Abstract: In this paper, the concepts of an operation $\gamma$ on a family of fuzzy pre-open sets in a fuzzy topological spaces $(X, T)$ is introduced. Using the operation $\gamma$ on FPO $(X)$ the concepts of fuzzy pre-$\gamma$-open sets, fuzzy pre-$\gamma$-border, fuzzy pre-$\gamma$-frontier, fuzzy pre-$(\gamma, \beta)$-continuous mappings, fuzzy pre-$\gamma$-normal spaces and fuzzy pre-$\gamma$-compact spaces are introduced. Some interesting properties and characterizations of them are investigated. Further, fuzzy pre-$\gamma$-$R_0$ and fuzzy pre-$\gamma$-$T_i$ $(i = 0, 1/2, 1, 2)$ spaces are introduced and interrelations among the spaces are discussed with relevant examples.

Key Words
Fuzzy pre-$\gamma$-open set, fuzzy pre-$\gamma$-border, fuzzy pre-$\gamma$-frontier, fuzzy pre-$(\gamma, \beta)$-continuous mapping, fuzzy pre-$\gamma$-normal space, fuzzy pre-$\gamma$-compact space, fuzzy pre-$\gamma$-$R_0$ space, fuzzy pre-$\gamma$-$T_i$ $(i = 0, 1/2, 1, 2)$ space.


1. INTRODUCTION AND PRELIMINARIES
The concept of fuzzy sets has invaded almost all branches of mathematics since the introduction of the concept by Zadeh [11]. Fuzzy sets have applications in many fields such as information [7] and control [8]. The theory of fuzzy topological spaces was introduced and developed by Chang [3]. The concept of
fuzzy pre-open sets and fuzzy pre-closed sets were introduced by Singal and Prakash [6]. The concept of fuzzy pre-continuity was introduced by Bin Shahna [1] and was studied by Uma, Roja and Balasubramanian [10]. By using the concepts of semi-γ-open sets introduced by Sai Sundara Krishnan, Ganster and Balachandran [4] and that of g-border and g-frontier introduced by Caldas, Jafari and Noiri [2], the concepts of fuzzy pre-γ-open set, fuzzy pre-γ-border, pre-γ-frontier, fuzzy pre-(γ, β)-continuous mappings, fuzzy pre-γ-normal spaces, fuzzy pre-γ-compact spaces, fuzzy pre-γ-T_i (i = 0, 1, 2) spaces and fuzzy pre-γ-R_0 space are introduced and interrelations among the spaces are discussed with relevant examples.

**Definition 1.1 [6]**

Let (X, T) be a fuzzy topological space. A fuzzy set λ in (X, T) is said to be fuzzy pre-open if λ ⊆ int cl (λ).

The complement of a fuzzy pre-open set is fuzzy pre-closed.

**Definition 1.2 [4]**

Let (X, T) be a fuzzy topological space. An operation γ on the topology T is a mapping from T into power set P(X) of X such that V ⊆ V' for each V ∈ T, where V' denotes the value of γ at V. It is denoted by γ : T → P(X).

**Definition 1.3 [4]**

A subset A of a topological space is called a γ-open set of (X, T) if for each x ∈ A there exists an open set U such that x ∈ U and U' ⊆ A. The complement of a γ-open set is said to be γ-closed.

**Notation 1.1 [4]**

SO(X) denotes the family of all semi-open sets of (X, T).

**Definition 1.4 [4]**

Let (X, T) be a topological space. An operation γ on the SO(X) is a mapping from SO(X) into a power set P(X) of X such that V ⊆ V' for each V ∈ SO(X) and V' denotes the value of γ at V. It is denoted by γ : SO(X) → P(X).

**Definition 1.5 [4]**

Let (X, T) be a topological space and γ be an operation on SO(X). Then a subset A of X is said to be a semi-γ-open set if for each x ∈ A, there exists a semi-open set U such that x ∈ U and U' ⊆ A. Also SO(X)_γ denotes the family of semi-γ-open sets in X.
Definition 1.6 [2]
Let \((X, T)\) be a topological space. For a subset \(A\) of \((X, T)\), \(b_g (A) = A - \text{int}_g (A)\) is said to be the \(g\)-border of \(A\) where \(\text{int}_g (A)\) is the set of all \(g\)-interior points of \(A\).

Definition 1.7 [2]
Let \((X, T)\) be a topological space. For a subset \(A\) of \((X, T)\), \(\text{Fr}_g (A) = \text{cl}_g (A) - \text{int}_g (A)\) is said to be the \(g\)-frontier of \(A\).

Definition 1.8 [9]
A topological space \((X, T)\) is said to be a fuzzy \(\text{pre-}T_{1/2}\) space if every \(gf\)-pre-closed set in \((X, T)\) is fuzzy closed in \((X, T)\).

Definition 1.9 [5]
A fuzzy set \(\lambda\) is quasi-coincident with a fuzzy set \(\mu\), denoted by \(\lambda \equiv \mu\), if there exists \(x \in X\) such that \(\lambda (x) + \mu (x) > 1\). Otherwise \(\lambda \not\equiv \mu\).

2. FUZZY PRE-\(\gamma\)-OPEN SETS

Definition 2.1
Let \((X, T)\) be a fuzzy topological space. Let \(\gamma : I^X \rightarrow T\) be an operation such that \(\lambda^\gamma = \land \mu\) where \(\lambda \leq \mu\), for each fuzzy open set \(\mu\) in \((X, T)\), \(\lambda \in I^X\) and \(\lambda^\gamma\) denotes the value of \(\gamma\) at \(\lambda\). That is, \(\lambda^\gamma = \gamma (\lambda)\).

Definition 2.2
Let \((X, T)\) be a fuzzy topological space. Let \(\gamma : I^X \rightarrow T\) be an operation. A fuzzy set \(\delta\) is said to be fuzzy-\(\gamma\)-open if for a fuzzy set \(\alpha\) with \(\alpha \leq \delta\), there exists a fuzzy open set \(\lambda\) such that \(\alpha \leq \lambda\) and \(\lambda^\gamma \leq \delta\).

The complement of a fuzzy \(\gamma\)-open-set is fuzzy-\(\gamma\)-closed.

Definition 2.3
Let \((X, T)\) be a fuzzy topological space. Let \(\gamma : I^X \rightarrow T\) be an operation. For any fuzzy set \(\lambda\), fuzzy-\(\gamma\)-interior of \(\lambda\) (briefly, \(\gamma\text{-int} (\lambda)\)) is defined as \(\gamma\text{-int} (\lambda) = \lor \{ \mu : \mu \leq \lambda \text{ and } \mu \text{ is fuzzy-\(\gamma\)-open} \}\).

Definition 2.4
Let \((X, T)\) be a fuzzy topological space. Let \(\gamma : I^X \rightarrow T\) be an operation. For any fuzzy set \(\lambda\), fuzzy-\(\gamma\)-closure of \(\lambda\) (briefly, \(\gamma\text{-cl} (\lambda)\)) is defined as \(\gamma\text{-cl} (\lambda) = \land \{ \mu : \mu \geq \lambda \text{ and } \mu \text{ is fuzzy-\(\gamma\)-closed} \}\).

Remark 2.1
\(\gamma \text{-int}(1 - \lambda) = 1 - (\gamma \text{-cl}(\lambda)).\)

**Notation 2.1**

FPO \((X)\) denotes the family of all fuzzy pre-open sets of \((X, T)\).

**Definition 2.5**

Let \((X, T)\) be a fuzzy topological space. Let \(\gamma : \text{FPO}(X) \rightarrow T\) be an operation such that \(\lambda^\gamma = \land \mu\), where \(\lambda \leq \mu\), for each fuzzy open set \(\mu\) in \((X, T)\) and \(\lambda \in \text{FPO}(X)\).

**Definition 2.6**

Let \((X, T)\) be a fuzzy topological space. Let \(\gamma\) be an operation on \(\text{FPO}(X)\). A fuzzy set \(\delta\) is called fuzzy pre-\(\gamma\)-open if for a fuzzy set \(\alpha\) with \(\alpha \leq \delta\), there exists a fuzzy pre-open set \(\lambda\) such that \(\alpha \leq \lambda\) and \(\lambda^\gamma \leq \delta\). The complement of a fuzzy pre-\(\gamma\)-open set is fuzzy pre-\(\gamma\)-closed.

**Definition 2.7**

Let \((X, T)\) be a fuzzy topological space. Let \(\gamma\) be an operation on \(\text{FPO}(X)\). The fuzzy pre-\(\gamma\)-interior of \(\delta\) (briefly, \(\gamma\text{-fp int}(\delta)\)) is defined by \(\gamma\text{-fp int}(\delta) = \lor \{ \mu : \mu \leq \delta\text{ and } \mu\text{ is fuzzy pre-}\gamma\text{-open} \}\).

**Definition 2.8**

Let \((X, T)\) be a fuzzy topological space. Let \(\gamma\) be an operation on \(\text{FPO}(X)\). The fuzzy pre-\(\gamma\)-closure of \(\delta\) (briefly, \(\gamma\text{-fp cl}(\delta)\)) is defined by \(\gamma\text{-fp cl}(\delta) = \land \{ \mu : \mu \geq \delta\text{ and } \mu\text{ is fuzzy pre-}\gamma\text{-closed} \}\).

**Remark 2.2**

\(\gamma\text{-fp int}(1 - \delta) = 1 - (\gamma\text{-fp cl}(\delta)).\)

**Remark 2.3**

Fuzzy pre-open set and fuzzy pre-\(\gamma\)-open set are independent notions.

**Example 2.1**

Let \(X = \{ a, b \}\). Define \(T = \{ 0, 1, \lambda_1, \lambda_2 \}\) where \(\lambda_1, \lambda_2 : X \rightarrow [0, 1]\) are defined as \(\lambda_1(a) = 0.3, \lambda_1(b) = 0.2, \lambda_2(a) = 0.45, \lambda_2(b) = 0.4\). Let \(\gamma : \text{FPO}(X) \rightarrow T\) be an operation. Let \(\mu, \delta, \eta : X \rightarrow [0, 1]\) be defined as \(\mu(a) = 0.4, \mu(b) = 0.3, \delta(a) = 0.45, \delta(b) = 0.3, \eta(a) = 0.55, \eta(b) = 0.65\). Now \(\text{cl}(\mu) \geq \mu\). Hence \(\mu\) is fuzzy pre-open but not fuzzy pre-\(\gamma\)-open. Now, for a fuzzy set \(\alpha\) with \(\alpha \leq \eta\), then \(\alpha \leq \mu\) and \(\mu' \leq \eta\). Hence \(\eta\) is fuzzy pre-\(\gamma\)-open but not fuzzy pre-open.
Let \((X, T)\) be a fuzzy topological space. Let \(\lambda\) and \(\mu\) be any two fuzzy pre-\(\gamma\)-open sets in \((X, T)\). Then \(\lambda \vee \mu\) (resp. \(\lambda \wedge \mu\)) is also a fuzzy pre-\(\gamma\)-open set in \((X, T)\).

**Proposition 2.2**

Let \((X, T)\) be a fuzzy topological space. For any two fuzzy sets \(\lambda, \mu\), the following statements hold:

a. If \(\lambda\) is fuzzy-\(\gamma\)-open then \(\lambda\) is fuzzy pre-\(\gamma\)-open.

b. \(\gamma\)-int \((\lambda)\) is fuzzy pre-\(\gamma\)-open.

c. \(\gamma\)-cl \((\lambda)\) is fuzzy pre-\(\gamma\)-closed.

d. is fuzzy pre-\(\gamma\)-open iff \(\lambda = \gamma\)-fp int \((\lambda)\).

e. is fuzzy pre-\(\gamma\)-closed iff \(\lambda = \gamma\)-fp cl \((\lambda)\).

f. \(\gamma\)-int \((\lambda)\) \(\leq\) \(\gamma\)-fp int \((\lambda)\) \(\leq\) \(\lambda\) \(\leq\) \(\gamma\)-fp cl \((\lambda)\) \(\leq\) \(\gamma\)-cl \((\lambda)\).

g. \(\gamma\)-cl \((\gamma\)-fp cl \((\lambda)\)) = \(\gamma\)-fp cl \((\lambda)\).

h. \(\gamma\)-cl \((\gamma\)-fp cl \((\lambda)\)) = \(\gamma\)-fp cl \((\gamma\)-cl \((\lambda)\)) = \(\gamma\)-cl \((\lambda)\).

i. \((\gamma\)-fp int \((\lambda)\)) \(\wedge\) \((\gamma\)-fp int \((\mu)\)) \(\geq\) \(\gamma\)-fp int \((\lambda \wedge \mu)\).

j. \((\gamma\)-fp int \((\lambda)\)) \(\vee\) \((\gamma\)-fp int \((\mu)\)) \(\leq\) \(\gamma\)-fp int \((\lambda \vee \mu)\).

**Definition 2.9**

Let \((X, T)\) be a fuzzy topological space and let \(\gamma : I^X \rightarrow T\) be an operation. For any fuzzy set \(\lambda\), fuzzy-\(\gamma\)-border of \(\lambda\) (briefly, \(\gamma\)-fb \((\lambda)\)) is defined as \(\gamma\)-fb \((\lambda)\) = \(\lambda\) \(-\) \((\gamma\)-int \((\lambda)\))

**Definition 2.10**

Let \((X, T)\) be a fuzzy topological space and let \(\gamma\) be an operation on FPO \((X)\) for any fuzzy set \(\lambda\), fuzzy pre-\(\gamma\)-border of \(\lambda\) (briefly, \(\gamma\)-fpb \((\lambda)\)) is defined as \(\gamma\)-fpb \((\lambda)\) = \(\lambda\) \(-\) \((\gamma\)-fp int \((\lambda)\))

**Definition 2.11**

Let \((X, T)\) be a fuzzy topological space and let \(\gamma : I^X \rightarrow T\) be an operation. For any fuzzy set \(\lambda\), fuzzy-\(\gamma\)-frontier of \(\lambda\) (briefly, \(\gamma\)-Fr \((\lambda)\)) is defined as \(\gamma\)-Fr \((\lambda)\) = \((\gamma\)-cl \((\lambda)\)) \(\wedge\) \((\gamma\)-int \((\lambda)\)).

**Definition 2.12**

Let \((X, T)\) be a fuzzy topological space and let \(\gamma\) be an operation on FPO \((X)\). For any fuzzy set \(\lambda\), fuzzy pre-\(\gamma\)-frontier of \(\lambda\) (briefly, \(\gamma\)-fp Fr \((\lambda)\)) is defined as \(\gamma\)-fp Fr \((\lambda)\) = \((\gamma\)-fp cl \((\lambda)\)) \(\wedge\) \((\gamma\)-fp int \((\lambda)\)).
Proposition 2.3
Let \((X, T)\) be a fuzzy topological space. For any two fuzzy sets \(\lambda, \mu\) the following statements hold:

a. \(\gamma\text{-fpb} (\lambda) \leq \gamma\text{-fp cl} (1 - \lambda)\).

b. \(\gamma\text{-fpb} (\lambda \lor \mu) \leq (\gamma\text{-fpb} (\lambda)) \lor (\gamma\text{-fpb} (\mu))\).

c. \(\gamma\text{-fpb} (\lambda \land \mu) \geq (\gamma\text{-fpb} (\lambda)) \land (\gamma\text{-fpb} (\mu))\).

d. \((\gamma\text{-int} (\lambda)) \lor (\gamma\text{-fb} (\lambda)) \geq \gamma\text{-int} (\lambda)\).

e. \((\gamma\text{-int} (\lambda)) \land (\gamma\text{-fb} (\lambda)) \leq \gamma\text{-int} (\lambda)\).

f. \(\gamma\text{-fp Fr} (\lambda) = \gamma\text{-fp Fr} (1 - \lambda)\).

g. \(\gamma\text{-fp Fr} (\gamma\text{-fp int} (\lambda)) \leq \gamma\text{-fp Fr} (\lambda)\).

h. \(\gamma\text{-fp Fr} (\gamma\text{-fp cl} (\lambda)) \leq \gamma\text{-fp Fr} (\lambda)\).

i. \(\lambda - (\gamma\text{-fp Fr} (\lambda)) \leq \gamma\text{-fp int} (\lambda)\).

j. \(\gamma\text{-fp Fr} (\lambda \lor \mu) \leq (\gamma\text{-fp Fr} (\lambda)) \lor (\gamma\text{-fp Fr} (\mu))\).

k. \(\gamma\text{-fp Fr} (\lambda \land \mu) \geq (\gamma\text{-fp Fr} (\lambda)) \land (\gamma\text{-fp Fr} (\mu))\).

3. FUZZY PRE-\(\gamma\)-\(T_1\) SPACES

Definition 3.1
A fuzzy topological space \((X, T)\) is called

a fuzzy pre-\(\gamma\)-\(T_0\) space iff for any two fuzzy sets \(\lambda, \mu\) with \(\lambda \not\leq \mu\), there exists a fuzzy pre-\(\gamma\)-open set \(\delta\) such that \(\lambda \leq \delta, \mu \not\leq \delta\) or \(\mu \leq \delta, \lambda \not\leq \delta\).

b. a fuzzy pre-\(\gamma\)-\(T_1\) space iff for any two fuzzy sets \(\lambda, \mu\) with \(\lambda \not\leq \mu\), there exist fuzzy pre-\(\gamma\)-open sets \(\delta, \eta\) such that either \(\lambda \leq \delta, \mu \not\leq \delta\) or \(\mu \leq \eta, \lambda \not\leq \eta\).

c. a fuzzy pre-\(\gamma\)-\(T_2\) space iff for any two fuzzy sets \(\lambda, \mu\) with \(\lambda \not\leq \mu\), there exist fuzzy pre-\(\gamma\)-open sets \(\delta, \eta\) such that \(\lambda \leq \delta, \mu \leq \eta\) and \(\delta \not\leq \eta\).

d. a fuzzy pre-\(\gamma\)-\(R_0\) space iff for any two fuzzy sets \(\lambda, \mu, \lambda \not\leq (\gamma\text{-fp cl} (\mu))\) implies that \(\mu \not\leq (\gamma\text{-fp cl} (\lambda))\).

Definition 3.2
Let \((X, T)\) be a fuzzy topological space and let \(\gamma\) be an operation on FPO\((X)\). A fuzzy set \(\lambda\) is called fuzzy pre-\(\gamma\)-\(g\) closed if \(\gamma\text{-fp cl} (\lambda) \leq \mu\) whenever \(\lambda \leq \mu\) and \(\mu\) is fuzzy pre-\(\gamma\)-open.

The complement of a fuzzy pre-\(\gamma\)-gclosed set is fuzzy pre-\(\gamma\)-g open.
Definition 3.3
A fuzzy topological space \((X, T)\) is called fuzzy pre-\(\gamma\)-\(T_{1/2}\) space if every fuzzy pre-\(\gamma\)-\(g\) closed set is fuzzy pre-\(\gamma\)-closed.

Remark 3.1
From the above definitions we have the following implications.

\[ \text{fuzzy pre-}\gamma\text{-}T_2 \space \Rightarrow \space \text{fuzzy pre-}\gamma\text{-}T_1 \space \Rightarrow \space \text{fuzzy pre-}\gamma\text{-}T_{1/2} \space \Rightarrow \space \text{fuzzy pre-}\gamma\text{-}T_0 \]  

The converse statements need not be true, as shown in the following examples.

Example 3.1
Let \(X = \{a, b\}\). Define \(T = \{0, 1, \lambda_1, \lambda_2, \lambda_3\}\) where \(\lambda_1, \lambda_2, \lambda_3 : X \rightarrow [0, 1]\) are defined as \(\lambda_1(a) = 0.5, \lambda_1(b) = 0.2, \lambda_2(a) = 0.6, \lambda_2(b) = 0.7, \lambda_3(a) = 0.8, \lambda_3(b) = 0.9\). Let \(\gamma : \text{FPO}(X) \rightarrow T\) be an operation. Let \(\alpha, \mu, \delta, \eta : X \rightarrow [0, 1]\) be defined as \(\alpha(a) = 0.5, \alpha(b) = 0.4, \mu(a) = 0.5, \mu(b) = 0.6, \delta(a) = 0.5, \delta(b) = 0.6, \eta(a) = 0.65, \eta(b) = 0.85\). Clearly \(\mu\) is a fuzzy pre-open set. Now, \(\alpha \leq \eta, \alpha \leq \delta\) and \(\alpha \geq \mu\). Further \(\mu' \leq \delta\) and \(\mu' \leq \eta\). Therefore \(\delta\) and \(\eta\) are fuzzy pre-\(\gamma\)-open sets. Let \(\theta, \lambda : X \rightarrow [0, 1]\) be such that \(\theta(a) = 0.3, \theta(b) = 0.1, \lambda(a) = 0.2, \lambda(b) = 0.7\). Then \(\theta \notin \lambda\). Further \(\theta \leq \delta, \lambda : \notin \delta\) and \(\lambda \leq \eta, \theta \notin \eta\). Hence \((X, T)\) is a fuzzy pre-\(\gamma\)-\(T_1\) space but not a fuzzy pre-\(\gamma\)-\(T_2\) space.

Example 3.2
Let \(X = \{a, b\}\). Define \(T = \{0, 1, \lambda_1, \lambda_2, \lambda_3\}\) where \(\lambda_1, \lambda_2, \lambda_3 : X \rightarrow [0, 1]\) are defined as \(\lambda_1(a) = 0.5, \lambda_1(b) = 0.6, \lambda_2(a) = 0.7, \lambda_2(b) = 0.75, \lambda_3(a) = 0.8, \lambda_3(b) = 0.9\). Let \(\gamma : \text{FPO}(X) \rightarrow T\) be an operation. The space \((X, T)\) is a fuzzy pre-\(\gamma\)-\(T_{1/2}\) space but not a fuzzy pre-\(\gamma\)-\(T_1\) space.

Example 3.3
Let \(X = \{a, b\}\). Define \(T = \{0, 1, \lambda_1, \lambda_2\}\) where \(\lambda_1, \lambda_2 : X \rightarrow [0, 1]\) are defined as \(\lambda_1(a) = 0.3, \lambda_1(b) = 0.2, \lambda_2(a) = 0.45, \lambda_2(b) = 0.4\). Let \(\gamma : \text{FPO}(X) \rightarrow T\) be an operation. Let \(\alpha, \mu, \delta, \eta : X \rightarrow [0, 1]\) be defined as \(\alpha(a) = 0.2, \alpha(b) = 0.3, \mu(a) = 0.4, \mu(b) = 0.5, \delta(a) = 0.55, \delta(b) = 0.65\). Clearly \(\mu\) is a fuzzy pre-open set. Now, \(\alpha \leq \delta, \alpha \leq \mu\) and \(\mu' \leq \delta\). Therefore \(\delta\) is a fuzzy pre-\(\gamma\)-open set. Let \(\theta, \rho : X \rightarrow [0, 1]\) be defined as \(\theta(a) = 0.3, \theta(b) = 0.4, \rho(a) = 0.4, \rho(b) = 0.2\). Then \(\theta \notin \rho\). Now, \(\theta \leq \delta\) and \(\rho \notin \delta\). Hence \((X, T)\) is a fuzzy pre-\(\gamma\)-\(T_0\) space. Let \(\lambda : X \rightarrow [0, 1]\) be defined as \(\lambda(a) = 0.5, \lambda(b) = 0.45\). Now, \(\lambda \leq \delta\) and \(\gamma-\)
Therefore \( \lambda \) is a fuzzy pre-\( \gamma \)-g closed set. But not a fuzzy pre-\( \gamma \)-closed set. Hence \((X, T)\) is not a fuzzy pre-\( \gamma \)-T1\(_2\) space.

**Proposition 3.1**

Let \((X, T)\) be a fuzzy topological space. Then

(a) for all fuzzy pre-\( \gamma \)-open set \( \lambda \) in \((X, T)\), \( \lambda \preceq \mu \) iff \( \lambda \preceq (\gamma \text{-fp cl} (\mu)) \), where \( \mu \) is any fuzzy set in \((X, T)\).

(b) \( \delta \preceq (\gamma \text{-fp cl} (\lambda)) \) iff \( \lambda \preceq \mu \), for all fuzzy pre-\( \gamma \)-open set \( \mu \) in \((X, T)\), with \( \delta \leq \mu \).

**Proof:**

(a) Let \( \lambda \) be a fuzzy pre-\( \gamma \)-open set such that \( \lambda \preceq \mu \). Then since \( \mu \leq (\gamma \text{-fp cl} (\mu)) \), \( \lambda \preceq (\gamma \text{-fp cl} (\mu)) \). Conversely let \( \lambda \) be a fuzzy pre-\( \gamma \)-open set in \((X, T)\) such that \( \lambda \preceq \mu \). Then \( \mu \leq 1 - \lambda \) and so \( (\gamma \text{-fp cl} (\mu)) \leq (1 - \lambda) = 1 - \lambda \). Thus \( \lambda \preceq (\gamma \text{-fp cl} (\mu)) \).

(b) Let \( \delta \preceq (\gamma \text{-fp cl} (\lambda)) \) and let \( \mu \) be a fuzzy pre-\( \gamma \)-open set in \((X, T)\) such that \( \delta \leq \mu \). Then \( \mu \preceq (\gamma \text{-fp cl} (\lambda)) \). By (a), \( \mu \preceq \lambda \) for all fuzzy pre-\( \gamma \)-open set \( \mu \) with \( \delta \leq \mu \). Conversely suppose that \( \delta \preceq (\gamma \text{-fp cl} (\lambda)) \). Then \( \delta \leq 1 - (\gamma \text{-fp cl} (\lambda)) \). Let \( \mu = 1 - (\gamma \text{-fp cl} (\lambda)) \). Then \( \mu \) is a fuzzy pre-\( \gamma \)-open set with \( \delta \leq \mu \). Since \( \lambda \leq (\gamma \text{-fp cl} (\lambda)) \), \( \mu = 1 - (\gamma \text{-fp cl} (\lambda)) \leq 1 - \lambda \). Therefore \( \lambda \preceq \mu \).

**Proposition 3.2**

Let \((X, T)\) be a fuzzy topological space. For any two fuzzy sets \( \delta, \rho \) in \((X, T)\), the following statements are equivalent:

(a) \((X, T)\) is a fuzzy pre-\( \gamma \)-R\(_0\) space.

(b) If \( \delta \preceq (\gamma \text{-fp cl} (\lambda)) \), where \( \lambda \) is any fuzzy set in \((X, T)\), then there exists a fuzzy pre-\( \gamma \)-open set \( \mu \) in \((X, T)\), such that \( \delta \preceq \mu \) and \( \lambda \leq \mu \).

(c) If \( \delta \preceq (\gamma \text{-fp cl} (\lambda)) \) then \( (\gamma \text{-fp cl} (\delta)) \preceq (\gamma \text{-fp cl} (\lambda)) \), where \( \lambda \) is any fuzzy set in \((X, T)\).

(d) If \( \delta \preceq (\gamma \text{-fp cl} (\rho)) \) then \( (\gamma \text{-fp cl} (\delta)) \preceq (\gamma \text{-fp cl} (\rho)) \).

**Proof:**

(a) \( \Rightarrow \) (b) Let \( \delta \preceq (\gamma \text{-fp cl} (\lambda)) \). Since \( \gamma \text{-fp cl} (\rho) \preceq (\gamma \text{-fp cl} (\lambda)) \), for each \( \rho \leq \lambda \), \( \delta \preceq (\gamma \text{-fp cl} (\rho)) \). Then by (a), \( \rho \preceq (\gamma \text{-fp cl} (\delta)) \). Then by (b) of Proposition 3.1, there exists a fuzzy pre-\( \gamma \)-open set \( \eta \) in \((X, T)\), such that \( \delta \preceq \eta \) and \( \rho \leq \eta \). Let \( \mu = \lor \{ \eta : \delta \preceq \eta \} \). Then \( \delta \preceq \mu \) and \( \lambda \leq \mu \), where \( \mu \) is fuzzy pre-\( \gamma \)-open in \((X, T)\).
(b) $\Rightarrow$ (c) Let $\delta \triangleleft \lambda = \gamma$-fp cl $(\chi)$. Then by (b), there exists a fuzzy pre-$\gamma$-open set $\mu$ in $(X, T)$, such that $\delta \triangleleft \mu$ and $\lambda \leq \mu$. Since $\delta \triangleleft \mu$, $\delta \leq 1 - \mu$. Therefore $\gamma$-fp cl $(\delta) \leq \gamma$-fp cl $(1 - \mu) = 1 - \mu \leq 1 - \lambda$.

Hence $(\gamma$-fp cl $(\delta)) \triangleleft \lambda = \gamma$-fp cl $(\lambda)$.

(c) $\Rightarrow$ (d) Let $\delta \triangleleft (\gamma$-fp cl $(\rho))$. since $\gamma$-fp cl $(\gamma$-fp cl $(\rho)) = \gamma$-fp cl $(\rho)$, by (c), $\gamma$-fp cl $(\delta) \triangleleft (\gamma$-fp cl $(\rho))$.

(d) $\Rightarrow$ (a) Let $\delta \triangleleft (\gamma - \text{fp cl}(\rho))$. Then by (d),

$\gamma$-fp cl $(\delta)) \triangleleft (\gamma$-fp cl $(\rho))$. Since $\rho \leq \gamma$-fp cl $(\rho)$, $\rho \triangleleft (\gamma$-fp cl $(\delta))$. Hence $(X, T)$ is a fuzzy pre-$\gamma$-R_0 space.

4. Fuzzy pre-$(\gamma, \beta)$-CONTINUOUS MAPPINGS

Let $(X, T), (Y, S)$ and $(Z, R)$ be any three fuzzy topological spaces and let $\gamma : \text{FP0}(X) \to T$, $\beta : \text{FP0}(Y) \to T$ and $\eta : \text{FP0}(Z) \to T$ be operations on $\text{FP0}(X), \text{FP0}(Y)$ and $\text{FP0}(Z)$ respectively.

Definition 4.1

Let $f : (X, T) \to (Y, S)$ be a mapping. Then

(a) $f$ is called fuzzy pre-$(\gamma, \beta)$-continuous iff for each fuzzy pre-$\beta$-open set $\mu$ in $(Y, S)$, $f^{-1}(\mu)$ is fuzzy pre-$\gamma$-open.

(b) $f$ is called fuzzy pre-$(\gamma, \beta)$-closed iff for each fuzzy pre-$\gamma$-closed set $\lambda$ in $(X, T)$, $f(\lambda)$ is fuzzy pre-$\beta$-closed.

(c) $f$ is called fuzzy pre-$(\gamma, \beta)$-g continuous iff for each fuzzy pre-$\beta$-g closed set $\mu$ in $(Y, S)$, $f^{-1}(\mu)$ is fuzzy pre-$\gamma$-g closed.

(d) $f$ is called fuzzy pre-$(\gamma, \beta)$-g closed iff for each fuzzy pre-$\gamma$-g closed set $\lambda$ in $(X, T)$, $f(\lambda)$ is fuzzy pre-$\beta$-g closed.

Proposition 4.1

A mapping $f : (X, T) \to (Y, S)$ is fuzzy pre-$(\gamma, \beta)$-continuous iff $f(\gamma$-fp cl $(\lambda)) \leq \beta$-fp cl $(f(\lambda))$, for each fuzzy set $\lambda$ in $(X, T)$.

Proposition 4.2

A mapping $f : (X, T) \to (Y, S)$ is fuzzy pre-$(\gamma, \beta)$-continuous iff $\gamma$-fp cl $(f^{-1}(\lambda)) \leq f^{-1}(\beta$-fp cl $(\lambda))$, for each fuzzy set $\lambda$ in $(Y, S)$.

Proposition 4.3
Let \( f : (X, T) \to (Y, S) \) be a fuzzy pre-\((\gamma, \beta)\)-continuous and \( g : (Y, S) \to (Z, R) \) be a fuzzy pre-\((\beta, \eta)\)-continuous mappings. Then \( g \circ f : (X, T) \to (Z, R) \) is fuzzy pre-\((\gamma, \eta)\)-continuous.

**Proposition 4.4**

Let \( f : (X, T) \to (Y, S) \) be a mapping. Then \( f \) is a fuzzy pre-\((\gamma, \beta)\)-closed mapping if \( \beta \)-fp cl \( (f(\lambda)) \) \( \leq \) \( f(\gamma \)-fp cl \( (\lambda) \) \), for each fuzzy set \( \lambda \) in \( (X, T) \).

**Definition 4.2**

Let \( f : (X, T) \to (Y, S) \) be a bijective mapping. If both \( f \) and \( f^{-1} \) are fuzzy pre-\((\gamma, \beta)\)-continuous, then \( f \) is called a fuzzy pre-\((\gamma, \beta)\)-homeomorphism.

**Proposition 4.5**

Let \( f : (X, T) \to (Y, S) \) be a bijective mapping. Then the following statements are equivalent:

(a) \( f \) is a fuzzy pre-\((\gamma, \beta)\)-homeomorphism.

(b) \( f \) is a fuzzy pre-\((\gamma, \beta)\)-continuous and fuzzy pre-\((\gamma, \beta)\)-open mapping.

(c) \( f \) is a fuzzy pre-\((\gamma, \beta)\)-continuous and fuzzy pre-\((\gamma, \beta)\)-closed mapping.

(d) \( f(\gamma \)-fp cl \( (\lambda) \) \( = \) \( \beta \)-fp cl \( f(\lambda) \) \), for each fuzzy set \( \lambda \) in \( (X, T) \).

**Proposition 4.6**

Let \( f : (X, T) \to (Y, S) \) be a fuzzy pre-\((\gamma, \beta)\)-continuous, fuzzy pre-\((\gamma, \beta)\)-g continuous and fuzzy pre-\((\gamma, \beta)\)-g closed mapping. Then the following statements hold:

(a) If \( f \) is injective and \((Y, S)\) is a fuzzy pre-\(\beta-T_{1/2}\) space, then \((X, T)\) is a fuzzy pre-\(\gamma-T_{1/2}\) space.

(b) If \( f \) is surjective and \((X, T)\) is a fuzzy pre-\(\gamma-T_{1/2}\) space, then \((Y, S)\) is a fuzzy pre-\(\beta-T_{1/2}\) space.

**Proof:**

(a) Let \( \lambda \) be a fuzzy pre-\(\gamma\)-g closed set in \((X, T)\). Since \( f \) is fuzzy pre-\((\gamma, \beta)\)-g closed, \( f(\lambda) \) is fuzzy pre-\(\beta\)-g closed. Since \((Y, S)\) is a fuzzy pre-\(\beta-T_{1/2}\) space, \( f(\lambda) \) is fuzzy pre-\(\beta\)-closed. Since \( f \) is fuzzy pre-\((\gamma, \beta)\)-continuous, \( f^{-1}(f(\lambda)) \) is fuzzy pre-\(\gamma\)-closed. Hence \((X, T)\) is a fuzzy pre-\(\gamma-T_{1/2}\) space.
b) Let $\mu$ be a fuzzy pre-$\beta$-g closed set in $(Y, S)$. Since $f$ is fuzzy pre-$\gamma$-$\beta$-g continuous, $f^{-1}(\mu)$ is a fuzzy pre-$\gamma$-g closed set. Since $(X, T)$ is a fuzzy pre-$\gamma$-T$_{1/2}$ space, $f^{-1}(\mu)$ is fuzzy pre-$\gamma$-closed. Therefore $\mu = f(f^{-1}(\mu))$ is a fuzzy pre-$\beta$-closed set. Hence $(Y, S)$ is a fuzzy pre-$\beta$-T$_{1/2}$ space.

**Proposition 4.7**

Let $f : (X, T) \rightarrow (Y, S)$ be a fuzzy pre-$\gamma$-$\beta$-continuous injective mapping. If $(Y, S)$ is a fuzzy pre-$\beta$-T$_2$ (resp. fuzzy pre-$\beta$-T$_1$) space then $(X, T)$ is a fuzzy pre-$\gamma$-T$_2$ (resp. fuzzy pre-$\gamma$-T$_1$) space.

**Proof:**

Let $(Y, S)$ be a fuzzy pre-$\beta$-T$_2$ space. Let $\lambda_1, \lambda_2$ be any two fuzzy sets in $(X, T)$ such that $\lambda_1 \not \subseteq \lambda_2$. Then there exist fuzzy pre-$\beta$-open sets $\lambda, \mu$ in $(Y, S)$ with $f(\lambda_1) \leq \lambda$ and $f(\lambda_2) \leq \mu$ such that $\lambda \not \subseteq \mu$. Then $\lambda \leq 1 - \mu$, which implies that $f^{-1}(\lambda) \not \subseteq f^{-1}(\mu)$.

Now, $\lambda_1 \leq f^{-1}(\lambda)$ and $\lambda_2 \leq f^{-1}(\mu)$. Since $f$ is fuzzy pre-$\gamma$-$\beta$-continuous, $f^{-1}(\lambda)$ and $f^{-1}(\mu)$ are fuzzy pre-$\gamma$-open sets such that $f^{-1}(\lambda) \not \subseteq f^{-1}(\mu)$. Hence $(X, T)$ is a fuzzy pre-$\gamma$-T$_2$ space. Similarly we prove the case of fuzzy pre-$\beta$-T$_1$ space.

5. Fuzzy pre-$\gamma$-normal and fuzzy pre-$\gamma$-compact spaces.

**Definition 5.1** A fuzzy topological space $(X, T)$ is said to be fuzzy pre-$\gamma$-normal if for every fuzzy pre-$\gamma$-closed set $\lambda$ and fuzzy pre-$\gamma$-open set $\mu$ in $(X, T)$ such that $\lambda \leq \mu$, there exists a fuzzy set $\delta$ such that $\lambda \leq \gamma$-fp $\text{int}(\delta) \leq \gamma$-fp $\text{cl}(\delta) \leq \mu$.

**Proposition 5.1** For any fuzzy topological space $(X, T)$ the following statements are equivalent:

(a) $(X, T)$ is fuzzy pre-$\gamma$-normal.

(b) For each fuzzy pre-$\gamma$-closed set $\lambda$ and each fuzzy pre-$\gamma$-open set $\mu$ in $(X, T)$ such that $\lambda \leq \mu$, there exists a fuzzy pre-$\gamma$-open set $\delta$ in $(X, T)$ such that $\gamma$-fp $\text{cl}(\lambda) \leq \delta \leq \gamma$-fp $\text{cl}(\delta) \leq \mu$.

(c) For each fuzzy pre-$\gamma$-closed set $\lambda$ and each fuzzy pre-$\gamma$-open set $\mu$ in $(X, T)$ such that $\lambda \leq \mu$, there exists a fuzzy pre-$\gamma$-open set $\delta$ in $(X, T)$ such that $\gamma$-fp $\text{cl}(\lambda) \leq \delta \leq \gamma$-fp $\text{cl}(\delta) \leq \mu$.

**Proof (a) $\Rightarrow$ (b)** The Proof is trivial.
(b) ⇒ (c) Let \( \lambda \) be any fuzzy pre-\( \gamma \)-closed set and \( \mu \) be any fuzzy pre-\( \gamma \)-open set in \((X, T)\) such that \( \lambda \leq \mu \). Since \( \lambda \) is fuzzy pre-\( \gamma \)-closed, \( \gamma \)-fp cl \((\lambda)\) ≤ \( \mu \). Now, \( \gamma \)-fp cl \((\lambda)\) is fuzzy pre-\( \gamma \)-closed and \( \mu \) is fuzzy pre-\( \gamma \)-open in \((X, T)\). By (b), there exists a fuzzy pre-\( \gamma \)-open set \( \delta \) in \((X, T)\) such that \( \gamma \)-fp cl \((\lambda)\) ≤ \( \delta \) ≤ \( \gamma \)-fp cl \((\delta)\) ≤ \( \mu \).

(c) ⇒ (a) The proof is trivial.

**Proposition 5.2** Let \((X, T)\) and \((Y, S)\) be any two fuzzy topological spaces. If \( f : (X, T) \rightarrow (Y, S) \) is a fuzzy pre-(\( \gamma \), \( \beta \))-homeomorphism and \((Y, S)\) is fuzzy pre-\( \beta \)-normal, then \((X, T)\) is fuzzy pre-\( \gamma \)-normal.

**Proposition 5.3** Let \( f : (X, T) \rightarrow (Y, S) \) be a fuzzy pre-(\( \gamma \), \( \beta \))-homeomorphism from a fuzzy pre-\( \gamma \)-normal space \((X, T)\) onto a fuzzy topological space \((Y, S)\). Then \((Y, S)\) is fuzzy pre-\( \beta \)-normal.

**Proof:** Let \( \lambda \) be any fuzzy pre-\( \beta \)-closed set and \( \mu \) be any fuzzy pre-\( \beta \)-open set in \((Y, S)\) such that \( \lambda \leq \mu \). Since \( f \) is fuzzy pre-(\( \gamma \), \( \beta \))-continuous, \( f^{-1}(\lambda) \) is fuzzy pre-\( \gamma \)-closed and \( f^{-1}(\mu) \) is fuzzy pre-\( \gamma \)-open in \((X, T)\). Since \((X, T)\) is fuzzy pre-\( \gamma \)-normal, there exists a fuzzy set \( \delta \) in \((X, T)\) such that

\[
f^{-1}(\lambda) \leq \gamma \text{-fp cl } (\delta) \leq \gamma \text{-fp cl } (\delta) \leq f^{-1}(\mu).
\]

Now, \( f(f^{-1}(\lambda)) = \lambda \leq f(\gamma \text{-fp cl } (\delta)) \leq f(\gamma \text{-fp cl } (\delta)) \leq f(f^{-1}(\mu)) = \mu.\)

That is, \( \lambda \leq \beta \text{-fp cl } (f(\delta)) \leq \beta \text{-fp cl } (f(\delta)) \leq \mu \). Therefore, \((Y, S)\) is fuzzy pre-\( \beta \)-normal.

**Definition 5.2** A collection \( \{\lambda_i\}_{i \in J} \) of fuzzy pre-\( \gamma \)-open sets (resp. fuzzy pre-\( \beta \)-open sets) of fuzzy topological space \((X, T)\) is called fuzzy pre-\( \gamma \) (resp. fuzzy pre-\( \beta \))-covering of \((X, T)\) if \( 1_X \leq \bigvee \lambda_i \).

A fuzzy topological space \((X, T)\) is called fuzzy pre-\( \gamma \) (resp. fuzzy pre-\( \beta \))-compact if every fuzzy pre-\( \gamma \) (resp. fuzzy pre-\( \beta \))-cover of \((X, T)\) has a finite subcover.

A collection \( \{\lambda_i\}_{i \in J} \) of fuzzy pre-\( \gamma \) (resp. fuzzy pre-\( \beta \))-open sets in \((X, T)\) is called fuzzy pre-\( \gamma \) (resp. fuzzy pre-\( \beta \))-cover of a fuzzy set \( \mu \) in \((X, T)\) if \( \mu \leq \bigvee \lambda_i \).

**Proposition 5.4** Let \( f : (X, T) \rightarrow (Y, S) \) be an fuzzy pre-(\( \gamma \), \( \beta \)) -continuous surjective function of a fuzzy pre-\( \gamma \)-compact space \((X, T)\) onto a fuzzy topological space \((Y, S)\). Then \((Y, S)\) is fuzzy pre-\( \beta \)-compact.
Proposition 5.5 Let $f : (X, T) \rightarrow (Y, S)$ be a fuzzy pre-$(\gamma, \beta)$-open bijective function and $(Y, S)$ be a fuzzy pre-\(\beta\)-compact space. Then $(X, T)$ is fuzzy pre-\(\gamma\)-compact.

REFERENCES