THE NEPALI MATHEMATICAL SCIENCES REPORT

INSTITUTE OF SCIENCE
DEAN'S OFFICE
TRIBHUVAN UNIVERSITY
KIRTIPUR
NEPAL

VOLUME 2, NUMBER 1

JANUARY 1977

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The Mathematical Science - Their Nature and Scope*

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Introduction

During the year 2032-2033 Nepal's first journal devoted to the mathematical sciences came into existence. It was given the title "The Nepali Mathematical Sciences Report." To many people the term "mathematical sciences" has brought great confusion. They think it means mathematics and science in totality, rather than the hybrid term which it is. It has come into existence, following two or three decades of activity, as a way of describing the new and broad role played by mathematics throughout the world. In this article our purpose is to describe the nature and scope of the mathematical sciences and to clarify the meaning of the term.

Some History

It is appropriate for us to begin with some history. For many centuries mathematics and the sciences, particularly the physical sciences, have lived and grown together, moving hand in hand, each serving the other and being served by the other. The movement has gone in cycles:

Practical Situations → Mathematical Formulations → Abstractions

↖ New applications ← New concepts and structures ↙

The great philosopher - mathematician Alfred North Whitehead has described the whole of education as being a cyclic action. What holds true there holds true here also. There are no sharp lines of division between stages. One flows into another the stages of a cycle being largely a matter of emphasis at a given time. The cycles overlap cycles. One development may be in one stage, while another is further on in the process. There are cycles within cycles as well. Out of it all arises a continuity of action, which has a rhythmic character.

For example, geometry began as "earth measure" with very real applications. Euclid organized it as a body of knowledge and a theory, with an axiomatic structure. From that organization and theory, over centuries of time, there have grown new approaches to geometric ideas, new uses of axiomatic structures within mathematics itself and within fields other than mathematics such as

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*This is a revised version of a lecture given to the Nepal Science Forum on May 28, 1976, with the title "The Emergence of the Mathematical Sciences."

biology, economics, education, industry, and government. New theories have developed in great numbers and depth. Whereas the value of geometry used to be thought of as the learning of logic and reasoning, now it is in the organization of data and the generation of ideas.

Mathematics and astronomy have been close allies for a long time. Out of their relationship have come the study of planetary motion, the works of Newton, Kepler, and others, studies of the structures of planetary systems, and on to the present day explorations of space, for which the computer is an indispensable tool.

The growth of Mathematics during the Renaissance period was related to the productive use and perfection of machines. This resulted in studies of motion and dynamical systems, the development of calculus and theoretical mechanics, differential equations, differential geometry, and other fields of endeavor. This relationship between mathematics and mechanics, which relates to industrial development of all kinds, exists today. For example, large automobile corporations often employ mathematicians to do work in approximation theory, statistics, and numerical analysis. Their work is related to design and manufacture of a automobiles, processes which begin with artists and clay models and which utilize sophisticated computing machinery.

To cite yet another example involving geometry, we can take the case of Riemannian geometry. It and its relative, non-Riemannian geometry, have had close ties to the theory of relativity, as expounded by Einstein. About twenty years ago, an offspring of this work came into existence in the form of geometrodynamics. Geometrodynamics is the study of curved empty space and the evolution of this geometry with time according to the equations of Einstein's 1916 general relativity theory. The theory, as it has been developed by Misner and Wheeler since the mid-1950's, demonstrates the sheer force of definition and deduction, since the work is remote from experiment and observation To illustrate this point, only those masses and fields are considered which can be regarded as built out of the geometry itself. Electromagnetism is thought of as a property of curved empty space. The electromagnetic field curves space in such a way that its footprints are themselves manifestations of the field. This new theory has proved itself to be very rich, and recently has been applied to work on human consciousness.

Many additional examples could be given. One more will suffice. To ease man's problems involving calculations, logarithms were invented. Then calculators came on the scene. In our present day, electronic calculating equipment in many varieties and forms is a very visible part of the scene. The development has brought with it new interest in numerical analysis and many new applications for mathematics via the computer.

Roles

What role or roles has mathematics played in these developments? There are several types of roles, as follows:

1. Quantification role.

2 Deductive role.

3. Synthesizing role.

4. Conceptualizing role.

Let us take a brief look at each. The "quantification" role played by mathematics involves the translation of ideas and attributes into numbers or numerical quantities. It is the arithmetic role, with which almost everyone is familiar. Indeed, for most people, mathematics is arithmetic, for they have never been taught differently nor experienced it differently. Quantification is part of everyone's life since it is vital to business and trading to running a family, to administrative work at all levels, and to one's livelihood. In recent years psychologists, educationists and social scientists of all types have begun to quantify things which had never been quantified before - intelligence, behavior, attitudes, and the like. Success in these developments has not always been good and has left many observers shaking their heads in anguish. Let it be said clearly that quantification, arithmetic if you like, is not mathematics, even though it is a crucial part of mathematics. Wherein does the distinction lie? The answer is in the other roles played by mathematics, all of which involve process. For example, the conceptualizing role of mathematics, that of generating new concepts in the process of mathematizing ideas and attributes, is in the long run a more useful role than that of making familiar notions operational through quantification.. Without the influx of ideas, the latter may not exist at all with any meaning or significance.

The deductive role played by mathematics has been most crucial to the historical developments we have mentioned. This role involves algorithms of reasoning, logic, and strengthening of observations through the use of more and more refined instrumentation. A part of it is confidence in basic laws, as given by Newton, Kepler and others, which have stood for many years.

In its synthesizing role, principles and relationships are the keys to development. Ideas do not live in isolation. Again Alfred North Whitehead speaks about this. He says "Knowledge is the reminiscence by the individual of the experience of the race. But reminiscence is never simple reproduction. The present reacts upon the past. It selects, it emphasizes, it adds. The additions are the new ideas by means of which the life of the present re-

flects itself upon the past." This is synthesis--bringing together, permitting a global view and a broad perspective, moving ahead.

As mentioned, of these four roles, it is the deductive role and the quantification role which have dominated developments in past history up to the time of 1945, the end of the second great world war. Evidence for this lies not only in the developments themselves but in school and university curricula and programs all over the world. Preparation for calculus, calculus, analysis, dynamics, differential equations, and their many applications have dominated the curricula. It is a direct outgrowth of the advances made in the Renaissance period and has been with us for so many years that we are hard put to think of a mathematics curriculum at any level which doesn't move in this direction. The confidence involved is as deep as that given to Euclid's geometry and to Aristotle's ideas, which went virtually unchallenged for nearly two centuries. In short, Mathematics has been largely deductive and quantitative in its uses.

New Directions

As we have seen, through the past centuries, mathematics has had its closest ties to the physical sciences and to engineering. Here deduction has been useful and instrumentation has been available. These sciences are frequently called the exact sciences. The phenomena which are considered are, by and large continuous phenomena.

In the past three decades, directions have changed. Mathematics has found new uses in the inexact sciences - sociology, psychology, economics, education, political science, and other areas. This development has been spurred on by the employment of professional mathematicians to solve strategic wartime problems, by Sputnik and all the activity related to space exploration and rocketry, and by the growth and refinement of computers.

Now the phenomena are discrete, the basic laws not so precise and perhaps not even known. Other than computers to handle large sets of data, instrumentation and measuring devices have, by complexity of the situations, lesser roles to play - hence "inexactness." Without well established basic laws, without use of precise measuring devices, without refined instrumentation, without reliance of past knowledge, deduction begins to play a back seat role, and induction, synthesis, and conceptualization, along with the ever necessary quantification, play the lead roles. The outcome lies in the development of new hybrid fields, all of which together constitute what are now called "the mathematical sciences", of which mathematics in its pure form is one part

What are the basic ingredients of mathematical approaches in these "inexact sciences"? First there is heavy use of mathematical models of new types. Start with a situation. Look at its components. Try to describe them mathematically. In the exact sciences this has been done for years with great success, both descriptively and predictively. In the inexact sciences the models frequently have a probabilistic or statistical base. Sometimes, as in linguistics or biology, they are structural. Here the descriptive aspects have been greater than the predictive aspects, unlike the situation in the exact sciences. By this we mean that we end up with an empirically suggested formula which has been fitted to a set of observations, which in turn link a set of variables. The laws and principles which underlie the system are not sufficiently precise, if known at all. Predictive capability is frequently hindered by the fact that human behavior is involved. Thus the model facilitates conceptualization and permits the formation of a simple picture of the situation which has been or is to be investigated, even though its predictive capability is limited.

Scientific investigation has three major aspects - observation, induction, and deduction. In the inexact sciences the role of mathematics in strengthening observation and induction is very great. Frequently observations are statistical in nature. Experiments are observations on random samples from populations in which a variable of interest is defined by a distribution. The degree of confidence is quantified. Principal components and correlations not visible to the eye nor observable via instruments are exposed by means of statistical techniques. Thus it is through synthesization by conceptualization that mathematics gets involved, makes a contribution, and generates a new hybrid field.

Let us consider an example from sociology, since investigations of this type are rather common place here. A sociogram is a graph or a picture, a mathematical model, of an aspect of the social profile of a set of individuals, who might live in a particular village or panchayat. This sociogram gives a description of the group of people under consideration - ethnic group, sex, age category, income, occupation, educational background, attitudes, and so on. It is used to map out an approach to development of that community through certain types of change agents. The data which is collected for what are called "base-line studies" is statistical in nature. Attitudes are usually quantified on scales of 1 to 7 or 1 to 5. Now this model, even though it is descriptive, is not predictive. It would be predictive if some behavior patterns could be predicted from it. Usually this cannot be done. However, this does not mean that there is no value in it. The mathematical approach has given the observations some strength. Some approaches may become evident which will improve the situation in desirable ways. In addition, certain factors may be observed which could affect communication capability, one way or the other.

As in the example cited, the models are frequently tailored to specific situations, but the prospects of synthesis and process development may be difficult to achieve because of human problems or governmental situations. Therefore, there is a danger involved if these models and methods are applied routinely and extended in ad hoc fashion to a variety of contexts without regard to the conditions which justified the model in the first place. For example, such application has already occurred in the use of statistical analysis, particularly factor analysis. It is based on the underlying distributions being normal distributions. This is usually ignored in applications, thus aborting the theoretical significance of the method. Factor analysis, by the way, is a classic example of a descriptive model which facilitates conceptualization. For instance, it has been used to advantage in studies in cognition and perception. This in turn has had an effect, in a very scientific way, on curriculum development in certain parts of the world.

The quantification and deductive roles of mathematics have been appreciated for a long time. The synthesizing and conceptualizing roles are only beginning to be appreciated in the same way. As we have already mentioned, synthesization and conceptualization are being made possible by new mathematical theories and hybrid fields of investigation. All are distinct from what has gone before in time. All are now lumped together under the heading "Mathematical Sciences." We mention a few of them and give a very brief description of each. Note that, in general, they derive not from the study of matter but from observations of activities of human beings - as in communications, planning, societal characteristics, and strategies of conflict.

1. Information Theory - analysis of communication processes, coding, storing, retrieval, exchange. Applications include studies in reaction times and memory spans.

2. Automata Theory - physical realizations of reasoning processes. Applications include input/output theories and memory banks.

3. Operations Research - literally theories of operations, such as in management problems. Conduct of and coordination of operations within an organization. Applications include transportation, management, and queueing.

4. Graph Theory - it begins with points, lines, and networks of these objects. Vast applications include circuitry, flows of all sorts.

5. Mathematical Linguistics - study of language structure and syntatic structures. Applications to human psychology.

6. Theory of Games - study of strategies of conflicts of all kinds, peaceful games as well as war games. Studies in alternate courses of action with optimization a key feature

And many others just emerging, such as mathematical Biology, Mathematical Sociology, Mathematical Psychology, Mathematical Economics, and Mathematical Chemistry. Finally, we must not forget to add to our list Statistics and Computing Science with their ever enlarging interest and use in the real world. Without the availability of computers many of the above fields of endeavor would be extremely hindered.

These are the Mathematical Sciences. Their nature has been described briefly. Their scope has been alluded to. Its full range has yet to be revealed, but clearly it will be vast, as will be its uses and its impact.

Implications

With the advent of this mathematization of so many areas of activity must come effects on nations and their peoples. In particular, what do these developments imply here? First, the school curriculum should provide a solid base in mathematics. This must deal not only with the deductive aspects of mathematics, as the present curriculum tends to do, but also with strong emphasis on observation, induction, and the study of patterns. In this way good foundations will be laid, the scope of thinking will be broadened, and compartmentalization will be reduced. Second, the University should make requisite courses in mathematics and statistics and basic courses in the social sciences easily accessible to students so that proper foundations can be laid for hybrid courses. Third, efforts can be made to develop capability in some of the areas of application which have been mentioned; that is, in transportation theories, queueing theories, and operations research work. Fourth, many of these areas can be employed in governmental and industrial work to better utilize manpower and facilities. Finally, in a conceptual way, bridges, links, and good relationships can be established and maintained in intellectual endeavors and their applications to the real world.

As Alfred North Whitehead has said many years ago, there is a rhythm to it all. It is cyclic in nature. It involves intuition, precision, and generalization as the essential stages. From it all comes growth, knowledge, learning, and development - individually and collectively. It is this that the Mathematical Sciences help to accomplish.

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Sampling Plan for Large Scale Survey

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Introduction

A sample enumeration, an enumeration of a fraction of the population, is possibly as old as human civilization itself. We have been using it as a decisive tool in our life in different situations. The technique has become so common to us that we frequently forget to pay our attention properly in the selection of samples. Thus, more often, we tend to loose the meaning of a sample. The logic involved behind the theory of sampling is the logic of induction; therefore, if a sample is not properly drawn from a population the findings cannot legitimately be generalized for the population under study. In a less developed country where the desiderata facilities such as funds, professional personnel, qualified enumerators etc. for complete enumeration are below the expected level, a sample enumeration becomes inevitable to make inference about the population. The main advantages of a sample enumeration over a complete enumeration are (i) it reduces cost, (ii) it provides information more quickly, (iii) it yields more comprehensive information and (iv) it gives more accurate results.

A large scale sample survey involves three main stages - (a) planning stage, (b) execution stage and (c) analysis and report writing stage. The planning stage further involves the following steps:

(i) Specification of the purpose of the survey.

(ii) Definition of the population.

(iii) Determination of the nature of the information to be collected.

(iv) Decision on the method of collecting the data.

(v) Choice of the sampling units and frame.

(vi) Designing the sample survey.

Designing a sample survey is the most important step in planning a sample survey. It includes the specification of the sampling techniques to be adopted, determination of the sample size and description of pilot or exploratory survey, if necessary, for the main design. The design of a sample survey is governed by the principles of (i) validity and (ii) optimization. The principle

of validity of a sample survey design refers to the selection of the sample in such a way that the findings could be interpreted in terms of "probability". This principle can be achieved by selecting a "probability sample", which assigns a definite probability for each individual of the population to be included in the sample. The adoption of this principle guarantees that the person's selection of the sample will not influence or bias the sample selection either conciously or unconciously. Moreover, a probability sample enables us to make estimates and their margin of errors which have a valid mathematical basis and hence decide whether the results are sufficiently accurate. The principle of optimization involves --- (i) precision and (ii) cost. Precision is measured by the inverse of the sampling mean square error (m.s.e.) of the estimate. The m.s.e. of the estimate t for the mean μ is equal to the sum of the variance of t and the square of its bias, i.e., m.s.e. of it = var $(t) + B[(t)]^2$. A sampling technique giving smaller m.s.e. is preferred to another with which m.s.e. is higher. Cost is measured by the expenditure incurred in terms of moneys. The m.s.e. can be reduced by taking large samples; but this consumes more time and money. Generally, the allotted fund for a sample survey would be fixed in advance.

Problem

Suppose, it is decided to conduct a national level sample survey in Nepal with a view to obtain reliable estimates of various economic characteristics of the household sector. It is a large scale sample survey which would require a carefully worked out sampling plan. The plan must --- (i) take into account the available resources, time and desired accuracy; and (ii) be able to generate estimating formulae, controlled by the notion of probability, of the population characteristics and mean square error of the estimates. The estimating formulae, if possible, must be unbiased. This paper is an attempt to derive a sampling plan for such a large scale survey which is strictly governed by the principle of validity and partially guided by the principle of optimization. The formulation of an appropriate and realistic sampling plan first of all demands to make a comparision between several valid sampling plans and suggests to select that plan for which either the cost is minimum for a specified level of precision or the precision is highest for a given cost.

A Tentative Sampling Plan

For this problem, the totality of all households of Nepal constitutes the target population. One cannot draw a random sample (one-stage) by assuming all these households as sampling frame units. Because the selection of random sample demands the sampling frame which is impossible to get in this case. Accordingly, one has to adopt multistage plan. The heterogeneity that exists within the target population also cannot be neglected in the selection of sample; otherwise the selected sample would become less repre-

sentative of the target population. To make a sample more representative of the target population one can stratify the target population into several mutually exclusive strata. Nepal displays a great variety of physical and economic conditions. Physically the country has been divided into three east-west contiguous regions viz., the Tarai region, the Mountainous region and the Hilly region. This extremity in physical condition has sufficient effect on the distribution of the population as well as their economic life style. One expects that the sample should be at least representative of the physically diverse economic conditions and livelihood of the population. So, at the first step one has to divide the target population into three mutually exclusive strata. Let N_1, N_2 and N_3 correspondingly denote the number of households in the Tarai, Hilly and Mountainous regions. Further stratification of the three subpopulation is possible if one gets some crdteria to do so.

Next, by adopting the multistage sampling plan one can draw three random samples one from each stratum. Let n_t denote the size of the sample to be drawn from the t^{th} straturm where $t= 1, 2, 3$. One cannot adopt one stage random sampling plan in each stratum for the selection of random sample, due to the afore-mentioned reason. The adoption of a two stage random sampling plan may also increase the cost of the survey. A three-stage random sampling plan can be adopted for the selection of three random samples one from each stratum. According to this plan districts within a stratum can be considered as first stage sampling units, panchayats within each district can be considered as second stage sampling units and lastly, the households within each panchayat can be considered as third stage sampling units. Sampling process has to be carried out in stages. In the first phase, few districts are to be selected from a stratum by random sampling method; in the second phase few panchayats are to be selected from each of the selected panchayats by random sampling method. The adoption of such stratified three stage random sampling plan would reduce the cost of travel and also that of the preparation of the frame.

<u>An Estimating Formula For Population Characteristics</u>

The following notations are used to denote the population constants of a fixed t^{th} stratum:

D = Number of districts;

P_i = Number of panchayats in the i^{th} district;

H_{ij} = Number of households in the j^{th} panchayat of the i^{th} district;

Y_{ijk} = Value of the characteristic "Y" of k^{th} household in the j^{th} panchayat of the i^{th} district; and d, p_α, $h_{\alpha\beta}$ and $y_{\alpha\beta\delta}$ corresponding values in the sample where $\delta = 1,2,\ldots h_{\alpha\beta}$, $\beta = 1,2,\ldots, p_\alpha$ and $\alpha = 1,2,\ldots d$.

$\mu_t = \frac{1}{H_{..}} \sum_{i,j,k} Y_{ijk}$ = Population mean per household of the characteristic "Y" in the t^{th} stratum. An unbiased estimate of μ_t is $\hat{\mu}_t$ where $\hat{\mu}_t$ is defined as

$$\hat{\mu}_t = \frac{1}{d}\left[\sum_{\alpha=1}^{d} \frac{H_{\alpha.}}{H_{..}}\left\{\frac{1}{p_\alpha \bar{H}_{\alpha.}} \sum_{\beta=1}^{p_\alpha} H_{\alpha\beta}\left(\frac{1}{h_{\alpha\beta}} \sum_{\delta=1}^{h_{\alpha\beta}} y_{\alpha\beta\delta}\right)\right\}\right]$$

Here,

$\bar{H}_{\alpha.} = \frac{1}{p_\alpha}\sum_{j=1}^{p_\alpha} H_{\alpha j}$ = Average households per panchayat in the selected district.

$H_{..} = \sum_{i,j} H_{ij}$ = Total household in the t^{th} stratum.

$\bar{H}_{..} = \frac{1}{D}.H_{..}$ = Average household per district.

A national level unbiased estimate of the population characteristic "Y" is $\hat{\mu}$ where $\hat{\mu}$ is defined as

$$\hat{\mu} = \frac{1}{N} \sum \hat{\mu}_t N_t$$

Here, $N = \sum_{t=1}^{3} N_t$ = Total number of households in Nepal.

Testing Constant Returns to Scale in Cobb-Douglas Production Function

Shankar P. Sharma

The Cobb-Douglas Production function is given by

$$Q = a\,L^b K^c \qquad (1)$$

where L and K stand for the labour and capital inputs and Q stands for the output. This can be transformed into the linear form and hence the equation becomes

$$\log Q = \log a + b \log L + c \log K \qquad (2)$$

Adding the disturbance term the model can be written as

$$Y = \beta_1 + \beta_2 X_2 + \beta_3 X_3 + u \qquad (3)$$

where $Y = \log Q$, $X_2 = \log L$, $X_3 = \log K$, $\beta_1 = \log a$, $\beta_2 = b$ and $\beta_3 = c$.

The disturbance term u satisfies the conditions $E(u) = 0$ and $\text{var}(u) = \sigma^2$.

For the model (3), define a n x 3 matrix $X = (X_1, X_2, X_3)$ where X_1, X_2, X_3 are column vectors and X_1 is a unit vector of order n, X_2 and X_3 are the vectors of order n (the total number of observations) each.

Hence the best linear unbiased estimate of the vector β of order 3 is given by

$$\hat{\beta} = \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{pmatrix} (X'X)^{-1} X'Y$$ where Y is a column vector of order n.

The coefficient of determination R^2 is defined as

$$R^2 = 1 - \frac{e^2}{(Y-\bar{Y})^2} \quad \text{where } e = Y - \hat{Y}$$

In the model defined in (1) b is the elasticity of output with respect to labour and c is the elasticity of output with respect to capital. From the economic point of view these elasticities cannot take the negative values. They show approximately the average percentage change in the total production for one percent change in the input concerned. As the sum of these elasti-

cities shows the returns to scale, there may arise the following conditions.

(i) $b + c = 1$, This means that if labour and capital inputs are increased by the same proportion, output also increases by that proportion. This is the case of constant return to scale.

(ii) $b + c > 1$ indicates that the proportion of output is more than that of the proportion of input and this is the case of increasing returns to scale.

(iii) $b + c < 1$ is the case of diminishing returns to scale and indicates that the proportion of output is less than that of the proportion of input.

Consider model (3), the linear form of equation (1) under the restriction $\beta_1 + \beta_2 = 1$ (i.e. $b + c = 1$ in the model (1))

The estimator subject to this restriction can be obtained by writting the restriction into the model, as follows:

$$Y = \beta_1 + \beta_2 X_2 + (1-\beta_2) X_3 + u$$

$$\text{or } (Y-X_3) = \beta_1 + \beta_2 (X_2-X_3) + u \qquad (4)$$

Now the regression of $(Y-X_3)$ on (X_2-X_3) would give the estimator $\hat{\beta}_{1R}$, $\hat{\beta}_{2R}$, and $\hat{\beta}_{3R}$ can be found as $\hat{\beta}_{3R} = 1 - \hat{\beta}_{2R}$

The problem now is to test the hypothesis $\beta_1 + \beta_2 = 1$ with the possible alternatives

$$H_{A_1} : \beta_1 + \beta_2 \neq 1$$

$$H_{A_2} : \beta_1 + \beta_2 > 1$$

$$H_{A_3} : \beta_1 + \beta_3 < 1$$

The test statistic can be constructed by calculating the sum of squares of the residuals in the restricted and unrestricted case.

The relevant quantity of interest in the unrestricted case would be

$$\sum e^2 = (Y - \hat{\beta}_1 - \hat{\beta}_2 X_2 - \hat{\beta}_3 X_3)^2 \qquad (5)$$

The corresponding quantity in case of restricted case is

$$\sum e_R^2 = \sum \{(Y-X_3) - \hat{\beta}_{1R} - \hat{\beta}_{2R}(X_2-X_3)\}^2 \qquad (6)$$

Denote the number of restriction by p, which in this case is one (i.e. $\beta_1 + \beta_2 = 1$). The number of degrees of freedom associated with the residual sum of squares in the unrestricted estimation is given by (n-k-1) where n is the no. of observations and k is the number of dependent variables or (k + 1) is the number of parameters in the model. Hence define the statistic

$$F = \frac{(S_R - S)\ /\ p}{S/\ (n-k-1)} \qquad (7)$$

where $S_R = \sum e_R^2$ and $S = \sum e^2$

Adding variables to any existing estimated relationship will decrease, or at least cannot increase the residual sum of squares. Conversely, removing variables will generally increase and cannot decrease the residual sum of squares[+]. Hence the restricted residual sum of squares S_R must be greater than the original residual sum of squares. So S_R is greater than S and (S_R-S) will be positive (sometimes S_R-S is zero which is not discussed here).

If the null hypothesis is true the statistic defined in (7) has the F distribution with p and (n-k-1) d.f.

Therefore the critical region for the two sided hypothesis (i.e. for the alternative hypothesis H_{A1}) at α% level of significance is

$$F \geq F_{\frac{1}{2}\alpha,\ p,\ n-k-1}$$

$$\leq F_{1-\frac{1}{2}\alpha,\ p,\ n-k-1}$$

The critical region for the one-sided hypothesis (i.e. for H_{A2}) at α% level of significance is

$$F \geq F_{\alpha,\ p,\ n-k-1}$$

and similarly for H_{A3} the critical region is

$$F \leq F_{1-\alpha,\ p,\ n-k-1}$$

[+]Jon Stewart - Understanding Econometrics - (Hutchinson of London - 1976), pp. 87.

So if our calculated value of F is in the critical region we reject the null hypothesis in favour of the corresponding alternative hypothesis.

Example+

Consider an example where Q is the total agricultural production (cereal crops) in kg, L is the land in hectares and K is the Budgeted money invested in Rs. for the production. Budgeted money gives the total of the variable cost (money used for seeds and fertilizer) and the other capital invested (mainly used for all kinds of labour) in the production of the crops. The data for these three variables were selected by a simple random sampling procedure from the data of the 52 households of village. The estimated value of $\hat{\beta}$ vector is given by

$$\hat{\beta} = (X'X)^{-1} (X'Y) = \begin{pmatrix} 2.477 \\ 0.224 \\ 0.828 \end{pmatrix}$$

a = Antilog 2.477 = 299.9

Hence the model becomes $Q = 299.9 \, L^{0.224} \, K^{0.848}$ (8)

The model (8) indicates that the increase of capital by one percent will, other things being equal, bring about an increase of 0.224 in the product. Similarly, if land increases by one percent, then other things being equal, bring about an increase by 0.828 of one percent.

The coefficient of determination comes to be equal

to $$R^2 = 1 - \frac{0.06738478}{176,339} = 0.948$$

It may be concluded that the production function (8) explains 94 percent of the variation in the dependent variable and can be considered as a better casual relationship.

+Data was collected by Gajurel N.H., student of Statistics Instruction Committee for his Dissertation and was guided by the contributor of this article. He is thankful to Mr. Gajurel as most of the calculations were done by him. The data was collected in Gotikada Village Panchayat of Surkhet district in the year 1975.

To test the hypothesis $H_o : b + c = 1$

against $H_A : b + c > 1$

The sum of the squared residuals using the restricted estimators

$\beta_{1R} = 2.746$, $\beta_{2R} = 0.142$ comes to be equal to 0.138241

$$\therefore F = \frac{(S_R - S)}{S} \quad \frac{(n - k - 1)}{p} = \frac{0.070856}{0.06738} \cdot \frac{11}{1} = 11.57$$

As this is one tailed test

$F_{.05,1,11} = 9.65$

Hence F lies in the critical region, and hence we reject H_o and accept H_A.

As the proportion of out-put is more than proportion of input in this region of study, the agriculture production in the sector can be raised by raising the investment in terms of land and capital. There is sufficient scope of raising production by the use of improved agro-techniques.

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Examples from Transformation Geometry

by Dr. Heinz Ruegger, IOE,
Kirtipur

1. Introduction

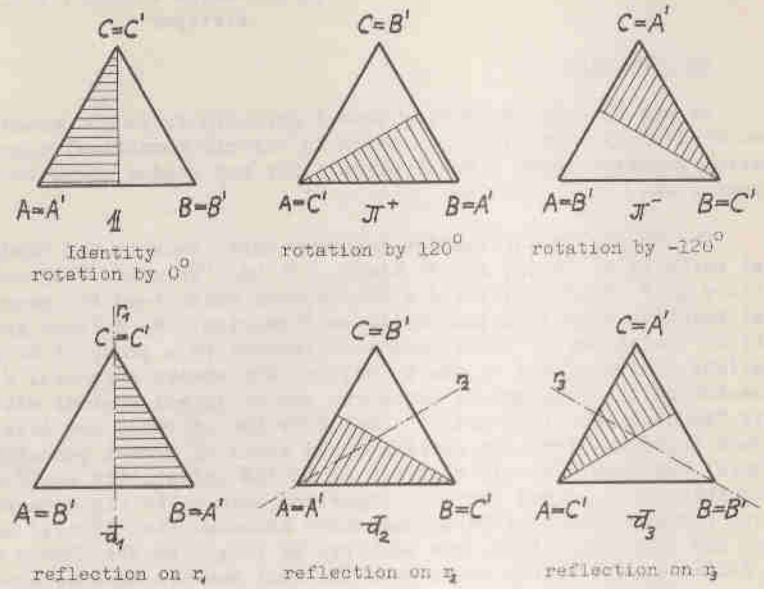
If you look how secondary school geometry is taught nowadays you will surely find that (at least in western countries) transformation geometry takes a broad place. You may wonder where this comes from.

The modern transformation geometry dates back to the fundamental works of A. Cayley and F. Klein. In his "Erlanger Program" of 1872 e.g. F. Klein initiated a development which lead the geometrical thinking away from the euclidean tradition. He defined geometry as the theory of invariants with respect to a group of transformations. After Klein it was D. Hilbert who opened a general discussion on the problems of axiomatic and of formal systems with his "Foundations of Geometry". But very few of these new ideas found their way into the curricula and texts of school geometry. There, the 2000 year old elements of Euclid and all its modified versions still played the most important role. The rigorous reform plans of Bourbaki/Dieudonné which demanded that "Euclid must go" and that plane Euclidean Geometry be taught as the theory of a 2-dimensional vector space over the real numbers were necessary to get things going. For a lot of teachers these plans were too extensive and so the transformation geometry emerged as a valuable alternative.

Instead of trying to give you a complete overview of topics which are taught under the heading 'Transformation Geometry', I shall discuss to some detail two selected examples. The first one shows how students can be led to the discovery of the group structure of transformations and the second one illustrates the invariant properties of different kinds of transformations.

2. Mappings of an Equilateral Triangle onto Itself

First have your students find all possible mappings which map a given equilateral triangle ABC onto itself. If they are familiar with the properties of (translations), rotations and reflections, they can easily do this job, finding that there are 6 possibilities: 3 rotations (including the identical mapping) and 3 reflections. You may illustrate them with the following figure:



$\mathbb{1}$ Identity rotation by 0° | $\mathcal{D}^+$ rotation by 120° | $\mathcal{D}^-$ rotation by -120°

$\mathcal{S}_1$ reflection on r_1 | $\mathcal{S}_2$ reflection on r_2 | $\mathcal{S}_3$ reflection on r_3

These mappings can be visualized further by using a cardboard model of the triangle.

From previous units your students know that mappings can be combined by applying a first mapping on a given figure getting a certain image and then map this image again by a second mapping. This allows us to define a connection between two elements of the set $G = \{\mathbb{1}, \mathcal{D}^+, \mathcal{D}^-, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3\}$ of transformations. We write

$$\mathcal{A} \circ \mathcal{B}$$

which means: apply first mapping $\mathcal{A}$ and then mapping $\mathcal{B}$. So let your students e.g. find the combination

$$\mathcal{S}_1 \circ \mathcal{D}^+ = ?$$

A look at the above figures or at the model shows him that he gets the same image if he uses the reflection $\mathcal{S}_3$ or if he first applies $\mathcal{S}_1$ and afterwards $\mathcal{D}^+$. Therefore it is

$$\mathcal{S}_1 \circ \mathcal{D}^+ = \mathcal{S}_3$$

The easiest way to form all possible combinations of 2 mappings is to write them in the following table:

$A \circ B$ $\;\; A \backslash B$	$\mathbb{1}$	D^+	D^-	S_1	S_2	S_3
$\mathbb{1}$	$\mathbb{1}$	D^+	D^-	S_1	S_2	S_3
D^+	D^+	D^-	$\mathbb{1}$	S_2	S_3	S_1
D^-	D^-	$\mathbb{1}$	D^+	S_3	S_1	S_2
S_1	S_1	S_3	S_2	$\mathbb{1}$	D^-	D^+
S_2	S_2	S_1	S_3	D^+	$\mathbb{1}$	D^-
S_3	S_3	S_2	S_1	D^-	D^+	$\mathbb{1}$

(It is of course the task of the students to complete this table) Now let us see what they can find if they carefully inspect the table:

1. There appear no other elements than the 6 mappings of G in the table. This illustrates that the set G is closed with respect to the connection $\circ$.

2. The first row and the first column contain exactly the set G i.e. the connection of any mapping of G with $\mathbb{1}$ does not change this mapping. ($\mathbb{1}$ plays the same role as 1 with respect to the ordinary multiplication; existence of an identity element in G.).

3. The element $\mathbb{1}$ appears in each row and column exactly once i.e. to each mapping A of G there exists exactly one mapping A^{-1} such that $A \circ A^{-1} = \mathbb{1}$ (existence of an inverse element).

4. Using the table they may verify that the connection $\circ$ is associative e.g.

$$\underbrace{(D^+ \circ S_1)}_{S_2} \circ S_3 = D^+ \circ \underbrace{(S_1 \circ S_3)}_{D^+}$$
$$S_2 \circ S_3 = D^+ \circ D^+$$
$$D^- = D^-$$

Summarized they found that [G, o] possesses all the properties required for a group.

You may further emphasize the facts that:

5. $A \circ B$ is not necessarily equal to $B \circ A$ (not commutative).

6. $\{\mathbb{1}, D^+, D^-\}$ alone has also group structure. It is the subgroup which we get if we restrict ourselves to the mappings of the equilateral triangle which leave the orientation of the triangle invariant. In this subgroup the commutative law is valid. (group table symmetric with respect to diagonal).

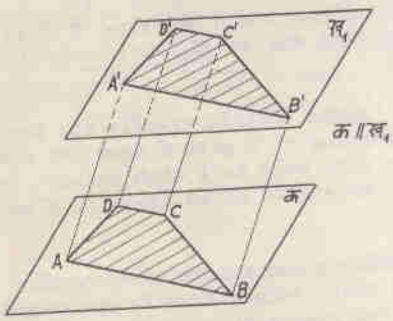
7. also $\{\mathbb{1}, S_1\}$, $\{\mathbb{1}, S_2\}$, $\{\mathbb{1}, S_3\}$ are subgroups. They are obtained if we require one fixpoint for the mapping.

8. the number of elements in these subgroups is a divisor of the number of elements in G.

Finally it should be noticed that this group [G, o] of mappings is isomorphic to the permutation group of P_3.

3. Invariant Properties of Transformations

This example should show how the ideas of Cayley and Klein, to classify geometry according to invariance properties with respect to certain transformation groups can be illustrated. For this purpose a trapezium ABCD in plane क is consequtively mapped by a parallel projection to a plane ख$_1$ ∥ क, a central projection to a plane ख$_2$ ∥ क, a parallel projection to a plane ख$_3$ ∦ क and finally by a central projection to a plane ख$_4$ ∦ क and it is observed which properties of the trapezium are invariant.

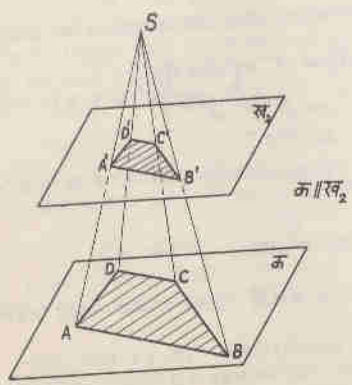


Congruence mapping

Trapezium ABCD is mapped by a parallel projection from plane क to ख$_1$ ∥ क onto the congruent trapezium A'B'C'D'

Invariants

size
angles
ratios
parallelism
coincidence

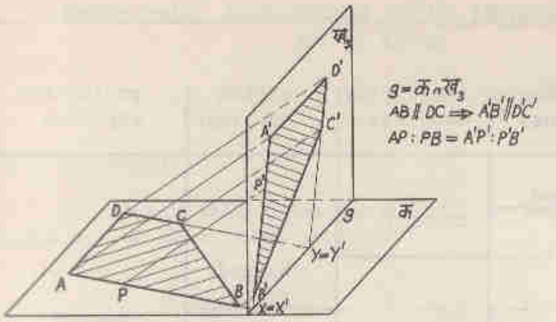


Similarity mapping

Trapezium ABCD is mapped by a central projection from plane क to ख$_2$ ∥ क onto the similar trapezium A'B'C'D'

Invariants

angles
ratios
parallelism
coincidence

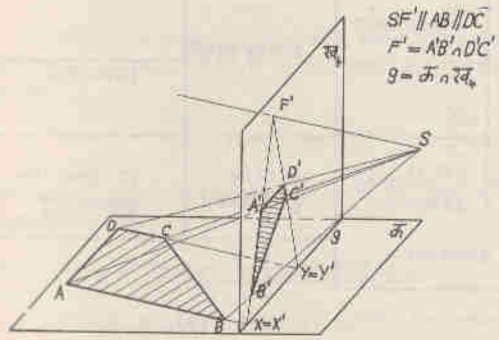


Affine mapping

Trapezium ABCD is mapped by a parallel projection from plane क to $ख_3$ onto the affine trapezium A'B'C'D'

Invariants

ratios
parallelism
coincidence



Projective mapping

Trapezium ABCD is mapped by a central projection from plane क to $ख_4$ onto quadrilateral A'B'C'D' which is no longer a trapezium.

Invariants

coincidence

Each of the figures gives only one representative of a whole group of respective mappings. The most comprehensive group of mappings are the projective mappings. Only coincidence properties are invariant under this group. If we want more properties to be invariant we have to admit special cases of the projective mappings only e.g. the affine mappings which form a subgroup of the projective mappings, or the similarity mappings which form a subgroup of the affine mappings or finally the congruence mappings which form a subgroup of the similarity mappings. By these transformation groups geometry can be classified into projective geometry discussing properties which are invariant under projective mappings i.e. coincidence, into affine geometry discussing properties invariant under affine mappings, into similarity geometry and into congruence geometry (which are usually combined into metric geometry) which are defined accordingly.

We may summarize this discussion with the following table:

<table>
<tr><td rowspan="2">Property</td><td colspan="4">GROUP OF THE</td></tr>
<tr><td>congruence mappings</td><td>similarity mappings</td><td>affine mappings</td><td>projective mappings</td></tr>
<tr><td>position</td><td>changed</td><td rowspan="2">changed</td><td rowspan="3">changed</td><td rowspan="5">changed</td></tr>
<tr><td>size</td><td rowspan="5">invariant</td></tr>
<tr><td>angles</td><td rowspan="4">invariant</td></tr>
<tr><td>ratios</td><td rowspan="3">invariant</td></tr>
<tr><td>parallelism</td></tr>
<tr><td>coincidence</td><td>invariant</td></tr>
<tr><td></td><td>congruence geometry</td><td>similarity geometry</td><td>affine geometry</td><td>projective geometry</td></tr>
<tr><td></td><td colspan="2">metric geometry</td><td></td><td></td></tr>
</table>

3.9.76 H. Ruegger IOE Kirtipur

Polynomial Associated With Legendre Polynomial

by Narayan B. Shrestha

1. Introduction

One of the well known polynomials is Legendre polynomial defined by the generating relation

$$(1-2xt+^{2})^{-\frac{1}{2}} = \sum_{n=o}^{\infty} P_n(x) \quad t^n \tag{1.1}$$

In this paper, we shall consider a set of polynomials $S_n(x)$ associated with $P_n(x)$ which may be defined by the generating relation

$$[1-2xt+t^2(2x-1)]^{-\frac{1}{2}} = \sum_{n=o}^{\infty} S_n(x)t^n \tag{1.2}$$

in which the particular branch $[1-2xt+t^2(2x-1)]^{-\frac{1}{2}}$ tends to one as tends to zero.

2. Finite Series Representations

Since ${}_1F_0(a;-;z) = (1-z)^{-a}$, we have, from (1.2)

$$[1-2xt+t^2(2x-1)]^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n \; (2xt-t^2(2x-1))^n}{n!}$$

which yields

$$S_n(x) = \sum_{k=o}^{\infty} \frac{(-1)^k \; (\frac{1}{2})_{n-k} \; (2x)^{n-2k} \; (2x-1)^k}{k! \; (n-2k)!} \tag{2.1}$$

Similarly, considering (I.2) and noting that

$$[1-2xt+t^2(2x-1)]^{-\frac{1}{2}} = [(1-xt)^2-t^2x^2 +t^2(2x-1)]^{-\frac{1}{2}}$$

we get another finite series representation as

$$S_n(x) = \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k \; (x-1)^{2k} \; x^{n-2k} \; n!}{k! \; (n-2k)! \; (2k)!} \tag{2.2}$$

Results thus obtained as the finite series representations for $S_n(x)$ is of particular interest to us in obtaining hypergeometric forms, integral representation and expansions in series for $S_n(x)$.

3. Hypergeometric Representations

From the relations (2·1) and (2.2), $S_n(x)$ can be put in the hypergeometric forms as

$$S_n(x) = \frac{(\frac{1}{2})_n\,(2x)^n}{n!}\,{}_2F_1\left[\begin{matrix} -\frac{1}{2}n,\ -\frac{1}{2}n+\frac{1}{2}; \\ \frac{1}{2}-n; \end{matrix}\ \frac{2x-1}{x^2}\right] \tag{3.1}$$

$$S_n(x) = x^n\ {}_2F_1\left[\begin{matrix} -\frac{1}{2}n,\ -\frac{1}{2}n+\frac{1}{2}; \\ 1;a \end{matrix}\ \frac{(x-1)^2}{x^2}\right] \tag{3.2}$$

4. Integral Representations

To obtain an integral representation of the polynomial $S_n(x)$ we consider the relation (2.2).

We note that

$$\frac{(\frac{1}{2})_k}{k!} = \frac{\Gamma(\frac{1}{2}+k)}{\Gamma(k+1)\Gamma(\frac{1}{2})}\cdot\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2})} = \frac{B(\frac{1}{2},\frac{1}{2}+k)}{[\Gamma(\frac{1}{2})]^2}$$

where $B(p,q) = \dfrac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$ is the Bete function defined for $Re(p)$ and $Re(q)$ greater than 0.

By integral definition of Beta function $B(p,q)$, we have

$$B(\tfrac{1}{2},\tfrac{1}{2}+k) = 2\int_0^{\pi/2} \cos^{2k}\phi\, d\phi \text{ and also } \Gamma(\tfrac{1}{2}) = (\pi)^{\frac{1}{2}}$$

which when used above yields

$$S_n(x) = \frac{1}{\pi}\int_0^{\pi}\sum_{k=0}^{[n/2]} \frac{n!\,(x-1)^{2k}\,x^{n-2k}\cos^{2k}\phi}{(2k)!\,(n-2k)!}\,d\phi$$

Term by term integration yields

$$S_n(x) = 1/\pi \sum_{k=0}^{[n/2]} \frac{n!\,(x-1)^{2k}\,x^{n-2k}}{(2k)!\,(n-2k)!}\int_0^{\pi}\cos^{2k}\phi\, d\phi$$

Since $\int_0^{\pi} \cos^m\phi \, d\phi = 0$ for odd m, we replace 2k by k in the summation in the right. Thus

$$S_n(x) = 1/\pi \sum_{k=0}^{n} \frac{n!\ (x-1)^k\ x^{n-k}}{k!\ (n-k)!} \int_0^{\pi} \cos^k\phi \, d\phi$$

in which each term involving an odd k is zero. This, in turn, gives an integral of the type called Laplace first integral

$$S_n(x) = \frac{1}{\pi} \int_0^{\pi} [x + (x - 1) \cos \phi]^n \, d\phi$$

5. <u>Generating Functions</u>

Some of the generating functions can be derived from the generating relation (1.2), this time we note that

$$[1 - 2xt + t^2 (2x-1)]^{-\frac{1}{2}} = [(1-t)^2 + (1-t)2t(1-x)]^{-\frac{1}{2}}$$

$$= (1 - t)^{-1} \left[1 + \frac{2t(1 - x)}{(1 - t)}\right]^{-\frac{1}{2}}$$

Expanding in ascending powers of t and equating the coefficients of t^n, we easily obtain

$$S_n(x) = \sum_{k=0}^{n} \frac{(\frac{1}{2})_k\ n!\ 2^k\ (x-1)^k}{(k\ !)^2\ (n-k)!} \qquad \ldots. (5.1)$$

which can be equally put in the hypergeometric form

$$S_n(x) = {}_2F_1 \left[\begin{matrix} \frac{1}{2}, -n; \\ 1; \end{matrix} \; 2(1-x) \right]$$

For arbitrary c, we have

$$\sum_{n=0}^{\infty} \frac{(c)_n S_n(x)\ t^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(c)_n\ (\frac{1}{2})_k\ [2(x - 1)]^k\ t^n}{k!\ (n-k)!\ k!}$$

from which we obtain

$$\sum_{n=o}^{\infty} \frac{(c)_n S_n(x)\ t^n}{n!} = (1-t)^{-c}\ {}_2F_1\left[\begin{matrix}1/2,\ c; & \frac{2t(x-1)}{(1-t)}\\ 1; & \end{matrix}\right] \quad (5.2)$$

Let us obtain a new generating function for the polynomial $S_n(x)$ using arbitrary c & (2.2) relation, we have

$$\sum_{n=o}^{\infty} \frac{(c)_n S_n(x)\ t^n}{n!} = \sum_{n=o}^{\infty}\sum_{k=o}^{[n/2]} \frac{(\tfrac{1}{2})_k\ (c)_n\ (x-1)^{2k} x^{n-2k} t^n}{k!\ (2k)!\ (n-2k)!}$$

Thus left side equals

$$= (1-xt)^{-c}\ {}_2F_1\left[\begin{matrix}\tfrac{1}{2}c, \tfrac{1}{2}c+\tfrac{1}{2}; & \\ & \frac{(x-1)^2t^2}{(1-xt)^2}\\ 1; & \end{matrix}\right] \quad (5.3)$$

Both the results (5.2) &(5.3) degenerate into the generating function used to define $S_n(x)$ when c is unity.

Starting from the finite series representation (2.2), we get another generating function which have the following form

$$\sum_{n=o}^{\infty} \frac{S_n(x)t^n}{n!} = e^{xt}\ {}_0F_1\left[-;1;\ \frac{(x-1)^2t^2}{(4)}\right] \quad (5.4)$$

since ${}_0F_0(-;\ -\ ;\ x) = e^x$. Again since ${}_0F_1$ is a Bessel function, (5.4) can equally well be written as

$$e^{xt}\ J_o[t(1-x)] = \sum_{n=o}^{\infty} \frac{S_n(x)t^n}{n!} \quad (5.5)$$

a result analogous to that obtained for classical Legendre polynomial [2]. Another result involving exponential function, is

$$\sum_{n=0}^{\infty} S_n(x)t^n = e^t\ {}_1F_1\ [\tfrac{1}{2};\ 1;\ 2t(x-1)] \quad /(.6)$$

Differentiating both sides of (1.2) w.r.t. t & then multipling both sides by 2t, we get

$$\frac{2xt - 2t^2(2x-1)}{[1 - 2xt + t^2(2x-1)]^{3/2}} = \sum_{n=0}^{\infty} (2n+1)\ S_n(x)t^n \quad (5.7)$$

Adding (1.2) & (5.7), we get R.H.S. of (5.7) equals

$$\frac{1 - t^2(2x - 1)}{[1 - 2xt + t^2(2x - 1)]^{3/2}} = \sum_{n = o}^{\infty} (2n + 1)\ S_n(x) t^n \qquad (5.8)$$

6. Recurrence relations: Different recurrence relations involving differential & different indices can be obtained in differential & pure forms from the defining relation. Differentiating both sides of (1.2), we obtain

$$[x - (2x - 1)t]\ [1 - 2xt + t^2(2x)\ 1)]^{-3/2} = \sum_{n=1}^{\infty} n\ S_n(x)\ t^{n-1} \qquad (6.1)$$

$$t\ (1 - 2t)\ [1 - 2xt + t^2(2x - 1)]^{-3/2} = \sum_{n=1}^{\infty} S_n'(x) t^n$$

Multiplying (6.1) by 2t & (6.2) by (2x - 1) & subtracting (6.2) from (6.1) & equating the coefficients of t^n, we arrive at

$$(1 + 4n\ x)\ S_n(x) - 2(n + 1)\ S_{n+1}(x) - 2(2x - 1)\ (n - 1)\ S_{n-1}(x)$$

$$= 2x(2x-1)S_n'(x) - (2x-1)^2 S_{n-1}'(x) - (2x-1)S_{n-1}'\ (x) \qquad (6.3)$$

Again with the help of (6.2), it can be obtained that

$$S_n(x) - S_{n-1}(x) = S_{n+1}'(x) - 2x\ S_n'(x) + (2x-1)\ S_{n-1}'(x) \qquad (6.4)$$

Preceding similarly, we can get the pure recurrence relation

$$(2n - 1)xS_{n-1}(x) - (n - 1)\ (2x - 1)\ S_{n-2}(x) = n\ S_n(x); \quad n \geq 2 \quad (6.5)$$

One of the result obtained in (5.8) above can be used to get the same pure recurrence relation obtained above in (6.5).

7. Particular cases: Particular cases suggested by $S_n(x)$ are as follows:

The pure recurrence relation obtained above gives fairly an easy method of computing successive new polynomials, which are

$S_0(x) = 1,$ $\quad S_1(x) = x,$ $\quad S_2(x)=\frac{3}{2}x^2 - x + 1/2,$

$S_3(x) = \frac{5}{2}x^3 - 3x^2 + \frac{3}{2}x,$ $\quad S_4(x)= \frac{35}{8}x^4 - \frac{15}{2}x^3 + \frac{21}{4}x^2 - \frac{3}{2}x + \frac{3}{8}$ etc.

These successive polynomials suggested by $S_n(x)$ can be used to study the nature of the polynomials. The graph of these four polynomials are given in Fig. 1.

8. Expansions in series: In this section, we shall obtain the expansions of the polynomial $S_n(x)$ in a series of Legendre, Hermite & Laguerre polynomials with known expansions of x^n[1]. We have, from [1],

$$x^n = \frac{n!}{2^n} \sum_{k=0}^{[\frac{n}{2}]} \frac{(2n-4k+1)\, P_{n-2k}(x)}{k!\ (\frac{3}{2})_{n-k}},$$

$$x^n = \sum_{k=0}^{[n/2]} \frac{n!\ H_{n-2k}(x)}{2^n\ k!\ (n-2k)!}$$

& $$x^n = \sum_{k=0}^{n} \frac{(-1)^k\ n!\ (1+a)_n\ L_k^a(x)}{(n-k)!\ (1+a)_k}$$

where $P_n(x)$, $H_n(x)$ & $L_n^{(a)}(x)$ are Legendre, Hermite & Laguerre polynomials respectively. The expansion formulas are

$$S_n(x) = \frac{(1/2)_n}{(3/2)_n} \sum_{k=o}^{[n/2]} {}_2F_1 \left[\begin{matrix} -k, -\frac{1}{2}-n+k; \\ \frac{1}{2}-n; \end{matrix}\ (1-2x)\right] \frac{(2n-4k+1)P_{n-2k}(x)}{k!\ (3/2+n)_k} \tag{8.1}$$

$$S_n(x) = \frac{(\frac{1}{2})_n}{n!} \sum_{k=0}^{[n/2]} {}_1F_1(-k;\ \tfrac{1}{2}-n\ ;\ (1-2x))\ \frac{(-n)_{2k}\ H_{n-2k}(x)}{k!} \tag{8.2}$$

and

$$S_n(x) = \frac{(\frac{1}{2})_n\ 2^n\ (1+a)_n}{n!} \sum_{s=o}^{n} \frac{(-1)^s L_s^{(a)}(x)(-n)_s}{(1+a)_s} {}_2F_3 \left[\begin{matrix} -(\frac{n}{2}+\frac{s}{2}), -\frac{1}{2}(n+s-1); \\ \\ -\frac{1}{2}(a+n+s), -\frac{1}{2}(a+n+s-1), \frac{1}{2}-n-s; \end{matrix}\ (2x-1)/4\right]$$

9. Acknowledgements: I am very much obliged to Dr. R.M. Shrestha, Reader, Mathematics Instruction Committee, Kirtipur Campus, T.U., for his guidance, time to time encouragements along with valuable suggestions in preparaing this paper. Lastly, I must acknowledge with thanks, the kind cooperations, encouragements offered by the teachers of Kirtipur Campus, T.U.

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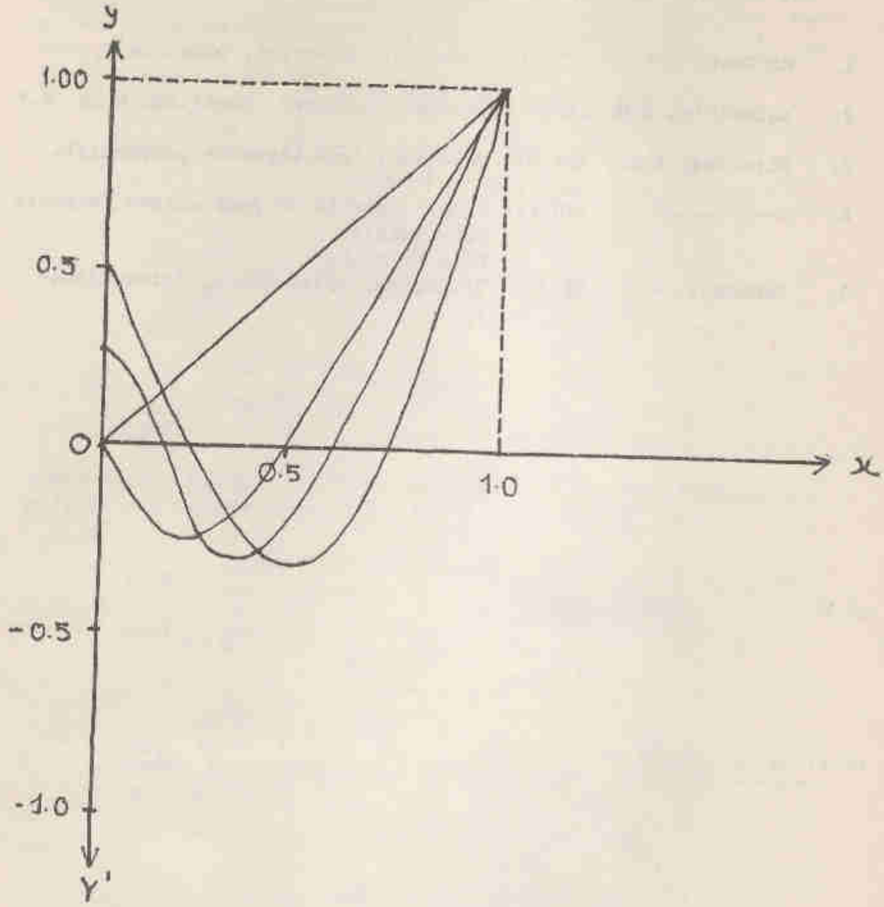

GRAPH OF THE POLYNOMINAL $S_n(x)$ SUGGESTED BY (7)

On A Class Of Convolution Transform*

By Bhanu Chandra Bajracharya

1. Introduction.

In this paper we shall study the inversion theory for the class of convolution transforms

$$(1.1)\qquad \begin{aligned} f(x) &= \int_{-\infty}^{\infty} G(x-t)e^{ct}\, d\alpha(t) \qquad (c:\text{real}) \\ f(x) &= \int_{-\infty}^{\infty} G(x-t)\, \phi(t)\, dt \end{aligned}$$

for which the kernel is of the form

$$(1.2)\qquad G(t) = (2\pi i)^{-1} \int_{-i\infty}^{i\infty} [E(s)]^{-1} e^{st}\, ds,$$

→ x

where $E(s)$ is a meromorphic function with real zeroes and poles only, and is of the form

$$(1.3)\qquad E(s) = \prod_{k=1}^{\infty} \frac{(1-s^2/a_k^2)(1-s^2/b_k^2)}{(1-s^2/c_k^2)}$$

with sequences of real constants $\{a_k\}$, $\{b_k\}$, $\{c_k\}$ satisfying

$$(1.4)\qquad \sum_{k=1}^{\infty} a_k^{-2} < \infty, \quad \sum_{k=1}^{\infty} b_k^{-2} < \infty, \quad \sum_{k=1}^{\infty} c_k^{-2} < \infty,$$

$$(1.5)\qquad a_k c_k > 0, \quad a_k b_k < 0 \quad \text{and} \quad 0 \leqq a_k/c_k < 1$$

for all positive integers k.

Here we shall permit (i) $c_k = \infty$ when $a_k > 0$ or $c_k = -\infty$ when $a_k < 0$ and (ii) $b_k = \infty$ when $a_k < 0$ or $b_k = -\infty$ when $a_k > 0$ by which we shall understand

$$1-s^2/c_k^2 \equiv 1 \quad \text{and} \quad 1-s^2/b_k^2 \equiv 1.$$

We shall consider the meromorphic function $E(s)$ as the ratio of two functions defined by

$$(1.6)\qquad \begin{aligned} E_1(s) &= \prod_{k=1}^{\infty} (1-s^2/a_k^2)\,(1-s^2/b_k^2) \\ E_2(s) &= \prod_{k=1}^{\infty} (1-s^2/c_k^2) \end{aligned}$$

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*The present paper is a part of the dissertation thesis submitted to the Department of Mathematics, Kirtipur Campus, as partial fulfillment of the requirement for the Degree in Mathematics.

This form E(s) is analogous to one in which $E_1(s)$ and $E_2(s)$ are the reciprocal of the generating functions of the kernels of class I and II treated in ([7], [8], [9], [10], [11], [3]).

The transform (1.1) reduces to the class of convolution transform studied by Yukichi Tanno [11] when b_k becomes infinite and to the convolution transforms considered by Widder [6] and John Dauns and Widder [2] when both b_k and c_k become infinite.

2. Properties of E(s)

An investigation on certain properties of E(s) will be useful for the later development. We define

$$(2.1)\quad \alpha_1 = \begin{cases} \max a_k & \text{if } b_k > 0 \\ a_k < 0 & \\ \max b_k & \text{if } a_k > 0 \\ b_k < 0 & \end{cases} \qquad \alpha_2 = \begin{cases} \min a_k & \text{if } b_k < 0 \\ a_k > 0 & \\ \min b_k & \text{if } a_k < 0 \\ b_k > 0 & \end{cases}$$

and

$$(2.2)\quad \alpha_1 = \max\{c_k, -\infty \mid c_k < 0\}, \alpha_2 = \min\{c_k, \infty \mid c_k > 0\}.$$

Theorem 2.1. If the real sequences $\{a_k\}$, $\{b_k\}$, $\{c_k\}$ satisfy (1.4) and (1.5) of §1, and if

$$E_m(s) = \prod_{k=m}^{\infty} \frac{(1-s^2/a_k^2)(1-s^2/b_k^2)}{(1-s^2/c_k^2)} \quad (m = 1,2,3,\ldots\ldots)$$

then

$$(2.3)\qquad [|E_m(\sigma+i\tau)|]^{-1} \leq [|E_m(\sigma)|]^{-1} \quad (m=1,2,3,\ldots\ldots).$$

Proof. Let $s = \sigma + i\tau$ and $(\alpha_1+\gamma_1)/2 \leq \sigma \leq (\alpha_2+\gamma_2)/2$. We have for any positive integer p

$$|E_m(\sigma+i\tau)|^2 = \prod_{k=m}^{m+p} \frac{[(1-\sigma/a_k)^2+\tau^2/a_k^2]\,[(1-\sigma/a_k)^2+\tau^2/a_k^2]\,[(1-\sigma/b_k)^2+\tau^2/b_k^2]\,[(1+\sigma/b_k)^2+\tau^2/b_k^2]}{[(1-\sigma/c_k)^2+\tau^2/c_k^2]/(1+\sigma/c_k)^2+\tau^2/c_k^2]}$$

By the conditions (1.5) and [1: lemma 9, § 9] the function

$$[(1\pm\sigma/a_k)^2+\tau^2/a_k^2]\;[(1\pm\sigma/c_k)^2+\tau^2/c_k^2]$$

has a single minimum at $\tau = o$ as a function of τ. Hence

$$|E_m(\sigma+i\tau)|^2 \geqq \prod_{k=m}^{m+p} \frac{(1-\sigma^2/a_k^2)^2(1-\sigma^2/b_k^2)^2}{(1-\sigma^2/c_k^2)^2} \tag{2.4}$$

$$(m = 1,2,3,\dots)$$

$$(p = 1,2,3,\dots)$$

Now, let $p \to \infty$. Both sides of the inequality (2.4) tend to limit by the assumption (1.4). In fact the limit we have is

$$|E_m(\sigma+i\tau)|^2 \geqq |E_m(\sigma)|^2$$

from which the inequality (2.3) is evident.

Theorem 2.2. If the hypotheses of the theorem 2.1 are satisfied, and if

$$E(s) = \prod_{k=1}^{\infty} \frac{(1-s^2/a_k^2)(1-s^2/b_k^2)}{(1-s^2/c_k^2)}$$

then for any positive numbers p and R

$$[\,|E(\sigma+i\tau)|\,]^{-1} = O([\,|\tau|^p]^{-1}) \qquad |\tau| \to \infty$$

uniformly in the strip $|\sigma| \leqq R$.

Proof. Let

$$E_N(s) = \prod_{k=N+1}^{\infty} \frac{(1-s^2/a_k^2)(1-s^2/b_k^2)}{(1-s^2/c_k^2)}$$

Then, if conditions (1.5) hold, the same arguments used in [4: pp. 52] and the theorem 2.1 yield

$$|E(\sigma+i\tau)| \geqq \frac{|\tau|^{2N}\, c_1^2.c_2^2. \dots\dots\dots c_N^2}{a_1^2.a_2^2 \dots\dots a_N^2.b_1^2.b_2^2 \dots\dots b_N^2}\, |E_N(\sigma)|$$

from which the conclusion for any positive number p not greater than 2N follows.

3. Properties of Kernels G(t)

Let us define

$$g_k^{(1)}(t) = \begin{cases} \dfrac{a_k}{2c_k^2}(c_k^2-a_k^2)\, e^{-a_k|t|} & \text{for } a_k > o,\ -\infty < t < \infty \\ -\dfrac{a_k}{2c_k^2}(c_k^2-a_k^2)\, e^{a_k|t|} & \text{for } a_k < o,\ -\infty < t < \infty \end{cases}$$

$$g_k^{(2)}(t) = \begin{cases} \dfrac{b_k}{2}\, e^{-b_k|t|} & \text{for } b_k > o,\ -\infty < t < \infty \\ -\dfrac{b_k}{2}\, e^{b_k|t|} & \text{for } b_k < o,\ -\infty < t < \infty \end{cases}$$

and

$$h_k^{(1)}(t) = \int_{-\infty}^{t} g_k^{(1)}(u)\, du + \frac{a_k^2}{c_k^2} J(t),$$

$$h_k^{(2)}(t) = \int_{-\infty}^{t} g_k^{(2)}(u)\, du ,$$

where $J(t)$ is a standard jump - function

$$J(t) = \begin{cases} o & \text{for } t < o \\ \tfrac{1}{2} & \text{for } t = o \\ 1 & \text{for } t > o \end{cases}$$

It can be easily verified that $h_k^{(1)}(t)$ is a normalised distribution function with mean zero and variance $2(a_k^{-2}-c_k^{-2})$, and

$$(3.1) \qquad \int_{-\infty}^{\infty} e^{-st}\, dh_k^{(1)}(t) = [(1-s^2/a_k^2)(1-s^2/c_k^2)]^{-1}$$

the bilateral Laplace transform converging absolutely for Re s $< a_k$ if $a_k > o$ and for Re s $> a_k$ if $a_k < o$, and that $h_k^{(2)}(t)$ is a normalised distribution function with mean zero and variance $2b_k^{-2}$, and

$$(3.2) \qquad \int_{-\infty}^{\infty} e^{-st}\, dh_k^{(2)}(t) = [(1-s^2/b_k^2)]^{-1}$$

the bilateral Laplace transform converging absolutely for Re s $< b_k$ if $b_k > o$ and for Re s $> b_k$ if $b_k < o$.

Theorem 3.1. If

(i) E(s) is defined by (1.3) and satisfies the hypothesis of the theorem 2.1, and

(ii) μ_1 = multiplicity of α_1 as zero of E(s)
μ_2 = multiplicity of α_2 as zero of E(s),

(iii) $$G(t) = (2\pi i)^{-1} \int_{-i\infty}^{i\infty} [E(s)]^{-1} e^{st}\, ds \qquad (-\infty < t < \infty)$$

then

(A) G(t) is a frequency function with mean zero and variance

$$2\left(\sum_{k=1}^{\infty} a_k^{-2} - \sum_{k=1}^{\infty} c_k^{-2}\right) + 2\sum_{k=1}^{\infty} b_k^{-2},$$

(B) $$\int_{-\infty}^{\infty} G(t)\, e^{-st} dt = [E(s)]^{-1}$$

the bilateral Laplace transform converging absolutely in the strip $\alpha_1 <$ Re s $< \alpha_2$, where α_1 and α_2 are given by (2·1),

(C) $$G(t) = p(t) e^{\alpha_1 t} + R_+(t)$$
$$G(t) = q(t) e^{\alpha_2 t} + R_-(t)$$

where p(t) and q(t) are real polynomials of degree $\mu_1 - 1$, $\mu_2 - 1$ respectively and

(D) $$[R_+(t)]^{(n)} = O(e^{(\alpha_1 - E)t}) \qquad (t \to \infty \quad n = o, 1)$$
$$[R_-(t)]^{(n)} = O(e^{(\alpha_2 + E)t}) \qquad (t \to -\infty, \ n = o, 1)$$

for some $\epsilon > o$.

Proof. If we set

$$H_n(t) = h_1^{(1)} \# h_1^{(2)} \# h_2^{(1)} \# h_2^{(2)} \# \ldots\ldots h_n^{(1)} \# h_n^{(2)} (t-b)$$

where the operation $\#$ denotes the Stieltjes convolution for distribution function. Then $H_n(t)$ is a distribution function with mean zero and variance

$$2\left(\sum_{k=1}^{\infty} a_k^{-2} - \sum_{k=1}^{\infty} c_k^{-2}\right) + 2\sum_{k=1}^{\infty} b_k^{-2} .$$

By the conditions (1.4) and (1.5) of § 1, it is easily seen from [9] that $H_n(t) = \lim_{n\to\infty} H_n(t)$ is twice differentiable and $G(t) = d(H(t))/dt$. Hence by [5 : pp. 257] we see that

$$\int_{-\infty}^{\infty} G(t)\, e^{-st}\, dt = [\, E(s)\,]^{-1},$$

the integral converging absolutely for $\alpha_1 < \operatorname{Re} s < \alpha_2$ where α_1 and α_2 are defined by (2.1). From which the conclusions (A) and (B) follow. For conclusions (C) and (D) we refer to [4: pp. 109].

With $E_1(s)$ and $E_2(s)$ given by (1.6) we define

$$G_1(t) = (2\pi i)^{-1} \int_{-i\infty}^{i\infty} [\, E_1(s)\,]^{-1} e^{st}\, ds$$

$$G_2(t) = (2\pi i)^{-1} \int_{-i\infty}^{i\infty} [E_2(s)\,]^{-1} e^{st}\, ds .$$

Then by the conditions $a_k c_k > o$ and $a_k b_k < o$, $G_1(t)$ belongs to class I and $G_2(t)$ belongs to class I, II and III according as

$$\sum_{k=1}^{\infty} c_k^{-2} = \infty \quad \text{and} \quad \sum_{k=1}^{\infty} c_k^{-2} < \infty \quad \text{provided } c_k > o .$$

The following two theorems are well known [4 : pp. 55, pp. 107].

Theorem 3.2.

(A) $G_1(t)$ is a frequency function with mean zero and variance $2\sum_{k=1}^{\infty} (a_k^{-2} + b_k^{-2})$,

(B) $$\int_{-\infty}^{\infty} e^{-st}\, G_1(t)\, dt = [\, E(s)\,]^{-1},$$

(C) $G_1(t) \in C^{\infty}(-\infty, \infty)$,

(D) $$\left(\frac{d}{dt}\right)^n G_1(t) = [P_1(t) e^{\alpha_1 t}]^{(n)} + O(e^{(\alpha_2-\epsilon)t}) \quad (t\to\infty)\ (n=o,1,2,\ldots)$$

$$\left(\frac{d}{dt}\right)^n G_2(t) = [q_1(t) e^{\alpha_2 t}]^{(n)} + O(e^{(\alpha_1+\epsilon)t}) \quad (t\to\infty)\ (n=o,1,2,\ldots)$$

for some $\varepsilon > 0$, where $p_1(t)$ and $q_1(t)$ are real polynomials of degree $\mu_1 - 1$ and $\mu_2 - 1$ respectively.

Theorem 3.3.

(A) $G_2(t)$ is a frequency function with mean zero and variance

$$2 \sum_{k=1}^{\infty} c_k^{-2},$$

(B)

$$\int_{-\infty}^{\infty} e^{st} G_2(t)\, dt = [E_2(s)]^{-1}$$

converging absolutely in the half strip $\gamma_1 < \operatorname{Re} s < \gamma_2$, where γ_1 and γ_2 are defined by (2.2),

(C) $G_2(t) \in C^{\infty}(-\infty, \infty)$

4. Convergence

We now determine the convergence behavior of the transform (1.1).

The following two theorems can be followed by [9].

Theorem 4.1. If $\alpha(t)$ is a function of bounded variation in every finite interval and

$$\int_{-\infty}^{\infty} G(x_0 - t) e^{ct}\, d\alpha(t) \qquad \left(\int_{-\infty}^{\infty} G_1(x_0 - t) e^{ct}\, d\alpha(t) \right)$$

converges (conditionally), then

$$\int_{-\infty}^{\infty} G(x-t) e^{ct}\, d\alpha(t) \qquad \left(G_1(x-t) e^{ct}\, d \quad (t) \right)$$

converges uniformly for every x in any finite interval. Theorem 4.2. The transform $\int_{-\infty}^{\infty} G(x-t) e^{ct}\, d\alpha(t)$ converges if, and only if the transform $\int_{-\infty}^{\infty} G_1(x-t) e^{ct} d\alpha(t)$ converges.

5. Inversion

In this section we give an inversion theorem.

We define

$$E_1(D) = \lim_{n \to \infty} E_{1,n}(D) = \lim_{n \to \infty} \prod_{k=1}^{n} (1 - D^2/a_k^2)(1 - D^2/a_k^2)$$

where D is the operator of differentiation and we interpret e^{tD}, the operator of translation through distance t. On the other hand, by virtue of equation in theorem 3.3 we have

$$[E_2(D)]^{-1}f(x) = \int_{-\infty}^{\infty} f(x-t)\, G_2(t)\, dt$$

whenever the integral converges.

By the condition (1.5) with theorems 3.1 and 3.2, it follows that the bilateral Laplace transforms of G(t) and $G_2(t)$ have a common region of absolute convergence and hence by the product theorem [5] we obtain

$$\frac{1}{E(s)} \cdot \frac{1}{E_2(s)} = \frac{1}{E_1(s)} = \int_{-\infty}^{\infty} e^{-st}G_1(t)\, dt \quad (s = \sigma + i\tau,\ -\infty < \tau < \infty)$$

$$(5.1) \qquad G_1(x) = \int_{-\infty}^{\infty} G(x-t)\, G_2(t)dt \qquad -\infty < x < \infty,$$

both integrals converging absolutely.

From the equation (5.1) and the definition of operator $[E_2(D)]^{-1}$ we have

$$\begin{aligned}[E_2(D)]^{-1}f(x) &= \int_{-\infty}^{\infty} G_2(t)dt \int_{-\infty}^{\infty} G(x-t-u)e^{cu}\, d\alpha(u) \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} G(x-t-u)G_2(t)\, dt \right) e^{cu}\, d\alpha(u) \\ &= \int_{-\infty}^{\infty} G_1(x-u)e^{cu}\, d\alpha(u).\end{aligned}$$

For the change of order of integration performed above see [9].

Now, in view of theorems 4.1 and 4.2, the relation (5.2) and the fact that G(t) and $G_2(t)$ have a common region of absolute convergence, it is sufficient to invert the transform $\int_{-\infty}^{\infty} G_1(x-u)e^{cu}\, d\alpha(u)$ where $G_1(t)$ belongs to class I kernels.

Theorem 5.1. If

(i) $f(x) = \int_{-\infty}^{\infty} G(x-t)e^{ct}\, d\alpha(t)$ converges,

(ii) $\alpha(t)$ is of bounded variation in every finite interval and continuous at x_1, x_2,

then

(A) $\alpha_1 < c < \alpha_2$ implies that

$$\lim_{n\to\infty}\int_{x_1}^{x_2} e^{-cx}\, E_{1,n}(D)\ ([E_2(D)]^{-1}f(x))\ dx = \alpha(x_2) - \alpha(x_1);$$

(B) $c \geqslant \alpha_2$ implies that $\alpha(+\infty)$ exists and that

$$\lim_{n\to\infty}\int_{x_1}^{\infty} e^{-cx}\, E_{1,n}(D)\ ([E_2(D)]^{-1}f(x))\ dx = \alpha(+\infty) - \alpha(x_1);$$

(C) $c \leqslant \alpha_1$ implies that $\alpha(-\infty)$ exists and that

$$\lim_{n\to\infty}\int_{-\infty}^{x_2} e^{-cx}\, E_{1,n}(D)\ ([E_2(D)]^{-1}f(x))\ dx = \alpha(x_2) - \alpha(-\infty).$$

The proof follows from [4: pp. 135].

Theorem 5.2. If

(i) $\phi(t)$ is integrable on every finite interval and is continuous at x,

(ii) $$f(x) = \int_{-\infty}^{\infty} G(x-t)\phi(t)\ dt \quad \text{converges,}$$

then

$$\lim_{n\to\infty} E_{1,n}(D)\ ([E_2(D)]^{-1}f(x)) = \phi(x).$$

The proof follows as in the preceding theorem.

I wish to express my sincere gratitude to my supervisor, Dr. R.P. Manandhar, for the help he has given me during the preparation of this paper.

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Problem Section

A section on problems has been included in each issue of the Report. A total of eight problems were presented for solution in Volume 1, Numbers 1 and 2. Here we provide hints to the solutions of those eight problems. You should use these and attempt to write complete solutions. No new problems will be given at this time.

A. Solutions to problems presented in Vol. I, Issue No. 1:

1. Show that a number consisting of 3^n equal digits is divisible by 3^n.

   Hint: Use induction. The case n=1 is easy. (M_1=aaa) For the next part you show that $M_{n+1} = M_n \cdot k$, where M_n consists of 3^n digits which are equal. M_{n+1} has 3 times as many digits as M_n. Show that k has a digital sum of 3, thus is divisible by 3.

2. Find the smallest natural number with the following property: If the first digit on the left is transferred to the right end, then the new number will be 1.5 times the old number.

   Hint: The answer is 1, 176, 470, 588, 235, 294. To get it, let N be the number. Suppose its base ten representation is $N = da_{n-1}a_{n-2} \ldots a_1 a_0$. Let $M = a_{n-1}a_{n-2}a_{n-3} \ldots a_1 a_0$. Then $N = d.10^n + M$ and $1.5\,N = 10M + d$, by the condition of the problem. Therefore by algebra $17\,M = d\,[3.10^n-2]$. This implies that 17 divides 3.10^n-2. Hence we must find the smallest natural number n such that $3.10^n \equiv 2 \pmod{17}$. To make n as small as possible, we select d = 1.

   Then $M = \frac{3.10^n-2}{17}$ and $N = 10^n + M$. Divide 30000... by 17 until a remainder 2 appears. Then n is found to be 15, and the solution is easily completed.

3. A set of n points in the plane has the following property:

   Any 3-subset of it forms a triangle of area $\leqq$ 1. Show that the set can be enclosed in a triangle of area. $\leqq$ 4.

   Hint: Select a 3-subset $\{A,B,C\}$ of the given set such that the triangle ABC has maximum area. Construct a new triangle $A_1B_1C_1$ such that A, B, and C are the midpoints of its sides. Show that the given set is enclosed by the triangle $A_1B_1C_1$ Use a method of contradiction at the last stage.

4. Let $o<a<b$. Ram chooses a number c in the interval [a,b] and Shyam must guess this number. Let Shyam's guess be d.

   Then the relative error $\frac{|d-c|}{c}$ is his loss to Ram. How should Shyam choose his guess so that his maximum possible loss is as small as possible?

   Hint: The relative error is largest if c=a or c=b. For c=a it is $\frac{d-a}{a}$. For c=b, it is $\frac{b-d}{b}$.

   You must find d such that $\max\limits_{a\le d\le b}\left(\frac{d-a}{a}, \frac{b-d}{b}\right)$ is a minimum.

   If d increases from a to b, then the first number increases and the second one decreases. Thus the minimum occurs when they are equal; that is, when

   $d= \frac{2ab}{a+b}$, the harmonic mean of a and b.

B. Solutions to problems presented in Vol. I, Issue No. 2:

5. (An Ancient Hindu Problem). Three men who had a monkey bought a pile of mangoes. At night one of the men came to the pile of mangoes while the others slept and, finding that there was just one more mango then could be divided exactly by three, tossed the extra mango to the monkey and took away one third of the remainder. Then he went back to sleep. Presently another of them awoke and went to the pile of mangoes. He also found that there were just one too many to be divided evenly by three, so he tossed the extra one to the monkey, took one third of the remainder, and returned to sleep. After a while the third man rose, and he too gave one mango to the monkey and took away the number of whole mangoes which represented precisely one third of the rest. Next morning the men got up and went to the pile. Again they found just one too many, so they gave one to the monkey and divided the rest evenly. What is the least number of mangoes with which this can be done?

   Hint: Let x= number of mangoes. By simple algebra, 1/3 (2x-2) mangoes remain after the first man takes some, (1/9)(4x-10) mangoes remain after the second man takes some, and (1/27)(8x-38) mangoes remain following the action of the third man. One less than this number is divisible by 3. Therefore, the complete solution is given by x=(1/8) (81t+65), where t assumes values so that x is a natural number. If we write x=10t+8+(1/8) (t+1), we wee that

this occurs when $t=8k-1$, for $k=1,2,3,\ldots$ Therefore the answer to the problem is given when $k=1$ and $t=7$, and is $x=79$ mangoes. The next largest answer is 160, and so on.

6. Show that all prime numbers, except 2 and 3, occur as terms of the sequence defined by $a_n=\sqrt{(24n+1)}$ for $n=1,2,3,\ldots$

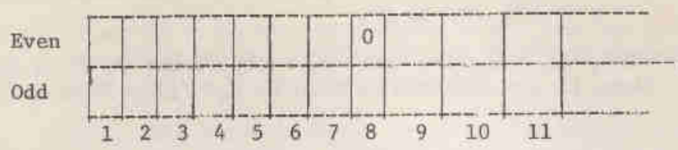
   Hint: Every prime from 5 on has the form $6k+1$ or $6k-1$. This is so since all natural numbers can be partitioned into the classes $6k$, $6k+1$, $6k+2$, $6k+3$, $6k+4$, and $6k+5$, that is, on division by 6 the remainders 0,1,2,3,4, and 5 are the only ones possible. Only the classes $6k+1$ and $6k+5$ can contain prime numbers. Modulo 6, $+5$ and -1 are of the same class. Show that $(6k\pm1)^2$ has the form $24n+1$ for a suitably chosen n. To do this, first find out for what values of n, a_n is a natural number. This is so if there is a natural number q so that $q^2 = 24n+1$. Then $n=\frac{q^2-1}{24}=\frac{(q-1)(q+1)}{24}$. For n to be a natural number, q must be odd. Then $q-1$ and $q+1$ are consecutive even numbers, and one of them is a multiple of 4. That is, the product $(q-1)(q+1)$ is divisible by 8. Also one $q-1$ or $q+1$ is divisible by 3. Thus there is a k so that $q\pm1 = 6k$ or $q=6k\pm1$. Then $n = \frac{k(3k\pm1)}{2}$ for $k=1,2,3,\ldots$ Therefore $a_n=6k\pm1$.

7. Let a,b,c,d, and e be five line segments. Any three of the segments can be used to construct a triangle. Show that at least one of these triangles has all acute angles.

   Hint: Let a,b,c be the sides of a triangle such that $a\geqq b \geqq c$. First show that the triangle is <u>not</u> acute if and only if $a^2\geqq b^2+c^2$. Let $a\geqq b\geqq c\geqq d\geqq e$. Assume that the triangles (a,b,c) and (c,d,e) are both not acute. Use of the inequalities $a^2\geqq b^2+c^2$ and $c^2\geqq d^2+e^2$ leads to $a^2 \geqq d^2+e^2+d^2+e^2 \geqq (d+e)^2+(d-e)^2$. Therefore, $a^2 \geqq (d+e)^2$ and $a\geqq d+e$, which contradicts the triangular inequality $a < d+e$.

8. In a game initially there are $2n+1$ counters. Two players take turns alternately selecting any number of counters from 1 to k. At the end one player has an even number of counters and the other an odd number. The one with the odd number is the winner. Find the losing positions for $k=4$ and for $k=3$.

Hint: Translate the game into a board game with two rows of squares as shown in the figure.



The position of the checker in the figure indicate that there are 8 counters in the pile, and the player to move next has an even number of counters. If he had an odd number of counters, the checker would be in the second (odd) row. The losing positions are

$k=4$ $(6i+5, \text{ODD})$, $(6i, \text{ODD})$, $(6i+1, \text{EVEN})$, for

$i=o,1,2,3,\ldots$

$k=3$ $(8i, \text{ODD})$, $(8i+5, \text{ODD})$, $(8i+1, \text{EVEN})$,

$(8i+4, \text{EVEN})$, for $i=0,1,2,\ldots$