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On the Logarithmic Proximate Type of an Entire Dirichlet Series

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1. Introduction

Let the function defined by $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$, where $s = \sigma + it$ and $\{\lambda_n\}$ is a sequence of real numbers such that $0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots < \lambda_n \uparrow \infty$ as $n \to \infty$ and

$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = D < \infty$, represent an entire function. The Ritt-order of $f(s)$ is defined [6, p. 77] as:

$$\limsup_{\sigma\to\infty} \frac{\log\log M(\sigma)}{\sigma} = \rho,\ 0 \leq \rho \leq \infty ,$$

where $M(\sigma) = \operatorname{l.u.b}_{-\infty<t<\infty} |f(\sigma+it)|$

Let $\mu(\sigma)$ be the maximum term in the representation of $\Sigma|a_n|\exp(\sigma\lambda_n)$ and call it as the maximum term of $f(s)$. Let $\lambda_{\nu(\sigma)}$ be that value of λ_n which makes $|a_n|\exp(\sigma\lambda_n)$ the maximum term and call $\lambda_{\nu(\sigma)}$ as the rank of $\mu(\sigma)$. Let us similarly correspond $\mu_{(m)}(\sigma)$ and $\lambda_{\nu^{(m)}(\sigma)}$ to $f^{(m)}(s)$, the mth derivative of $f(s)$ as we have done about $\mu(\sigma)$ and $\lambda_{\nu(\sigma)}$ connecting them with $f(s)$, where $\mu_{(o)}(\sigma) \equiv \mu(\sigma)$, $\lambda_{\nu(o)(\sigma)} \equiv \lambda_{\nu(o)}$. It is well-known that [9, p. 67; 4, p. 2]

(1.1) $\log \mu(\sigma) = \log \mu(\sigma_o) + \int_{\sigma_o}^{\sigma} \lambda_{\nu(x)}\, dx,\ \sigma > \sigma_o.$

We shall always take $D = 0$. Then, for functions of finite Ritt-order ρ, we have [9]

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(1.2) $\log M(\sigma) \simeq \log \mu(\sigma)$.

Let the order of f(s) be zero. Then, we define the logarithmic order /5/ of f(s) as :

$$(1.3)\ \limsup_{\sigma \to \infty} \frac{\log \log M(\sigma)}{\log \sigma} = \rho^{*},\ 1 \leq \rho^{*} \leq \infty$$

If 1 ρ^* , then Awasthi /1/ has introduced the notion of logarithmic proximate order $\rho^*(\sigma)$; according to him it is function of σ satisfying the following relations:

(1.4) $\rho^*(\sigma)$ is continuous, piecewise differentiable for

$\sigma > \sigma_0$ (2) .

(1.5) $\rho^*(\sigma) \to \rho^*$ as $\sigma \to \infty$,

(1.6) $\sigma\,(\rho^*(\sigma))' \log \sigma \to 0$ as $\sigma \to \infty$,

where $(\rho^*(\sigma))'$ is either the left or right hand derivatives at points where they are different.

Clearly, for a given entire function $\rho^*(\sigma)$ is not unique. For example, if $\rho^*(\sigma)$ be a logarithmic proximate order of f(s), then $\rho^*(\sigma) + \frac{c \log \sigma}{\sigma}$ (c = constant) is also a logarithmic proximate order of f(s).

The logarithmic proximate type T* and lower logarithmic proximate type t* of f(s) with respect to logarithmic proximate order $\rho^*(\sigma)$ are defined as:

$$(1.7)\ \lim_{\sigma \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log M(\sigma)}{\sigma^{\rho^*(\sigma)}} = \begin{matrix} T^* \\ t^* \end{matrix},\ 0 \leq t^* \leq T^* \leq \infty .$$

Our aim, in this paper, is to study a few growth properties of f(s) with the help of logarithmic proximate order and logarithmic proximate type. Throughout this paper we shall assume 1 $\rho^* < \infty$. First, we show

2. Theorem 1. Let $L(\sigma) = \sigma^{\rho^*(\sigma) - \rho^*}$, then

$$\lim_{\sigma \to \infty} \left\{ \frac{L(\sigma + k)}{L(\sigma)} \right\}^{\sigma/k} = 1,\ k > 0.$$

---

(2) σ_0 is a fixed large number and need not be the same at each occurrence.

Proof. Let us set $\emptyset(\sigma) = \rho^*(\sigma) \log \sigma$, then

$$\log \left\{ \frac{L(\sigma+k)}{L(\sigma)} \right\} = \left\{ \rho^*(\sigma+k) - \rho^* \right\} \log (\sigma+k) - \left\{ \rho^*(\sigma) - \rho^* \right\} \log\sigma$$

$$= \left\{ \emptyset(\sigma+k) - \emptyset(\sigma) \right\} - \rho^* \log \left(1+\frac{k}{\sigma}\right)$$

$$= k\, \emptyset'(\eta) - \frac{k\rho^*}{\rho}(1-o(1)), \quad \sigma_0 < \sigma < \eta < \sigma+k,$$

$$= k \left\{ \frac{1}{\sigma}\rho^*(\sigma) + (\rho^*(\sigma))' \log \sigma \right\} - \frac{k\rho^*}{\sigma}(1-o(1))$$

or, $\frac{\sigma}{k} \log \left\{ \frac{L(\sigma+k)}{L(\sigma)} \right\} = \left\{ \rho^*(\sigma) - \rho^* + \sigma (\rho^*(\sigma))' \log \sigma \right\} + \rho^* - \rho^*(1-o(1))$.

From the conditions (1.5) and (1.6) of logarithmic proximate order, we have,

$$| \rho^*(x) - \rho^* | \quad \frac{\varepsilon}{2} \text{ and } | x(\rho^*(x))' \log x | \quad \frac{\varepsilon}{2},$$

where ε being an arbitrarily taken small positive number. Using this, we get,

$$\lim_{\sigma \to \infty} \log \left\{ \frac{L(\sigma+k)}{L(\sigma)} \right\}^{\sigma/k} = 0$$

Hence,

$$\lim_{\sigma \to \infty} \left\{ \frac{L(\sigma+k)}{L(\sigma)} \right\}^{\sigma/k} = 1.$$

This proves the theorem 1.

Corollary 1. For all large values of σ, we have

$$1 - \varepsilon < \left\{ \frac{(\sigma+k)^{\rho^*(\sigma+k)-\rho^*}}{\sigma^{\rho^*(\sigma)-\rho^*}} \right\}^{\sigma/k} < 1 + \varepsilon, \quad \varepsilon > 0.$$

Theorem 2. If $\emptyset(\sigma)$ is a bounded function on each finite interval and

(2.1) $$\lim_{\sigma \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\emptyset(\sigma)}{\sigma^{\rho^*(\sigma)-1}} = \begin{matrix} \alpha^* \\ \beta^* \end{matrix}, \quad 0 \leq \beta^* \leq \alpha^* \leq \infty,$$

then

(2.2) $$\limsup_{\sigma \to \infty} \left\{ \frac{1}{\sigma^{\rho^*(\sigma)}} \int_{\sigma_0}^{\sigma} \emptyset(\sigma)\, d\sigma \right\} \leq \frac{\alpha^*}{\rho^*}$$

and

(2.3) $$\liminf_{\sigma\to\infty} \left\{ \frac{1}{\sigma^{\rho^*(\sigma)}} \int_{\sigma_o}^{\sigma} \phi(\sigma)\, d\sigma \right\} \geqslant \frac{\beta^*}{\rho^*},$$

where $\sigma > \sigma_o$.

To prove this theorem we need the following lemma:

Lemma 1. For $\sigma > \sigma_o$,

(2.4) $$\int_{\sigma_o}^{\sigma} x^{\rho^*(x)-1}\, dx = \frac{1}{\rho^*}\, \sigma^{\rho^*(\sigma)} + o(\sigma^{\rho^*(\sigma)})$$

Proof. We have

$$\int_{\sigma_o}^{\sigma} x^{\rho^*(x)-1}\, dx = \int_{\sigma_o}^{\sigma} x^{\rho^*(x)-\rho^*}\, x^{\rho^*-1}\, dx$$

$$= \left[x^{\rho^*(x)-\rho^*}\, \frac{x^{\rho^*}}{\rho^*} \right]_{\sigma_o}^{\sigma}$$

$$- \frac{1}{\rho^*} \int_{\sigma_o}^{\sigma} x^{\rho^*(x)-1}\, (\rho^*(x)-\rho^*) + x(\rho^*(x))' \log x\, dx$$

Using (1.5) and (1.6), we get

$$\int_{\sigma_o}^{\sigma} x^{\rho^*(x)-1}\, dx = \frac{1}{\rho^*}\, \sigma^{\rho^*(\sigma)} + O(1) - o(1) \int_{\sigma_o}^{\sigma} x^{\rho^*(x)-1}\, dx$$

or, $$(1+o(1)) \int_{\sigma_o}^{\sigma} x^{\rho^*(x)-1}\, dx = \frac{1}{\rho^*}\, \sigma^{\rho^*(\sigma)} + O(1)$$

which gives,

$$\int_{\sigma_o}^{\sigma} x^{\rho^*(x)-1}\, dx = \frac{1}{\rho^*}\, \sigma^{\rho^*(\sigma)} + o(\sigma^{\rho^*(\sigma)}) .$$

Proof of theorem 2. From (2.1), we have, for any $\varepsilon > 0$,

$$\phi(\sigma) < (\alpha^* + \varepsilon)\, \sigma^{\rho^*(\sigma)-1}$$

Therefore, $\int_{\sigma_0}^{\sigma} \phi(\sigma)\, d\sigma = \int_{\sigma_0}^{\sigma_0'} \phi(\sigma)\, d\sigma + \int_{\sigma_0'}^{\sigma} \phi(\sigma)\, d\sigma$, $\sigma > \sigma_0'$

$$\leq O(1) + (\alpha^* + \varepsilon)\int_{\sigma_0'}^{\sigma} \sigma^{\rho^*(\sigma)-1}\, d\sigma$$

$$= O(1) + (\alpha^* + \varepsilon)\left\{\frac{1}{\rho^*}\sigma^{\rho^*(\sigma)} + o(\sigma^{\rho^*(\sigma)})\right\}$$, using

Lemma 1.

or, $$\frac{1}{\sigma^{\rho^*(\sigma)}}\int_{\sigma_0}^{\sigma} \phi(\sigma)\, d\sigma \leq \frac{\alpha^* + \varepsilon}{\rho^*} + o(\sigma^{\rho^*(\sigma)})$$

or, $$\limsup_{\sigma \to \infty} \frac{1}{\sigma^{\rho^*(\sigma)}}\int_{\sigma_0}^{\sigma} \phi(\sigma)\, d\sigma \leq \frac{\alpha^*}{\rho^*} .$$

Similarly, considering the definition of β^*, we can easily prove (2.3).

Corollary 2. If $\phi(\sigma)$ is bounded on each finite interval and

$$\lim_{\sigma \to \infty} \frac{\phi(\sigma)}{\sigma^{\rho^*(\sigma)-1}} = \Delta$$, then

(2.5) $$\lim_{\sigma \to \infty} \left\{\frac{1}{\sigma^{\rho^*(\sigma)}}\int_{\sigma_0}^{\sigma} \phi(\sigma)\, d\sigma\right\} = \frac{\Delta}{\rho^*} , \ \sigma > \sigma_0 .$$

3. Theorem 3. Let $f(s)$ be an entire function of logarithmic order ρ^*. Then, with respect to logarithmic proximate order $\rho^*(\sigma)$, $f(s)$ is of,

(i) minimal logarithmic proximate type when $A^* = 0$

(ii) maximal logarithmic proximate type when $B^* = \infty$

(iii) normal logarithmic proximate type when $B^* \neq 0$ and $A^* \neq \infty$

where,

(3.1) $$\lim_{\sigma \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\lambda(\sigma)}{\sigma^{\rho^*(\sigma)-1}} = \begin{matrix} A^* \\ B^* \end{matrix} , \ 0 \leq B^* \leq A^* \leq \infty$$

Proof. It is well-known that [2] the function $\lambda_{\nu(\sigma)}$ is bounded in finite intervals, has an enumerable set of discontinuities and changes values at these discontinuities only.

Hence, from (1.1), (1.2) and (2.2), we find

$$T^* = \limsup_{\sigma\to\infty} \frac{\log M(\sigma)}{\sigma^{\rho^*(\sigma)}} = \limsup_{\sigma\to\infty} \frac{\log \mu(\sigma)}{\sigma^{\rho^*(\sigma)}}$$

$$= \limsup_{\sigma\to\infty} \left[\frac{1}{\sigma^{\rho^*(\sigma)}} \left\{ \log \mu(\sigma_0) + \int_{\sigma_0}^{\sigma} \lambda_{\nu(x)}\, dx \right\} \right]$$

$$= \limsup_{\sigma\to\infty} \left\{ \frac{1}{\sigma^{\rho^*(\sigma)}} \int_{\sigma_0}^{\sigma} \lambda_{\nu(x)}\, dx \right\} \leq \frac{A^*}{\rho^*} .$$

Hence,

(3.2) $\rho^* T^* \leq A^*$.

Similarly, proceeding for lower logarithmic proximate type t^* and applying (2.3), we obtain

(3.3) $\rho^* t^* \geq B^*$.

Combining (3.2) and (3.3), we get

(3.4) $B^* \leq \rho^* t^* \leq \rho^* T^* \leq A^*$.

All the three results of theorem 3 now follow from (3.4).

Corollary 3. If $\lim_{\sigma\to\infty} \frac{\lambda_{\nu(\sigma)}}{\sigma^{\rho^*(\sigma)-1}}$ exists and equals to Δ^*, then $A^* = B^* = \Delta^*$. It follows from (3.4) that $\rho^* t^* = \rho^* T^* = \Delta^*$, that is

(3.5) $\rho^* T^* = \lim_{\sigma\to\infty} \frac{\lambda_{\nu(\sigma)}}{\sigma^{\rho^*(\sigma)-1}}$

and in this case

(3.6) $t^* = T^* = \lim_{\sigma\to\infty} \frac{\log M(\sigma)}{\sigma^{\rho^*(\sigma)}}$.

Thus existence of $\lim_{\sigma\to\infty} \frac{\lambda_{\nu(\sigma)}}{\sigma^{\rho^*(\sigma)-1}}$ implies the existence of $\lim_{\sigma\to\infty} \frac{\log M(\sigma)}{\sigma^{\rho^*(\sigma)}}$.

Theorem 4. Let T^* and t^* be the logarithmic proximate type and lower logarithmic proximate type, respectively, of $f(s)$ with respect to logarithmic proximate order $\rho^*(\sigma)$. If ρ^* be the logarithmic order of $f(s)$ and

$$(3.7)\quad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{M'(\sigma)/M(\sigma)}{\sigma^{\rho^*(\sigma)-1}} = \begin{matrix}\gamma^*\\ \delta^*\end{matrix}, \quad 0 \le \delta^* \le \gamma^* \le \infty,$$

where $M'(\sigma)$ is the derivative of $M(\sigma)$, then

$$(3.8)\quad \delta^* \le \rho^* t^* \le \rho^* T^* \le \gamma^* .$$

Proof. It is well-known that $\log M(\sigma)$ is an increasing convex function of σ [3, p. 138]. This implies that $\log M(\sigma)$ is differentiable almost everywhere with an increasing derivative; the set of points where the left hand derivative is less than the right hand derivative is of measure zero. Hence, we can write

$$\log M(\sigma) = \log M(\sigma_o) + \int_{\sigma_o}^{\sigma} \frac{M'(x)}{M(x)}\, dx .$$

But $\frac{M'(x)}{M(x)}$ is bounded on each finite interval. Hence (3.8) follows from (2.2) and (2.3), if we proceed on the lines of the proof of theorem 3.

Corollary 4. If $\lim_{\sigma\to\infty} \left\{ \frac{M'(\sigma)/M(\sigma)}{\sigma^{\rho^*(\sigma)-1}} \right\}$ exists, then

$$\lim_{\sigma\to\infty} \left\{ \frac{M'(\sigma)/M(\sigma)}{\sigma^{\rho^*(\sigma)-1}} \right\} = \rho^* T^* .$$

4. Results involving derivatives of $f(s)$:

We have already spoken in the introduction about $\mu_{(m)}(\sigma)$ and $\lambda_{\nu^{(m)}(\sigma)}$. In this section, we assume that $\lim_{\sigma\to\infty} \frac{\lambda_{\nu^{(m)}(\sigma)}}{\sigma^{\rho^*(\sigma)-1}}$ exists for all non-negative integers m.

Theorem 5. The logarithmic proximate type of the derivatives of $f(s)$ is the same as that of $f(s)$ with respect to logarithmic proximate order $\rho^*(\sigma)$.

Proof. We have [7, p. 200]

$$(4.1) \quad \limsup_{n \to \infty} \frac{\phi(\lambda_n)}{\log |a_n|^{-1/\lambda_n}} = \frac{\rho^*}{\rho^*-1} (\rho^* T^*)^{1/(\rho^*-1)},$$

where $\phi/(t)$ is the unique solution $(t > t_o)$ of the equation $t = \sigma^{\rho^*(\sigma)-1}$.

Let T^*_m be the logarithmic proximate type of the mth derivative $f^{(m)}(s) = \sum_{n=1}^{\infty} a_n \lambda^m_n e^{s\lambda_n}$ with respect to logarithmic proximate order $\rho^*(\sigma)$. It is well-known that the logarithmic order of $f^{(m)}(s)$ is equal to that of $f(s)$. Hence

$$\frac{\rho^*}{\rho^*-1} (\rho^* T^*_m)^{1/(\rho^*-1)} = \limsup_{n \to \infty} \frac{\phi(\lambda_n)}{\log a_n^{-1/} n + \log(^{1/}_n n)^{-m}}$$

$$= \limsup_{n \to \infty} \frac{\phi(\ _n)}{\log an^{-1/} n + \log(^{1/}_n n)^{-m}}$$

$$= \limsup_{n \to \infty} \frac{\phi(\ _n)}{\log |a_n|^{-1/\lambda_n}} = \frac{\rho^*}{\rho^*-1} (\rho^* T^*)^{1/(\rho^*-1)}.$$

This gives

$$(4.2) \quad T^*_m = T^*$$

Finally, we prove:

Theorem 6. If ρ^* be the logarithmic order and T^* be the logarithmic proximate type with respect to logarithmic proximate order $\rho^*(\sigma)$ of an entire function $f(s)$, then

$$(4.3) \quad \lim_{\sigma \to} \left[\left\{ \frac{\mu_{(m)}(\sigma)}{\mu(\sigma)} \right\}^{1/m} \Big/ \sigma^{\rho^*(\sigma)-1} \right] = \rho^* T^*.$$

Before, we start with actual proof, we first consider a lemma which will be needed in the proof of this theorem.

Lemma 2. For $m = 1,2,3,\ldots$, we have

$$\lim_{\sigma \to \infty} \frac{\lambda_{\nu(\sigma)}}{\sigma^{\rho^*(\sigma)-1}} = \lim_{\sigma \to \infty} \frac{\lambda_{\nu^{(m)}(\sigma)}}{\sigma^{\rho^*(\sigma)-1}}.$$

Proof. We know

$$\log \mu(\sigma) = \log \mu(\sigma_o) + \int_{\sigma_o}^{\sigma} \lambda_{\nu(x)}\, dx$$

and $\log \mu_{(m)}(\sigma) = \log \mu_{(m)}(\sigma_o) + \int_{\sigma_o}^{\sigma} \lambda_{\nu^{(m)}(x)}\, dx, \ \sigma > \sigma_o.$

Also, $\lim\limits_{\sigma \to \infty} \dfrac{\lambda_{\nu^{(m)}(\sigma)}}{\sigma^{\rho^*(\sigma)-1}}$ exists for all non-negative integers m. Hence, from (3.5), we obtain

$$(4.4) \quad \rho^* T^* = \lim_{\sigma \to \infty} \frac{\lambda_{\nu(\sigma)}}{\sigma^{\rho^*(\sigma)-1}}$$

and

$$(4.5) \quad \rho^* T^*_m = \lim_{\sigma \to \infty} \frac{\lambda_{\nu^{(m)}(\sigma)}}{\sigma^{\rho^*(\sigma)-1}}$$

The lemma 2 now follows from (4.2), (4.4) and (4.5).

Proof of theorem 5. It is known [8, p. 708] that

$$(4.6) \quad \lambda_{\nu(\sigma)} \leq \frac{\mu_{(1)}(\sigma)}{\mu(\sigma)} \leq \lambda_{\nu^{(1)}(\sigma)} \leq \frac{\mu_{(2)}(\sigma)}{\mu_{(1)}(\sigma)} \leq \cdots \leq \lambda_{\nu^{(m-1)}(\sigma)}$$

$$\leq \frac{\mu_{(m)}(\sigma)}{\mu_{(m-1)}(\sigma)} \leq \lambda_{\nu^{(m)}(\sigma)}$$

Multiplying the ratios involving these μ's one finds that

$$\frac{\mu_{(m)}(\sigma)}{\mu(\sigma)} \geq \lambda_{\nu^{(m-1)}(\sigma)} \cdots\cdots \lambda_{\nu(\sigma)} \geq \left\{\lambda_{\nu(\sigma)}\right\}^m$$

Also, from (4.6), we have

$$\frac{\mu_m(\sigma)}{\mu(\sigma)} \leq \lambda_{\nu^{(1)}(\sigma)} \cdots\cdots \lambda_{\nu^{(m)}(\sigma)} \leq \left\{\lambda_{\nu^{(m)}(\sigma)}\right\}^m .$$

Thus

$$\lambda_{\nu(\sigma)} \leq \left\{ \frac{\mu_{(m)}(\sigma)}{\mu(\sigma)} \right\}^{1/m} \leq \lambda_{\nu^{(m)}(\sigma)} . \tag{4.7}$$

(4.7) and lemma 2 give the desired result.

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On Pseudo N-Ideals

Nirmesh Kumar

Kumar [5] defined a new type of algebraic structure and called it N-Ring. This paper illustrates some more interesting properties of N-Ring. First of all we give some definitions followed by examples.

Definition 1: Let R be an N-Ring such that $a * b = b * a$, for all $a, b \in R$, then R is called a commutative N-Ring.

Example 1: Consider the set Z of integers. We define two binary operations 'o' and '*' as follows:

$$a \circ b = ab, \text{ for all } a, b, \in Z$$

and $$a * b = a^2 b^2, \text{ for all } a, b \in Z.$$

It may be verified that $(Z, o, *)$ is a commutative N-Ring.

In example 1 in [5], R is a non-commutative N-Ring.

Definition 2: An N-Ring R is called associative if $a * (b * c) = (a * b) * c$, for all $a, b, c \in R$.

Example 2: In [5, example 1] $(R, o, *)$ is an associative N-Ring, but in example 1, $(Z, +, -)$ is not associative.

Definition 3: Let $(R, o, *)$ be an N-Ring and $S \subseteq R$ be a non-empty subset of R. If the system $(S, o, *)$ is itself an N-Ring, then $(S, o, *)$ is said to be a sub N-Ring of the N-Ring $(R, o, *)$.

Example 3: We have seen that $(R, o, *)$ as defined in [5] is an N-Ring. Consider now the set

$$S = \left\{ \frac{p^2}{q^2} : p \text{ and } q \text{ are non zero integers} \right\}.$$

Evidently, S is a subset of R. It may be easily verified that $(S, o, *)$ is a sub N-Ring of the N-Ring R.

Example 4: Consider the N-Ring $(Z, +, -)$. Also Z_e, the set of even integers forms an N-Ring under + and − . Consequently $(Z_e, +, -)$ is a sub N-Ring of the N-Ring $(Z, +, -)$.

Definition 4: A sub N-Ring K of an N-Ring R is said to be a left N-ideal of R if $r \in R$ and $k \in K$ imply $r * k \in K$.

In a similar manner, we may define right N-ideal.

When $r \in R$ and $k \in K$ imply both $r * k$ and $k * r \in K$, then K is said to be a two sided N-ideal.

Example 5: Consider the set

$R = \{(p, q) : p$ and q are non-zero positive integers$\}$. We define the two binary operations 'o' and '*' of R as follows:

$$(p_1, q_1) \circ (p_2, q_2) = (p_1p_2, q_1q_2)$$

and $$(p_1, q_1) * (p_2, q_2) = (p_1q_2, q_1p_2)$$

Obviously, (R, o, *) is an N-Ring.

It may be verified that

$K = \{(2p, q) : p$ and q are non zero positive integers$\}$ is a right N-ideal of R.

Definition 5: An N-Ring (F, o, *) is said to be an N-field if it (i) is commutative, (ii) has unity, (iii) is such that for each non zero element $a \in F$ there exists an element $b \in F$ such that $a * b = 1$, the unity of F.

Example 6: Consider the set

$F = \left\{ \frac{p}{q} : p \text{ and } q \text{ are non zero positive integers} \right\}$.

We define two binary operations 'o' and '*' as follows:

$$\frac{p_1}{q_1} \circ \frac{p_2}{q_2} = \frac{p_1p_2}{q_1q_2}$$

and $$\frac{p_1}{q_1} * \frac{p_2}{q_2} = \frac{p_1^2p_2^2}{q_1^2q_2^2}$$

It may be verified that (F, o, *) is an N-field.

Example 7: Consider the set Z of integers. Let

$$z_1 \circ z_2 = z_1 + z_2$$

and $z_1 * z_2 = nz_1 + nz_2$ where n is an integer and $n \geq 2$. It may be verified that (Z, o, *) is an N-field.

In definition 5, if (1) is not satisfied then F is called a skew N-field. In example 1 in [5] R is a skew N-field.

Sen [1] gave the idea of pseudo ideal in case of semi-groups. We have made an effort to extend this idea in case of N-rings.

Let (R, o, *) be an associative N-ring and K be a non-empty subset of R such that

(i) $a \circ b \in K$ when $a, b \in K$

(ii) $x^2 * K \subseteq K$ for every element $x \in R$.

Then (K, o, *) is known as a left pseudo N-ideal of the N-ring (R, o, *).

In a similar way, we may define a right pseudo N-ideal.

K is a pseudo N-ideal if it is both left pseudo N-ideal and right N-ideal.

Example 8: Consider the N-ring (R, o, *) where

$R = \{(p, q) :$ p and q are non zero positive integers$\}$.

We consider the subset K of R as

$K = \{(ap^2, bq^2) :$ a, p, b and q are non zero posit integers and $p, q \neq 1\}$.

It may be verified that (K, o, *) is a pseudo N-ideal of R.

From the definition of pseudo N-ideal of an N-ring it is clear that every N-ideal is a pseudo N-ideal. But it is interesting to note that every pseudo N-ideal is not necessarily an N-ideal. We will authenticate this fact by considering the following example.

Example 9: We consider the N-ring (R, o, *) where

$R = \left\{ \frac{p}{q} :\right.$ p and q are non zero integers$\left.\right\}$.

We now consider the subset K of R as

$K = \left\{ \frac{r}{s} :\right.$ r and s are non zero positive integers$\left.\right\}$.

Obviously (K, o, *) is a pseudo N-ideal, but not an N-ideal because

$$\frac{r}{s} * \frac{-p}{q} = \frac{rq}{-sp} \notin K \text{ for } \frac{r}{s} \in K \text{ and } \frac{-p}{q} \in R$$

We can now prove the following propositions.

Proposition 1

The intersection of any collection of left pseudo N-ideals (right pseudo N-ideal) is a left pseudo N-ideal (right pseudo N-ideal).

Proposition 2

Let (R, o, *) be an N-ring and A and B be any two pseudo N-ideals of R. Then A * B will be a pseudo N-ideal of R if * is self distributive in R.

In the above proposition, the condition '* is self distributive in R' can be replaced by the condition '* is associative in R'.

Proposition 3

In an associative N-ring, a non-empty set A, satisfying $A * x = x * A$, for all $x \in R$, is a right pseudo N-ideal if, and only if, A is a left pseudo N-ideal.

Proposition 4

Let (R, o, *) be an N-ring and A and B be any two N-ideals of R then A * B is a pseudo N-ideal if * obeys symmetric law.

Proposition 5

Let (R, o, *) be an associative N-ring and $A \subset R$ be a left N-ideal of R. Let $\alpha(A)$ denote the set of those elements $a \in R$ such that for each 'a' there exists $y \in R$ such that $y * a \in R$. Then $\alpha(A)$ is a left N-ideal and consequently also a left pseudo N-ideal.

Obviously $\alpha(A)$ is closed under o. Let r be any element belonging to R. Further suppose that

$a \in \alpha(A)$

$\Rightarrow y \in R$ such that $y * a \in A$

Now $r \in R$ and $y * a \in A$, therefore

$r * (y * a) \in A$

$(r * y) * a \in A$

or $y * a \in \alpha(A)$

This completes the proof.

Proposition 6

Let (S, o, *) be a sub N-ring of an N-ring (R, o, *) which is both commutative and associative, then S is a pseudo N-ideal of R iff $x * A * x \subset A$ for all $x \in R$.

Proposition 7

If (R, o, *) is a commutative N-ring with identity in which $r \circ r = r$ for every $r \in R$, then the following conditions are equivalent.

(a) R is an N-field

(b) $x * (A \sim B) * x \subseteq A * B$ for every element $x \in R$ when $A \sim B$ is a non-empty, where A and B are two pseudo N-ideals of R. $A \sim B$ denotes the complement of B in A.

(c) $A' * x^2 \subset A'$ when A is a pseudo N-ideal of R, where A' denotes the complement of A in R.

Proof

(a) $\Longrightarrow$ (b)

Let $a \in A \sim B \Longrightarrow a \in A$ and $a \notin B$

For every $x \in R$, we have $x * a * x \in A$

If we suppose that $a * a * x \in B$ then

$a = x^{-1} * (x * a * x) * x^{-1} \in x^{-1} * B * x^{-1} \subset B$

which is contrary to our assumption.

(b) $\Longrightarrow$ (c)

If $a \in A'$ then $x^2 * (R \sim A) * x^2 \subset R \sim A$ ... (1)

Suppose $a * x^2 \notin R \sim A$

Since A is a pseudo N-ideal, therefore $x^2 * a * x^2 \in A$ which contradicts (1). Hence the result.

(c) $\Longrightarrow$ (a)

Let $a \in R$, then $a * R$ is a pseudo N-ideal of R

Suppose $a * R \neq R$ and $b \in R$ $a * R$, then $a^2 * b \in R \sim a * R$

also $a^2 * b \in a * R$

This contradiction implies that $a * r = R$

Hence R is an N-field.

Proposition 8

Suppose (R, o, *) is a commutative N-ring with an identity and $r \circ r = r$ for all $r \in R$. Let $\beta(A)$ denote the set of all those elements $x \in R$ such that for each x there exists $y \in R$ such that $y^2 * x \in A$. Then R will be an N-field iff $\beta(A) = A$.

Proof

Let $a \in \beta(A)$

$\Rightarrow \exists\ x \in R$ such that $x^2 * a \in A$

$\Rightarrow (x-1)^2 * (x^2 * a) \in A \Rightarrow a \in A$

Therefore, $\beta(A) \subseteq A$

Obviously $A \subseteq \beta(A)$

Thus $\beta(A) = A$.

Conversely, if a is any element of R, then $a * R$ is a pseudo N-ideal of R. Suppose $b \in R$, then $a^2 * b \in a * R$. This shows that $b \in \beta(a * R)$. Therefore, $R \subseteq \beta(a * R)$.

Obviously, $(a * R) = R$. So, we have

$a * R = R$. This completes the proof.

Proposition 9

Suppose (R, o, *) is an N-ring. Let A be a pseudo N-ideal of R and $\bar{A} = \{a^2, a \in A\}$. Then $\bar{A} * A$ is a pseudo N-ideal of R if (R, *) is a symmetric semi-group.

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On Some Postulates for Quasi—Groups

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1. Introduction

Let S be a system of elements a, b, c,... closed with respect to a multiplicative operation. For any a, b, c $\in$ S, consider the following postulates:

(A) $(ab)a = b$ (A') $a(ba) = b$

(B) $a(ab) = b$

(C) $(ba)a = b$

(D) $c(b(a(ac))) = b$.

In this paper, we have formulated some postulates which lead us to the study of quasi-groups and commutative quasi-groups. Higman and Newmann [1], Padmanabhan [2] and Sholander [3] gave some identities (single equational axioms) to characterize abelian groups. Here, we also have given one identity (D) to characterize a commutative quasi-group.

2. Some propositions

We prove the following:

Proposition (2.1) (A) $\Longleftrightarrow$ (A').

Proof: Replacing a by ba in (A), we get

$$((ba)\,b)(ba) = b$$

or $$a(ba) = b.$$

Conversely, replacing a by ab in (A'), we obtain (A).

Proposition (2.2) (A, B)* $\Longleftrightarrow$ (A, C).

Proof: Replacing a by ba in (B) and using (A), we find

$$(ba)((ba)\,b) = b,$$

or $$(ba)a = b,$$

which is (C).

*Here and onwards, the notation (A, B) will mean the postulates (A) and (B) together.

Conversely, replacing a by ab in (C) and using the proposition (2.1), we obtain

$$a(ab) = b ,$$

which is (B).

Proposition (2.3) $(B, C) \Longleftrightarrow (A, B)$.

Proof: Replacing a by ba in (C) and using (b), we get

$$(b(ba))(ba) = b ,$$

or $$a(ba) = b ,$$

which, in view of the proposition (2.1), is equivalent to

$$(ab)a = b .$$

The converse is obvious from the proposition (2.2).

3. Quasi-groups

Following Hall [4], a quasi-group Q is a system of elements a, b, c,... in which a binary operation $(a, b) \longrightarrow ab$ is defined, such that, in ab = c, any two of the elements a, b, c determine the third uniquely as an element of Q. Now, we prove the following theorems:

Theorem (3.1). The system S together with the postulate (A) forms a quasi-group.

Proof: If ab = c, the proposition (2.1) yields

$$b = ca \quad \text{and} \quad a = bc,$$

from which it follows that given c and a or given c and b, there is atmost one b or atmost one a respectively. In other words, a and b determine c uniquely.

Theorem (3.2). The system S together with the postulates (A, B) forms a commutative quasi-group.

Proof: From (B) and (A), we have

$$ba = (a(ab))a$$
$$= ab .$$

Thus, in view of the theorem (3.1), the system S together with (A, B) is a commutative quasi-group.

<u>Theorem</u> (3.3). The system S is a commutative quasi-group if any one of the following is satisfied:

(i) (A, B)

(ii) (A, C)

(iii) (B, C)

Proof: The proof follows immediately from the propositions (2.2), (2.3) and the theorem (3.2).

<u>Theorem</u> (3.4). The system S together with the identity relation (D) characterizes a commutative quasi-group.

Proof: Putting $c = ab$ in (D), we find

$$(ab)(b(a(a(ab)))) = b \ . \tag{3.1}$$

Now, if we replace first b by a and then c by b in (D), we get

$$b(a(a(ab))) = a \ . \tag{3.2}$$

Thus, by virtue of (3.2), (3.1) yields

$$(ab)a = b \ . \tag{3.3}$$

Again, replacing a by cb in (D) and using (3.3), we find

$$c(b((cb)((cb)c))) = b$$

or

$$c(b((cb)b)) = b,$$

which, in view of (3.3) and consequently the proposition (2.1) yields

$$c(cb) = b \ . \tag{3.4}$$

Thus, we conclude that (3.3) and (3.4) are the same as the postulates (A) and (B) and therefore, in view of the theorem (3.2), the system S forms a commutative quasi-group.

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Homotopy and Isotopy of Topological Groupoids and Quasi—Groups

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Abstract

This paper deals with the notions of homotopy and isotopy for topological groupoids and quasi-groups and establishes certain results concerning them. Similar notions in algebra have worked as motivation.

1. Introduction

The notion of homotopy was introduced in algebra by A.C. Choudhury (1948) in his paper entitled Quasi-groups and non-associative systems [1]. This is a sort of generalization of the concept of isotopy [2]. An ordered triple $t = (\alpha, \beta, \gamma)$ of mappings of a groupoid (G, o) onto a groupoid (H,.) is said to be a homotopy of (G, o) onto (H,.) if $(a \circ b)\gamma = (a\alpha).(b\beta)$ for all $a, b \in G$. If α, β, γ are bijective, the homotopy is an isotopy. If $\alpha = \beta = \gamma$, homotopy and isotopy reduce to a homomorphism and an isomorphism respectively. These algebraic notions have been carried over to the topological groupoids and quasi-groups and some -esults have been established.

2. Some preliminary definitions

A topological algebraic structure is a pair consisting of an algebraic structure over a non-empty set G and a topology over the same set such that the algebraic compositions are continuous in the given topology.

The only composition in a groupoid is $(a, b) \rightarrow ab$. A quasi-group has three compositions, namely $(a, b) \rightarrow ab$, $(a, b) \rightarrow a/b$ (left division, $a/b = x$ if and only if $ax = b$) and (a, b) — b/a (right division, $\frac{b}{a} = x$ if and only if $xa = b$).

Multiplication $(a, b) \rightarrow ab$ in a groupoid is continuous in the given topology if for any neighbourhood W of ab, we can find neighbourhoods U and V of a and b respectively such that $UV \subseteq W$. The left division $(a, b) \rightarrow a/b$ in a quasi-group is continuous if for every neighbourhood W of a/b, we can find neighbourhoods U and V of a and b respectively such that $V \subseteq UW$. The case $V \subseteq WU$ is the case of continuity of right division.

3. Definitions of Homotopy and Isotopy

Definition 1. Let (G, o) and(H,.) be two topological groupoids. Then an ordered triple $t = (\alpha, \beta, \gamma)$ of continuous mappings of G onto H will be said to be a homotopy of (G, o) onto (H,.) if $(a \circ b)\gamma = (a\alpha).(b\beta)$ for all $a, b \in G$.

In this event, we say that (G, o) is homotopic to (H,.) and (H,.) is a homotope of (G, o).

Definition 2. If γ be a homeomorphism, t is said to be a principal homotopy and if α, β be homeomorphisms, it is said to be a simple homotopy.

Definition 3. If α, β and γ be homeomorphisms, then t is said to be an isotopy of (G, o) onto (H,.).

In this event, we say that (G, o) is isotopic to (H,.) and (H,.) is an isotope of (G, o).

Definition 4. If I_G be the identity mapping of G, then an isotopy (α, β, I_G) of a topological groupoid (G, o) onto a topological groupoid (G,.) is said to be a principal isotopy of (G, o) onto (G,.).

4. Some Theorems

Theorem 1. Let (H,.) be a topological groupoid and (G, o), a groupoid as well as a topological space. If

(i) $t = (\alpha, \beta, \gamma)$ be a homotopy of (G, o) onto (H,.) in the algebraic sense,

(ii) α and β are continuous,

and (iii) γ is a homeomorphism,

then t is a principal homotopy of (G, o) onto (H,.).

Proof: It is sufficient to show that multiplication in (G, o) is continuous. Let a, b $\in$ G and lit W be any neighbourhood of a o b in G. Since γ is a homeomorphism, $W' = W\gamma$ is a neighbourhood of $(a \text{ o } b)\gamma = (a\alpha).(b\beta)$ in H. Since multiplication in H is continuous, there exist neighbourhoods U' and V' of $a\alpha$ and $b\beta$ in H such that $U'.V' \subseteq W'$. Also by the continuity of α and β, we can find neighbourhoods U and V of a and b respectively in G such that $U\alpha \subseteq U'$ and $V\beta \subseteq V'$. Now it is simple to see that $U \text{o} V \subseteq W$. So multiplication in (G, o) is continuous and hence the theorem follows.

Theorem 2. Let (G, o) be a topological groupoid and Let (H,.) be a groupoid as well as a topological space. If

(i) $t = (\alpha, \beta, \gamma)$ be a homotopy of (G, o) onto (H,.) in the algebraic sense,

(ii) α and β are open,

and (iii) α, β and γ are continuous,

then t is a homotopy of (G, o) onto (H,.).

Proof: It is sufficient to prove that the multiplication in $(H,.)$ is continuous in the topological space H. Let $a, b \in H$ and let W be any neighbourhood of $a.b$ in H. If $a'\alpha = a$ and $b'\beta = b$, then W is a neighbourhood of $(a'\alpha), (b'\beta) = (a'ob')\gamma$ in H. Since γ is continuous, we can find a neighbourhood W' of $a'ob'$ in G such that $W'\gamma \subseteq W$. Since multiplication in (G, o) is continuous, we can find neighbourhoods U' and V' of a' and b' respectively such that $U'.V' \subseteq W'$. Since α and β are open, there exist neighbourhoods U and V, of a and b respectively in H such that $U \subseteq U'\alpha$ and $V \subseteq V'\beta$. Now it can be easily seen that $U.V \subseteq W$, proving that multiplication in $(H,.)$ is continuous. Hence the theorem.

Theorem 3. Let $t = (\alpha, \beta, I_G)$ be a principal homotopy of a topological groupoid (G, o) onto a topological groupoid $(G,.)$. If (G, o) contains the identity element e, then t is a principal isotopy.

Proof: In fact, we are to prove that α and β are homeomorphisms. Let $e\alpha = a$ and $e\beta = b$. If $x \in G$, $x\, I_G = (eox)\, I_G = (e\alpha).(x\beta) = a.(x\beta) = (x\beta)\, L_a$, where $x\, L_a = a.x$, that is, $x\, I_G = x\,(\beta\, L_a)$, proving that $I_G = \beta\, L_a$, or $L_a = \beta^{-1}$ and so β is bijective. Since L_a is continuous, it follows that β is a homeomorphism of the topological space G onto itself.

Similarly, we can see that α is a homeomorphism. Hence the theorem.

Theorem 4. Let $t = (\alpha, \beta, \gamma)$ be a principal homotopy of a topological groupoid (G, o) onto a topological quasi-group $(H,.)$. If (G, o) be a left topological quasi-group, that is, if the composition of left division be defined in G and the mapping $(a, b) \rightarrow a/b$ be continuous, then β is a homeomorphism.

Proof: β is injective, for if $x_1 \neq x_2 \Rightarrow x_1\beta = x_2\beta = c$, (say) let $a'.c = b'$. Also let $a\alpha = a'$ and $b\gamma = b'$. Then it can be seen that $aox_1 = b = aox_2$, which contradicts the fact that (G, o) is a left quasi-group.

To show that β is a homeomorphism, it remains now to show that β is open. Let $x \in G$ and W be any neighbourhood of x. If a, b in G be such that $aox = b$, then there exist neighbourhoods, U' and V' of a and b such that $V' \subseteq U'oW$. Since γ is a homeomorphism, $V'\gamma$ is a neighbourhood of $b\gamma = (aox)\gamma = (a\alpha).(x\beta)$ in H. Since multiplication in $(H,.)$ is continuous, we can find neighbourhoods U, V of $a\alpha$, $x\beta$ in H such that $U.V \subseteq V'\gamma$. It can now be seen that $V \subseteq W\beta$, proving the openness of β.

It may similarly be shown that if (G, o) be a right topological quasi-group, then α is a homeomorphism.

Corollary. If (G, o) be a topological quasi-group, then t is an isotopy.

Proof: It is sufficient to prove that the multiplication in $(H,.)$ is continuous in the topological space H. Let $a, b \in H$ and let W be any neighbourhood of a.b in H. If $a'\alpha = a$ and $b'\beta = b$, then W is a neighbourhood of $(a'\alpha), (b'\beta) = (a'ob')\gamma$ in H. Since γ is continuous, we can find a neighbourhood W' of a'ob' in G such that $W'\gamma \subseteq W$. Since multiplication in (G, o) is continuous, we can find neighbourhoods U' and V' of a' and b' respectively such that $U'.V' \subseteq W'$. Since α and β are open, there exist neighbourhoods U and V, of a and $\overline{b}$ respectively in H such that $U \subseteq U'\alpha$ and $V \subseteq V'\beta$. Now it can be easily seen that $U.V \subseteq W$, proving that multiplication in $(H,.)$ is continuous. Hence the theorem.

Theorem 3. Let $t = (\alpha, \beta, I_G)$ be a principal homotopy of a topological groupoid (G, o) onto a topological groupoid $(G,.)$. If (G, o) contains the identity element e, then t is a principal isotopy.

Proof: In fact, we are to prove that α and β are homeomorphisms. Let $e\alpha = a$ and $e\beta = b$. If $x \in G$, $x\, I_G = (eox)\, I_G = (e\alpha).(x\beta) = a.(x\beta) = (x\beta)\, L_a$, where $x\, L_a = a.x$, that is, $x\, I_G = x\, (\beta\, La)$, proving that $I_G = \beta\, La$, or $L_a = \beta^{-1}$ and so β is bijective. Since L_a is continuous, it follows that β is a homeomorphism of the topological space G onto itself.

Similarly, we can see that α is a homeomorphism. Hence the theorem.

Theorem 4. Let $t = (\alpha, \beta, \gamma)$ be a principal homotopy of a topological groupoid (G, o) onto a topological quasi-group $(H,.)$. If (G, o) be a left topological quasi-group, that is, if the composition of left division be defined in G and the mapping $(a, b) \longrightarrow a/b$ be continuous, then β is a homeomorphism.

Proof: β is injective, for if $x_1 \neq x_2 \Rightarrow x_1\beta = x_2\beta = c$, (say) let $a'.c = b'$. Also let $a\alpha = a'$ and $b\gamma = b'$. Then it can be seen that $aox_1 = b = aox_2$, which contradicts the fact that (G, o) is a left quasi-group.

To show that β is a homeomorphism, it remains now to show that β is open. Let $x \in G$ and W be any neighbourhood of x. If a, b in G be such that $aox = b$, then there exist neighbourhoods, U' and V' of a and b such that $V' \subseteq U'oW$. Since γ is a homeomorphism, $V'\gamma$ is a neighbourhood of $b\gamma = (aox)\gamma = (a\alpha).(x\beta)$ in H. Since multiplication in $(H,.)$ is continuous, we can find neighbourhoods U, V of $a\alpha$, $x\beta$ in H such that $U.V \subseteq V'\gamma$. It can now be seen that $V \subseteq W\beta$, proving the openness of β.

It may similarly be shown that if (G, o) be a right topological quasi-group, then α is a homeomorphism.

Corollary. If (G, o) be a topological quasi-group, then t is an isotopy.

Theorem 5. A simple homotopy $t = (\alpha, \beta, \gamma)$ of a topological quasi-group (G, o) onto a topological quasi-group (H,.) is an isotopy.

Proof: γ is injective, for let $x_1 \neq x_2 \Rightarrow x_1\gamma = x_2\gamma$. Let $x_1 = yoz_1$ and $x_2 = yoz_2$. Then $(yoz_1)\gamma = (yoz_2)\gamma$, or $(y\alpha).(z_1\beta) = (y\alpha).(z_2\beta) \Rightarrow z_1\beta = z_2\beta \Rightarrow z_1 = z_2 \Rightarrow x_1 = x_2$. It is now sufficient to show that γ is open. Let $x = yoz \in C$ and W be any neighbourhood of x. Since multiplication in (G, o) is continuous, we can find neighbourhoods U and V of y and z respectively in G such that $UoV \subseteq W$. Since α and β are homeomorphisms, $U\alpha$ and $V\beta$ are neighbourhoods of $y\alpha$ and $z\beta$ in H. Now $W' = (U\alpha).(V\beta)$ is a neighbourhood of $(y\alpha).(x\beta) = (yoz)\gamma = x\gamma$ in H. It can now be easily been that $W' \subseteq W$, proving that γ is open. Hence the theorem.

Theorem 6. Let $t = (\alpha, \beta, \gamma)$ be a simple homotopy of a topological groupoid (G, o) onto a topological quasi-group (H,.). If (G, o) contains a left (or a right) identity, then t is an isotopy.

Proof: Let e be a left identify of (G, o). Now γ is injective, for $x_1 \neq x_2 \Rightarrow x_1\beta \neq x_2\beta$

$$\Rightarrow (e\alpha).(x_1\beta) \neq (e\alpha).(x_2\beta)$$

$$\Rightarrow (eox_1)\gamma \neq (eox_2)\gamma \Rightarrow x_1\gamma \neq x_2\gamma .$$

If remains now to show that γ is open. This follows from the previous theorem if we note that $x = eox$.

We can similarly prove the result if e is a right identity.

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Flow Near an Oscillating Porous Flat Plate

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Abstract

This paper presents the solution for the flow of an incompressible visco-elastic (Revlin-Ericksen model) fluid near an infinite porous flat plate which executes linear harmonic oscillation in its own plane.

1. Introduction

The solution for the flow of a viscous incompressible fluid near an infinite oscillating solid flat plate has been obtained by Stokes [3] and later by Rayleigh [1]. In this paper we extend the analysis to the flow of an incompressible visco-elastic (Revlin-Ericksen model) fluid near an infinite oscillating porous flat plate when there is uniform suction imposed over the plate.

2. Formulation of the problem

Consider the flow of an incompressible visco-elastic fluid about an infinite porous flat plate which executes linear harmonic oscillation with velocity $U\cos nt$. Let x denote the coordinate parallel to the direction of motion and y the coordinate perpendicular to the plate. Since the plate is infinite in length and uniform suction is imposed over it, all the physical variables depend only on y and t, the time. The pressure p in the fluid is assumed constant. If V represents the velocity of suction or injection at the plate, the equation of continuity

$$(1) \qquad \frac{\partial v}{\partial y} = 0$$

with the condition $v = V$ at $y = 0$, yields $v = V$ everywhere.

The governing equation describing the flow of an incompressible visco-elastic fluid [1] is

$$(2) \qquad \frac{\partial u}{\partial t} + V\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + \beta\frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial y}\right)$$

with the boundary conditions

$$(3) \qquad \left.\begin{array}{l} u = U\cos nt \quad \text{at } y = 0, \\ u = 0 \quad \text{as } y \longrightarrow \infty \end{array}\right\} \quad t > 0,$$

where ν is the kinematic viscosity, β, the kinematic visco-elasticity.

Introducing

$$\bar{t} = nt,$$

$$\eta = y\sqrt{\frac{n}{\nu}},$$

$$\lambda = \frac{V}{n\nu}\ ;\ \begin{matrix}\lambda > 0 \text{ (injection)}\\ \lambda < 0 \text{ (suction)}\end{matrix}$$

$$S = \frac{n}{\beta\nu},$$

the equation (2) transforms to

$$\frac{\partial u}{\partial \bar{t}} + \lambda\frac{\partial u}{\partial \eta} = \frac{\partial^2 u}{\partial \eta^2} + S\frac{\partial^2}{\partial \eta^2}\left(\frac{\partial u}{\partial \bar{t}} + \lambda\frac{\partial u}{\partial \eta}\right) \tag{4}$$

with the conditions

$$\begin{aligned} u &= U\cos\bar{t} \text{ at } \eta = 0,\\ u &= 0 \text{ as } \eta \longrightarrow \infty . \end{aligned} \tag{5}$$

3. Solution

Assuming the solution in the form

$$u = W(\eta)\exp(i\bar{t}) \tag{6}$$

and substituting in (4) we get

$$S\lambda W''' + (1 \pm iS)W'' - \lambda W' - iW = 0. \tag{7}$$

The corresponding boundary conditions are

$$\begin{aligned} W &= U \text{ at } \eta = 0,\\ W &= 0 \text{ as } \eta \longrightarrow \infty . \end{aligned} \tag{8}$$

Since the equation (7) is the differential equation of order three and we have only two boundary conditions, therefore we solve the equation regarding the elastic parameter S is small. This is consistent with the derivation of Revlin-Ericksen constitutive equation where S is taken as the perturbation parameter. Thus we set

$$W = W_o + SW_1 + o(S^2). \tag{9}$$

Substituting (9) in (7) and equating the coefficients of like powers of S, we get

$$(10) \qquad W_o'' - \lambda W_o' - iW_o = 0,$$

$$(11) \qquad W_1'' - \lambda W_1' - iW_1 = -\lambda W_o''' - iW_o''.$$

The corresponding boundary conditions are

$$(12) \qquad \begin{aligned} &W_o = U,\ W_1 = 0 \quad \text{at } \eta = 0, \\ &W_o = W_1 = 0 \quad \text{as } \eta \longrightarrow \infty. \end{aligned}$$

Solving (10) and (11) with the boundary conditions (12), we get

$$(13) \qquad W_o = U \exp\,(b\eta)$$

and

$$(14) \qquad W = \frac{U(\lambda b + i)b^2}{\sqrt{\lambda^2 + 4i}}\,\eta \exp\,(b\eta),$$

where

$$b = \frac{1}{2}\,(\lambda - \sqrt{\lambda^2 + 4i}).$$

Hence from (6), (9), (13) and (14) we get

$$(15) \qquad \frac{u}{U} = \left\langle \exp\,(b\eta) + S\left\{ \frac{(\lambda b + i)b^2}{\sqrt{{}^2 + 4i}}\,\eta \exp\,(b\eta) \right\} \right\rangle \exp\,(i\bar{t})$$

$$= (F_r + iF_i) \exp\,(i\bar{t}),$$

where

$$F_r = \left\{ \cos\frac{q\eta}{2} + S\eta\left(M\cos\frac{q\eta}{2} + N\sin\frac{q\eta}{2}\right) \right\} \exp\left(\frac{\lambda - p}{2}\right)\eta,$$

$$F_i = \left\{ -\sin\frac{q\eta}{2} + S\eta\left(N\cos\frac{q\eta}{2} - M\sin\frac{q\eta}{2}\right) \right\} \exp\left(\frac{\lambda - p}{2}\right)\eta,$$

$$M = \frac{Cp + Dq}{p^2 + q^2}, \qquad N = \frac{Dp - Cq}{p^2 + q^2},$$

$$C = \frac{\lambda(\lambda - p)\left\{(\lambda - p)^2 - q^2\right\}}{8} - \frac{(2 - \lambda q)(\lambda - p)q}{4},$$

$$D = \frac{(2 - \lambda q)\left\{(\lambda - p)^2 - q^2\right\}}{8} + \frac{\lambda(\lambda - p)^2 q}{4},$$

$$p = \frac{1}{\sqrt{2}} (\sqrt{\lambda^4 + 16} + \lambda^2)^{\frac{1}{2}},$$

$$q = \frac{1}{\sqrt{2}} (\sqrt{\lambda^4 + 16} - \lambda^2)^{\frac{1}{2}}.$$

Therefore, from (15), the real part of $u\ (\eta,\ t)$ is

$$(16) \qquad u = U \langle \cos\ (\bar{t} - \frac{q\eta}{2}) + S\eta \{ M \cos\ (\bar{t} - \frac{q\eta}{2}) + N \sin\ (\bar{t} - \frac{q\eta}{2}) \} \rangle \ \exp\ (\frac{\lambda - p}{2})\eta$$

Particular cases:

i) For $S \neq 0,\ \lambda = 0$,

$$u = U \exp\ (-y\sqrt{\frac{n}{2}}) \langle \cos\ (nt - y\sqrt{\frac{n}{2\nu\nu}} + Sy\sqrt{\frac{n}{2\nu}}\ [\cos\ (nt - y\sqrt{\frac{n}{2\nu}}) - \sin\ (nt - y\sqrt{\frac{n}{2\nu}})] \rangle ,$$

which is the solution for the flow of an incompressible visco-elastic fluid near an infinite oscillating solid flat plate.

ii) For $S = 0,\ \lambda \neq 0$,

$$u = U \exp \{ \lambda - \frac{1}{\sqrt{2}} (\sqrt{\lambda^4 + 16} + \lambda^2)^{\frac{1}{2}} \} \frac{\eta}{2} . \cos \{ \bar{t} - \frac{\eta}{2\sqrt{2}} (\sqrt{\lambda^4 + 16} - \lambda^2)^{\frac{1}{2}} \},$$

which is the solution for the flow of viscous incompressible fluid near an oscillating porous flat plate.

iii) For $\lambda = 0,\ S = 0$,

$$u = U \exp\ (-y\sqrt{\frac{n}{2\nu}})\ \cos\ (nt - y\sqrt{\frac{n}{2\nu}}),$$

which is the solution for the flow of a viscous incompressible fluid near an infinite oscillating solid flat plate [3].

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On KH—Structure Manifold

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Summary

In this paper we have established some theorems on a differentiable manifold equipped with KH-structure. Besides this some curvature tensors are defined and a relationship among them is obtained.

1. Introduction

Let us consider a n-dimensional differentiable manifold M^n. Let there exist in M^n a vector valued linear function F such that

$$F^2 \lambda = a^2 \lambda , \tag{1.1}$$

for an arbitrary vector field λ in M^n and any complex number a. Then F gives to M^n a GF-structure [1]. Let this structure be endowed with a Riemannian metric G' such that

$$G'(F\lambda , F\mu) = a^2 G'(\lambda , \mu) , \tag{1.2}$$

for arbitrary vector fields λ , μ in M^n. Then M^n is called an H-structure manifold [3].

In M^n, let

$$(E_\lambda F)\mu = 0 , \tag{1.3}$$

where E is the Riemannian connection then M^n is called a KH-structure manifold [3].

2. Sub-manifold of co-dimension 2.

In this section, we will assume that M^n is a H-structure manifold. Let M^{n-2} be a submanifold of co-dimension 2 with the immersion $b: M^{n-2} \to M^n$ such that a point $p \in M^{n-2} \Longrightarrow b_p \in M^n$. Let B be the corresponding Jacobian map such that X at p in $M^{n-2} \Longrightarrow$ BX at b_p in M^n. If g is the induced metric tensor in M^{n-2}, N and N* are two mutually orthogonal unit normals to M^{n-2}, then

$$\text{(a)} \quad G'(BX, BY) \circ b = g(X, Y),$$

$$\text{(b)} \quad G'(N, N^*) \circ b = \delta^N_{N^*} , \tag{2.1}$$

(c) (i) $G'(BX, N) \circ h = 0$,

(ii) $G'(BX, N^*) \circ h = 0$.

Let us express the transformations of BX, N and N^* by F as the sum of tangential and normal parts in the form

$$FBX = BfX + A_1(X)N + A_2(X)N^* , \tag{2.2}$$

$$FN = - BT_1 + aN^* , \tag{2.3}$$

$$FN^* = - BT_2 - AN , \tag{2.4}$$

where f is a tensor field of type $(1, 1)$, A_1 and A_2 are 1-forms, T_1 and T_2 are vector fields.

<u>Theorem</u> (2.1). In order that a submanifold M^{n-2} of co-dimension 2 of an H-structure manifold admits an H-structure, it is necessary and sufficient that

$$A_1(X)T_1 + A_2(X)T_2 = 0 \tag{2.5}$$

and

$$A_1(X)A_1(Y) + A_2(X)A_2(Y) = 0 . \tag{2.6}$$

<u>Proof</u>. Let us consider the transformations (2.2), (2.3) and (2.4) and multiply by F.

Using (1.1), (2.2), (2.3) and (2.4) and collecting the tangential and normal parts, we get

$$\begin{aligned}
&\text{(a)} \quad f^2X = a^2X + A_1(X)T_1 + A_2(X)T_2, \\
&\text{(b)} \quad fT_1 = - aT_2, \quad fT_2 = aT_1, \\
&\text{(c)} \quad A_1(fX) = a\, A_2(X) , \quad A_2(fX) = - a\, A_1(X) , \\
&\text{(d)} \quad A_1(T_1) = - 2a^2 , \quad A_2(T_2) = - 2a^2, \\
&\text{(e)} \quad A_1(T_2) = A_2(T_1) = 0.
\end{aligned} \tag{2.7}$$

Also making use of (2.1) a, b, c in (1.2), we get

$$g(fX, fY) = - a^2g(X, Y) - A_1(X)A_1(Y) - A_2(X)A_2(Y). \tag{2.8}$$

Equations (2.7) and (2.8) yield

$$(2.9) \qquad f^2X = a^2X ,$$

$$(2.10) \qquad g(fX, fY) = -a^2g(X, Y)$$

respectively if and only if

$$(2.11) \qquad \begin{aligned} &(a) \quad A_1(X)T_1 + A_2(X)T_2 = 0 , \\ &(b) \quad A_1(X)A_1(Y) + A_2(X)A_2(Y) = 0 . \end{aligned}$$

Now we consider Gauss and Weigarten equations [2]

$$(2.12) \qquad E_{BX}BY = BD_XY + H(X, Y)N^* + K(X, Y)N ,$$

$$(2.13) \qquad E_{BX}N = -B'K(X) - 'L(X)N^* ,$$

$$(2.14) \qquad E_{BX}N^* = -B'H(X) + 'L(X)N ,$$

where E and D are the Riemannian connection in M^n and M^{n-2} respectively; H and K are geometric bilinear functions in M^n and M^{n-2} respectively such that [2]

$$(2.15) \qquad g('H(X), Y) = H(X, Y) ,$$

$$(2.16) \qquad g('K(X), Y) = K(X, Y)$$

and $'L(X)$ is the third fundamental tensor.

Let M^n be a KH-structure manifold [3], then

$$(2.17) \qquad E_{BX}FBY = FE_{BX}BY .$$

Using (2.2), (2.3), (2.4) and (2.12) in (2.17) and comparing the tangential and normal parts, we have

$$(2.18) \qquad \begin{aligned} (D_Xf)Y = &- A_1(Y)'KX - 'HX\, A_2(Y) \\ &- H(X, Y)T_2 - K(X, Y)T_1 , \end{aligned}$$

$$(2.19) \qquad (D_XA_1)Y = -\, 'L(X)A_2(Y) + K(X, fY) - aH(X, Y) ,$$

$$(2.20) \qquad (D_XA_2)Y = +\, 'L(X)A_1(Y) + H(X, fY) + aK(X, Y) .$$

Definition (2.1). When $S(X, Y)$ vanishes then M^n is said to be normal.

Theorem (2.2). Let M^{n-2} be an H-structure submanifold of a KH-structure manifold M^n. Let $'H$ and $'K$ both commute with f, then M^{n-2} is normal if

$$(2.21) \qquad 'L(X)A_2(Y)T_1 - 'L(Y)A_2(X)T_1 = 'L(X)A_1(Y)T_2 - 'L(Y)A_1(X)T_2$$

Proof. We define a tensor field $S(X, Y)$ as

$$(2.22) \qquad S(X, Y) = N(X, Y) + dA_1(X, Y)T_1 + dA_2(X, Y)T_2,$$

where $N(X, Y)$ is the Nijenhuis tensor and is equal to

$$N(X, Y) = (D_{fX}f)Y - (D_{fY}f)X + f(D_Yf)X - f(D_Xf)Y$$

and

$$dA_1(X, Y)T_1 = \left\{ (D_XA_1)Y - (D_YA_1)X \right\} T_1,$$

$$dA_2(X, Y)T_2 = \left\{ (D_XA_2)Y - (D_YA_2)X \right\} T_2.$$

Substituting the values from (2.18), (2.19), (2.20) in (2.22), we get

$$(2.23) \qquad \begin{aligned} S(X, Y) = {} & A_1(Y)(-\,'KfX + f'KX) + A_1(X)('KfY - f'KY) \\ & + A_2(Y)(-\,'HfX + f'HX) + A_2(X)('HfY - f'HY) \\ & - H(fX, Y)T_2 - K(fX, Y)T_1 + H(fY, X)T_2 + K(fY, X)T_1 \\ & - H(Y, X)fT_2 - K(Y, X)fT_1 + fH(X, Y)T_2 + fK(X, Y)T_1 \\ & - 'L(X)A_2(Y)T_1 + K(X, fY)T_1 + aH(X, Y)T_1 \\ & + 'L(Y)A_2(X)T_1 - K(Y, fX)T_1 - aH(Y, X)T_1 \\ & + 'L(X)A_1(Y)T_2 + H(X, fY)T_2 + aK(X, Y)T_2 \\ & - 'L(Y)A_1(X)T_2 - H(Y, fX)T_2 - aK(Y, X)T_2. \end{aligned}$$

Using (2.7b) in (2.23), we get the required result (2.21).

3. Curvature tensors

In this section, we have defined some curvature tensors and obtained a relationship among them.

Let R and Ric be the curvature and Ricci Tensors of M^n equipped with KH-structure. Then for arbitrary vector fields λ, μ, ν, ρ of M^n, we have [5]

$$(3.1)\quad \begin{aligned} &(a)\quad R(\lambda, \mu, \bar{\nu}) = \overline{R(\lambda, \mu, \nu)},\\ &(b)\quad \overline{R(\lambda, \mu, \bar{\nu})} = a^2 R(\lambda, \mu, \nu),\\ &(c)\quad {}'R(\lambda, \mu, \bar{\nu}, \bar{\rho}) = -a^2\, {}'R(\lambda, \mu, \nu, \rho)\\ &\qquad\qquad = {}'R(\bar{\lambda}, \bar{\mu}, \nu, \rho), \end{aligned}$$

where

$$(a)\quad {}'R(\lambda, \mu, \nu, \rho) \overset{\text{def}}{=\!=\!=} G'(R(\lambda, \mu, \nu), \rho)$$

and

$$(3.2)\quad \begin{aligned} &(a)\quad Ric(\bar{\lambda}, \bar{\mu}) = -a^2 Ric(\lambda, \mu),\\ &(b)\quad Ric(\bar{\lambda}, \mu) = -Ric(\lambda, \bar{\mu}). \end{aligned}$$

The Pseudo H-Projective curvature tensor $P(\lambda, \mu, \nu)$ in M^n equipped with KH-structure is given by [3]

$$\begin{aligned} P(\lambda, \mu, \nu) = R(\lambda, \mu, \nu) - \frac{1}{(n+2)a^2}\Big[a^2 Ric(\mu, \nu)\lambda \\ (3.3)\qquad - a^2 Ric(\lambda, \nu)\mu - Ric(\bar{\mu}, \nu)\bar{\lambda} \\ + Ric(\bar{\lambda}, \nu)\bar{\mu} - 2\, Ric(\lambda, \bar{\mu})\bar{\nu} \quad . \end{aligned}$$

Weyl curvature tensor W, conformal curvature tensor V, COn-harmonic curvature tensor L and COn-circular curvature tensor C are given by [4]

$$(3.4)\quad \begin{aligned} W(\lambda, \mu, \nu) &= R(\lambda, \mu, \nu)\\ &- \frac{1}{(n-1)}\big[\lambda\, Ric(\mu, \nu) - \\ &- \mu Ric(\lambda, \nu)\big] , \end{aligned}$$

$$(3.5)\quad \begin{aligned} V(\lambda, \mu, \nu) &= R(\lambda, \mu, \nu)\\ &- \frac{1}{(n-2)}\big[Ric(\mu, \nu)\lambda - Ric(\lambda, \nu)\mu\\ &+ G'(\lambda, \nu)\, R(\mu) - G'(\mu, \nu)\, R(\lambda)\big]\\ &+ \frac{1}{(n-2)(n-1)}\quad G'(\mu, \nu)\lambda - G'(\lambda, \nu)\mu\big] , \end{aligned}$$

(3.6) $$L(\lambda, \mu, \gamma) = R(\lambda, \mu, \gamma) - \frac{1}{(n-2)} \Big[G'(\mu, \gamma) R(\lambda) - G'(\lambda, \gamma) R(\mu) + Ric(\mu, \gamma)\lambda - Ric(\lambda, \gamma)\mu\Big],$$

(3.7) $$C(\lambda, \mu, \gamma) = R(\lambda, \mu, \gamma) - \frac{r}{n(n-1)} \Big[G'(\mu, \gamma)\lambda - G'(\lambda, \gamma)\mu\Big].$$

Eliminating R from (3.3), (3.4), (3.5), (3.6) and (3.7), and making use of (3.2)b we get the following expression:

(3.8) $$P(\lambda, \mu, \gamma) = W(\lambda, \mu, \gamma) + \frac{3}{(n-1)(n+2)} \Big[Ric(\mu, \gamma)\lambda - Ric(\lambda, \gamma)\mu\Big] - \frac{1}{(n+2)a^2} \Big[Ric(\mu, \bar{\gamma})\bar{\lambda} - Ric(\lambda, \bar{\gamma})\bar{\mu} - 2\,Ric(\lambda, \bar{\mu})\bar{\gamma}\Big],$$

(3.9) $$P(\lambda, \mu, \gamma) = V(\lambda, \mu, \gamma) + \frac{4}{(n-2(n+2)} \Big[Ric(\mu, \gamma)\lambda - Ric(\lambda, \gamma)\mu\Big] + \frac{1}{(n-2)} \Big[G'(\mu, \gamma) R(\lambda) - G'(\lambda, \gamma) R(\mu)\Big] - \frac{1}{(n-2)(n-1)} \Big[G'(\mu, \gamma)\lambda - G'(\lambda, \gamma)\mu\Big] - \frac{1}{(n+2)a^2} \Big[Ric(\mu, \bar{\gamma})\bar{\lambda} - Ric(\lambda, \bar{\gamma})\bar{\mu} - 2\,Ric(\lambda, \bar{\mu})\bar{\gamma}\Big],$$

(3.10) $$P(\lambda, \mu, \gamma) = L(\lambda, \mu, \gamma) + \frac{4}{(n-2)(n+2)} \Big[Ric(\mu, \gamma)\lambda - Ric(\lambda, \gamma)\mu\Big] + \frac{1}{(n-2)} \Big[G'(\mu, \gamma) R(\lambda) - G'(\lambda, \gamma) R(\mu)\Big]$$

$$- \frac{1}{(n+2)a^2} \Big[\mathrm{Ric}(\mu, \bar{\gamma})\bar{\lambda} - \mathrm{Ric}(\lambda, \bar{\gamma})\bar{\mu}$$

$$- 2\, \mathrm{Ric}(\lambda, \bar{\mu})\bar{\gamma} \Big]$$

and

$$(3.11) \qquad P(\lambda, \mu, \gamma) = C(\lambda, \mu, \gamma)$$

$$+ \frac{r}{n(n-1)} \Big[G'(\mu, \gamma)\lambda - G'(\lambda, \gamma)\mu \Big]$$

$$- \frac{1}{(n+2)a^2} \Big[a^2\, \mathrm{Ric}(\mu, \gamma)\lambda - a^2\, \mathrm{Ric}(\lambda, \gamma)\mu$$

$$+ \mathrm{Ric}(\mu, \bar{\gamma})\bar{\lambda} - \mathrm{Ric}(\lambda, \bar{\gamma})\bar{\mu} - 2\, \mathrm{Ric}(\lambda, \bar{\mu})\bar{\gamma} \Big].$$

Theorem (3.1). In a KH-structure manifold M^n, we have

$$(3.12) \qquad W(\lambda, \mu, \gamma)$$

$$- C(\lambda, \mu, \gamma)$$

$$= \frac{r(n-2)}{n} \Big[V(\lambda, \mu, \gamma)$$

$$- L(\lambda, \mu, \gamma) \Big]$$

if

$$\mathrm{Ric}(\mu, \gamma)\lambda = \mathrm{Ric}(\lambda, \gamma)\mu .$$

Proof. The theorem follows in view of equations (3.8), (3.9), (3.10) and (3.11).

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Fixed Point Theorems for Nearly Densifying Maps

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In this paper, we improve upon the fixed point theorems of Madhusudana Rao [1].

In what follows, (X, d) is a complete metric space; for a subset A of X, $\delta(A)$ is the diameter of A, $\bar{A}$ is the closure of A and when A is bounded

$$(A) = \inf \{ \varepsilon > 0 \mid A \text{ admits a finite cover consisting of sets with diameter less than } \varepsilon \}$$

is the measure of non-compactness of A.

When A, B are bounded subsets of X, we have

$$0 \leq \alpha(A) \leq \delta(A)$$

$$\alpha(A) = 0 \text{ if and only if } A \text{ is precompact}$$

$$\alpha(A \cup B) = \max \{ \alpha(A), \alpha(B) \}$$

and

$$\alpha(A) = \alpha(B) \text{ if } B \subset A \text{ and } A - B \text{ is finite.}$$

f, g are self maps on X;

for $x \in X$, $I(x) = \{ y \mid y = x \text{ or } y = hx \text{ for some } h \text{ in } S \}$,
where S is the subsemigroup generated by f and g in the semigroup of all self maps on X with composition operation.

Definition 1. f is said to be densifying if

$$\alpha(f(A)) < \alpha(A) \text{ whenever } \alpha(A) > 0.$$

f is said to be nearly densifying if

$$\alpha(f(A)) < \alpha(A) \text{ whenever } \alpha(A) > 0 \text{ and } A \text{ is } f\text{-invariant}.$$

Theorem 2. Let f, g be continuous and nearly densifying. Let F be a symmetric continuous real valued function on $X \times X$. Suppose that

$$\text{(2.1)} \qquad F(fx, gy) < \max \{ F(x, y), F(x, fx), F(y, gy) \}$$

whenever $x, y \in X$, $x \neq y$, $x \neq fx$, $y \neq gy$ and $fx \neq gy$ and $I(x_o)$ is bounded for some $x_o \in X$. Then either f or g has a fixed point.

Proof. Write I for $I(x_o)$. We have $I = f(I) \cup g(I) \cup \{x_o\}$ so that $\alpha(I) = \max \{\alpha(f(I)), \alpha(g(I))\}$. Since f and g are nearly densifying and I is invariant under both f and g, it follows that $\alpha(I) = 0$, so that I is precompact and hence by the completeness of X, $\bar{I}$ is compact. By the continuity of f and g and the invariance of I under f and g, we have $f(\bar{I}) \subset \bar{I}$ and $g(\bar{I}) \subset \bar{I}$. By the continuity of F, f and g it follows that the real valued function H on X defined by

$$Hx = \min \{ F(x, fx), F(x, gx) \}$$

is continuous. By the compactness of $\bar{I}$, there exists $z \in \bar{I}$ such that H(z) is the minimum of H on $\bar{I}$. Suppose that

(2.2) $\quad fz \neq z,\ gz \neq z,\ g(fz) \neq fz$ and $f(gz) \neq gz.$

when $\quad Hz = F(z, fz)$, we have, from (2.1), that

$$H(fz) \leq F(fz, g(fz)) < F(z, fz) = Hz$$

and when $\quad Hz = F(z, gz)$ we have again from (2.1) that

$$H(gz) \leq F(gz, f(gz)) < F(z, gz) = Hz$$

so that we have a contradiction to the minimality of z in either case. Hence (2.2) cannot hold, consequently the result follows.

From the above theorem, we can deduce the following:

Theorem 3. Let f, g be continuous and nearly densifying. Suppose that $I(x_o)$ is bounded for some $x_o \in X$. Let G be a real valued continuous function on X x X. Suppose that

$$\max \{ G(fx, gy), G(gy, fx)\} < \max \{G(x, y), G(y, x),$$
$$G(x, fx), G(fx, x), G(y, gy), G(gy, y)\}$$

whenever $x, y \in X$, $x \neq y$, $x \neq fx$, $y \neq gy$ and $fx \neq gy$. Then either f or g has a fixed point.

Proof. Define $F(x, y) = \max \{ G(x, y), G(y, x)\}$ for all $x, y \in X$. Then F is a real valued, symmetric, continuous function on X x X satisfying (2.1).

Corollary 4. (Madhusudana Rao [1], Theorem 1) Let f, g be continuous and densifying. Suppose that $I(x_o)$ is bounded for some $x_o \in X$. Let G be a real valued continuous function on X x X. Suppose that

$\max \{ G(fx, gy), G(gy, fx) \} < 1/2 [G(x, fx) + G(y, gy)]$ whenever $x, y \in X$, $x \neq fx$ and $y \neq gy$. Then either f or g has a fixed point.

Theorem 5. Let f, g be continuous and nearly densifying. Suppose that $\overline{I(x_0)}$ is bounded for some $x_0 \in X$. Let F be a continuous semi-metric on X. Suppose that

$$(5.1) \qquad F(fx, gy) < \max \{ F(x, y), F(x, fx), F(y, gy), 1/2 [F(x, gy) + F(fx, y)] \}$$

whenever $x, y \in X$, $x \neq y$, $x \neq fx$ and $y \neq gy$. Then either f or g has a fixed point.

If, in addition, (5.1) holds

(i) Whenever $x \neq fx$ or $y \neq gy$ then the fixed point sets of f and g are non-empty and are the same.

(ii) Whenever $fx \neq gy$ then each of f and g has a unique fixed point and these concide.

(iii) Whenever $x \neq y$ and the fixed point sets of f and g are non-empty then each is a singleton set and these concide.

Proof. While the proof of the first part of the theorem is analogous to that of Theorem 2, the second part is trivial.

Remark. By taking the metric d in the place of F, we obtain Theorems 2, 3 and 4 of Madhusudana Rao [1] as corollaries of the above theorem.

The following example shows that the average of $F(x, gy)$ and $F(fx, y)$ in (5.1) cannot be replaced by the maximum of these two, even when $F = d$.

Example 6. ([2], Example 6). Let $X = \{ 1, 2, 3, 4 \}$;
$d(1, 2) = d(3, 4) = 2$, $d(1, 3) = d(2, 4) = 1$, $d(1, 4) = d(2, 3) = 3/2$;
$f1 = f4 = 2$, $f2 = f3 = 1$; $g1 = g3 = 4$, $g2 = g4 = 3$.
$d(fx, gy) < \max \{ d(x, fx), d(x, gy), d(fx, y) \}$ for all x, y.
Neither f nor g has a fixed point.

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GLOSSARY OF MATHEMATICAL TERMS
(Proposed)
(Q)

Quadrangle चतुष्कोण

Quadrangular चतुष्कोणिय, चतुर्भुजीय

Quadrangularly चतुष्कोण रूपले

Quadraut चतुर्थांश

Quadrantal वृत्तपादीय

Quadrat क्वाड्राट्

Quadrate क्वाड्रेट, समचतुर्भुजाकार बनाउनु

Quadratic वर्ग, वर्गात्मक
द्विघात, द्विघातिय

Quadrature क्षेत्रकलन

Quadric द्विघात द्विघातीय

Quadrilateral चतुर्भुज

Quadrillion एकहजारको पाचौंघात

Quadrinomial चतुष्पदी

Quadruple चौगुना, चतु:

Quantic समघाती

Quantile विभाजक

Quantisation राशिकरण

Quantitative परिमाणात्मक

Quantity परिमाण, राशि

Quantum क्वान्टम

Quarter चौथाई, चतुर्थांश

Quartic चतुर्घाती

Quartile चतुर्थक

Quaternary चतुर्थ

Queue पंक्ति

Quintic पंचघात, पंचघाति

Quintile पंचमक

Quintillion एकहजारको छैठौं घात

Quintuple पाँचगुना

Quintuplet पंचक

Quotient योगफल, विभाग

(R)

Radial त्रिज्य, अर्धव्यासिय

Radially अर्धव्यासरूपले

Radian रेडियन

Radical मूलक, मूलसम्बन्धि

Radius त्रिज्या, अर्धव्यास, व्यासार्ध

Radix उद्गम, मूलांक

Random संयोगिक, यादृच्छिक

Randomisation यादृच्छिकरण

Randomised यादृच्छिकृत

Range श्रेणी, परास, फैलाव

Rank कोटि, रायङ्क

Rate दर

Ratio अनुपात

Rational परिमेय

यो शब्दावली त्रि.वि., कीर्तिपुर बहुमुखी क्याम्पस, गणित तथा नेपाली शिक्षण शिक्षण समितिले संयुक्त रूपमा तयार गरिएको हो ।

Rationalise परिमेयकरण गर्नु
Rationalisation परिमेयकरण
Ray अर्धरेखा
Reaction प्रतिक्रिया
Real वास्तविक
Rebound प्रतिक्षेप
Reciprocal व्युत्क्रम, प्रतिलोम
Recoil प्रतिक्षिप्त
Recovery उपलब्धि
Rectangle आयात, समकोण चतुर्भुज
Rectangular आयाताकार
Rectifiability चापकलनियता
Rectification चापकलन
Rectified चापकलित
Rectilinear सरलरेखी, सरल
Rectilineal रेखात्मक
Recur दोहरिनु
Recurrence दोहराई, पुनरावृत्ति
Recurring आवर्त, आवर्ति
Recurrent पुनरावर्ती
Reduce कम गर्नु
Reduced लघुकृत, कम गरिएको
Reduction न्यूनीकरण, समानयन
Reducible लघुकरणीय, न्यूनकरणीय
Reducibility खंडनीयता, समानेयता
Reference निर्देश, संदर्भ
Refinement अधिशोधन
Reflex प्रतिवर्ति
Reflexive स्वतुल्य
Reflexivity स्वतुल्यता
Region क्षेत्र
Regression समाश्रयण
Regular सम, नियमित
Regularity नियमितता
Reinforced प्रबलित
Related सम्बन्धित
Relative आपेक्षिक, सापेक्ष
Relatively सापेक्षत
Relativity आपेक्षिकी, सापेक्षिकी
Relativitistic आपेक्षिकीय
Remainder बाँकी, शेष, शेषफल
Removable अपनेय
Renormalization पुन: प्रसामान्यीकरण
Rent कर
Repeat दोह-याउनु, पुनरावर्तन
Repeated बारंबार, पुनरावृत्त
Repetition पुनरावृत्ति
Replace प्रतिस्थापन गर्नु
Replacement प्रतिस्थापन
Represent निरूपित गर्नु
Represented निरूपित
Representation निरूपण

Repulse प्रतिकर्षण गर्नु

Repulsion प्रतिकर्षण

Residual अवशिष्ट

Residue अवशेष

Resolute वियोजित

Resolution वियोजन, खंडन

Resolve खंड गर्नु, वियोजित गर्नु

Response अनुक्रिया

Rest विराम

Restitution प्रत्यानयन

Restore पुन: स्थापना गर्नु

Restricted प्रतिबन्धित

Resubmission पुन: प्रस्तुतीकरण

Restriction प्रतिबंधन, नियंत्रण

Result परिणाम, फल, नतीजा

Resultant परिणामी

Retardation मंदन

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Return प्रतिगमन, प्रत्यागमन

Reverse उत्क्रम, प्रतिलोम

Reversibility उत्क्रमणीयता

Revolution परिक्रमण

Revolve परिक्रमण गर्नु

Rhombic समचतुर्भुजी

Rhomboid समानान्तर असमचतुर्भुज्य

Rhombus समचतुर्भुज्य

Rho रो (ρ)

Right दक्षिण

Rigid दृढ

Rigidity दृढता

Rigorous परिशुद्ध

Ring वलय, रिंग

Rise उत्थान, चढाव

Roaster रोष्टर

Root मूल

Rotation घुमाई

Round गोलो

Row पंक्ति

Rule नियम

Run धावन

(S)

Sag आनमन, दुबाव

Sagitta सैजिता

Sale विक्रय

Saltus विसातत्य

Same समान

Sample नमूना, प्रतिदर्श

Sampling प्रतिचयन, प्रतिदर्शी

Satisfy मान्य हुनु

Satisfactory मान्य

Satisfied मान्य भएको

Scalar स्केलार

Scale स्केल, मापक

Scalene विषम बाहु

Scaling सोपानी

Schlicht सरल
Scatter प्रकिर्ण
Screw पेच

Schedule अनुसूची

Score समंक

Secant कोटिजा, सेकेण्ट, छेदक

Second सेकेन्द, दोश्रो

Secondary द्वितियक, गौण

Section खण्ड, सेक्शन, अनुभाग

Sectional अनुभागीय, परिछेदी

Sectionally खंडत:

Sector सेक्टर, त्रिज्यखंड

Segment खंड

Select चुन्नु, छान्नु

Selection छनाई, चुनाई, वरण

Semi अर्ध

Separate पृथक

Separable पृथक्करणीय

Sequence अनुक्रम

Sequential अनुक्रमिक

Series श्रेणी, माला

Serial क्रमिक, अनुक्रमी

Set सेट्, समुच्चय

Seven सात

Seventh सातौं

Seventeen सत्र

Seventeenth सत्रौं

Seventy सत्तरी

Seventieth सत्तरीऔं

Shear अपरूपण, कर्तन

Shearing अपरूपण

Side पक्ष, भुजा

Sigma सिग्मा (σ , Σ)

Sign राशि, चिन्ह

Signature सिग्नेचर

Similar समरूप, सदृश

Similarly समरूपी,

Similarity सादृश्यता, समरूपता

Simple सरल, साधारण

Simplex सिम्प्लेक्स, प्रसमुच्चय

Simulate अनुकरण गर्नु

Simulation अनुकरण

Simultaneous अनुकरण

Sine साइन, ज्या

Single एक, एउटा

Singular विचित्र

Singularity विचित्रता, सिंगुलरीती

Sink अभिगम, डुब्नु

Six छ

Sixth छैथौं

Sixteen सोह्र

Sixteenth सोह्रौं

Sixty साठ्ठी

Sixtieth साठ्ठीऔं

Size साइज

Skew स्क्यू, विषमतलीय

Skewness वैषम्य

Slide स्लाइड, घच्रनु

Slope ढाल, प्रवणता

Slit स्लीट, प्वाल

Smooth चिप्लो, स्मूथ

Solid ठोस

Solve हलगर्नु, साधना गर्नु

Solvability साधनीयता

Solvable साधनीय

South दक्षिण

Source उद्गम, स्रोत

Space आकाश, समष्टि

Span विस्तृति, स्पायन्

Spatial आकाशिय, स्थानिक

Special विशिष्ट, विशेष

Specified विनिर्दिष्ट

Specific विशिष्ट, आपेक्षिक

Specification विनिर्देश

Spectral मानावलिय

Speed गति, चाल

Sphere गोला, गोल

Spherical गोलाकार, गोलीय

Spherically गोलत:

Spheroid गोलाभ

Spheroidal गोलाभ

Spin स्पिन, प्रचक्रण

Spinor स्पाइनर, संदिश

Spiral सर्पिल

Split विपाट

Splitting स्प्लिटिंग, विपाटन

Spring कमानी, स्प्रिंग

Square वर्ग

Squared वर्गिकृत

Squaring वर्गिकरण

Stable स्थिर

Stabilise स्थायीकृत गर्नु

Stabilization स्थिरीकरण

Stability स्थायित्व

Standard प्रमाणित, मानक

Start आरंभ गर्नु

State स्थिति, अवस्था

Static स्थिर, स्थैतिक

Statics स्थिति-विज्ञान

Statical स्थैतिक

Stationary अचर, स्थिर

Statistic प्रतिदर्शज

Statistical सांख्यिकीय

Statistics सांख्यिकी, तथ्यांक, आँकडा, तथ्यांक शास्त्र

Straight सरल, सीधा

Step चरण, पद

Strain तनाव

Stress प्रतिबल

String तार, डोरी

Subcover उपावरण

Substitute विस्थापन गर्नु, प्रतिस्थापना गर्नु

Substitution विस्थापन, प्रतिस्थापना

Subtract घटाउनु

Subtraction घटाउ

Subtrahend व्यवकलित

Successor परवर्ती

Sufficient पर्याप्त

Sufficiency पर्याप्तता

Suffix अनुलग्न

Sum जोड, योग

Subfield उपक्षेत्र

Sublayer उपस्तर

Summand योग खण्ड

Summandable संकलनिय

Summability संकलनियता

Support आधार

Surd सर्ड, करणी

Surface सतह, धरातल

Stochastic स्टोकास्टिक, प्रसंभाव्य

Superficial सतही, पृष्ठिय

Surrounding परिवेश, परिगामि

Suspend निलंबन गर्नु, लटकाउनु

Suspension निलबन, लटकाव

Swing डोल

Symbol चिन्ह, प्रतीक

Symbolical प्रतिकात्मक

Symmetry समिमिति

Symmetric समित

Symmetrical समिमितिय

System प्रणाली

Systematic क्रमवद्ध

Instructions to Contributors

"The Nepali Mathematical Sciences Report" will publish the original research papers, short notes and survey articles in different branches of Mathematics and Mathematical Sciences.

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References should be listed at the end of the paper in alphabetical order and each reference is to be arranged in the following form:

1. Erdelyi, A., Tables of Integral Transforms, Vol. II, McGraw Hill, (1954
2. Stokes, G.G., On the effect of the internal friction on the motion of pendulums, Trans. Camb. Phil. Soc., 9 (1851), 8-106.

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