Infinitesimal Variation of Hypersurfaces of an Almost $r$-Contact Hyperbolic Structure Manifold

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Summary: The infinitesimal variation of the structure tensors of an almost contact metric structure induced on the hyper surface of a Kahlerian manifold under various conditions has been studied by Yano. In this paper we have studied the infinitesimal variation of the structure tensors of an almost $r$-contact hyperbolic structure induced on the hyper surface of a differentiable manifold equipped with an almost $r$-contact hyperbolic structure.

1. Introduction:

Let $M^{n+r}$ be an $(n+r)$ dimensional differentiable manifold of differentiability class $C^\infty$. Let there exist on $M^{n+r}$ a $C^\infty$ vector valued linear function $F$, an $rC^\infty$ linearly independent and non-zero contravariant vector fields $T^1, T^2, \ldots, T^r$ such that

$$F^2 X = X + \sum_{i=1}^r A_i(x) T^i$$

for arbitrary vector field $X$ on $M^{n+r}$. Also

$$F(X) = \overline{X}$$

In view of (1.1), let $M^{n+r}$ be endowed with the Riemannian metric $G$ such that it satisfies the following condition.

$$G(\overline{X}, \overline{Y}) + G(X, Y) + \sum_{i=1}^r A_i(X) A_i(Y) = 0$$
Thus $M^{n+r}$ satisfying the conditions (1.1) and (1.3) will be called an almost $r$-contact hyperbolic structure manifold [2].

In $M^{n+r}$ the following results hold

\begin{align*}
(1.4) & \quad T^l = 0, \\
(a) \\ 
(1.5) & \quad A_l (\bar{X}) = 0, \text{ for arbitrary vector field } X. \\
(b) \\ 
(1.6) & \quad T^l (T^m) + \delta^l_m = 0,
\end{align*}

Where $\delta^l_m$ is Kronecker delta and $l,m$ take the values $1,2,\ldots,r$.

Let us imbed a hypersurface $M^{n+r-1}$ into $M^{n+r}$ by the isometric immersion $b : M^{n+r-1} \to M^{n+r}$. Corresponding to this we have the Jacobian $b^*$ of $b$ denoted by $B$ which carries $T_q (M^{n+r-1})$ into $T_b (M^{n+r})$ injectively. Since the immersion is isometric, we have

\begin{align*}
(1.7) & \quad G(BX, BY) = g(X,Y),
\end{align*}

g being the metric induced on the hyper surface and $X,Y$ denote arbitrary vector fields. We have

\begin{align*}
(1.8) & \quad G(BX, N) = 0, \\
(1.9) & \quad G(N, N) = 1.
\end{align*}

The transformation equations are

\begin{align*}
(1.10) & \quad FBX = BfX + \alpha (X) N, \\
(1.11) & \quad FN = B\eta + \eta N,
\end{align*}

where $f$ is a tensor field of type (1.1) and $\alpha$ is a 1-form on $M^{n+r-1}$. From equation (1.8) and the relations

\begin{align*}
(1.12) & \quad T^l = Bt_l + \delta^l_1, \\
(a) \\ 
(1.13) & \quad A_l (BX) = \alpha r (X), \\
(b) \\ 
(1.14) & \quad \alpha (X) P = 0,
\end{align*}

we get

\begin{align*}
(1.15) & \quad f^2 X = X + \sum_{l=1}^r \alpha_l (X) t_l, \\
\end{align*}

The metric $g$ in (1.5) is found to satisfy

\begin{align*}
(1.16) & \quad g(fX, fY) + g(X, Y) + \sum_{l=1}^r \alpha_l (X) \alpha_l (Y) = 0.
\end{align*}
Consequently an almost r-contact hyperbolic structure gets induced on $M^{n+r-1}$.

Let $D$ be the Riemannina connexion induced on $M^{n+r-1}$. Then we have the Gauss and Weingarten equations [1].

\begin{align}
E_{BY} BY &= BD_{X}Y + H(X,Y)N, \\
E_{BX} N &= -B'HX,
\end{align}

Where $H$ is the 2nd fundamental form of $M^{n+r-1}$ and $'H$ is a tensor field of type (1,1) associated with $H$. Let $\kappa$ and $\kappa'$ stand for the curvature tensors of the hyper surface and the enveloping manifold. Then we have Gauss and Codazzi equations.

\begin{align}
\kappa(BX, BY, BZ, BU) &= \kappa(X,Y,Z,U) - H(Y,Z)H(X,U) \\
&\quad + H(X,Z)H(Y,U) \\
\kappa'(BX, BY, BZ, N) &= (D_{X}H)(Y,Z) - (D_{Y}H)(X,Z),
\end{align}

Where $\kappa$ and $\kappa'$ are the associate covariant curvature tensors of $M^{n+r-1}$ and $M^{n+r}$. Now let us differentiate equation (1.8) along the hyper surface and use $E_{X}F = 0$ hence

$$E_{BX}BY = F(E_{BY}BY) - ((D_{X}A)Y + A(D_{X}Y))N - A(Y)E_{BX}N.$$

In view of (1.9), (1.13) and (1.14) we get

\begin{align}
(D_{X}f)Y &= H(X,Y)P + \alpha(Y)'HX, \\
(D_{X}\alpha)Y &= H(X,Y)\eta - H(X,f\eta).
\end{align}

Covariant differentiation of (1.9) along $M^{n+r-1}$ yields

\begin{align}
D_{X}P &= \eta 'HX - 'HfX.
\end{align}

**Definition 1.1** An almost r-contact hyperbolic structure is said to be normal if

\begin{align}
S(X,Y) &= N(X,Y) + \sum_{i=1}^{r}((D_{X}\alpha)Y - (D_{Y}\alpha)X)t^{i} = 0,
\end{align}

Where

$$N(X,Y) = (D_{[X,f]}f)Y - (D_{f}[X,Y])X + f(D_{X}f)X - f(D_{X}f)Y + \sum_{i=1}^{r}a_{i}[X,Y]t^{i}.$$
Therefore it follows that 
(1.22) 
So that the normality condition (1.20) takes the form
\[
S(X,Y) = (D_X f)Y - (D_Y f)X + f(D_f X)
- f(D_X f)Y + \sum_{i=1}^{r} a_i [X,Y] t^i .
+ \sum_{i=1}^{r} ((D_X \alpha) Y - (D_Y \alpha) X) t^i = 0.
\]
If almost r-contact hyperbolic structure induces on \( M^{n+r} \) be normal, from the last equation and from (1.17) and (1.18) we obtain
\[
\alpha(X) \{ Hf - f 'H \} Y - \alpha(Y) \{ Hf - f 'H \} X = 0
\]
(1.21) 
\( 'Hf = f 'H \)
Therefore it follows that [1]
(1.22) 
\( H(P,P) = 'HP \)
Showing that \( H(P,P) \) is an eigen value of \( 'H \) and the corresponding eigen vector is \( P \). Let us denote \( H(P,P) \) by \( \tau \).

**Definition 1.2.** An almost r-contact hyperbolic structure is called r-hyperbolic Sasakian if

(1.23) 
\[
\sum_{i=1}^{r} ((D_X \alpha_i) Y - (D_Y \alpha_i) X) = r'f(X,Y).
\]
We have,
\( 'f(X,Y) = g(fX,Y) \).
More generally in a normal r-contact hyperbolic structure hyper surface of \( M^{n+r} \) we assume that [3]

(1.24) 
\[
\sum_{i=1}^{r} ((D_X \alpha_i) Y - (D_Y \alpha_i) X) = r \beta 'f(X,Y).
\]
Applying (1.18) to the above equation we have

(1.25) 
\( 'Hf = 'HF = -r \beta 'f \).
Thus we obtain

(1.26) 
\( 'HX = -r \beta x + (\tau + r' \beta) \alpha(X) P \).
Equation (1.17), (1.18), (1.19) then transform as

(1.27) 
\( (D_X f) Y = -r \beta \{ g(X,Y)P + \alpha(Y)X \} + 2(\tau + r' \beta) \alpha(X) \alpha(Y), \)
(1.28) \[(D_X \alpha)Y = r' \beta f(X,Y),\]
(1.29) \[D_X P = -(\eta - f) r' \beta X.\]

Let \( \beta \) be a constant so that from (1.27) and (1.29) we obtain
\[K(X, Y, P) = -r'^2 \beta^2 \eta (\alpha(Y)X - \alpha(X)Y),\]
which shows that for a normal \( r \)-contact hyperbolic structure hypersurface satisfying (1.24) and involving constant \( r' \beta \), the sectional curvature with respect to a plane section containing \( P \) is \( r'^2 \beta^2 \).

Let us call such a structure a normal \( r \)-contact hyperbolic structure with \( f \) sectional curvature \( r'^2 \beta^2 \).

2. Infinitesimal Variation of a Hypersurface of an Almost \( r \)-contact Hyperbolic Structure Manifold

Let us take the restriction of an almost decomposable Killing vector field \( U \) on the enveloping manifold of the hypersurface. According the variation of the differential of imbeddimg is given by [4].

\[(\delta B)X = \varepsilon E_{ax} U\]

where \( \varepsilon \) is infinitesimally small number. Splitting \( U \) into its tangential and normal parts as
\[U = BV + \lambda N\]
and from (1.13), (1.14) we express (2.1) as
\[(\delta B)(X) = \varepsilon \{B(D_X V - \lambda HX) + (X \lambda + H(X,Y))N\}.

Infinitesimal Variation of \( N \) is given by [5]
\[(2.4) \delta N = \varepsilon L_{\lambda} N = \varepsilon BW\]

The Lie derivative of \( N \) (i.e., \( L_{\lambda} N \)) being orthogonal to \( N \). Infinitesimal variation of equation (1.6) yields
\[G(BD_X V + H(X,Y)N + (X \lambda)N - \lambda B'(HX,N)) = -G(BX, BW)\]
which implies that \( W = -(\beta HV + \lambda) \)

Where \( \lambda \) stands for the vector field associate to the gradient of \( \lambda \). Thus we have
\[\delta N = -\varepsilon B'(HV + \lambda)\]

Now varying equation (1.8) infinitesimally, we get
\((\delta B)(fX) + B(\delta f)X = F((\delta B)X) - (\delta N) \alpha(X) - \delta \alpha(X)N.\)

Making use of (1.8), (2.3) and (2.4) in it we find
\[
B(\delta f)X + (\delta \alpha)(X)N = \epsilon\left\{ B\left[ (D_X V - \lambda'HX)N + \alpha(D_X V - \lambda'HX)N \right.ight.
\]
\[
\left. + (\lambda + H(X))B + \eta N \right) + \alpha(X)B \left[ (\lambda + \eta N) \right] \right\} - B \left[ (D_X V - \lambda'HF)X \right] + (fX')(X' + H(fX')N).
\]

Comparing the tangential and normal components, we have
\[
(\delta f)X = \epsilon \left\{ f(D_X V - \lambda'HX) + (H(X,V) + \lambda')P \right. + \alpha(X) \left[ (\lambda'P + \lambda'HfX) \right].
\]

and
\[
(\delta \alpha)(X) = \epsilon \left\{ (D_X V - \lambda'HX) + \eta(HX + H(X,V) - X\lambda - H(fX,V)) \right\}
\]

Since the derivative of \(f\) along \(V\) is given by
\[
(L_{fX}f)X = L_f(X) - f(L_fX)
\]
\[
= D_f(X) - D_{fX}V - f(D_fX - D_XV).
\]

Therefore equation (2.5) assumes the following form
\[
(\delta f)X = \epsilon \left\{ (L_{fX}f)X + \lambda(HV - f')X + X\lambda P + \alpha(X)\lambda + 2H(X,V)P \right\}
\]

Applying equation (1.8) and the definition
\[
(L_f\alpha)(X) \overset{\text{def}}{=} (D_f\alpha)(X) + (D_X V)
\]
\[
(\delta \alpha)(X) = \epsilon\left\{ (L_{fX}\alpha)(X) - \alpha \left[ \lambda'(HX - (fX'V) \right.ight.
\]
\[
\left. \left. + 2H(X,V)\eta + 2h(V,fX) \right) \right\}
\]

Next varying equation (1.9) infinitesimally, we get
\[
-\epsilon FB \left[ (\lambda'HV + \lambda) \right] = \left\{ B(\delta P) + \epsilon \left\{ B(D_fV - \lambda'H_P) + P\lambda + H(P,V)N \right\} \right\}
\]
\[
- \epsilon \eta B \left[ (\lambda'HV + \lambda) \right].
\]

Which by virtue of (1.8) and (2.3) yields
\[
B \delta P = \epsilon \left\{ B(D_PV - \lambda'H_P) + (P\lambda + H(P,V)N) - \epsilon \eta B \left[ (\lambda'HV + \lambda) \right] \right\}
\]
\[
= - \epsilon \left\{ Bf \left( (\lambda'HV + \lambda) + \alpha'(\lambda'HV + \lambda)N \right) \right\},
\]

whose tangential part reduces in virtue of (1.19) to the form
\[
(2.9) \quad \delta P = \varepsilon [ \lambda 'HP + L_\mathcal{U} P + \Lambda (\eta - f)].
\]

Again varying equation (1.5) infinitesimally, we get
\[
(2.10) \quad (\delta g)(X,Y) = G ((\delta B)X, BY) + G (BX, (\delta B)Y),
\]

which in virtue of (2.3) reduces to
\[
(2.11) \quad (\delta g)(X,Y) + \in \{(L_\mathcal{V}, g)(X,Y) - 2\lambda H(X,Y)\}.
\]

Thus we establish the following theorem.

**Theorem 2.1.** When a hyper surface of an almost r-contact hyperbolic structure manifold varied infinitesimally by means of a vector field \( U = BV + \lambda N \) the structure tensors of almost r-contact hyperbolic structure hypersurface vary according to equations (2.7), (2.8), (2.9) and (2.10).

**Corollary 2.1.** When a hypersurface of an almost r-contact hyperbolic structure manifold is given infinitesimally tangential variation by means of \( BV \), the variation of the induced almost r-constant hyperbolic structure tensors on the hypersurface are given by their Lie-derivatives along \( V \).

**Corollary 2.2.** When a hypersurface of an almost r-contact hyperbolic structure manifold is given infinitesimal normal variation by means of \( \lambda N \), the variation of the induced almost r-contact hyperbolic structure tensors on the hyper surface are given by
\[
(2.11) \quad \begin{align*}
(a) & \quad (\delta f)(X) = \in [\lambda ('H f - f 'H)X + X \lambda P + \alpha(X)\Lambda + 2H(X, V)P], \\
(b) & \quad (\delta \alpha)(X) = \in [-\alpha \lambda 'HX - f X \lambda + 2H(X, V) \eta + 2H(V, f X)], \\
(c) & \quad (\delta P) = \in [\lambda 'HP + \Lambda (\eta - f)]. \\
(d) & \quad (\delta g)(X,Y) = -2 \in \lambda H(X,Y).
\end{align*}
\]

The infinitesimal variation is said to be parallel when \( BX \) and \( B \bar{X} \) are both parallel equivalently and when \( (\delta B) \lambda \mathcal{X} \) is tangential to the original hyper surface. Since
\[
(\delta B)X = \in [B (D_X V - \lambda 'H X) + (X \lambda + H(X, V) N].
\]

Therefore for an infinitesimal parallel variation it is necessary and sufficient that
\[
(2.12) \quad X \lambda + H(X, V) = 0.
\]

**Corollary 2.3.** When a hyper surface of an almost r-contact hyperbolic structure manifold is given infinitesimal parallel variation the hypersurface variation the hypersurface
Corollary 2.4. Let the structure induced on a hypersurface of an almost r-contact hyperbolic structure manifold be a normal r-contact hyperbolic structure with f-sectional curvature $r^2 \beta^2$ then the infinitesimal normal parallel variation of the hypersurface makes the structure tensor vary as

\[(\delta f)X = \alpha(X)\Lambda,\]
\[(\delta \alpha)X = -\lambda \tau P,\]
\[(\delta \beta)X = -2 \varepsilon \lambda \tau P,\]
\[(\delta g)(X,Y) = -2 \varepsilon \lambda \{-r'\beta g(X,Y) + (\tau + r'\beta)\alpha(X)\alpha(Y)\}.

3. Variation of r–Hyperbolic Sasakian Hypersurface with f–Sectional Curvature $r^2 \beta^2$

We now assume that an almost r-contact hyperbolic structure induced on the hypersurface is a r-hyperbolic Sasakian structure with f-sectional curvature $r^2 \beta$, we have [1]

\[(3.1) \quad H(X, HY) = r^2 \beta^2 g(X, Y) + (r^2 + r^2 \beta^2)\alpha(X)\alpha(Y)\]
and
\[(3.2) \quad H(X, Y) = -r'\beta g(X, Y) - r'\beta(\delta g)(X,Y) + \delta(\tau + r'\beta)\alpha(X)\alpha(Y).

The variation in the connections and the second fundamental form are given by [1].

\[(3.3) \quad (\delta D)(X,Y) = \varepsilon\{((D_X D)(X,Y) - (D_Y D)(X,Y) - (D_{(X,Y)}D)(X,Y)) + H(X, Y) + \lambda H^*(X, Y)\}

where
\[g H^*(X,Y) = (D_Z H)(X, Y)\]
and
\[(3.4) \quad (\delta H)(X,Y) = \varepsilon\{(L_Y H)(X,Y) - \lambda H(X, HY) + X\lambda - (D_Y \lambda)\lambda + \lambda K(N, BX, BY, N)\}

If the infinitesimal variation of the hypersurface are normal the variation of D would be given by [1].
\[ (\delta D)(X,Y) = e[X,Y] + (D_X Y) \lambda + K(N,BX,BY,N) - \lambda H(X'HY) \].

Varying equation (3.2) infinitesimally, we have

\[ (\delta H)(X,Y) = -(\delta r'\beta)g(X,Y) - r'\beta(\delta g)(X,Y) \]
\[ + \delta(\pi + r'\beta)(\alpha(X)\alpha(Y)) \]
\[ + (\pi + r'\beta)((\delta \alpha)(X)\alpha(Y) + \alpha(X)(\delta \alpha)(Y)). \]

which with the help of equations (2.8), (2.9), (2.10), (3.5) and

\[ (L_{f'} H)(X,Y) = -r'\beta(L_{f'} g)(X,Y) + ((L_{f'} H)(P,P) \]
\[ + 2H(L_{f'} P,P)\alpha(X)\alpha(Y) \]
\[ + (\pi + r'\beta)((L_{f'} \alpha)(X)\alpha(Y) \]
\[ + \alpha(X)(L_{f'} \alpha)(Y)). \]

becomes

\[ e\{X,Y\lambda - (D_X Y)\lambda + \lambda K(N,BX,BY,N) - \lambda H(X'HY)\} \]
\[ = -2r\beta e\lambda H(X,Y) + e\{PP\lambda - (D_P P)\lambda \}
\[ - \lambda H(P,HP) - 2 H(P,\lambda HP - \Lambda(\eta - f)) + \delta r'\beta l \in \} \alpha(Y) \]
\[ + e(\pi + r'\beta)(-\alpha\lambda HX - fX\lambda + 2H(X,V)\eta \]
\[ + 2H(V,fX)\alpha(Y) + (-\alpha\lambda HY + fY\lambda \]
\[ + 2H(Y,V) + 2 H(V,fY) \alpha(X). \]

Conversely if \( \lambda \) satisfies the differential equation (3.8) then by retreating the steps we get (3.3).

Hence we have the following theorem

**Theorem 3.1.** In order that for an infinitesimal variation (2.1) may have the \( \alpha \)
\( r \)-hyperbolic Sasakian hypersurface with \( f \)-sectional curvature \( -r'^2\beta^2 \) in a 
\( r \)-hyperbolic Sasakian with \( f \)-sectional curvature \( -r'\beta^2 - \delta r'^2\beta^2 \), It is
necessary and sufficient that the function \( \lambda \) satisfies the relation

\[ e\{XY\lambda - (D_X Y)\lambda + \lambda (K(N,BX,BY,N)) + r'^2\beta^2 (g(X,Y) - \alpha(X) - \alpha(Y)) \]
\[ + (PP\lambda - D_P P)\lambda\alpha(X)\alpha(Y) + (\pi + r'\beta)(-fX\lambda\alpha(Y) \]
\[ - fY\lambda\alpha(X)) \}
\[ f\{2H(X,V) + 2H(X,fY)\alpha(X) \]
\[ + \{2H(Y,V) + 2H(V,fY)\alpha(X) \]
\[ = \delta r'\beta(\alpha(X)\alpha(Y) - g(X,Y)). \]
Corollary 3.1. The infinitesimal normal parallel variation carries a normal $\gamma$-hyperbolic Sasakian hypersurface with $f$-sectional curvature $-r^2 \beta^2$ to a normal $\gamma$-hyperbolic Sasakian hypersurface with $f$-sectional curvature $-r^2 \beta^2 - \delta r^2 \beta^2$ if and only if

\[ \lambda \in \{ K(N, BX, BY, N) + r^2 \beta^2 (g(X,Y) - \alpha(X) X(Y)) \} \]

\[ = \{ \alpha(X) \alpha(Y) - g(X,Y) \delta r^2 \beta \} \]

Corollary 3.2. If the enveloping manifold of corollary (3.1) be flat the condition reduced to $\delta r^2 \beta = - \lambda \in r^2 \beta^2$.

Hence the proof is obvious.

REFERENCES


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