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Concening an Ancient Chinese Formula for the Area of a Circular Segment

Howard Eves
University of Maine
U.S.A.

The second oldest of the extant Chinese texts on mathematics is the K'ui-ch'ang Suan-shu, or Arithmetic in Nine Sections. Neither the original author nor the original date of composition of the work is known. It was certainly written before 213 B.C., the fateful year in which the egotistical and despotic Emperor Shi Huang-ti ordered all books to be burned and all scholars to be buried. It has naturally become difficult to date precisely works written prior to that infamous occurrence, but it is believed that the Arithmetic in Nine Sections had already been long in existence. Shi died in 210 B.C., and, soon after, copies of many of the condemned works were either smuggled out of hiding or restored from memory. It was in one of these ways that the Arithmetic in Nine Sections was resurrected -- through a revised and expanded treatment made by Chang T'sang, and somewhat later added to by Ching Ch'ou-Ch'ang. These enlargements took place in the early part of the Han Dynasty, which began with the accession of Emperor Kao-tsu to the throne in 202 B.C. It is not known just what revisions and/or additions were made by Chang and Ching, nor can we differentiate between the possible contributions of the two scholars.

The Arithmetic in Nine Sections, as it has come down to us, is a collection of 246 problems on agriculture, business procedures, engineering, and surveying — involving formulas for an assortment of areas and volumes, the arithmetic of fractions, square and cube roots, proportion and percentage, the solution of systems of linear equations, and properties of right triangles.

The area formulas (or rules) appear early in the first section of the work, under the heading of field mensuration. Here we find

I. The area of a triangle is given by half its base multiplied by its altitude.

II. The area of a trapezoid, isosceles or otherwise, is half the sum of its bases multiplied by its altitude.

III. The area of a circle is given by any one of the four formulas

$$\text{(i) } (p/2)(d/2), \quad \text{(ii) } pd/4, \quad \text{(iii) } (3/4)d^2, \quad \text{(iv) } (1/12)p^2,$$

where p and d respectively represent the perimeter (circumference) and diameter of the circle.

IV. The area of a segment of a circle is given by

$$[(\text{chord} \times \text{altitude}) + (\text{altitude})^2]/2.$$

V. The area of a sector of a circle is one fourth the product of arc and diameter.

VI. An annular area is found by taking half the sum of the inner and outer circumferences and multiplying by the width of the ring between the two circumferences.

It is to be noted that all of the above formulas, with the exception of IV, III (iii), and III (iv), are exact; III (iii) and III (iv), however, are consistent with III (i) and III (ii) if, as is done in the Arithmetic in Nine Sections, p is taken equal to 3d that is, if π is taken as 3.

Of particular interest is the approximate formula IV for the area of a circular segment in terms of the chord and altitude (or sagitta) of the segment. Denoting the chord by c and the sagitta by s, the formula may be written as

$$S = (c + s)s/2. \tag{1}$$

It is not surprising that the formula offered for this area is only approximate, for the exact formula,

$$S = \frac{(4s^2 + c^2)^2}{32s^2} \arctan\left(\frac{2s}{c}\right) - \frac{c^3}{16s} + \frac{cs}{4}, \tag{2}$$

is not only disappointingly complex, but involves an inverse trigonometric function — a concept not yet developed at that early time.

We are not told in the Arithmetic in Nine Sections how the approximate formula for the area of a circular segment was determined, but we can make some reasonable guesses. The procedure had to be an empirical one, and the formula might well have been arrived at using the known formula A = bh/2 for the area of a triangle with base b and height h. Specifically, see Figure 1, when the secant lines are drawn so as to

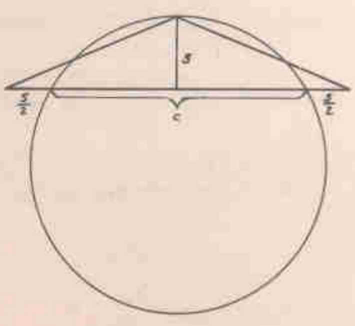

Fig. 1

make the area of the isosceles triangle appear by eye to be equal to that of the circular segment, these lines seem to cut the segment's chord prolonged in each direction a distance closely equal to s/2.

Another, but perhaps less likely, explanation can be based upon Figure 2 and the known formula $A = (b_1 + b_2)h/2$ for the area of a trape-

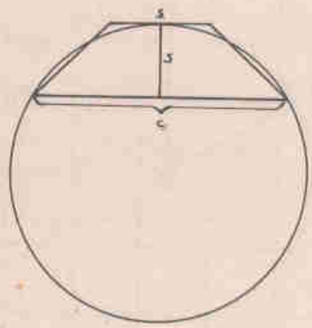

Fig. 2

zoid with bases b_1 and b_2 and height h. Here, to construct the isosceles trapezoid so that it appears by eye to have the same area as that of the circular segment, it seems the upper base must be taken closely equal to the sagitta in length.

Or, it could be that the formula was first obtained for the semicircle and then inductively extended to apply to all circul. segments. For the semicircle we have, by III (iii) (where, in conjunction with the rest of the Arithmetic in Nine Sections, π is taken as 3),

$$\text{area} = (3/2)r^2 .$$

But

$$(3/2)r^2 = 3r^2/2 = (3r)r/2 = (d + r)r/2.$$

Here $\underline{d}$ is the chord and $\underline{r}$ is the sagitta of the semicircle. An inductive generalization of this leads to the formula (1) for the area of a general circular segment in terms of the chord c and the sagitta s of the segment.

In the Arithmetic in Nine Sections there are no proofs in the Greek sense, and so the author must have resorted to some sort of empirical procedure like the above. Formulas obtained by empirical procedures sometimes turn out to be exact, but frequently, as in the present instance, they turn out to be merely approximate.

One naturally wonders how well does the ancient Chinese formula approximate the true area of the circular segment. An interesting way to express this approximation is to note how the value taken by π varies as the central angle subtended by the segment grows from 0^o to 360^o. We have seen that for the semicircle, where the central angle is 180^o, this value is 3. An approach along this line may be made as follows. Denote the radius of the circle by r and let half the concerned central angle be k^o. One then easily finds (see Figure 3):

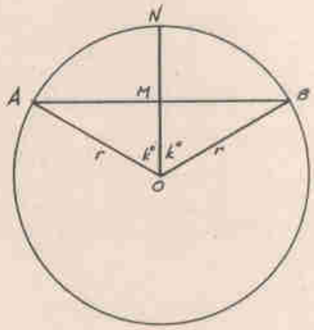

Fig. 3.

exact area = area of sector AOB - area of triangle AOB

$$= r^2\left(\frac{k\pi}{180} - \sin k^o \cos k^o\right), \tag{3}$$

approximate area = (AB + MN)MN/2 = (2MB + ON - OM) (ON - OM)/2

$$= r^2\left(\frac{1}{2} + \sin k^o - \cos k^o - \sin k^o \cos k^o + \frac{\cos^2 k^o}{2}\right). \tag{4}$$

It follows, by equating the two formulas (3) and (4), that for an approximation π' of π, we have

$$\frac{k\pi'}{90} = 1 + 2\sin k^o - 2\cos k^o + \cos^2 k^o,$$

or

$$\pi' = \frac{90}{k}\left(1 + 2\sin k^o - 2\cos k^o + \cos^2 k^o\right). \tag{5}$$

One can now construct a table giving the values of π' for circular segments of central angles $2k^o$. Thus, if k = 90, we find from (5),

$$\pi' = 1 + 2 = 3,$$

and once again we see that for the semicircle the ancient Chinese formula is equivalent to taking $\pi = 3$.

From such a table as described above, one obtains the graph pictured in Figure 4, which shows how π' varies with the central angle. It is to

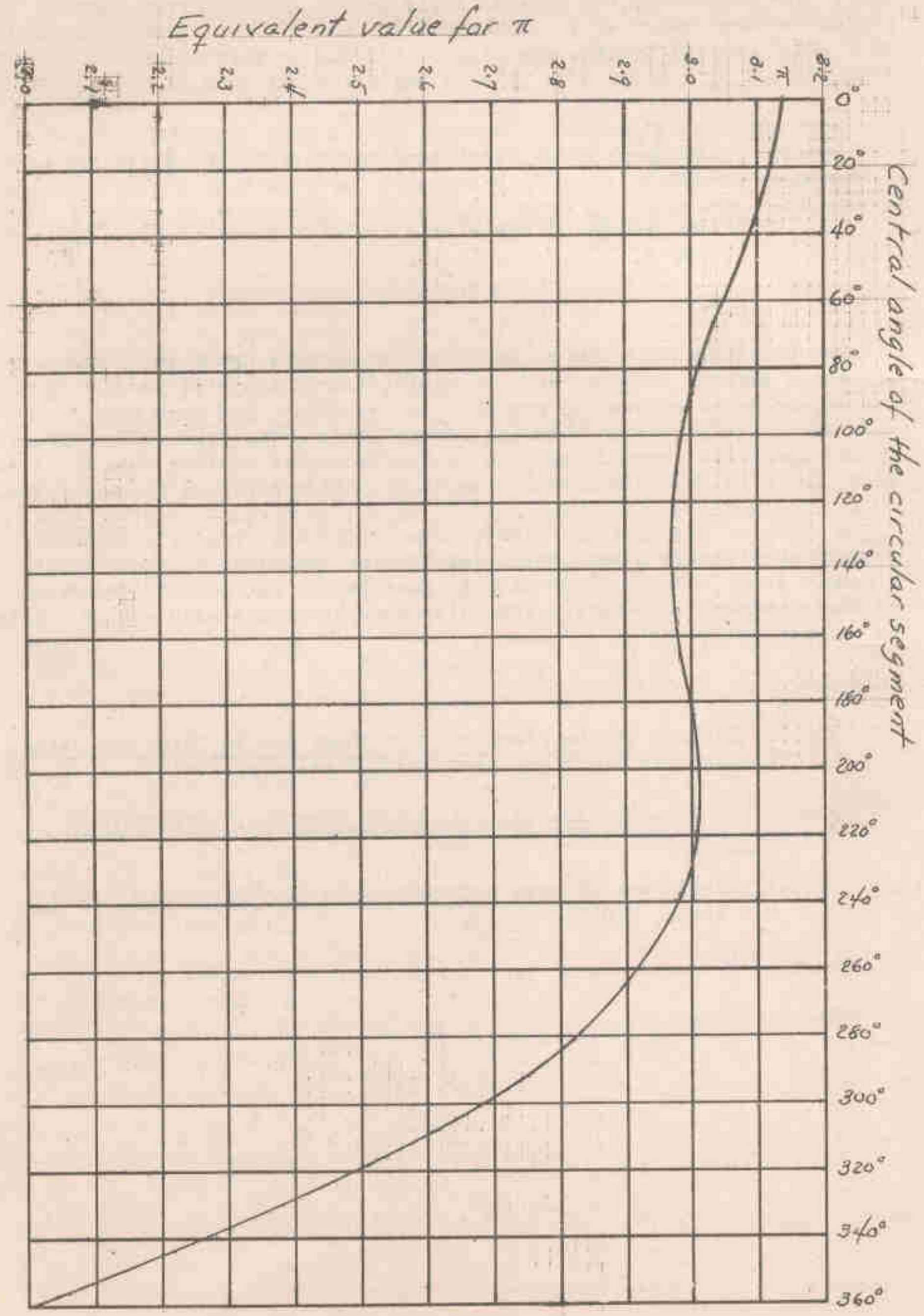

Fig. 4

be noticed that the value 3 for π' occurs in two other instances besides the semicircle, namely for a circular segment having a central angle of 90°, and a circular segment having a central angle close to 230°. The case where the central angle is 90° is, of course, easily checked without recourse to trigonometric tables. For circular segments with small central angles, the Chinese formula is very accurate. In fact, from (5) and by l'Hospital's rule of beginning calculus,

$$\lim_{k \to 0} \pi' = \lim_{k \to 0} \left[\frac{90}{k} (1 + 2 \sin k^{o} - 2 \cos k^{o} + \cos^{2} k^{o})\right]$$
$$= \lim_{k \to 0} \left[90 \frac{\pi}{180} (2 \cos k^{o} + 2 \sin k^{o} - 2 \cos k^{o} \sin k^{o})\right]$$
$$= \pi .$$

One concludes that the ancient Chinese formula has much to recommend it: it is simple, compact, easy to apply, fairly accurate for all circular segments having central angles less than 240°, and much more accurate for circular segments having central angles less than 40°. Though in all cases the formula leads to an area somewhat smaller than the true area, for a quick estimation of the area it leaves little to be desired.

A study of the ancient Chinese formula for the area of a circular segment constitutes a good topic for "junior" research at the college freshman level. Readers may wish to consult [3] for an extended survey of early Chinese mathematics that discusses at length many topics treated in undergraduate college courses.

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University of Maine
U.S.A.

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Generalised Directed-Divergence Involving More Than Two Distributions

—Parvinder Kaur

Abstract

A new concept of generalized directed divergence involving more than two distributions is introduced here which generalizes the non-additive generalized directed divergence of order α. Also this measure will be characterized using a functional equation and using a functional inequality.

1. Definition

Let $P = (p_1, p_2, \ldots, p_n)$, $p_i \geqslant 0$, $\sum_{i=1}^{n} p_i = 1$,

$Q = (q_1, q_2, \ldots, q_n)$, $q_i \geqslant 0$, $\sum_{i=1}^{n} q_i = 1$ and $R = (r_1, r_2, \ldots, r_n)$

$r_i \geqslant 0$, $\sum_{i=1}^{n} r_i = 1$ be three finite discrete probability distributions.

Then we define the 'Generalized directed divergence involving more than two distributions' for the distributions P, Q and R as

$$I^{(\alpha,\beta)}(P:Q:R) = (2^{\alpha-1} - 2^{\beta-1})^{-1} \sum_{i=1}^{n} (p_i q_i^{\alpha-1} r_i^{1-\alpha} - p_i q_i^{\beta-1} r_i^{1-\beta}) \qquad \alpha \neq \beta \quad (1.1)$$

For $\beta = 1$, (1.1) reduces to $(2^{\alpha-1}-1)^{-1} \left(\sum_{i=1}^{n} p_i q_i^{\alpha-1} r_i^{1-\alpha} - 1\right)$ (1.2)

(Kannappan and Rathie 1972) which again reduces to $\sum p_i \log q_i/r_1$ (1.3)

(Theil (1967)) as $\alpha \longrightarrow 1$.

2. Characterizations:

The first characterization of (1.1) is based on the solution of the functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{n} F(x_i y_j,\ u_i v_j,\ t_i s_j) = \sum_{i=1}^{m}\sum_{j=1}^{n} x_i u_i^{\alpha-1} t_i^{1-\alpha} F(y_j, v_j, s_j)$$

$$+ \sum_{i=1}^{m}\sum_{j=1}^{n} y_i v_j^{\beta-1} s_j^{1-\beta} F(x_i, u_i, t_i) \qquad (2.1)$$

where $x_i\ y_j \geqslant 0, \sum_{i=1}^{n} x_i = \sum_{j=1}^{n} y_j = 1,\ u_i,\ v_j \geqslant 0$

$t_i,\ s_j \geqslant 0, \sum_{i=1}^{n} u_i \leqslant 1,\ \sum_{j=1}^{n} v_j \leqslant 1,\ \sum_{i=1}^{n} t_i \leqslant 1,\ \sum_{j=1}^{n} s_j \leqslant 1,\ \alpha \neq \beta .$

Theorem 1:

Let $F: [0,1] \times [0,1] \times [0,1] \longrightarrow R$ (reals) be a continuous function. If for all positive integers m and n, F satisfies the functional equation (2.1), then under the condition $F(1, \frac{1}{2}, \frac{1}{4}) = 1$,

$$\sum_{i=1}^{n} F(p_i, q_i, r_i) = (2^{\alpha-1} - 2^{\beta-1})^{-1} \sum_{i=1}^{n} (p_i q_i^{\alpha-1}\ r_i^{1-\alpha} - p_i q_i^{\beta-1}\ r_i^{1-\beta})$$

Proof:

We shall first find the solution of the functional equation (2.1),

Let m, n, u, v, t, s be any positive integers such that $1 \leqslant m \leqslant u$, $1 \leqslant m \leqslant t$, $1 \leqslant n \leqslant v$, $1 \leqslant n \leqslant s$

setting

$$x_i = \frac{1}{m},\ u_i = \frac{1}{u},\ t_i = \frac{1}{t} \qquad (i = 1,2,\ldots,m)$$

$$y_j = \frac{1}{n},\ v_j = \frac{1}{v},\ s_j = \frac{1}{s} \qquad (j = 1,2,\ldots,n)$$

in (2.1), we get

$$mn\ F(\frac{1}{mn}, \frac{1}{uv}, \frac{1}{ts}) = mn\ (\frac{1}{m})\ (\frac{1}{u})^{\alpha-1}\ (\frac{1}{t})^{1-\alpha} F(\frac{1}{n}, \frac{1}{v}, \frac{1}{s})$$

$$+\ mn(\frac{1}{n})\ (\frac{1}{v})^{\beta-1}\ (\frac{1}{s})^{1-\beta} F(\frac{1}{m}, \frac{1}{u}, \frac{1}{t})$$

or

$$F(ab,\ cd,\ ef) = ac^{\alpha-1}\ e^{1-\alpha}\ F(b,\ d,\ f) + bd^{\beta-1}\ f^{1-\beta} F(a,\ c,\ e).$$

where $a = \frac{1}{m},\ b = \frac{1}{n},\ c = \frac{1}{u},\ d = \frac{1}{v},\ e = \frac{1}{t},\ f = \frac{1}{s}$

Also

$$F(ab,\ cd,\ ef) = F(ba,\ dc,\ fe)$$

which with the above equation gives

$$ac^{\alpha-1}e^{1-\alpha}F(b, d, f) + bd^{\beta-1}f^{1-\beta}F(a, c, e) =$$

$$bd^{\alpha-1}f^{1-\alpha}F(a, c, e) + ac^{\beta-1}e^{1-\beta}F(b, d, f)$$

so that

$$\frac{ac^{\alpha-1}e^{1-\alpha} - ac^{\beta-1}e^{1-\beta}}{F(a, c, e)} = \frac{bd^{\alpha-1}f^{1-\alpha} - bd^{\beta-1}f^{1-\beta}}{F(b, d, f)} = A(\text{say})$$

implying

$$F(a, c, e) = A^{-1}\ (ac^{\alpha-1}e^{1-\alpha} - ac^{\beta-1}e^{1-\beta}) \tag{2.2}$$

where $A\ (\neq 0)$ is a constant.

The result (2.2) can be extended to the case when a,c,e are rational numbers such that $a, c, e \in [0,1]$

For this let $x = \frac{m}{n}\ (m < n)$, $u = \frac{p}{q}\ (p < q)$

$t = \frac{g}{h}\ (g < h)$ be three rational numbers. Let k be a positive integer sufficiently large such that

$$kp \geq m,\ kg \geq m,\ kq \geq n, \quad kh \geq n, \quad k \geq \frac{q(n-m)}{n(q-p)} \text{ and } k \geq \frac{h(n-m)}{n(h-g)}$$

Taking m as n-m+1 and n as m and setting

$$x_1 = \frac{m}{n},\ x_2 = \dots = x_{n-m+1} = \frac{1}{n}$$

$$y_1 = y_2 = \dots\dots = y_m = \frac{1}{m}$$

$$u_1 = \frac{p}{q},\ u_2 = \dots = u_{n-m+1} = \frac{1}{kn}$$

$$v_1 = v_2 = \dots\dots = v_m = \frac{1}{pk}$$

$$t_1 = \frac{g}{h},\ t_2 = \dots = t_{n-m+1} = \frac{1}{kn}$$

$$s_1 = s_2 = \dots\dots = s_m = \frac{1}{gk}$$

in (2.1), we get

$$m\,F\left(\frac{1}{n}, \frac{1}{qk}, \frac{1}{hk}\right) + m\,(n-m)\,F\left(\frac{1}{mn}, \frac{1}{pnk^2}, \frac{1}{gnk^2}\right) =$$

$$m\left[\left(\frac{m}{n}\right)\left(\frac{p}{q}\right)^{\alpha-1}\left(\frac{g}{h}\right)^{1-\alpha} + (n-m)\left(\frac{1}{n}\right)\left(\frac{1}{kn}\right)^{1-\alpha}\left(\frac{1}{kn}\right)^{1-\alpha}\right] F\left(\frac{1}{m}, \frac{1}{pk}, \frac{1}{gk}\right)$$

$$+ m\left[F\left(\frac{m}{n}, \frac{p}{q}, \frac{g}{h}\right) + (n-m)\,F\left(\frac{1}{n}, \frac{1}{kn}, \frac{1}{kn}\right)\right]\left(\frac{1}{m}\right)\left(\frac{1}{pk}\right)^{\beta-1}\left(\frac{1}{gk}\right)^{1-\beta}$$

This equation with (2.2) gives

$$F\left(\frac{m}{n}, \frac{p}{q}, \frac{g}{h}\right) = A^{-1}\left[\left(\frac{m}{n}\right)\left(\frac{p}{q}\right)^{\alpha-1}\left(\frac{g}{h}\right)^{1-\alpha} - \frac{m}{n}\left(\frac{p}{q}\right)^{\beta-1}\left(\frac{g}{h}\right)^{1-\beta}\right]$$

or

$$F(x,u,t) = A^{-1}\left[x\,u^{\alpha-1}\,t^{1-\alpha} - xu^{\beta-1}\,t^{1-\beta}\right] \tag{2.3}$$

for all rational $x, u, t \in [0,1]$

From the continuity of F we can say that (2.3) is valid for all real $x, u, t, \in [0,1]$

Now the condition $F(1, \frac{1}{2}, \frac{1}{4}) = 1$, with (2.3) gives

$$A = (2^{\alpha-1} - 2^{\beta-1})$$

so that

$$F(x,u,t) = (2^{\alpha-1} - 2^{\beta-1})^{-1}\,(xu^{\alpha-1}\,t^{1-\alpha} - xu^{\beta-1}\,t^{1-\beta}),\ \alpha \neq \beta$$

Therefore,

$$\sum_{i=1}^{n} F(p_i, q_i, r_i) = (2^{\alpha-1} - 2^{\beta-1})^{-1} \sum_{i=1}^{n} (p_i q_i^{\alpha-1}\, r_i^{1-\alpha} - p_i q_i^{\beta-1}\, r_i^{1-\beta})$$

$$\alpha \neq \beta$$

This proves the theorem.

Second characterization:

Theorem 2:

If a function K_n, $n > 2$, satisfies the postulates

$$C_1 : K_n\,(P:Q:R) = K_n\,(p_1, p_2, \ldots, p_n : q_1, q_2, \ldots, q_n : r_1, r_2, \ldots, r_n)$$
$$= \sum_{i=1}^{n} p_i \left(\frac{q_i}{r_i}\right)^{\beta-1}\left[\frac{f(q_i)}{f(r_i)} - 1\right]$$

$C_2 : K_n (P:Q:R) \geqslant 0$

and

$C_3 : K_2 (1,0; \frac{1}{2}, \frac{1}{2} : \frac{1}{4}, \frac{3}{4}) = (2^{\alpha-1} - 2^{\beta-1})$

then

$$(2^{\alpha-1} - 2^{\beta-1})^{-1} K_n(P:Q:R) = (2^{\alpha-1} - 2^{\beta-1})^{-1} \sum_{i=1}^{n} (p_i q_i^{\alpha-1} r_i^{1-\alpha} - p_i q_i^{\beta-1} r_i^{1-\beta}), \alpha \neq \beta$$

$$= I^{(\alpha,\beta)} (P:Q:R)$$

Proof:

Before proving the theorem we shall find the differentiable solution of the functional inequality

$$\sum_{i=1}^{n} p_i \left(\frac{q_i}{r_i}\right)^{\beta-1} \left[\frac{f(q_i)}{f(r_i)} - \right] \geqslant 0 \tag{2.4}$$

Taking $p_i = q_i = r_i$, $i = 3,4,\ldots,n$, in (2.4), we get

$$p_1 \left(\frac{q_1}{r_1}\right)^{\beta-1} \left[\frac{f(q_1) - f(r_1)}{f(r_1)}\right] + p_2 \left(\frac{q_2}{r_2}\right)^{\beta-1} \left[\frac{f(q_2) - f(r_2)}{f(r_2)}\right] \geqslant 0 \tag{2.5}$$

for $p_1 + p_2 = q_1 + q_2 = r_1 + r_2 < 1$

Putting $p_1 = r_1 + \delta$, $p_2 = r_2 - \delta$, $\delta > 0$, $q_1 = r_1 + \lambda$, $q_2 = r_2 - \lambda$, $0 < \lambda$ and dividing both sides of (2.5) by λ, we get

$$\frac{(r_1 + \delta)}{f(r_1)} \left(\frac{r_1 + \lambda}{r_1}\right)^{\beta-1} \left[\frac{f(r_1 + \lambda) - f(r_1)}{\lambda}\right] + \frac{(r_2 - \beta)}{f(r_2)} \left(\frac{r_2 - \lambda}{r_2}\right)^{\beta-1} \cdot \left[\frac{f(r_2 - \lambda) - f(r_2)}{\lambda}\right] \geqslant 0 \tag{2.6}$$

Let r_1 and r_2 be the points where f is differentiable taking $\delta \longrightarrow 0$, $\lambda \longrightarrow 0$ in (2.6) and using symmetry in r_1 and r_2, we have

$$\frac{r\, f'(r)}{f(r)} = a$$

so that

$$f(r) = br^{a} \tag{2.7}$$

where a and b are constants.

Now postulates C_1 and C_2 imply

$$K_n(P:Q:R) = \sum_{i=1}^{n} p_1 \left(\frac{q_i}{r_i}\right)^{\beta-1} \left[\left(\frac{q_i}{r_i}\right)^{a} - 1\right] \geq 0 \tag{2.8}$$

From (2.7) and (2.8)

$$K_n(P:Q:R) = \sum_{i=1}^{n} p_i \left(\frac{q_i}{r_i}\right)^{\beta-1} \left[\left(\frac{q_i}{r_i}\right)^{a} - 1\right] \tag{2.9}$$

This with postulates C_3 yields

$$a = \alpha - \beta$$

so that

$$\begin{aligned}(2^{\alpha-1} - 2^{\beta-1})^{-1} K_n(P:Q:R) &= (2^{\alpha-1} - 2^{\beta-1})^{-1} \sum_{i=1}^{n} (p_i q_i^{\alpha-1} r_i^{1-\alpha} - p_i q_i^{\beta-1} r_i^{1-\beta}) \\ &= I^{(\alpha,\beta)}(P:Q:R), \quad \alpha \neq \beta .\end{aligned}$$

This completes the proof of theorem.

Note:

Since the generalized directed divergence involving more than two distributions (1.1) reduces to the non-additive directed divergence of order α involving more than two distributions (1.2) for $\beta = 1$, the above theorems 1 and 2, also characterize (1.2) for $\beta = 1$,

that is, the functional equation

$$\begin{aligned}\sum_{i=1}^{n} \sum_{j=1}^{n} F(x_i y_j, u_i v_j, t_i s_j) &= \sum_{i=1}^{m} \sum_{j=1}^{n} x_i u_i^{\alpha-1} t_i^{1-\alpha} F(y_j, v_j, s_j) \\ &+ \sum_{i=1}^{n} \sum_{j=1}^{n} y_j F(x_i, u_i, t_i)\end{aligned}$$

where $x_i, y_j, u_i, v_j, t_i, s_j \geq 0$, $\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j = 1$

$$\sum_{i=1}^{n} u_i \leq 1, \ \sum_{j=1}^{n} v_j \leq 1, \ \sum_{i=1}^{m} t_i \leq 1, \ \sum_{j=1}^{n} s_j \leq 1, \quad \alpha \neq \beta$$

characterizes (1.2) under the condition

$$F\left(1, \frac{1}{2}, \frac{1}{4}\right) = 1$$

and similarly the functional inequality.

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Mailing Address:
Parvinder Kaur
Anand Bhawan
89-B, Subhash Nagar
Alambagh
Lucknow-226005
U.P., INDIA.

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Hypersurface of Quasi-C-Reducible Finsler Spaces

—U. P. Singh and R. S. Prasad

A quasi-C-reducible Finsler spaces has been introduced by Matsumoto and Shimada (1977). In this paper, the properties of the hypersurfaces imbedded in quasi-C-reducible F_n have been studied. It has been proved that the hypersurface of a quasi-C-reducible F_n is also quasi-C-reducible. The conditions under which the hypersurface of a quasi-C-reducible Landsberg space is a Landsberg space have been obtained. The conditions for the p-reducibility of the hypersurface of a p-reducible quasi-C-reducible F_n have also been deduced. Further after applying so-called T-conditions it has been shown that if F_n is Landsberg so is its hypersurface provided both F_n and F_{n-1} satisfy T-conditions.

1. Introduction

Let F_n be a n-dimensional Finsler space with fundamental function $F(x, \dot{x})$, the fundamental tensor g_{ij} and the angular metric tensor $h_{ij} = g_{ij} - l_i l_j$ $(l_i = \frac{\partial F}{\partial \dot{x}^i})$.

Let F_{n-1} be a hypersurface of F_n which is represented parametrically by the equations

$$(1.1) \quad x^i = x^i(u^\alpha) \quad (i = 1,2,\cdots n, \ \alpha = 1,2,\cdots n-1)$$

in which u^α denote the parameters of the hypersurface. It is assumed throughout, the functions (1.1) are of class C^2 and the matrix of the projection factors

$$(1.2) \qquad B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}$$

has the rank (n-1). We shall write $B^i_{\alpha\beta} = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}$. The induced metric tensor of F_{n-1} is defined as

$$(1.3) \qquad g_{\alpha\beta} = g_{ij} B^{ij}_{\alpha\beta}$$

The inverse of (1.3) is denoted by $g^{\alpha\beta}$ by means of which we define a set of quantities

(1.4) $$B_i^{\alpha} = g^{\alpha\beta} g_{ij} B^j_{\beta}$$

which gives

(1.5) $$B_i^{\alpha} B^i_{\alpha} = \delta^{\alpha}_{\beta}$$

At any point P of the hypersurface, a unit normal vector $N^i(x, \dot{x})$ is defined by equations

(1.6) $$N_i B^i_{\alpha} = 0, \; g^{ij} N_i N_j = 1$$

which satisfy the identity

(1.7) $$B_i^{\alpha} B^j_{\alpha} = \delta^j_i - N^j N_i$$

where $N^i = g^{ij} N_j$

A useful set of relations is ([5], p. 156)

(1.8) $$g^{ij} = g^{\alpha\beta} B^{ij}_{\alpha\beta} + N^i N^j$$

(1.9) $$g_{ij} N^j B^i_{\beta} = 0$$

(1.10) $$C_{\alpha\beta\gamma} = C_{ijk} B^{ijk}_{\alpha\beta\gamma}$$

Following are the useful tensors which vanish identically in a locally Euclidian (or Riemannian) space

(1.11) $$M_{\alpha\beta} = C_{ijk} B^{ij}_{\alpha\beta} N^k$$

(1.12) $$M_{\alpha} = C_{ijk} B^i_{\alpha} N^j N^k$$

The induced connection coefficients of F_{n-1} defined by the relation ([5], p. 160)

(1.13) $$\Gamma^{*\alpha}_{\beta\gamma} = B^{\alpha}_i \left(B^i_{\beta\gamma} + \Gamma^{*i}_{hk} B^{hk}_{\beta\gamma}\right)$$

While the normal curvature vector is given by ([5], p. 193)

$$(1.14) \qquad I^{\;i}_{\alpha\beta} = B^{i}_{\alpha\beta} - B^{i}_{\gamma}\Gamma^{*\gamma}_{\alpha\beta} + \Gamma^{*i}_{hk}\, B^{hk}_{\alpha\beta}$$

which gives ([5], p. 194)

$$(1.15) \qquad I^{\;i}_{\alpha\beta} = \Omega_{\alpha\beta} N^{i}$$

where $\Omega_{\alpha\beta}$ are the components of the second fundamental tensor of F_{n-1}

Definition:

A Finsler space F_n is said to be quasi-C-reducible (Matsumoto and Shimada [1]) if the tensor C_{ijk} is given by

$$(1.16) \qquad C_{ijk} = (A_{ij}C_k + A_{jk}C_i + A_{ik}C_j)$$

where A_{ij} is of the form

$$A_{ij} = \lambda h_{ij} + \mu C_i C_j$$

λ, μ being scalar functions.

The following relation is an immediate consequence of (1.16)

$$(1.17) \qquad (n+1)\lambda + 3\mu C^2 = 1 \quad (C^2 = g_{ij}C^iC^j)$$

We shall now prove the following lemma.

Lemma 1

The necessary and sufficient conditions for a quasi-C-reducible F_n to be Berwald space (Landsberg Space) are

$C_{i|h} = o$ and $\lambda_{|h} = o$ ($C_{i|j}\,\dot{x}^j = o$ and $\lambda_{|i}\,\dot{x}^i = o$).

Proof

The h-covariant derivatives of (1.16) and (1.17) are written in the forms

$$(1.18)\quad C_{ijk|h} = \mathfrak{S}_{(ijk)}\, h_{ij}\left\{ (\lambda_{|h} C_k + C_{k|h}) + 3\mu C_{i\,h} C_j C_k \right\} + + 3\mu_{|h}\, C_i C_j C_k$$

$$(1.19)\quad (n+1)\,\lambda_{|h} + 3\mu_{|h}\, C^2 + 3\,\mu C^2_{|h} = o$$

where the notation $\mathfrak{S}_{(ijk)}$ denotes the cyclic permutation of the indices i, j, k and summation.

$C_{i|h} = o$ implies that $C^2_{|h} = o$. In view of this condition and $\lambda_{|h} = o$ the relation (1.19) gives $\mu_{|h} = o$. Thus $C_{i|h} = o$ and $\mu_{|h} = o$ imply $C_{ijk|h} = o$ by (1.16). Conversely the condition $C_{ijk|h} = o$ implies $C_{i|h} = o$ and this reduces (1.18) to the form

$$\lambda_{|h}(h_{ij}C_k + h_{jk}C_i + h_{ik}C_j) + 3\mu_{|h}\, C_i C_j C_k = o$$

Contracting this equation by $C^j C^k$, substituting the value of $\mu_{|h}$ from (1.19) and noting the relation $C^2_{|h} = o$ we get $\lambda_{|h} = o$. This proves the lemma.

The following lemma has been proved by Matsumoto and Shimada [1]

Lemma 2

A non-Riemannian quasi-C-reducible $F_n (n \geqslant 4)$ is p-reducible if and only if the following relation holds

$$C_j\, \mu_{|i}\, \dot{x}^i + 3\,\mu C_{j|i}\, \dot{x}^i = o$$

2. Hypersurfaces of a quasi-C-reducible Finsler space.

From the equations (1.10), (1.16) and the facet $h_{\alpha\beta} = g_{\alpha\beta} - l_\alpha l_\beta$ we have

$$(2.1)\quad C_{\alpha\beta\gamma} = \lambda\left\{ h_{\alpha\beta}\,(C_k B^k_\gamma) + h_{\beta\gamma}(C_i B^i_\alpha) + h_{\alpha\gamma}(C_j B^j_\beta) \right. + 3\mu\,(B^i_\alpha C_i)\,(B^j_\beta C_j)\,(B^k_\gamma C_k)$$

In view of (1.8), (1.12) and (1.16) we get

(2.2) $C = (1-\lambda-3\mu\rho^2)\, B^i_\alpha C_i \qquad (\rho = C_i N^i)$

where $C_\alpha = g^{\beta\gamma} C_{\alpha\beta\gamma}$

with the help of (2.2), the equation (2.1) reduces to

(2.3) $C_{\alpha\beta\gamma} = p\,(h_{\alpha\beta}C_\gamma + h_{\beta\gamma}C_\alpha + h_{\alpha\gamma}C_\beta) + 3qC_\alpha C_\beta C_\gamma$

where

(2.3a) $$p = \frac{\lambda}{(1-\lambda-3\mu\rho^2)} \text{ and } q = \frac{\mu}{(1-\lambda-3\mu\rho^2)^3}$$

equation (2.3) can also be written as

(2.4) $C_{\alpha\beta\gamma} = (A_{\alpha\beta}C_\gamma + A_{\beta\gamma}C_\alpha + A_{\alpha\gamma}C_\beta)$

where $A_{\alpha\beta} = p\,h_{\alpha\beta} + q\,C_\alpha C_\beta$

It gives the following theorem.

Theorem 2.1

A hypersurface of a quasi-C-reducible Finsler space is also a quasi-C-reducible Finsler space.

The difference between the intrinsic and induced connection parameters $\hat{\Gamma}^\alpha_{\beta\gamma}$ and $\Gamma^{*\alpha}_{\beta\gamma}$ is given by ([5], p. 214)

$$\Gamma^\alpha_{\beta\gamma} - \Gamma^{*\alpha}_{\beta\gamma} = \Lambda^\alpha_{\beta\gamma}$$

where

(2.5) $$\Lambda_{\alpha\beta\gamma} = g_{\beta\delta}\Lambda^\delta_{\alpha\gamma} = M_{\beta\gamma}\Omega_{\alpha o} + M_{\alpha\beta}\Omega_{\gamma o} - M_{\alpha\gamma}\Omega_{\beta o}$$
$$- (M_{\alpha\delta}C^\delta_{\beta\gamma} + M_{\gamma\delta}C^\delta_{\alpha\beta} - M_{\beta\delta}C^\delta_{\alpha\gamma})\,\Omega_{oo}$$

where $\Omega_{\alpha o}$ denotes the contraction of $\Omega_{\alpha\beta}$ by u^β. If the space F_n is quasi-C-reducible, then by virtue of (1.11) and (1.16) we get

(2.6) $M_{\alpha\beta} = \rho(\lambda h_{\alpha\beta} + \mu' C_\alpha C_\beta)$

where $\mu' = \dfrac{3\mu}{(1-\lambda-3\mu\rho^2)^2}$

Substituting the values of $M_{\alpha\beta}$ in (2.5) and simplifying with the help of (2.4) we get.

$$
\begin{aligned}
(2.7)\quad \Lambda_{\alpha\beta\gamma} = \rho\Big[& \lambda h_{\alpha\beta}\{\Omega_{\gamma o} - p\, C_\gamma \Omega_{oo}\} + \lambda h_{\beta\gamma}\{\Omega_{\alpha o} - \\
& - p\, C_\alpha \Omega_{oo}\} - \lambda h_{\alpha\gamma}\{\Omega_{\beta o} + p\, C_\beta \Omega_{oo} + \mu' C_\alpha \cdot \\
& \cdot\{\Omega_{\gamma o} C_\beta - C_\delta C^\delta_{\beta\gamma}\Omega_{oo}\} + \mu' C_\beta\{C_\gamma \Omega_{\alpha o} + C_\delta C^\delta_{\alpha\gamma}\Omega_{oo}\} \\
& - \mu' C_\gamma\{C_\alpha \Omega_{\beta o} + C_\delta C^\delta_{\alpha\beta}\Omega_{oo}\} - 3q\lambda\, C_\alpha C_\beta C_\gamma \Omega_{oo}\Big]
\end{aligned}
$$

This equation leads us to the following theorem.

Theorem 2.2

The necessary and sufficient condition that the induced and intrinsic connection parameters of a hypersurface of the quasi-C-reducible Finsler space coincide is that either the vector C_i is tangential to the hypersurface or $\Omega_{\alpha o} = o$ provided that $p \neq 1/2$.

Proof:

The sufficient part is obvious from the eqn. (2.7). For the necessary condition, $\Lambda_{\alpha\beta\gamma} = o$ implies that either $\rho = o$ or the expression within the bracket [ ] in (2.7) is zero which after contraction by $\dot{u}^\alpha$ yields

$$(\lambda h_{\beta\gamma} + \mu' C_\beta C_\gamma)\,\Omega_{oo} = o$$

The condition $(\lambda h_{\beta\gamma} + \mu' C_\beta C_\gamma) = o$ gives $n = 3$ which contradicts the assumption $(n \geqslant 4)$. Hence $\Omega_{oo} = o$.

Substituting $\Omega_{oo} = o$ in the expression within the bracket [ ] in (2.7), we get an expression which after contraction by

$g^{\alpha\beta}$ and equating to zero gives

(2.8) $\quad \Omega_{\gamma o} \left\{ \lambda(n-2) + \mu' \, C^2 \right\} = o$

(2.8a) $\quad$ where $\; C^{-2} = g^{\alpha\beta} \, C_\alpha C_\beta$

A simple calculation based on (1.8), (2.2), the eqn. (2.8a) gives

(2.9) $\quad \bar{C}^2 = (1 - \lambda - 3\mu p^2)^2 \, (C^2 - p^2)$

Substituting (2.9) in (2.8) we get either $\Omega_{\gamma o} = o$ or $p = 1/2$ and hence the theorem is proved.

In view of (2.3a), eqn (2.2) takes the form

(2.10) $\quad C_\alpha = \frac{\lambda}{p} \, B^i_\alpha \, C_i$

In order to establish the condition under which the hypersurface of a quasi-C-reducible Landsberg space be a Landsberg space we take the induced covariant differentiation of (2.10) which after the use of (1.15) reads

(2.11) $$C_{\alpha|\beta} = \frac{\lambda}{p} \left\{ C_{i|h} \, B^{ih}_{\alpha\beta} + \frac{\partial C_1}{\partial \dot{u}^\alpha} \Omega_{o\beta} \, N^1 + \Omega_{\alpha\beta} \, p \right\} + \frac{1}{p} \left\{ \lambda_{|h} \, B^h_\beta + \frac{\partial \lambda}{\partial \dot{x}^1} \Omega_{o\beta} \, N^1 \right\} B^i_\alpha C_i - \frac{\lambda}{p^2} \, p_{|\beta} \, B^i_\alpha \, C_i$$

where we have used the fact that $\frac{\partial C_i}{\partial x^1}$ is symmetric in the indices i and 1.

The contraction of (2.11) by $\dot{u}^\beta$ yields

(2.12) $$C_{\alpha|o} = \frac{\lambda}{p} \left\{ C_{i|h} \dot{x}^h \, B^i_\alpha + \frac{\partial C_1}{\partial \dot{u}^\alpha} \Omega_{oo} \, N^1 + \Omega_{\alpha o} \, p \right\} + \frac{1}{p} \left\{ \lambda_{/h} \, \dot{x}^h + \frac{\partial \lambda}{\partial \dot{x}^1} \Omega_{oo} \, N^1 \right\} B^i_\alpha \, C_i - \frac{\lambda}{p^2} \, p_{|o} \, B^i_\alpha \, C_i$$

It is now assumed that the intrinsic and induced connection parameters are identical and $\rho \neq o$, $p \neq 1/2$). Therefore by theorem (2.2) $\Omega_{\alpha o} = o$

If $\rho \neq o$ and $p \neq 1/2$, the eqn (2.12) now reduces to

$$(2.13)\quad C_{\alpha|o} = \frac{\lambda}{p} C_{i|h}\dot{x}^h B^i_\alpha + \frac{1}{p}\lambda_{|h}\dot{x}^h B^i_\alpha C_i - \frac{\lambda}{p^2} p_{|o} B^i_\alpha C_i$$

which after using the Lemma 1, gives the following theorem.

<u>Theorem 2.3</u>

If the induced and intrinsic connection parameters of a quasi-C-reducible Landsberg space coincide and $p \neq 1/2$ and ρ is non-zero, then the hypersurface is Landsberg if and only if $C_{\alpha|o} = o$.

Again if $\Omega_{\alpha o} = o$, the equation (2.6) shows that the tensor $M_{\alpha\beta}$ vanishes identically. According to Brown [7], the partial derivative of the normal in this case is given by

$$\frac{\partial N^1}{\partial \dot{u}^\alpha} = -M_\alpha N^1$$

which in view of $\rho = C_1 N^1 = o$ reduces to

$$(2.14)\quad \frac{\partial C_1}{\partial \dot{u}^\alpha} N^1 = -C_1 \frac{\partial N^1}{\partial \dot{u}^\alpha} = (C_1 N^1)\, M_\alpha = o$$

Assuming that λ is the function of the coordinates only, from first of (2.3a) we have

$$(2.15)\quad p_{|o} = \frac{\lambda_{|h}\dot{x}^h (1+p)}{(1-\lambda)}$$

On substitution from (2.14) and (2.15) in (2.12) we get

$$C_{\alpha|o} = \frac{\lambda}{p} B^i_\alpha C_{i|h}\dot{x}^h + \frac{1}{p}\lambda_{|h}\dot{x}^h B^i_\alpha C_i - \frac{\lambda_{|h}\dot{x}^h \lambda}{p^2 (1-\lambda)} (1+p) B^i_\alpha C^i$$

This equation by virtue of Lemma 1, gives the following theorem.

<u>Theorem 2.4</u>

If the induced and intrinsic connection parameters of the hypersurface of a quasi-C-reducible Landsberg space are identical, $\Omega_{\alpha o} \neq o$

and the scalar λ is the function of the coordinates only, the hypersurface is also a Landsberg space.

In order to deduce the condition under which the hypersurface of a quasi-C-reducible and p-reducible be a p-reducible space, we take the induced covariant derivatives of the characteristic scalar q in (2.3a) and the torsion tensor C_α in (2.2). These derivatives are given as

$$(2.16)\quad q_{|\alpha} C_\beta \dot{u}^\alpha = \frac{(n+1)^2}{S^3} C_j B^j_\beta \dot{u}^\alpha \Big[n\mu_{|\alpha} - 3\mu^2 \big\{ C^2_{|\alpha} - (n+1)\,\rho^2_{|\alpha} \big\} \Big]$$

where $S = n + 3\mu\, C^2 + (n+1)\,\rho^2$

$$(2.17)\quad 3q\, C_{\beta|\alpha} \dot{u}^\alpha = \frac{3\mu(n+1)^2}{S^3} C_j B^j_\beta \dot{u}^\alpha \Big[3\mu_{|\alpha} \big\{ C^2 - (n+1)\rho^2 \big\} + 3\mu \big\{ C^2_{|\alpha}$$
$$- (n+1)\rho^2_{|\alpha} \big\} \Big] + \frac{3\mu(n+1)^2}{S^2} \Big\{ C_{j|i} B^i_\alpha \dot{u}^\alpha + \frac{\partial C_j}{\partial \dot{x}^l} \Omega_{oo} N^l \Big\} B^j_\beta + \frac{3\mu}{S^2}(n+1)^2 \Omega_{\beta o}\, \rho$$

Adding (2.16) and (2.17) we get

$$(2.18)\quad q_{|o} C_\beta + 3q\, C_{\beta|o} = \frac{(n+1)^2}{S^2} \Big[C_j \mu_{|i} \dot{x}^i + 3\mu C_{j|i} \dot{x}^i \Big] + G_\alpha$$

where

$$(2.19)\quad G_\alpha = \frac{(n+1)^2}{S^2} \Big\{ 3\mu\, \Omega_{\beta o}\, \rho + \frac{\partial \mu}{\partial \dot{x}^l} N^l \Omega_{oo} C_j B^j_\beta +$$
$$+ 3\mu B^J_\beta \frac{\partial C_j}{\partial \dot{x}^l} \Omega_{oo} N^l \Big\} + \frac{2(n+1)^2}{S^3} C_j B^j_\beta S_{|o} \,.$$

In view of Lemma 2 equation (2.18) gives the following theorem.

Theorem 2.5

The hypersurface of a quasi-C-reducible and p-reducible Finsler space is p-reducible if and only if G_α in (2.19) vanishes.

From the above discussion it is clear that the theorems (2.2), (2.3) and (2.4) are true if $\rho = o$ or $\Omega_{\alpha o} = o$ and $p \neq 1/2$. We study the properties of the hypersurfaces in these cases in the following two sections.

3. The case $\rho = C_i N^i = o$

According to Rund ([5], p. 160), the induced connection coefficients are given by

$$\overset{*}{\Gamma}{}^{\alpha}_{\beta\gamma} = B^{\alpha}_{i}\left(B^{i}_{\beta\gamma} + \overset{*}{\Gamma}{}^{i}_{hk}\, B^{hk}_{\beta\gamma}\right)$$

which on assumption that F_n is Landsberg, are independent of the direction if and only if B^{α}_{i} are independent of the direction. The directional derivative of B^{α}_{i} calculated with the help of (1.4), (1.7), (1.10), (1.11) and (1.16) is given by

$$(3.1)\quad \frac{\partial B^{\alpha}_{i}}{\partial \dot{u}^{\gamma}} = 2\,\rho(\lambda h^{\alpha}_{\gamma} + 3\mu' C^{\alpha} C_{\gamma})\, N_i$$

Here $(\lambda h^{\alpha}_{\gamma} + 3\mu' C^{\alpha} C_{\gamma}) = o$ implies that $n = 3$ which contradicts the assumption $(n \geqslant 4)$. Therefore we have the following:

Theorem 3.1

The induced connection parameters of the hypersurface of a quasi-C-reducible Landsberg space are independent of the direction if and only if the vector C_i is tangential to the hypersurface.

Combining theorems (3.1), (2.2) and (2.4) we get

Theorem 3.2

If the induced connection parameters of the hypersurface of a quasi-C-reducible Landsberg space are independent of the direction, then the intrinsic and induced connection parameters are coincidence and the hypersurface is Landsberg space provided the scalar λ is the function of the coordinates only.

4. The Case $\mu_{\alpha o} = o$

Two forms of normal curvature vector of the hypersurface have so far been given. One is due to Rund based on locally Minkowaskian theory and other is due to Davies based on locally Ecludian theory. These curvatures are denoted by $\bar{I}^{i}_{\alpha\beta}$ and $H^{oi}_{\alpha\beta}$ respectively. A relation between them has been given by Rund ([5], p. 193) which reads

$$(4.1)\quad \overset{o}{H}{}^{i}_{\alpha\beta} = I^{i}_{\alpha\beta} + N^{i} N_{j}\, C^{j}_{hk}\, B^{h}_{\beta}\, H^{ok}_{\alpha\lambda}\, \dot{u}^{\lambda}$$

Multiplication by $\dot{u}^{\rho}$ and use of (1.15) gives

$$(4.2)\quad H^{oi}_{\alpha\beta}\dot{u}^{\beta} = I^{i}_{\alpha\beta}\,\dot{u}^{\beta} = \Omega_{\alpha o}\,N^{i}$$

By virtue of eqns (1.16), (2.2), and (4.2) the eqn (4.1) reduces to

$$(4.3)\quad H^{oi}_{\alpha\beta} = I^{i}_{\alpha\beta} + \frac{\lambda + 3\ \ o^{2}}{1-\lambda - 3\mu\rho^{2}}\, C_{\beta}\,\Omega_{\alpha o}\,N^{i}$$

We assume that the hypersurface F_{n-1} is non-Riemannian i.e. $C_{\beta} \neq o$. Therefore for the equality of two curvature vectors either $\Omega_{\alpha o} = o$ or $\lambda + 3\mu\rho^{2} = o$. The latter condition in view of (2.3a) shows that $p = \lambda$. Hence we have the following theorem.

Theorem 4.1

If the hypersurface is non-Riemannian, then Rund's and Davies' normal curvature vectors are identical if and only if either $\Omega_{\alpha o} = o$ or the characteristic scalars λ and p are equal.

From theorems (2.2), (2.3) and (4.1) we get the following:

Theorem 4.2

If Rund's and Davies' normal curvature vectors of the hypersurface of a quasi-C-reducible F_n are equal and $p \neq \lambda$, then the induced and intrinsic connection parameters are equal.

Theorem 4.3

If Rund's and Davies' normal curvature vectors of a hypersurface of a quasi-C-reducible Landsberg space are equal and $p \neq \lambda$, then the hypersurface is also a Landsberg space if and only if $C_{\alpha|o} = o$.

5. T-Conditions

According to (Matsumoto (1974) and Kawaguchi (1972))* the so-called T-tensor is given by

$$(5.1)\quad T_{hijk} = C_{hij|k} + C_{hij}{}^{1k} + C_{hik}{}^{1T} + C_{hkj}{}^{1i} + C_{ikj}{}^{1h}$$

---

*The expression for T_{hijk} differs slightly from what Matsumoto has given in which the first term in the right handside is $FC_{hij|k}$. This difference is due to the fact that we have used Rund's notation for v-covariant differentiation $C_{hij|k}$ of C_{hij}.

where $C_{hij|k}$ denotes the v-covariant derivative of C_{hij}. The corresponding expression for the tensor $T_{\alpha\beta\gamma\delta}$ of the space F_{n-1} is given by

$$(5.2)\quad T_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma|\delta} + C_{\alpha\beta\gamma}\, l_{\delta} + C_{\alpha\beta\delta}\, l_{\gamma} + C_{\alpha\gamma\delta}\, l_{\beta} + C_{\beta\gamma\delta}\, l_{\alpha}$$

The v-covariant derivative of the relation $C_{\alpha\beta\gamma} = C_{hij}\, B^{hij}_{\alpha\beta\gamma}$ yields

$$(5.3)\quad C_{\alpha\beta\gamma/\delta} = C_{hij|\delta}\, B^{hij}_{\alpha\beta\gamma} + C_{hij}\, Z^{h}_{\alpha\delta}\, B^{ij}_{\beta\gamma} + C_{hij}\, Z^{i}_{\beta\delta} B^{hj}_{\alpha\gamma} + C_{hij}\, Z^{j}_{\gamma\delta}\, B^{hi}_{\alpha\beta}$$

where $Z^{h}_{\alpha\delta}$ stands for $B^{h}_{\alpha}\backslash\delta$

A direct calculation gives

$$(5.4)\quad C_{\alpha\beta\gamma|\delta} = C_{hij|k}\, B^{hijk}_{\alpha\beta\gamma\delta}$$

$$(5.5)\quad \text{and}\quad Z^{h}_{\alpha\delta} = M_{\alpha\delta} N^{h}$$

In view of eqns (1.11), (1.16) and (5.5) we get

$$(5.6)\quad C_{hij}\, Z^{h}_{\alpha\delta}\, B^{ij}_{\beta\gamma} = \lambda^2\rho^2\, h_{\beta\gamma} h_{\alpha\delta} + \frac{3\mu\rho^2\lambda}{(1-\lambda-3\mu\rho^2)^2}\,(h_{\beta\gamma} C_{\alpha} C_{\delta} + h_{\alpha\delta} C_{\beta} C_{\gamma}) + \frac{9\mu^2\rho^2}{(1-\lambda-3\mu\rho^2)^4}\, C_{\alpha} C_{\beta} C_{\gamma} C_{\delta}\ .$$

Substituting the values from (5.4) and (5.6) in (5.3) and simplifying we get

$$(5.3a)\quad C_{\alpha\beta\gamma|\delta} = C_{hij|k}\, B^{hijk}_{\alpha\beta\gamma\delta} + \lambda^2\rho^2\,(h_{\beta\gamma} h_{\alpha\delta} + h_{\alpha\gamma} h_{\beta\delta} + h_{\alpha\beta} h_{\gamma\delta}) + \frac{3\mu\lambda\rho^2}{(1-\lambda-3\mu\rho^2)^2}\,(h_{\beta\gamma} C_{\alpha} C_{\delta} + h_{\alpha\gamma} C_{\beta} C_{\delta} + h_{\alpha\beta} C_{\gamma} C_{\delta} + h_{\alpha\delta} C_{\beta} C_{\gamma} + h_{\beta\delta} C_{\gamma} C_{\alpha} + h_{\gamma\delta} C_{\alpha} C_{\beta}) + \frac{27\mu^2\rho^2}{(1-\lambda-3\mu\rho^2)^4}\, C_{\alpha} C_{\beta} C_{\gamma} C_{\delta}\ .$$

On substituting (5.3a) in (5.2) we find

$$(5.7)\quad T_{\alpha\beta\gamma\delta} = T_{hijk}\, B^{hijk}_{\alpha\beta\gamma\delta} + \rho^2 \Big[\lambda^2 (h_{\beta\gamma} h_{\alpha\delta} + h_{\alpha\beta} h_{\gamma\delta} + h_{\alpha\gamma} h_{\beta\delta}) + \frac{3\mu\lambda}{(1-\lambda-3\mu\rho^2)^2} (h_{\beta\gamma} C_\alpha C_\delta + h_{\alpha\gamma} C_\beta C_\delta + h_{\alpha\beta} C_\gamma C_\delta + h_{\alpha\delta} C_\beta C_\gamma + h_{\beta\delta} C_\alpha C_\gamma + h_{\gamma\delta} C_\alpha C_\beta) + \frac{27\mu^2}{(1-\lambda-3\mu\rho^2)^4} C_\alpha C_\beta C_\gamma C_\delta \Big]$$

The space F_n is said to satisfy T-condition if T_{hijk} vanishes.

Theorem 5.1

If a quasi-C-reducible F_n satisfies T-condition, then the necessary and sufficient condition for its hypersurface also to satisfy T-condition is that the vector C_i is tangential to the hypersurface.

Proof:

From eqn. (5.7) it follows that if $T_{hijk} = o$, the tensor $T_{\alpha\beta\gamma\delta}$ vanishes if and only if either $\rho = o$ or the expression within the bracket [ ] in (5.7) is zero.

The contraction by $g^{\alpha\beta}$ of the expression within the bracket [ ] in (5.7) and use of (2.9) yields

$$(5.8)\quad \{\lambda^2 n + 3\mu\lambda(C^2 - \rho^2)\}\, h_{\gamma\delta} + \frac{3\mu}{(1-\lambda-3\mu\rho^2)^2} \{\lambda(n+2) + 9\mu(C^2-\rho^2)\}\, C_\gamma C_\delta = o$$

From (5.8) it follows that $\lambda^2 n + 3\mu\lambda\ (C^2 - \rho^2) = o$ implies that n=1 and if $\lambda^2 n + 3\mu\lambda(C^2 - \rho^2)$ is non-zero then $h_{\gamma\delta}$ and $C_\gamma C_\delta$ are proportional and eqn (5.8) gives n=3. In both the cases the values of n contradict the assumption $n \geq 4$ and hence the theorem.

Combining theorems (3.1), (3.2) and (5.1) we get the following theorem.

Theorem 5.2

If the quasi-C-reducible Landsberg space F_n and its hypersurface F_{n-1} satisfy T-conditions and λ is the function of coordinates only, then the hypersurface is Landsberg whose induced and intrinsic connection parameters are coincidence and independent of the direction element $\dot{u}^{\alpha}$

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Department of Mathematics
University of Gorakhpur
Gorakhpur-273001

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Some Problems on Hyperbolic Kahlerian Recurrent Space

—C. P. Awasthi

1. Introduction

Yano [2] studied H-projective transformation in an almost complex space. Later on, Provanovic' Mileva [1] studied the H-projective transformation in a locally product space and obtained some results on "Hyperbolic Kähler Space" defined by Rasevaski [6]. Using these results, we have defined hyperbolic Kählerian recurrent and hyperbolic Kählerian semi-recurrent spaces and established some results connecting these spaces. The last section of this paper is devoted to the study of some properties of conformally recurrent hyperbolic Kähler space.

An n-dimensional manifold M_n is called a locally product space [2], if in M_n a tensor field $F_j^i \neq \delta_j^i$ is given, satisfying the conditions

$$F_j^i F_i^k = \delta_j^k , \tag{1.1}$$

$$F_{j,k}^i = 0 , \tag{1.2}$$

here (,) is the operator of covariant differentiation with respect to the symmetric affine connection Γ_{ij}^h.

Let us suppose that in an n-dimensional locally product space, the Riemannian metric

$$ds^2 = g_{ij} \, dx^i \, dx^j , \tag{1.3}$$

satisfying

$$F_j^t F_i^s g_{ts} = - g_{ji} , \tag{1.4}$$

is given. We also assume that covariant differentiation is taken with respect to the Christoffel symbols calculated for the tensor g_{ij}. If we put

(1.5) $$F_{ji} = F_j^a \, g_{ai},$$

we have

(1.6) $$F_{ji} = - F_{ij}, \; F_{ij,k} = 0 \text{ and } g_{ij,k} = 0 \, .$$

A locally product space satisfying above conditions, is called a hyperbolic Kähler space. Rasevaski was first to consider such spaces [6].

In a hyperbolic Kähler space, we have the following relation [1]

(1.7) $$R_{kt} \, F_p^t = - R_{tp} \, F_k^t \, ,$$

(1.8) $$R_k^s \, F_{sh} = - \frac{1}{2} R_{srkh} \, F^{sr} \, ,$$

(1.9) $$R_{ki} = - R_{st} \, F_k^s \, F_i^t \, .$$

Holomorphic projective curvature tensor of hyperbolic Kähler space is [1]

(1.10) $$P_{kji}^h = R_{kji}^h + \frac{1}{n+2} \Big[R_{ki} \, \delta_j^h - R_{ji} \, \delta_k^h + R_{ka} \, F_i^a \, F_j^h - R_{ja} \, F_i^a \, F_k^h + 2R_{ka} \, F_j^a \, F_i^h \Big].$$

2. Hyperbolic Kählerian recurrent and hyperbolic Kählerian semi-recurrent spaces

Now we define the following:

Definition (2.1): A hyperbolic Kähler space is called hyperbolic Kählerian recurrent if it satisfies

$$R_{kjih,l} = \lambda_l \, R_{kjih} \, ,$$

for some non-zero vector λ_l. It is called hyperbolic Kählerian semi-recurrent space, if it satisfies

(2.2) $$R_{ij,l} = \lambda_l \, R_{ij} \, .$$

From this definition, we can check that every hyperbolic Kählerian recurrent space is hyperbolic Kählerian semi-recurrent but the converse is not true.

Definition (2.2): A hyperbolic Kähler space is called H-projective recurrent, if it satisfies the relation

$$(2.3) \qquad P_{kjih,l} = \lambda_l P_{kjih} .$$

The H-projective curvature tensor in hyperbolic Kähler space is

$$(2.4) \qquad P_{kjih} = R_{kjih} + \frac{1}{n+2}\Big[g_{jh} R_{ki} - R_{ji} g_{kh} + R_{ka} g_{th} F_i^a F_j^t - R_{ja} g_{mh} F_i^a F_k^m + 2R_{ka} g_{hb} F_j^a F_i^b \quad .$$

Differentiating (2.4) covariantly with respect to x^l and making use of (1.6), we get

$$(2.5) \qquad P_{kjih,l} = R_{kjih,l} + \frac{1}{n+2}\Big[g_{jh}R_{ki,l} - g_{kh}R_{ji,l} + R_{ka,l}g_{th}F_i^aF_j^t - R_{ja,l}g_{mh}F_i^aF_k^m + 2R_{ka,l}g_{hb}F_j^aF_i^b\Big] .$$

Multiplying (2.4) by λ_l and subtracting it from (2.5), we get

$$(2.6) \qquad P_{kjih,l} - \lambda_l P_{kjih} = R_{kjih,l} - \lambda_l R_{kjih} + \frac{1}{n+2}\Big[g_{jh}(R_{ki,l} - \lambda_l R_{ki}) - g_{kh}(R_{ji,l} - \lambda_l R_{ji}) + g_{th}F_i^aF_j^t(R_{ka,l} - \lambda_l R_{ka}) - g_{mh}F_i^aF_k^m(R_{ja,l} - \lambda_l R_{ja}) + 2g_{hb} F_j^a F_i^b (R_{ka,l} - \lambda_l R_{ka})\Big] .$$

If the hyperbolic Kähler space is hyperbolic Kählerian semi-recurrent, in view of (2.2), equation (2.6) takes the form

$$(2.7) \qquad P_{kjih,l} - \lambda_l P_{kjih} = R_{kjih,l} - \lambda_l R_{kjih} .$$

Conversely, if in a hyperbolic Kähler space (2.7) is true, we get

$$(2.8)\qquad \begin{aligned} & g_{jh}(R_{ki,l} - \lambda_l R_{ki}) - g_{kh}(R_{ij,l} - \lambda_l R_{ji}) + \\ & + g_{th} F_i^a F_j^t (R_{ka,l} - \lambda_l R_{ka}) - g_{mh} F_i^a F_k^m (R_{ja,l} - \lambda_l R_{ja}) + \\ & + 2g_{hb} F_j^a F_i^b (R_{ka,l} - \lambda_l R_{ka}) = 0 . \end{aligned}$$

Performing the contraction with g^{jh} in (2.8) and making use of (1.1), (1.6) and (1.9), we obtain

$$(2.9)\qquad (n+2)\ (R_{ki,l} - \lambda_l R_{ki}) = 0 .$$

Hence (2.9) yields

$$R_{ki,l} = \lambda_l R_{ki}, \qquad \text{since } n \neq -2 ,$$

which shows that hyperbolic Kähler space is hyperbolic Kählerian semi-recurrent.

Theorem (2.1): A necessary and sufficient condition for a hyperbolic Kähler space to be hyperbolic Kählerian semi-recurrent is

$$P_{kjih,l} - \lambda_l P_{kjih} = R_{kjih,l} - \lambda_l R_{kjih}$$

If in a hyperbolic Kähler space, H-projective curvature tensor vanishes from (2.7), we get

$$R_{kjih,l} = \lambda_l R_{kjih} ,$$

which shows that space is hyperbolic Kählerian recurrent.

Theorem (2.2): An H-projectively flat hyperbolic Kählerian semi-recurrent space is hyperbolic Kählerian recurrent.

From (2.7), it follow that every hyperbolic Kählerian recurrent space is H-projectively recurrent. Hence the following theorems can easily be proved.

Theorem (2.3): A necessary and sufficient condition for a hyperbolic Kählerian semi-recurrent space to be hyperbolic Kählerian recurrent is that it must be H-projectively recurrent.

Theorem (2.4): A sufficient condition for a hyperbolic Kähler space to be hyperbolic Kählerian recurrent is that

(a) it is H-projectively recurrent,

(b) (2.8) is satisfied.

Theorem (2.5): In a hyperbolic Kähler space, if any two of the following hold in (2.6), the third one also holds

(i) the space is hyperbolic Kählerian recurrent,

(ii) the space is hyperbolic Kählerian semi-recurrent,

(iii) the space is H-projectively recurrent.

3. Conformal recurrent hyperbolic Kähler space

A hyperbolic Kähler space which satisfies

$$C^h_{kji,l} = \lambda_l C^h_{kji} , \tag{3.1}$$

where λ_l is a non-zero vector and C^h_{kji} is the conformal curvature tensor, will be called a conformally recurrent [4] hyperbolic Kähler space and λ_l will be called its vector of recurrence.

Let R^h_{kji} be the curvature tensor and $R_{ji} = R^h_{hji}$ be the Racci tensor. Also let

$$R_{kjih} = R^r_{kji}\, g_{rh} , \tag{3.2}$$

$$R = R_{ji}\, g^{ji} \tag{3.3}$$

and

$$H_{ij} \overset{\text{def}}{=\!=} -\frac{1}{2} R_{ijkl}\, F^{kl} \tag{3.4}$$

Then we have the following relations:

$$(3.5)\qquad \text{(a)}\quad H_{ij} = -H_{ji}$$

$$\text{(b)}\quad H_{ki} = -R_{kp}\, F^{p}_{i}$$

$$\text{(c)}\quad R_{kj} = -H_{kt}\, F^{t}_{j}$$

$$\text{(d)}\quad R = -H_{kj}\, F^{kj}\,.$$

Conformal curvature tensor is given by

$$(3.6)\quad C_{kjih} = R_{kjih} + g_{kh} Z_{ji} - g_{jh} Z_{ki} + g_{ji} Z_{kh} - g_{ki} Z_{jh}\,,$$

$$(3.7)\qquad Z_{ji} = \frac{R_{ji}}{n-2} - \frac{R\, g_{ji}}{2(n-1)(n-2)}\,.$$

We can write (3.1) with the help of (3.6) as follows:

$$R_{kjih,l} + g_{kh} Z_{ji,l} - g_{jh} Z_{ki,l} + g_{ji} Z_{kh,l} - g_{ki} Z_{jh,l}$$

$$= \lambda_l \left[R_{kjih} + g_{kh} Z_{ji} - g_{jh} Z_{ki} + g_{ji} Z_{kh} - g_{ki} Z_{jh} \right].$$

Transvecting (3.8) by F^{ih} and making use of (3.4) and (3.5)b, we get

$$(3.9)\quad -2H_{kj,l} - \frac{4}{n+2} H_{jk,l} - \frac{4}{2(n-1)(n-2)} F_{kj}\, R_{,l}$$

$$= \lambda_l \left[-2\, H_{kj} - \frac{4}{n-2} H_{jk} - \frac{4}{2(n-1)(n-2)} R\, F_{kj} \right].$$

Again transvecting (3.9) by F^{kj} and using (3.5)d, we obtain

$$(3.10)\qquad 2R_{,l} + \frac{4}{n-2} R_{,l} - \frac{4}{2(n-1)(n-2)} F_{kj}\, F^{kj}\, R_{,l}$$

$$= \lambda_l \left[2R + \frac{4}{n-2} R - \frac{4}{2(n-1)(n-2)} F_{kj}\, F^{kj} \right].$$

From (3.10), we get

$$(3.11) \qquad (R_{,1} - \lambda_1 r) \left[n(n-1) - F_{kj} F^{kj} \right] = 0 .$$

Equation (3.11) yields

$$(3.12) \qquad R_{,1} = \lambda_1 R,$$

since $F_{kj} F^{kj} \neq n(n-1)$.

Equation (3.9), with the help of (3.10) and (3.12), yields

$$(3.13) \qquad H_{kj,1} = \lambda_1 H_{kj} .$$

From the relation (3.5)b and (3.13), we have

$$(3.14) \qquad R_{kp,1} F^p_j F^j_m = - \lambda_1 H_{kj} F^j_m ,$$

which by virtue of (3.5)c, yields

$$(3.15) \qquad R_{km,1} = \lambda_1 R_{km} .$$

Using (3.12) and (3.15) in (3.7), we obtain

$$(3.16) \qquad Z_{ji,1} = \lambda_1 Z_{ji} .$$

Hence, we have from (3.6)

$$(3.17) \qquad R_{kjih,1} = \lambda_1 R_{kjih} ,$$

by virtue of (3.16).

If we use theorem 1 due to Roter [5], we get from last equation

$$(3.18) \qquad R_{kjih} R^{kjih} = R^2 .$$

From Walker's theorem, it is known that in a non-simple recurrent space the vector of recurrence is null. Since g_{ij} is not positive definite and therefore $\lambda_i \lambda^i$ may be equal to zero. If $\lambda_i \lambda^i = 0$, then $\lambda_i = 0$.

Thus when $R \neq 0$, the space under consideration is non-simple. Hence we have

Theorem (3.1): A conformally recurrent hyperbolic Kähler space is a non-simple recurrent space if its scalar curvature is different from zero.

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Department of Mathematics
J.N. Degree College
Lucknow 226001
India

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Characterizations of a Sub-additive Measure of Relative Information

—Parvinder Kaur

Summary

Sharma and Gupta (1976) defined a subadditive measure of relative information which involves sine functions in its definition. Its particular case is considered here which is characterized with the help of a functional inequality and also using a functional equation.

1. Introduction

Let $P = (p_1, p_2, \ldots, p_n)$, $p_i \geq 0$ $\sum_{i=1}^{n} p_i = 1$ and $Q = (q_1, q_2, \ldots, q_n)$, $q_i \geq 0$, $\sum_{i=1}^{n} q_i = 1$ be two finite discrete probability distributions. Then Sharma and Gupta (1976) defined a subadditive measure of relative information as

$$\hat{H}^s(P:Q;\alpha,\beta,\gamma) = \frac{2^{\beta}}{\sin\gamma} \sum_{i=1}^{n} p_i^{\alpha} q_i^{\beta} \sin\left(\gamma \log\frac{p_i}{q_i}\right) \quad \ldots \quad (1.1)$$

$$\alpha > 0, \quad \beta > 0, \quad \gamma \neq 0$$

where α, β, γ are real arbitary constants and logarithm is taken with base 2.

For $\alpha = 1$, $\beta = 0$, (1.1) reduces to

$$\hat{H}^s(P:Q, 1, 0, \gamma) = \frac{1}{\sin\gamma} \sum_{i=1}^{n} p_i \sin\left(\gamma \log\frac{p_i}{q_i}\right) \quad \ldots \quad (1.2)$$

Further it can be easily seen that

$$\lim_{\gamma \to 0} \hat{H}^s(P:Q, 1, 0, \gamma) = \sum_{i=1}^{n} p_i \log\frac{p_i}{q_i} \quad \ldots \quad (1.3)$$

which is well known Kullback's (1959) directed divergence.

2. Characterizations

First characterization:

To characterize (1.2) we shall first find the solution of a functional inequality in the following lemma.

Lemma 1: Every solution of the functional inequality

$$\sum_{i=1}^{n} p_i\, f(p_i)\, g(q_i) \geqslant \sum_{i=1}^{n} p_i\, f(q_i)\, g(p_i) \qquad \ldots \quad (2.1)$$

for $n > 2$ under the condition

$$f^2(p) + g^2(p) = 1 \qquad \ldots \quad (2.2)$$

for every p, is given by

$$\left.\begin{aligned} f(p_i) &= \sin\,(\gamma \log_2 p_i) \\ &\text{and} \\ g(p_i) &= \cos\,(\gamma \log_2 p_i) \end{aligned}\right\} \qquad \ldots \quad (2.3)$$

where γ is an arbitrary constant.

Proof. Taking $p_3 = q_3 \ldots\ldots p_n = q_n$ in (2.1), we get

$$p_1[f(p_1)\, g(q_1) - f(q_1) g(p_1)] + p_2[f(p_2) g\,(q_2) - f(q_2)\, g(p_2)] \geqslant 0 \quad \ldots (2.4)$$

for $p_1 + p_2 = q_1 + q_2 < 1$

Let $p_1 = q_1 + \delta$, $p_2 = q_2 - \delta$, $\delta > o$ in (2.4), we have

$$(q_1 + \delta)\, [f(q_1 + \delta)\, g(q_1) - f(q_1)\, g(q_1 + \delta)] +$$

$$(q_2 - \delta)\, [f(q_2 - \delta)\, g(q_2) - f(q_2)\, g(q_2 - \delta)] \geqslant o$$

or $(q_1 + \delta)\, [f(q_1 + \delta)\, g(q_1) - f(q_1)\, g(q_1) + f(q_1)\, g(q_1) - f(q_1) g(q_1 + \delta)]$

$$+ (q_2 - \delta)\, [f(q_2 - \delta)\, g(q_2) - f(q_2)\, g(q_2) + f(q_2)\, g(q_2)$$

$$- f(q_2)\, g(q_2 - \delta)] \geqslant o \quad \ldots \quad (2.5)$$

Let q_1, q_2 be the points where f and g are differentiable. Then dividing by δ and taking limit $\delta \longrightarrow o$ in (2.5), we get

$$q_1 [g(q_1)f'(q_1) - f(q_1)g'(q_1)] - q_2[g(q_2)f'(q_2) - f(q_2)g'(q_2)] \geq o$$

or

$$q_1[g(q_1)f'(q_1)-f(q_1)g'(q_1)] \geq q_2[g(q_2)f'(q_2)-f(q_2)g'(q_2)] \quad \ldots \quad (2.6)$$

Interchanging the roles of q_1 and q_2 due to symmetry, in (2.6) one gets

$$q[g(q)\ f'(q) - f(q)\ g'(q)] = a \quad \ldots \quad (2.7)$$

where a is some constant.

From (2.7) we have, on integration

$$\int g^2(q) \frac{d}{dq} \left[\frac{f(q)}{g(q)}\right] dq = a \int \frac{dq}{q} \quad \ldots \quad (2.8)$$

Using the condition $g^2(q) + f^2(q) = 1$, (2.8) can be written as

$$\int \frac{g^2(q)}{f^2(q) + g^2(q)} \frac{d}{dq} \left[\frac{f(q)}{g(q)}\right] dq = a \log q$$

or

$$\tan^{-1} \left[\frac{f(q)}{g(q)}\right] = a \log q$$

giving that

$$\frac{f(q)}{g(q)} = \tan (a \log q) = \frac{\sin(a \log q)}{\cos(a \log q)} \quad \ldots \quad (2.9)$$

Now (2.9) with the help of (2.2) gives

$$\left.\begin{aligned} f(q) &= \sin (a \log q) \\ g(q) &= \cos (a \log q) \end{aligned}\right\} \quad \ldots \quad (2.10)$$

which is precisely (2.3)

Hence the lemma is proved.

Theorem 1: The function $\hat{H}^s$ (P:Q) satisfying the postulates

$$P_1 : \hat{H}^s(P:Q) = \frac{1}{\sin\gamma} \sum_{i=1}^{n} p_i[f(p_i)\, g(q_i) - f(q_i)\, g(p_i)]$$

with $f^2(p) + g^2(p) = 1$ for every p

$$P_2 = \hat{H}^s\ (P:Q) \geqslant o$$

is the sub-additive measure of relative information given by

$$\hat{H}^s(P:Q) = \hat{H}^s\ (P:Q;\ 1,\ 0,\ \gamma) = \frac{1}{\sin\gamma} \sum_{i=1}^{n} p_i\ \sin(\gamma \log_2 \frac{p_i}{q_i})$$

where γ ($\neq$ o) is an arbitrary constant.

Proof: Postulates P_1 and P_2 are equivalent to (2.1) and (2.2) and so from P_1, P_2 and (2.3) we have

$$\hat{H}^s(P:Q) = \frac{1}{\sin\gamma} \sum_{i=1}^{n} p_i[\sin(\gamma\log p_i)\ \cos\ (\gamma\log q_i) - \sin(\gamma\log q_i)\ \cos(\ \log p_i)]$$

$$= \frac{1}{\sin\gamma} \sum_{i=1}^{n} p_i\ \sin(\gamma\log \frac{p_i}{q_i}) \qquad \ldots \quad (2.11)$$

which is precisely (1.2)

Hence the theorem is proved.

Second characterization: This characterization is based on the solution of the functional equation

$$\sum_{i=1}^{m}\sum_{j=1}^{n} G(x_i\, y_j,\ u_i\, v_j) = \sum_{i=1}^{m}\sum_{j=1}^{n} G(x_i,\ u_i) + \sum_{i=1}^{m}\sum_{j=1}^{n} G(y_j,\ v_j) \ldots (2.12)$$

where $x_i,\ y_j \geqslant o,\ u_i,\ v_i \geqslant o,\ \sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j = 1,\ \sum_{i=1}^{m} u_i \leqslant 1,\ \sum_{j=1}^{n} v_j \leqslant 1$

under some boundary conditions.

<u>Lemma 2</u>: Let $G : I \times I \longrightarrow R$ be a continuous function where $I = [0,1]$ and R be the set of real numbers. Then for all positive integers m and n the only continuous solution of the functional equation (2.12) is given by

$$G(x,u) = A \log x + B \log u \qquad \ldots \quad (2.13)$$

where A and B are some constants.

Proof: Let m, n, r, s be any positive integers such that $1 \leq m \leq r$
$1 \leq n \leq s$

Putting

$$x_i = \frac{1}{m},\ u_i = \frac{1}{r}\ (i = 1,2\ldots m)$$

$$y_j = \frac{1}{n},\ v_j = \frac{1}{s}\ (j = 1,2\ldots n)$$

in (2.12), we get

$$mn\ G\left(\frac{1}{mn}, \frac{1}{rs}\right) = mn\ G\ \left(\frac{1}{m}, \frac{1}{r}\right) + mn\ G\ \left(\frac{1}{n}, \frac{1}{s}\right)$$

or

$$G(ab,\ cd) = G(a,c) + G(b,d) \qquad \ldots (2.14)$$

where $\quad a = \frac{1}{m},\ b = \frac{1}{n},\ C = \frac{1}{r},\ d = \frac{1}{s}$

Taking $c = d = 1$ in (2.14) we get

$$G(ab,\ 1) = G(a,\ 1) + g(b,1)$$

which is well known Cauchy's functional equation having the continuous solution

$$G(x,\ 1) = A \log x \qquad \ldots (2.15)$$

where A is any arbitrary constant.

Taking $a = b = 1$ in (2.14), we get

$$G(1,\ cd) = G(1,c) + G(1,d)$$

giving that

$$G(1,x) = B \log x \qquad \ldots (2.16)$$

where B is an arbitrary constant.

Now taking $b = c = 1$ in (2.14) we have

$$G(a,d) = G(a,1) + G(1,d)$$

$$= A \log a + B \log d \text{ (using (2.15) and (2.16))} \quad \ldots (2.17)$$

The result (2.17) can be extended to the case when a and d are rational numbers. For this let $x = \frac{m}{n}$, $(m < n)$ $u = \frac{p}{q}$ $(p < q)$ be two rational numbers. Let K be a sufficiently larger integer such that $Kp \geq m$, $Kq \geq n$, $K \geq \frac{q(n-m)}{n(q-p)}$

Taking m as $n - m + 1$ and n as m and

setting

$$x_1 = \frac{m}{n},\ x_2 = \ldots\ldots = x_{n-m+1} = \frac{1}{n}$$

$$y_1 = y_2 = \ldots\ldots = y_m = \frac{1}{m}$$

$$u_1 = \frac{p}{q},\ u_2 \ldots\ldots = u_{n-m+1} = \frac{1}{Kn}$$

$$v_1 = \ldots\ldots\ldots = v_m = \frac{1}{pK}$$

in (2.12), we have

$$mG\left(\frac{1}{n}, \frac{1}{qK}\right) + m(n-m)\ G\left(\frac{1}{mn}, \frac{1}{pnK^2}\right) = m(n-m+1)\ G\left(\frac{1}{m}, \frac{1}{pK}\right) + mG\left(\frac{m}{n}, \frac{p}{q}\right)$$
$$+ m(n-m)\ G\left(\frac{1}{n}, \frac{1}{Kn}\right) \quad \ldots (2.18)$$

The equation (2.18) together with (2.17) gives

$$G\left(\frac{m}{n}, \frac{p}{q}\right) = A \log \frac{m}{n} + B \log \frac{p}{q}$$

or

$$G(x,u) = A \log x + B \log u \text{ for all rational } x,u, \in [o,1] \ldots (21.19)$$

From the continuity of G, it follows that (2.19) is valid for all real $x,\ u \in [o,1]$

This proves the lemma.

In terms of the solution $G(x,u)$, we define the subadditive directed divergence as

$$\hat{H}^s (P:Q;\ 1,o,\gamma) = \frac{1}{\sin \gamma} \sum_{i=1}^{n} p_i \sin G(p_i,\ q_i) \qquad \ldots (2.20)$$

under the conditions that $G(\frac{1}{2},\ \frac{1}{2}) = 0$, $G(1,\ \frac{1}{2}) = \gamma$

Theorem 2: The subadditive directed divergence between the distributions P and Q under the conditions $G(\frac{1}{2},\ \frac{1}{2}) = o$,

$G\ (1,\frac{1}{2}) = \gamma\ (\neq o)$ corresponding to the continuous solution (2.13) of the functional equation (2.12) is

$$\hat{H}^s (P:Q;\ 1,\ o,\gamma) = \frac{1}{\sin \gamma} \sum_{i=1}^{n} p_i \sin\ (\gamma \log \frac{p_i}{q_i})$$

Proof. Taking $x = 1/2$, $u = 1/2$ in (2.13), we get on using the condition that $G(\frac{1}{2},\ \frac{1}{2}) = o$

$$A = -\ B \qquad \ldots (2.21)$$

Now condition $G(1,\ \frac{1}{2}) = \gamma$, with (2.13) gives

$$-\ B = \gamma \qquad \ldots (2.22)$$

so that

$$G(x,u) = \gamma \log \frac{x}{u} \qquad \ldots (2.23)$$

The result now follows from (2.20) and (2.23).

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Mailing Address: Parvinder Kaur, Anand Bhawan, 89-B
Subhas Nagar, Alambagh
Lucknow-296005 (INDIA)

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