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A History of Fixed Point Theorems

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Abstract: In this paper, we present a brief historical account of the development of Fixed Point Theorems (*fpths*), There are about seven thousand results on *fpths* and this paper includes almost all initial generalizations and extensions of major and interesting results on *fpths*, through different types of mappings, like Contraction; Non-expansive: Multifunctions; Family of mappings, Set valued mappings, etc.

Keywords and Phrases: Fixed points, contraction mapping, non-expansive mapping, multifunctions, commuting mappings.

1. Introduction:

Let T be a self mapping on a set X. An element u in X is said to be a fixed point of the mapping T if $Tu = u$. The ***fpth*** is a statement which asserts that under certain conditions (on the mapping T and on the space X), a mapping T of X into itself admits one or more fixed points. History is a meaningful record of man's achievement & historical research is the application of scientific method to the description and analysis of past events. In this paper, we have tried briefly to present a history of ***fpths***. There are plenty of results on different cases of ***fpths*** and this paper is basically a survey work which deals with almost all earlier settings of ***fpths***, with suitable examples. This paper considers some short abbreviations like ***fpths***, ***Bfth***, ***Sfpth***, & **CMTH** for fixed point theorems, **Brouwer's** fixed point theorem, Schauder's fixed point theorem & Contraction Mapping Theorem respectively.

Historically, the most important result in the ***fpth*** is the famous theorem of **L.E.J. Brouwer** which says that every continuous self-mapping of the closed unit ball in $\mathbb{R}^n$, the n-dimensional Euclidean space, possesses a fixed point. This result, published by **Brouwer** (1910), was previously known to **H. Pointcaré** in an equivalent form. In 1986, **Pointcaré** proved the following result: If $f: E_n \to E_n$ is any continuous function with the property that, for some $r > 0$ and any $\alpha > 0, f(x) + \alpha\, x \neq 0, \| x \| = r$ then there exists a point x_0 , $||x_0|| \leq r$ such that $f(x_0) = x_0$. Now it is known that this assertion is equivalent to the ***Bfpth***. Another interesting fact is that the **Poincaré** theorem was also rediscovered by **P.Bohl** (1904). Also, **A.L. Cauchy** (1844) was the first mathematician to give a proof for the existence and uniqueness of the solution of the differential equations $\frac{dy}{dx} = f(x,y)$;

$y(x_0) = y_0$, when f is a continuous differentiable function. **R. Lipschitz** (1877) simplified **Cauchy's** proof using which is known today as the 'Lipschitz Condition'. Latter **G.Peano** (1890) established a deeper result, supposing only the continuity of **F. Eeano's** approach is more related to modern ***fpth***, which is used to obtain existence theorem.

Also, **E.Sperner** (1928) proved the combinatorial geometric lemma on the decomposition of a triangle, which plays an important role in the theory of fixed points. Due to its wide applications, there are plenty of results and still more results to come on ***fpths***. These are the most important tools for proving the existence and uniqueness of solutions to various mathematical models (differential, integral, ordinary and partial differential equations, variational inequalities). Other fields are Steady-state temperature distribution, Chemical reactions, Neutron transport theory, Economic theory, game theory, Epidemics, Flow of fluids, Optimal control theory, Fractals, etc.

2. Brouwer's Schauder's and Tychonoff's Epths:

Brouwer proved his famous theorem in 1912. As such theorems, where the spaces are subsets of are $\mathbb{R}^n$ are not of much use in Functional analysis where one is generally concerned with infinite dimensional subset of some function spaces. The first infinite dimensional ***fpth*** was investigated by **C.D. Birkhoff & O.D. Kellogg** (1922). There exist many proofs of the original ***Bfpth***. **Birkhoff & Kellogg** gave one proof of ***Bfpth*** with the assumption about convexity and compactness. **P.J. Schauder** in 1927 extended the **Birkhoff-Kellogg** theorem to metric linear space and in 1930, **Schauder** extended ***Rfpth*** to the result that every compact convex set in a Banach space has the fixed point property for continuous mapping, as well as that every weakly compact convex set in a separable Banach space has the fixed point property for weakly continuous mapping. An improvement of the last assertion was obtained by **M. Krein & V. Smulian** (1940).

Among the several proofs of ***Bfpth*** namely topological, analytic and degree theoretic, the proof of ***Bfpth*** depending on various definitions of the degree of a mapping (i.e. rotation of a vector field) were given by **Brouwer** (1910, 1912); **J.W. Alexander** (1922), **S. Lefschetz** (1926); **H Hopf** (1929); **J. Leray & J.Schauder** (1934); **E.Rothe** (1937); **S. Kakutani** (1943); **J. Leray** (1950); **M. Nagumo** (1951); **J.Dugundji** (1951); **V.L.Klee** (1960); **A. Granas** (1962); **P.Whittlesey** (1963); **J Cronin** (1964); **E. Fadell** (1970); **F.E. Browder & J.A.B. Potter** (1972); **Alexander** (1922), under the impression about ***Bfpth*** was proved for homeomorphism only, gave a new proof. The first continuos theorems applicable to non-linear problems were due to **Leray & Schaunder** (1934), known as "the Leray-Schaunder theorem", using the linearisation trick. But this theorem cannot be stated or applied without a knowledge of degree theory. Various attempts have been made to replace Leray-Schauder theorem by theorems in which the degree is not used. These theorems use conditions which are less general but more easily established in applications. The most useful result is that of **Schaefer** (1955) and **Browder** (1966). In 1926, **Lefcshetz** gave an extension of ***Bfpth*** to orientable n-manifolds without boundary, using what is called now the Lefschetz number $\wedge(f)$, and also extended this to the case of n-manifolds with boundary. This result was further extended to finite polyhedra by **Hopf** (1929). **Lefschetz** (1930,42) gave ***fpth*** for compact contractible sets. **E.Spanier** (1966) gave a modern proof of **Lefschetz**'s result in 1942. **S.Kinoshita** (1953). **E.F. Whittlesey** (1963); **R.H. Bing** (1969), **E. Fadell** (1970) have given ***fpths*** on contractible sets with many interesting examples and references.

Another proof of ***Bfpth*** depending on classical method (Calculus & determinants) were given by **Birkhoff & Kellog** (1922) the most proof of ***Bfpth*** is by simplicial subdivision of an n-simplex due to **B Knaster, C. Kuratwoski & S. Mazurkiewicz** (1929); **C. Kuratwoski** (1933) and **L.M. Graves** (1946). **Hirsch's** (1963). **Hirsch's** proof plays an important role in the algorithms for fixed point in *Bfpth*. Unlike **CMT**, **Bfpth** .does not give any computational scheme for obtaining a *fpth*. However, in 1967, **H. Scarf** gave some sort of algorithm for computing a fixed point of a mapping with some additional conditions. This gives a new proof of *Bfpth*. We find many other algorithms in a book edited by **S.Karamardian** (1977). **H. Robbins** (1967) gave compliments of ***Bfpth***. The most interesting generalization of ***Bfpth*** is the so called **K. Borsuk- Ulam theorem** and Borsuk's theorem about antipodal points the proof of **Borsuk-Ulam theorem** is that of **M.D.Meyerson & A.L Wright** (1979).

The condition of compactness in ***Sfpth*** was a very strong condition. As many problems in analysis do not have compact setting, it was natural to modify this theorem by relaxing the condition of compactness. **A.N. Tyechonoff** (1935) proved a generalizations of *Sfpth* for the case of compact operators on locally convex linear spaces, and **M. Hukuhara** in 1950. **Tychonoff** need simplicial subdivision method to prove his ***fpth***. An interesting extension was obtained by **Browder** (1959), under some deep conditions for the iterations of the mappings. **H.H. Schaeffer** (1955) gave a slight but very useful variation of ***Sfpth*** for compact mapping on Banach space. Browder extends ***Sfpth*** for the compact sets. **Rothe** extended it in 1937 and it was latter proved by **Potter** (1972) to the more general case of convexity, adopting the argument of **Browder Branas** (1962) considered a general region for the same theorem and need the method of **Potter**.

The generalization of ***Sfpth*** concerning set valued mappings was proved by **Ky Fan & I. Glicksbert** (1952). A proof of **Tychonoff's** theorem using the fixed point property for the Hilbert cube is given by **N. Dunford & J.T. Schwartz** (1958). An interesting generalization of both ***Sfpth*** and Tychonoff's theorem was obtained by **Ky Fan** (1961). The extension given by **Ky Fan** depends upon a lemma which is essentially the infinite dimensional version of the **Knaster-Kuratowski-Mazurkiewicz** theorem (1929). Also, a new and important step in extending the **Sfpth** to more general class of mappings was made by **G.Darbo** (1955). In 1967, **V.N.Sdovski**, using a new measure of non compactness, proved a generalization of the ***Sfpth*** for mappings which are known as condensing or densifying. **M Volato** (1953) extended ***Sfpth*** to mappings without strong relation with compactness. **G.Jones** (1953) extended ***Sfpth*** which was generalized by **V. Istrătescu** (1978). Also, **M.A. Krasnoselskii** (1953). **M.Altman** (1957) and **W.V. Petryshin** (1967) proposed some conditions for the mapping for the computation of fixed point. **Altmann** proved his ***fpth*** using ***Sfpth***. A proof of altmann's theorem using the concept of degree theory was found by **M.S. Berger & M. Berger** (1968). Latter on, **M. Edelstein** (1966) has generalized the theorem of **Krasnoselskii**. **Browder** (1970) used the homeomorphism as initial condition. **Sadvskii** (1972) developed the generalized degree theorem who extended the concept of degree to the class of limit compact operators. **B.V. Singbal** has shown that the ***Sfpths*** is true for locally convex spaces in its full generality, using a technique due to **Nagumo** (1951). A paper of **K.L.Stepaneek** (1957) includes different kind of generalizations of ***Sfpth***. **M.G. Krein & M.A.Rutman** gave results related to the transition from nonlinear to linear problem. Also **N. Aronszajn** (1942) gives general regular condition on T sufficient to establish that the set of its fixed points is a homeomorphic image of the intersection of decreasing sequence of absolute retracts.

3. Banach Contraction Principle:

There are ***fpths*** that can be approached without any combinatorial topology as background. One result applies to contraction that is, distance diminishing mappings of a complete metric space into itself. The concept of Banach space was introduced by **Stefen Banach** and obtained a ***fpth*** for contraction mappings in 1922, famous as Banach Contraction Principle **(BCP)** or **CMTH**. Recently there have been numerous generalization of **BCP** by weakening its hypothesis while retaining the convergence property of the successive iterates to the unique fixed point of the mapping. One result is due to **R.Caccioppoli** in 1930. **BCP** is very useful in the existence and uniqueness theories. **S. C. Chu & J.B. Diaz** in 1964, 65 gave one generalization of **BCP**. The result due to **Chu & Diaz** has been further extended by **V. M. Sehegal** (1969). **Krasnoseiskii** generalized **CMT** in 1964. Also **E. Rakotch** (1962), **D.Boyd & J.S.W. Wong** (1969) and **Browder** (1968) have attempted to generalize **BCP** by replacing the <u>Lipschitz constant</u> by some real valued function whose values are less than 1. It is noted that the class of **Boyd & Wong** is strictly larger than the class of **Rakotch**, **A.Meir & E.Keeler** (1969) has generalized **BCP** for the case of weakly uniformly strict contraction.

Some significant generalizations of **Boyd & Wong** theorem are those due to **S.Park & B.E.Rhoades** (1981); **S.L. Singh & S. Kasahara** (1982); **S.A. Hussain**

& V.M. Sehegal (1975); **S.P. Singh & B.A. Meade** (1977); **J. Jachymski** (1994) and **R.P. Pant** (1996). Similarly, some of the well known generalizations of the **Meir & Keeler** theorem are those due to **S.Park & J. S. Bae** (1981); **Park & Rhoades** (1981); **I. H. N. Rao** (1985); **Pant** (1986); **G.Jungck** (1986); **Jungck, K.B. Moon, Park & Fhoades** (1993). **Edelstein** in 1962 has shown that compactness of metric space X will guarantee a unique fixed point for a contractive mapping on **S. D. F. Bailey** (1966) extended this result for contraction mappings. **Edelstein** (1961) gave a local version of the **CMTH**. The generalization of the **BCP** to a class of mappings on C-chainable space is due to **Edelstein (1966)** and **Sehgal** (1969). **S.P. Singh & Zorzitto** (1971) have obtained more general results by replacing the metric by some real valued function with continuity condition. **Wong** in 1972 proved the result for lower semi continuous mapping **on a compact** Hausdorff space.

R. Kannan (1968); **Hussain & Sehgal** (1975) and **J.V. Caristi** (1975) have considered several generalizations of contraction mappings. **Rhoades** (1977) have given more results on contractive mappings and its generalizations. **G.E.hardy & T.D. Regers** proved a ***fpth*** concerning Kannan-Reich type mapping in 1972. **Hussain & Sehgal** (1975) proved a ***fpth*** which generalizes the Kannan-Reich's and Ciric type of generalized **CMTHS**. An extension of **Hussain & Sehgal's** result was obtained by **Singh & Meade** in 1977. Both results deal with common ***fpth*** of a pair of mappings. **Singh & Meade** proved ***fpth*** under the assumption that Φ is upper semi continuous. We have some generalizations due to **S.Reich** (1971), **Maia** (1968), **Singh** (1970) and **Hardy & Rogers** (1973) for two mappings on a complete metric space. **Kannan** (1969) proved a theorem in which the completeness of the space is not required. **Kannan's** results have been generalized by **Singh** in 1969. We have fixed point results from **L.P. Belluce & W.A. Kirk** (1969) and **Fukushima** (1970) on diminishing orbital diameters. **S.B. Nadler** (1969) represented the extension of the **BCP** to the case of set-valued contraction mapping.

Browder (1965) gave a result which did not assume compactness. The result remains true for the case of a uniformly convex Banach space. We have some more results on ***fpth*** due to **Browder** (1965), **Kirk** (1965) or **K. Goebel** (1969). Converses of **CMTH** have been discussed by **P.R.Meyers** (1967), **L.Janos** (1967) and **Edelstein** (1969). In 1970, **L.F. Guseman Jr**. gave a ***fpth*** that was first proved by **Sehgal** in 1972. In 1976, **Caristi** rediscovered independently a **fpth** which turned out to be an abstraction of a Lemma of **E.Bishop & P.R. Phelps** (1963). Its applications were discussed by **Kirk & Caristi** (1974); **Kirk** (1975). **D. downing & Kirk** (1977). **Caristi's** proof involves transfinite induction. The proof of **Caristi's** theorem is given by **Kirk** (1976) and implicit in a paper by **A. BrΦnsted** (1974). The localization of contractive mapping condition was given by **R.D. Holmes** (1976). **David Hilbert** (1895) introduced a metric which is interesting in its own right but also applications to analysis, as was proved by **Birkhoff** (1957) and by

U.Urabe (1956) . It is possible to show that, after a suitable change of the metric, the mapping is actually a contraction mapping. The first result of this type seems to be that of **C Bessage** in 1959. Generalizations of **Bessage's** result as well as of related results were obtained by **Meyers** (1970). **S.Leader** (1977), **I.Rosenholtz** (1976), etc. Various applications of the **CMTH** have been given in **Copson** [pp 111-136]; **Heuser** [pp 17-23]; **Martin** [pp 114-117]; **Pitts** [pp 88-89]; **Simmons** [pp 339-340], **Singh** [pp 10-11], **Smart** [pp 41-52].

4. FPTHS For Non- Expansive Mappings:

As the fundamental properties of contraction mapping do not extend to non-expansive mappings, so it is of great importance in applications to find out if non-expansive mappings have fixed points. The study of non-expansive mappings has been one of the main features in recent development of fixed point properties. Contractive mappings, isometries and orthogonal projections are all non-expansive mappings. The problem of the existence of an extension for non-expansive mappings on $\mathbb{R}^n$ was first considered by **M.D Kirszbraun** (1934) .**M. Markov & Kakutani** in 1938 referred to simultaneous fixed points of suitable families of continuous mappings of compact convex subset of a topological vector space into itself. The mappings in the Markov-Kakutani theorem must satisfy a condition close to line .linearity .the first important result in the theory of fixed points for non-expansive mapping was obtained by **R. de Marr** in 1963 who has proved an interesting extensive of the famous result of Markov-Kakutani. This result greatly influenced to the development of fixed point theory . **R.de Marr** gave various ***fpths*** concerning families of mappings which need not to be affine, using downward induction argument

Kirk in 1963 proved a ***fpth*** using a characterization of reflexive due to **V.Smulian** and a concept (normal structure) of **M.S. Brodski & D.P. Milman** in 1948 to prove the ***fpth*** for mapping which do not increase distances. **Brodski** and **Milman** gave conditions under which a convex set in a Banach space has a point invariant under all isometric self mappings. **E.W. Cheney & Goldstein** in 1959 have given the results for a non-expansive mapping in a metric space. **Kirk** in 1965 proved ***fpth*** for a non-expansive self mapping of a bounded, closed, convex subset of a reflexive Banach space. An immediate consequence of Kirk's theorem was proved independently by **Browder (1965)**; **D.Gohde & Kirk** (1965). They proved that a non-expansive self mapping of a bounded closed convex subset of a uniformly convex Banach space has a fixed point. Latter on, **Kirk** in 1970 proved the same result under slightly weaker assumptions that the space is reflexive and a bounded closed convex subset has normal structure.

Theorems for approximating fixed point concerning the convergence of some sequence defined using iteration techniques for general non-expansive mappings are given by **Browder & Petryshin** (1966, 67); **Edelstein** (1966), **Diaz & F.T. Metcalf** (1967); A. **Pazy** (1971); **S. Kaniel** (1971); **H.F. Senter & W.G. Dotson Jr.** (1974); **Reich** (1976); **Browder** (1976). In recent works include papers of **Browder** (1966) dealing with the relationship of non-expansive mappings to the theory of monotone operators in Hilbert space and, in more general setting to the theory of J-monotone operators and accretive operators. **Browder & Petryshin** (1966) proved ***fpth*** for non-expansive and asymptotically regular mappings in Banach space. Petryshin (1966) proved that the class of demi compact operators is more general than the compact operators. In 1967, **Browder** proved a well known result in strictly convex Banach space. **Browder**'s result is false in the most general case of Banach space and a beautiful example for it is due to **de Marr**. The weak convergence of successive approximations for a non-expansive mappings is dealt by **Z. Opial** in 1967.

Edelstein in 1966 proved some interesting results in uniformly convex Banach spaces. In 1972, he gave the original notion of asymptotic centre and proved some of its properties and used it to prove a ***fpth*** for a class of mappings which includes non-expansive mappings. A semi contraction is a generalization obtained by intertwining of non-expansive mappings with strongly continuous mappings. **Browder**'s result is false in the most general case of Banach space and a beautiful example for it is due to de **Marr**. The weak convergence of successive approximations for a non-expansive mappings is dealt by **Z. Opial** in 1967.

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(Mann type) of a quasi-non-expansive mappings converge to a fixed points of the mapping. **Bose & Mukherjee** in 1981 considered approximation of fixed points of generalized non-expansive mapping.

Reich in 1976 considered the iteration scheme for non-expansive mappings in uniformly convex space with a Fréchet differentiable norm. For a commuting family of non-expansive mappings, we have results from **Belluce & Kirk** in 1966 and **P.K.F. Kuhfitty** in 1980. **Ng** in 1970 gave a result for non-expansive mapping on cluster set. **Caristi** (1975,76) ; **Lim** (1980); **Downing & Kirk** (1977) and **Yanagi** (1980) considered inward mappings and proved ***fpth*** for such mappings (both single valued and multivalued). **B.Halpern** in 1965 first considered inward mapping in his Ph.D. thesis. **Goebel & Kirk** in 1972 proved **fpth** for asymptotically non-expansive mappings. **S.C. Bose** in 1978 proved a ***fpth*** as an extension of **Opial's** convergence theorem in 1967 for non-expansive mappings to the class of asymptotically non-expansive mappings. In 1982. **G.B. Passty** extended **Bose's** result. **J. Lindenstraus** in 1975 has constructed an example of a non-expansive mapping defined no a closed, convex and bounded set of a **Banach** space such that the sequence does not converge. An interesting class of non-expansive mappings for which the **Cauchy-Picard** sequence of iterations converges, was discovered by **J.J. Moreau** in 1978. His result refers to the **Hilbert** space nonlinear mappings. The extension of **Moreau's** result to the case of uniformly convex **Banach** spaces was obtained by **B.Beauzamy** in 1978. **D. de Figueiredo** and **Petryshin** in 1967 computed fixed point for class of non-expansive mappings.

5. Fpths For Many Values Mappings :

If each point *x* of a set *m* is mapped onto a set U(x) then U is called a many valued mappings (multi functions). The study of fixed point problem of multifunctions was initiated by **Kakuntani** in 1941, when U(x) is compact & convex in finite dimensional spaces. This is the extension of ***Bfpth*** to the point compact convex set-valued mappings on a compact convex set in Euclidean space. It was extended to infinite dimensional Banach spaces by **Bohnenblust & Karlin** in 1950 by a method similar to **Schauder**'s proof in 1930 and to locally convex spaces by **Ky Fan** in 1952 and by **Glicksberg** in 1952. **S. Eilenberg & D.Montgomery** in 1946 allowed U(x) to be the acyclic (homologically trivial); so did **de Begle** (1950) and **Górniewiez & Granas** (1970). **Ky Fan** in 1961 merely requires U(x) to be compact but here U(x) must depend continuously on *x*. **R.E. Smithson** in 1965 considers cases where U(x) is finite-valued. Among two significant sets of methods in the fixed points of multivalued mappings, the first homological method started in 1946 by **Eilnberge & Montgomery** whereas the second method started in 1935 by **J. Von Neumann.**

Fan's result also generalizes **Schauder-Tychonoff's theorem. J.P. Dauer** in 1972 have considered ***fpth*** for multifunctions providing natural settings for many problems in Control theory involving differential equations. The developments of geometric ***fpth*** for multifunctions was initiated by **Nadler Jr.** In 1969 and subsequently persued by **J.T.Markin** (1973); **Browder** (1968); **N.A. Assad & Kirk** (1972); **Goebel, E. Lami-Dogo** (1973) and others. **S.C.J. Himmelbert** in 1972 generalized **Fan**'s result and **Sehgal & E.A. Morrison** has further generalized **Himmelberg**'s work in 1973. **R.L. Plluket** (1956) and **L.E. Ward Jr.** (1961) have shown the spaces which have fixed point property for multivalued contraction mappings. These theorems do not place servere restrictions on the images of points and, in general, the space is required to be complete metric space. **Fleischman** (1970) and **Smithson** (1972) have given various recent related contributions.

Assad & Kirk and **Markin** worked on multivalued contraction, while **Smithson** worked on contractive multifunctions in 1971, which extends **Edeltein's** ***fpth*** for contractive single valued mappings to multifunctions. **Reich** (1971) and **Bose & Mukherjee** (1977) have extended the work of **Nadler Jr.** and obtained ***fpths*** for generalized multivalued contraction mappings. In 1980, **Bose & Mukherjee** proved ***fpth*** wihich's a generalization of a theorem of **Iseki**'s result for single valued mappings. They also gave a generalization of a theorem of **Wong** to multi-valued functions. **S.Itoh & W.Takahashi** (1977) and **Yanagi** (1980) proved ***fpth*** of multivalued non-expansive mappings on non-convex domain, more precisely on star-shaped domains. **Downing & Kirk** (1977) proved a ***fpth*** in conjunction with an elegant approach of **Goebel**. For single valued mappings, it is known that a non-expansive mapping is a pseudo-contractive. **Dowing & W.O.Ray** (1981) showed that the same is not true in the set valued case. **J.P. Aubin & J.Siegel** in 1980 proved a ***fpth*** that has relevance in Control theory. For the case of pseudo contractive mappings. **Browder & Petryshin** in 1967 gave methods to compute fixed points. We have some more results for ***fpth*** of pseude-contractive mappings due to **N.G. Crandall & Pazy** (1969); **T.Kato** (1970); **J. Reinermann & Schoneberg** (1976), **Kirk & R.Schoneberg** (1977) and **Kirk & Ray** (1979).

6. Special Cases on FPTHS:

The first theorem regarding to the continuity of fixed points of contraction mappings was proved by **F.F. Bonsall** in 1962. Subsequently, **Nadler Jr.** In 1968 obtained results concerning sequences of contraction mapping and also gave an application suggested by **Dorroh**. The first result about the convergence of sequence for the case s =1/2 in (0.1) was obtained by **Krasnoselski** (1955). This result was extended by **Schaefer** (1957) by proving the convergence for any fixed s in (0,1) and then weakening the assumptions about the mapping. The second result of **Schaefer** was extended by **Edlstein** (1966) to the case of rotund spaces.

The first result about fixed points for family of mappings was proved by **Markov** in 1936, depending on **Tychonoffs** theorem (1935). **Kakutani** in 1938 found a direct proof of **Markov**'s result and also proved a ***fpth*** for groups of affine equicontinuous mappings. Thus, for affine mappings, fixed points have a natural geometric significance and we have famous Markov-Kakutani theorem. Latter **Edward** in 1965 gave a result which includes both Markov-Kakutani theorem and the case of a solvable group of affine mappings. **M.M.Day**'s theorem in 1961 is still more general. **Day** gave a connection between amenability of a group of mappings of semi groups and fixed points. An important extension of **Markov-Kakutani** results was given by **C.Ryll-Nardzewski** in 1967. It is interesting to note that the proof given by **Nardzewski** was probabilities in nature. Latter he found a proof which do not use probabilistic ideas, and **Asplund & Namioka** in 1967 have found a simplified proof. We find some more fixed point results on families from **N.W. Riekert** (1967); **F.F. Greenleaf** (1969); **R.E.Huff** (1970) and **T.Mitchell** (1970).

The first extension of topological fixed point theory of continuous mappings to the case of set-valued mappings was made by **John Von Newman** in 1937 in connection with the proof of the fundamental theorem of Game theory. The behaviour of the fixed points of set valued mappings has been considered by **Nadler Jr.** (1969) and **Markin** (1973). Both established conditions implying the strong convergence of the fixed points of a sequence of set valued contractions. These results were extended further by **Nadler & Fraser** in 1969 and **H. Covitz & Nadler**. Using **Urysohn**'s lemma and ***Bfpth***, we obtain ***fpth*** for a class of set valued mappings which generalizes and extends the results of **Kakutani**, **Bohnenblust**, **Karlin**, **Blicksberg** and **Fan**. Also, using the Liapunov function, **Sehgal & Smithson** were able to extend many results from the single valued mappings to set-valued mappings. In 1966, **Ky Fan** gave an analytic formulation of his ***fpth***, using so called quasi-concave functions. The weak convergence of fixed points of set-valued non-expansive mappings in a Banach space was obtained by **Markin** in 1978 who used it to obtain a stability result for generalized differential equations. We have more results related to this mapping due to Ng (1968); **M. Furi & A. Vignoli** (1969); **Singh & Russel** (1969); **Singh** (1970); **Reich** (1971); **G.W. Collins** (1973); **Dube & Singh** (1973).

Eldon Dyer (1954), **Allon Schields** (1955) and **Lester Dubins** asked the following question that whether any two continuous commuting self functions defined on [0,1] have a common fixed point or not ? An interesting problem related to fixed points for families of commuting mappings on [0,1] was noted by **Isbel** in 1957. This was a main source of inspiration for a decade and several mathematicians tried to solve this problem. However, in 1967, **W.M. Boyee & Huneke** independently disproved the conjecture. **Kakutani** produced an example that this theorem does not hold for infinite dimensional spaces. **Ryll-Nardzewski** in 1966 proved a more general form of common fixed points in which norm topology is replaced by any locally convex topology. **Folkmann** in 1966 gave results for common ***fpth***. In 1975, **Hussain & Sehgal** proved a common ***fpth*** and later on, it was improved upon by **Singh & B.A. Meade** in 1977 in a slightly different form. An iteration scheme which converges strongly in one case and weakly in another case to a common fixed point of a finite family of non-expansive mapping were obtained by **Kuhfitting** in 1981. Another iteration scheme converging weakly in more general setting then Kuhfitting's were proved by **R.K.Bose & D.Sahani** in 1984. The study of common fixed points of non-commuting generalized contraction mappings was initiated by **S.Sessa** in 1982, introducing the notion of weakly commuting mapping.

In ordered Banach space, we find fixed point results due to **Krasnoselskii** (1964); **J.A. Gatica & H.L. Smith** (1977) ; **Gustufson & Schmitt** (1976); **R.W. Leggett & L.R. Williams** (1980); **H. Amann** (1976) ; **Turner** (1975). **Amann** has shown that using the asymptotic behaviour of a mapping, the existence of fixed points of the mapping can be derived. **Williams & Leggett** obtained some multiple ***fpths*** which they applied to problems in Chemical reactor theory. The theory of measure of non-compactness and densifying operators have applications in general topology, geometry of Banach spaces and the theory of differential equations. The most widely used measure of non-compactness on metric spaces are the α-measure introduced by **Kuratowskii** (1958) and used by **Darbo** (1955); **Furi & Vignoli** (1969); **Nussbaum** (1970); **Petryshin** (1971) and others. In 1972, **Sadovskii** introduced the concept of a condensing operator for mappings defined on subsets of a Banach space ad there by obtained a generalization of ***sfpth***. **Petryshym** (1971) points out that α and the Banach space are different although they have a good deal in common. We find some further results involving densifying mapping by **Diaz & Metcalf** (1969); **Kirk** (1971); **Singh & Guerra** (1971); **Singh & Riggio** (1972); **Singh** (1972); **Yadav** (1972); **Singh & Yadav** (1973). **J. Halle** in 1974 obtained a result concerning the continuous dependence of fixed points for densifying mappings.

In many problems of analysis, one encounters operators which may be expressed in the form $T=A+B$, where A is a contraction mapping and B is compact and T itself neither of these properties ? Thus neither the **BCP** nor the ***Sfpth*** applies directly and it becomes desirable to develop fixed point for such situations. **Krasnoselskii** first introduced well known theorem of this kind in 1955. In 1967, **Zabreiko & Krasnoselskii** proved the stronger variation of **Kransnoselskii**'s theorem. **Nashed & Wong** in 1969 gave extensions of **Kransnoselskii's** theorem. Also we have related results due to **Sadovskii** (1967); **Nussbaum** (1969); **Furi & Vignoli** (1970); **Srinivasa Chargulu** (1971); **Singh** (1973). **Browder** (1965) gave example which illustrates that a non-expansive mapping under perturbation by a compact mapping loses it fixed point. Then the question arises, does a non-expansive mapping have a fixed point under any perturbation ? This question has answered affirmatively by **Edmunds** (1967). **Zabreiko, Krasnoselskii & Kachurovskii** (1967) and **Reinermann** (1971).

A comprehensive account of history, properties and applications of convex functions upto 1946 has been given by **Beckenback** (1948). **J.W. Green** (1954) published an elegant article on convex function. He remarked that convex functions have been consistently of value of Analysis, Geometry and other branches of mathematics, notably Mathematical Economics. **Belluce & Kirk** (1969) **and Singh & Veitch** have given fixed point results for convex functions. **Poljak** (1966) gave a useful fixed point result for strongly quasi convexity that is natural assumption for minimization problems. The concept of associating a distribution function with the point pairs was first introduced by **K. Menger** (1942) under the name "Statistical metric spaces". Shortly, it is named as "Probabilistic Metric Spaces" or **PM-Spaces.** The important paper of **B.Schweizer** and **A.Sklar** in 1960 has given a new impulse to the theory of **PM-Spaces**, introducing the notion of convergence in **PM-Spaces.** The notion of a contraction mapping defined on a **PM-Space** was first defined by **Sehgal** who has also proved that every contraction mapping in a complete **Nebger** Space has a unique fixed points. The notion of C-chainabililty for **PM-Spaces** is first defined by **Sehgal** and **Barucha-Reid**. The notion of **Kuratowski** probabilistic measure of non-compactness was introduced by **Bocsan & Constantin** in 1873, under the name of Kuratowski functions. **V. Istrătescu** in 1974 has given a detailed discussion of **PM-Spaces.**

Finally the existence theorems for ordinary differential equations are due to **Cauchy-Lipschitz**, **Nirenberg** (1953) ; **Peano**, **Picard & Bass** (1958); **Stokes** (1960); **Cronin** (1964); **Edwards** (1965); **Jones** (1965); **Güssefeldt**(1970); **Browder** (1973). Similarly, the existence theorems for partial differential equations are due to **Caccioppoli** (1930) and **Nemyckii** (1936). A fixed point method for finding periodic solutions of dynamical problems was used by Poincare in 1912. In 1920, **Lawson** gave a formulation as an implicit function theorem in abasract spaces and this result is universal, in the sense that the convergence is proved for any arbitrary initial value. A local version of Lawson's theorem was obtained by **Hildebrandt & Grave in** 1929.

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