$

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we have

$$\frac{m_1 \lambda_1^{(1)} + m_2 \lambda_1^{(2)}}{m_1 + m_2} \leq \lambda_1 \leq \rho_1 \leq \frac{m_1 \rho_1^{(1)} + m_2 \rho_1^{(2)}}{m_1 + m_2} \tag{2.1}$$

Proof: Using the relation (1.1) for $f_1(z)$, we have

$$\frac{\log n_1(r)}{\log r} < (\rho_1^{(1)} + \varepsilon) \tag{2.2}$$

for $r > r_1$, and

$$\frac{\log n_1(r)}{\log r} > (\lambda_1^{(1)} - \varepsilon) \tag{2.3}$$

for $r > r_2$.

Similarly for $f_2(z)$, we have

$$\frac{\log n_2(r)}{\log r} < (\rho_1^{(2)} + \varepsilon) \tag{2.4}$$

for $r > r_3$ and

$$\frac{\log n_2(r)}{\log r} > (\lambda_1^{(2)} - \varepsilon) \tag{2.5}$$

for $r > r_4$,

from (2.2) and (2.4), we have

$$\frac{m_1 \log n_1(r) + m_2 \log n_2(r)}{(m_1 + m_2) \log r} < \frac{m_1(\rho_1^{(1)} + \varepsilon) + m_2(\rho_1^{(2)} + \varepsilon)}{(m_1 + m_2)}$$

since $\log n(r) \sim \dfrac{m_1 \log n_1(r) + m_2 \log n_2(r)}{(m_1 + m_2)}$

and therefore,

$$\limsup_{r\to\infty} \frac{\log n(r)}{\log r} \leqslant \frac{m_1 \rho_1^{(1)} + m_2 \rho_1^{(2)}}{m_1 + m_2},$$

Hence

$$\rho_1 \leqslant \frac{m_1 \rho_1^{(1)} + m_2 \rho_1^{(2)}}{m_1 + m_2}, \tag{2.6}$$

similarly from (2.3) and (2.5), we get

$$\lambda_1 \geqslant \frac{m_1 \lambda_1^{(1)} + m_2 \lambda_1^{(2)}}{m_1 + m_2} \tag{2.7}$$

From (2.6) and (2.7), the result follows:

Corollary 1:

Let $n_1(r), n_2(r), \ldots , n_s(r), n(r)$ denote respectively the numbers of zeros of integral function $f_1(z), f_2(z), \ldots , f_s(z), f(z)$, each having at least one zero in $|z| \leq r$.

Also, let $\rho_1^{(1)}, \rho_1^{(2)}, \ldots , \rho_1^{(s)}, \rho_1$ and $\lambda_1^{(1)}, \lambda_1^{(2)}, \ldots , \lambda_1(s) , \lambda_1$ denote the exponent of convergence and lower orders of the zeros of $f_1(z), f_2(z), \ldots , f_s(z), f(z)$ respectively. Then if,

$$\log \ n(r) \sim \frac{m_1 \log n_1(r) + m_2 \log n_2(r) + .. + m_s \log n_s(r)}{m_1 + m_2 + \ldots + m_s}.$$

for $r\to\infty$, we have

$$\frac{m_1 \lambda_1^{(1)} + m_2 \lambda_2^{(2)} + \ldots + m_s \lambda_1^{(s)}}{m_1 + m_2 + \ldots + m_s} \leqslant \lambda_1 \leq \rho_1 \leqslant \frac{m_1 \rho_1^{(1)} + m_2 \rho_1^{(2)} + .. + m_s \rho_1^{(s)}}{m_1 + m_2 + \ldots + m_s}$$

Corollary 2:

If in the above theorem $m_1 = m_2$, then (2.1) reduces to

$$\frac{\lambda_1^{(1)} + \lambda_1^{(2)}}{2} \leq \lambda_1 \leq \rho_1 \leqslant \frac{\rho_1^{(1)} + \rho_1^{(2)}}{2}$$

3. Theorem 2:

Let $n_1(r)$, $n_2(r)$, $n(r)$ denote respectively the number of zeros of integral functions $f_1(z)$, $f_2(z)$, $f(z)$, each having at least one zero in $|z| \leqslant r$. Also let $\rho_1^{(1)}$, $\rho_1^{(2)}$, ρ_1 and $\lambda_1^{(1)}$, $\lambda_1^{(2)}$, λ_1 denote the exponents of convergence and lower orders of zeros of $f_1(z)$, $f_2(z)$, $f(z)$ respectively. Then if,

$$\log n(r) \sim \left[\log n_1(r)\right]^{\frac{m_1}{m_1+m_2}} \left[\log n_2(r)\right]^{\frac{m_2}{m_1+m_2}} \quad \text{for } r \to \infty ,$$

we have,

$$(3.1) \qquad \left\{\lambda_1^{(1)}\right\}^{\frac{m_1}{m_1+m_2}} \left\{\lambda_1^{(2)}\right\}^{\frac{m_2}{m_1+m_2}} \leqslant \rho_1 \leqslant \left\{\rho_1^{(1)}\right\}^{\frac{m_1}{m_1+m_2}} \left\{\rho_1^{(2)}\right\}^{\frac{m_2}{m_1+m_2}}$$

Proof: From (2.2) and (2.4), we have

$$\frac{\left[\log n_1(r)\right]^{\frac{m_1}{m_1+m_2}} \left[\log n_2(r)\right]^{\frac{m_2}{m_1+m_2}}}{\log r}$$

$$< (\rho_1^{(1)} + \varepsilon)^{\frac{m_1}{m_1+m_2}} (\rho_1^{(2)} + \varepsilon)^{\frac{m_2}{m_1+m_2}} .$$

Hence

$$\limsup_{r \to \infty} \frac{\log n(r)}{\log r} \leqslant (\rho_1^{(1)})^{\frac{m_1}{m_1+m_2}} (\rho_1^{(2)})^{\frac{m_2}{m_1+m_2}} ,$$

since,

$$\log n(r) \sim \left[\log n_1(r)\right]^{\frac{m_1}{m_1+m_2}} \left[\log n_2(r)\right]^{\frac{m_2}{m_1+m_2}} \quad \text{as } r \to \infty ,$$

and therefore

$$(3.2) \qquad \rho_1 \leq \left\{\rho_1^{(1)}\right\}^{\frac{m_1}{m_1+m_2}} \left\{\rho_1^{(2)}\right\}^{\frac{m_1}{m_1+m_2}}.$$

Similarly from (2.3) and (2.5), we get

$$(3.3) \qquad \lambda_1 \geqslant \left\{\lambda_1^{(1)}\right\}^{\frac{m_1}{m_1+m_2}} \left\{\lambda_1^{(2)}\right\}^{\frac{m_1}{m_1+m_2}}$$

and therefore,

$$\left\{\lambda_1^{(1)}\right\}^{\frac{m_1}{m_1+m_2}} \left\{\lambda_1^{(2)}\right\}^{\frac{m_2}{m_1+m_2}} \leqslant \lambda_1 \leqslant \rho_1 \leqslant \left\{\rho_1^{(1)}\right\}^{\frac{m_1}{m_1+m_2}} \left\{\rho_1^{(2)}\right\}^{\frac{m_2}{m_1+m_2}}$$

<u>Corollary 1:</u>

Let $n_1(r)$, $n_2(r), \ldots, n_s(r)$, $n(r)$ denote respectively the number of zeros of Integral functions $f_1(z)$, $f_2(z), \ldots, f_s(z)$, $f(z)$, each having at least one zero in $|z| \leqslant r$. Also, let $\rho_1^{(1)}$, $\rho_2^{(2)}$, ..., $\rho_1^{(s)}$, ρ_1 and $\lambda_1^{(1)}$, $\lambda_1^{(2)}$, ... $\lambda_1^{(s)}$ λ_1 denote the exponents of convergence and lower orders of zeros of $f_1(z)$, $f_2(z)$, ... , $f_z(z)$, $f(z)$, respectively then if,

$$\left[\log n(r)\right] \sim \left[\log n_1(r)\right]^{\frac{m_1}{m_1+m_2+\ldots+m_s}} \left[\log n_2(r)\right]^{\frac{m_2}{m_1+m_2+\ldots+m_s}} \ldots \left[\log n_s(r)\right]^{\frac{m_s}{m_1+m_2+\ldots+m_s}}$$

as $r \to \infty$. Then we have

$$\left\{\lambda_1^{(1)}\right\}^{\frac{m_1}{m_1+m_2+\ldots+m_s}} \left\{\lambda_1^{(2)}\right\}^{\frac{m_2}{m_1+m_2+\ldots+m_s}} \ldots\ldots \left\{\lambda_1^{(s)}\right\}^{\frac{m_s}{m_1+m_2+\ldots+m_s}} \leq \lambda_1 \leq \rho_1$$

$$\left\{\rho_1^{(1)}\right\}^{\frac{m_1}{m_1+m_2+\ldots+m_s}} \left\{\rho_1^{(2)}\right\}^{\frac{m_2}{m_1+m_2+\ldots+m_s}} \ldots \left\{\rho_1^{(s)}\right\}^{\frac{m_s}{m_1+m_2+\ldots+m_s}}$$

Corollary2:

If in the above theorem $m_1 = m_2$, then (3.1) reduces to

$$\sqrt{\lambda_1^{(1)} \lambda_1^{(2)}} \leq \lambda_1 \leq \rho_1 \leq \sqrt{\rho_1^{(1)} \rho_1^{(2)}}$$

4. Theorem 3:

Let $n_1(r)$, $n_2(r)$, $n(r)$ denote respectively the number of zeros of Integral functions $f_1(z)$, $f_2(z)$, $f(z)$, each having at least one zero in $|z| \leq r$. Also let $\rho_1^{(1)}$, $\rho_1^{(2)}$, ρ_1 and $\lambda_1^{(1)}$, $\lambda_1^{(1)}$, λ_1 denote the exponents of convergence and lower orders of $f_1(z)$, $f_2(z)$, $f(z)$ respectively. Then if, $\frac{m_1+m_2}{\log n(r)} \sim \frac{m_1}{\log n_1(r)} + \frac{m_2}{\log n_2(r)}$ as $r \to \infty$,

we have

$$\frac{m_1}{\rho_1^{(1)}} + \frac{m_2}{\rho_1^{(2)}} \leq \frac{m_1+m_2}{\rho_1} \leq \frac{m_1+m_2}{\lambda_1} \leq \frac{m_1}{\lambda_1^{(1)}} + \frac{m_2}{\lambda_1^{(2)}} . \tag{4.1}$$

Proof: From the relation (1.1), we have

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log r}{\log n(r)} = \begin{matrix}1/\lambda_1\\ 1/\rho_1\end{matrix} . \tag{4.2}$$

Now applying the relation (4.2) for $f_1(z)$, we have,

$$(4.3) \qquad \frac{\log r}{\log n_1(r)} < \frac{1}{\lambda_1^{(1)}} + \varepsilon ,$$

for $r > r_1$ and

$$(4.4) \qquad \frac{\log r}{\log n_1(r)} > \frac{1}{\rho_1^{(1)}} - \varepsilon ,$$

for $r > r_2$.

Similarly for $f_2(z)$, we have

$$(4.5) \qquad \frac{\log r}{\log n_2(r)} < \frac{1}{\lambda_1^{(2)}} + \varepsilon ,$$

for $r > r_3$ and

$$(4.6) \qquad \frac{\log r}{\log n_2(r)} > \frac{1}{\rho_1^{(2)}} - \varepsilon ,$$

for $r > r_4$

From (4.3) and (4.5), we have

$$\log r \left\{ \frac{m_1}{\log n_1(r)} \right\} + \log r \left\{ \frac{m_2}{\log n_2(r)} \right\}$$

$$< m_1 \left\{ \frac{1}{\lambda_1^{(1)}} + \varepsilon \right\} + m_2 \left\{ \frac{1}{\lambda_1^{(2)}} + \varepsilon \right\}$$

and hence

$$\lim_{r \to \infty} \sup (m_1 + m_2) \frac{\log r}{\log n(r)} \leqslant \frac{m_1}{\lambda_1^{(1)}} + \frac{m_2}{\lambda_1^{(2)}} ,$$

Since

$$\frac{m_1+m_2}{\log n(r)} \sim \frac{m_1}{\log n_1(r)} + \frac{m_2}{\log n_2(r)} \text{ as } r \to \infty$$

and therefore

$$(4.7) \qquad \frac{(m_1+m_2)}{\lambda_1} \leqslant \frac{m_1}{\lambda_1^{(1)}} + \frac{m_2}{\lambda_1^{(2)}} .$$

Similarly from (4.4) and (4.6), we have

$$(4.8) \qquad \frac{(m_1+m_2)}{\rho_1} \geqslant \frac{m_1}{\rho_1^{(1)}} + \frac{m_2}{\rho_1^{(2)}} ,$$

and therefore

$$\frac{m_1}{\rho_1^{(1)}} + \frac{m_2}{\rho_1^{(2)}} \leqslant \frac{(m_1+m_2)}{\rho_1} \leqslant \frac{(m_1+m_2)}{\lambda_1} \quad \frac{m_1}{\lambda_1^{(1)}} + \frac{m_2}{\lambda_1^{(2)}} .$$

<u>Corollary 1:</u>

Let $n_1(r)$, $n_2(r), \ldots, n_s(r)$, $n(r)$ denote respectively the number of zeros of integral functions $f_1(z)$, $f_2(z), \ldots, f_s(z)$, $f(z)$ each having at least one zero in $|z| \leqslant r$. Also let, $\rho_1^{(1)}$, $\rho_1^{(2)}, \ldots, \rho_1^{(s)}$, ρ_1 and $\lambda_1^{(1)}$, $\lambda_1^{(2)}, \ldots, \lambda_1^{(s)}$, λ_1 denote exponents of convergence and lower orders of zeros of $f_1(z)$, $f_2(z), \ldots, f_s(z)$, $f(z)$ respectively. Then if,

$$\frac{m_1+m_2+\ldots+m_s}{\log n(r)} \sim \frac{m_1}{\log n_1(r)} + \frac{m_2}{\log n_2(r)} + \ldots + \frac{m_s}{\log n_s(r)} \text{ as } r \to \infty ,$$

we have,

$$\frac{m_1}{\rho_1^{(1)}} + \frac{m_2}{\rho_1^{(2)}} + \ldots + \frac{m_s}{\rho_1^{(s)}} \leqslant \frac{m_1+m_2+\ldots+m_s}{\rho_1} \leqslant \frac{m_1+m_2+\ldots+m_s}{\lambda_1}$$

$$\leqslant \frac{m_1}{\lambda_1^{(1)}} + \frac{m_2}{\lambda_1^{(2)}} + \ldots + \frac{m_s}{\lambda_1^{(s)}} .$$

Corollary 2:

If in the above theorem $m_1 = m_2$, then (4.1) reduces to

$$\frac{1}{\rho_1^{(1)}} + \frac{1}{\rho_1^{(2)}} \leqslant \frac{2}{\rho_1} \leq \frac{2}{\lambda_1} \leqslant \frac{1}{\lambda_1^{(1)}} + \frac{1}{\lambda_1^{(2)}} .$$

Theorem 4:

Let $n_1(r)$, $n_2(r)$, $n(r)$ denote respectively the number of integral functions $f_1(z)$, $f_2(z)$, $f(z)$, in $|z| \leq r$ each having the same exponent of convergence ρ_1, and let Δ_1, Δ_2, Δ denote the upper densities of the zeros of $f_1(z)$, $f_2(z)$, $f(z)$ respectively. Then if,

$$n(r) \sim \frac{m_1 . n_1(r) + m_2 . n_2(r)}{(m_1+m_2)} \text{ as } r \to \infty ,$$

we have,

$$(5.1) \qquad \Delta \leqslant \frac{m_1 \Delta_1 + m_2 \Delta_2}{m_1+m_2} ,$$

Proof: Using the relation (2.1) for $f_1(z)$, we have

$$(5.2) \qquad \frac{n_1(r)}{r^{\rho_1}} \quad (\Delta_1 + \varepsilon)$$

for $r > r_1$.

Similarly for $f_2(z)$, we have

$$(5.3) \qquad \frac{n_2(r)}{r^{\rho_1}} \quad (\Delta_2 + \varepsilon)$$

for $r > r_2$.

Then from (5.2) and (5.3), we have

$$\frac{m_1.n_1(r) + m_2.n_2(r)}{r^{\rho_1}.(m_1+m_2)} < \frac{m_1}{m_1+m_2}(\Delta_1 + \varepsilon) + \frac{m_2}{(m_1+m_2)}(\Delta_2 + \varepsilon).$$

Hence,

$$\lim_{r\to\infty} \sup \frac{n(r)}{r^{\rho_1}} \leqslant \frac{m_1 \Delta_1 + m_2 \Delta_2}{m_1 + m_2},$$

since $n(r) \sim \dfrac{m_1.n_1(r) + m_2.n_2(r)}{(m_1+m_2)}$ as $r\to\infty$,

Therefore $\Delta \leqslant \dfrac{m_1 \Delta_1 + m_2 \Delta_2}{m_1 + m_2}$.

Corollary 1:

Let $n_1(r)$, $n_2(r)$, ... , $n_s(r)$ denote respectively the number of zeros of integral functions $f_1(z)$, $f_2(z)$, ... , $f_s(z)$, $f(z)$ in $|z| \leq r$, each having the same exponent of convergence ρ_1 and let Δ_1, Δ_2, ----, Δ_s, Δ denote the upper densities of the zeros of integral functions $f_1(z)$, $f_2(z)$, ... , $f_s(z)$, $f(z)$ respectively. Then if

$$n(r) \sim \frac{m_1.n_1(r) + m_2.n_2(r) + \dots + m_s.n_s(r)}{m_1 + m_2 + \dots + m_s},$$

we have,

$$\Delta \leqslant \frac{m_1 \Delta_1 + m_2 \Delta_2 + \dots + m_s \Delta_s}{m_1 + m_2 + \dots + m_s}.$$

Corollary 2:

If in the above theorem $m_1 = m_2$, the result (5.1) reduces to

$$\Delta \leqslant \frac{\Delta_1 + \Delta_2}{2}.$$

6. Theorem 5:

Let $n_1(r)$, $n_2(r)$, $n(r)$ denote respectively the number of zeros of integral functions $f_1(z)$, $f_2(z)$, $f(z)$ in $|z| \leqslant r$, each having the same exponent of convergence ρ_1, and let Δ_1, Δ_2 denote the upper densities of the zeros of $f_1(z)$, $f_2(z)$, $f(z)$ respectively, then if,

$$n(r) \sim \left[n_1(r)\right]^{\frac{m_1}{m_1+m_2}} \left[n_2(r)\right]^{\frac{m_2}{m_1+m_2}}$$

we have,

$$\Delta \leqslant \Delta_1^{\frac{m_1}{m_1+m_2}} \Delta_2^{\frac{m_2}{m_1+m_2}} \tag{6.1}$$

Proof: From the relations (5.2) and (5.3), we get,

$$\frac{\left[n_1(r)\right]^{\frac{m_1}{m_1+m_2}} \left[n_2(r)\right]^{\frac{m_2}{m_1+m_2}}}{r^{\rho_1}} < (\Delta_1 + \varepsilon)^{\frac{m_1}{m_1+m_2}}$$

Hence,

$$\lim_{r \to \infty} \sup \frac{n(r)}{r^{\rho_1}} \leqslant \Delta_1^{\frac{m_1}{m_1+m_2}} \Delta_2^{\frac{m_2}{m_1+m_2}}$$

since,

$$n(r) \sim \left[n_1(r)\right]^{\frac{m_1}{m_1+m_2}} \left[n_2(r)\right]^{\frac{m_2}{m_1+m_2}} \text{ as } r \to \infty .$$

and therefore,

$$\Delta \leqslant \Delta_1^{\frac{m_1}{m_1+m_2}} \Delta_2^{\frac{m_2}{m_1+m_2}}$$

Corollary 1:

Let $n_1(r), n_2(r), \ldots, n_s(r), n(r)$ denote respectively the number of zeros of integral functions $f_1(z), f_2(z), \ldots, f_s(z), f(z)$ in $|z| \leqslant r$, each having the same exponent of convergence ρ_1, and let $\Delta_1, \Delta_2, \ldots, \Delta_s, \Delta$ denote the upper densities of zeros of $f_1(z)$, $f_2(z), \ldots, f_s(z), f(z)$, respectively, then if,

$$n(r) \sim \left[n_1(r)\right]^{\frac{m_1}{m_1+m_2+\ldots+m_s}} \left[n_2(r)\right]^{\frac{m_2}{m_1+m_2+\ldots+m_s}} \ldots \left[n_s(r)\right]^{\frac{m_s}{m_1+m_2+\ldots+m_s}} \quad \text{as } r \to \infty,$$

we have,

$$\Delta \leqslant \Delta_1^{\frac{m_1}{m_1+m_2+\ldots+m_s}} \Delta_2^{\frac{m_2}{m_1+m_2+\ldots+m_s}} \ldots\ldots \Delta_s^{\frac{m_s}{m_1+m_2+\ldots+m_s}}.$$

Corollary 2:

If in the above theorem $m_1 = m_2$, the result (6.1) reduces to

$$\Delta \leqslant \sqrt{\Delta_1 \Delta_2}.$$

7. Theorem 6:

Let $n_1(r), n_2(r), n(r)$ denote respectively the number of zeros of integral functions $f_1(z), f_2(z), f(z)$ in $|z| \leqslant r$, each having the same exponents of convergence ρ_1, and let $\Delta_1, \Delta_2, \Delta$ denote the upper densities of zeros of $f_1(z), f_2(z), f(z)$, respectively, then if

$$\frac{m_1+m_2}{n(r)} \sim \frac{m_1}{n_1(r)} + \frac{m_2}{n_2(r)} \quad \text{as } r \to \infty,$$

we have

$$(7.1) \qquad \frac{m_1+m_2}{\Delta} \geqslant \frac{m_1}{\Delta_1} + \frac{m_2}{\Delta_2}$$

Proof: From the relation (1.2), we have

$$(7.2) \qquad \liminf_{r \to \infty} \frac{r^{\rho_1}}{n(r)} = \frac{1}{\Delta}$$

Applying the relation (7.2) for $f_1(z)$, we get

$$(7.3) \qquad \frac{r^{\rho_1}}{n_1(r)} > \left(\frac{1}{\Delta_1} - \varepsilon\right) \quad \text{for } r > r_1 ,$$

Similarly for $f_2(z)$, we get

$$(7.4) \qquad \frac{r^{\rho_1}}{n_2(r)} > \left(\frac{1}{\Delta_2} - \varepsilon\right)$$

for $r > r_2$.

Now from (7.3) and (7.4), we have

$$r^{\rho_1}\left[\frac{m_1}{n_1(r)} + \frac{m_2}{n_2(r)}\right] > m_1\left(\frac{1}{\Delta_1} - \varepsilon\right) + m_2\left(\frac{1}{\Delta_2} - \varepsilon\right)$$

Hence,

$$\liminf_{r \to \infty} (m_1+m_2) \frac{r^{\rho_1}}{n(r)} \geqslant \frac{m_1}{\Delta_1} + \frac{m_2}{\Delta_2} ,$$

since

$$\frac{m_1+m_2}{n(r)} \sim \frac{m_1}{n_1(r)} + \frac{m_2}{n_2(r)} \quad \text{as } r \longrightarrow \infty$$

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