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Amazing Roles of Diagonal Edges on the Reducibility Problem of Open Shop Sequences Minimizing the Makespan

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Abstract: In this survey, we investigate the solution spaces of open shop irreducible sequences, which contain an optimal element for arbitrary processing times. The objective function is the minimization of makespan. The computational status of deciding whether a given sequence is irreducible remained unsolved for last 20 years. Here, we explore the roles of diagonal edges in the given sequence, which play vital roles for the conclusion of this decision problem.

Keywords: Scheduling, sequencing, open shop problem, reducibility, complexity.

1. Introduction

We consider the strongly *NP*-hard nonpreemptive open shop problem $O||C_{\max}$, [13]. Assume that, at a time, each job $i \in I = \{1, \ldots, n\}$ has to be processed on each machine $j \in J = \{1, \ldots, m\}$ exactly once for the positive time such that each machine can process at most one job and each job can be processed on at most one machine. The 2-jobs m-machines problem $O|n = 2|C_{max}$ is solvable in time $O(m)$, [13, 5]. Let $OIJ = I \times J$, $P = [p_{ij}]_{n\times m}$ and $C = [c_{ij}]_{n\times m}$ be the sets of all operations o_{ij}, matrix of processing times p_{ij} and matrix of completion times c_{ij}, respectively. The objective function is $C_{max} = \max_{i\in I} C_i$, where C_i is the completion time of job i. All machine orders (jobs processed on machines) and job orders (machines processing jobs) are arbitrarily. The $n \times m$ matrices of all job orders and machine orders are denoted by JO and MO, respectively. An interest is to find an optimal schedule (solution) $C = (A, P)$ which minimizes C_{max} (A) for given P. A schedule C is the time table corresponding to a sequence A (a feasible combination of all processing orders).

Informally, a decision problem is said to be in the class P if there exists a deterministic algorithm which solves the problem in polynomial time. A decision problem is in *NP* if there exists a nondeterministic polynomial time algorithm solving it. The co-*NP* class contains the decision problems whose complements are in *NP*. A decision problem is called *NP*-complete if the problem belongs to *P*, then $NP = P$ holds. A decision problem in *NP* which is neither polynomial solvable nor *NP*-complete is called *NP*-incomplete. We refer to [10] for a systematic analysis of the complexity classes.

The set of all semiactive schedules in which each operation is started as early as possible with respect to the given processing orders is sufficient for an optimal solution. An infinite set of schedules can be assigned to each sequence. We can define an equivalence relation on the set of all schedules decomposing the set into finite number of equivalence classes. Two schedules belong to the same class if and only if they base on the same sequence. The semiactive schedules under unit processing times, i.e., a finite set of all sequences, are used to find a set of distinct representatives. Obtaining the associated semiactive schedule (a scheduling problem) for a given sequence is an easy problem. Therefore, investigations on the theory of irreducibility are focused in obtaining an optimal sequence (a sequencing problem). For such difficult problems, the study concentrated either on the determination of polynomial solvable subproblems or on the development of an algorithm for an approximate solution.

A set of sequences is called a solution space if it contains an optimal element for arbitrary processing times. Obviously, the set of all sequences, i.e. of all semiactive schedules, is the largest solution space. A study focused on searching such potentially (universally) optimal solution spaces of smaller cardinalities. But, the obtained results show that the existence of unique minimal one is unlikely in general, [8]. The concept of potentially optimal solution spaces is very applicable when the processing times are erroneous, difficult to find out in advance or simply unknown, for instance, in manufacturing and service industries, satellite communications, examination scheduling and teacher class assignments, [2, 17].

A sequence A is called reducible to a sequence *B*, we write $B \preceq A$, if $Cmax\ (B) \leq C_{max}\ (A)$ for all $P \in \mathbf{P}nm$. It is called strongly reducible, denoted by $B \prec A$, if $B \preceq A$ but not $A \preceq B$. They are called similar, denoted by $A \simeq B$, if $B \preceq A$ and $A \preceq B$. A sequence is irreducible if there exists no other non-similar sequence to which it can be reduced. The dominance relation $\preceq$ on the set of all sequences with fixed format $n \times m$, introduced in [15], determines the minimal sequences with respect to the partial order $\prec$ independent of the given processing times. The solution set of all these locally optimal sequences is potentially optimal of smaller cardinality, however, still not the minimal. Two algorithms (one polynomial time and the other exponential) are proposed in [1]. They base on the characteristics of the diagonal edges of the associated H- comparability graph [7]. A number of open problems are raised, for instance strong conditions under which both algorithms coincide. A key role lies on the diagonal edges while resolving the conflicts.

In this paper, we consider the following open questions: Does there exist a polynomial time algorithm for the decision whether a given sequence is irreducible? What is the computational complexity status of this decision problem? We characterize the H-comparability graphs in Section 2. We give different mathematical formulations of the problem of reducibility in Section 3. This section also presents the main characteristics of the diagonal edges in connection. Section 4 briefly overviews the status of the theory of irreducibility. The final section concludes the paper.

2. Comparability Graphs

A comparability graph is an undirected graph $G = (V, E)$ that has a transitive orientation $G^{tr} = (V, E^{tr})$. It is prime if it is uniquely orientable, [11]. We denote the Hamming graph $K_n \times K_m$ by G_{IJ}. A Hamming graph restricted on a partial operation set $PIJ \subseteq OIJ$ is called H-graph. A comparability graph $G = (PIJ, E)$ which contains an H-graph is called H-comparability graph. There exist Hamming graphs which are not comparability graphs, for instance $K_2 \times K_3$.

For any pair (MO, JO) in the open shop, define the shop graph $G_{MO,JO} = (PIJ, E_{MO,JO})$ where the arc set reflects the union of all machine orders and all job orders, [9]. An acyclic shop graph is called a sequence graph that is an acyclic orientation of the disjunctive graph. For each sequence graph $G_{MO,JO}$ we can describe the sequence (MO, JO) by a special latin rectangle $A = [a_{ij}]$, where $a_{ij} = \text{rank}\ (o_{ij})$ (the number of vertices on a longest path from a source to o_{ij}), such that for each integer $a_{ij} > 1$ there exists $a_{ij} - 1$ in row i or in column j or in both. An arc from o_{ij} to o_{kl} exists if and only if $i = k$ or $j = l$ and $a_{ij} < a_{kl}$ hold. There is a one-to-one correspondence between the sets of all sequences and all sequence graphs which can be done in linear time on the number of operations, see [9].

A sequence graph, denoted by $G_A = (PIJ, E)$, is an acyclic orientation of the H-graph G_{IJ}. For the sequence A, we denote the transitive orientation of a sequence graph and its symmetric closure by G^{tr}_A and $[G^{tr}_A]$, respectively. Given a sequence, both graphs can be determined in polynomial time $O(n^2m^2)$, [16]. Let $E_{r(A)}$ and $E_{d(A)}$ represent the sets of all regular edges (edges in H-graph GIJ) and diagonal edges, respectively. Here, $[G^{tr}_A] = (OIJ, G^{tr}_A + (G^{tr}_A)^{-1}) = (OIJ, E_{r(A)} \cup E_{d(A)})$ is undirected graph, where G^{-1} denotes the reversed graph of a graph G with all arcs in the reversed direction.

Consider a 3-jobs 4-machines open shop with machine order $J_1 : M_2 \to M_4 \to M_1$, J_2: $M_3 \to M_2 \to M_4 \to M_1$, $J_3 : M_1 \to M_3 \to M_4$ and job orders $M_1 : J_3 \to J_1 \to J_2$,

$M_2 : J_1 \to J_2$, $M_3 : J_2 \to J_3$, $M_4 : J_1 \to J_3 \to J_2$. The corresponding and rank matrices and the graphs are given.

$$MO = \begin{bmatrix} 3 & 1 & & 2 \\ 4 & 2 & 1 & 3 \\ 1 & & 2 & 3 \end{bmatrix} \quad JO = \begin{bmatrix} 2 & 1 & & 1 \\ 3 & 2 & 1 & 3 \\ 1 & & 2 & 2 \end{bmatrix} \quad A = \begin{bmatrix} 3 & 1 & & 2 \\ 5 & 2 & 1 & 4 \\ 1 & & 2 & 3 \end{bmatrix}$$

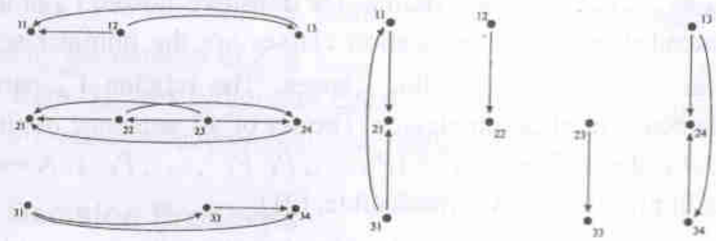

Figure 1: Machine order graph G_{MO} and job order graph G_{JO}.

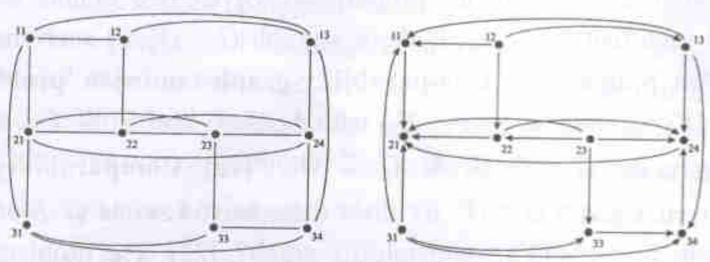

Figure 2: The H-graph $G_{IJ}(A)$ and the sequence graph G_A.

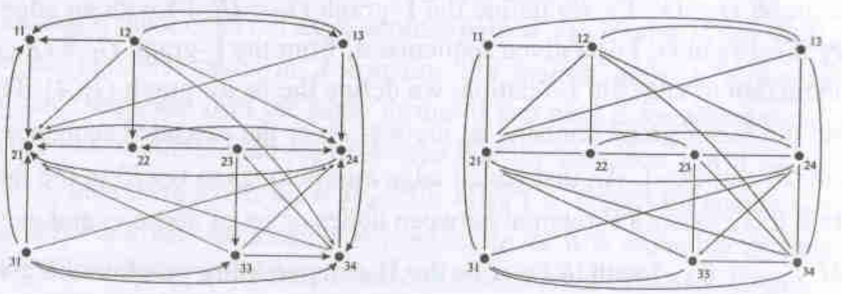

Figure 3: The transitive closure G_A^{tr} and the symmetric closure $[G_A^{tr}]$.

For two edges ab, cd in a graph $G = (V, E)$ a Γ-relation, denoted by $ab\Gamma cd$, is defined if and only if either $a = c$, $bd \notin E$ or $b = d$, $ac \notin E$ or $ab = cd$. The transitive relation Γ^{tr} decompose the set of all edges into equivalent implication classes in the comparability graph. The set of all implication classes of the sequence A is denoted by $\mathcal{I}_{[G_A^{tr}]} = \{I_1, \ldots, I_l, I_1^{-1}, \ldots, I_l^{-1}\}$. Note that a graph is a comparability graph if and only if there is no implication class containing both an arc and its reverse. A sequence with only one implication class is irreducible, [3]. Two edges ab, $cd \in G_A$ are said to be in Γ_A-relation, denoted by $ab\Gamma_A\, cd$, if and only if $ab\Gamma cd$ in $[G_A^{tr}]$. Two edges e, $e' \in E_{r(A)}$ are connected by a Γ_A-path if there exist $e = e_0, e_1, \ldots, e_m, e_{m+1} = e'$ from $E_{r(A)}$ such that $e\Gamma_A e_1\, \Gamma_A\, e_2 \ldots e_m\Gamma_A\, e'$ which defines the transitive closure Γ_A^{tr} of the Γ_A-relation on $E_{r(A)}$. The extended sequence implication classes are the minimal sets containing all transitive edges of the corresponding classes. The relation Γ_A^{tr} partitions $E_{r(A)}$ into equivalent sequence implication classes. The set of all sequence implication classes of the sequence A is denoted by $\mathcal{P}_{[G_A^{tr}]} = \{P_1, \ldots, P_k, P_1^{-1}, \ldots, P_k^{-1}\}$. A sequence with only one sequence implication class is irreducible, [21].

The problem of irreducibility is closely related to the following problems (see Section 3).

The graph sandwich problem for property P: Given two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ such that $E_1 \subseteq E_2$, is there a graph $G = (V, E)$ such that $E_1 \subseteq E \subseteq E_2$ which satisfies property P? **Comparability-graph-sandwich problem:** Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$, does there exist a comparability graph G with $G_1 \subseteq G \subseteq G_2$?, [12]. **Comparability-graph-deletion problem:** Given a graph $G = (V, E)$, does there exist a set $M \subseteq E$ of at most k edges deletion of which yields G a comparability-graph?, [22]. The problems comparability-graph-sandwich and comparability-graph-deletion are NP-complete. The problem of irreducibility is a special case with $G_1 = K_n \times K_m$ and $G_2 = [G_A^{tr}]$.

Given a graph $G = (V, E)$, we define the Γ-graph $G_\Gamma = (E, \Gamma)$ with an edge $e_1e_2 \in \Gamma$ if and only if $e_1\Gamma e_2$ in G. For a given sequence A, from the Γ-graph $G_\Gamma = (E_{r(A)} + E_{d(A)}, \Gamma)$ with contraction of edges in Γ-relation, we define the factor graph $G_{\mathfrak{F}}(A)$. The vertex set $G_{\mathfrak{F}}(V)$ in the factor graph contains an arc $v \in E_d$ or the extended sequence implication classes in $\mathcal{P}_A$ and $\mathcal{P}_A$–1. An undirected edge e_1e_2 belongs to $G_{\mathfrak{F}}(E)$ in the factor graph if and only if there exists a Γ-relation between nodes or set of nodes e_1 and e_2.

Let $(OIJ, E_{r(A)} + E_{d(A)})$ with $|E_d| = d$ be the H-comparability graph to the given $A \in SIJ$, $\mathcal{P}_A = \{P_1, \ldots, P_k\}$ and $E_{r(A)} = P_1 + \ldots + P_k + P_1^{-1} + \ldots P_k^{-1}$. The consequence graph $G_k(A) = (V_K, E_K)$ is defined as follows. The set of nodes is $V_k \subseteq E_{d(A)} + \mathcal{P}_A + \mathcal{P}_A$–1. Two edges e' and e'' from V_K are connected by an undirected edge of color $i \in \{1, \ldots, d\}$ when the removal of $e_i \in E_d$ forms a Γ-relation between e' and e'' or between the sequence implication classes they represent, respectively, i.e., $E_K = \{e'{}^i e''$ with color i

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3. Modeling

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$|e'\Gamma e''$ in $[G_A^{tr}] - e_i, e_i \in E_{d(A)}\}$. The set G_{Ki} represents the subgraph of G_K with i[th] color such that $G_K = G_{k1} + \ldots + G_{Kd}$.

For a given set $M \subseteq E_{d(A)}$, the reduction graph $G_{R_M}(A) = (V_{R_M}, E_{R_M})$ is defined by inserting into $G_{\mathfrak{F}}$ all edges from G_K which are colored from M and deleting the nodes which represent edges in M as $G_{R_M}(A) = [G_{\mathfrak{F}} + \text{U}_{e\in M}\, G_{K_e}] - M$. The removal of an edge result new Γ-relations. The consequence graph informs which sequence implication classes are merged by the removal of $e_i \in E_{d(S)}$. The reduction graph informs about the deletion of nodes from $G_{\mathfrak{F}}$ and addition of edges between the remaining nodes in $G_{\mathfrak{F}}$ which induce a new Γ-relation between sequence implication classes.

3. Modeling Decision Problems

Given a sequence $A \in$ SIJ on the same operation set OIJ for the open shop sequencing problem $O \parallel C_{max}$, we reformulate different versions of the following recognition problem. This question has not been completely answered since 1990's raised implicitly in [15]. Does there exist a polynomial time algorithm for the decision whether a given sequence is irreducible? What is the computational complexity status of this decision problem? The irreducibility and reducibility are complement decision problems. Reducing is the constructive optimization problem to the decision problem reducibility.

Irreducibility 1 *Is the sequence A irreducible?*

Reducibility 1 *Does there exist a sequence $B \in SIJ$ such that $B \prec A$?*

Reducing: Find a sequence B, if it exists, such that $B \prec A$.

A path w_A with vertex set $V(w_A)$ in the sequence graph G_A (equivalently, in the sequence A) is called maximal if there does not exist another path w^*_A in it with $V(w_A) \subset V(w^*_A)$. The set W_A of all maximal paths in A contains the longest path. A sequence A is reducible to another sequence B of the same format if and only if for all maximal paths w_B in B, there exists a maximal path w_A in A such that the inclusion $V(w_B) \subseteq V(w_A)$ is satisfied. If $B \prec A$, then there exists $w_B \in W_B$ with $V(w_B) \subset V(w_A)$ for some $w_A \in \mathbf{W}_A$. The decision whether a given sequence A is irreducible, or it is reducible or similar to another sequence B simply by using the related definitions takes exponential time.

Lemma 3.1 [18] Let be the operation sets such that $OIJ' \subseteq OIJ$. Then there exists a path w_A in the closure A^{tr} with $V(w_A) = OIJ'$ if and only if OIJ' is a clique in $[G_A^{tr}]$. Moreover, such a path is unique for the clique OIJ' in $[G_A^{tr}]$.

Since the subgraphs can be tested and the transitive closures can be constructed with the same time complexity $O(n^2m^2)$ for given sequences of the same size $n \times m$, Theorem 3.1

yields an answer in polynomial time to the question of irreducibility, reducibility or similarity between two given sequences.

Theorem 3.1 [3] Let A, $B \in SIJ$ be on the same operation set OIJ for $O||C_{max}$. The sequence A is reducible, strongly reducible or similar to the sequence B if and only if $[G_B^{tr}] \subseteq [G_A^{tr}]$, $[G_B^{tr}] \subset [G_A^{tr}]$ or $[G_B^{tr}] = [G_A^{tr}]$ for the corresponding H-comparability *graphs, respectively.*

For a reducible sequence, the reducibility can be proved with nondeterministic polynomial time. As this proof is constructive, such a procedure answers not only to the reducibility but also to the problem reducing. The problem reducibility is in N P and the problem irreducibility is in co-*NP*. Furthermore, if there exists a *NP*-test for irreducibility, then this problem is either polynomially solvable or *NP*-incomplete, as far as $P \neq NP$ holds.

The relation $\prec$ induces a partial order on the set of all sequences of the same operation set *SIJ*. The irreducible sequences are the minimal elements of this half-order on H-comparability graph $[G_A^{tr}]$ containing H-graph G_{IJ}, for given $A \in SIJ$. Let $A \in SIJ$ on the operation set *OIJ* for the open shop sequencing problem $O||C_{max}$. Let $[G_A^{tr}]$ be the corresponding H-comparability graph containing the H-graph G_{IJ} on the same *OIJ*. With this the problems of irreducibility and reducibility can be reformulated as the question of the existence of an H-comparability graph G as follows.

Irreducibility 2 Is there no H-comparability graph G with $G_{IJ} \subseteq G \subset [G_A^{tr}]$?

Reducibility 2 Does there exist an H-comparability graph G with $G_{IJ} \subseteq G \subset [G_A^{tr}]$?

The theory of reducibility concerns the reduction of a sequence through the reversion of an implication class in its transitive closure. One of the most important fundamental properties states that a sequence A whose H-comparability graph $[G_A^{tr}]$ is not prime is either reducible or is similar to an irreducible sequence B with $B \neq A$ and $B \neq A^{-1}$, [21].

A sequence can be obtained from every transitive orientation of an H-comparability graph. If the H-comparability graph G has a sequence orientation G_A^{tr}, then $G = [G_A^{tr}]$ is the H-comparability graph to a sequence $A \in SIJ$. A transitive orientation $T \in T_{[G_A^{tr}]}$ is called a sequence orientation if every diagonal edges in T is transitive. If a transitive orientation G_B^{tr} of $[G_A^{tr}]$ is not a sequence orientation, then some diagonal edges of $[G_A^{tr}]$ are not in the orientation G_B^{tr}, and $B \prec A$.

One may reduce the given sequence by reversing an implication class. One method to reverse the implication classes is the deletion of a single diagonal edge. Deletion of an edge from a transitive reduction can be done easily. However, if $[G_A^{tr}]$ can be transitively oriented such that neither e nor e^{-1} are transitive edges, then the edge $\hat{e}$ can be deleted and the graph $[G_A^{tr}] - \hat{e}$ is a comparability graph whose sequence orientation reduces A strongly. As transitive orientation of an H-comparability graph can be found in polynomial time and the number of diagonal edges for an n × m open shop sequencing problem is of order $O(n^2m^2)$, it can be tested in polynomial time whether a give sequence can be strongly reduced by deleting a diagonal edge.

Theorem 3.2 [21] If there exists $\hat{e} \in E_{d(A)}$ in $[G_A^{tr}]$ such that $[G_A^{tr}] - \hat{e}$ is a comparability graph, then ever transitive orientation of $[G_A^{tr}] - \hat{e}$ induces a sequence which strongly reduces the sequence A.

Thus a sequence $A \in SIJ$ can be strongly reduced to a sequence $B \in SIJ$ which cannot be further reduced by reversing an arbitrary implication class. This can be done in polynomial time. The H-comparability graph $[B^{tr}]$ is then either prime or there exist similar sequences to B other than B^{-1}. The set of all such reducible sequences cannot be obtained in polynomial time as the recombination of all implication classes is of size $O(2^k)$ for k implication classes and every edge may represent an implication class in the worst case. The reversion of only implication classes and their recombination does not generate the sequence space.

Not every recombination of the sequence implication classes of a sequence A is acyclic, and it yields a sequence B if it is acyclic. The set of all recombination of the sequence implication classes is sufficient. Therefore, taking sequence implication classes as basis for the space of sequences, we reformulate

Irreducibility 3 *Does every feasible recombination of the sequence implication classes of A produce a sequence B similar to A?*

Reducibility 3 *Does there exist a feasible recombination of the sequence implication classes of A where at least one diagonal edges of A is missing?*

Removal of one edge may not yield a strongly reduced sequence but with more than two edges removed may yield. A removable set with respect to a given sequence A is a set of undirected diagonal edges $M \subseteq E_{d(A)}$. The set M is called feasible if $[G_A^{tr}] - M$ is an H-comparability graph, and it is called feasibly extendable if there exists a feasible removable set M^* of diagonal edges of $[G_A^{tr}]$ such that $M \subset M^*$. The set M is called infeasible if it is not feasibly extendable. A removable set which is not feasible may not be necessarily infeasible. A removable set can be feasible and, in addition, feasibly extendable, too. We reformulate the problem of reducibility as follows.

Irreducibility 4 Is every removable set $M \subseteq E_{d(A)}$ in $[G_A^{tr}]$ infeasible?

Reducibility 4 Does there exist a feasibly extendable removable set $M \subseteq E_{d(A)}$ in $[G_A^{ir}]$?

In any $[G_A^{ir}]$, the implication classes which consist exclusively of diagonal edges can be deleted and the reduction through the reversion of group of implication classes can be done in polynomial time. A sequence is called normal if it cannot be reduced in either of these ways. Since any sequence can thus be reduced to a normal sequence in polynomial time, we restrict the space of sequences into the class of normal sequences [1].

For a reduction of a normal sequence by the reversion of a sequence implication class P_1 against the sequence implication class P_2 from the same implication class, all Γ-paths between them which contains at least one diagonal edge have to be cut keeping the comparability property. For feasibility of M each such path has to be broken, in order to avoid a connection in $[G_A^{ir}] - M$. Not all Γ-paths are destroyed if P_1 and P_2 belong to one connected component of $G_F - M$.

With the help of factor, consequence and reduction graphs, one can recognize a feasible removable set. These graphs inform existence of a transitive orientation of $[G_A^{ir}] - M$, if M terns out to be feasible. However, if this is not the case, the question remains how a none feasible but feasibly extendable set M can be expanded or to prove that the set M is not only none feasible but is infeasible.

The way to decide which additional diagonal edges should be added to a none feasible but feasibly extendable removable set in order to get a feasible removable set is another problem. One of the main issues while doing this process is that the diagonal edges, which belong to the irreducible sequences, must not be removed and the problem of merging two implication classes occurs when an edge play a role of conflict.

If there exists a path $W \subseteq V_{RM}$ from a sequence implication class $P_i \in V_{RM}$ to its reversion $P_i^{-1} \in V_{RM}$, we call it a conflict in $G_{RM}(A)$. The number $l \geq 0$ of the diagonal edges contained in the inclusion-minimal path W is called the order of the conflict. A direct conflict is a conflict with $l = 0$. Clearly, every conflict in G_{R_M} reflects a Γ-path in $[G_A^{ir}] - M$ from P_i to P_i^{-1}. For M, in order to be a feasibly extendable, all these conflicts must be dissolved and every one of these Γ-paths must be broken. A Γ-path between two edges in a graph will only be destroyed removing edges from this graph when at least one edge from the Γ-path is removed.

A diagonal edge is called stable if it is contained in every irreducible sequence of A. A diagonal edge is called trivial-stable if it in an extended sequence implication class. A stable diagonal edge which is not in an extended sequence implication class is called non-trivial-stable. If all edges in $[G_A^{ir}]$ are trivial-stable, then the irreducibility of a sequence is decidable in polynomial time.

We construct a sequence with two sequence implication classes such that a diagonal edge is contained between the classes which is transitive in each combination of these classes.

$$B = \begin{bmatrix} 2 & & & \\ 1 & 2 & 3 & \\ & 1 & 4 & 2 \\ & & 1 & \end{bmatrix} \quad C = \begin{bmatrix} 2 & & & \\ 1 & 3 & 4 & \\ & 2 & 1 & 3 \\ & & 2 & \end{bmatrix}$$

The H-comparability graph of the sequence B is uniquely orientable and this transitive orientation contains three diagonal edges $\{(o_{32}, o_{23}), (o_{22}, o_{33}), (o_{21}, o_{33})\}$. The set of regular edges form two sequence implication classes, say P_1 and P_2. The four inner nodes $\{o_{22}, o_{23}, o_{32}, o_{33}\}$ form the complete graph K_4 which contains trivial-stable edge $\{o_{32}, o_{23}\}$ and nontrivial-stable edge $\{o_{22}, o_{33}\}$. If we consider the combination $P_1 + P_2^{-1}$, then the resulting graph of the irreducible sequence C removes the third diagonal edge $\{o_{21}, o_{33}\}$ and converts the nontrivial-stable edge $\{o_{22}, o_{33}\}$ into trivial-stable edge.

A diagonal edge $e \in [G_d^{tr}]\,(A)$ is called magic-stable with respect to M if it does not lead to a direct conflict in $G_{R_{M^*}}$, with $M + e \subseteq M^*$, through a series of conflicts of order 1. It has not been found any sequence yet which contains a magic-stable edge. If one could prove that there exists no magic-stable edges, then the problem of irreducibility is polynomially solvable. Therefore, the difficulty of the problem of irreducibility depends upon the existence or non-existence of magic-stable edges in a given sequence graph. For recent algorithms and some conjectures on the complexity status of this problem based on the category of diagonal stable edges, we refer to [1].

4. Results on Irreducibility

We briefly review the status of irreducibility in the open shop $O||C_{max}$, see [8, 1] for details. A sequence B is optimal if $B \preceq A$ for all sequences A. It holds $C_{max}(A) = C_{max}(A^{-1})$. Moreover, $B \preceq A$ implies $B \preceq A^{-1}$, $B^{-1} \preceq A$ and $B^{-1} \preceq A^{-1}$.

An unavoidable set of sequences is computed for small formats, [19]. This approach formulates the dominance relation as a mixed integer programming. Among seven classes of all 3×3 irreducible sequences only three of them are unavoidable in the sense that these together with their reverses form unique minimal optimal set ensuring of at least one optimal sequence. A sequence of biggest rank five among all irreducible sequences cannot be missed. For the problem $O3|n = 2|C_{max}$, the minimal cardinality of two distinct potentially optimal solution sets is 3. Thus, the minimal set is not unique. The properties of sequence isomorphisms play important roles for the enumeration of open shop sequences, [14]. The sequence A is irreducible if and only if the sequence B is also irreducible in the same isomorphism class.

The set of all irreducible sequences for $O2||C_{max}$ is presented in [5]. For every irreducible sequence A, there exists a $k \in \{2, \ldots, m\}$ such that A can be obtained by a permutation of the columns of the sequence A_k . Any sequence in this class is irreducible and any sequence not belonging to this class reduces to a sequence belonging to this class. This can be done in polynomial time. Note that the latter sequence A_k is irreducible having only one implication class. This result also holds even if the sets of operations is partial.

$$A_k = \begin{pmatrix} 1 & \ldots & k-1 & k & k+1 & \ldots m \\ m-k+2 & \ldots & m & 1 & 2 & \ldots m-k+1 \end{pmatrix}$$

The asymptotic ratio of all irreducible sequences to all sequences for $O|n = 2|C_{max}$ is $\lim_{m \to \infty} \frac{m!(m-1)}{m!(m! + \sum_{k=1}^{m} \binom{m}{k} \frac{m!}{k!})} = 0$, motivating a research in the narrow search space.

There are several sufficient conditions for the reducibility which can be tested without computing the transitive closures. An $n \times m$ sequence $[a_{ij}]$, where $\min\{n, m\} \geq 3$, having an operation o_{ij} with $a_{ij} \geq nm - 2$ is strongly reducible. Any sequence with o_{ij} such that o_{ij} has at least one successor but none of its successors in row i or column j has a direct predecessor outside row i and column j, respectively, is strongly reducible. If A be a sequence such that each job i is first processed on the same machine j, then $B \prec A$ for some sequence B. We refer to [3, 14] for more polynomial time test conditions, enumeration algorithms and computations of irreducible sequences for small formats. A motivating result is that only a very small fraction of all sequences has been found irreducible.

The irreducible sequences for the open shop on an operation set with spanning tree structure are studied in [5]. They describe in detail the set of all locally minimal elements for $O2||C_{max}$. A necessary and sufficient condition for the irreducibility can be tested in polynomial time on tree-like operation sets, [20]. Given $OIJ \subseteq I \times J$, the bipartite graph is defined as follows. For each node $i \in I$ and $j \in J$, there exists an edge between them if and only if the operation $o_{ij} \in OIJ$. If this graph is a tree then the operation set is called tree-like. The problem of irreducibility in this structure is polynomially solvable as all diagonal edges in a sequence of this structure are trivial-stable [1].

A test whether a given sequence can be strongly reduced to another sequence by deleting an operation and reinserting it as a sink or a source, it has to be ensured that at least one path is destroyed in the former and no new path is created in the latter. This can be done in $O(n^2m^2)$ time and space, given an $n \times m$ sequence, [3].

An enumeration algorithm computes all irreducible sequences constructing inclusion minimal comparability graphs by successively inserting diagonal arcs into $[G_A]$, [4]. Each sequence in such a set is similar to exactly one sequence in this class, namely its reverse one. This algorithm constructs graphs G such that $G = [G_A^{tr}]$ for some sequence A. For $\min\{n, m\} \geq 2$, a lower bound on the number of diagonal arcs of an $n \times m$ sequence on the complete operation set is $\frac{1}{4}n(n-2)\binom{m}{2}$.

An insertion method for the enumeration of all sequences can be found in [6]. This method is modified and a new enumeration algorithm is presented in [3]. In their algorithm, a set of nonisomorphic sequences is computed and, thereafter tested for irreducibility. One sequence per isomorphic class is sufficient. They compute that the ratio between the number of irreducible sequences and all sequences decreases with growing n and m. There exist irreducible sequences that are not rank minimal, [3].

A generalized decomposition on irreducibility is introduced in [9]. For this, he considers an underlying 2×2 open shop problem by the assignment of an operation to each part. This approach invents the properties of irreducibility on the sequences of larger sizes based on similar properties on the sequences of smaller sizes.

The concept of irreducibility for arbitrary numerical input data, release time $r_i \geq 0$, due date $d_i \geq 0$, weight $w_i \geq 0$ and the processing times $p_{ij} \geq 0$ with respect to some additional regular objective functions γ can be found in [21].

A sequence A is called general-reducible to another sequence B, written as $B \preceq_g A$, if $C_i(B) \leq C_i(A)$ holds for all jobs i and all possible instances of numerical data N_D; release times, due dates, weights and processing times. Obviously, if $C_i(B) \leq C_i(A)$ for all jobs and all numerical data, then $\gamma(B) \leq \gamma(A)$ holds for all regular γ.

Theorem 4.1 Let $A, B \in SIJ$, and let the set $v_i(A)$ of operations contains the predecessors of an operation of the job i. Then it holds

$$B \preceq_g A \;\Leftrightarrow\; ([G_B^{tr}] \subseteq [G_A^{tr}]) \wedge (v_i(B) \subseteq v_i(A) \;\; for\ all\ i \in I).$$

Proof. First we show that the condition $([G_B^{tr}] \subseteq [G_A^{tr}]) \wedge (v_i(B) \subseteq v_i(A))$ for all jobs implies that $B \preceq_g A$. On the contrary assume that there exists a matrix P of processing times and a job k such that $C_k(B) > C_k(A)$. Let $w_B = (o_{ij}, \ldots, o_{kl})$ be a path in sequence B which ends at an operation o_{kl} of the job k with $\Sigma o_{uv \in w_B}\, p_{uv} = C_k(B)$. All nodes in the path w_B are contained in $v_k(B)$ and hence are contained in $v_k(A)$ because $v_k(B) \subseteq v_k(A)$ holds.

Because $B \preceq A$, the nodes $o_{ij}, \ldots, o_{kl}$ form a clique in A, and therefore, there exists a path w_A in A which contains all nodes of w_B and ends at an operation of job k. Then the inequality $\Sigma_{o_{uv} \in w_A} p_{uv} \leq C_k(A)$ contradicts to the assumption $C_k(B) > C_k(A)$.

On the other hand, we assume that $B \preceq_g A$ and show the validity of the condition $([G_B^{tr}] \subseteq [G_A^{tr}]) \wedge (v_i(B) \subseteq v_i(A))$ for all jobs. If $B \npreceq A$, then there exists an arc $e = (o_{uv}, o_{kl}) \in [G_B^{tr}]$ but not in $[G_A^{tr}]$. We define the matrix such that

$$p_{ij} = \begin{cases} 1 & \text{if } o_{ij} \in \{o_{uv}, o_{kl}\} \\ 0 & \text{otherwise} \end{cases}$$

This follows that $C_k(B) = 2$ and $C_k(A) = 1$, which is a contradiction.

If $v_k(B) \nsubseteq v_k(A)$ for a job k, then there exists an operation o_{uv} which is a predecessor of an operation of job k in B but not in A. If we define a matrix $P = [p_{ij}]$ as follows:

$$p_{ij} = \begin{cases} 1 & \text{if } o_{ij} = o_{uv} \\ 0 & \text{otherwise,} \end{cases}$$

the inequality $C_k(B) > C_k(A)$ follows immediately. □

A sequence A is called r-reducible to another sequence B, denoted by $B \preceq_r A$, if $C_{max}(B) \leq C_{max}(A)$ for all $P = [p_{ij}]$ and all $r = [r_i]$.

Along with a number of results in terms of the comparability graphs and precedence relations between operations, the interesting relations between the general-reducibility and r-reducibility are established.

Theorem 4.2 For any $A, B \in SIJ$, it holds $B \preceq_r A$ if and only if $B^{-1} \preceq_g A^{-1}$ holds.

Proof. If $B^{-1} \preceq_g A^{-1}$, then $v_i(B^{-1}) \subseteq v_i(A^{-1})$ for all jobs i and $[G_B^{tr}] \subseteq [G_A^{tr}]$. Furthermore, each induced subgraph from $v_i(B^{-1})$ of $[G_B^{tr}]$ is also an induced subgraph from $v_i(A^{-1})$ of $[G_A^{tr}]$. Because the maximal completion time of jobs in A corresponds to the maximal sum of one of the release times r_i and the maximal clique value of the induced subgraph from $v_i(A^{-1})$ of $[G_A^{tr}]$, the completion time of jobs in B must not be greater than that of in A.

On the other hand, if $B^{-1} \npreceq A^{-1}$ then it is easy to verify that there exists a matrix of processing times such that $C_{max}(B) > C_{max}(A)$.

Similarly, if $v_k(B^{-1}) \nsubseteq v_k(A^{-1})$ for some job k, then there exists an operation o_{uv} which is a successor of the job k in B but not in A. Then the following settings

$$p_{ij} = \begin{cases} 1 & \text{if } o_{ij} = o_{uv} \\ 0 & \text{otherwise} \end{cases} \text{ and } r_i = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

implies that $C_{max}(B) = 2$ and $C_{max}(A) = 1$. □

The definitions of strong reducibility, similarity and irreducibility have been extended similarly for general case and the r-reducibility.

5. Conclusions

In this paper we consider the problem: does there exist a polynomial time algorithm for the decision whether a given sequence is irreducible? Then we review all mathematical formulations and all solution approaches of the problem.

We mainly deal with the makespan objective function in the open shop. The polynomial solvability of this problem depends on the diagonal edge as all regular edges are stable.

One of the main issues raised from this research is whether the concept of stability of the diagonal edges in the corresponding H-comparability graph of the sequencing problem with makespan objective can be extended for other regular objectives. The results are likely to be similar, however, some extensions have to be made.

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