New subclass of univalent function defined by using generalized Salagean operator

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Abstract:
In this paper, we have introduced and studied a new subclass $TD_a(\alpha, \beta, \xi;n)$ of univalent functions defined by using generalized Salagean operator in the unit disk $U=\{z:|z|<1\}$. We have obtained among others results like, coefficient inequalities, distortion theorem, extreme points, neighbourhood and Hadamard product properties.

Key Words
Univalent function, Distortion theorem, Neighbourhood, Hadamard product 2000
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1. INTRODUCTION

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$  \hspace{1cm} (1.1)

which are analytic in the unit disk $U = \{z : |z| < 1\}$. In [4], Al-oboudi defined a differential operator as follows, for a function $f(z) \in A$,

$$D^0 f(z) = f(z),$$

$$Df(z) = D^1 f(z) = (1 - \lambda)f(z) + \lambda z f'(z) = D_1 f(z), \quad \lambda \geq 0$$  \hspace{1cm} (1.2)

in general

$$D^n f(z) = D_1 \left( D^{n-1} f(z) \right).$$  \hspace{1cm} (1.3)

If $f(z)$ is given by (1.1), then from (1.2) and (1.3) we observe that

$$D^n f(z) = z + \sum_{k=2}^{\infty} \left[ 1 + (k-1)\lambda \right] a_k z^k$$  \hspace{1cm} (1.4)

when $\lambda = 1$, we get Salagean differential operator [7]. Further, let $T$ denote the subclass of $A$ which consists of functions of the form

$$f(z) = z - \sum_{k=1}^{\infty} a_k z^k, \quad a_k \geq 0.$$  \hspace{1cm} (1.5)

A function $f(z)$ belonging to $A$ is in the class $D_1 (\alpha, \beta, \xi, n)$, if and only if

$$\left| \frac{(D^n f(z))^{-1} - 1}{2^\xi \left[ (D^n f(z))' - \alpha \right] - (D^n f(z))^{-1}} \right| < \beta$$  \hspace{1cm} (1.6)

where $0 \leq \alpha < 1/2 \xi$, $0 < \beta \leq 1$, $1/2 \leq \xi \leq 1$, $n \in \mathbb{N} \cup \{0\}$, $z \in U$. 

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Let \( TD_\lambda(\alpha, \beta, \xi; n) = T \cap D_\lambda(\alpha, \beta, \xi; n) \) \quad (1.7)

## 2. MAIN RESULTS

**Theorem 2.1.** Let \( f(z) \) be defined by (1.5). Then \( f(z) \in TD_\lambda(\alpha, \beta, \xi; n) \), if and only if

\[
\sum_{k=2}^{\infty} \left[ 1 + (k-1) \lambda \right]^n k \left[ 1 + \beta (2 \xi - 1) \right] a_k \leq 2 \beta \xi (1 - \alpha) \quad (2.1)
\]

\[ 0 \leq \alpha < 1/2 \xi, \quad 0 < \beta \leq 1, \quad 1/2 \leq \xi \leq 1, \quad n \in \mathbb{N} \cup \{0\}, \quad \lambda \geq 0. \]

**Proof.** For \( |z| = 1 \), we get

\[
\left| \left( D^\nu f(z) \right)' - 1 - \beta \left[ D^\nu f(z) \right]' - \alpha \right| - \left[ (D^\nu f(z))' - 1 \right] = \sum_{k=2}^{\infty} \left[ 1 + (k-1) \lambda \right]^n k a_k z^{k-1} - \beta \left[ 2 \xi (1 - \alpha) - 2 \xi \sum_{k=2}^{\infty} \left[ 1 + (k-1) \lambda \right]^n k a_k z^{k-1} \right]
\]

\[ + \sum_{k=2}^{\infty} \left[ 1 + (k-1) \lambda \right]^n k a_k z^{k-1} \]

\[ \leq \sum_{k=2}^{\infty} \left[ 1 + (k-1) \lambda \right]^n k \left[ 1 + \beta (2 \xi - 1) \right] a_k - 2 \beta \xi (1 - \alpha) \leq 0, \]

by hypothesis. Thus by maximum modulus theorem, we have \( f(z) \in TD_\lambda(\alpha, \beta, \xi; n) \).

Conversely, suppose that \( f(z) \in TD_\lambda(\alpha, \beta, \xi; n) \), hence the condition (1.6) gives us

\[
\left| \left( D^\nu f(z) \right)' - 1 \right| - \left| 2 \xi \left[ (D^\nu f(z))' - \alpha \right] - (D^\nu f(z))' - 1 \right|
\]
Since $|\text{Re}(z)| < |z|$ for all $z$, we obtain
\[
\text{Re} \left\{ \frac{\sum_{k=2}^{n}[1+(k-1)\lambda]^k a_z z^{k-1}}{2\xi(1-\alpha)-(2\xi-1)\sum_{k=2}^{\infty}[1+(k-1)\lambda]^k a_z z^{k-1}} \right\} < \beta.
\]
Letting $z \to 1$ through real values, we get (2.1). The result is sharp for the function
\[
f(z) = z - \frac{2\beta\xi(1-\alpha)}{[1+(k-1)\lambda]^k[1+\beta(2\xi-1)]} z^k, \quad k \geq 2.
\]

**Corollary 2.1.** Let $f(z) \in T$ belong to the class $TD_{\lambda}(\alpha, \beta, \xi; n)$, then
\[
a_k \leq \frac{2\beta\xi(1-\alpha)}{[1+(k-1)\lambda]^k[1+\beta(2\xi-1)]}, \quad k \geq 2. \tag{2.2}
\]

**Theorem 2.2.** Let $f(z) \in T$ belong to the class $TD_{\lambda}(\alpha, \beta, \xi; n)$, then for $|z| \leq r < 1$, we have
\[
1-r^2 - \frac{\beta\xi(1-\alpha)}{1+\beta(2\xi-1)} \leq |D^n f(z)| \leq 1+r^2 - \frac{\beta\xi(1-\alpha)}{1+\beta(2\xi-1)} \tag{2.3}
\]
\[
1-r - \frac{2\beta\xi(1-\alpha)}{1+\beta(2\xi-1)} \leq \left|D^n f(z)\right| \leq 1+r - \frac{2\beta\xi(1-\alpha)}{1+\beta(2\xi-1)}. \tag{2.4}
\]
The bounds given by (2.3) and (2.4) are sharp.
Proof. By Theorem 2.1, we have
\[
\sum_{k=2}^{n} \left[ 1 + (k-1) \lambda \right] k \left[ 1 + \beta (2 \xi - 1) \right] a_k \leq 2 \beta \xi (1 - \alpha)
\]
then, we have
\[
2 (1 + \lambda)^n \left[ 1 + \beta (2 \xi - 1) \right] a_k \leq \sum_{k=2}^{n} \left[ 1 + (k-1) \lambda \right] k \left[ 1 + \beta (2 \xi - 1) \right] a_k \leq 2 \beta \xi (1 - \alpha),
\]
thus,
\[
\sum_{k=2}^{n} a_k \leq \frac{2 \beta \xi (1 - \alpha)}{2 (1 + \lambda)^n \left[ 1 + \beta (2 \xi - 1) \right]}
\]
Hence
\[
|D^n f(z)| \leq |z| + \sum_{k=2}^{n} \left[ 1 + (k-1) \lambda \right] a_k z^k
\]
\[
\leq |z| + |z|^2 (1 + \lambda)^n \sum_{k=2}^{n} a_k
\]
\[
\leq r + r^2 (1 + \lambda)^n \sum_{k=2}^{n} a_k
\]
\[
\leq r + r^2 \frac{\beta \xi (1 - \alpha)}{1 + \beta (2 \xi - 1)}
\]
and
\[
|D^n f(z)| \geq |z| - \sum_{k=2}^{n} \left[ 1 + (k-1) \lambda \right] a_k z^k
\]
\[
\geq |z| - |z|^2 (1 + \lambda)^n \sum_{k=2}^{n} a_k
\]
\[
\geq r - r^2 (1 + \lambda)^n \sum_{k=2}^{n} a_k
\]
\[
\geq r - r^2 \frac{\beta \xi (1 - \alpha)}{1 + \beta (2 \xi - 1)}
\]
thus (2.3) is true. Further, 
\[
\left| D^n f(z) \right| \leq 1 + 2r (1 + \lambda)^n \sum_{k=2}^{\infty} a_k
\]
\[
\leq 1 + r \frac{2\beta \xi (1 - \alpha)}{1 + \beta (2\xi - 1)}
\]
and 
\[
\left| D^n f(z) \right| \geq 1 - 2r (1 + \lambda)^n \sum_{k=2}^{\infty} a_k
\]
\[
\geq 1 - r \frac{2\beta \xi (1 - \alpha)}{1 + \beta (2\xi - 1)}.
\]
The result is sharp for the function \( f(z) \) defined by
\[
f(z) = z - \frac{2\beta \xi (1 - \alpha)}{1 + \beta (2\xi - 1)} z^2, \quad z = \pm r.
\]

Theorem 2.3. Let \( n \in \mathbb{N} \cup \{0\}, \lambda \geq 0, 0 \leq \alpha_1 \leq \alpha_2 < 1/2\xi, 0 < \beta \leq 1, 1/2 \leq \xi \leq 1. \) Then \( TD_z (\alpha, \beta, \xi; n) \subset TD_z (\alpha_1, \beta, \xi; n). \)

Proof. By assumption we have
\[
\frac{2\beta \xi (1 - \alpha_1)}{[1 + (k-1)\lambda]^n k [1 + \beta (2\xi - 1)]} \leq \frac{2\beta \xi (1 - \alpha)}{[1 + (k-1)\lambda]^n k [1 + \beta (2\xi - 1)]}.
\]
Thus, \( f(z) \in TD_z (\alpha, \beta, \xi; n) \) implies that
\[
\sum_{k=2}^{\infty} \left[1 + (k-1)\lambda\right]^n a_k \leq \frac{2\beta \xi (1 - \alpha)}{k [1 + \beta (2\xi - 1)]} \leq \frac{2\beta \xi (1 - \alpha_1)}{k [1 + \beta (2\xi - 1)]}
\]
then \( f(z) \in TD_z (\alpha_1, \beta, \xi; n). \)
Theorem 2.4. The set $TD_{\lambda}(\alpha, \beta, \xi;n)$ is the convex set.

Proof. Let $f_i(z) = z - \sum_{k=2}^{\infty} a_k z^k$ ($i = 1, 2$) belong to $TD_{\lambda}(\alpha, \beta, \xi;n)$ and let $g(z) = \zeta_1 f_1(z) + \zeta_2 f_2(z)$, with $\zeta_1$ and $\zeta_2$ non-negative and $\zeta_1 + \zeta_2 = 1$.

We can write

$$g(z) = z - \sum_{k=2}^{\infty} \left( \zeta_1 a_{k,1} + \zeta_2 a_{k,2} \right) z^k.$$

It is sufficient to show that $g(z) \in TD_{\lambda}(\alpha, \beta, \xi;n)$ that means

$$\sum_{k=2}^{\infty} \left[ 1 + (k-1) \lambda \right]^{\eta} k \left[ 1 + \beta (2\xi - 1) \right] \left( \zeta_1 a_{k,1} + \zeta_2 a_{k,2} \right)$$

$$= \zeta_1 \sum_{k=2}^{\infty} \left[ 1 + (k-1) \lambda \right]^{\eta} k \left[ 1 + \beta (2\xi - 1) \right] a_{k,1} + \zeta_2 \sum_{k=2}^{\infty} \left[ 1 + (k-1) \lambda \right]^{\eta} k \left[ 1 + \beta (2\xi - 1) \right] a_{k,2}$$

$$\leq \zeta_1 \left( 2\beta \xi (1-\alpha) \right) + \zeta_2 \left( 2\beta \xi (1-\alpha) \right) = (\zeta_1 + \zeta_2) \left( 2\beta \xi (1-\alpha) \right) = 2\beta \xi (1-\alpha).$$

Thus $g(z) \in TD_{\lambda}(\alpha, \beta, \xi;n)$.

We shall now present a result on extreme points in the following theorem.

Theorem 2.5. Let $f_i(z) = z$ and

$$f_k(z) = z - \frac{2\beta \xi (1-\alpha)}{\left[ 1 + (k-1) \lambda \right]^{\eta} k \left[ 1 + \beta (2\xi - 1) \right]} z^k$$

for all $k \geq 2$, $n \in N \cup \{0\}$, $\lambda \geq 0$, $0 \leq \alpha < 1/2\xi$, $0 < \beta \leq 1$, $1/2 \leq \xi \leq 1$.

Then $f(z)$ is in the subclass $TD_{\lambda}(\alpha, \beta, \xi;n)$, if and only if it can be expressed in the form $f(z) = \sum_{k=2}^{\infty} \gamma_k z^k$ where $\gamma_k \geq 0$ and $\sum_{k=2}^{\infty} \gamma_k = 1$ or $1 = \gamma_1 + \sum_{k=2}^{\infty} \gamma_k$. 
Proof. Let \( f(z) = \sum_{k=2}^{\infty} \gamma_k z^k \) where \( \gamma_k \geq 0 \) and \( \sum_{k=2}^{\infty} \gamma_k = 1. \) Thus
\[
f(z) = z - \sum_{k=2}^{\infty} \frac{2\beta_\xi(1-\alpha)}{[1+(k-1)\lambda]^n k[1+\beta(2\xi-1)]} \gamma_k z^k
\]
and we obtain
\[
\sum_{k=2}^{\infty} \frac{[1+(k-1)\lambda]^n k[1+\beta(2\xi-1)]}{2\beta_\xi(1-\alpha)} \gamma_k \times \frac{2\beta_\xi(1-\alpha)}{[1+(k-1)\lambda]^n k[1+\beta(2\xi-1)]} = \sum_{k=2}^{\infty} \gamma_k = 1 - \gamma_1 \leq 1.
\]
In view of Theorem (2.1), this show that \( f(z) \in TD_\lambda(\alpha, \beta, \xi; n). \)

Conversely, suppose that \( f(z) \) of the form (1.5) belong to \( TD_\lambda(\alpha, \beta, \xi; n) \) then
\[
a_k \leq \frac{2\beta_\xi(1-\alpha)}{[1+(k-1)\lambda]^n k[1+\beta(2\xi-1)]}, \quad k \geq 2.
\]
Putting
\[
\gamma_k = \frac{[1+(k-1)\lambda]^n k[1+\beta(2\xi-1)]}{2\beta_\xi(1-\alpha)}
\]
and \( \gamma_1 = 1 - \sum_{k=2}^{\infty} \gamma_k, \) then we have \( f(z) = \gamma_1 f_1(z) + \sum_{k=2}^{\infty} \gamma_k f_k(z). \)

This completes the proof.

3. NEIGHBOURHOOD AND HADAMARD PRODUCT PROPERTIES

Definition 3.1. [6]. Let \( \gamma_k \geq 0 \) and \( f(z) \in T \) of the form (1.5).

The \((k, \gamma)\)-neighbourhood of a function \( f(z) \) defined by
\[
N_{(k, \gamma)}(f) = \left\{ g \in T : g(z) = z - \sum_{k=2}^{\infty} b_k z^k \text{ and } \sum_{k=2}^{\infty} k |a_k - b_k| \leq \gamma \right\}, \quad (3.1)
\]
For the identity function \( e(z) = z \), we have
\[
N_{\lambda, \gamma}(e) = \left\{ g \in T : g(z) = z - \sum_{k=2}^{\infty} b_k z^k \text{ and } \sum_{k=2}^{\infty} k |b_k| \leq \gamma \right\}. \tag{3.2}
\]

**Theorem 3.1.** Let \( \gamma = \frac{2\beta \xi (1 - \alpha)}{(1 + \lambda)^{\xi} \left[ 1 + \beta (2 \xi - 1) \right]} \). Then \( TD_\lambda (\alpha, \beta, \xi; n) \subset N_{\lambda, \gamma}(e) \).

**Proof.** Let \( f(z) \in TD_\lambda (\alpha, \beta, \xi; n) \) then we have
\[
2(1 + \lambda)^{\xi} \left[ 1 + \beta (2 \xi - 1) \right] \sum_{k=2}^{\infty} a_k \leq \sum_{k=2}^{\infty} \left[ 1 + (k - 1) \lambda \right]^{\xi} k \left[ 1 + \beta (2 \xi - 1) \right] a_k \leq 2\beta \xi (1 - \alpha),
\]
therefore
\[
\sum_{k=2}^{\infty} a_k \leq \frac{\beta \xi (1 - \alpha)}{(1 + \lambda)^{\xi} \left[ 1 + \beta (2 \xi - 1) \right]} \tag{3.3}
\]
also we have for \( |z| < r \)
\[
|f'(z)| \leq 1 + |z| \sum_{k=2}^{\infty} k a_k \leq 1 + r \sum_{k=2}^{\infty} k a_k.
\]

In view of (3.3), we have
\[
|f''(z)| \leq 1 + r \frac{2\beta \xi (1 - \alpha)}{(1 + \lambda)^{\xi} \left[ 1 + \beta (2 \xi - 1) \right]}. \tag{3.4}
\]

From above inequalities we get
\[
\sum_{k=2}^{\infty} k a_k \leq \frac{2\beta \xi (1 - \alpha)}{(1 + \lambda)^{\xi} \left[ 1 + \beta (2 \xi - 1) \right]} = \gamma,
\]
therefore, \( f(z) \in N_{\lambda, \gamma}(e) \).

**Definition 3.2.** The function \( f(z) \) defined by (1.5) is said to be a member of the subclass \( TD_\lambda (\alpha, \beta, \xi; n) \) if there exists a function \( g(z) \in TD_\lambda (\alpha, \beta, \xi; n) \) such that
Theorem 3.2. Let $g(z) \in TD_{\lambda}(\alpha, \beta, \xi; n)$ and

$$\zeta = 1 - \frac{\gamma}{2} d(\alpha, \beta, \xi; n).$$

(3.4)

Then $N_{\alpha, \xi}(g) \subset TD_{\lambda}(\alpha, \beta, \xi, \zeta; n)$ where $n \in N \cup \{0\}$, $\lambda \geq 0$, $0 \leq \alpha < 1/2\xi$, $0 < \beta \leq 1$, $1/2 \leq \xi \leq 1$, $0 \leq \zeta < 1$ and

$$d(\alpha, \beta, \xi, n) = \frac{(1+\lambda)^{\nu}[1+\beta(2\xi-1)]}{(1+\lambda)^{\nu}[1+\beta(2\xi-1)] - \beta \xi(1-\alpha)}.$$

Proof. Let $f(z) \in N_{\alpha, \xi}(g)$, then by (3.3) we have $\sum_{k=2}^{\infty} k|a_k - b_k| \leq \gamma$, then

$$\sum_{k=2}^{\infty} |a_k - b_k| \leq \gamma/2.$$

Since $g(z) \in TD_{\lambda}(\alpha, \beta, \xi; n)$, we have

$$\sum_{k=2}^{\infty} b_k \leq \frac{\beta \xi(1-\alpha)}{(1+\lambda)^{\nu}[1+\beta(2\xi-1)]},$$

therefore,

$$\left| \frac{f(z)}{g(z)} - 1 \right| \leq \frac{\sum_{k=2}^{\infty} |a_k - b_k|}{1 - \sum_{k=2}^{\infty} b_k} \leq \frac{\gamma}{2} \frac{(1+\lambda)^{\nu}[1+\beta(2\xi-1)]}{(1+\lambda)^{\nu}[1+\beta(2\xi-1)] - \beta \xi(1-\alpha)} = \frac{\gamma}{2} d(\alpha, \beta, \xi, n) = 1 - \zeta.$$

Then by definition 3.2, we get $f(z) \in TD_{\lambda}(\alpha, \beta, \xi, \zeta; n)$.

Theorem 3.3. Let $f(z)$ and $g(z) \in TD_{\lambda}(\alpha, \beta, \xi; n)$ be of the form (1.5) such that
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\[ f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad g(z) = z - \sum_{k=2}^{\infty} b_k z^k, \text{where} \ a_k, b_k \geq 0. \text{Then the Hadamard product} \ h(z) \ \text{defined by} \ h(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k \ \text{is in the subclass} \]

\[ \mathcal{T}D_{n}(\alpha, \beta, \xi; n) \]

where

\[ \alpha_2 \leq \frac{1 + \beta (2 \xi - 1)}{2 \beta \xi (1 - \alpha_1)} \]

**Proof.** By Theorem 2.1, we have

\[ \sum_{k=2}^{\infty} \frac{1 + (k-1) \lambda^n}{2 \beta \xi (1 - \alpha_1)} a_k \leq 1 \quad \text{(3.5)} \]

and

\[ \sum_{k=2}^{\infty} \frac{1 + (k-1) \lambda^n}{2 \beta \xi (1 - \alpha_1)} b_k \leq 1. \quad \text{(3.6)} \]

We have only to find the largest \( \alpha_2 \) such that

\[ \sum_{k=2}^{\infty} \frac{1 + (k-1) \lambda^n}{2 \beta \xi (1 - \alpha_1)} a_k b_k \leq 1. \]

Now, by Cauchy-Schwarz inequality, we obtain

\[ \sum_{k=2}^{\infty} \frac{1 + (k-1) \lambda^n}{2 \beta \xi (1 - \alpha_1)} \sqrt{a_k b_k} \leq 1, \quad \text{(3.7)} \]

we need only to show that

\[ \frac{1 + (k-1) \lambda^n}{2 \beta \xi (1 - \alpha_1)} a_k b_k \leq \frac{1 + \beta (2 \xi - 1)}{2 \beta \xi (1 - \alpha_1)} \sqrt{a_k b_k} \]

equivalently,
\[ \sqrt{a_k b_k} \leq \frac{[1 + (k-1) \lambda]^n}{2\beta \xi (1 - \alpha_1)} \times \frac{2\beta \xi (1 - \alpha_2)}{[1 + (k-1) \lambda]^n [1 + \beta (2\xi - 1)]} \]

Consequently, we need to prove that

\[ \frac{2\beta \xi (1 - \alpha_1)}{[1 + (k-1) \lambda]^n [1 + \beta (2\xi - 1)]} \leq \frac{1 - \alpha_2}{1 - \alpha_1}. \]

or equivalently, that

\[ \alpha_2 \leq \frac{[1 + (k-1) \lambda]^n [1 + \beta (2\xi - 1)] - 2\beta \xi (1 - \alpha_1)^2}{[1 + (k-1) \lambda]^n [1 + \beta (2\xi - 1)]} \]

**Theorem 3.4.** Let \( f(z) \in TD_\alpha(\alpha, \beta, \xi; n) \) be defined by (1.5) and \( c \) any real number with \( c > -1 \) than the function \( G(z) \) defined as

\[ G(z) = \frac{c+1}{z^c} \int_0^z s^{c-1} f(s) ds, \quad c > -1, \text{ also belongs to } TD_\alpha(\alpha, \beta, \xi; n). \]

**Proof.** By virtue of \( G(z) \) it follows from (1.5) that

\[ G(z) = \frac{c+1}{z^c} \int_0^z \left( s^c - \sum_{k=2}^n a_k s^{k+c-1} \right) ds \]

\[ = z - \sum_{k=2}^n \left( \frac{c+1}{c+k} \right) a_k z^k. \]
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But
\[ \sum_{k=2}^{n} \frac{\left[1+(k-1)\lambda\right]^k \left[1+\beta (2\xi-1)\right]}{2\beta \xi (1-\alpha)} \frac{(c+1)}{(c+k)} a_k \leq 1. \]

Since \( \frac{c+1}{c+k} \leq 1 \) and by Theorem 2.1, so the proof is complete.

**Theorem 3.5.** Let \( f(z) \in TD_1 (\alpha, \beta, \xi; n) \) be defined by (1.5) and

\[ F_{\mu}(z) = (1-\mu)z + \mu \int_0^z \frac{f(s)}{s} ds \quad (\mu \geq 0, z \in U). \]

Then \( F_{\mu}(z) \) is also in \( TD_1 (\alpha, \beta, \xi; n) \) if \( 0 \leq \mu \leq 2 \).

**Proof.** Let \( f(z) \) defined by (1.5) then

\[ F_{\mu}(z) = (1-\mu)z + \mu \int_0^z \left( s - \sum_{k=2}^{n} \frac{a_k s^k}{s} \right) ds \]

\[ = z - \sum_{k=2}^{n} \frac{\mu a_k z^k}. \]

By Theorem 2.1 and since \( \left( \frac{\mu}{k} \leq 1 \right) \) we have

\[ \sum_{k=2}^{n} \frac{\left[1+(k-1)\lambda\right]^k \left[1+\beta (2\xi-1)\right]}{2\beta \xi (1-\alpha)} \frac{(\mu)}{(k)} a_k \leq 1. \]

Then \( F_{\mu}(z) \) is in \( TD_1 (\alpha, \beta, \xi; n) \).
4. REFERENCES


